ON SERIES IN WHICH THE PRODUCTS FROM TWO CONTIGUOUS TERMS CONSTITUTE A GIVEN PROGRESSION

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The question consists of this, from some proposed progression of numbers

$$\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F}\ \text{etc.},$$

which I have put in place to be examined here, in order that a series of this kind may be found

$$a,\ b,\ c,\ d,\ e,\ f\ \text{etc.},$$

in which there shall be

$$ab=\mathrm{A},\ bc=\mathrm{B},\ cd=\mathrm{C},\ de=\mathrm{D},\ ef=\mathrm{E},\ fg=\mathrm{F}\ \text{etc.},$$

where, even if the numbers A, B, C, D etc. may be rational and proceed to satisfy a simple law, generally it will usually be the case that the numbers *a, b, c, d* etc. thus may emerge to be transcending mainly. Moreover it is clear the whole problem can be defined by a single term of this series, certainly with which known all the remaining terms may be defined easily; indeed with the first term a found, thus all the remaining terms will themselves be obtained:

$$b=\frac{\mathrm{A}}{a},\ c=\frac{\mathrm{B}}{b},\ d=\frac{\mathrm{C}}{c},\ e=\frac{\mathrm{D}}{d}\ \text{etc}.$$

But I have observed a twofold way to be apparent for the solution of this question, of which the one may be resolved by a certain interpolation of the same series, but the other which may appear to be more direct, may be deduced according to continued fractions ; which two methods since they may complete the matter plainly in a different manner, the combining of these will reveal certain commendable prosperities. Therefore I will set out each method separately, then we are going to compare between themselves the matters which were elicited.

THE FIRST METHOD DEPENDING ON INTERPOLATION

1. We may consider the series sought being formed in this manner

$$\begin{matrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a, & ab, & abc, & abcd, & abcde, & abcdef, & abcdefg & \text{etc.,}\end{matrix}$$

which on account of

$$ab=\mathrm{A},\ bc=\mathrm{B},\ cd=\mathrm{C},\ de=\mathrm{D},\ ef=\mathrm{E}\ \text{etc.},$$

will be changed into this form

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & \mathrm{A}, & a\mathrm{B}, & \mathrm{AC}, & a\mathrm{BD}, & \mathrm{ACE}, & a\mathrm{BDF} \text{ etc.,} \end{matrix}$$

of which therefore the terms in equivalent places are to be constituted from the given progression A, B, C, D etc. are themselves to become known.

2. Therefore since the progression of the alternate terms

$$\mathrm{A},\ \mathrm{AC},\ \mathrm{ACE},\ \mathrm{ACEG}\ \text{etc.}$$

shall be known, its interpolation will lead to the true value of the term sought a. But that same progression has been prepared thus always, so that on being continued indefinitely it may be put together by another simple progression of this kind, of which the interpolation shall be liable to no greater difficulty. But generally that progression produced indefinitely is accustomed to become geometric, thus so that the mean proportionals between two contiguous terms shall be required to be interpolated.

3. Therefore if we may regard the whole series

$$a,\ \mathrm{A},\ a\mathrm{B},\ \mathrm{AC},\ a\mathrm{BD},\ \mathrm{ACE}\ \text{etc.}$$

as geometric, and thence we may define the mean terms, initially we may stray perhaps greatly from the truth; but where we may proceed longer, there we may approach closer to the truth, as finally clearly we may reach infinity. Hence the following determinations will approach continually towards the truth :

$$\begin{matrix} aa = \frac{\mathrm{AA}}{\mathrm{B}}, & aa = \frac{\mathrm{AAC}}{\mathrm{BB}}, \\ aa = \frac{\mathrm{AACC}}{\mathrm{BBD}}, & aa = \frac{\mathrm{AACE}}{\mathrm{BBDD}}, \\ aa = \frac{\mathrm{AACEE}}{\mathrm{BBDDF}}, & aa = \frac{\mathrm{AACCEEG}}{\mathrm{BBDDFF}} \\ \text{etc.} & \text{etc.} \end{matrix}$$

And thus finally on progressing to infinity there will become :

$$a = \mathrm{A} \cdot \frac{\mathrm{AC}}{\mathrm{BB}} \cdot \frac{\mathrm{CE}}{\mathrm{DD}} \cdot \frac{\mathrm{EG}}{\mathrm{FF}} \cdot \frac{\mathrm{GI}}{\mathrm{HH}} \cdot \frac{\mathrm{IL}}{\mathrm{KK}} \cdot \text{etc.}$$

4. This infinite expression shows the true value of a, as often as the progression of numbers A, B, C, D etc. has been prepared thus, so that the terms may maintain the ratio of equality infinitesimally close between themselves , and the factors of this expression finally may become unity. Just as if the series of natural numbers may be taken for the series A, B, C, D etc., so that there shall become :

$$ab=1,\ bc=2,\ cd=3,\ de=4,\ ef=5,\ fg=6 \text{ etc.},$$

there will be :

$$aa=\tfrac{1\cdot 3}{2\cdot 2}\cdot\tfrac{3\cdot 5}{4\cdot 4}\cdot\tfrac{5\cdot 7}{6\cdot 6}\cdot\tfrac{7\cdot 9}{8\cdot 8}\cdot\tfrac{9\cdot 11}{10\cdot 10}\cdot\tfrac{11\cdot 13}{12\cdot 12}\cdot \text{etc.}$$

But it is agreed this infinite product to be $=2:\pi$, with the ratio of the diameter to the periphery $=1:\pi$, thus so that there shall become :

$$a=\sqrt{\tfrac{2}{\pi}},$$

and hence

$$b=\sqrt{\tfrac{\pi}{2}},\quad c=\tfrac{2\sqrt{2}}{\sqrt{\pi}},\quad d=\tfrac{3\sqrt{\pi}}{2\sqrt{2}}\quad \text{etc.}$$

5. Therefore this series of transcendental numbers proceeds by a certain law of uniformity, which numbers may be set out by approximation and it will help to have noted the differences of these :

	diff. 1	diff. 2	diff. 3
$a=0,7978846$			
	4554295		
$b=1,2533141$		1129745	
	3424550		547216
$c=1,5957691$		582529	
	2842021		217718
$d=1,87997121$		364811	
	2477210		110319
$e=2,1276922$		254492	
	2222718		64440
$f=2,3499640$		190052	
	2032666		41327
$g=2,5532306$		148725	
	1883941		
$h=2,7416247$			

Indeed if some other number may be assumed for *a* and the following may be defined from that, large leaps my become apparent in the differences.

6. The matter proceeds in the same manner, if some arithmetical progression may be assumed for the numbers A, B, C, D etc. For the series *a, b, c, d, e* etc. shall be required to be found, thus so that there shall be

$$ab=p,\ bc=p+q,\ cd=p+2q,\ de=p+3q \text{ etc.},$$

and since the terms approach infinitesimally to the ratio of equality, there will become

$$aa = p\cdot\frac{p(p+2q)}{(p+q)(p+q)}\cdot\frac{(p+2q)(p+4q)}{(p+3q)(p+3q)}\cdot\frac{(p+4q)(p+6q)}{(p+5q)(p+5q)}\cdot\text{etc.},$$

the value of which expression can be shown by integral formulas, so that there shall become :

$$aa = p\cdot\frac{\int z^{p+q-1}dz:\sqrt{(1-z^{2q})}}{\int z^{p-1}dz:\sqrt{(1-z^{2q})}}$$

on putting $z=1$ after each integration [see E122].

7. If a mixed progression may be assumed for A, B, C, D etc. from arithmetical and harmonic progressions, so that such a series of numbers *a, b, c, d, e* etc. shall be required to be investigated :

$$ab = \frac{p}{r},\quad bc = \frac{p+q}{r+s},\quad cd = \frac{p+2q}{r+2s},\quad de = \frac{p+3q}{r+3s}\quad \text{etc.},$$

and because here the numbers A, B, C, D etc. converge to the ratio of equality, there will become

$$aa = \frac{p}{r}\cdot\frac{p(r+s)(p+2q)(r+s)}{r(p+q)(r+2s)(p+q)}\cdot\frac{(p+2q)(r+3s)(p+4q)(r+3s)}{(r+2s)(p+3q)(r+4s)(p+3q)}\cdot\text{etc.},$$

the value of which can be deduced as above

$$aa = \frac{p}{r}\cdot\frac{\int z^{p+q-1}dz:\sqrt{(1-z^{2q})}\cdot\int z^{r-1}dz:\sqrt{(1-z^{2s})}}{\int z^{p-1}dz:\sqrt{(1-z^{2q})}\cdot\int z^{r+s-1}dz:\sqrt{(1-z^{2s})}}$$

where again after the integration it will be required to put $z=1$.

8. If there were $s=q$, the numbers A, B, C, D etc. continually approach to the number one and finally will become equal to that. From which since it may be agreed the terms of the series

$$a,\ \text{A},\ a\text{B},\ \text{AC},\ a\text{BD},\ \text{ACE}\ \text{ etc.}$$

to be infinitesimally equal between themselves, thence it will be concluded

$$a = \frac{p}{r}\cdot\frac{(p+2q)(r+q)}{(p+q)(r+2q)}\cdot\frac{(p+4q)(r+3q)}{(p+3q)(r+4q)}\cdot\text{etc.},$$

which expression can be referred to thus :

$$a=\frac{p(r+q)}{r(p+q)}\cdot\frac{(p+2q)(r+3q)}{(r+2q)(p+3q)}\cdot\frac{(p+4q)(r+5q)}{(r+4q)(p+5q)}\cdot\text{etc.},$$

the value of which, by the integral formulas, is :

$$a=\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}.$$

9. Hence also the case, where *s* and *q* are unequal, can be set out more clearly. For there shall be $s=nnq$ and there may be put $r=nnt$; then truly there may be put

$$a=\frac{\alpha}{n},\quad b=\frac{\beta}{n},\quad c=\frac{\gamma}{n},\quad d=\frac{\delta}{n},\quad e=\frac{\varepsilon}{n},\quad \text{etc.}$$

and there will become, by the prescribed condition

$$\alpha\beta=\frac{p}{t},\quad \beta\gamma=\frac{p+q}{t+q},\quad \gamma\delta=\frac{p+2q}{t+2q},\quad \delta\varepsilon=\frac{p+3q}{t+3q}\quad \text{etc.},$$

from the agreement of which with the preceding there becomes :

$$\alpha=\frac{p(t+q)}{t(p+q)}\cdot\frac{(p+2q)(t+3q)}{(t+2q)(p+3q)}\cdot\frac{(p+4q)(t+5q)}{(t+4q)(p+5q)}\cdot\text{etc.},$$

and thus

$$\alpha=\frac{\int z^{t-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}.$$

10. Therefore since there shall be

$$n=\sqrt{\frac{s}{q}},\quad t=\frac{qr}{s}\quad \text{and}\quad a=\frac{\alpha\sqrt{q}}{\sqrt{s}},$$

for the case established in § 7 there will be :

$$a=\frac{\sqrt{q}}{\sqrt{s}}\cdot\frac{p(r+s)}{r(p+q)}\cdot\frac{(p+2q)(r+3s)}{(r+2s)(p+3q)}\cdot\frac{(p+4q)(r+5s)}{(r+4s)(p+5q)}\cdot\text{etc.},$$

and by the integral formulas :

$$a=\frac{\sqrt{q}}{\sqrt{s}}\cdot\frac{\int z^{\frac{qr}{s}-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}};$$

where, if in the numerator for z^q there may be written z^s there becomes

$$a=\frac{\sqrt{s}}{\sqrt{q}}\cdot\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2s}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}};$$

therefore by necessity the square of which will be equal to the formula found above, thus so that there shall become

$$\frac{s}{q}\cdot\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2s}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}=\frac{p}{r}\cdot\frac{\int z^{p+q-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{r+s-1}dz:\sqrt{\left(1-z^{2s}\right)}}.$$

11. Therefore the agreement of these formulas for the case, where after the integration $z=1$ is established, supplies us with the following theorem

$$pq\int\frac{z^{p-1}dz}{\sqrt{\left(1-z^{2q}\right)}}\cdot\int\frac{z^{p+q-1}dz}{\sqrt{\left(1-z^{2q}\right)}}=rs\int\frac{z^{r-1}dz}{\sqrt{\left(1-z^{2s}\right)}}\cdot\int\frac{z^{r+s-1}dz}{\sqrt{\left(1-z^{2s}\right)}},$$

the truth of which indeed I have now given a demonstration elsewhere from other principles [E223]. Hence therefore it follows, by taking $r=s=1$, to become

$$pq\int\frac{z^{p-1}dz}{\sqrt{\left(1-z^{2q}\right)}}\cdot\int\frac{z^{p+q-1}dz}{\sqrt{\left(1-z^{2q}\right)}}=\frac{\pi}{2},$$

on account of

$$\int\frac{dz}{\sqrt{(1-zz)}}=\frac{\pi}{2}\quad\text{et}\quad\int\frac{zdz}{\sqrt{(1-zz)}}=1.$$

12. Therefore we may consider some examples.

I. If there should be

$$ab=1,\ bc=2,\ cd=3,\ de=4,\ ef=5\ \text{ etc.},$$

there will become

$$aa=1\cdot\frac{1\cdot3}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}\cdot\frac{5\cdot7}{6\cdot6}\cdot\frac{7\cdot9}{8\cdot8}\cdot\text{etc.}$$

and

$$aa=\frac{\int zdz:\sqrt{(1-zz)}}{\int dz:\sqrt{(1-zz)}}=\frac{2}{\pi}.$$

II. If there should be

$$ab=1,\ \ bc=3,\ \ cd=5,\ \ de=7,\ \ ef=9\ \text{ etc.},$$

there will become

$$aa=\frac{1\cdot5}{3\cdot3}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{9\cdot13}{15\cdot15}\cdot\text{etc.}$$

or

$$aa = \frac{\int zzdz:\sqrt{(1-z^4)}}{\int dz:\sqrt{(1-z^4)}}.$$

Truly since there shall be from the theorem expressed in the manner

$$\frac{\pi}{4} = \int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zzdz}{\sqrt{(1-z^4)}},$$

there is deduced :

$$a = \sqrt{\frac{2}{\pi}} \int \frac{zzdz}{\sqrt{(1-z^4)}}.$$

III. If there should be

$$ab = 1, \quad bc = 4, \quad cd = 7, \quad de = 10, \quad ef = 13 \text{ etc.},$$

there will be

$$aa = \frac{1 \cdot 7}{4 \cdot 4} \cdot \frac{7 \cdot 13}{10 \cdot 10} \cdot \frac{13 \cdot 19}{16 \cdot 16} \cdot \text{etc.}$$

or

$$aa = \frac{\int z^3 dz:\sqrt{(1-z^6)}}{\int dz:\sqrt{(1-z^6)}},$$

and hence

$$a = \frac{\sqrt{6}}{\sqrt{\pi}} \int \frac{z^3 dz}{\sqrt{(1-z^6)}}.$$

IV. If there should be more generally,

$$ab = p, \quad bc = p+q, \quad cd = p+2q, \quad de = p+3q, \quad ef = p+4q \text{ etc.},$$

by the reduction, we will deduce with the aid of the above theorem being put in place:

$$a = \frac{p\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}dz}{\sqrt{(1-z^{2q})}}.$$

13. These examples have been chosen from arithmetical progressions; to which we may add a few, in which the progression of the numbers A, B, C, D etc. is a mixture from arithmetic and harmonic progressions.

I. If there should be

$$ab=\tfrac{1}{2},\;\; bc=\tfrac{2}{3},\;\; cd=\tfrac{3}{4},\;\; de=\tfrac{4}{5},\;\; ef=\tfrac{5}{6}\;\; \text{etc.},$$

on account of

$$p=1,\; q=1,\; r=2,\; \mathrm{s}=1$$

there will be

$$a=\tfrac{1\cdot 3}{2\cdot 2}\cdot\tfrac{3\cdot 5}{4\cdot 4}\cdot\tfrac{5\cdot 7}{6\cdot 6}\cdot\tfrac{7\cdot 9}{8\cdot 8}\cdot \text{etc}\;.$$

or

$$a=\frac{\int zdz:\sqrt{(1-zz)}}{\int dz:\sqrt{(1-zz)}}=\frac{2}{\pi}$$

II. If there should be

$$ab=\tfrac{1}{2},\;\; bc=\tfrac{3}{4},\;\; cd=\tfrac{5}{6},\;\; de=\tfrac{7}{8},\;\; ef=\tfrac{9}{10}\;\; \text{etc.},$$

on account of

$$p=1,\; q=2,\; r=2,\; \mathrm{s}=2$$

there will be

$$a=\tfrac{1\cdot 4}{2\cdot 3}\cdot\tfrac{5\cdot 8}{6\cdot 7}\cdot\tfrac{9\cdot 12}{10\cdot 11}\cdot\tfrac{13\cdot 16}{14\cdot 15}\cdot\tfrac{17\cdot 20}{18\cdot 19}\cdot \text{etc}\;.$$

or

$$a=\frac{\int zdz:\sqrt{\left(1-z^4\right)}}{\int dz:\sqrt{\left(1-z^4\right)}}=\frac{\pi}{4}:\int\frac{dz}{\sqrt{\left(1-z^4\right)}}=\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}}.$$

III. If there should be

$$ab=\tfrac{1}{1},\;\; bc=\tfrac{2}{3},\;\; cd=\tfrac{3}{5},\;\; de=\tfrac{4}{7},\;\; ef=\tfrac{5}{9}\;\; \text{etc.},$$

on account of

$$p=1,\; q=1,\; r=1,\; \mathrm{s}=2$$

there will be

$$a=\tfrac{1}{\sqrt{2}}\cdot\tfrac{1\cdot 3}{1\cdot 2}\cdot\tfrac{3\cdot 7}{5\cdot 4}\cdot\tfrac{5\cdot 11}{9\cdot 6}\cdot\tfrac{7\cdot 15}{13\cdot 8}\cdot\tfrac{9\cdot 19}{17\cdot 10}\cdot \text{etc}\;.$$

or

$$a=\frac{\sqrt{2}}{1}\cdot\frac{\int dz:\sqrt{\left(1-z^4\right)}}{\int dz:\sqrt{(1-zz)}}=\frac{2\sqrt{2}}{\pi}\cdot\int\frac{dz}{\sqrt{\left(1-z^4\right)}}=\frac{1}{\sqrt{2}}:\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}}.$$

Moreover the product from this value and the preceding clearly is $=\frac{1}{\sqrt{2}}$.

