ANALYTICAL OBSERVATIONS
[*regarding continued fractions.*]

[E553]

Opuscula analytica 1, 1783, p. 85-120

1. Among the different matters, which I have commented on concerning continued fractions from time to time, this form may be seen to be noteworthy :

$$1+\cfrac{n}{2+\cfrac{n+1}{3+\cfrac{n+2}{4+\cfrac{n+3}{5+\cfrac{n+4}{6+\text{etc.}}}}}}.$$

the value of which, whenever n is a whole number, can be shown in the following way, with e denoting the number, of which the logarithm is unity, so that there shall become

$$e = 2,718281828459045:$$

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\text{etc.}}}}} = \frac{1}{e-2},$$

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{etc.}}}}} = e-1,$$

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\text{etc.}}}}} = 2,$$

$$1+\cfrac{4}{2+\cfrac{5}{3+\cfrac{6}{4+\cfrac{7}{5+\text{etc.}}}}} = \frac{9}{4}$$

$$1+\cfrac{5}{2+\cfrac{6}{3+\cfrac{7}{4+\cfrac{8}{5+\text{etc.}}}}} = \frac{52}{21},$$

$$1+\cfrac{6}{2+\cfrac{7}{3+\cfrac{8}{4+\cfrac{9}{5+\text{etc.}}}}}=\frac{365}{136},$$

$$1+\cfrac{7}{2+\cfrac{8}{3+\cfrac{9}{4+\cfrac{10}{5+\text{etc.}}}}}=\frac{3006}{1045},$$

$$1+\cfrac{8}{2+\cfrac{9}{3+\cfrac{10}{4+\cfrac{11}{5+\text{etc.}}}}}=\frac{28357}{9276},$$

where the way certainly arises for a special use, as the first two forms involve the transcending number *e*, while all the following are expressed by rational numbers.

2. This may be seen therefore to be even more amazing, because also the preceding cases, where either zero or negative numbers may be put in place for *n*, may be held by rational values, indeed for which cases the form of the continued fraction itself may be broken off. Indeed there will be :

$$1+\cfrac{0}{2+\cfrac{1}{3+\cfrac{2}{4+\text{etc.}}}}=1,$$

$$1-\cfrac{1}{2+\cfrac{0}{3+\cfrac{1}{4+\text{etc.}}}}=\frac{1}{2},$$

$$1-\cfrac{2}{2+\cfrac{1}{3+\cfrac{0}{4+\text{etc.}}}}=-\frac{1}{5},$$

$$1-\cfrac{3}{2-\cfrac{2}{3-\cfrac{1}{4+\cfrac{0}{5+\text{etc.}}}}}=-\frac{19}{14},$$

$$1-\cfrac{4}{2-\cfrac{3}{3-\cfrac{2}{4-\cfrac{1}{5+0}}}}=-\frac{151}{37},$$

$$1-\cfrac{5}{2-\cfrac{4}{3-\cfrac{3}{4-\cfrac{2}{5-\cfrac{1}{6+0}}}}}=-\frac{1091}{34}.$$

Therefore I am considering how to show, on considering this matter, by which rule both these values as well as the preceding themselves may be able to be combined together. But initially it will help to have the method set out, by which these values may be able to be found.

3. Therefore I note in the first place, if for some number n the value of the continued fraction may be indicated thus :

$$f(n)=1+\cfrac{n}{2+\cfrac{n+1}{3+\cfrac{n+2}{4+\cfrac{n+3}{5+\text{etc.}}}}}$$

there become :

$$f(n+1)=\frac{n(f(n)+1)}{f(n)+n-1},$$

the truth of which is evident in the values indicated, since there shall be :

$$f(1)=\tfrac{1}{e-2},\ \ f(2)=e-1,\ \ f(3)=2,\ \ f(4)=\tfrac{9}{4},$$
$$f(5)=\tfrac{52}{21},\ f(6)=\tfrac{365}{136},\ f(7)=\tfrac{3006}{1045},\ f(8)=\tfrac{28357}{9276}$$

and for the preceding :

$$f(0)=1,\ \ f(-1)=\tfrac{1}{2},\ \ f(-2)=-\tfrac{1}{5},\ f(-3)=-\tfrac{19}{14},$$
$$f(-4)=-\tfrac{151}{37},\ \ f(-5)=-\tfrac{1091}{34},\ \ f(-6)=-\tfrac{7841}{887}.$$

This relation between two neighbouring values does not stand in the way of intermediate values, unless they shall be transcending, as in the cases $n=1$ and $n=2$. For on putting

$$n=0$$

there becomes

$$f(1)=\frac{0(1+1)}{1+0-1}=\frac{0}{0},$$

which expression cannot be applied to the value

$$\frac{1}{e-2},$$

thence even if this may be unable to be elicited. Then on putting

$$n=2$$

there will be produced :

$$f(3)=\frac{2\left(f(2)+1\right)}{f(2)+1}=2,$$

thus so that the value

$$f(2) = e - 1$$

may be arrive at here without computation.

4. But the investigation of these values cannot be seen without a little hardship ; whereby, I will establish these clearly, just as I came upon them, since the method which we have assumed, may be more widely apparent and perhaps it may be able to deduce other precluded speculations. Therefore I have taken two indefinite numbers *m* and *n* and I have considered a certain function of these, which shall be *p*, from which similar functions of the same numerators one and increased by unity several times have been formed, which with the letter φ taken for the designation of this function on being represented thus :

$$\begin{array}{lll}
p = \varphi(m \text{ and } n), & p' = \varphi(m \text{ and } n+1), & p'' = \varphi(m \text{ and } n+2), \\
q = \varphi(m+1 \text{ and } n), & q' = \varphi(m+1 \text{ and } n+1), & q'' = \varphi(m+1 \text{ and } n+2), \\
r = \varphi(m+2 \text{ and } n), & r' = \varphi(m+2 \text{ and } n+1), & r'' = \varphi(m+2 \text{ and } n+2), \\
s = \varphi(m+3 \text{ and } n) & s' = \varphi(m+3 \text{ and } n+1) & s'' = \varphi(m+3 \text{ and } n+2) \\
\text{etc.} & \text{etc.} & \text{etc.}
\end{array}$$

Moreover I establish the function φ to be of this nature, so that there shall be

$$p = Am + Bn + C + \frac{Dnn+En+F}{p'};$$

there will become

$$q = A(m+1) + Bn + C + \frac{Dnn+En+F}{q'},$$

$$r = A(m+2) + Bn + C + \frac{Dnn+En+F}{r'},$$

$$s = A(m+3) + Bn + C + \frac{Dnn+En+F}{s'}$$

etc.

5. Therefore, since *p'*, *q'*, *r'*, *s*' etc. shall arise from *p, q, r, s* etc., if one of the number *m* or *n* may be increased by one from the observed rules, in a similar manner there will be

$$p' = Am + B(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{p''};$$

$$q' = A(m+1) + B(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{q''},$$

$$r' = A(m+2) + B(n+1) + C + \frac{D(n+1)^2 + E(n+1) + F}{r''}$$

etc.

then truly by the same reasoning

$$p'' = Am + \quad B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{p'''};$$

$$q'' = A(m+1) + B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{q'''},$$

$$r'' = A(m+2) + B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{r'''}$$

etc.

and

$$p''' = Am + \quad B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{p''''};$$

$$q''' = A(m+1) + B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{q''''},$$

$$r''' = A(m+2) + B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{r''''}$$

etc.

and so on, thus by progressing further.

6. Hence the function *p* may be expressed in the following manner by the fraction continued indefinitely :

$$p = Am + Bn + C + \cfrac{Dn^2 + En + F}{Am + B(n+1) + C + \cfrac{D(n+1)^2 + E(n+1) + F}{Am + B(n+2) + C + \cfrac{D(n+2)^2 + E(n+2) + F}{Am + B(n+3) + C + \text{etc.}}}}$$

from which with *n* remaining constant, if in place of *m* the numbers $m+1,\ m+2,\ m+3\ etc.$ may be written successively, the values of the functions *q*, *r*, *s*, *t* etc. will be produced to be expressed by similar continued fractions. Therefore now a relation of some kind is sought, connecting the functions *p* and *q*. With which found by the above analogy, likewise the relation between all the functions shown here will be put in place. Which since it may be seen with great difficulty from the previous determination, I consider this to be the conjecture required to be used.

7. Therefore we may observe, a relation of this kind may be able to be put in place now between *p* and *q*:

$$\left(p + (\alpha - A)m + (\beta - B)n + \gamma - C\right)\left(q + (\delta - A)m + (\varepsilon - B)n + \zeta - A - C\right)$$
$$= \lambda mm + \mu m + \nu,$$

from which with *m* remaining constant, if in place of *n* there may be written $n+1$, there will be

$$(p' + (\alpha - A)m + (\beta - B)(n+1) + \gamma - C)$$
$$\times(q' + (\delta - A)m + (\varepsilon - B)(n+1) + \zeta - A - C)$$
$$= \lambda mm + \mu m + \nu$$

But there, if in place of p and q, p' and q' may be substituted for the above values, there will be produced:

$$(\alpha m + \beta n + \gamma + F + \tfrac{Dnn+En+F}{p'})(\delta m + \varepsilon n + \zeta + \tfrac{Dnn+En+F}{q'})$$
$$= \lambda mm + \mu m + \nu,$$

which is transformed into this :

$$(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta)p'q' - (\lambda mm + \mu m + \nu)p'q'$$
$$+(\alpha m + \beta n + \gamma)(Dnn + En + F)p'$$
$$+(\delta m + \varepsilon n + \zeta)(Dnn + En + F)q'$$
$$+(Dnn + En + F)^2 = 0,$$

which must agree with that. From which it is observed there is required to be

$$(\alpha m + \beta n + \gamma)(\delta m + \varepsilon n + \zeta) - \lambda mm - \mu m - \nu$$
$$= \theta(Dnn + En + F),$$

so that on dividing by $\theta(Dnn + En + F)$ there may be put in place :

$$p'q' + \tfrac{1}{\theta}(\alpha m + \beta n + \gamma)p' + (\delta m + \varepsilon n + \zeta)q'$$
$$+ \tfrac{1}{\theta}(Dnn + En + F) = 0,$$

which thus may be shown by the factors represented

$$(p' + \tfrac{\delta m+\varepsilon n+\zeta}{\theta})(q' + \tfrac{\alpha m+\beta n+\gamma}{\theta})$$
$$= \tfrac{(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)}{\theta\theta} - \tfrac{1}{\theta}(Dnn + En + F),$$

or

$$(p' + \tfrac{\delta m+\varepsilon n+\zeta}{\theta})(q' + \tfrac{\alpha m+\beta n+\gamma}{\theta})$$
$$= \tfrac{\lambda mm+\mu m+\nu}{\theta\theta}.$$

8. This form may be compared with the former:

$$\begin{aligned}&(p'+(\alpha-A)m+(\beta-B)n+\gamma-B-C)\\&\times(q'+(\delta-A)m+(\varepsilon-B)n+\varepsilon+\zeta-A-B-C)\\&=\lambda mm+\mu m+v,\end{aligned}$$

from which there is deduced at once:

$$\theta\theta=1$$

and thus either

$$\theta=1 \text{ or } \theta=-1.$$

Then truly there must be:

$$\begin{aligned}&\delta=\theta(\alpha-A),\ \varepsilon=\theta(\beta-B),\ \zeta=\theta(\beta+\gamma-B-C),\\&\alpha=\theta(\delta-A),\ \beta=\theta(\varepsilon-B),\ \gamma=\theta(\varepsilon+\zeta-A-B-C).\end{aligned}$$

Therefore since the value $\theta=1$ is not compatible, we may put

$$\theta=-1,$$

so that we may have

$$\alpha+\delta=A,\ \beta+\varepsilon=B,\ \beta+\gamma+\zeta=B+C \text{ and } \gamma+\varepsilon+\zeta=A+B+C$$

and hence

$$\varepsilon-\beta=\mathrm{A};$$

therefore

$$\beta=\tfrac{1}{2}(B-A),\ \varepsilon=\tfrac{1}{2}(A+B) \text{ and } \gamma+\zeta=\tfrac{1}{2}\left(A+B\right)+C.$$

Truly besides, this condition is to be satisfied :

$$\begin{aligned}&(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)=\lambda mm+\mu m+v-Dnn-En-F\\&=a\delta mm+a\varepsilon mn+\alpha\zeta m+\beta\zeta n+\gamma\zeta+\beta\varepsilon nn+\beta\delta mn+\gamma\delta m+\gamma\varepsilon n.\end{aligned}$$

Therefore there will become:

$$\begin{aligned}&\lambda=\alpha\delta,\ \mu=\alpha\zeta+\gamma\delta,\ D=-\beta\varepsilon,\ E=-\beta\zeta-\gamma\varepsilon,\\&v-F=\gamma\zeta \text{ and } \alpha\varepsilon+\beta\delta=0,\end{aligned}$$

from which initially there becomes

$$D=-\beta\varepsilon=\tfrac{1}{4}(AA-BB),$$

then

$$\tfrac{1}{2}\alpha(A+B)+\tfrac{1}{2}\delta(B-A)=0$$

or

$$\delta=\tfrac{A+B}{A-B}\alpha$$

and thus

$$\alpha=\tfrac{1}{2}(B-A) \text{ and } \delta=\tfrac{1}{2}(A+B).$$

Then truly there will be

$$E+\tfrac{1}{2}\zeta(B-A)+\tfrac{1}{2}\gamma(A+B)=0$$

or

$$E+\tfrac{1}{4}B(A+B)+\tfrac{1}{2}BC+\tfrac{1}{2}A(\gamma-\zeta)=0$$

and hence

$$\gamma-\zeta=\tfrac{BC}{A}+\tfrac{B(A+B)}{2A}+\tfrac{2E}{A};$$

therefore

$$\zeta=\tfrac{1}{4}(A+B)+\tfrac{1}{2}C+\tfrac{BC}{2A}+\tfrac{B(A+B)}{4A}+\tfrac{E}{A},$$
$$\gamma=\tfrac{1}{4}(A+B)+\tfrac{1}{2}C-\tfrac{BC}{2A}-\tfrac{B(A+B)}{4A}-\tfrac{E}{A},$$

or, if in this manner,

$$\zeta=\tfrac{1}{4}(A+B+2C)(1+\tfrac{B}{A})+\tfrac{E}{B}=\tfrac{(A+B)(A+B+2C)}{4A}+\tfrac{E}{A},$$
$$\gamma=\tfrac{1}{4}(A+B+2C)(1-\tfrac{B}{A})-\tfrac{E}{A}=\tfrac{(A-B)(A+B+2C)}{4A}-\tfrac{E}{A}.$$

9. Therefore the relation between p and q cannot be assumed to remain, unless there shall be

$$D=\tfrac{1}{4}(AA-BB);$$

which value, if it may be attributed to D , the following letters themselves thus will be had :

$$\alpha=\tfrac{1}{2}(A-B),\ \delta=\tfrac{1}{2}(A+B),\ \gamma=\tfrac{(A-B)(A+B+2C)}{4A}-\tfrac{E}{A},$$
$$\beta=-\tfrac{1}{2}(A-B),\ \varepsilon=\tfrac{1}{2}(A+B),\ \zeta=\tfrac{(A+B)(A+B+2C)}{4A}+\tfrac{E}{A},$$
$$\lambda=\tfrac{1}{4}(AA-BB)=D,\ \mu=\tfrac{(AA-BB)(A+B+2C)}{4A}-\tfrac{BE}{A},$$
$$\nu=\tfrac{(AA-BB)(A+B+2C)^2}{16AA}-\tfrac{BE(A+B+2C)}{2AA}-\tfrac{EE}{AA}+F$$

and hence again

$$\alpha-A=-\tfrac{1}{2}(A+B),\ \beta-B=-\tfrac{1}{2}(A+B),$$
$$\gamma-C=\tfrac{AA-BB}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A},$$
$$\delta-A=-\tfrac{1}{2}(A-B),\ \varepsilon-B=\tfrac{1}{2}(A-B)$$
$$\zeta-A-C=-\tfrac{(A-B)(3A+B)}{4A}-\tfrac{C(A-B)}{2A}+\tfrac{E}{A},$$

from which this equation emerges between p and q :

$$(p-\tfrac{1}{2}(A+B)(m+n)+\tfrac{AA-BB}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A})$$
$$\times(q-\tfrac{1}{2}(A-B)(m-n)-\tfrac{(A-B)(3A+B)}{4A}-\tfrac{C(A+B)}{2A}+\tfrac{E}{A})$$
$$=\lambda mm+\mu m+\nu.$$

10. We may put as an abbreviation :

$$P=\tfrac{(A+B)(A-B)}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A},$$
$$Q=\tfrac{(A-B)(3A+B)}{4A}+\tfrac{C(A-B)}{2A}-\tfrac{E}{A},$$

so that there shall become :

$$(p-\tfrac{1}{2}(A+B)(m+n)+P)\times(q-\tfrac{1}{2}(A-B)(m-n)-Q)$$
$$=\lambda mm+\mu m+\nu;$$

there will be :

$$p=\tfrac{1}{2}(A+B)(m+n)-P+\frac{\lambda mm+\mu m+\nu}{q-\frac{1}{2}(A-B)(m-n)-Q}.$$

In a similar manner there is :

$$q=\tfrac{1}{2}(A+B)(m+n)+\tfrac{1}{2}(A+B)-P+\frac{\lambda(m+1)^2+\mu(m+1)+\nu}{r-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q},$$
$$r=\tfrac{1}{2}(A+B)(m+n)+\ (A+B)-P+\frac{\lambda(m+2)^2+\mu(m+2)+\nu}{s-\frac{1}{2}(A-B)(m-n)-(A-B)-Q},$$

from which there becomes:

$$q-\tfrac{1}{2}(A-B)(m-n)-Q=Bm+An+\tfrac{1}{2}(A+B)-P-Q$$
$$+\frac{\lambda(m+1)^2+\mu(m+1)+\nu}{r-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q},$$
$$r-\tfrac{1}{2}(A-B)(m-n)-\tfrac{1}{2}(A-B)-Q=Bm+An+\tfrac{1}{2}(A+3B)-P-Q$$
$$+\frac{\lambda(m+2)^2+\mu(m+2)+\nu}{s-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q}.$$

Truly there is:

$$P+Q=\tfrac{(A-B)(2A+B)}{2A}-\tfrac{BC}{A}-\tfrac{2E}{A},$$

and hence

$$q-\tfrac{1}{2}(A-B)(m-n)-Q=Bm+An+B+\tfrac{BB-AA+2BC+4E}{2A}.$$

Whereby if for the sake of brevity there may be put

$$\frac{BB-AA+2BC+4E}{2A}=G,$$

there will become :

$$p=\frac{1}{2}(A+B)(m+n)+\frac{BB-AA}{4A}+\frac{C(A+B)}{2A}+\frac{E}{A}$$
$$+\cfrac{\lambda m^2+\mu m+\nu}{B(m+1)+An+G+\cfrac{\lambda(m+1)^2+\mu(m+1)+\nu}{B(m+2)+An+G+\text{etc.}}}$$

or by introducing the value G :

$$p=\frac{1}{2}(A+B)(m+n)+\frac{1}{2}(C+G)$$
$$+\cfrac{\lambda m^2+\mu m+\nu}{B(m+1)+An+G+\cfrac{\lambda(m+1)^2+\mu(m+1)+\nu}{B(m+2)+An+G+\cfrac{\lambda(m+2)^2+\mu(m+2)+\nu}{B(m+3)+An+G+\text{etc.}}}}$$

Moreover truly there will be

$$P=-\tfrac{1}{2}(C+G)\ \text{ and }\ Q=\tfrac{1}{2}(A-B)+\tfrac{1}{2}(C-G)$$

and thus

$$\left(p-\tfrac{1}{2}(A+B)(m+n)-\tfrac{1}{2}(C+G)\right)\left(q-\tfrac{1}{2}(A-B)(m+1-n)-\tfrac{1}{2}(C-G)\right)$$
$$=\lambda mm+\mu m+\nu.$$

11. Therefore if a proposed infinite continued fraction of this kind were proposed :

$$p=Am+Bn+C$$
$$+\cfrac{Dn^2+En+F}{Am+B(n+1)+C+\cfrac{D(n+1)^2+E(n+1)+F}{Am+B(n+2)+C+\cfrac{D(n+2)^2+E(n+2)+F}{Am+B(n+3)+C+\text{etc.}}}}$$

in which there shall be

$$D=\tfrac{1}{4}(AA-BB),$$

and thence this other fraction may be formed :

$$q=A(m+1)+Bn+C$$
$$+\cfrac{Dn^2+En+F}{A(m+1)+B(n+1)+C+\cfrac{D(n+1)^2+E(n+1)+F}{A(m+1)+B(n+2)+C+\cfrac{D(n+2)^2+E(n+2)+F}{A(m+1)+B(n+3)+C+\text{etc.}}}}$$

while in place of *m* there may be written $m+1$ everywhere, the first relation between p and q can be assigned in this manner: For the sake of brevity there may be put

$$\frac{BB-AA+2BC+4E}{2A}=G,$$

and

$$\lambda=\tfrac{1}{2}(AA-BB),$$
$$\mu=\tfrac{1}{4}(AA-BB)+\tfrac{1}{2}(AC-BG),$$
$$\nu=\tfrac{1}{4}CC+(C-G)(A+B)-\tfrac{1}{4}GG+F$$

or

$$\nu=F+\tfrac{1}{4}(C-G)(A+B+C+G)$$

and this relation will become

$$\left(p-\tfrac{1}{2}(A+B)(m+n)-\tfrac{1}{2}(C+G)\right)\left(q-\tfrac{1}{2}(A-B)(m+1-n)-\tfrac{1}{2}(C-G)\right)$$
$$=\lambda mm+\mu m+\nu.$$

then truly besides the function *p* also may be equal to this other continued fraction

$$p=\frac{1}{2}(A+B)(m+n)+\frac{1}{2}(C+G)$$
$$+\cfrac{\lambda m^2+\mu m+\nu}{B(m+1)+An+G+\cfrac{\lambda(m+1)^2+\mu(m+1)+\nu}{B(m+2)+An+G+\cfrac{\lambda(m+2)^2+\mu(m+2)+\nu}{B(m+3)+An+G+\text{etc.}}}}$$

12. The above examples proposed may arise hence, if there may be put in place

$$D=\tfrac{1}{4}(AA-BB)=0.$$

Therefore there shall become :

$$B=A,$$

so that these continued fractions may be had:

$$p=A(m+n)+C+\cfrac{En+F}{A(m+n+1)+C+\cfrac{E(n+1)+F}{A(m+n+2)+C+\cfrac{E(n+2)+F}{A(m+n+3)+C+\text{etc.}}}}$$

$$q=A(m+n+1)+C+\cfrac{En+F}{A(m+n+2)+C+\cfrac{E(n+1)+F}{A(m+n+3)+C+\cfrac{E(n+2)+F}{A(m+n+4)+C+\text{etc.}}}}$$

of which with the relation put in place :

$$G=C+\frac{2E}{A}$$

and

$$\mu=\tfrac{1}{2}A\left(C-G\right),\ \nu\ =\ F+\tfrac{1}{4}(C-G)(2A+C+G)$$

or

$$\mu = -E,\ \nu = F - \tfrac{2E}{A}(2A + C + G)$$

there will become:

$$\left(p - A(m+n) - \tfrac{1}{2}(C+G)\right)\left(q - \tfrac{1}{2}(C-G)\right)$$
$$= \mu m + \nu,$$

and the quantity *p* also may be expressed in the other way by the infinite continued fraction:

$$p = A(m+n) + \frac{1}{2}(C+G) + \cfrac{\mu m+\nu}{A(m+n+1)+G+\cfrac{\mu(n+1)+\nu}{A(m+n+2)+G+\cfrac{\mu(m+2)+\nu}{A(m+n+3)+G+\text{etc.}}}}$$

13. Therefore here since both the numerators as well as the denominators constitute an arithmetical progression, of which we may return a simpler form and the two infinite continued fractions shall be :

$$p = a + \cfrac{f}{a+b+\cfrac{f+g}{a+2b+\cfrac{f+2g}{a+3b+\cfrac{f+3g}{a+4b+\text{etc.}}}}}$$

$$q = a + b + \cfrac{f}{a+2b+\cfrac{f+g}{a+3b+\cfrac{f+2g}{a+4b+\cfrac{f+3g}{a+5b+\text{etc.}}}}}$$

thus so that *q* arises from the fraction *p*, if $a+b$ may be written in place of a . Therefore there will become:

$$A(m+n) + C = a,\ En + F = f,\ A = b,\ E = g$$

and hence

$$C = a - b(m+n) \text{ and } F = f - gn,$$

from which there is made:

$$G = a - b(m+n) + \tfrac{2g}{b},\ C - G = -\tfrac{2g}{b}$$
$$C + G = 2a - 2b(m+n) + \tfrac{2g}{b};$$

then

$$\mu = -g,\ \nu = f - gn - \tfrac{g}{b}\left(a - b(m+n)C + \tfrac{g}{b}\right)$$

or

$$\nu = f - \tfrac{g(a+b)}{b} + gm - \tfrac{gg}{bb};$$

therefore

$$\mu m+v=f-\tfrac{g(a+b)}{b}-\tfrac{gg}{bb},$$

$$A(m+n)+\tfrac{1}{2}(C+G)=a+\tfrac{g}{b}$$

and

$$\tfrac{1}{2}(C-G)=-\tfrac{g}{b}.$$

Whereby this relation may be found between p and q :

$$(p-a-\tfrac{g}{b})(q+\tfrac{g}{b})=f-\tfrac{g(a+b)}{b}-\tfrac{gg}{bb};$$

or

$$pq+\tfrac{g}{b}p-(a+\tfrac{g}{b})q=f-g,$$

from which the continued fraction is obtained for p also :

$$p=a+\frac{g}{b}+\cfrac{f-\frac{g(a+b)}{b}-\frac{gg}{bb}}{a+b+\frac{2g}{b}+\cfrac{f-\frac{g(a+2b)}{b}-\frac{gg}{bb}}{a+2b+\frac{2g}{b}+\cfrac{f-\frac{g(a+3b)}{b}-\frac{gg}{bb}}{a+3b+\frac{2g}{b}+\text{etc.}}}}$$

14. So that we may remove the fractions, we may put $g=bh$, so that we may have these continued fractions :

$$p=a+\cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\cfrac{f+3bh}{a+4b+\text{etc.}}}}}$$

$$q=a+b+\cfrac{f}{a+2b+\cfrac{f+bh}{a+3b+\cfrac{f+2bh}{a+4b+\cfrac{f+3bh}{a+5b+\text{etc.}}}}}$$

the relation of which may be had, so that there shall be

$$(p-a-h)(q+h)=f-(a+b)h-hh$$

or

$$pq+hp-(a+h)q=f-bh,$$

from which for p this continued fraction also is elicited :

$$p = a + h + \cfrac{f-(a+b)h-hh}{a+b+2h+\cfrac{f-(a+2b)h-hh}{a+2b+2h+\cfrac{f-(a+3b)h-hh}{a+3b+2h+\text{etc.}}}}$$

Therefore since this continued fraction shall be equal to the first, this moreover is being interrupted, just as there will have been

$$f = (a+ib)h + hh$$

with i denoting some positive integer, as often as the value of the first can be assigned rationally.

15. From the relation found between p and q, also q can be expressed in terms of p thus :

$$q = -h + \frac{f-(a+b)h-hh}{-a-h+p},$$

and since p may arise from q, if in place of a there may be written $a-b$, if the terms of the preceding series p, q, r etc. shall be o, n, m etc., there will become:

$$p = -h + \frac{f-ah-hh}{-a+b-h+o},$$
$$o = -h + \frac{f-(a-b)h-hh}{-a+2b-h+n},$$
$$n = -h + \frac{f-(a-2b)h-hh}{-a+3b-h+m},$$
etc.,

from which also this continued fraction for p is obtained :

$$p = -h + \cfrac{f-ah-hh}{b-a-2h+\cfrac{f+(b-a)h-hh}{2b-a-2h+\cfrac{f+(2b-a)h-hh}{3b-a-2h+\text{etc.}}}}$$

the same as we may come upon from our general formulas, if above in § 12 we may have substituted $B=-A$. Whereby also the value can be expressed rationally, as often as there were

$$hh = (ib-a)h + f,$$

unless perhaps in these cases where the denominator for that vanishing numerator itself may be subjected to vanish too.