THE SECOND METHOD BY CONTINUED FRACTIONS

14. Thus we may represent the series required to be found with these indices :

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & & n & n+1 \\ a, & b, & c, & d, & e & \text{etc.......} & x, & y \end{array}$$

and initially we may investigate that series, in which there shall be

$$ab = p,\ bc = p+q,\ cd = p+2q,\ de = p+3q \ \text{etc.},$$

so that by the preceding method, there shall be

$$aa = p\cdot\frac{p(p+2q)}{(p+q)(p+q)}\cdot\frac{(p+2q)(p+4q)}{(p+3q)(p+3q)}\cdot\frac{(p+4q)(p+6q)}{(p+5q)(p+5q)}\cdot\text{etc.},$$

and

$$aa = p\cdot\frac{\int z^{p+q-1}dz:\sqrt{(1-z^{2q})}}{\int z^{p-1}dz:\sqrt{(1-z^{2q})}}$$

or

$$a = \frac{p\sqrt{2q}}{\sqrt{\pi}}\cdot\int\frac{z^{p+q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}}:\int\frac{z^{p-1}dz}{\sqrt{(1-z^{2q})}},$$

hence

$$b = \frac{(p+q)\sqrt{2q}}{\sqrt{\pi}}\cdot\int\frac{z^{p+2q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}}:\int\frac{z^{p+q-1}dz}{\sqrt{(1-z^{2q})}},$$

and

$$c = \frac{(p+2q)\sqrt{2q}}{\sqrt{\pi}}\cdot\int\frac{z^{p+3q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}}:\int\frac{z^{p+2q-1}dz}{\sqrt{(1-z^{2q})}}$$

and thus

$$x = \frac{(p+nq)\sqrt{2q}}{\sqrt{\pi}}\cdot\int\frac{z^{p+(n+1)q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}}:\int\frac{z^{p+nq-1}dz}{\sqrt{(1-z^{2q})}}.$$

15. Therefore since for this series there shall be in general

$$xy = p+nq,$$

the quantity x must be a function of the index n of this kind, so that by putting $n+1$ into that equation in place of n, y may be produced and the product becomes

$$xy = p + nq\ ;$$

which since it may be turned towards rationality, it is agreed to search for the values of the squares xx and yy from the equation :

$$xxyy = pp + 2npq + nnqq,$$

since that account of the function may be apparent also for squares. This investigation therefore may be extended further conveniently to the resolution of this equation :

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma\ ;$$

from which the value of xx can be reduced to continued fractions in several ways, which depend on the following lemmas.

LEMMA I

16. From this proposed equation :

$$\left(\mathrm{X} + \lambda n + \mu\right)\left(\mathrm{Y} + \lambda n + \nu\right) = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

in which Y is defined from $n+1$ and X likewise from n , there may be put

$$\mathrm{X} + \lambda n + \mu = \zeta n + f + \tfrac{k}{\mathrm{X}'}$$

and

$$\mathrm{Y} + \lambda n + \nu = \zeta n + g + \tfrac{k}{\mathrm{Y}'},$$

so that there shall be

$$\mathrm{X} = \left(\zeta - \lambda\right)n + f - \mu + \tfrac{k}{\mathrm{X}'}$$

and

$$\mathrm{Y} = \left(\zeta - \lambda\right)n + g - \nu + \tfrac{k}{\mathrm{Y}'},$$

where now X' and Y' shall be new similar functions of n and $n+1$; and by necessity there shall be

$$g - \nu = \zeta - \lambda + f - \mu \ \text{ or } g = \zeta - \lambda - \mu + \nu + f.$$

$$[i.e.\ \mathrm{Y} = \left(\zeta - \lambda\right)n + \left(\zeta - \lambda\right) + f - \mu + \tfrac{k}{\mathrm{Y}'}$$
$$= \left(\zeta - \lambda\right)\left(n+1\right) + f - \mu + \tfrac{k}{\mathrm{Y}'} = \left(\zeta - \lambda\right)n + g - \nu + \tfrac{k}{\mathrm{Y}'}.]$$

17. With this in place, the prescribed equation will be changed into this :

$$\zeta\zeta nn + \zeta\left(f+g\right)n + fg + \frac{k\left(\zeta n + f\right)}{\mathrm{Y}'} + \frac{k\left(\zeta n + g\right)}{\mathrm{X}'} + \frac{kk}{\mathrm{X}'\mathrm{Y}'} = \zeta\zeta nn + 2\zeta\eta n + \theta.$$

On putting

$$f+g=2\eta \text{ and } k=fg-\theta,$$

so that there may be produced:

$$\text{X'Y'}+(\zeta n+f)\text{X'}+(\zeta n+g)\text{Y'}+fg-\theta=0,$$

or

$$(\text{X'}+\zeta n+g)(\text{Y'}+\zeta n+f)=\zeta\zeta nn+\zeta(f+g)n+\theta,$$

which is similar to the form proposed. But on account of $f+g=2\eta$ there will be had

$$\zeta-\lambda-\mu+v+2f=2\eta,$$

$[$*i.e.* since $f=2\eta-g=2\eta-\zeta+\lambda+\mu-v-f]$

$$f=\eta+\tfrac{\lambda-\zeta+\mu-v}{2}$$

and

$$g=\eta-\tfrac{\lambda-\zeta+\mu-v}{2}$$

and hence

$$k=fg-\theta=\eta\eta-\tfrac{1}{4}(\lambda-\zeta+\mu-v)^2-\theta.$$

18. On account of which the proposed equation

$$(\text{X}+\lambda n+\mu)(\text{Y}+\lambda n+v)=\zeta\zeta nn+2\zeta\eta n+\theta,$$

with the aid of this substitution

$$\text{X}=(\zeta-\lambda)n+\eta+\tfrac{\lambda-\zeta-\mu-v}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-v)^2-\theta.}{\text{X'}},$$
$$\text{Y}=(\zeta-\lambda)n+\eta-\tfrac{\lambda-\zeta+\mu+v}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-v)^2-\theta.}{\text{Y'}},$$

is reduced to this similar proposed equation

$$\left(\text{X'}+\xi n-\tfrac{\lambda-\zeta+\mu-v}{2}\right)\left(\text{Y'}+\zeta n+\tfrac{\lambda-\xi+\mu-v}{2}\right)=\zeta\zeta nn+2\zeta\eta n+\theta.$$

19. In a similar manner the same proposed equation,

$$(\text{X}+\lambda n+\mu)(\text{Y}+\lambda n+v)=\zeta\zeta nn+2\zeta\eta n+\theta,$$

with these substitutions made :

$$\mathrm{X} = (\zeta - \lambda)n + \eta + \frac{\lambda-\zeta-\mu-\nu}{2} + \frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2 - \eta\eta + \theta.}{\mathrm{X}'},$$

$$\mathrm{Y} = (\zeta - \lambda)n + \eta - \frac{\lambda-\zeta+\mu+\nu}{2} + \frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2 - \eta\eta + \theta}{\mathrm{Y}'},$$

is reduced to this form similar to itself :

$$\left(\mathrm{X}' - \xi n - \eta + \frac{\lambda-\zeta+\mu-\nu}{2}\right)\left(\mathrm{Y}' - \zeta n - \eta - \frac{\lambda-\zeta+\mu-\nu}{2}\right) = \zeta\zeta nn + 2\zeta\eta n + \theta.$$

LEMMA II

20. With this equation proposed

$$(\mathrm{X} + \lambda n + \mu)(\mathrm{Y} + \lambda n + \nu) = \zeta\zeta nn + 2\zeta\eta n + \theta\,,$$

in which X depends on *n* and likewise Y depends on $n+1$, there may be put

$$\mathrm{X} + \lambda n + \mu = \zeta n + f + \frac{hn+k}{\mathrm{X}'},$$

$$\mathrm{Y} + \lambda n + \nu = \zeta n + g + \frac{hn+h+k}{\mathrm{Y}'},$$

where on account of the similarity of the functions there must be as before

$$g = \zeta - \lambda - \mu + \nu + f.$$

21. Again with the substitution of these values made we will have

$$\zeta\zeta nn + \zeta(f+g)n + fg + \frac{(\zeta n+f)(hn+h+k)}{\mathrm{Y}'} + \frac{(\zeta n+g)(hn+k)}{\mathrm{X}'}$$
$$+\frac{(hn+k)(hn+h+k)}{\mathrm{X'Y'}} = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

from which there becomes

$$\big(\zeta(f+g-2\eta)n + fg - \theta\big)\mathrm{X'Y'} + (\zeta n + f)(hn+h+k)\mathrm{X}'$$
$$+(\xi n + g)(hn+k)\mathrm{Y}' + (hn+k)(hn+h+k) = 0,$$

which, as it shall be similar to the proposed form, must be divisible by

$\zeta(f+g-2\eta)n + fg - \theta$;

to which quantity therefore either $hn+k$ or $hn+h+k$, or a multiple will be required to be put equal

22. Initially, let

$$hn+k=\alpha\zeta\left(f+g-2\eta\right)n+\alpha\left(fg-\theta\right)$$

and $\zeta n+f$ will be required to be a sub multiple of $\zeta\left(f+g-2\eta\right)n+fg-\theta$; from which there becomes

$$f\left(f+g-2\eta\right)=fg-\theta \text{ or } ff=2\eta f-\theta$$

and hence

$$ff=\eta+\sqrt{(\eta\eta-\theta)} \text{ and } g=\zeta-\lambda-\mu+v+\eta+\sqrt{(\eta\eta-\theta)};$$

whereby again

$$h=\alpha\zeta\left(f+g-2\eta\right) \text{ and } k=\alpha\left(fg-\theta\right),$$

and the resulting equation emerges

$$\begin{aligned}&\mathrm{X'Y'}+\tfrac{\alpha\zeta(f+g-2\eta)n+\alpha\zeta(f+g-2\eta)+\alpha(fg-\theta)}{f+g-2\eta}\mathrm{X'}\\&+\alpha\left(\zeta n+g\right)\mathrm{Y'}+\alpha\alpha\left(\xi\left(f+g-2\eta\right)n+\zeta\left(f+g-2\eta\right)+fg-\theta\right)=0.\end{aligned}$$

23. So that we may remove fractions, we may put

$$\alpha=f+g-2\eta=\zeta-\lambda-\mu+v+2\sqrt{(\eta\eta-\theta)}$$

and thus there will become

$$\begin{aligned}&\mathrm{X'Y'}+\left(\zeta\left(f+g-2\eta\right)n+\zeta\left(f+g-2\eta\right)+fg-\theta\right)\mathrm{X'}\\&+\left(\zeta\left(f+g-2\eta\right)n+g\left(f+g-2\eta\right)\right)\mathrm{Y'}\\&+\left(f+g-2\eta\right)^2\left(\zeta\left(f+g-2\eta\right)n+\zeta\left(f+g-2\eta\right)+fg-\theta\right)=0.\end{aligned}$$

Truly if we cannot take care of the fractions, we will have

$$\begin{aligned}&\mathrm{X'Y'}+\alpha\left(\zeta n+\zeta+\tfrac{fg-\theta}{(f+g-2\eta)}\right)\mathrm{X'}+\alpha\left(\zeta n+g\right)\mathrm{Y'}\\&+\alpha\alpha\left(f+g-2\eta\right)\left(\zeta n+\zeta+\tfrac{fg-\theta}{f+g-2\eta}\right)=0,\end{aligned}$$

which equation on putting for the sake of brevity

$$\frac{fg-\theta}{f+g-2\eta}=\varepsilon\,,$$

is reduced to this proposed similar equation,

$$\left(\mathrm{X}'+\alpha\left(\zeta n+g\right)\right)\left(\mathrm{Y}'+\alpha\left(\zeta n+\zeta+\varepsilon\right)\right)=\alpha\alpha\left(\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta\right)$$
$$+\left(\xi+\varepsilon\right)\left(2\eta-f\right)=\alpha\alpha\left(\zeta n+\zeta+\varepsilon\right)\left(\zeta n+2\eta-f\right).$$

24. Therefore the proposed equation

$$\left(\mathrm{X}+\lambda n+\mu\right)\left(\mathrm{Y}+\lambda n+v\right)=\zeta\zeta nn+2\zeta\eta n+\theta$$

if there may be put for the sake of brevity

$$f=\eta+\sqrt{(\eta\eta-\theta)},\ g=\zeta-\lambda-\mu+v+\eta+\sqrt{(\eta\eta-\theta)}$$

and

$$\frac{fg-\theta}{f+g-2\eta}=\varepsilon,$$

by substituting the following:

$$\mathrm{X}=\left(\zeta-\lambda\right)n+f-\mu+\frac{\zeta\left(f+g-2\eta\right)+fg-\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=\left(\zeta-\lambda\right)n+g-v+\frac{\zeta\left(f+g-2\eta\right)\left(n+1\right)+fg-\theta}{\mathrm{Y}'}$$

will supply the following equation similar to the proposed :

$$\left(\mathrm{X}'+\zeta n+g\right)\left(\mathrm{Y}'+\xi\left(n+1\right)+\varepsilon\right)=\zeta\zeta nn+\zeta\left(\zeta+\varepsilon-f+2\eta\right)n+\left(\zeta+\varepsilon\right)\left(2\eta-f\right).$$

25. Since here we have assumed $\alpha=1$, thus by putting $\alpha=-1$ with the same abbreviations remaining, this substitution will give

$$\mathrm{X}=\left(\zeta-\lambda\right)n+f-\mu+\frac{\zeta\left(2\eta-f-g\right)n-fg+\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=\left(\zeta-\lambda\right)n+g-v+\frac{\zeta\left(2\eta-f-g\right)\left(n+1\right)-fg+\theta}{\mathrm{Y}'},$$

from which this equation arises similar to the proposed

$$\left(\mathrm{X}'-\zeta n-g\right)\left(\mathrm{Y}'-\zeta\left(n+1\right)-\varepsilon\right)=\zeta\zeta nn+\zeta\left(\zeta+\varepsilon-f+2\eta\right)n+\left(\zeta+\varepsilon\right)\left(2\eta-f\right).$$

26. Again we may put to be

$$hn+h+k=\zeta\left(f+g-2\eta\right)n+fg-\theta,$$

so that there shall become:

$$h=\zeta\left(f+g-2\eta\right) \text{ and } k=fg-\theta\ -\zeta\left(f+g-2\eta\right),$$

and it is necessary that there becomes :

$$\zeta\left(f+g-2\eta\right)n+fg-\theta=\left(f+g-2\eta\right)\left(\zeta n+g\right)$$

and thus

$$g\left(f+g-2\eta\right)=fg-\theta \text{ or } g=\eta+\sqrt{(\eta\eta-\theta)}$$

and hence

$$f=\lambda-\zeta+\mu-\nu+\eta+\sqrt{(\eta\eta-\theta)}.$$

Moreover the resulting equation will be

$$\mathrm{X'Y'}+(\zeta n+f)\mathrm{X'}+\left(\xi n-\zeta+\tfrac{fg-\theta}{f+g-2\eta}\right)\mathrm{Y'}+\left(f+g-2\eta\right)\left(\zeta\left(n-1\right)+\tfrac{fg-\theta}{f+g-2\eta}\right)=0,$$

which on putting $\frac{fg-\theta}{f+g-2\eta}=\varepsilon$ will be changed into this :

$$\begin{aligned}&\left(\mathrm{X'}+\zeta n-\zeta+\varepsilon\right)\left(\mathrm{Y'}+\zeta n+f\right)=\zeta\zeta nn+\zeta n\left(2\eta-g-\zeta+\varepsilon\right)+\left(\varepsilon-\zeta\right)\left(2\eta-g\right)\\&=\left(\xi n-\zeta+\varepsilon\right)\left(\zeta n+2\eta-g\right).\end{aligned}$$

27. Therefore with this equation proposed :

$$\left(\mathrm{X}+\lambda n+\mu\right)\left(\mathrm{Y}+\lambda n+\nu\right)=\zeta\zeta nn+2\zeta\eta n+\theta$$

if there may be put for the sake of brevity

$$f=\lambda-\zeta+\mu-\nu+\eta+\sqrt{(\eta\eta-\theta)},\ g=\eta+\sqrt{(\eta\eta-\theta)}$$

and

$$\varepsilon=\tfrac{fg-\theta}{f+g-2\eta},$$

on substituting the following:

$$\mathrm{X}=\left(\zeta-\lambda\right)n+f-\mu+\tfrac{\zeta\left(f+g-2\eta\right)\left(n-1\right)+fg-\theta}{\mathrm{X'}},$$

$$\mathrm{Y}=\left(\zeta-\lambda\right)n+g-\nu+\tfrac{\zeta\left(f+g-2\eta\right)n+fg-\theta}{\mathrm{Y'}}$$

this equation similar to the proposed will appear :

$$\left(\mathrm{X}'+\xi n-\zeta+\varepsilon\right)\left(\mathrm{Y}'+\zeta n+f\right)=\zeta\zeta nn+\zeta\left(2\eta-g-\zeta+\varepsilon\right)n+\left(\varepsilon-\zeta\right)\left(2\eta-g\right).$$

28. In a similar manner, with the same abbreviations remaining, the same proposed equation with the aid of the substitution

$$\mathrm{X}=\left(\zeta-\lambda\right)n+f-\mu+\frac{\zeta\left(2\eta-f-g\right)\left(n-1\right)-fg+\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=\left(\zeta-\lambda\right)n+g-\nu+\frac{\zeta\left(2\eta-f-g\right)n-fg+\theta}{\mathrm{Y}'}$$

will be reduced to this equation similar to the proposed :

$$\left(\mathrm{X}'-\zeta n+\zeta-\varepsilon\right)\left(\mathrm{Y}'-\zeta n-f\right)=\zeta\zeta nn+\zeta\left(2\eta-g-\zeta+\varepsilon\right)n+\left(\varepsilon-\zeta\right)\left(2\eta-g\right).$$

Therefore with the aid of these six reductions examined in § 18, 19, 24, 25, 27, 28 all the equations of this kind will be able to be resolved by continued fractions in an infinite number of ways.

THE RESOLUTION OF EQUATION $xxyy=aann+2a\beta n+\gamma$ BY § 18.