16. Moreover from that proposed continued fraction :

$$p=a+\cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\text{etc.}}}}$$

another at once equal to this can be deduced in this manner. Since indeed there shall be

$$p=a+\frac{f}{p'},\ p'=a+b+\frac{f+bh}{p''},\ p''=a+2b+\frac{f+2bh}{p'''}\ \text{etc.},$$

there will be by going backwards

$$p_{,}=a-b+\frac{f-bh}{p},\ \ p_{,,}=a-2b+\frac{f-2bh}{p_{,}},\ \ p_{,,,}=a-3b+\frac{f-3bh}{p_{,,}}\ \text{etc.},$$

and hence

$$p=\frac{f-bh}{b-a+p_{,}},\ \ p_{,}=\frac{f-2bh}{2b-ap_{,,}},\ \ p_{,,}=\frac{f-3bh}{3b-a+p_{,,,}}\ \ \text{etc.},$$

from which we conclude:

$$p=\cfrac{f-bh}{b-a+\cfrac{f-2bh}{2b-a+\cfrac{f-3bh}{3b-a+\cfrac{f-4bh}{4b-a+\text{etc.}}}}}$$

thus so that also in the cases $f=ibh$ the value may be able to be shown rationally.

17. Behold therefore the four continued fractions are equal to each other:

$$\text{I. } p = a + \cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\cfrac{f+3bh}{a+4b+\text{etc.}}}}}$$

$$\text{II. } p = \cfrac{f-bh}{b-a+\cfrac{f-2bh}{2b-a+\cfrac{f-3bh}{3b-a+\cfrac{f-4bh}{4b-a+\text{etc.}}}}}$$

$$\text{III. } p = a+h+\cfrac{f-(a+b)h-hh}{a+b+2h+\cfrac{f-(a+2b)h-hh}{a+2b+2h+\cfrac{f-(a+3b)h-hh}{a+3b+2h+\text{etc.}}}}$$

$$\text{IV. } p = -h+\cfrac{f-ah-hh}{b-a-2h+\cfrac{f+(b-a)h-hh}{2b-a-2h+\cfrac{f+(2b-a)h-hh}{3b-a-2h+\text{etc.}}}}$$

18. So that we may approach closer to the initial form proposed, there shall become

$$a = m,\ f = n,\ b = 1 \text{ and } h = 1$$

and we will have :

$$\text{I. } p = m + \cfrac{n}{m+1+\cfrac{n+1}{m+2+\cfrac{n+2}{m+3+\cfrac{n+3}{m+\text{etc.}}}}}$$

$$\text{II. } p = \cfrac{n-1}{-m+1+\cfrac{n-2}{-m+2+\cfrac{n-3}{-m+3+\cfrac{n-4}{-m+4+\text{etc.}}}}}$$

$$\text{III. } p = m+1+\cfrac{n-m-2}{m+3+\cfrac{n-m-3}{m+4+\cfrac{n-m-4}{m+5+\cfrac{n-m-5}{m+6+\text{etc.}}}}}$$

$$\text{IV. } p = -1+\cfrac{n-m-1}{-m-1+\cfrac{n-m}{-m+1+\cfrac{n-m+2}{-m+2+\text{etc.}}}}$$

Whereby with *i* denoting a positive integer, with zero not excluded, the value of our repeated fraction will be able to be expressed in these cases :

$$\text{I. } n = i,\ \text{II. } n = m+2+i,\ \text{III. } n = m+1-i,$$

unless perhaps the inconvenient situation mentioned above may be come upon.

19. From the rare cases $n=i$ the value sought will be found from the inconvenience mentioned, where also the denominator will become zero. If indeed $n=1$, where there becomes

$$p=m+\cfrac{1}{m+1+\cfrac{2}{m+2+\cfrac{3}{m+3+\cfrac{4}{m+4 \text{ etc.}}}}}$$

certainly there is not $p=0$, and if that is considered to be shown from the second form, from which we can confirm to be

$$0=1-m-\cfrac{1}{2-m-\cfrac{2}{3-m-\cfrac{3}{4-m-\cfrac{4}{5-m-\text{etc.}}}}}$$

But if the first general form may be adapted to this case, there will be

$$a=1-m,\ b=1,\ f=-1 \text{ and } h=-1,$$

from which the second gives

$$p=\cfrac{1-1}{m+\cfrac{1}{m+1+\cfrac{2}{m+2+\text{etc.}}}}$$

truly the third :

$$p=-m-\cfrac{m}{-m+\cfrac{1-m}{1-m+\cfrac{2-m}{2-m \text{ etc.}}}}$$

and the fourth

$$p=+1-\cfrac{1-m}{m-2-\cfrac{2-m}{m-1-\cfrac{3-m}{m-\text{etc.}}}}$$

which therefore are equal to zero.

20. Since there shall be from the second form of § 18 :

$$\frac{n-1}{p}=1-m+\cfrac{n-2}{2-m+\cfrac{n-3}{3-m+\cfrac{n-4}{4-m+\text{etc.}}}}$$

if this may be compared with the general form, there will be :

$$a=1-m,\ b=1,\ f=n-2 \text{ and } h=-1,$$

from which the third form presents

$$\frac{n-1}{p} = -m + \cfrac{n-m-1}{-m+\cfrac{n-m}{1-m+\cfrac{n-m+1}{2-m+\cfrac{n-m+2}{3-m+\text{ etc.}}}}}$$

and the fourth

$$\frac{n-1}{p} = 1 + \cfrac{n-m-2}{m+2+\cfrac{n-m-3}{m+3+\cfrac{n-m-4}{m+4+\text{ etc.}}}}$$

and thus two new expressions may be had for p ; and many others are able to be shown in a like manner.

21. But from the value p found this continued fraction is readily defined :

$$x = m + \cfrac{n+1}{m+1+\cfrac{n+2}{m+2+\cfrac{n+3}{m+3+\text{ etc.}}}}$$

For $m-1$ may be written everywhere in place of m and there may be put

$$q = m-1 + \cfrac{n}{m+\cfrac{n+1}{m+1+\cfrac{n+2}{m+2+\text{ etc.}}}} = m-1+\tfrac{n}{x};$$

but from the third there will be :

$$q = m + \cfrac{n-m-1}{m+2+\cfrac{n-m-2}{m+3+\cfrac{n-m-3}{m+4+\text{ etc.}}}} = m + \frac{n-m-1}{p+1},$$

from which two equal values of q there becomes :

$$-1+\frac{n}{x} = \frac{n-m-1}{p+1}$$

or

$$\frac{n}{x} = \frac{n-m-p}{p+1}$$

and thus

$$x = \frac{n(p+1)}{p-m+n}.$$

22. Therefore since on putting $n = m+2$ there shall be $p = m+1$, there will be

$$m+1 = p = m + \cfrac{m+2}{m+1+\cfrac{m+3}{m+2+\cfrac{m+4}{m+3+\text{ etc.}}}}$$

and likewise in a similar manner, on putting $n = m+3,\ n = m+4$ etc.

$$m+1+\frac{1}{m+3} = m+\cfrac{m+3}{m+1+\cfrac{m+4}{m+2+\cfrac{m+5}{m+3+\text{ etc.}}}} = q,$$

$$m+1+\cfrac{2}{m+3+\cfrac{1}{m+4}} = m+\cfrac{m+4}{m+1+\cfrac{m+5}{m+2+\cfrac{m+6}{m+3+\text{ etc.}}}} = r,$$

$$m+1+\cfrac{3}{m+3+\cfrac{2}{m+4+\cfrac{1}{m+5}}} = m+\cfrac{m+5}{m+1+\cfrac{m+6}{m+2+\cfrac{m+7}{m+3+\text{ etc.}}}} = s,$$

$$m+1+\cfrac{4}{m+3+\cfrac{3}{m+4+\cfrac{2}{m+5+\cfrac{1}{m+6}}}} = m+\cfrac{m+6}{m+1+\cfrac{m+7}{m+3+\cfrac{m+8}{m+4+\text{ etc.}}}} = t,$$

from which on putting $m = 1$ the rational cases observed above follow. But these values progress thus, so that there shall be

$$p = m+1, \qquad q = \frac{(m+2)(p+1)}{p+2},$$

$$r = \frac{(m+3)(q+1)}{q+3}, \qquad s = \frac{(m+4)(r+1)}{r+4}$$

or

$$q = (m+2)(1-\tfrac{1}{p+2}),$$

$$r = (m+3)(1-\tfrac{2}{q+3}),$$

$$s = (m+4)(1-\tfrac{3}{r+4}),$$

etc.,

which expressions also can be shown thus :

$$q = m+2-\frac{m+2}{m+3}$$

$$r = m+3-\cfrac{2(m+3)}{m+5-\cfrac{m+2}{m+3}}$$

$$s = m+4-\cfrac{3(m+4)}{m+7-\cfrac{2(m+3)}{m+5-\cfrac{m+2}{m+3}}}$$

But from the expansion made there is found :

$$p = m+1,$$
$$q = \frac{(m+2)(m+2)}{(m+3)},$$
$$r = \frac{(m+3)(mm+5m+7)}{mm+7m+13},$$
$$s = \frac{(m+4)(m^3+9m^2+29m+34)}{\left(m^3+12mm+50m+73\right)},$$
$$t = \frac{(m+5)(m^4+14m^3+77m^2+200m+209)}{m^4+18m^3+125m^2+400m+501}.$$

23. The continued fractions shown in the preceding paragraph may be specified by p, q, r, s, t etc. , and there shall be

$$p=(m+1)\tfrac{a}{\alpha},\ q=(m+2)\tfrac{b}{\beta},\ r=(m+3)\tfrac{c}{\gamma}\ \text{etc.};$$

there will be

$$a=1,\ \alpha=1,$$

truly the remaining letters thus in turn depend on these, so that there shall be

$$\begin{array}{ll} b=(m+1)a+\alpha, & \beta=(m+1)a+2\alpha, \\ c=(m+2)b+\beta, & \gamma=(m+2)b+3\beta, \\ d=(m+3)c+\gamma, & \delta=(m+3)c+4\gamma \\ e=(m+4)d+\delta, & \varepsilon=(m+4)d+5\delta \\ \text{etc.} & \text{etc.,} \end{array}$$

from which the relation between the Latin and Greek letters is deduced separately :

$$\begin{array}{ll} b=(m+2)a, & \beta=\left(m+3\right)\alpha,. \\ c=(m+4)b-1(m+1)\alpha, & \gamma=(m+5)\beta-1(m+2)\alpha, \\ d=(m+6)c-2(m+2)b, & \delta=(m+7)\gamma-2(m+3)\beta, \\ e=(m+8)d-3(m+3)c & \varepsilon=(m+9)\delta-3(m+4)\gamma \\ \text{etc.} & \text{etc.} \end{array}$$

But with the denominators $\alpha,\ \beta,\ \gamma,\ \delta$ etc. found the numerators will be

$$b=\beta-\alpha,\ c=\gamma-2\beta,\ d=\delta-3\gamma,\ e=\varepsilon-4\delta\ \text{etc.}$$

But if the numerators *a, b, c, d, e* now shall have been found, the denominators will be

$$\alpha = a,\ \beta = 2b-(m+1)a,\ \gamma = 3c-2(m+2)b,$$
$$\delta = 4d-3(m+3)c,\ \varepsilon = 5e-4(m+4)d \text{ etc.}$$

But we have seen to be :

$$a=1, \qquad \alpha = 1,$$
$$b=m+2, \qquad \beta = m+3,$$
$$c=m^2+5m+7, \qquad \gamma = m^2+7m+13,$$
$$d=m^3+9m^2+29m+24, \qquad \delta = m^3+12m^2+50m+73,$$
$$e=m^4+14m^3+77m^2+200m+209, \qquad \varepsilon = m^4+18m^3+125m^2+400m+501.$$

24. From the value of each continued fraction in § 22 also the value of the preceding can be defined in this manner

$$p=\frac{m+2-2q}{q-(m+2)},\ q=\frac{m+3-3r}{r-(m+3)},\ r=\frac{m+4-4s}{s-(m+4)} \text{ etc.,}$$

from which, if the continued fractions may be designated in the preceding order by the letters *O*, *N*, *M* etc., there will be

$$O=\frac{m+1-p}{p-(m+1)},\ N=\frac{m}{O-m},\ \ M=\frac{m-1+N}{N-(m-1)},\ L=\frac{m-2+2M}{M-(m-2)} \text{ etc.}$$

But there is

$$O=m+\cfrac{m+1}{m+1+\cfrac{m+2}{m+2+\cfrac{m+3}{m+3+\cfrac{m+4}{m+4+\text{etc.}}}}}$$

of which the remaining equivalent values are :

$$O=\cfrac{m}{1-m+\cfrac{m-1}{2-m+\cfrac{m-2}{3-m+\cfrac{m-3}{4-m+\text{etc.}}}}}$$

$$O=m+1-\cfrac{1}{m+3-\cfrac{2}{m+4-\cfrac{3}{m+5-\cfrac{4}{m+6-\cfrac{5}{m+7-\text{etc.}}}}}}$$

Hence moreover a finite value of *O* is not allowed to be expected, since in the case $m=1$ it shall certainly be transcending, which we may set out, just as it is required to be found.

25. These formulas will arise from the first form :

$$O=m+\frac{m+1}{A},\ A=m+1+\frac{m+2}{B},\ \ B=m+2+\frac{m+3}{C}\ \text{etc.}$$

and there will become :

$$OA=mA+m+1,\ \ AB=(m+1)B+m+2\ \ \text{etc.}$$

There may be put in place:

$$O=-1+\frac{1}{\omega},\ A=-1+\frac{1}{\alpha},\ \ B=-1+\frac{1}{\beta}\ \text{etc.}$$

and these formulas will be found :

$$\alpha+(m+1)\omega=1,\ \beta+(m+2)\alpha=1,\ \gamma+(m+3)\beta=1\ \text{etc.}$$

or

$$\omega=\frac{1}{m+1}-\frac{\alpha}{m+1},\ \ \alpha=\frac{1}{m+2}-\frac{\beta}{m+2},\ \ \beta=\frac{1}{m+3}-\frac{\gamma}{m+3},$$

from which there becomes by the customary series :

$$\omega=\frac{1}{m+1}-\frac{1}{(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}-\frac{1}{(m+1)(m+2)(m+3)(m+4)}+\text{etc.},$$

of which the value is :

$$\omega=\frac{1}{e}\int e^{z}z^{m}dz$$

with this integral thus taken, so that it may vanish on putting $z=0$, with which done there must be put in place $z=1$. Hence in the cases, in which m is a whole number, will be :

if $m=0,\ \omega=\frac{e-1}{e}$ and $O=\frac{1}{e-1};$

if $m=1,\ \omega=\frac{1}{e}$ and $O=e-1;$

if $m=2,\ \omega=\frac{e-2}{e}$ and $O=\frac{2}{e-2};$

if $m=3,\ \omega=\frac{-2e+6}{e}$ and $O=\frac{3e-6}{6-2e}=\frac{3(e-2)}{2(3-e)};$

if $m=4,\ \omega=\frac{9e-24}{e}$ and $O=\frac{24-8e}{9e-24}=\frac{4(6-2e)}{3(3e-8)}=\frac{8(3-e)}{3(3e-8)};$

if $m=5,\ \omega=\frac{120-44e}{e}$ and $O=\frac{45e-120}{120-44e}=\frac{5(9e-24)}{4(30-11e)}=\frac{15(3e-8)}{4(30-11e)};$

if $m=6,\ \omega=\frac{265e-720}{e}$ and $O=\frac{720-264e}{265e-720}=\frac{6(120-44e)}{5(53e-144)}=\frac{24(30-11e)}{5(53e-144)};$

if $m=7,\ \omega=\frac{5040-1854e}{e}$ and $O=\frac{1855e-5040}{5040-1854e}=\frac{7(265-720)}{6(720-309e)}=\frac{35(53e-144)}{6(720-309e)}.$

The value of O cannot be expressed unless m is a whole number, in which case it is expressed in terms of the number e, of which the logarithm $=1$.

26. We may put

$$m=0,$$

of which all the cases can be referred to, in which m is a whole number, and there will be

$$O=\frac{1}{e-1},\quad N=0,\quad M=-1,\quad L=\frac{-2+2M}{M+2}=-4,$$
$$K=\frac{-3+3L}{L+3}=15,\quad I=\frac{-4+4K}{K+4}=\frac{56}{19},$$
$$H=\frac{-5+5I}{I+5}=\frac{185}{151},\quad G=\frac{-6+6H}{H+6}=\frac{204}{1091},$$

from which the following continued fractions arise:

$$1=0+\cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\text{etc.}}}}\qquad \frac{1}{-1}=0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}\qquad \frac{4}{3}=0+\cfrac{3}{1+\cfrac{4}{2+\cfrac{5}{3+\text{etc.}}}}$$

$$0=0+\cfrac{0}{1+\cfrac{1}{2+\cfrac{2}{3+\text{etc.}}}}\qquad \frac{21}{13}=0+\cfrac{4}{1+\cfrac{5}{2+\cfrac{6}{3+\text{etc.}}}}\qquad -1=0-\cfrac{1}{1+\cfrac{0}{2+\cfrac{1}{3+\text{etc.}}}}$$

$$\frac{136}{73}=0+\cfrac{5}{1+\cfrac{6}{2+\cfrac{7}{3+\text{etc.}}}}\qquad -4=0-\cfrac{2}{1-\cfrac{1}{2+\cfrac{0}{3+\text{etc.}}}}\qquad \frac{1045}{501}=0+\cfrac{6}{1+\cfrac{7}{2+\cfrac{8}{3+\text{etc.}}}}$$

$$15=0-\cfrac{3}{1-\cfrac{2}{2-\cfrac{1}{3-\text{etc.}}}}$$

to which these must be added :

$$\frac{1}{e-1}=0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}\qquad \frac{1}{e-2}=1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}\qquad \frac{1}{2e-5}=2+\cfrac{1}{3+\cfrac{2}{4+\cfrac{3}{5+\text{etc.}}}}$$

$$\frac{1}{6e-16}=3+\cfrac{1}{4+\cfrac{2}{5+\cfrac{3}{6+\text{etc.}}}}\qquad \frac{1}{24e-65}=4+\cfrac{1}{5+\cfrac{2}{6+\cfrac{3}{7+\text{etc.}}}}\qquad \frac{1}{120e-326}=5+\cfrac{1}{6+\cfrac{2}{7+\cfrac{3}{8+\text{etc.}}}}$$

If indeed there shall be

$$x = m + \cfrac{1}{m+1+\cfrac{2}{m+2+\cfrac{3}{m+3+\text{etc.}}}} = m+1 - \cfrac{m+1}{m+3-\cfrac{m+2}{m+4-\cfrac{m+3}{m+5-\text{etc.}}}}$$

$$y = m+1 + \cfrac{1}{m+2+\cfrac{2}{m+3+\cfrac{3}{m+4+\text{etc.}}}} = m+2 - \cfrac{m+2}{m+4-\cfrac{m+3}{m+5-\cfrac{m+4}{m+6-\text{etc.}}}}$$

there will be

$$y = \frac{x}{m+1-x}.$$

27. Truly our investigations extend much more widely ; so that we may set these out more accurately, we may revert to the formulas in § 11, which lose nothing from their size, even if we may put the numbers *m* and *n* equal to zero. Whereby the continued fractions being considered will be these :

$$p = C + \cfrac{F}{C+B+\cfrac{F+E+D}{C+2B+\cfrac{F+2E+4D}{C+3B+\cfrac{F+3E+9D}{C+4B+\text{etc.}}}}}.$$

$$q = C+A+\cfrac{F}{C+A+B+\cfrac{F+E+D}{C+A+2B+\cfrac{F+2E+4D}{C+A+3B+\cfrac{F+3E+9D}{C+4B+\text{etc.}}}}}.$$

$$r = C+2A+\cfrac{F}{C+2A+B+\cfrac{F+E+D}{C+2A+2B+\cfrac{F+2E+4D}{C+2A+3B+\cfrac{F+3E+9D}{C+2A+4B+\text{etc.}}}}}$$

which forms may be continued further continually by writing $C+A$ in place of *C*. Therefore in each the denominators constitute an arithmetic progression, truly the numerators a progression of which the second orders are constants. But here we assume to be :

$$D = \tfrac{1}{4}(AA-BB).$$

So that if now for the sake of brevity, we may put

$$G = \frac{BB-AA+2BC+4E}{2A} = \frac{BC-2D+2E}{A}$$

and

$$\lambda = \tfrac{1}{4}(AA-BB) = D,$$
$$\mu = \tfrac{1}{4}(AA-BB) + \tfrac{1}{2}(AC-BG) = \frac{(A+B+2C)D-BE}{A},$$
$$v = F + \tfrac{1}{4}(C-G)(A+B+C+G)$$

or

$$v = F + \frac{CD(A+C)+DD-EE}{AA} + \frac{B(A+B+2C)(D-E)}{2AA},$$

there will be

$$\left(p - \frac{(A+B)(C+D-2E)}{2A}\right)\left(q - \frac{(A-B)(A+C)-2C(D-E)}{2A}\right) = v$$

and hence by the other continued fraction

$$p = \frac{1}{2}(C+G) + \cfrac{v}{G+B+\cfrac{v+\mu+\lambda}{G+2B+\cfrac{v+2\mu+4\lambda}{G+3B+\text{etc.}}}}$$

28. Therefore some continued fraction of this form :

$$p = a + \cfrac{f}{a+b+\cfrac{f+g}{a+2b+\cfrac{f+2g+h}{a+3b+\cfrac{f+3g+3h}{a+4b+\cfrac{f+4g+6h}{a+5b+\text{etc.}}}}}}$$

this can be transformed into another equal to itself. For with a comparison made there is :

$$C = a,\ B = b,\ F = f,\ E = g - \tfrac{1}{2}h,\ D = \tfrac{1}{2}h.$$

Therefore there may be taken

$$A = \sqrt{(bb+2h)},$$

then truly

$$G = \frac{ab+2g-2h}{A},$$

$$\lambda = \tfrac{1}{2}h,\quad \mu = \tfrac{1}{2}h + \tfrac{1}{2}(Aa - Gb) = \tfrac{1}{2}h + \frac{ah-b(g-h)}{A}$$

$$v = f + \frac{ah-b(g-h)}{2A} + \frac{aah-bb(g-h)-2ab(g-h)-2g(g-h)}{2AA}$$

and hence there becomes

$$p = \frac{1}{2}(a+G) + \cfrac{v}{G+b+\cfrac{v+\mu+\lambda}{G+2b+\cfrac{v+2\mu+4\lambda}{G+3b+\cfrac{v+3\mu+9\lambda}{G+4b+\text{etc.}}}}}$$

29. Therefore if there were $f = 0$, the value of this latter continued fraction certainly is $= a$, whatever numbers may be attributed to the remaining letters. Therefore there may be put $g - h = k$, so that there shall be

$$A = \sqrt{(bb+2h)},\ G = \frac{ab+2k}{A},$$

$$\lambda = \tfrac{1}{2}h, \quad \mu = \tfrac{1}{2}h + + \frac{ah-bk}{2A} \text{ and } \nu = \frac{ah-bk}{2A} + \frac{aah-bbk-2abk-2hk-2kk}{2AA},$$

and there will be

$$\tfrac{1}{2}(a+G) = G + \cfrac{\nu}{G+b+\cfrac{\nu+\mu+\lambda}{G+2b+\cfrac{\nu+2\mu+4\lambda}{G+3b+\cfrac{\nu+3\mu+9\lambda}{G+4b+\text{etc.}}}}}$$

where if the letters a, b, A and G may be had as given, there will become:

$$\lambda = \frac{AA-bb}{4}, \quad \mu = \lambda + \frac{Aa-Gb}{2} = \frac{AA-bb}{4} + \frac{Aa-Gb}{2}$$
$$\nu = \frac{aa+ab+Aa+Bb-AG-GG}{4} = \tfrac{1}{4}(A-G)(a+b+G),$$

hence

$$\nu + \mu + \lambda = \tfrac{1}{4}(a - b + A - G)(a+2b+2A+G),$$
$$\nu + 2\mu + 4\lambda = \tfrac{1}{4}(a-2b+2A-G)(a+3b+3A+G),$$
$$\nu + 3\mu + 9\lambda = \tfrac{1}{4}(a-3b+3A-G)(a+4b+4A+G).$$

There may be put, contracting the formula

$$a-G = 2\alpha, \; A-b = 2\gamma, \; a+G = 2\beta, \; A+b = 2\delta \text{ etc.};$$

there will become:

$$a = \cfrac{\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}}$$

the truth of which may be elucidated at once in many examples.

30. If the same positions may be retained, but the number f may be taken not equal to zero, this continued fraction will be obtained :

$$p = \alpha + \beta + \cfrac{f}{\alpha+\beta+(\delta-\gamma)+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta+2(\delta-\gamma)+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta+3(\delta-\gamma)+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}}$$

which is transformed into this equal to itself

$$p=\beta+\cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+3(\delta-\gamma)+\text{etc.}}}}}$$

from which if either γ or δ may be taken as vanishing, the case treated before arises. But this includes within itself the equality of all the pairs of continued fractions, which have been set out at this stage.

31. From these the forms arise, which we have denoted by the letter *q*, if in place of α and β we may write $\alpha+\gamma$ and $\beta+\delta$, thus so that there shall become :

$$q=\alpha+\beta+\gamma+\delta$$
$$+\cfrac{f}{\alpha+\beta+2\delta+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta-\gamma+3\delta+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta-2\gamma+4\delta+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta-3\gamma+5\delta+\text{etc.}}}}}$$

and likewise :

$$q=\beta+\delta$$
$$+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}$$

thus so that this relation exists between these two expressions

$$(p-\beta)(q-\alpha-\gamma)=f+\alpha(\beta+\delta)$$

or

$$pq-(\alpha+\gamma)p-\beta q+\beta\gamma-\alpha\delta=f,$$

with the aid of which the equality can best be shown by the method of substitution of the two superior formulas, which we have used above .

32. If we may put

$$f+\alpha(\beta+\delta)=g,$$

so that there shall be:

$$f=g-\alpha(\beta+\delta),$$

the first form will be had thus :

$$p=\alpha+\beta+\cfrac{g-\alpha(\beta+\delta)}{\alpha+\beta+(\delta-\gamma)+\cfrac{g-(\alpha-\gamma)(\beta+2\delta)}{\alpha+\beta+2(\delta-\gamma)+\cfrac{g-(\alpha-2\gamma)(\beta+3\delta)}{\alpha+\beta+3(\delta-\gamma)+\cfrac{g-(\alpha-3\gamma)(\beta+4\delta)}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}},$$

to which this is equal :

$$p=\beta+\cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}},$$

And these forms may be considered to be especially suitable, the equality of which may be able to be investigated and demonstrated by the direct method. Moreover such a method is desired even now. Moreover there is no doubt, why that may not be expected to reveal many outstanding advances in analysis. Therefore since the first form may escape a finite form, if there were

$$g=(\alpha-i\gamma)(\beta+(i+1)\delta),$$

we understand the value of the latter can be expressed rationally, whenever there were

$$f=(\alpha-i\gamma)(\beta+(i+1)\delta)-\alpha(\beta+\delta)$$

or

$$f=i(\alpha\delta-\beta\gamma-(i+1)\gamma\delta)$$

with *i* denoting some whole number.

OBSERVATIONES ANALYTICAE

Commentatio 553 indicis ENESTROEMIANI
Opuscula analytica 1, 1783, p. 85-120

1. Inter alia, quae passim de fractionibus continuis sum commentatus, notatu digna videtur haec forma

$$1+\cfrac{n}{2+\cfrac{n+1}{3+\cfrac{n+2}{4+\cfrac{n+3}{5+\cfrac{n+4}{6+\text{etc.}}}}}}.$$

cuius valor, quoties *n* est numerus integer, sequenti modo exhiberi potest denotante *e* numerum, cuius logarithmus est unitas, ut sit

$$e=2,718281828459045:$$

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\text{etc.}}}}}=\frac{1}{e-2},$$

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\text{etc.}}}}}=e-1,$$

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\text{etc.}}}}}=2,$$

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\text{etc.}}}}}=2,$$

$$1+\cfrac{4}{2+\cfrac{5}{3+\cfrac{6}{4+\cfrac{7}{5+\text{etc.}}}}}=\frac{9}{4}$$

$$1+\cfrac{5}{2+\cfrac{6}{3+\cfrac{7}{4+\cfrac{8}{5+\text{etc.}}}}}=\frac{52}{21},$$

$$1+\cfrac{6}{2+\cfrac{7}{3+\cfrac{8}{4+\cfrac{9}{5+\text{etc.}}}}}=\frac{365}{136},$$

$$1+\cfrac{7}{2+\cfrac{8}{3+\cfrac{9}{4+\cfrac{10}{5+\text{etc.}}}}}=\frac{3006}{1045},$$

$$1+\cfrac{8}{2+\cfrac{9}{3+\cfrac{10}{4+\cfrac{11}{5+\text{etc.}}}}}=\frac{28357}{9276},$$

ubi modo prorsus singulari usu venit, ut binae priores numerum transcendentem *e* implicent, dum sequentes omnes numeris rationalibus exprimuntur.