29. Since here there shall be

$$\mathrm{X}=xx,\ \mathrm{Y}=yy,\ \lambda=0,\ \mu=0,\ \nu=0,\ \xi=\alpha,\ \eta=\beta\ \text{ et }\ \theta=\gamma,$$

this substitution will arise

$$xx=\alpha n+\beta-\tfrac{1}{2}\alpha+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}'},$$

$$yy=\alpha n+\beta+\tfrac{1}{2}\alpha+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}'},$$

which leads to this following equation

$$\left(\mathrm{X}'+\alpha n+\beta+\tfrac{1}{2}\alpha\right)\left(\mathrm{Y}'+\alpha n+\beta-\tfrac{1}{2}\alpha\right)=\alpha\alpha nn+2\alpha\beta n+\gamma.$$

30. For this requiring to be resolved in a similar manner, on account of

$$\lambda=\alpha,\ \mu=\beta+\tfrac{1}{2}\alpha,\ \nu=\beta-\tfrac{1}{2}\alpha,\ \xi=\alpha,\ \eta=\beta,\ \theta=\gamma,$$

we will attend to this with the substitution

$$\mathrm{X}'=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}''},\ \mathrm{Y}'=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}''}$$

which leads to this third equation

$$\left(\mathrm{X}''+\alpha n+\beta-\tfrac{1}{2}\alpha\right)\left(\mathrm{Y}''+\alpha n+\beta+\tfrac{1}{2}\alpha\right)=\alpha\alpha nn+2\alpha\beta n+\gamma.$$

But this again provides these substitutions

$$\mathrm{X}''=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}'''},\ \mathrm{Y}''=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}'''}$$

from which since $\mathrm{X}'''=\mathrm{X}'$ and $\mathrm{Y}'''=\mathrm{Y}'$, nothing further can be concluded.

RESOLUTION OF THE EQUATION $xxyy=\alpha\alpha nn+2\alpha\beta n+\gamma$ BY § 19.

31. With these substitutions made :

$$xx=\alpha n-\tfrac{1}{2}\alpha+\beta+\frac{\frac{1}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{X}},$$

$$yy=\alpha n+\tfrac{1}{2}\alpha+\beta+\frac{\frac{1}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{Y}},$$

this equation is come upon:

$$\left(\mathrm{X}-\alpha n-\tfrac{1}{2}\alpha-\beta\right)\left(\mathrm{Y}-\alpha n+\tfrac{1}{2}\alpha-\beta\right)=\alpha\alpha nn+2\alpha\beta n+\gamma,$$

which reduced following § 19, on account of

$$\lambda=-\alpha,\ \mu=-\tfrac{1}{2}\alpha-\beta,\ \nu=\tfrac{1}{2}\alpha-\beta,\ \zeta=\alpha,\ \eta=\beta,\ \theta=\gamma$$

gives these substitutions

$$\mathrm{X}=2\alpha n-\alpha+2\beta+\frac{\frac{9}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{X}'},$$

$$\mathrm{Y}=2\alpha n+\alpha+2\beta+\frac{\frac{9}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{Y}'},$$

from which this new equation arises :

$$\left(X' - \alpha n - \tfrac{3}{2}\alpha - \beta\right)\left(Y' - \alpha n + \tfrac{3}{2}\alpha - \beta\right) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

32. This equation may be reduced further, and on account of

$$\lambda = -\alpha,\ \mu = -\tfrac{3}{2}\alpha - \beta,\ \nu = \tfrac{3}{2}\alpha - \beta,\ \xi = \alpha,\ \eta = \beta,\ \theta = \gamma$$

we will have these substitutions

$$X' = 2\alpha n - \alpha + 2\beta + \frac{\tfrac{25}{4}\alpha\alpha - \beta\beta + \gamma}{X''},$$

$$Y' = 2\alpha n + \alpha + 2\beta + \frac{\tfrac{25}{4}\alpha\alpha - \beta\beta + \gamma}{Y''},$$

and hence this new equation

$$\left(X'' - \alpha n - \tfrac{5}{2}\alpha - \beta\right)\left(Y'' - \alpha n + \tfrac{5}{2}\alpha - \beta\right) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

from which the following substitutions are deduced easily.

33. Therefore if it may be required to be abbreviated, there may be put

$$\alpha n - \tfrac{1}{2}\alpha + \beta = N \quad \text{et} \quad \beta\beta - \gamma = B,$$

the value of xx may be expressed by the following continued fraction :

$$xx = N + \cfrac{\frac{1}{4}\alpha\alpha - B}{2N + \cfrac{\frac{9}{4}\alpha\alpha - B}{2N + \cfrac{\frac{25}{4}\alpha\alpha - B}{2N + \cfrac{\frac{49}{4}\alpha\alpha - B}{2N + \cfrac{\frac{81}{4}\alpha\alpha - B}{2N + \text{etc.}}}}}}$$

which agrees with the proposed equation

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma\ .$$

RESOLUTION OF THE EQUATION $xxyy = aann + 2a\beta n + \gamma$ BY § 24.

34. Since here there shall be

$$\lambda = 0,\ \mu = 0,\ \nu = 0,\ \zeta = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

there will be

$$f = \beta + \sqrt{(\beta\beta - \gamma)},\ g = \alpha + \beta + \sqrt{(\beta\beta - \gamma)},$$

hence

$$fg - \theta = \alpha\beta + 2\beta\beta - 2\gamma + (\alpha + 2\beta)\sqrt{(\beta\beta - \gamma)}$$

and

$$f + g - 2\eta = \alpha + 2\sqrt{(\beta\beta - \gamma)}.$$

Therefore there may be put

$$\frac{fg-\theta}{f+g-2\eta} = \beta + \sqrt{(\beta\beta - \gamma)} = \varepsilon,$$

thus so that there shall be

$$\varepsilon = f \text{ et } g = \alpha + f,$$

from which these substitutions arise :

$$xx = \alpha n + f + \frac{(f+g-2\eta)(\alpha n+f)}{\mathrm{X}} = (\alpha n + f)\left(1 + \frac{\alpha+2\sqrt{(\beta\beta-\gamma)}}{\mathrm{X}}\right),$$

$$yy = \alpha n + g + \frac{(f+g-2\eta)(\alpha n+g)}{\mathrm{Y}} = (\alpha n + g)\left(1 + \frac{\alpha+2\sqrt{(\beta\beta-\gamma)}}{\mathrm{Y}}\right),$$

and thence this new equation :

$$(\mathrm{X} + \alpha n + g)(\mathrm{Y} + \alpha n + \alpha + f) = \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + f)(2\beta - f).$$

35. We may put

$$\beta + \sqrt{(\beta\beta-\gamma)} = \delta ,$$

so that there shall become:

$$\lambda = \delta,\ g = \alpha + \delta \text{ and } f + g - 2\eta = \alpha - 2\beta + 2\delta,$$

and thus with the substitutions :

$$xx = \alpha n + \delta + \tfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{X},$$

$$yy = \alpha(n+1) + \delta + \tfrac{(\alpha - 2\beta + 2\delta)(\alpha(n+1) + \delta)}{Y},$$

they will give this equation

$$(X + \alpha n + \alpha + \delta)(Y + \alpha n + \alpha + \delta) = \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + \delta)(2\beta - \delta).$$

36. For the reduction of this equation there is :

$$\lambda = \alpha,\ \mu = \alpha + \delta,\ v = \alpha + \delta,\ \zeta = \alpha,\ \eta = \beta + \tfrac{1}{2}\alpha,\ \theta = (\alpha + \delta)(2\beta - \delta),$$

from which on account of

$$ff - (2\beta + \alpha)f + (\alpha + \delta)(2\beta - \delta) = 0$$

there will be either

$$f = \alpha + \delta$$

or

$$f = 2\beta - \delta$$

but the first position leads to nothing further, while by using the latter there will become :

$$f = 2\beta - \delta,\ g = 2\beta - \delta \text{ and } \varepsilon = 2\beta - \delta,$$

and thus these substitutions may be obtained :

$$X = 2\beta - \alpha - 2\delta + \tfrac{(2\beta - \alpha - 2\delta)(\alpha n + 2\beta - \delta)}{X'},$$

$$Y = 2\beta - \alpha - 2\delta + \tfrac{(2\beta - \alpha - 2\delta)(\alpha(n+1) + 2\beta - \delta)}{Y'},$$

which gives rise to this equation

$$\begin{aligned}&(X' + \alpha n + 2\beta - \delta)(Y' + \alpha n + \alpha + 2\beta - \delta)\\&= aann + 2\alpha(\alpha + \beta)n + (\alpha + \delta)(\alpha + 2\beta - \delta).\end{aligned}$$

37. Therefore there is now :

$$\lambda = \alpha,\ \mu = 2\beta - \delta,\ v = \alpha + 2\beta - \delta,\ \zeta = \alpha,\ \eta = \alpha + \beta,\ \theta = (\alpha + \delta)(\alpha + 2\beta - \delta),$$

from which on account of

$$ff - 2(\alpha+\beta)f + (\alpha+\delta)(\alpha+2\beta-\delta) = 0$$

the value may be taken:

$$f = \alpha + 2\beta - \delta;$$

there will become

$$g = 2\alpha + 2\beta - \delta$$

and

$$\varepsilon = \frac{(\alpha+2\beta-\delta)(\alpha+2\beta-2\delta)}{\alpha+2\beta-2\delta} = \alpha + 2\beta - \delta.$$

Whereby this substitution

$$\text{X}' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\text{X}''},$$

$$\text{Y}' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{\text{Y}''}$$

will give this equation

$$\begin{aligned}&(\text{X}''+\alpha n+2\alpha+2\beta-\delta)(\text{Y}''+\alpha n+2\alpha+2\beta-\delta)\\&= aann + \alpha(3\alpha+2\beta)n + (2\alpha+2\beta-\delta)(\alpha+\delta).\end{aligned}$$

38. If we may use the other value

$$f = \alpha + \delta$$

there becomes

$$g = 2\alpha + \delta \text{ and } \varepsilon = \alpha + \delta$$

and with the substitution made

$$\text{X}' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{\text{X}''},$$

$$\text{Y}' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\text{Y}''}$$

we obtain this equation

$$\begin{aligned}&(\text{X}''+\alpha n+2\alpha+\delta)(\text{Y}''+\alpha n+2\alpha+\delta)\\&= \alpha\alpha nn + \alpha(3\alpha+2\beta)n + (2\alpha+\delta)(\alpha+2\beta-\delta).\end{aligned}$$

39. We may pursue this latter equation, because it is more similar to the second, since it arises from that on putting $\delta+\alpha$ for δ and $\beta+\alpha$ for β, from which this substitution is produced

$$\mathrm{X}'' = 2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+\alpha-\delta)}{\mathrm{X}'''},$$

$$\mathrm{Y}'' = 2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{\mathrm{Y}'''},$$

which leads to this equation

$$\begin{aligned}&(\mathrm{X}'''+\alpha n+2\beta+\alpha-\delta)(\mathrm{Y}'''+\alpha n+2\beta+\alpha-\delta)\\&=\alpha\alpha nn+2\alpha(2\alpha+\beta)n+(2\alpha+\delta)(2\alpha+2\beta-\delta).\end{aligned}$$

40. This equation again treated as in § 38 with the aid of these substitutions

$$\mathrm{X}''' = \alpha-2\beta+2\delta+\frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\mathrm{X}''''},$$

$$\mathrm{Y}''' = \alpha-2\beta+2\delta+\frac{(\alpha-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{\mathrm{Y}''''}$$

is reduced to this:

$$\begin{aligned}&(\mathrm{X}''''+\alpha n+3\alpha+\delta)(\mathrm{Y}''''+\alpha n+3\alpha+\delta)\\&=\alpha\alpha nn+\alpha(5\alpha+2\beta)n+(3\alpha+\delta)(2\alpha+2\beta-\delta).\end{aligned}$$

and these further by these substitutions

$$\mathrm{X}'''' = 2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{\mathrm{X}'''''},$$

$$\mathrm{Y}'''' = 2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+3\alpha-\delta)}{\mathrm{Y}'''''}$$

is reduced to this same

$$\begin{aligned}&(\mathrm{X}'''''+\alpha n+2\beta+2\alpha-\delta)(\mathrm{Y}'''''+\alpha n+3\alpha+2\beta-\delta)\\&=\alpha\alpha nn+2\alpha(3\alpha+\beta)n+(3\alpha+\delta)(3\alpha+2\beta-\delta).\end{aligned}$$

41. Hence therefore the value of xx from this equation

$$xxyy = aann+2a\beta n+\gamma$$

on putting for brevity

$$\beta + \sqrt{(\beta\beta - \gamma)} = \delta \text{ and } \alpha - 2\beta + 2\delta = \mathrm{A}$$

will be

$$xx = an + \delta + \cfrac{\mathrm{A}(\alpha n + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + \alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + \alpha + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + 2\alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 2\alpha + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + 3\alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 3\alpha + 2\beta - \delta)}{\mathrm{A} + \text{etc.}}}}}}}}}$$

But this expression set out presents that same product from infinitely many factors for xx agreeing with that, which was deduced by the former method.

42. This continued fraction can be expressed more easily in this manner:

$$xx = an + \delta - \cfrac{(\alpha n + \delta)}{1 + \cfrac{(\alpha n + 2\beta - \delta)}{\mathrm{A} - \cfrac{(\alpha n + \alpha + \delta)}{1 + \cfrac{(\alpha n + \alpha + 2\beta - \delta)}{\mathrm{A} - \cfrac{(\alpha n + 2\alpha + \delta)}{1 + \cfrac{(\alpha n + 2\alpha + 2\beta - \delta)}{\mathrm{A} - \cfrac{(\alpha n + 3\alpha + \delta)}{1 + \cfrac{(\alpha n + 3\alpha + 2\beta - \delta)}{\mathrm{A} - \cfrac{(\alpha n + 4\alpha + \delta)}{1 + \text{etc.}}}}}}}}}}$$

But if the formulas of § 37 may be reduced further in this manner, this irregular expression is found from the beginning.

43. If each expression with a common heading may be truncated, for the value assumed 2β the value $\alpha + 2\delta - \mathrm{A}$ may be substituted, and in addition for $an + \alpha + \delta$ there may be written N, this equality will be obtained :

$$\mathrm{A}-\cfrac{\mathrm{N}}{1+\cfrac{\mathrm{N}+\alpha-\mathrm{A}}{\mathrm{A}-\cfrac{\mathrm{N}+\alpha}{1+\cfrac{\mathrm{N}+2\alpha-\mathrm{A}}{\mathrm{A}-\cfrac{\mathrm{N}+2\alpha}{1+\cfrac{\mathrm{N}+3\alpha-\mathrm{A}}{\mathrm{A}-\text{etc.}}}}}}} \quad =\alpha-\cfrac{\mathrm{N}+\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}}{\mathrm{A}-\cfrac{\mathrm{N}+2\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}+\alpha}{\mathrm{A}-\cfrac{\mathrm{N}+3\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}+2\alpha}{\mathrm{A}-\text{etc.}}}}}}}$$

where some numbers can be taken for A, α and N.

RESOLUTION OF THE EQUATION $xxyy=\alpha\alpha nn+2\alpha\beta n+\gamma$
WITH THE AID OF § 25.

44. First the substitution desired from the preceding resolution [§ 35] with X and Y assumed negative

$$xx=\alpha n+\delta+\tfrac{(2\beta-\alpha-2\delta)(\alpha n+\delta)}{\mathrm{X}},$$

$$yy=\alpha(n+1)+\delta+\tfrac{(2\beta-\alpha-2\delta)(\alpha n+\alpha+\delta)}{\mathrm{Y}},$$

on putting

$$\delta=\beta+\sqrt{(\beta\beta-\gamma)}$$

leads to this equation

$$\begin{aligned}&(\mathrm{X}-\alpha n-\alpha-\delta)(\mathrm{Y}-\alpha n-\alpha-\delta)\\&=\alpha\alpha nn+\alpha(\alpha+2\beta)n+(\alpha+\delta)(2\beta-\delta),\end{aligned}$$

which compared with § 24 gives

$$\begin{aligned}&\lambda=-\alpha,\quad \mu=-\alpha-\delta,\quad \nu=-\alpha-\delta,\\&\zeta=\alpha,\quad \eta=\tfrac{1}{2}\alpha+\beta,\quad \theta=(\alpha+\delta)(2\beta-\delta)\end{aligned}$$

from which there is deduced

$$ff-(\alpha+2\beta)f+(\alpha+\delta)(2\beta-\delta)=0.$$

There shall be

$$f=\alpha+\delta;$$

there will be

$$g=3\alpha+\delta \quad\text{and}\quad \varepsilon=\alpha+\delta$$

and hence this substitution arises :

$$\mathrm{X} = 2\alpha n + 2\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{\mathrm{X}'},$$

$$\mathrm{Y} = 2\alpha n + 4\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{\mathrm{Y}'},$$

which leads to the following equation

$$\begin{aligned}&(\mathrm{X}' - \alpha n - 3\alpha - \delta)(\mathrm{Y}' - \alpha n - 2\alpha - \delta)\\&= \alpha\alpha nn + \alpha(2\alpha + 2\beta)n + (2\alpha + \delta)(2\beta - \delta).\end{aligned}$$

45. This equation may be treated in a similar manner following § 24, and on account of the values

$$\begin{aligned}&\lambda = -\alpha,\ \mu = -3\alpha - \delta,\ \nu = -2\alpha - \delta,\\&\zeta = \alpha,\ \eta = \alpha + \beta,\ \ \theta = (2\alpha + \delta)(2\beta - \delta)\end{aligned}$$

there will be

$$ff - (2\alpha + 2\beta)f + (2\alpha + \delta)(2\beta - \delta) = 0,$$

from which there may be assumed

$$f = 2\alpha + \delta,$$

and there becomes

$$g = 5\alpha + \delta \ \text{ and } \ \varepsilon = 2\alpha + \delta.$$

Therefore this substation will be found :

$$\mathrm{X}' = 2\alpha n + 5\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{\mathrm{X}''},$$

$$\mathrm{Y}' = 2\alpha n + 7\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{\mathrm{Y}''},$$

which provides this equation

$$\begin{aligned}&(\mathrm{X}'' - \alpha n - 5\alpha - \delta)(\mathrm{Y}'' - \alpha n - 3\alpha - \delta)\\&= \alpha\alpha nn + \alpha(3\alpha + 2\beta)n + (3\alpha + \delta)(2\beta - \delta).\end{aligned}$$

46. Now therefore there will be in the same manner :

$$\lambda = -\alpha,\ \mu = -5\alpha - \delta,\ \nu = -3\alpha - \delta,$$

$$\zeta = \alpha,\ \eta = \tfrac{3}{2}\alpha + \beta,\quad \theta = (3\alpha + \delta)(2\beta - \delta),$$

from which on account of

$$f = 3\alpha + \delta$$

there may be deduced :

$$g = 7\alpha + \delta \text{ and } \varepsilon = 3\alpha + \delta.$$

Therefore the substitution

$$\text{X}'' = 2\alpha n + 8\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{\text{X}'''},$$

$$\text{Y}' = 2\alpha n + 10\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 4\alpha + \delta)}{\text{Y}'''},$$

will give this equation

$$(\text{X}''' - \alpha n - 7\alpha - \delta)(Y''' - \alpha n - 4\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(4\alpha + 2\beta) + (4\alpha + \delta)(2\beta - \delta).$$

47. Since the law of progression shall be shown here well enough, it is concluded easily to become

$$xx = an + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + 2\alpha + 2\delta - \cfrac{(3\alpha n - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{2\alpha n + 5\alpha + 2\delta - \cfrac{(5\alpha n - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{2\alpha n + 8\alpha + 2\delta - \cfrac{(7\alpha n - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{2\alpha n + 11\alpha + 2\delta - \text{etc.}}}}}$$

where it is required to be observed from the proposed equation

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

δ to be given in a twofold manner, since there shall be

$$\delta = \beta \pm \sqrt{(\beta\beta - \gamma)},$$

and thus a twofold series of this kind will be obtained, the second of which will be going to be produced , if everywhere we may have assume the other values for f .