2. Hoc eo magis mirum videtur, quod etiam casus praecedentes, ubi pro *n* vel cyphra vel numeri negativi ponuntur, valoribus rationalibus continentur, quibus quidem casibus ipsa fractionis continuae forma abrumpitur.
Erit enim

$$1+\cfrac{0}{2+\cfrac{1}{3+\cfrac{2}{4+\text{etc.}}}}=1,$$

$$1-\cfrac{1}{2+\cfrac{0}{3+\cfrac{1}{4+\text{etc.}}}}=\frac{1}{2},$$

$$1-\cfrac{2}{2+\cfrac{1}{3+\cfrac{0}{4+\text{etc.}}}}=-\frac{1}{5},$$

$$1-\cfrac{3}{2-\cfrac{2}{3-\cfrac{1}{4+\cfrac{0}{5+\text{etc.}}}}}=-\frac{19}{14},$$

$$1-\cfrac{4}{2-\cfrac{3}{3-\cfrac{2}{4-\cfrac{1}{5+0}}}}=-\frac{151}{37},$$

$$1-\cfrac{5}{2-\cfrac{4}{3-\cfrac{3}{4-\cfrac{2}{5-\cfrac{1}{6+0}}}}}=-\frac{1091}{34}.$$

Qua igitur lege tam hi valores quam praecedentes inter se cohaereant, haud abs re fore arbitror ostendisse. Imprimis autem iuvabit methodum exposuisse, qua isti valores investigari queant.

3. Primum igitur observo, si pro numero quocunque *n* valor fractionis continuae ita indicetur

$$f(n)=1+\cfrac{n}{2+\cfrac{n+1}{3+\cfrac{n+2}{4+\cfrac{n+3}{5+\text{etc.}}}}}$$

fore

$$f(n+1)=\frac{n(f(n)+1)}{f(n)+n-1},$$

cuius veritas in valoribus indicatis perspicitur, cum sit

$$f(1)=\tfrac{1}{e-2},\ \ f(2)=e-1,\ \ f(3)=2,\ \ f(4)=\tfrac{9}{4},$$
$$f(5)=\tfrac{52}{21},\ \ f(6)=\tfrac{365}{136},\ \ f(7)=\tfrac{3006}{1045},\ \ f(8)=\tfrac{28357}{9276}$$

et pro praecedentibus

$$f(0)=1,\ \ f(-1)=\tfrac{1}{2},\ \ f(-2)=-\tfrac{1}{5}, f(-3)=-\tfrac{19}{14},$$
$$f(-4)=-\tfrac{151}{37},\ \ f(-5)=-\tfrac{1091}{34},\ \ f(-6)=-\tfrac{7841}{887}.$$

Haec relatio inter binos valores contiguos intercedens non impedit, quominus casibus $n=1$ et $n=2$ sint transcendentes. Posito enim

$$n=0$$

fit

$$f(1)=\frac{0(1+1)}{1+0-1}=\frac{0}{0},$$

quae expressio valori

$$\frac{1}{e-2}$$

non adversatur, etiamsi hic inde elici nequeat. Deinde posito

$$n=2$$

prodit

$$f(3)=\frac{2(f(2)+1)}{f(2)+1}=2,$$

ita ut ipse valor

$$f(2)=e-1$$

hic non in computum veniat.

4. Investigatio autem horum valorum haud parum ardua videtur; quare, quemadmodum ad eos pervenerim, dilucide exponam, quandoquidem methodus, qua sumusus, multo latius patet ac fortasse ad alias praeclusas speculationes deducere potest. Sumsi igitur binos numeros indefinitos *m* et *n* eorumque certam quandam functionem, quae sit *p*, sum contemplatus, unde similes functiones eorundem numerorum una pluribusve unitatibus auctorum formavi, quas sumta littera φ pro signo huius functionis ita repraesento:

$$\begin{array}{lll} p=\varphi(m \text{ et } n), & p'=\varphi(m \text{ et } n+1), & p''=\varphi(m \text{ et } n+2), \\ q=\varphi(m+1 \text{ et } n), & q'=\varphi(m+1 \text{ et } n+1), & q''=\varphi(m+1 \text{ et } n+2), \\ r=\varphi(m+2 \text{ et } n), & r'=\varphi(m+2 \text{ et } n+1), & r''=\varphi(m+2 \text{ et } n+2), \\ s=\varphi(m+3 \text{ et } n) & s'=\varphi(m+3 \text{ et } n+1) & s''=\varphi(m+3 \text{ et } n+2) \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

Functionem autem φ eius indolis esse statuo, ut sit

$$p=Am+\quad Bn+C+\frac{Dnn+En+F}{p'};$$

erit

$$q=A(m+1)+Bn+C+\frac{Dnn+En+F}{q'},$$
$$r=A(m+2)+Bn+C+\frac{Dnn+En+F}{r'},$$
$$s=A(m+3)+Bn+C+\frac{Dnn+En+F}{s'}$$
etc.

5. Cum igitur *p', q', r', s'* etc. oriantur ex *p, q, r, s* etc., si servato numero *m* alter *n* unitate augeatur, erit simili modo

$$p'=Am+\quad B(n+1)+C+\frac{D(n+1)^2+E(n+1)+F}{p''};$$
$$q'=A(m+1)+B(n+1)+C+\frac{D(n+1)^2+E(n+1)+F}{q''},$$
$$r'=A(m+2)+B(n+1)+C+\frac{D(n+1)^2+E(n+1)+F}{r''}$$
etc.

tum vero ob eandem rationem

$$p'' = Am + \quad B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{p'''};$$

$$q'' = A(m+1) + B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{q'''},$$

$$r'' = A(m+2) + B(n+2) + C + \frac{D(n+2)^2 + E(n+2) + F}{r'''}$$

etc.

atque

$$p''' = Am + \quad B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{p''''};$$

$$q''' = A(m+1) + B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{q''''},$$

$$r''' = A(m+2) + B(n+3) + C + \frac{D(n+3)^2 + E(n+3) + F}{r''''}$$

etc.

sicque porro ulterius progrediendo.

6. Hinc functio p sequenti modo per fractionem continuam infinitam exprimetur

$$p = Am + Bn + C + \cfrac{Dn^2 + En + F}{Am + B(n+1) + C + \cfrac{D(n+1)^2 + E(n+1) + F}{Am + B(n+2) + C + \cfrac{D(n+2)^2 + E(n+2) + F}{Am + B(n+3) + C + \text{etc.}}}}$$

unde servato n, si loco m successive scribantur numeri $m+1,\ m+2,\ m+3$ *etc.*, prodibunt valores functionum q, r, s, t etc. per similes fractiones continuas expressi. Nunc igitur quaeritur, cuiusmodi relatio sit intercessura inter functiones p et q. Qua inventa per superiorem analogiam simul relatio inter omnes functiones hic exhibitas constabit. Quod cum a priori determinatu nimis difficile videatur, coniectura utendum censeo.

7. Videamus ergo, num inter p et q huiusmodi relatio statui queat

$$(p + (\alpha - A)m + (\beta - B)n + \gamma - C)(q + (\delta - A)m + (\varepsilon - B)n + \zeta - A - C)$$
$$= \lambda mm + \mu m + \nu,$$

unde servato m, si loco n scribatur $n+1$, erit

$$(p' + (\alpha - A)m + (\beta - B)(n+1) + \gamma - C)$$
$$\times (q' + (\delta - A)m + (\varepsilon - B)(n+1) + \zeta - A - C)$$
$$= \lambda mm + \mu m + \nu$$

At si ibi pro p et q superiores valores per p' et q' substituantur, prodibit

$$(\alpha m+\beta n+\gamma+F+\tfrac{Dnn+En+F}{p'})(\delta m+\varepsilon n+\zeta+\tfrac{Dnn+En+F}{q'})$$
$$=\lambda mm+\mu m+\nu,$$

quae evolvitur in hanc

$$\begin{aligned}&(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)p'q'-(\lambda mm+\mu m+\nu)p'q'\\&+(\alpha m+\beta n+\gamma)(Dnn+En+F)p'\\&+(\delta m+\varepsilon n+\zeta)(Dnn+En+F)q'\\&+(Dnn+En+F)^2=0,\end{aligned}$$

quae cum illa congruere debet. Unde perspicuum est esse oportere

$$\begin{aligned}&(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)-\lambda mm-\mu m-\nu\\&=\theta(Dnn+En+F),\end{aligned}$$

ut divisione per $\theta(Dnn+En+F)$ instituta fiat

$$\begin{aligned}&p'q'+\tfrac{1}{\theta}(\alpha m+\beta n+\gamma)p'+(\delta m+\varepsilon n+\zeta)q'\\&+\tfrac{1}{\theta}(Dnn+En+F)=0,\end{aligned}$$

quae per factores repraesentata ita exhibeatur

$$\begin{aligned}&(p'+\tfrac{\delta m+\varepsilon n+\zeta}{\theta})(q'+\tfrac{\alpha m+\beta n+\gamma}{\theta})\\&=\tfrac{(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)}{\theta\theta}-\tfrac{1}{\theta}(Dnn+En+F),\end{aligned}$$

seu

$$\begin{aligned}&(p'+\tfrac{\delta m+\varepsilon n+\zeta}{\theta})(q'+\tfrac{\alpha m+\beta n+\gamma}{\theta})\\&=\tfrac{\lambda mm+\mu m+\nu}{\theta\theta}.\end{aligned}$$

8. Comparetur haec forma cum priori

$$\begin{aligned}&(p'+(\alpha-A)m+(\beta-B)n+\gamma-B-C)\\&\times(q'+(\delta-A)m+(\varepsilon-B)n+\varepsilon+\zeta-A-B-C)\\&=\lambda mm+\mu m+\nu,\end{aligned}$$

unde statim colligitur

$$\theta\theta=1$$

ideoque vel

$$\theta=1 \text{ vel } \theta=-1.$$

Tum vero esse debet

$$\delta=\theta(\alpha-A),\ \varepsilon=\theta(\beta-B),\ \zeta=\theta(\beta+\gamma-B-C),$$
$$\alpha=\theta(\delta-A),\ \beta=\theta(\varepsilon-B),\ \gamma=\theta(\varepsilon+\zeta-A-B-C).$$

Quia ergo valor $\theta=1$ non convenit, ponamus

$$\theta=-1,$$

ut habeamus

$$\alpha+\delta=A,\ \beta+\varepsilon=B,\ \beta+\delta+\zeta=B+C \text{ et } \gamma+\varepsilon+\zeta=A+B+C$$

hincque

$$\varepsilon-\beta=\mathrm{A};$$

ergo

$$\beta=\tfrac{1}{2}(B-A),\ \varepsilon=\tfrac{1}{2}(A+B) \text{ et } \gamma+\zeta=\tfrac{1}{2}\left(A+B\right)+C.$$

Praeterea vero haec conditio est ad implenda

$$(\alpha m+\beta n+\gamma)(\delta m+\varepsilon n+\zeta)=\lambda mm+\mu m+\nu-Dnn-En-F$$
$$=a\delta mm+a\varepsilon mn+\alpha\zeta m+\beta\zeta n+\gamma\zeta+\beta\varepsilon nn+\beta\delta mn+\gamma\delta m+\gamma\varepsilon n.$$

Erit ergo

$$\lambda=\alpha\delta,\ \mu=\alpha\zeta+\gamma\delta,\ D=-\beta\varepsilon,\ E=-\beta\zeta-\gamma\varepsilon,$$
$$\nu-F=\gamma\zeta \text{ et } \alpha\varepsilon+\beta\delta=0,$$

unde primo fit

$$D=-\beta\varepsilon=\tfrac{1}{4}(AA-BB),$$

deinde

$$\tfrac{1}{2}\alpha(A+B)+\tfrac{1}{2}\delta(B-A)=0$$

seu

$$\delta=\tfrac{A+B}{A-B}\alpha$$

ideoque

$$\alpha=\tfrac{1}{2}(B-A) \text{ et } \delta=\tfrac{1}{2}(A+B).$$

Tum vero erit

$$E+\tfrac{1}{2}\zeta(B-A)+\tfrac{1}{2}\gamma(A+B)=0$$

seu

$$E+\tfrac{1}{4}B(A+B)+\tfrac{1}{2}BC+\tfrac{1}{2}A(\gamma-\zeta)=0$$

hincque

$$\gamma-\zeta=\tfrac{BC}{A}+\tfrac{B(A+B)}{2A}+\tfrac{2E}{A};$$

ergo

$$\zeta=\tfrac{1}{4}(A+B)+\tfrac{1}{2}C+\tfrac{BC}{2A}+\tfrac{B(A+B)}{4A}+\tfrac{E}{A},$$
$$\gamma=\tfrac{1}{4}(A+B)+\tfrac{1}{2}C-\tfrac{BC}{2A}-\tfrac{B(A+B)}{4A}-\tfrac{E}{A},$$

sive hoc modo

$$\zeta = \tfrac{1}{4}(A+B+2C)(1+\tfrac{B}{A})+\tfrac{E}{B} = \tfrac{(A+B)(A+B+2C)}{4A}+\tfrac{E}{A},$$
$$\gamma = \tfrac{1}{4}(A+B+2C)(1-\tfrac{B}{A})-\tfrac{E}{A} = \tfrac{(A-B)(A+B+2C)}{4A}-\tfrac{E}{A}.$$

9. Relatio ergo inter *p* et *q* assumta subsistere nequit, nisi sit

$$D = \tfrac{1}{4}(AA-BB);$$

qui valor si ipsi *D* tribuatur, sequentes litterae ita se habebunt:

$$\alpha = \tfrac{1}{2}(A-B),\ \delta = \tfrac{1}{2}(A+B),\ \gamma = \tfrac{(A-B)(A+B+2C)}{4A}-\tfrac{E}{A},$$
$$\beta = -\tfrac{1}{2}(A-B),\ \varepsilon = \tfrac{1}{2}(A+B),\ \zeta = \tfrac{(A+B)(A+B+2C)}{4A}+\tfrac{E}{A},$$
$$\lambda = \tfrac{1}{4}(AA-BB) = D,\ \mu = \tfrac{(AA-BB)(A+B+2C)}{4A}-\tfrac{BE}{A},$$
$$\nu = \tfrac{(AA-BB)(A+B+2C)^2}{16AA}-\tfrac{BE(A+B+2C)}{2AA}-\tfrac{EE}{AA}+F$$

hincque porro

$$\alpha - A = -\tfrac{1}{2}(A+B),\ \beta - B = -\tfrac{1}{2}(A+B),$$
$$\gamma - C = \tfrac{AA-BB}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A},$$
$$\delta - A = -\tfrac{1}{2}(A-B),\ \varepsilon - B = \tfrac{1}{2}(A-B)$$
$$\zeta - A - C = -\tfrac{(A-B)(3A+B)}{4A}-\tfrac{C(A-B)}{2A}+\tfrac{E}{A},$$

unde inter *p* et *q* haec resultat aequatio

$$(p-\tfrac{1}{2}(A+B)(m+n)+\tfrac{AA-BB}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A})$$
$$\times(q-\tfrac{1}{2}(A-B)(m-n)-\tfrac{(A-B)(3A+B)}{4A}-\tfrac{C(A+B)}{2A}+\tfrac{E}{A})$$
$$= \lambda mm + \mu m + \nu.$$

10. Ponamus ad abbreviandum

$$P = \tfrac{(A+B)(A-B)}{4A}-\tfrac{C(A+B)}{2A}-\tfrac{E}{A},$$
$$Q = \tfrac{(A-B)(3A+B)}{4A}+\tfrac{C(A-B)}{2A}-\tfrac{E}{A},$$

ut sit

$$(p-\tfrac{1}{2}(A+B)(m+n)+P)\times(q-\tfrac{1}{2}(A-B)(m-n)-Q)$$
$$=\lambda mm+\mu m+v;$$

erit

$$p=\tfrac{1}{2}(A+B)(m+n)-P+\frac{\lambda mm+\mu m+v}{q-\frac{1}{2}(A-B)(m-n)-Q}.$$

Simili vero modo est

$$q=\tfrac{1}{2}(A+B)(m+n)+\tfrac{1}{2}(A+B)-P+\frac{\lambda(m+1)^2+\mu(m+1)+v}{r-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q},$$
$$r=\tfrac{1}{2}(A+B)(m+n)+\ (A+B)-P+\frac{\lambda(m+2)^2+\mu(m+2)+v}{s-\frac{1}{2}(A-B)(m-n)-(A-B)-Q},$$

unde fit

$$q-\tfrac{1}{2}(A-B)(m-n)-Q=Bm+An+\tfrac{1}{2}(A+B)-P-Q$$
$$+\frac{\lambda(m+1)^2+\mu(m+1)+v}{r-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q},$$
$$r-\tfrac{1}{2}(A-B)(m-n)-\tfrac{1}{2}(A-B)-Q=Bm+An+\tfrac{1}{2}(A+3B)-P-Q$$
$$+\frac{\lambda(m+2)^2+\mu(m+2)+v}{s-\frac{1}{2}(A-B)(m-n)-\frac{1}{2}(A-B)-Q}.$$

Est vero

$$P+Q=\frac{(A-B)(2A+B)}{2A}-\frac{BC}{A}-\frac{2E}{A}$$

hincque

$$q-\tfrac{1}{2}(A-B)(m-n)-Q=Bm+An+B+\frac{BB-AA+2BC+4E}{2A}.$$

Quare si brevitatis gratia ponatur

$$\frac{BB-AA+2BC+4E}{2A}=G,$$

erit

$$p=\tfrac{1}{2}(A+B)(m+n)+\frac{BB-AA}{4A}+\frac{C(A+B)}{2A}+\frac{E}{A}$$
$$+\cfrac{\lambda m^2+\mu m+v}{B(m+2)+An+G+\cfrac{\lambda(m+2)^2+\mu(m+2)+v}{B(m+3)+An+G+\text{etc.}}}$$

vel valorem *G* introducendo

$$p=\tfrac{1}{2}(A+B)(m+n)+\tfrac{1}{2}(C+G)$$
$$+\cfrac{\lambda m^2+\mu m+v}{B(m+1)+An+G+\cfrac{\lambda(m+1)^2+\mu(m+1)+v}{B(m+2)+An+G+\cfrac{\lambda(m+2)^2+\mu(m+2)+v}{B(m+3)+An+G+\text{etc.}}}}$$

Tum vero etiam erit

$$P=-\tfrac{1}{2}(C+G)\ \text{ et }\ Q=\tfrac{1}{2}(A-B)+\tfrac{1}{2}(C-G)$$

ideoque

$$(p-\tfrac{1}{2}(A+B)(m+n)-\tfrac{1}{2}(C+G))(q-\tfrac{1}{2}(A-B)(m+1-n)-\tfrac{1}{2}(C-G))$$
$$=\lambda mm+\mu m+\nu.$$

11. Quodsi ergo proposita fuerit huiusmodi fractio continua infinita

$$p = Am+Bn+C+\cfrac{Dm^2+En+F}{Am+B(n+1)+C+\cfrac{D(n+1)^2+E(n+1)+F}{Am+B(n+2)+C+\cfrac{D(n+2)^2+E(n+2)+F}{Am+B(n+3)+C+\text{etc.}}}}$$

in qua sit

$$D=\tfrac{1}{4}(AA-BB),$$

indeque haec altera formetur

$$q = A(m+1)+Bn+C+\cfrac{Dn^2+En+F}{A(m+1)+B(n+1)+C+\cfrac{D(n+1)^2+E(n+1)+F}{A(m+1)+B(n+2)+C+\cfrac{D(n+2)^2+E(n+2)+F}{A(m+1)+B(n+3)+C+\text{etc.}}}}$$

dum loco *m* ubique scribitur $m+1$, primo relatio inter *p* et *q* assignari potest hoc modo: Ponatur brevitatis gratia

$$\frac{BB-AA+2BC+4E}{2A}=G,$$

et

$$\lambda=\tfrac{1}{2}(AA-BB),$$
$$\mu=\tfrac{1}{4}(AA-BB)+\tfrac{1}{2}(AC-BG),$$
$$\nu=\tfrac{1}{4}CC+(C-G)(A+B)-\tfrac{1}{4}GG+F$$

seu

$$\nu=F+\tfrac{1}{4}(C-G)(A+B+C+G)$$

eritque ista relatio

$$(p-\tfrac{1}{2}(A+B)(m+n)-\tfrac{1}{2}(C+G))(q-\tfrac{1}{2}(A-B)(m+1-n)-\tfrac{1}{2}(C-G))$$
$$=\lambda mm+\mu m+\nu.$$

tum vero praeterea functio *p* etiam huic alteri fractioni continuae aequatur

$$p = \frac{1}{2}(A+B)(m+n) + \frac{1}{2}(C+G)$$
$$+\cfrac{\lambda m^2+\mu m+\nu}{B(m+1)+An+G+\cfrac{\lambda(m+1)^2+\mu(m+1)+\nu}{B(m+2)+An+G+\cfrac{\lambda(m+2)^2+\mu(m+2)+\nu}{B(m+3)+An+G+\text{etc.}}}}$$

12. Exempla supra proposita hinc nascuntur, si statuatur

$$D = \tfrac{1}{4}(AA-BB) = 0.$$

Sit ergo

$$B = A,$$

ut habeantur istae fractiones continuae

$$p = A(m+n)+C+\cfrac{En+F}{A(m+n+1)+C+\cfrac{E(n+1)+F}{A(m+n+2)+C+\cfrac{E(n+2)+F}{A(m+n+3)+C+\text{etc.}}}}$$

$$q = A(m+n+1)+C+\cfrac{En+F}{A(m+n+2)+C+\cfrac{E(n+1)+F}{A(m+n+3)+C+\cfrac{E(n+2)+F}{A(m+n+4)+C+\text{etc.}}}}$$

quarum relatio posito

$$G = C + \tfrac{2E}{A}$$

et

$$\mu = \tfrac{1}{2}A\big(C-G\big),\ \nu = F + \tfrac{1}{4}(C-G)(2A+C+G)$$

seu

$$\mu = -E,\ \nu = F - \tfrac{2E}{A}(2A+C+G)$$

erit

$$(p - A(m+n) - \tfrac{1}{2}(C+G))(q - \tfrac{1}{2}(C-G))$$
$$= \mu m + \nu,$$

et quantitas *p* etiam alio modo per fractionem continuam infinitam exprimetur:

$$p = A(m+n) + \frac{1}{2}(C+G) +$$
$$\cfrac{\mu m+\nu}{A(m+n+1)+G+\cfrac{\mu(n+1)+\nu}{A(m+n+2)+G+\cfrac{\mu(m+2)+\nu}{A(m+n+3)+G+\text{etc.}}}}$$

13. Cum igitur hic tam numeratores quam denominatores progressionem arithmeticam constituant, eorum formam simpliciorem reddamus sintque binae fractiones continuae infinitae

$$p = a + \cfrac{f}{a+b+\cfrac{f+g}{a+2b+\cfrac{f+2g}{a+3b+\cfrac{f+3g}{a+4b+\text{etc.}}}}}$$

$$q = a + b + \cfrac{f}{a+2b+\cfrac{f+g}{a+3b+\cfrac{f+2g}{a+4b+\cfrac{f+3g}{a+5b+\text{etc.}}}}}$$

ita ut ex fractione p nascatur q, si loco a scribatur $a+b$. Erit ergo

$$A(m+n)+C=a,\ En+F=f,\ A=b,\ E=g$$

hincque

$$C=a-b(m+n) \text{ et } F=f-gn,$$

unde conficitur

$$G=a-b(m+n)+\frac{2g}{b},\ C-G=-\frac{2g}{b}$$

$$C+G=2a-2b(m+n)+\frac{2g}{b};$$

tum

$$\mu=-g,\ \nu=f-gn-\frac{g}{b}(a-b(m+n)C+\frac{g}{b})$$

seu

$$\nu=f-\frac{g(a+b)}{b}+gm-\frac{gg}{bb};$$

ergo

$$\mu m+\nu=f-\frac{g(a+b)}{b}-\frac{gg}{bb},$$

$$A(m+n)+\tfrac{1}{2}(C+G)=a+\frac{g}{b}$$

et

$$\tfrac{1}{2}(C-G)=-\frac{g}{b}.$$

Quare inter p et q haec habetur relatio

$$(p-a-\tfrac{g}{b})(q+\tfrac{g}{b})=f-\frac{g(a+b)}{b}-\frac{gg}{bb};$$

seu

$$pq+\frac{g}{b}p-(a+\frac{g}{b})q=f-g,$$

unde pro p etiam haec habetur fractio continua

$$p=a+\frac{g}{b}+\cfrac{f-\frac{g(a+b)}{b}-\frac{gg}{bb}}{a+b+\frac{2g}{b}+\cfrac{f-\frac{g(a+2b)}{b}-\frac{gg}{bb}}{a+2b+\frac{2g}{b}+\cfrac{f-\frac{g(a+3b)}{b}-\frac{gg}{bb}}{a+3b+\frac{2g}{b}+\text{etc.}}}}$$

14. Ut fractiones tollamus, ponamus $g=bh$, ut habeamus has fractiones continuas

$$p=a+\cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\cfrac{f+3bh}{a+4b+\text{etc.}}}}}$$

$$q=a+b+\cfrac{f}{a+2b+\cfrac{f+bh}{a+3b+\cfrac{f+2bh}{a+4b+\cfrac{f+3bh}{a+5b+\text{etc.}}}}}$$

quarum relatio ita se habet, ut sit

$$(p-a-h)(q+h)=f-(a+b)h-hh$$

seu

$$pq+hp-(a+h)q=f-bh,$$

unde pro p elicitur haec quoque fractio continua

$$p=a+h+\cfrac{f-(a+b)h-hh}{a+b+2h+\cfrac{f-(a+2b)h-hh}{a+2b+2h+\cfrac{f-(a+3b)h-hh}{a+3b+2h+\text{etc.}}}}$$

Cum igitur haec fractio continua primae sit aequalis, haec autem abrumpatur, quoties fuerit

$$f=(a+ib)h+hh$$

denotante i numerutn integrum positivum, toties valor primae rationaliter assignari potest.

15. Ex relatione inter p et q inventa per p quoque q ita exprimitur

$$q=-h+\frac{f-(a+b)h-hh}{-a-h+p},$$

et cum p oriatur ex q, si loco a scribatur $a-b$, si seriei p, q, r etc. termini praecedentes sint o, n, m etc., erit

$$p=-h+\frac{f-ah-hh}{-a+b-h+o},$$
$$o=-h+\frac{f-(a-b)h-hh}{-a+2b-h+n},$$
$$n=-h+\frac{f-(a-2b)h-hh}{-a+3b-h+m},$$
etc.,

unde pro p etiam haec fractio continua obtinetur

$$p=-h+\cfrac{f-ah-hh}{b-a-2h+\cfrac{f+(b-a)h-hh}{2b-a-2h+\cfrac{f+(2b-a)h-hh}{3b-a-2h+\text{etc.}}}}$$

quam eandem ex nostris formulis generalibus invenissemus, si supra § 12 posuissemus $B=-A$. Quare etiam valor rationaliter exprimi poterit, quoties fuerit

$$hh=(ib-a)h+f,$$

nisi forte his casibus denominator isti numeratori evanescenti subiectus quoque evanescat.

16. Ex ipsa autem fractione continua proposita

$$p=a+\cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\text{etc.}}}}$$

alia immediate hoc modo deduci potest ipsi aequalis. Cum enim sit

$$p=a+\frac{f}{p'},\ p'=a+b+\frac{f+bh}{p''},\ p''=a+2b+\frac{f+2bh}{p'''}\ \text{etc.},$$

erit regrediendo

$$p_{,}=a-b+\frac{f-bh}{p},\ \ p_{,,}=a-2b+\frac{f-2bh}{p_{,}},\ \ p_{,,,}=a-3b+\frac{f-3bh}{p_{,,}}\ \text{etc.},$$

hincque

$$p=\frac{f-bh}{b-a+p_{,}},\ \ p_{,}=\frac{f-2bh}{2b-ap_{,,}},\ \ p_{,,}=\frac{f-3bh}{3b-a+p_{,,,}}\ \ \text{etc.},$$

unde concludimus

$$p=\cfrac{f-bh}{b-a+\cfrac{f-2bh}{2b-a+\cfrac{f-3bh}{3b-a+\cfrac{f-4bh}{4b-a+\text{etc.}}}}}$$

ita ut etiam casibus $f = ibh$ valor rationaliter exhiberi queat.