ANOTHER RESOLUTION BY INTERCHANGING THE TWO VALUES OF f.

48. We may take this in the resolution § 44

$$f = 2\beta - \alpha,$$

so that there shall be

$$g = 2\alpha + 2\beta - \delta \text{ and } \varepsilon = 2\beta - \delta;$$

substituting, there shall be

$$\mathrm{X} = 2\alpha n + \alpha + 2\beta - \frac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{\mathrm{X'}},$$

$$\mathrm{Y} = 2\alpha n + 3\alpha + 2\beta - \frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\mathrm{Y'}},$$

from which this equation emerges :

$$\left(\mathrm{X'} - \alpha n - 2\alpha - 2\beta + \delta\right)\left(\mathrm{Y'} - \alpha n - \alpha - 2\beta + \delta\right)$$
$$= \alpha\alpha nn + \alpha\left(2\alpha + 2\beta\right)n + \left(\alpha + \delta\right)\left(\alpha + 2\beta - \delta\right),$$

which arises from the above, if there for δ there may be written $-\alpha + 2\beta - \delta$, with which value retained in the following there becomes :

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + 2\beta - \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n + 2\beta - \delta)}{2\alpha n + 3\alpha + 4\beta - 2\delta - \cfrac{(3\alpha + 2\beta - 2\delta)(\alpha n + \alpha + 2\beta - \delta)}{2\alpha n + 6\alpha + 4\beta - 2\delta - \cfrac{(5\alpha + 2\beta - 2\delta)(\alpha n + 2\alpha + 2\beta - \delta)}{2\alpha n + 9\alpha + 4\beta - 2\delta - \text{etc.}}}}}$$

49. But the equation elicited collated with § 25 gives :

$$\lambda = -\alpha,\ \mu = -2\alpha - 2\beta + \delta,\ \nu = -\alpha - 2\beta + \delta,$$
$$\zeta = \alpha,\ \eta = \alpha + \beta,\ \ \theta = \left(\alpha + \delta\right)\left(\alpha + 2\beta - \delta\right),$$

and

$$ff - 2\left(a + \beta\right)f + \left(\alpha + \delta\right)\left(\alpha + 2\beta - \delta\right) = 0.$$

Here if we may accept

$$f = \alpha + 2\beta - \delta,$$

we will have the formula in the manner found. Therefore there shall be

$$f=\alpha+\delta\,;$$

there will be

$$g=4\alpha+\delta \text{ and } \varepsilon=\alpha+\delta,$$

and thus

$$\mathrm{X'}=2\alpha n+3\alpha+2\beta-\frac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{\mathrm{X''}},$$

$$\mathrm{Y'}=2\alpha n+5\alpha+2\beta-\frac{(3\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\mathrm{Y''}},$$

and hence the equation arises

$$\begin{aligned}&\left(\mathrm{X''}-\alpha n-4\alpha-\delta\right)\left(\mathrm{Y''}-\alpha n-2\alpha-\delta\right)\\&=\alpha\alpha nn+\alpha\left(3\alpha+2\beta\right)n+\left(2\alpha+\delta\right)\left(\alpha+2\beta-\delta\right),\end{aligned}$$

which arises from the preceding [§ 45], if there for δ there may be written $-\alpha+\delta$ *;* and thus there will be

$$xx=an+\delta-\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+6\alpha+2\delta-\text{etc.}}}}}$$

50. Truly that equation resolved in the other manner on account of the values

$$\begin{aligned}&\lambda=-\,\alpha,\ \mu=-4\alpha-\delta,\ \nu=-2\alpha-\delta,\\&\zeta=\alpha,\ \eta=\tfrac{3}{2}\alpha+\beta,\ \ \theta=\left(2\alpha+\delta\right)\left(\alpha+2\beta-\delta\right),\end{aligned}$$

gives

$$ff-\left(3\alpha+2\beta\right)f+\left(2\alpha+\delta\right)\left(\alpha+2\beta-\delta\right)=0\,;$$

with which we may take now

$$f=\alpha+2\beta-\delta\,,$$

so that there shall become

$$g=5\alpha+2\beta-\delta \text{ and } \varepsilon=\alpha+2\beta-\delta\,,$$

and this substitution will be produced

$$\text{X}'' = 2\alpha n + 5\alpha + 2\beta - \frac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\text{X}'''},$$

$$\text{Y}'' = 2\alpha n + 7\alpha + 2\beta - \frac{(3\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{\text{Y}'''},$$

which leads to this equation

$$\left(\text{X}''' - \alpha n - 5\alpha - 2\beta + \delta\right)\left(\text{Y}''' - \alpha n - 2\alpha - 2\beta + \delta\right)$$
$$= \alpha\alpha nn + \alpha\left(4\alpha + 2\beta\right)n + \left(2\alpha + \delta\right)\left(2\alpha + 2\beta - \delta\right).$$

51. Hence now since the law of the progression may be able to be deduced, we will have

$$xx = an + \delta - \cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta - \cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta - \cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+5\alpha+2\beta - \cfrac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{2\alpha n+7\alpha+2\beta - \text{etc.}}}}}$$

which continued fraction on account of a concise enough law of the progression is noteworthy.

RESOLUTION OF THE EQUATION $xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$ BY § 28.

52. On putting

$$\delta = \beta + \sqrt{(\beta\beta - \gamma)}$$

on account of

$$\lambda = 0,\quad \mu = 0,\quad \nu = 0,\quad \zeta = \alpha,\quad \eta = \beta,\quad \theta = \gamma$$

there will be

$$g = \delta,\quad f = -\alpha + \delta \quad \text{and} \quad \varepsilon = \delta,$$

from which on substituting

$$xx = \alpha n - \alpha + \delta + \frac{(-\alpha-2\beta+2\delta)(\alpha n-\alpha+\delta)}{\text{X}},$$

$$yy = \alpha n + \delta + \frac{(-\alpha-2\beta+2\delta)(\alpha n+\delta)}{\text{Y}}$$

this equation will give, from § 27

$$\left(\mathrm{X}+\alpha n-\alpha+\delta\right)\left(\mathrm{Y}+\alpha n+\delta\right)$$
$$=\alpha\alpha nn+\alpha\left(2\beta-\alpha\right)n+\left(\delta-\alpha\right)\left(2\beta-\delta\right).$$

But with X and Y assumed negative, so that there shall be, from § 28

$$xx=\alpha n-\alpha+\delta+\frac{(\alpha+2\beta-2\delta)(\alpha n-\alpha+\delta)}{\mathrm{X}},$$

$$yy=\alpha n+\delta+\frac{(\alpha+2\beta-2\delta)(\alpha n+\delta)}{\mathrm{Y}},$$

there will be had

$$\left(\mathrm{X}+\alpha n+\alpha-\delta\right)\left(\mathrm{Y}-\alpha n-\delta\right)$$
$$=\alpha\alpha nn-\alpha\left(2\beta-\alpha\right)n-\left(\delta-\alpha\right)\left(2\beta-\delta\right).$$

53. This equation treated again following the same will provide

$$\lambda=-\ \alpha,\ \mu=\alpha-\delta,\ \nu=-\delta,$$
$$\zeta=\alpha,\ \eta=-\tfrac{1}{2}\alpha+\beta,\ \ \theta=\left(\delta-\alpha\right)\left(2\beta-\delta\right),$$

from which there becomes

$$gg-\left(2\beta-\alpha\right)g+\left(\delta-\alpha\right)\left(2\beta-\delta\right)=0,$$

therefore either

$$g=\delta-\alpha\ \text{ or }\ g=2\beta-\delta$$

and both

$$f=-a+g\ \text{ and }\ e=g.$$

Whereby the substitution will be

$$\mathrm{X}=2\alpha n-2\alpha+g+\delta+\frac{(2\beta-2g)(\alpha n-\alpha+g)}{\mathrm{X'}},$$

$$\mathrm{Y}=2\alpha n+g+\delta+\frac{(2\beta-2g)(\alpha n+g)}{\mathrm{Y'}},$$

which leads to this equation

$$\left(\mathrm{X}'-\alpha n+\alpha-g\right)\left(\mathrm{Y}'-\alpha n+\alpha-g\right)$$
$$=\alpha\alpha nn+\alpha\left(2\beta-2\alpha\right)n+\left(g-\alpha\right)\left(2\beta-\alpha-g\right).$$

54. We may retain the letter g involving a twofold value and we may indicate the following by $g',\ g''$. Therefore since here there shall be

$$\lambda = -\alpha,\ \mu = \alpha - g,\ \nu = \alpha - g,$$
$$\zeta = \alpha,\ \eta = -\alpha + \beta,\ \ \theta = (g-\alpha)(2\beta - \alpha - g),$$

there will be

$$g' = g - \alpha \ \text{ or } \ g' = 2\beta - \alpha - g$$

and hence

$$f = -2\alpha + g' \ \text{ et } \ \varepsilon = g'$$

and thus

$$\mathrm{X}' = 2\alpha n - 3\alpha + g + g' + \frac{(2\beta - 2g')(\alpha n - \alpha + g')}{\mathrm{X}''},$$

$$\mathrm{Y}' = 2\alpha n - \alpha + g + g' + \frac{(2\beta - 2g')(\alpha n + g')}{\mathrm{Y}''},$$

from which this equation arises

$$(\mathrm{X}'' - \alpha n + \alpha - g')(\mathrm{Y}'' - \alpha n + 2\alpha - g')$$
$$= \alpha\alpha nn + \alpha(2\beta - 3\alpha)n + (g' - \alpha)(2\beta - \alpha - g').$$

55. Now therefore again there will be

$$\lambda = -\alpha,\ \mu = \alpha - g',\ \nu = 2\alpha - g',$$
$$\zeta = \alpha,\ \eta = \beta - \tfrac{3}{2}\alpha,\ \ \theta = (g' - \alpha)(2\beta - \alpha - g')$$

and hence either

$$g'' = g' - \alpha \ \text{ or } \ g'' = 2\beta - 2\alpha - g'$$

and

$$f = -3\alpha + g'' \text{, and } \varepsilon = g''.$$

Whereby the substitution

$$\mathrm{X}'' = 2\alpha n - 4\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n - \alpha + g'')}{\mathrm{X}'''},$$

$$\mathrm{Y}'' = 2\alpha n - 2\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n + g'')}{\mathrm{Y}'''}$$

will give this equation

$$(\mathrm{X}''' - \alpha n + \alpha - g'')\ (\mathrm{Y}''' - \alpha n + 3\alpha - g'')$$
$$= \alpha\alpha nn + \alpha(2\beta - 4\alpha)n + (g'' - \alpha)(2\beta - 3\alpha - g'').$$

56. Now for the resolution of this equation there is :

$$\lambda = -\alpha,\ \mu = \alpha - g'',\ \nu = 3\alpha - g'',$$
$$\zeta = \alpha,\ \eta = \beta - 2\alpha,\ \ \theta = (g'' - \alpha)(2\beta - 3\alpha - g''),$$

from which either

$$g''' = g'' - \alpha \ \text{ or } \ g''' = 2\beta - 3\alpha - g'',$$
$$f = -4\alpha + g''' \text{ and } \ \varepsilon = g'''.$$

Whereby from the substitution

$$\mathrm{X}''' = 2\alpha n - 5\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n - \alpha + g''')}{\mathrm{X}''''},$$

$$\mathrm{Y}'' = 2\alpha n - 3\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n + g''')}{\mathrm{Y}'''}$$

this equation may arise

$$(\mathrm{X}'''' - \alpha n + \alpha - g''')(\mathrm{Y}''' - \alpha n + 4\alpha - g''')$$
$$= \alpha\alpha nn + \alpha(2\beta - 5\alpha)n + (g''' - \alpha)(2\beta - 4\alpha - g''').$$

57. Therefore with these being deduced, from the proposed equation

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

putting

$$\delta = \beta + \sqrt{(\beta\beta - \gamma)},$$

if for the following values for the letters $g,\ g',\ g'',\ g'''$ etc., two may be assumed :

$$g = \begin{Bmatrix} \delta - \alpha \\ 2\beta - \delta \end{Bmatrix},\ g' = \begin{Bmatrix} g - \alpha \\ 2\beta - \alpha - g \end{Bmatrix},\ g'' = \begin{Bmatrix} g' - \alpha \\ 2\beta - 2\alpha - g' \end{Bmatrix},\ g''' = \begin{Bmatrix} g'' - \alpha \\ 2\beta - 3\alpha - g'' \end{Bmatrix} \text{etc.,}$$

the following value may be deduced for xx :

$$xx = \alpha n - \alpha + \delta + \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{2\alpha n - 2\alpha + \delta + g + \cfrac{(2\beta - 2g)(\alpha n - \alpha + g)}{2\alpha n - 3\alpha + g + g' + \cfrac{(2\beta - 2g')(\alpha n - \alpha + g')}{2\alpha n - 4\alpha + g' + g'' + \text{etc.}}}}$$

which therefore on account of the two individual values $\delta,\ g,\ g',\ g'',\ g'''$ etc. can be varied indefinitely.

58. If the same equation in a similar manner with all the interchangeable values retained may be resolved according to § 25 and by taking

$$\delta=\beta\pm\sqrt{(\beta\beta-\gamma)},$$

there may be put

$$f=\begin{Bmatrix}\alpha+\delta\\ 2\beta-\delta\end{Bmatrix},\ f'=\begin{Bmatrix}\alpha+f\\ \alpha+2\beta-f\end{Bmatrix},\ f''=\begin{Bmatrix}\alpha+f'\\ 2\alpha+2\beta-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}\alpha+f''\\ 3\alpha+2\beta-f''\end{Bmatrix}\ \text{etc.},$$

there will become :

$$xx=\alpha n+\delta-\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+\delta+f-\cfrac{(\alpha-2\beta+2f)(\alpha n+f)}{2\alpha n+2\alpha+f+f'+\cfrac{(\alpha-2\beta+2f')(\alpha n+f')}{2\alpha n+3\alpha+f'+f''-\cfrac{(\alpha-2\beta+2f'')(\alpha n+f'')}{2\alpha n+4\alpha+f'+f''-\text{etc.}}}}}.$$

59. By treating the proposed equation in a similar manner according to § 24 , if on putting

$$\delta=\beta\pm\sqrt{(\beta\beta-\gamma)},$$

there may be put as before :

$$f=\begin{Bmatrix}\alpha+\delta\\ 2\beta-\delta\end{Bmatrix},\ f'=\begin{Bmatrix}\alpha+f\\ \alpha+2\beta-f\end{Bmatrix},\ f''=\begin{Bmatrix}\alpha+f'\\ 2\alpha+2\beta-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}\alpha+f''\\ 3\alpha+2\beta-f''\end{Bmatrix}\text{etc.},$$

there will become:

$$xx=\alpha n+\delta-\cfrac{(2\beta-\alpha-2\delta)(\alpha n+\delta)}{-\alpha-\delta+f-\cfrac{(2\beta+\alpha-2f)(\alpha n+f)}{-f+f'-\cfrac{(2\beta+\alpha-2f')(\alpha n+f')}{-\alpha-f'+f''-\cfrac{(2\beta+3\alpha-2f'')(\alpha n+f'')}{-f'+f''-\cfrac{(2\beta+3\alpha-2f''')(\alpha n+f''')}{-\alpha-f'''+f'''-\cfrac{(2\beta+5\alpha-2f'''')(\alpha n+f'''')}{-f''''+f''''-\text{etc.}}}}}}}.$$

60. Again from § 27 if after

$$\delta=\beta\pm\sqrt{(\beta\beta-\gamma)},$$

there may be put

$$g=\begin{Bmatrix}\delta-\alpha\\ 2\beta-\delta\end{Bmatrix},\ g'=\begin{Bmatrix}g-\alpha\\ 2\beta-\alpha-g\end{Bmatrix},\ g''=\begin{Bmatrix}g'-\alpha\\ 2\beta-2\alpha-g'\end{Bmatrix},\ g'''=\begin{Bmatrix}g''-\alpha\\ 2\beta-3\alpha-g''\end{Bmatrix}\text{etc.},$$

there will become :

$$xx=\alpha n-\alpha+\delta-\cfrac{(2\beta-2\delta+\alpha)(\alpha n+\alpha+\delta)}{-g-\delta-\cfrac{(2\beta-2g)(\alpha n-\alpha+g)}{g'-g+\alpha-\cfrac{(2\beta-2g'-2\alpha)(\alpha n-\alpha+g')}{g''-g'-\cfrac{(2\beta-2g''-2\alpha)(\alpha n-\alpha+g'')}{g'''-g''+\alpha-\cfrac{(2\beta-2g'''-4\alpha)(\alpha n-\alpha+g''')}{g''''+g'''-\cfrac{(2\beta-2g''''-4\alpha)(\alpha n-\alpha+g'''')}{g'''''-g''''+\alpha-\text{etc.}}}}}}}.$$

61. But with these reductions being mixed together innumerable other continued fractions may be able to be elicited, which all may express the value of xx ; truly from these four general forms, to which we may acquiesce the first shown in § 33 can be added, and we may adapt these to determining some case. Evidently if $xxyy=nn$ or a series of this kind is sought

$$a,\ b,\ c,\ d,\ e,\ f\ \text{etc.},$$

so that there shall be

$$ab=1,\ bc=2,\ cd=3,\ de=4,\ ef=5\ \text{etc.},\ \ldots.\ xy=n,$$

and now we have noted to become (§ 12) :

$$aa=\tfrac{2}{\pi},\ \ bb=\tfrac{\pi}{2},\ \ cc=\tfrac{2}{1}\cdot\tfrac{2}{\pi},\ \ dd=\tfrac{3}{2}\cdot\tfrac{\pi}{2},\ \ ee=\tfrac{2\cdot4}{1\cdot3}\cdot\tfrac{2}{\pi}\ \ \text{etc.}$$

Then truly from § 6 there is deduced :

$$xx=n\cdot\frac{\int z^{n}dz:\sqrt{(1-zz)}}{\int z^{n-1}dz:\sqrt{(1-zz)}},$$

or, by the infinite product

$$xx=n\cdot\frac{n(n+2)}{(n+1)^{2}}\cdot\frac{(n+2)(n+4)}{(n+3)^{2}}\cdot\frac{(n+4)(n+6)}{(n+5)^{2}}\cdot\frac{(n+6)(n+8)}{(n+7)^{2}}\cdot\text{etc.}$$

Therefore this value of *xx,* as we may observe, may be able to be expressed by continued fractions.

62. Therefore since for the equation

$$xxyy=nn$$

there shall be $\alpha=1,\ \beta=0,\ \gamma=0$, following § 33 there will be

$$\mathrm{N}=n-\tfrac{1}{2}\ \text{ and }\ \mathrm{B}=0,$$

from which there becomes

$$xx=n-\frac{1}{2}+\cfrac{1:4}{2n-1+\cfrac{9:4}{2n-1+\cfrac{25:4}{2n-1+\cfrac{49:4}{2n-1+\cfrac{1}{1+\text{etc.}}}}}}$$

or

$$2xx=2n-1+\cfrac{1}{2(2n-1)+\cfrac{9}{2(2n-1)+\cfrac{25}{2(2n-1)+\cfrac{49}{2(2n-1)+\cfrac{81}{2(2n-1)+\text{etc.}}}}}}$$

63. Again from §59 on account of

$$\beta=0,\ \gamma=0,\ \text{and}\ \ \delta=0,$$

if there may be taken

$$f=\begin{Bmatrix}1\\0\end{Bmatrix},\ f'=\begin{Bmatrix}1+f\\1-f\end{Bmatrix},\ f''=\begin{Bmatrix}1+f'\\2-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}1+f''\\3-f''\end{Bmatrix},\ f''''=\begin{Bmatrix}1+f'''\\4-f'''\end{Bmatrix},$$

there will be

$$xx=n+\cfrac{n}{f-1-\cfrac{(1-2f)(n+f)}{f'-f-\cfrac{(1-2f')(n+f')}{f''-f'-1-\cfrac{(3-2f'')(n+f'')}{f'''-f''+\cfrac{(3-2f''')(n+f''')}{f''''-f'''-1-\cfrac{(5-2f'''')(n+f'''')}{f'''''-f''''-\cfrac{(5-2f''''')(n+f''''')}{f''''''-f'''''-\text{etc.}}}}}}}}$$

or from § 58 under the same denominations

$$xx=n-\cfrac{n}{2n+1+f-\cfrac{(1+2f)(n+f)}{2n+2+f+f'-\cfrac{(1+2f')(n+f')}{2n+3+f'+f''-\cfrac{(1+2f'')(n+f'')}{2n+4+f''+f'''--\cfrac{(1+2f''')(n+f''')}{2n+5+f'''+f''''-\text{etc.}}}}}}$$

64. Then on putting

$$g=\begin{Bmatrix}-1\\0\end{Bmatrix},\ g'=\begin{Bmatrix}g-1\\-1-g\end{Bmatrix},\ g''=\begin{Bmatrix}g'-1\\-2-g'\end{Bmatrix},\ g'''=\begin{Bmatrix}g''-1\\-3-g''\end{Bmatrix}\ \ \text{etc.},$$

there will be from § 60

$$xx=n-1-\cfrac{n-1}{g+\cfrac{2g(n-1+g)}{g'-g+1+\cfrac{2(1+g')(n-1+g')}{g''-g'+1+\cfrac{2(1+g'')(n-1+g'')}{g'''-g''+1+\cfrac{2(1+g''')(n-1+g''')}{g''''-g'''+\text{etc.}}}}}}.$$

and from §57

$$xx = n-1-\cfrac{n-1}{2n-2+g-\cfrac{2g(n-1+g)}{2n-3+g+g'-\cfrac{2g'(n-1+g')}{2n-4+g'+g''-\cfrac{2g''(n-1+g'')}{2n-5+g''+g'''-\text{etc.}}}}}$$

65. Therefore generally for the series a, b, c, d etc., in which there shall be

$$ab = p,\ bc = p+q,\ cd = p+2q,\ de = p+3q \text{ etc., } \ldots\ xy = p+nq,$$

from the above [§ 6] there is agreed to be

$$xx = (p+nq)\cdot\frac{(p+nq)(p+(n+2)q)}{(p+(n+1)q)(p+(n+1)q)}\cdot\frac{(p+(n+2)q)(p+(n+4)q)}{(p+(n+3)q)(p+(n+3)q)}\cdot\frac{(p+(n+4)q)(p+(n+6)q)}{(p+(n+5)q)(p+(n+5)q)}\cdot\text{etc.},$$

and by the integral formulas

$$xx = (p+nq)\cdot\frac{\int z^{p+(n+1)q-1}dz:\sqrt{(1-z^{2q})}}{\int z^{p+nq-1}dz:\sqrt{(1-z^{2q})}}$$

on putting $z=1$. Now on account of

$$xxyy = qqnn + 2pqn + pp$$

we will have $a = q,\ \beta = p$ and $\gamma = pp,$ hence $\delta = p.$ Whereby from § 33 there will become

$$\mathrm{N} = nq - \tfrac{q}{2} + p \text{ and } \mathrm{B} = 0$$

and thus

$$xx = p+q\left(n-\frac{1}{2}\right)+\cfrac{\frac{1}{4}qq}{2p+q(2n-1)+\cfrac{\frac{9}{4}qq}{2p+q(2n-1)+\cfrac{\frac{25}{4}qq}{2p+q(2n-1)+\cfrac{\frac{49}{4}qq}{2p+q(2n-1)+\text{etc.}}}}}$$

66. But for the remaining formulas, if may put initially

$$f = \begin{Bmatrix} q+p \\ p \end{Bmatrix},\ f' = \begin{Bmatrix} q+f \\ q+2p-f \end{Bmatrix},\ f'' = \begin{Bmatrix} 1+f' \\ 2q+2p-f' \end{Bmatrix},\ f''' = \begin{Bmatrix} q+f'' \\ 3q+2p-f'' \end{Bmatrix}, \text{ etc.}$$

we will have from § 59

$$xx = qn + p + \cfrac{q(qn+p)}{f-p-q-\cfrac{(q+2p-2f)(qn+f)}{f'-f-\cfrac{(q+2p-2f')(qn+f')}{f''-f'-q-\cfrac{(3q+2p-2f'')(qn+f'')}{f'''-f''-\cfrac{(3q+2p-2f''')(qn+f''')}{f''''-f'''-q-\text{etc.}}}}}}$$

and from §58

$$xx = qn + p - \cfrac{q(qn+p)}{2qn+q+p+f-\cfrac{(q-2p+2f)(qn+f)}{2qn+2q+f+f'-\cfrac{(q-2f+2f')(qn+f')}{2qn+3q+f+f'-\text{etc.}}}}$$

where from the three numbers given *p, q, n* any two can be assumed negative, just as the equation being resolved hence permits no change.

67. Then if we may put

$$g = \begin{Bmatrix} p-q \\ p \end{Bmatrix},\ g' = \begin{Bmatrix} g-q \\ 2p-q-g \end{Bmatrix},\ g'' = \begin{Bmatrix} g'-q \\ 2p-2q-g' \end{Bmatrix},\ g''' = \begin{Bmatrix} g''-q \\ 2p-3q-g'' \end{Bmatrix}, \text{ etc.}$$

there will be by § 60

$$xx = qn - q + p - \cfrac{q(qn-q+p)}{g-p-\cfrac{2(p-g)(qn-q+g)}{g'-g+q-\cfrac{2(p-g'-q)(qn-q-g')}{g''-g'-\cfrac{2(p-g''-q)(qn-q-g'')}{g'''-g''+q-\text{etc.}}}}},$$

by §57

$$xx = qn - q + p - \cfrac{q(qn-q+p)}{2qn-2q+p+g+\cfrac{2(p-g)(qn-q+g)}{2qn-3q+g+g'+\cfrac{2(p-g')(qn-q+g)}{2qn-4q+g'+g''+\text{etc.}}}}$$

68. Truly from these expressions departing to infinity, it is understood in many occasions these are equivalent to divergent series, thus so that further we may deduce more values of these, with which therefore we will depart more from the truth; yet which inconvenience does not usually arise in the first expression. On account of which, with the same situation arising for these cases, I have now set out regarding the nature of divergent series [see E247], evidently these are required to be regarded as infinite formulas arising from the establishment of some finite formulas, nevertheless for the sum of these which may be obtained, even if we stop somewhere in the summation of the parts, truly we attain nothing.

69. Also we will examine this series

$$a,\ b,\ c,\ d\ \text{ etc.},$$

in which there shall be

$$ab = \frac{\beta}{\gamma},\ \ bc = \frac{\alpha+\beta}{\alpha+\gamma},\ \ cd = \frac{2\alpha+\beta}{2\alpha+\gamma},\ \ de = \frac{3\alpha+\beta}{3\alpha+\gamma}\ \text{ etc.}$$

and hence in general

$$xy = \frac{\alpha n+\beta}{\alpha n+\gamma},$$

and there will be obtained:

$$x = \frac{\alpha n+\beta}{\alpha n+\gamma}\cdot\frac{\alpha(n+1)+\gamma}{\alpha(n+1)+\beta}\cdot\frac{\alpha(n+2)+\beta}{\alpha(n+2)+\gamma}\cdot\frac{\alpha(n+3)+\gamma}{\alpha(n+3)+\beta}\cdot\text{etc.}$$

or

$$x = \frac{\int z^{\alpha n+\gamma-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\alpha n+\beta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}.$$

Since now, if there shall become $n=\infty$, there becomes $x=y=1$, the values of x and y will continually approach more towards unity; whereby we may put

$$x = 1+\frac{\mathrm{A}}{\mathrm{X}} \text{ et } y = 1+\frac{\mathrm{A}}{\mathrm{Y}}$$

and there will become

$$(\mathrm{X}+\mathrm{A})(\mathrm{Y}+\mathrm{A}) = \frac{\alpha n+\beta}{\alpha n+\gamma}\mathrm{XY}$$

or

$$(\beta-\gamma)\mathrm{XY}-\mathrm{A}(\alpha n+\gamma)\mathrm{X}-\mathrm{A}(\alpha n+\gamma)\mathrm{Y} = \mathrm{AA}(\alpha n+\gamma)$$

Let

$$\mathrm{A} = \beta-\gamma$$

or

$$x = 1+\frac{\beta-\gamma}{\mathrm{X}} \text{ and } y = 1+\frac{\beta-\gamma}{\mathrm{Y}}$$

and there will be had

$$\mathrm{XY}-(\alpha n+\gamma)\mathrm{X}-(\alpha n+\gamma)\mathrm{Y} = (\beta-\gamma)(\alpha n+\gamma)$$

or

$$(\mathrm{X}-\alpha n-\gamma)(\mathrm{Y}-\alpha n-\gamma) = (\alpha n+\beta)(\alpha n+\gamma).$$

70. Now from this equation the values X and Y can be shown to be given in an infinite number of ways, from which § 19 may suffice to be maximally convergent.
But since there shall be

$$\lambda = -\alpha,\ \mu = -\gamma,\ \nu = -\gamma,\ \zeta = \alpha,\ \eta = \frac{\beta+\gamma}{2} \text{ and } \theta = \beta\gamma,$$

there will become

$$\mathrm{X} = 2\alpha n-\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma+\frac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{X}'}$$

$$\mathrm{Y} = 2\alpha n+\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma+\frac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{Y}'}$$

and hence this new equation emerges

$$\left(X'-\alpha n-\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)\left(Y'-\alpha n+\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\right)\gamma=(\alpha n+\beta)(\alpha n+\gamma).$$

71. But if this equation may be set out anew, on account of

$$\lambda=-\alpha,\ \mu=-\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,\ v=\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,$$
$$\zeta=\alpha,\ \eta=\tfrac{\beta+\gamma}{2}\ \text{and}\ \ \theta=\beta\gamma,$$

so that there shall become $\eta\eta-\theta=\frac{1}{4}(\beta-\gamma)^2$, this substitution may arise

$$X'=2\alpha n-\alpha+\beta+\gamma+\frac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{X''},$$

$$Y'=2\alpha n+\alpha+\beta+\gamma+\frac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{Y''},$$

which provides this same equation

$$\left(X''-\alpha n-2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)\left(Y''-\alpha n+2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\right)\gamma=(\alpha n+\beta)(\alpha n+\gamma).$$

72. Now since here there shall be

$$\lambda=-\alpha,\ \mu=-2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,\ v=2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,$$
$$\zeta=\alpha,\ \eta=\tfrac{\beta+\gamma}{2}\ \text{et}\ \theta=\beta\gamma,$$

this substitution will arise:

$$X''=2\alpha n-\alpha+\beta+\gamma+\frac{9\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{X'''},$$

$$Y''=2\alpha n+\alpha+\beta+\gamma+\frac{9\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{Y'''}$$

and hence this equality

$$\left(X'''-\alpha n-3\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)\left(Y'''-\alpha n+3\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)=(\alpha n+\beta)(\alpha n+\gamma).$$

73. By progressing in this manner finally we will come upon the resolution of this equation proposed :

$$xy=\frac{\alpha n+\beta}{\alpha n+\gamma}$$

$$x=1+\cfrac{\beta-\gamma}{2\alpha n-\alpha+\frac{1}{2}\beta+\frac{3}{2}\gamma+\cfrac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\cfrac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\cfrac{9\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\text{etc.}}}}}$$

thus so that there shall become

$$x=\frac{\int z^{\alpha n+\gamma-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\alpha n+\beta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}.$$

74. We may run through some examples and at first there shall be

$$\alpha=1,\ \beta=\ 2\ \text{ and }\ \gamma=0;$$

there will become

$$x=\frac{\int z^{n-1}dz:\sqrt{(1-zz)}}{\int z^{n+1}dz:\sqrt{(1-zz)}}=\frac{n+1}{n}$$

and thus

$$x=1+\cfrac{2}{2n+\cfrac{1-1}{2n+1+\text{etc.}}}=1+\frac{1}{n},$$

as is apparent. And in general, if $\alpha=1$, whenever $\beta-\gamma$ is an even number, the value of x is expressed rationally.

75. With $\alpha=1$remaining, there shall be

$$\beta=1\ \text{ and }\ \gamma=0;$$

there will become

$$x=\frac{\int z^{n-1}dz:\sqrt{(1-zz)}}{\int z^{n}dz:\sqrt{(1-zz)}},$$

and from the equation

$$xy=\frac{n+1}{n},$$

but by the continued fraction :

$$x = 1 + \cfrac{1}{2n - \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{2n + \cfrac{9 - \frac{1}{4}}{2n + \text{etc.}}}}} = 1 + \cfrac{2}{4n - 1 + \cfrac{1 \cdot 3}{4n + \cfrac{3 \cdot 5}{4n + \cfrac{5 \cdot 7}{4n + \text{etc.}}}}}$$

But by taking

$$\beta = 0 \text{ and } \gamma = 1$$

the reciprocal value produces

$$\frac{1}{x} = 1 - \cfrac{1}{2n + \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{4n + \text{etc.}}}} = 1 - \cfrac{2}{4n + 1 + \cfrac{1 \cdot 3}{4n + \cfrac{3 \cdot 5}{4n + \cfrac{5 \cdot 7}{4n + \text{etc.}}}}}$$

the agreement of which with the preceding is readily apparent.