17. En ergo quatuor fractiones continuas inter se aequales:

$$\text{I. } p = a + \cfrac{f}{a+b+\cfrac{f+bh}{a+2b+\cfrac{f+2bh}{a+3b+\cfrac{f+3bh}{a+4b+\text{etc.}}}}}$$

$$\text{II. } p = \cfrac{f-bh}{b-a+\cfrac{f-2bh}{2b-a+\cfrac{f-3bh}{3b-a+\cfrac{f-4bh}{4b-a+\text{etc.}}}}}$$

$$\text{III. } p = a+h+\cfrac{f-(a+b)h-hh}{a+b+2h+\cfrac{f-(a+2b)h-hh}{a+2b+2h+\cfrac{f-(a+3b)h-hh}{a+3b+2h+\text{etc.}}}}$$

$$\text{IV. } p = -h + \cfrac{f-ah-hh}{b-a-2h+\cfrac{f+(b-a)h-hh}{2b-a-2h+\cfrac{f+(2b-a)h-hh}{3b-a-2h+\text{etc.}}}}$$

18. Quo ad formam initio propositam propius accedamus, sit

$$a = m,\ f = n,\ b = 1 \text{ et } h = 1$$

atque habebimus:

$$\text{I. } p = m + \cfrac{n}{m+1+\cfrac{n+1}{m+2+\cfrac{n+2}{m+3+\cfrac{n+3}{m+\text{etc.}}}}}$$

$$\text{II. } p = \cfrac{n-1}{-m+1+\cfrac{n-2}{-m+2+\cfrac{n-3}{-m+3+\cfrac{n-4}{-m+4+\text{etc.}}}}}$$

$$\text{III. } p = m+1+\cfrac{n-m-2}{m+3+\cfrac{n-m-3}{m+4+\cfrac{n-m-4}{m+5+\cfrac{n-m-5}{m+6+\text{etc.}}}}}$$

$$\text{IV. } p = -1 + \cfrac{n-m-1}{-m-1+\cfrac{n-m}{-m+1+\cfrac{n-m+2}{-m+2+\text{etc.}}}}$$

Quare denotante *i* numerum integrum positivum, cyphra non exclusa, fractionis nostrae continuae valor rationaliter exprimi poterit his casibus

$$\text{I. } n=i, \text{II. } n=m+2+i, \text{III. } n=m+1-i,$$

nisi forte incommodum supra memoratum locum inveniat.

19. Ex casibus $n=i$ raro valor quaesitus reperitur ob memoratum incommodum, quo etiam denominator in nihilum abit. Si enim $n=1$, quo fit

$$p=m+\cfrac{1}{m+1+\cfrac{2}{m+2+\cfrac{3}{m+3+\cfrac{4}{m+4 \text{ etc.}}}}}$$

certe non est $p=0$, etsi secunda forma id ostendere videtur, unde affirmare possumus esse

$$0=1-m-\cfrac{1}{2-m-\cfrac{2}{3-m-\cfrac{3}{4-m-\cfrac{4}{5-m-\text{etc.}}}}}$$

Quodsi prima forma generalis ad hunc casum accommodetur, erit

$$a=1-m,\ b=1,\ f=-1 \text{ et } h=-1,$$

unde secunda dat

$$p=\cfrac{1-1}{m+\cfrac{1}{m+1+\cfrac{2}{m+2+\text{etc.}}}}$$

tertia vera

$$p=-m-\cfrac{m}{-m+\cfrac{1-m}{1-m+\cfrac{2-m}{2-m \text{ etc.}}}}$$

et quarta

$$p=+1-\cfrac{1-m}{m-2-\cfrac{2-m}{m-1-\cfrac{3-m}{m-\text{etc.}}}}$$

quae ergo nihilo sunt aequales.

20. Cum ex forma secunda § 18 sit

$$\frac{n-1}{p}=1-m+\cfrac{n-2}{2-m+\cfrac{n-3}{3-m+\cfrac{n-4}{4-m+\text{etc.}}}}$$

si haec cum prima generali comparetur, erit

$$a = 1 - m,\ b = 1,\ f = n - 2 \text{ et } h = -1,$$

unde forma tertia praebet

$$\frac{n-1}{p} = -m + \cfrac{n-m-1}{-m + \cfrac{n-m}{1-m + \cfrac{n-m+1}{2-m + \cfrac{n-m+2}{3-m + \text{etc.}}}}}$$

et quarta

$$\frac{n-1}{p} = 1 + \cfrac{n-m-2}{m+2 + \cfrac{n-m-3}{m+3 + \cfrac{n-m-4}{m+4 + \text{etc.}}}}$$

sicque duae novae expressiones pro p habentur; similique modo plures aliae exhiberi possunt.

21. Invento autem valore p facile definitur haec fractio continua

$$x = m + \cfrac{n+1}{m+1 + \cfrac{n+2}{m+2 + \cfrac{n+3}{m+3 + \text{etc.}}}}$$

Scribatur enim ibi $m-1$ loco m ponaturque

$$q = m - 1 + \cfrac{n}{m + \cfrac{n+1}{m+1 + \cfrac{n+2}{m+2 + \text{etc.}}}} = m - 1 + \frac{n}{x};$$

at ex tertia erit

$$q = m + \cfrac{n-m-1}{m+2 + \cfrac{n-m-2}{m+3 + \cfrac{n-m-3}{m+4 + \text{etc.}}}} = m + \frac{n-m-1}{p+1},$$

quibus binis valoribus ipsius q aequatis fit

$$-1 + \frac{n}{x} = \frac{n-m-1}{p+1}$$

et

$$\frac{n}{x} = \frac{n-m-p}{p+1}$$

sicque

$$x = \frac{n(p+1)}{p-m+n}.$$

22. Cum igitur posito $n = m + 2$ sit $p = m + 1$, erit

$$m+1=p=m+\cfrac{m+2}{m+1+\cfrac{m+3}{m+2+\cfrac{m+4}{m+3+\text{ etc.}}}}$$

similique modo ponendo $n=m+3,\ n=m+4$ etc.

$$m+1+\frac{1}{m+3}=m+\cfrac{m+3}{m+1+\cfrac{m+4}{m+2+\cfrac{m+5}{m+3+\text{ etc.}}}}=q,$$

$$m+1+\cfrac{2}{m+3+\cfrac{1}{m+4}}=m+\cfrac{m+4}{m+1+\cfrac{m+5}{m+2+\cfrac{m+6}{m+3+\text{ etc.}}}}=r,$$

$$m+1+\cfrac{3}{m+3+\cfrac{2}{m+4+\cfrac{1}{m+5}}}=m+\cfrac{m+5}{m+1+\cfrac{m+6}{m+2+\cfrac{m+7}{m+3+\text{ etc.}}}}=s,$$

$$m+1+\cfrac{4}{m+3+\cfrac{3}{m+4+\cfrac{2}{m+5+\cfrac{1}{m+6}}}}=m+\cfrac{m+6}{m+1+\cfrac{m+7}{m+3+\cfrac{m+8}{m+4+\text{ etc.}}}}=t,$$

unde posito $m=1$ casus rationales supra observati consequuntur. Hi autem valores ita progrediuntur, ut sit

$$p=m+1,\qquad q=\frac{(m+2)(p+1)}{p+2},$$

$$r=\frac{(m+3)(q+1)}{q+3},\qquad s=\frac{(m+4)(r+1)}{r+4}$$

seu

$$q=(m+2)(1-\tfrac{1}{p+2}),$$

$$r=(m+3)(1-\tfrac{2}{q+3}),$$

$$s=(m+4)(1-\tfrac{3}{r+4}),$$

etc.,

quae expressiones etiam ita exhiberi possunt:

$$q=m+2-\frac{m+2}{m+3}$$

$$r=m+3-\cfrac{2(m+3)}{m+5-\cfrac{m+2}{m+3}}$$

$$s=m+4-\cfrac{3(m+4)}{m+7-\cfrac{2(m+3)}{m+5-\cfrac{m+2}{m+3}}}$$

Facta autem evolutione invenitur

$$p = m+1,$$
$$q = \frac{(m+2)(m+2)}{(m+3)},$$
$$r = \frac{(m+3)(mm+5m+7)}{mm+7m+13},$$
$$s = \frac{(m+4)(m^3+9m^2+29m+34)}{\left(m^3+12mm+50m+73\right)},$$
$$t = \frac{(m+5)(m^4+14m^3+77m^2+200m+209)}{m^4+18m^3+125m^2+400m+501}.$$

23. Denotent *p, q, r, s, t* etc. fractiones continuas in paragrapho praecedente exhibitas sitque

$$p = (m+1)\tfrac{a}{\alpha},\ q = (m+2)\tfrac{b}{\beta},\ r = (m+3)\tfrac{c}{\gamma}\ \text{etc.};$$

erit

$$a = 1,\ \alpha = 1,$$

reliquae vero litterae ita a se invicem pendent, ut sit

$$\begin{aligned} b &= (m+1)a+\alpha, & \beta &= (m+1)a+2\alpha,\\ c &= (m+2)b+\beta, & \gamma &= (m+2)b+3\beta,\\ d &= (m+3)c+\gamma, & \delta &= (m+3)c+4\gamma\\ e &= (m+4)d+\delta, & \varepsilon &= (m+4)d+5\delta\\ &\text{etc.} & &\text{etc.,} \end{aligned}$$

unde relatio inter litteras latinas et graecas seorsim colligitur

$$\begin{aligned} b &= (m+2)a, & \beta &= \left(m+3\right)\alpha,.\\ c &= (m+4)b-1(m+1)\alpha, & \gamma &= (m+5)\beta-1(m+2)\alpha,\\ d &= (m+6)c-2(m+2)b, & \delta &= (m+7)\gamma-2(m+3)\beta,\\ e &= (m+8)d-3(m+3)c & \varepsilon &= (m+9)\delta-3(m+4)\gamma\\ &\text{etc.} & &\text{etc.} \end{aligned}$$

Inventis autem denominatoribus $\alpha,\ \beta,\ \gamma,\ \delta$ etc. erunt numeratores

$$b = \beta-\alpha,\ c = \gamma-2\beta,\ d = \delta-3\gamma,\ e = \varepsilon-4\delta\ \text{etc.}$$

At si numeratores *a, b, c, d, e* iam sint inventi, erunt denominatores

$$\alpha = a,\ \beta = 2b - (m+1)a,\ \gamma = 3c - 2(m+2)b,$$
$$\delta = 4d - 3(m+3)c,\ \varepsilon = 5e - 4(m+4)d \text{ etc.}$$

Vidimus autem esse

$$\begin{array}{ll} a = 1, & \alpha = 1, \\ b = m+2, & \beta = m+3, \\ c = m^2+5m+7, & \gamma = m^2+7m+13, \\ d = m^3+9m^2+29m+24, & \delta = m^3+12m^2+50m+73, \\ e = m^4+14m^3+77m^2+200m+209, & \varepsilon = m^4+18m^3+125m^2+400m+501. \end{array}$$

24. Ex valore cuiusque fractionis continuae § 22 definiri quoque potest valor praecedentis hoc modo

$$p = \frac{m+2-2q}{q-(m+2)},\ q = \frac{m+3-3r}{r-(m+3)},\ r = \frac{m+4-4s}{s-(m+4)} \text{ etc.,}$$

unde, si fractiones continuae ordine praecedentes designentur litteris O, N, M etc., erit

$$O = \frac{m+1-p}{p-(m+1)},\ N = \frac{m}{O-m},\ M = \frac{m-1+N}{N-(m-1)},\ L = \frac{m-2+2M}{M-(m-2)} \text{ etc.}$$

At est

$$O = m + \cfrac{m+1}{m+1+\cfrac{m+2}{m+2+\cfrac{m+3}{m+3+\cfrac{m+4}{m+4+\text{etc.}}}}}$$

cuius reliqui valores aequivalentes sunt

$$O = \cfrac{m}{1-m+\cfrac{m-1}{2-m+\cfrac{m-2}{3-m+\cfrac{m-3}{4-m+\text{etc.}}}}}$$

$$O = m+1-\cfrac{1}{m+3-\cfrac{2}{m+4-\cfrac{3}{m+5-\cfrac{4}{m+6-\cfrac{5}{m+7-\text{etc.}}}}}}$$

Hinc autem finitum valorem ipsius *O* expectare non licet, cum casu $m=1$ certe sit transcendens, qui quemadmodum sit investigandus, exponamus.

25. Ex forma ergo prima formentur hae formulae

$$O = m + \frac{m+1}{A},\ A = m+1+\frac{m+2}{B},\ B = m+2+\frac{m+3}{C} \text{ etc.}$$

eritque

$$OA = mA + m + 1,\ \ AB = (m+1)B + m + 2\ \text{ etc.}$$

Statuatur

$$O = -1 + \tfrac{1}{\omega}, A = -1 + \tfrac{1}{\alpha},\ \ B = -1 + \tfrac{1}{\beta}\ \text{etc.}$$

ac reperientur hae formulae

$$\alpha + (m+1)\omega = 1,\ \beta + (m+2)\alpha = 1,\ \gamma + (m+3)\beta = 1\ \text{etc.}$$

seu

$$\omega = \tfrac{1}{m+1} - \tfrac{\alpha}{m+1},\ \ \alpha = \tfrac{1}{m+2} - \tfrac{\beta}{m+2},\ \ \beta = \tfrac{1}{m+3} - \tfrac{\gamma}{m+3},$$

unde per seriem consuetam fit

$$\omega = \tfrac{1}{m+1} - \tfrac{1}{(m+1)(m+2)} + \tfrac{1}{(m+1)(m+2)(m+3)} - \tfrac{1}{(m+1)(m+2)(m+3)(m+4)} + \text{etc.},$$

cuius valor est

$$\omega = \tfrac{1}{e}\int e^z z^m dz$$

integrali hoc ita sumto, ut evanescat posito $z = 0$, quo facto statui debet $z = 1$. Hinc casibus, quibus m est numerus integer, erit;

$$\begin{array}{lll}
\text{si } m = 0,\ \omega = \tfrac{e-1}{e} & \text{et} & O = \tfrac{1}{e-1};\\
\text{si } m = 1,\ \omega = \tfrac{1}{e} & \text{et} & O = e - 1;\\
\text{si } m = 2,\ \omega = \tfrac{e-2}{e} & \text{et} & O = \tfrac{2}{e-2};\\
\text{si } m = 3,\ \omega = \tfrac{-2e+6}{e} & \text{et} & O = \tfrac{3e-6}{6-2e} = \tfrac{3(e-2)}{2(3-e)};\\
\text{si } m = 4,\ \omega = \tfrac{9e-24}{e} & \text{et} & O = \tfrac{24-8e}{9e-24} = \tfrac{4(6-2e)}{3(3e-8)} = \tfrac{8(3-e)}{3(3e-8)};\\
\text{si } m = 5,\ \omega = \tfrac{120-44e}{e} & \text{et} & O = \tfrac{45e-120}{120-44e} = \tfrac{5(9e-24)}{4(30-11e)} = \tfrac{15(3e-8)}{4(30-11e)};\\
\text{si } m = 6,\ \omega = \tfrac{265e-720}{e} & \text{et} & O = \tfrac{720-264e}{265e-720} = \tfrac{6(120-44e)}{5(53e-144)} = \tfrac{24(30-11e)}{5(53e-144)};\\
\text{si } m = 7,\ \omega = \tfrac{5040-1854e}{e} & \text{et} & O = \tfrac{1855e-5040}{5040-1854e} = \tfrac{7(265-720)}{6(720-309e)} = \tfrac{35(53e-144)}{6(720-309e)}.
\end{array}$$

Nisi *m* est numerus integer, valor ipsius *O* per numerum *e*, cuius logarithmus $= 1$, exprimi nequit.

26. Ponamus

$$m = 0,$$

quorsum omnes casus, quibus *m* est numerus integer, referri possunt, eritque

$$O=\frac{1}{e-1},\quad N=0,\quad M=-1,\quad L=\frac{-2+2M}{M+2}=-4,$$
$$K=\frac{-3+3L}{L+3}=15,\quad I=\frac{-4+4K}{K+4}=\frac{56}{19},$$
$$H=\frac{-5+5I}{I+5}=\frac{185}{151},\quad G=\frac{-6+6H}{H+6}=\frac{204}{1091},$$

unde sequentes oriuntur fractiones continuae:

$$1=0+\cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\text{etc.}}}}\qquad \frac{1}{-1}=0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}\qquad \frac{4}{3}=0+\cfrac{3}{1+\cfrac{4}{2+\cfrac{5}{3+\text{etc.}}}}$$

$$0=0+\cfrac{0}{1+\cfrac{1}{2+\cfrac{2}{3+\text{etc.}}}}\qquad \frac{21}{13}=0+\cfrac{4}{1+\cfrac{5}{2+\cfrac{6}{3+\text{etc.}}}}\qquad -1=0-\cfrac{1}{1+\cfrac{0}{2+\cfrac{1}{3+\text{etc.}}}}$$

$$\frac{136}{73}=0+\cfrac{5}{1+\cfrac{6}{2+\cfrac{7}{3+\text{etc.}}}}\qquad -4=0-\cfrac{2}{1-\cfrac{1}{2+\cfrac{0}{3+\text{etc.}}}}\qquad \frac{1045}{501}=0+\cfrac{6}{1+\cfrac{7}{2+\cfrac{8}{3+\text{etc.}}}}$$

$$15=0-\cfrac{3}{1-\cfrac{2}{2-\cfrac{1}{3-\text{etc.}}}}$$

quibus adiungi debent istae:

$$\frac{1}{e-1}=0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}\qquad \frac{1}{e-2}=1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}\qquad \frac{1}{2e-5}=2+\cfrac{1}{3+\cfrac{2}{4+\cfrac{3}{5+\text{etc.}}}}$$

$$\frac{1}{6e-16}=3+\cfrac{1}{4+\cfrac{2}{5+\cfrac{3}{6+\text{etc.}}}}\qquad \frac{1}{24e-65}=4+\cfrac{1}{5+\cfrac{2}{6+\cfrac{3}{7+\text{etc.}}}}\qquad \frac{1}{120e-326}=5+\cfrac{1}{6+\cfrac{2}{7+\cfrac{3}{8+\text{etc.}}}}$$

Si enim sit

$$x=m+\cfrac{1}{m+1+\cfrac{2}{m+2+\cfrac{3}{m+3+\text{etc.}}}}\quad =m+1-\cfrac{m+1}{m+3-\cfrac{m+2}{m+4-\cfrac{m+3}{m+5-\text{etc.}}}}$$

$$y=m+1+\cfrac{1}{m+2+\cfrac{2}{m+3+\cfrac{3}{m+4+\text{etc.}}}}\quad =m+2-\cfrac{m+2}{m+4-\cfrac{m+3}{m+5-\cfrac{m+4}{m+6-\text{etc.}}}}$$

erit

$$y=\frac{x}{m+1-x}.$$

27. Verum nostrae investigationes multo latius patent; quas ut accuratius evolvamus, ad formulas § 11 revertamur, quae nihil de sua amplitudine amittunt, etiamsi numeros *m* et *n* nihilo aequales statuamus. Quare fractiones continuae considerandae erunt hae:

$$p = C + \cfrac{F}{C+B+\cfrac{F+E+D}{C+2B+\cfrac{F+2E+4D}{C+3B+\cfrac{F+3E+9D}{C+4B+\text{etc.}}}}}$$

$$q=C+A+\cfrac{F}{C+A+B+\cfrac{F+E+D}{C+A+2B+\cfrac{F+2E+4D}{C+A+3B+\cfrac{F+3E+9D}{C+4B+\text{etc.}}}}}$$

$$r = C + 2A + \cfrac{F}{C+2A+B+\cfrac{F+E+D}{C+2A+2B+\cfrac{F+2E+4D}{C+2A+3B+\cfrac{F+3E+9D}{C+2A+4B+\text{etc.}}}}}$$

quae formae continuo ulterius continuantur scribendo $C+A$ loco *C*. In singulis ergo denominatores progressionem arithmeticam, numeratores vero progressionem secundi ordinis constituunt, cuius differentiae secundae sunt constantes. Hic autem assumimus esse

$$D = \tfrac{1}{4}(AA-BB).$$

Quodsi iam brevitatis gratia ponamus

$$G = \frac{BB-AA+2BC+4E}{2A} = \frac{BC-2D+2E}{A}$$

et

$$\lambda = \tfrac{1}{4}(AA-BB) = D,$$

$$\mu = \tfrac{1}{4}(AA-BB)+\tfrac{1}{2}(AC-BG) = \frac{(A+B+2C)D-BE}{A},$$

$$v = F + \tfrac{1}{4}(C-G)(A+B+C+G)$$

sive

$$v = F + \frac{CD(A+C)+DD-EE}{AA} + \frac{B(A+B+2C)(D-E)}{2AA},$$

erit

$$\left(p - \frac{(A+B)(C+D-2E)}{2A}\right)\left(q - \frac{(A-B)(A+C)-2C(D-E)}{2A}\right) = v$$

hincque per aliam fractionem continuam

$$p=\tfrac{1}{2}(C+G)+\cfrac{v}{G+B+\cfrac{v+\mu+\lambda}{G+2B+\cfrac{v+2\mu+4\lambda}{G+3B+\text{etc.}}}}$$

28. Proposita ergo fractione continua quacunque huius formae

$$p=a+\cfrac{f}{a+b+\cfrac{f+g}{a+2b+\cfrac{f+2g+h}{a+3b+\cfrac{f+3g+3h}{a+4b+\cfrac{f+4g+6h}{a+5b+\text{etc.}}}}}}$$

ea in aliam sibi aequalem transmutari potest. Comparatione enim facta est

$$C=a,\ B=b,\ F=f,\ E=g-\tfrac{1}{2}h,\ D=\tfrac{1}{2}h.$$

Capiatur ergo

$$A=\sqrt{(bb+2h)},$$

tum vero

$$G=\frac{ab+2g-2h}{A},$$

$$\lambda=\tfrac{1}{2}h,\quad \mu=\tfrac{1}{2}h+\tfrac{1}{2}(Aa-Gb)=\tfrac{1}{2}h+\frac{ah-b(g-h)}{A}$$

$$\nu=f+\frac{ah-b(g-h)}{2A}+\frac{aah-bb(g-h)-2ab(g-h)-2g(g-h)}{2AA}$$

hincque fiet

$$p=\tfrac{1}{2}(a+G)+\cfrac{\nu}{G+b+\cfrac{\nu+\mu+\lambda}{G+2b+\cfrac{\nu+2\mu+4\lambda}{G+3b+\cfrac{\nu+3\mu+9\lambda}{G+4b+\text{etc.}}}}}$$

29. Si ergo fuerit $f=0$, huius postremae fractionis continuae valor certe est $=a$, quicunque numeri reliquis litteris tribuantur. Statuatur ergo $g-h=k$, ut sit

$$A=\sqrt{(bb+2h)},\ G=\frac{ab+2k}{A},$$

$$\lambda=\tfrac{1}{2}h,\quad \mu=\tfrac{1}{2}h++\frac{ah-bk}{2A}\ \text{ et }\ \nu=\frac{ah-bk}{2A}+\frac{aah-bgk-2abk-2hk-2kk}{2AA},$$

eritque

$$\tfrac{1}{2}(a+G)=G+\cfrac{\nu}{G+b+\cfrac{\nu+\mu+\lambda}{G+2b+\cfrac{\nu+2\mu+4\lambda}{G+3b+\cfrac{\nu+3\mu+9\lambda}{G+4b+\text{etc.}}}}}$$

ubi si litterae *a*, *b*, *A* et *G* pro datis habeantur, erit

$$\lambda = \frac{AA-bb}{4},\quad \mu = \lambda + \frac{Aa-Gb}{2} = \frac{AA-bb}{4} + \frac{Aa-Gb}{2}$$
$$\nu = \frac{aa+ab+Aa+Bb-AG-GG}{4} = \frac{1}{4}(A-G)(a+b+G),$$

hinc

$$\nu + \mu + \lambda = \tfrac{1}{4}(a - b + A - G)(a + 2b + 2A + G),$$
$$\nu + 2\mu + 4\lambda = \tfrac{1}{4}(a - 2b + 2A - G)(a + 3b + 3A + G),$$
$$\nu + 3\mu + 9\lambda = \tfrac{1}{4}(a - 3b + 3A - G)(a + 4b + 4A + G).$$

Ponatur ad formulam contrahendam

$$a - G = 2\alpha,\ A - b = 2\gamma,\ a + G = 2\beta,\ A + b = 2\delta \text{ etc.};$$

[erit]

$$a = \cfrac{\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}}$$

cuius veritas in pluribus exemplis sponte elucet.

30. Si eaedem positiones retineantur, numerus autem *f* non nihilo aequalis capiatur, habebitur haec fractio continua

$$p = \alpha + \beta + \cfrac{f}{\alpha+\beta+(\delta-\gamma)+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta+2(\delta-\gamma)+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta+3(\delta-\gamma)+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}}$$

quae transformatur in hanc sibi aequalem

$$p = \beta + \cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+3(\delta-\gamma)+\text{etc.}}}}}$$

unde si vel γ vel δ evanescens capiatur, casus ante tractatus exsurgit. Haec autem binarum fractionum continuarum aequalitas omnia, quae hactenus sunt exposita, in se complectitur.

31. Ex his oriuntur formae, quas littera *q* denotavimus, si loco α et β scribamus $\alpha+\gamma$ et $\beta+\delta$, ita ut sit

$$q=\alpha+\beta+\gamma+\delta+\cfrac{f}{\alpha+\beta+2\delta+\cfrac{f+(\beta\gamma-\alpha\delta)+2\gamma\delta}{\alpha+\beta-\gamma+3\delta+\cfrac{f+2(\beta\gamma-\alpha\delta)+6\gamma\delta}{\alpha+\beta-2\gamma+4\delta+\cfrac{f+3(\beta\gamma-\alpha\delta)+12\gamma\delta}{\alpha+\beta-3\gamma+5\delta+\text{etc.}}}}}$$

itemque

$$q=\beta+\delta+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\beta-\alpha+4(\delta-\gamma)+\text{etc.}}}}$$

ita ut inter has binas expressiones subsistat haec relatio

$$(p-\beta)(q-\alpha-\gamma)=f+\alpha(\beta+\delta)$$

seu

$$pq-(\alpha+\gamma)p-\beta q+\beta\gamma-\alpha\delta=f,$$

cuius ope aequalitas binarum superiorum formularum methodo substitutionum, qua supra usi sumus, demonstrari potest.

32. Si ponamus

$$f+\alpha(\beta+\delta)=g,$$

ut sit

$$f=g-\alpha(\beta+\delta),$$

prior forma ita se habebit

$$p=\alpha+\beta+\cfrac{g-\alpha(\beta+\delta)}{\alpha+\beta+(\delta-\gamma)+\cfrac{g-(\alpha-\gamma)(\beta+2\delta)}{\alpha+\beta+2(\delta-\gamma)+\cfrac{g-(\alpha-2\gamma)(\beta+3\delta)}{\alpha+\beta+3(\delta-\gamma)+\cfrac{g-(\alpha-3\gamma)(\beta+4\delta)}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}},$$

cui aequalis est ista

$$p=\beta+\cfrac{f+\alpha(\beta+\delta)}{\beta-\alpha+(\delta-\gamma)+\cfrac{f+(\alpha+\gamma)(\beta+2\delta)}{\beta-\alpha+2(\delta-\gamma)+\cfrac{f+(\alpha+2\gamma)(\beta+3\delta)}{\beta-\alpha+3(\delta-\gamma)+\cfrac{f+(\alpha+3\gamma)(\beta+4\delta)}{\alpha+\beta+4(\delta-\gamma)+\text{etc.}}}}},$$

Atque hae formae maxime idoneae videntur, quarum aequalitas methodo directa explorari ac demonstrari possit. Talis autem methodus etiamnunc desideratur. Nullum autem est dubium, quin ea patefacta multa praeclara incrementa Analyseos expectare liceat. Cum igitur prior forma finita evadat, si fuerit

$$g=(\alpha-i\gamma)(\beta+(i+1)\delta),$$

intelligimus etiam posterioris valorem rationaliter exprimi posse, quoties fuerit

$$f=(\alpha-i\gamma)(\beta+(i+1)\delta)-\alpha(\beta+\delta)$$

seu

$$f=i(\alpha\delta-\beta\gamma-(i+1)\gamma\delta)$$

denotante *i* numerum integrum quemcunque.