76. But of now the successive numbers 1, 2, 3, 4 etc. may be substituted for n, the following continued fractions will be found

$$\frac{\pi}{2} = 1 + \cfrac{2}{3 + \cfrac{1 \cdot 3}{4 + \cfrac{3 \cdot 5}{4 + \cfrac{5 \cdot 7}{4 + \cfrac{7 \cdot 9}{4 + \text{etc.}}}}}}$$

$$\frac{2}{1} \cdot \frac{2}{\pi} = 1 + \cfrac{2}{7 + \cfrac{1 \cdot 3}{8 + \cfrac{3 \cdot 5}{8 + \cfrac{5 \cdot 7}{8 + \cfrac{7 \cdot 9}{8 + \text{etc.}}}}}}$$

$$\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} = 1 + \cfrac{2}{11 + \cfrac{1 \cdot 3}{12 + \cfrac{3 \cdot 5}{12 + \cfrac{5 \cdot 7}{12 + \cfrac{7 \cdot 9}{12 + \text{etc.}}}}}}$$

$$\frac{2 \cdot 2 \cdot 4}{1 \cdot 3 \cdot 3} \cdot \frac{2}{\pi} = 1 + \cfrac{2}{15 + \cfrac{1 \cdot 3}{16 + \cfrac{3 \cdot 5}{16 + \cfrac{5 \cdot 7}{16 + \cfrac{7 \cdot 9}{16 + \text{etc.}}}}}}$$

77. Hence also in turn the values of the continued fractions of this kind will be able to be investigated. For let this fraction be proposed :

$$s=\cfrac{\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{4\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{9\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{16\alpha\alpha-\delta\delta}{m-\alpha+\text{etc.}}}}}$$

there will be [§ 73]

$$\beta-\gamma=2\delta \text{ and } 2\alpha n+\beta+\gamma=m$$

from which

$$\beta=2\delta+\gamma \text{ and } 2\alpha n=m-2\delta-2\gamma$$

and thus

$$x=1+\frac{2\delta}{m-\alpha-\delta+s}=\frac{m-\alpha+\delta+s}{m-\alpha-\delta+s}$$

therefore

$$s=\frac{2\delta}{x-1}-m+\alpha+\delta.$$

Truly there is :

$$x=\frac{\int z^{\frac{1}{2}m-\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\frac{1}{2}m+\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}},$$

from which, if there may be put

$$\int z^{\frac{1}{2}m-\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}=\mathrm{P} \text{ and } \int z^{\frac{1}{2}m+\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}=Q,$$

there will become:

$$s=\frac{(m-\alpha+\delta)\mathrm{Q}-(m-\alpha-\delta)\mathrm{P}}{\mathrm{P}\text{-}\mathrm{Q}}.$$

78. Hence if there may be put

$$\delta=\varepsilon\sqrt{-1},$$

so that there shall be

$$s=\cfrac{\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{4\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{9\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{16\alpha\alpha+\varepsilon\varepsilon}{m+\text{etc.}}}}}$$

on account of

$$z^{-\delta}=z^{-\varepsilon\sqrt{-1}}=e^{-\varepsilon\sqrt{-1}\cdot lz}=\cos.\varepsilon lz-\sqrt{-1}\cdot\sin.\varepsilon lz$$

and

$$z^{\delta}=\cos.\varepsilon lz-\sqrt{-1}\cdot\sin.\varepsilon lz,$$

there may be put

$$\int \frac{z^{\frac{1}{2}m-1} dz \cos.\varepsilon lz}{\sqrt{(1-z^{2\alpha})}} = \mathrm{R}$$

and

$$\int \frac{z^{\frac{1}{2}m-1} dz \sin.\varepsilon lz}{\sqrt{(1-z^{2\alpha})}} = \mathrm{S},$$

there will become

$$\mathrm{P} = \mathrm{R} - \mathrm{S}\sqrt{-1} \text{ and } \mathrm{Q} = \mathrm{R} + \mathrm{S}\sqrt{-1}$$

and thus

$$s = \frac{2(m-\alpha)\mathrm{S}\sqrt{-1} + 2\varepsilon \mathrm{R}\sqrt{-1}}{-2\mathrm{S}\sqrt{-1}} = \alpha - m - \varepsilon \frac{\mathrm{R}}{\mathrm{S}},$$

where it is requires to be observed the integrals R and S must be taken thus, so that they may vanish on putting $z = 0$, then truly to be put in place $z = 1$.

DE SERIEBUS IN QUIBUS PRODUCTA EX BINIS TERMINIS CONTIGUIS DATAM CONSTITUUNT PROGRESSIONEM

Opuscula analytica 1, 1788, p. 3-47

Proposita progressione numerorum quacunque

$$\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F} \text{ etc.}$$

quaestio, quam hic tractare statui, in hoc consistit, ut inveniatur eiusmodi series

$$a,\ b,\ c,\ d,\ e,\ f \text{ etc.},$$

in qua sit

$$ab = \mathrm{A},\ bc = \mathrm{B},\ cd = \mathrm{C},\ de = \mathrm{D},\ ef = \mathrm{E},\ fg = \mathrm{F} \text{ etc.},$$

ubi, etsi numeri A, B, C, D etc. sint rationales satisque simplici lege procedant, plerumque fieri solet, ut numeri *a, b, c, d* etc. evadant adeo maxime transcendentes. Evidens autem est totum negotium ad unicum terminum huius seriei revocari, quippe quo cognito reliqui omnes facillime definientur; invento enim primo *a* reliqui ita se habebunt:

$$b = \frac{\mathrm{A}}{a},\ c = \frac{\mathrm{B}}{b},\ d = \frac{\mathrm{C}}{c},\ e = \frac{\mathrm{D}}{d} \text{ etc}.$$

Duplicem autem ad solutionem huius quaestionis patere viam observavi, quarum altera interpolatione certae cuiusdam seriei absolvatur, altera autem, quae magis directa videatur, ad fractiones continuas perducat; quae duae methodi cum diverso plane modo negotium conficiant, earum collatio haud contemnendas proprietates patefaciet. Utramque igitur methodum seorsim exponam, deinceps, quae fuerint eruta, inter se comparaturus.

METHODUS PRIOR INTERPOLATIONE INNIXA

1. Consideretur series ex quaesita hoc modo formanda

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & ab, & abc, & abcd, & abcde, & abcdef, & abcdefg \text{ etc.,} \end{array}$$

quae ob

$$ab = \mathrm{A},\; bc = \mathrm{B},\; cd = \mathrm{C},\; de = \mathrm{D},\; ef = \mathrm{E} \text{ etc.}$$

in hanc abibit formam

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & \mathrm{A}, & a\mathrm{B}, & \mathrm{AC}, & a\mathrm{BD}, & \mathrm{ACE}, & a\mathrm{BDF} \text{ etc.,} \end{array}$$

cuius ergo termini locis paribus constituti ob progressionem A, B, C, D etc. datam per se innotescunt.

2. Cum ergo progressio terminorum alternorum

$$\mathrm{A},\; \mathrm{AC},\; \mathrm{ACE},\; \mathrm{ACEG} \text{ etc.}$$

sit cognita, eius interpolatio ad verum valorem termini quaesiti a manuducet.At ista progressio semper ita est comparata, ut in infinitum continuata cum eiusmodi progressione simplici confundatur, cuius interpolatio nulli amplius difficultati sit obnoxia. Plerumque autem illa progressio in infinitum productam geometricam abire solet, ita ut interpolandi sint medii proportionales inter binos contiguos.

3. Si ergo seriem

$$a, \mathrm{A}, a\mathrm{B}, \mathrm{AC}, a\mathrm{BD}, \mathrm{ACE} \text{ etc.}$$

totam ut geometricam spectemus indeque terminos medios definiamus, ab initio multum fortasse a veritate aberrabimus; sed quo longius progrediamur,eo propius ad veritatem accedemus, quam tandem in infinito plane assequemur.Hinc sequentes determinationes ad verum continuo magis appropinquabunt:

$$\begin{array}{ll} aa = \frac{\mathrm{AA}}{\mathrm{B}}, & aa = \frac{\mathrm{AAC}}{\mathrm{BB}}, \\ aa = \frac{\mathrm{AACC}}{\mathrm{BBD}}, & aa = \frac{\mathrm{AACE}}{\mathrm{BBDD}}, \\ aa = \frac{\mathrm{AACEE}}{\mathrm{BBDDF}}, & aa = \frac{\mathrm{AACCEEG}}{\mathrm{BBDDFF}} \\ \text{etc.} & \text{etc.} \end{array}$$

Sicque revera in infinitum progrediendo erit

$$a = \mathrm{A}\cdot\frac{\mathrm{AC}}{\mathrm{BB}}\cdot\frac{\mathrm{CE}}{\mathrm{DD}}\cdot\frac{\mathrm{EG}}{\mathrm{FF}}\cdot\frac{\mathrm{GI}}{\mathrm{HH}}\cdot\frac{\mathrm{IL}}{\mathrm{KK}}\cdot\text{etc.}$$

4. Expressio haec infinita verum valorem ipsius *a* exhibet, quoties progressio numerorum A, B, C, D etc. ita est comparata, ut termini infinitesimi inter se rationem aequalitatis teneant illiusque expressionis factores tandem in unitatem abeant. Veluti si pro A, B, C, D etc. series numerorum naturalium accipiatur, ut sit

$$ab = 1,\ bc = 2,\ cd = 3,\ de = 4,\ ef = 5,\ fg = 6 \text{ etc.},$$

erit

$$aa = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\text{etc.}$$

Constat autem hoc productum infinitum posita ratione diametri ad peripheriam $=1:\pi$ esse $=2:\pi$, ita ut sit

$$a = \sqrt{\frac{2}{\pi}}$$

hincque

$$b = \sqrt{\frac{\pi}{2}},\quad c = \frac{2\sqrt{2}}{\sqrt{\pi}},\quad d = \frac{3\sqrt{\pi}}{2\sqrt{2}}\quad \text{etc.}$$

5. Haec ergo series numerorum transcendentalium lege quadam uniformitatis procedit, quos numeros per approximationem evolvisse eorumque differentias notasse iuvabit:

	diff. 1	diff. 2	diff. 3
$a = 0,7978846$			
	4554295		
$b = 1,2533141$		1129745	
	3424550		547216
$c = 1,5957691$		582529	
	2842021		217718
$d = 1,87997121$		364811	
	2477210		110319
$e = 2,1276922$		254492	
	2222718		64440
$f = 2,3499640$		190052	
	2032666		41327
$g = 2,5532306$		148725	
	1883941		
$h = 2,7416247$			

Si enim pro *a* alius quicunque numerus assumeretur ex eoque sequentes definirentur, in differentiis ingentes saltus essent apparituri.

6. Eodem modo negotium procedit, si pro numeris A, B, C, D etc. quaecunque capiatur progressio arithmetica. Quaerenda enim sit series *a, b,c, d, e* etc., ita ut sit

$$ab = p,\ bc = p + q,\ cd = p + 2q,\ de = p + 3q \text{ etc.,}$$

et quia termini infinitesimi ad rationem aequalitatis accedunt, erit

$$aa = p \cdot \frac{p(p+2q)}{(p+q)(p+q)} \cdot \frac{(p+2q)(p+4q)}{(p+3q)(p+3q)} \cdot \frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \cdot \text{etc.,}$$

cuius expressionis valor ita per formulas integrales exhiberi potest, ut sit

$$aa = p \cdot \frac{\int z^{p+q-1} dz : \sqrt{(1-z^{2q})}}{\int z^{p-1} dz : \sqrt{(1-z^{2q})}}$$

posito post utramque integrationem $z = 1$.

7. Si pro A, B, C, D etc. sumatur progressio mixta ex arithmetica et harmonica, ut talis series numerorum *a, b, c, d, e* etc. sit investiganda

$$ab = \frac{p}{r},\quad bc = \frac{p+q}{r+s},\quad cd = \frac{p+2q}{r+2s},\quad de = \frac{p+3q}{r+3s} \quad \text{etc.,}$$

quia et hic numeri A, B, C, D etc. ad rationem aequalitatis convergunt, erit

$$aa = \frac{p}{r} \cdot \frac{p(r+s)(p+2q)(r+s)}{r(p+q)(r+2s)(p+q)} \cdot \frac{(p+2q)(r+3s)(p+4q)(r+3s)}{(r+2s)(p+3q)(r+4s)(p+3q)} \cdot \text{etc.,}$$

cuius valor ut supra colligitur

$$aa = \frac{p}{r} \cdot \frac{\int z^{p+q-1} dz : \sqrt{(1-z^{2q})} \cdot \int z^{r-1} dz : \sqrt{(1-z^{2s})}}{\int z^{p-1} dz : \sqrt{(1-z^{2q})} \cdot \int z^{r+s-1} dz : \sqrt{(1-z^{2s})}}$$

ubi iterum post integrationem poni oportet $z = 1$.

8. Si fuerit $s = q$, numeri A, B, C, D etc. continuo propius ad unitatem accedunt eique tandem fient aequales. Unde cum seriei

$$a,\ \text{A},\ a\text{B},\ \text{AC},\ a\text{BD},\ \text{ACE} \ \text{ etc.}$$

termini infinitesimi inter se aequales sint censendi, inde concludetur

$$a = \frac{p}{r} \cdot \frac{(p+2q)(r+q)}{(p+q)(r+2q)} \cdot \frac{(p+4q)(r+3q)}{(p+3q)(r+4q)} \cdot \text{etc.,}$$

quae expressio etiam ita referri potest

$$a=\frac{p(r+q)}{r(p+q)}\cdot\frac{(p+2q)(r+3q)}{(r+2q)(p+3q)}\cdot\frac{(p+4q)(r+5q)}{(r+4q)(p+5q)}\cdot\text{etc.},$$

cuius valor per formulas integrales est

$$a=\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}.$$

9. Hinc etiam casus, quo s et q sunt inaequales, facilius expediri potest. Sit enim $s=nnq$ ac ponatur $r=nnt$; tum vero statuatur

$$a=\frac{\alpha}{n},\quad b=\frac{\beta}{n},\quad c=\frac{\gamma}{n},\quad d=\frac{\delta}{n},\quad e=\frac{\varepsilon}{n},\quad \text{etc.}$$

eritque per conditionem praescriptam

$$\alpha\beta=\frac{p}{t},\quad \beta\gamma=\frac{p+q}{t+q},\quad \gamma\delta=\frac{p+2q}{t+2q},\quad \delta\varepsilon=\frac{p+3q}{t+3q}\quad \text{etc.},$$

ex cuius convenientia cum praecedenti est

$$\alpha=\frac{p(t+q)}{t(p+q)}\cdot\frac{(p+2q)(t+3q)}{(t+2q)(p+3q)}\cdot\frac{(p+4q)(t+5q)}{(t+4q)(p+5q)}\cdot\text{etc.},$$

ideoque

$$\alpha=\frac{\int z^{t-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}.$$

10. Cum igitur sit

$$n=\sqrt{\frac{s}{q}},\quad t=\frac{qr}{s}\quad \text{et}\quad a=\frac{\alpha\sqrt{q}}{\sqrt{s}},$$

erit pro casu in § 7 exposito

$$a=\frac{\sqrt{q}}{\sqrt{s}}\cdot\frac{p(r+s)}{r(p+q)}\cdot\frac{(p+2q)(r+3s)}{(r+2s)(p+3q)}\cdot\frac{(p+4q)(r+5s)}{(r+4s)(p+5q)}\cdot\text{etc.},$$

ac per formulas integrales

$$a=\frac{\sqrt{q}}{\sqrt{s}}\cdot\frac{\int z^{\frac{qr}{s}-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}};$$

ubi si in numeratore pro z^q scribatur z^s fiet

$$a=\frac{\sqrt{s}}{\sqrt{q}}\cdot\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2s}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}};$$

cuius ergo quadratum aequetur necesse est formulae supra inventae, ita ut sit

$$\frac{s}{q}\cdot\frac{\int z^{r-1}dz:\sqrt{\left(1-z^{2s}\right)}}{\int z^{p-1}dz:\sqrt{\left(1-z^{2q}\right)}}=\frac{p}{r}\cdot\frac{\int z^{p+q-1}dz:\sqrt{\left(1-z^{2q}\right)}}{\int z^{r+s-1}dz:\sqrt{\left(1-z^{2s}\right)}}.$$

11. Harum ergo formularum consensus casu, quo post integrationem statuitur $z=1$, sequens nobis suppeditat theorema

$$pq\int\frac{z^{p-1}dz}{\sqrt{\left(1-z^{2q}\right)}}\cdot\int\frac{z^{p+q-1}dz}{\sqrt{\left(1-z^{2q}\right)}}=rs\int\frac{z^{r-1}dz}{\sqrt{\left(1-z^{2s}\right)}}\cdot\int\frac{z^{r+s-1}dz}{\sqrt{\left(1-z^{2s}\right)}},$$

cuius veritatem quidem iam alibi ex aliis principiis demonstratam dedi. Hinc ergo sequitur sumendo $r=s=1$ fore

$$pq\int\frac{z^{p-1}dz}{\sqrt{\left(1-z^{2q}\right)}}\cdot\int\frac{z^{p+q-1}dz}{\sqrt{\left(1-z^{2q}\right)}}=\frac{\pi}{2},$$

ob

$$\int\frac{dz}{\sqrt{(1-zz)}}=\frac{\pi}{2}\quad\text{et}\quad\int\frac{zdz}{\sqrt{(1-zz)}}=1.$$

12. Contemplemur igitur aliquot exempla.

I. Si esse debeat

$$ab=1,\ bc=2,\ cd=3,\ de=4,\ ef=5\ \text{ etc.},$$

erit

$$aa=1\cdot\frac{1\cdot3}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}\cdot\frac{5\cdot7}{6\cdot6}\cdot\frac{7\cdot9}{8\cdot8}\cdot\text{etc.}$$

et

$$aa=\frac{\int zdz:\sqrt{(1-zz)}}{\int dz:\sqrt{(1-zz)}}=\frac{2}{\pi}.$$

II. Si esse debeat

$$ab=1,\ \ bc=3,\ \ cd=5,\ \ de=7,\ \ ef=9\ \text{ etc.},$$

erit

$$aa=\frac{1\cdot5}{3\cdot3}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{9\cdot13}{15\cdot15}\cdot\text{etc.}$$

seu

$$aa = \frac{\int zzdz:\sqrt{(1-z^4)}}{\int dz:\sqrt{(1-z^4)}}.$$

Cum vero sit ex theoremate modo exposito

$$\frac{\pi}{4} = \int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zzdz}{\sqrt{(1-z^4)}},$$

colligitur

$$a = \sqrt{\frac{2}{\pi}} \int \frac{zzdz}{\sqrt{(1-z^4)}}.$$

III. Si esse debeat

$$ab = 1, \quad bc = 4, \quad cd = 7, \quad de = 10, \quad ef = 13 \text{ etc.},$$

erit

$$aa = \frac{1 \cdot 7}{4 \cdot 4} \cdot \frac{7 \cdot 13}{10 \cdot 10} \cdot \frac{13 \cdot 19}{16 \cdot 16} \cdot \text{etc.}$$

seu

$$aa = \frac{\int z^3 dz:\sqrt{(1-z^6)}}{\int dz:\sqrt{(1-z^6)}},$$

hincque

$$a = \frac{\sqrt{6}}{\sqrt{\pi}} \int \frac{z^3 dz}{\sqrt{(1-z^6)}}.$$

IV. Si generalius esse debeat

$$ab = p, \quad bc = p+q, \quad cd = p+2q, \quad de = p+3q, \quad ef = p+4q \text{ etc.},$$

per reductionem ope theorematis superioris instituendam colligemus

$$a = \frac{p\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1} dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1} dz}{\sqrt{(1-z^{2q})}}.$$

13. Haec exempla ex progressione arithmetica sunt desumta; quibus adiungamus aliquot, in quibus numerorum A, B, C, D etc. progressio est mixta ex arithmetica et harmonica.

I. Si esse debeat

$$ab = \frac{1}{2}, \quad bc = \frac{2}{3}, \quad cd = \frac{3}{4}, \quad de = \frac{4}{5}, \quad ef = \frac{5}{6} \text{ etc.},$$

ob

$$p = 1, \; q = 1, \; r = 2, \; \text{s} = 1$$

erit

$$a=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\text{etc .}$$

seu

$$a=\frac{\int zdz:\sqrt{(1-zz)}}{\int dz:\sqrt{(1-zz)}}=\frac{2}{\pi}$$

II. Si esse debeat

$$ab=\frac{1}{2},\quad bc=\frac{3}{4},\quad cd=\frac{5}{6},\quad de=\frac{7}{8},\quad ef=\frac{9}{10}\quad\text{etc.,}$$

ob

$$p=1,\ q=2,\ r=2,\ \mathrm{s}=2$$

erit

$$a=\frac{1\cdot 4}{2\cdot 3}\cdot\frac{5\cdot 8}{6\cdot 7}\cdot\frac{9\cdot 12}{10\cdot 11}\cdot\frac{13\cdot 16}{14\cdot 15}\cdot\frac{17\cdot 20}{18\cdot 19}\cdot\text{etc .}$$

seu

$$a=\frac{\int zdz:\sqrt{\left(1-z^4\right)}}{\int dz:\sqrt{\left(1-z^4\right)}}=\frac{\pi}{4}:\int\frac{dz}{\sqrt{\left(1-z^4\right)}}=\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}}.$$

III. Si esse debeat

$$ab=\frac{1}{1},\quad bc=\frac{2}{3},\quad cd=\frac{3}{5},\quad de=\frac{4}{7},\quad ef=\frac{5}{9}\quad\text{etc.,}$$

ob

$$p=1,\ q=1,\ r=1,\ \mathrm{s}=2$$

erit

$$a=\frac{1}{\sqrt{2}}\cdot\frac{1\cdot 3}{1\cdot 2}\cdot\frac{3\cdot 7}{5\cdot 4}\cdot\frac{5\cdot 11}{9\cdot 6}\cdot\frac{7\cdot 15}{13\cdot 8}\cdot\frac{9\cdot 19}{17\cdot 10}\cdot\text{etc .}$$

vel

$$a=\frac{\sqrt{2}}{1}\cdot\frac{\int dz:\sqrt{\left(1-z^4\right)}}{\int dz:\sqrt{(1-zz)}}=\frac{2\sqrt{2}}{\pi}\cdot\int\frac{dz}{\sqrt{\left(1-z^4\right)}}=\frac{1}{\sqrt{2}}:\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}}.$$

Productum autem ex hoc valore et praecedente manifesto est $=\frac{1}{\sqrt{2}}$.

METHODUS ALTERA PER FRACTIONE CONTINUAS

14. Seriem inveniendam ita cum indicibus repraesentemus

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & & n & n+1 \\ a, & b, & c, & d, & e & \text{etc.......} & x, & y \end{array}$$

ac primo investigemus eam seriem, in qua sit

$$ab=p,\ bc=p+q,\ cd=p+2q,\ de=p+3q\ \ \text{etc.,}$$

ut sit per methodum praecedentem

$$aa = p \cdot \frac{p(p+2q)}{(p+q)(p+q)} \cdot \frac{(p+2q)(p+4q)}{(p+3q)(p+3q)} \cdot \frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \cdot \text{etc.},$$

et

$$aa = p \cdot \frac{\int z^{p+q-1}dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}dz : \sqrt{(1-z^{2q})}}$$

seu

$$a = \frac{p\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}dz}{\sqrt{(1-z^{2q})}},$$

hinc

$$b = \frac{(p+q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+2q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+q-1}dz}{\sqrt{(1-z^{2q})}},$$

et

$$c = \frac{(p+2q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+3q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+2q-1}dz}{\sqrt{(1-z^{2q})}}$$

ideoque

$$x = \frac{(p+nq)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+(n+1)q-1}dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+nq-1}dz}{\sqrt{(1-z^{2q})}}.$$

15. Cum ergo pro hac serie in genere sit

$$xy = p + nq,$$

quantitas x eiusmodi functio indicis n esse debet, ut posito in ea $n+1$ loco n prodeat y fiatque productum

$$xy = p + nq\,;$$

quod cum rationalitati adversetur, quaeri convenit valores quadratorum xx et yy ex aequatione

$$xxyy = pp + 2npq + nnqq,$$

quandoquidem ratio illa functionum etiam ad quadrata patet. Haec igitur investigatio commode latius extendetur ad resolutionem huius aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma\,;$$

unde valor ipsius xx pluribus modis ad fractiones continuas reduci potest, qui sequentibus lemmatibus innituntur.

LEMMA I

16. Proposita hac aequatione

$$(X+\lambda n+\mu)(Y+\lambda n+v)=\xi\xi nn+2\xi\eta n+\theta,$$

in qua Y perinde ex $n+1$ atque X ex n definitur, ponatur

$$X+\lambda n+\mu=\xi n+f+\tfrac{k}{X'}$$

et

$$Y+\lambda n+v=\xi n+g+\tfrac{k}{Y'},$$

ut sit

$$X=(\xi-\lambda)n+f-\mu+\tfrac{k}{X'}$$

et

$$Y=(\xi-\lambda)n+g-v+\tfrac{k}{Y'},$$

ubi iam X' et Y' sint novae functiones similes ipsarum n et $n+1$; atque necesse est sit

$$g-v=\xi-\lambda+f-\mu \text{ seu } g=\xi-\lambda-\mu+v+f.$$

17. Hoc posito aequatio praescripta abit in hanc

$$\xi\xi nn+\xi(f+g)n+fg+\tfrac{k(\xi n+f)}{Y'}+\tfrac{k(\xi n+g)}{X'}+\tfrac{kk}{X'Y'}=\xi\xi nn+2\xi\eta n+\theta.$$

Statuatur

$$f+g=2\eta \text{ et } k=fg-\theta,$$

ut prodeat

$$X'Y'+(\xi n+f)X'+(\xi n+g)Y'+fg-\theta=0,$$

seu

$$(X'+\xi n+g)(Y'+\xi n+f)=\xi\xi nn+\xi(f+g)n+\theta,$$

quae similis est formae propositae. At ob $f+g=2\eta$ habebitur

$$\xi-\lambda-\mu+v+2f=2\eta,$$

$$f=\eta+\tfrac{\lambda-\xi+\mu-v}{2}$$

et

$$g=\eta-\tfrac{\lambda-\xi+\mu-v}{2}$$

hincque

$$k=fg-\theta=\eta\eta-\tfrac{1}{4}(\lambda-\xi+\mu-v)^2-\theta.$$

18. Quocirca aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+v)=\xi\xi nn+2\xi\eta n+\theta,$$

ope huius substitutionis

$$X=(\xi-\lambda)n+\eta+\tfrac{\lambda-\xi-\mu-v}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\xi+\mu-v)^2-\theta.}{X'},$$

$$Y=(\xi-\lambda)n+\eta-\tfrac{\lambda-\xi+\mu+v}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\xi+\mu-v)^2-\theta.}{Y'},$$

reducitur ad hanc aequationem ipsi propositae similem

$$\left(X'+\xi n-\tfrac{\lambda-\xi+\mu-v}{2}\right)\left(Y'+\xi n+\tfrac{\lambda-\xi+\mu-v}{2}\right)=\xi\xi nn+2\xi\eta n+\theta.$$

19. Simili modo eadem aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+v)=\xi\xi nn+2\xi\eta n+\theta$$

factis his substitutionibus

$$X=(\xi-\lambda)n+\eta+\tfrac{\lambda-\xi-\mu-v}{2}+\frac{\frac{1}{4}(\lambda-\xi+\mu-v)^2-\eta\eta+\theta.}{X'},$$

$$Y=(\xi-\lambda)n+\eta-\tfrac{\lambda-\xi+\mu+v}{2}+\frac{\frac{1}{4}(\lambda-\xi+\mu-v)^2-\eta\eta+\theta}{Y'}.$$

reducitur ad hanc sui similem

$$\left(X'-\xi n-\eta+\tfrac{\lambda-\xi+\mu-v}{2}\right)\left(Y'-\xi n-\eta-\tfrac{\lambda-\xi+\mu-v}{2}\right)=\xi\xi nn+2\xi\eta n+\theta.$$

LEMMA II

20. Proposita hac aequatione

$$(X+\lambda n+\mu)(Y+\lambda n+v)=\xi\xi nn+2\xi\eta n+\theta,$$

in qua Y perinde ab $n+1$ atque X ab n pendet, ponatur

$$X+\lambda n+\mu=\xi n+f+\tfrac{hn+k}{X'},$$

$$Y+\lambda n+V=\xi n+g+\tfrac{hn+h+k}{Y'},$$

ubi ob similitudinem functionum esse debet ut ante

$$g=\xi-\lambda-\mu+\nu+f.$$

21. Porro substitutione horum valorum facta habebimus

$$\xi\xi nn+\xi(f+g)n+fg+\frac{(\xi n+f)(hn+h+k)}{\mathrm{Y}'}+\frac{(\xi n+g)(hn+k)}{\mathrm{X}'}$$
$$+\frac{(hn+k)(hn+h+k)}{\mathrm{X}'\mathrm{Y}'}=\xi\xi nn+2\xi\eta n+\theta,$$

unde fit

$$(\zeta(f+g-2\eta)n+fg-\theta)\mathrm{X}'\mathrm{Y}'+(\zeta n+f)(hn+h+k)\mathrm{X}'$$
$$+(\xi n+g)(hn+k)\mathrm{Y}'+(hn+k)(hn+h+k)=0,$$

quae, ut similis sit formae propositae, divisibilis esse debet per $\xi(f+g-2\eta)n+fg-\theta$; cui quantitati ergo vel $hn+k$ vel $hn+h+k$ aequale vel multiplum statui oportet.

22. Sit primo

$$hn+k=\alpha\xi(f+g-2\eta)n+\alpha(fg-\theta)$$

et $\xi n+f$ submultiplum ipsius $\xi(f+g-2\eta)n+fg-\theta$ esse oportet; unde fit

$$f(f+g-2\eta)=fg-\theta \text{ seu } ff=2\eta f-\theta$$

hineque

$$ff=\eta+\sqrt{(\eta\eta-\theta)} \text{ et } g=\xi-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)};$$

quare porro

$$h=\alpha\xi(f+g-2\eta) \text{ et } k=\alpha(fg-\theta),$$

et aequatio resultans evadet

$$\mathrm{X}'\mathrm{Y}'+\frac{\alpha\zeta(f+g-2\eta)n+\alpha\zeta(f+g-2\eta)+\alpha(fg-\theta)}{f+g-2\eta}\mathrm{X}'$$
$$+\alpha(\zeta n+g)\mathrm{Y}'+\alpha\alpha\big(\xi(f+g-2\eta)n+\zeta(f+g-2\eta)+fg-\theta\big)=0.$$

23. Ut fractiones tollamus, ponamus

$$\alpha=f+g-2\eta=\xi-\lambda-\mu+\nu+2\sqrt{(\eta\eta-\theta)}$$

sicque fiet

$$\mathrm{X}'\mathrm{Y}'+\left(\zeta\left(f+g-2\eta\right)n+\zeta\left(f+g-2\eta\right)+fg-\theta\right)\mathrm{X}'$$

$$+\left(\zeta\left(f+g-2\eta\right)n+g\left(f+g-2\eta\right)\right)\mathrm{Y}'$$
$$+\left(f+g-2\eta\right)^2\left(\zeta\left(f+g-2\eta\right)n+\zeta\left(f+g-2\eta\right)+fg-\theta\right)=0.$$

Verum si fractiones non curemus, habebimus

$$\mathrm{X}'\mathrm{Y}'+\alpha\left(\xi n+\xi+\tfrac{fg-\theta}{(f+g-2\eta)}\right)\mathrm{X}'+\alpha\left(\xi n+g\right)\mathrm{Y}'$$
$$+\alpha\alpha\left(f+g-2\eta\right)\left(\xi n+\xi+\tfrac{fg-\theta}{f+g-2\eta}\right)=0,$$

quae aequatio posito brevitatis gratia

$$\tfrac{fg-\theta}{f+g-2\eta}=\varepsilon,$$

reducitur ad hanc propositae similem,

$$\left(\mathrm{X}'+\alpha\left(\zeta n+g\right)\right)\left(\mathrm{Y}'+\alpha\left(\zeta n+\zeta+\varepsilon\right)\right)=\alpha\alpha\left(\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta\right)$$
$$+\left(\xi+\varepsilon\right)\left(2\eta-f\right)=\alpha\alpha\left(\zeta n+\zeta+\varepsilon\right)\left(\zeta n+2\eta-f\right).$$

24. Proposita ergo aequatione

$$\left(\mathrm{X}+\lambda n+\mu\right)\left(\mathrm{Y}+\lambda n+\nu\right)=\xi\xi nn+2\xi\eta n+\theta$$

si brevitatis gratia ponatur

$$f=\eta+\sqrt{(\eta\eta-\theta)},\ g=\zeta-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)}$$

atque

$$\tfrac{fg-\theta}{f+g-2\eta}=\varepsilon,$$

sequens substitutio

$$\mathrm{X}=\left(\xi-\lambda\right)n+f-\mu+\frac{\xi\left(f+g-2\eta\right)+fg-\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=\left(\xi-\lambda\right)n+g-\nu+\frac{\xi\left(f+g-2\eta\right)\left(n+1\right)+fg-\theta}{\mathrm{Y}'}$$

suppeditabit sequentem aequationem propositae similem

$$\left(\mathrm{X}'+\xi n+g\right)\left(\mathrm{Y}'+\xi\left(n+1\right)+\varepsilon\right)=\xi\xi nn+\xi\left(\xi+\varepsilon-f+2\eta\right)n+\left(\xi+\varepsilon\right)\left(2\eta-f\right).$$

25. Quemadmodum hic sumsimus $\alpha=1$, ita positio $\alpha=-1$ manentibus iisdem abbreviationibus dabit hanc substitutionem

$$\mathrm{X}=(\zeta-\lambda)n+f-\mu+\frac{\zeta(2\eta-f-g)n-fg+\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=(\zeta-\lambda)n+g-\nu+\frac{\zeta(2\eta-f-g)(n+1)-fg+\theta}{\mathrm{Y}'},$$

unde oritur haec aequatio similis propositae

$$(\mathrm{X}'-\xi n-g)(\mathrm{Y}'-\xi(n+1)-\varepsilon)=\xi\xi nn+\xi(\xi+\varepsilon-f+2\eta)n+(\xi+\varepsilon)(2\eta-f).$$

26. Ponamus porro esse

$$hn+h+k=\xi(f+g-2\eta)n+fg-\theta,$$

ut sit

$$h=\xi(f+g-2\eta)\ \text{ et } k=fg-\theta\ -\xi(f+g-2\eta),$$

ac necesse est, ut fiat

$$\xi(f+g-2\eta)n+fg-\theta=(f+g-2\eta)(\xi n+g)$$

ideoque

$$g(f+g-2\eta)=fg-\theta\ \text{ seu }\ g=\eta+\sqrt{(\eta\eta-\theta)}$$

hincque

$$f=\lambda-\xi+\mu-\nu+\eta+\sqrt{(\eta\eta-\theta)}$$

Aequatio autem resultans erit

$$\mathrm{X}'\mathrm{Y}'+(\zeta n+f)\mathrm{X}'+\left(\xi n-\zeta+\frac{fg-\theta}{f+g-2\eta}\right)\mathrm{Y}'+(f+g-2\eta)\left(\zeta(n-1)+\frac{fg-\theta}{f+g-2\eta}\right)=0,$$

quae posito $\frac{fg-\theta}{f+g-2\eta}=\varepsilon$ abit in hanc

$$\begin{aligned}(\mathrm{X}'+\xi n-\xi+\varepsilon)(\mathrm{Y}'+\xi n+f)&=\xi\xi nn+\xi n(2\eta-g-\xi+\varepsilon)+(\varepsilon-\xi)(2\eta-g)\\&=(\xi n-\xi+\varepsilon)(\xi n+2\eta-g).\end{aligned}$$

27. Proposita ergo aequatione

$$(\mathrm{X}+\lambda n+\mu)(\mathrm{Y}+\lambda n+\nu)=\xi\xi nn+2\xi\eta n+\theta$$

si ponatur brevitatis gratia

$$f=\lambda-\zeta+\mu-\nu+\eta+\sqrt{(\eta\eta-\theta)},\; g=\eta+\sqrt{(\eta\eta-\theta)}$$

atque

$$\varepsilon=\frac{fg-\theta}{f+g-2\eta},$$

sequens substitutio

$$\mathrm{X}=(\zeta-\lambda)n+f-\mu+\frac{\zeta(f+g-2\eta)(n-1)+fg-\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=(\zeta-\lambda)n+g-\nu+\frac{\zeta(f+g-2\eta)n+fg-\theta}{\mathrm{Y}'}$$

praebebit hanc aequationem propositae similem

$$(\mathrm{X}'+\xi n-\xi+\varepsilon)(\mathrm{Y}'+\xi n+f)=\xi\xi nn+\xi(2\eta-g-\xi+\varepsilon)n+(\varepsilon-\xi)(2\eta-g).$$

28. Simili modo manentibus iisdem abbreviaturis eadem aequatio proposita ope harum substitutionum

$$\mathrm{X}=(\xi-\lambda)n+f-\mu+\frac{\xi(2\eta-f-g)(n-1)-fg+\theta}{\mathrm{X}'},$$

$$\mathrm{Y}=(\xi-\lambda)n+g-\nu+\frac{\xi(2\eta-f-g)n-fg+\theta}{\mathrm{Y}'}$$

reducetur ad hanc aequationem propositae similem

$$(\mathrm{X}'-\xi n+\xi-\varepsilon)(\mathrm{Y}'-\xi n-f)=\xi\xi nn+\xi(2\eta-g-\xi+\varepsilon)n+(\varepsilon-\xi)(2\eta-g).$$

Ope ergo harum senarum reductionum in § 18, 19, 24, 25, 27, 28 traditarum omnes huiusmodi aequationes infinitis modis per fractiones continuas resolvi poterunt.

RESOLUTIO AEQUATIONIS $xxyy=aann+2a\beta n+\gamma$ PER § 18.

29. Cum hic sit

$$\mathrm{X}=xx,\; \mathrm{Y}=yy,\; \lambda=0,\; \mu=0,\; \nu=0,\; \xi=\alpha,\; \eta=\beta \text{ et } \theta=\gamma,$$

prodibit haec substitutio

$$xx=\alpha n+\beta-\tfrac{1}{2}\alpha+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}'},$$

$$yy=\alpha n+\beta+\tfrac{1}{2}\alpha+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}'},$$

quae deducit ad hanc secundam aequationem

$$\left(\mathrm{X}'+\alpha n+\beta+\tfrac{1}{2}\alpha\right)\left(\mathrm{Y}'+\alpha n+\beta-\tfrac{1}{2}\alpha\right)=\alpha\alpha nn+2\alpha\beta n+\gamma.$$

30. Ad hanc simili modo resolvendam ob

$$\lambda=\alpha,\ \mu=\beta+\tfrac{1}{2}\alpha,\ \nu=\beta-\tfrac{1}{2}\alpha,\ \xi=\alpha,\ \eta=\beta,\ \theta=\gamma$$

consequemur hanc substitutionem

$$\mathrm{X}'=0\ +\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}''},\ \mathrm{Y}'=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}''}$$

quae deducit ad hanc tertiam aequationem

$$\left(\mathrm{X}''+\alpha n+\beta-\tfrac{1}{2}\alpha\right)\left(\mathrm{Y}''+\alpha n+\beta+\tfrac{1}{2}\alpha\right)=\alpha\alpha nn+2\alpha\beta n+\gamma.$$

Haec autem porro istas substitutiones praebet

$$\mathrm{X}''=0\ +\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{X}'''},\ \mathrm{Y}''=0+\frac{\beta\beta-\frac{1}{4}\alpha\alpha-\gamma}{\mathrm{Y}'''}$$

unde ob $\mathrm{X}'''=\mathrm{X}'$ et $\mathrm{Y}'''=\mathrm{Y}'$ nihil ultra concludi potest.

RESOLUTIO AEQUATIONIS $xxyy=\alpha\alpha nn+2\alpha\beta n+\gamma$ PER § 19.

31. Factis his substitutionibus

$$xx=\alpha n-\tfrac{1}{2}\alpha+\beta+\frac{\frac{1}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{X}},$$

$$yy=\alpha n+\tfrac{1}{2}\alpha+\beta+\frac{\frac{1}{4}\alpha\alpha-\beta\beta+\gamma}{\mathrm{Y}},$$

pervenitur ad hanc aequationem

$$\left(\mathrm{X}-\alpha n-\tfrac{1}{2}\alpha-\beta\right)\left(\mathrm{Y}-\alpha n+\tfrac{1}{2}\alpha-\beta\right)=\alpha\alpha nn+2\alpha\beta n+\gamma,$$

quae secundum § 19 reducta ob

$$\lambda=-\alpha,\ \mu=-\tfrac{1}{2}\alpha-\beta,\ \nu=\tfrac{1}{2}\alpha-\beta,\ \xi=\alpha,\ \eta=\beta,\ \theta=\gamma$$

dat has substitutiones

$$\mathrm{X} = 2\alpha n - \alpha + 2\beta + \frac{\frac{9}{4}\alpha\alpha - \beta\beta + \gamma}{\mathrm{X}'},$$

$$\mathrm{Y} = 2\alpha n + \alpha + 2\beta + \frac{\frac{9}{4}\alpha\alpha - \beta\beta + \gamma}{\mathrm{Y}'},$$

unde nascitur haec nova aequatio

$$\left(\mathrm{X}' - \alpha n - \tfrac{3}{2}\alpha - \beta\right)\left(\mathrm{Y}' - \alpha n + \tfrac{3}{2}\alpha - \beta\right) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

32. Haec aequatio ulterius reducatur et ob

$$\lambda = -\alpha,\ \mu = -\tfrac{3}{2}\alpha - \beta,\ \nu = \tfrac{3}{2}\alpha - \beta,\ \xi = \alpha,\ \eta = \beta,\ \theta = \gamma$$

habebimus has substitutiones

$$\mathrm{X}' = 2\alpha n - \alpha + 2\beta + \frac{\frac{25}{4}\alpha\alpha - \beta\beta + \gamma}{\mathrm{X}''},$$

$$\mathrm{Y}' = 2\alpha n + \alpha + 2\beta + \frac{\frac{25}{4}\alpha\alpha - \beta\beta + \gamma}{\mathrm{Y}''},$$

hincque hanc aequationem novam

$$\left(\mathrm{X}'' - \alpha n - \tfrac{5}{2}\alpha - \beta\right)\left(\mathrm{Y}'' - \alpha n + \tfrac{5}{2}\alpha - \beta\right) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

unde sequentes substitutiones facile colliguntur.

33. Quodsi ergo ad abbreviandum ponatur

$$\alpha n - \tfrac{1}{2}\alpha + \beta = \mathrm{N} \quad \text{et} \quad \beta\beta - \gamma = \mathrm{B},$$

valor ipsius xx sequenti fractione continua exprimetur

$$xx = \mathrm{N} + \cfrac{\frac{1}{4}\alpha\alpha - \mathrm{B}}{2\mathrm{N} + \cfrac{\frac{9}{4}\alpha\alpha - \mathrm{B}}{2\mathrm{N} + \cfrac{\frac{25}{4}\alpha\alpha - \mathrm{B}}{2\mathrm{N} + \cfrac{\frac{49}{4}\alpha\alpha - \mathrm{B}}{2\mathrm{N} + \cfrac{\frac{81}{4}\alpha\alpha - \mathrm{B}}{2\mathrm{N} + \text{etc.}}}}}}$$

qui convenit aequationi propositae

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma .$$

RESOLUTIO AEQUATIONIS $xxyy = aann + 2a\beta n + \gamma$ PER § 24.

34. Cum hic sit

$$\lambda = 0,\ \mu = 0,\ \nu = 0,\ \xi = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

erit

$$f = \beta + \sqrt{(\beta\beta - \gamma)},\ g = \alpha + \beta + \sqrt{(\beta\beta - \gamma)},$$

hinc

$$fg - \theta = \alpha\beta + 2\beta\beta - 2\gamma + (\alpha + 2\beta)\sqrt{(\beta\beta - \gamma)}$$

et

$$f + g - 2\eta = \alpha + 2\sqrt{(\beta\beta - \gamma)}.$$

Ponatur ergo

$$\frac{fg-\theta}{f+g-2\eta} = \beta + \sqrt{(\beta\beta - \gamma)} = \varepsilon,$$

ita ut sit

$$\varepsilon = f \text{ et } g = \alpha + f,$$

unde oriuntur hae substitutiones

$$xx = \alpha n + f + \frac{(f+g-2\eta)(\alpha n+f)}{\mathrm{X}} = (\alpha n + f)\left(1 + \frac{\alpha+2\sqrt{(\beta\beta-\gamma)}}{\mathrm{X}}\right),$$

$$yy = \alpha n + g + \frac{(f+g-2\eta)(\alpha n+g)}{\mathrm{Y}} = (\alpha n + g)\left(1 + \frac{\alpha+2\sqrt{(\beta\beta-\gamma)}}{\mathrm{Y}}\right),$$

indeque haec aequatio nova

$$(\mathrm{X} + \alpha n + g)(\mathrm{Y} + \alpha n + \alpha + f) = \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + f)(2\beta - f).$$

35. Ponamus

$$\beta + \sqrt{(\beta\beta-\gamma)} = \delta$$

ut sit

$$\lambda = \delta,\ g = \alpha + \delta \text{ et } f + g - 2\eta = \alpha - 2\beta + 2\delta,$$

sicque substitutiones

$$xx = \alpha n + \delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{\mathrm{X}},$$

$$yy = \alpha(n+1) + \delta + \frac{(\alpha-2\beta+2\delta)(\alpha(n+1)+\delta)}{\mathrm{Y}},$$

dabunt hanc aequationem

$$(\mathrm{X}+\alpha n+\alpha+\delta)(\mathrm{Y}+\alpha n+\alpha+\delta)=\alpha\alpha nn+\alpha(\alpha+2\beta)n+(\alpha+\delta)(2\beta-\delta).$$

36. Pro huius aequationis reductione est

$$\lambda=\alpha,\ \mu=\alpha+\delta,\ \nu=\alpha+\delta,\ \xi=\alpha,\ \eta=\beta+\tfrac{1}{2}\alpha,\ \theta=(\alpha+\delta)(2\beta-\delta),$$

unde ob

$$ff-(2\beta+\alpha)f+(\alpha+\delta)(2\beta-\delta)=0$$

erit vel

$$f=\alpha+\delta$$

vel

$$f=2\beta-\delta$$

at prior positio non ulterius deducit, unde posteriori utendo erit

$$f=2\beta-\delta,\ \ g=2\beta-\delta\ \text{ et }\ \varepsilon=2\beta-\delta,$$

sicque hae obtinentur substitutiones

$$\mathrm{X}=2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta-\delta)}{\mathrm{X}'},$$

$$\mathrm{Y}=2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha(n+1)+2\beta-\delta)}{\mathrm{Y}'},$$

quae hanc praebent aequationem

$$(\mathrm{X}'+\alpha n+2\beta-\delta)(\mathrm{Y}'+\alpha n+\alpha+2\beta-\delta)$$
$$=aann+2\alpha(\alpha+\beta)n+(\alpha+\delta)(\alpha+2\beta-\delta).$$

37. Nunc igitur est

$$\lambda=\alpha,\ \mu=2\beta-\delta,\ \nu=\alpha+2\beta-\delta,\ \xi=\alpha,\ \eta=\alpha+\beta,\ \theta=(\alpha+\delta)(\alpha+2\beta-\delta),$$

unde ob

$$ff-2(\alpha+\beta)f+(\alpha+\delta)(\alpha+2\beta-\delta)=0$$

sumatur valor

$$f=\alpha+2\beta-\delta;$$

erit

$$g=2\alpha+2\beta-\delta$$

atque

$$\varepsilon=\frac{(\alpha+2\beta-\delta)(\alpha+2\beta-2\delta)}{\alpha+2\beta-2\delta}=\alpha+2\beta-\delta.$$

Quare haec substitutio

$$\mathrm{X}'=\alpha+\frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\mathrm{X}''},$$

$$\mathrm{Y}'=\alpha+\frac{(\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{\mathrm{Y}''}$$

dabit hanc aequationem

$$\begin{aligned}&(\mathrm{X}''+\alpha n+2\alpha+2\beta-\delta)(\mathrm{Y}''+\alpha n+2\alpha+2\beta-\delta)\\&=aann+\alpha(3\alpha+2\beta)n+(2\alpha+2\beta-\delta)(\alpha+\delta).\end{aligned}$$

38. Si utamur altero valore

$$f=\alpha+\delta$$

fit

$$g=2\alpha+\delta \text{ et } \varepsilon=\alpha+\delta$$

et facta substitutione

$$\mathrm{X}'=\alpha-2\beta+2\delta+\frac{(\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{\mathrm{X}''},$$

$$\mathrm{Y}'=\alpha-2\beta+2\delta+\frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\mathrm{Y}''}$$

nanciscimur hanc aequationem

$$\begin{aligned}&(\mathrm{X}''+\alpha n+2\alpha+\delta)(\mathrm{Y}''+\alpha n+2\alpha+\delta)\\&=\alpha\alpha nn+\alpha(3\alpha+2\beta)n+(2\alpha+\delta)(\alpha+2\beta-\delta).\end{aligned}$$

39. Prosequamur hanc posteriorem aequationem, quia magis similis est secundae, cum ex ea nascatur ponendo $\delta+\alpha$ pro δ et $\beta+\alpha$ pro β, unde prodit haec substitutio

$$\mathrm{X}''=2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+\alpha-\delta)}{\mathrm{X}'''},$$

$$\mathrm{Y}''=2\beta-\alpha-2\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{\mathrm{Y}'''},$$

quae ducit ad hanc aequationem

$$\begin{aligned}&(\mathrm{X}'''+\alpha n+2\beta+\alpha-\delta)(\mathrm{Y}'''+\alpha n+2\beta+\alpha-\delta)\\&=\alpha\alpha nn+2\alpha(2\alpha+\beta)n+(2\alpha+\delta)(2\alpha+2\beta-\delta).\end{aligned}$$

40. Haec aequatio porro uti in § 38 tractata ope harum substitutionum

$$\mathrm{X}''' = \alpha - 2\beta + 2\delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{\mathrm{X}''''},$$

$$\mathrm{Y}''' = \alpha - 2\beta + 2\delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{\mathrm{Y}''''}$$

reducitur ad hanc

$$\left(\mathrm{X}'''' + \alpha n + 3\alpha + \delta\right)\left(\mathrm{Y}'''' + \alpha n + 3\alpha + \delta\right)$$
$$= \alpha\alpha nn + \alpha\left(5\alpha + 2\beta\right)n + \left(3\alpha + \delta\right)\left(2\alpha + 2\beta - \delta\right).$$

haecque ulterius per has substitutiones

$$\mathrm{X}'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + 2\beta + 2\alpha - \delta)}{\mathrm{X}'''''},$$

$$\mathrm{Y}'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + 2\beta + 3\alpha - \delta)}{\mathrm{Y}'''''}$$

ad istam reducitur

$$\left(\mathrm{X}''''' + \alpha n + 2\beta + 2\alpha - \delta\right)\left(\mathrm{Y}''''' + \alpha n + 3\alpha + 2\beta - \delta\right)$$
$$= \alpha\alpha nn + 2\alpha\left(3\alpha + \beta\right)n + \left(3\alpha + \delta\right)\left(3\alpha + 2\beta - \delta\right).$$

41. Hinc ergo valor ipsius *xx* ex hac aequatione

$$xxyy = aann + 2a\beta n + \gamma$$

posito brevitatis causa

$$\beta + \sqrt{(\beta\beta - \gamma)} = \delta \quad \text{et} \quad \alpha - 2\beta + 2\delta = \mathrm{A}$$

erit

$$xx = an + \delta + \cfrac{\mathrm{A}(\alpha n + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + \alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + \alpha + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + 2\alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 2\alpha + 2\beta - \delta)}{\mathrm{A} + \cfrac{\mathrm{A}(\alpha n + 3\alpha + \delta)}{-\mathrm{A} - \cfrac{\mathrm{A}(\alpha n + 3\alpha + 2\beta - \delta)}{\mathrm{A} + \text{etc.}}}}}}}}}$$

Haec autem expressio evoluta praebet pro *xx* ipsum illud productum ex infinitis factoribus constans, quod per methodum priorem elicitur.

42. Ista fractio continua simplicius hoc modo exprimi potest

$$xx = an + \delta - \cfrac{(\alpha n+\delta)}{1+\cfrac{(\alpha n+2\beta-\delta)}{\mathrm{A}-\cfrac{(\alpha n+\alpha+\delta)}{1+\cfrac{(\alpha n+\alpha+2\beta-\delta)}{\mathrm{A}-\cfrac{(\alpha n+2\alpha+\delta)}{1+\cfrac{(\alpha n+2\alpha+2\beta-\delta)}{\mathrm{A}-\cfrac{(\alpha n+3\alpha+\delta)}{1+\cfrac{(\alpha n+3\alpha+2\beta-\delta)}{\mathrm{A}-\cfrac{(\alpha n+4\alpha+\delta)}{1+\text{etc.}}}}}}}}}}$$

Sin autem formulae § 37 hoc modo ulterius reducentur, invenitur haec expressio ab initio irregularis

$$xx=an+\delta-\cfrac{\alpha n+\delta}{1+\cfrac{\alpha n+2\beta-\delta}{\alpha-\cfrac{\alpha n+\alpha+2\beta-\delta}{1+\cfrac{\alpha n+\alpha+\delta}{\mathrm{A}-\cfrac{\alpha n+2\alpha+2\beta-\delta}{1+\cfrac{\alpha n+2\alpha+\delta}{\mathrm{A}-\cfrac{\alpha n+3\alpha+2\beta-\delta}{1+\cfrac{\alpha n+2\alpha+\delta}{\mathrm{A}-\cfrac{\alpha n+3\alpha+2\beta-\delta}{1+\cfrac{\alpha n+3\alpha+\delta}{\mathrm{A}-\text{etc.}}}}}}}}}}}$$

43. Si utraque expressio capite communi truncetur, pro 2β valor assumtus $\alpha+2\delta-\mathrm{A}$ substituatur insuperque pro $an+\alpha+\delta$ scribatur N, habebitur haec aequalitas

$$\mathrm{A}-\cfrac{\mathrm{N}}{1+\cfrac{\mathrm{N}+\alpha-\mathrm{A}}{\mathrm{A}-\cfrac{\mathrm{N}+\alpha}{1+\cfrac{\mathrm{N}+2\alpha-\mathrm{A}}{\mathrm{A}-\cfrac{\mathrm{N}+2\alpha}{1+\cfrac{\mathrm{N}+3\alpha-\mathrm{A}}{\mathrm{A}-\text{etc.}}}}}}} \qquad =\alpha-\cfrac{\mathrm{N}+\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}}{\mathrm{A}-\cfrac{\mathrm{N}+2\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}+\alpha}{\mathrm{A}-\cfrac{\mathrm{N}+3\alpha-\mathrm{A}}{1+\cfrac{\mathrm{N}+2\alpha}{\mathrm{A}-\text{etc.}}}}}}}$$

ubi pro A, α et N numeri quicunque assumi possunt.

RESOLUTIO AEQUATIONIS $xxyy=\alpha\alpha nn+2\alpha\beta n+\gamma$ OPE § 25

44. Prima substitutio ex resolutione praecedente sumendis X et Y negativis petita [§ 35]

$$xx=\alpha n+\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+\delta)}{\mathrm{X}},$$

$$yy=\alpha(n+1)+\delta+\frac{(2\beta-\alpha-2\delta)(\alpha n+\alpha+\delta)}{\mathrm{Y}},$$

posito

$$\delta=\beta+\sqrt{(\beta\beta-\gamma)}$$

deducit ad hanc aequationem

$$\left(\mathrm{X}-\alpha n-\alpha-\delta\right)\left(\mathrm{Y}-\alpha n-\alpha-\delta\right)$$
$$=\alpha\alpha nn+\alpha\left(\alpha+2\beta\right)n+\left(\alpha+\delta\right)\left(2\beta-\delta\right),$$

quae cum § 24 comparata praebet

$$\lambda=-\alpha,\quad \mu=-\alpha-\delta,\quad \nu=-\alpha-\delta,$$
$$\xi=\alpha,\quad \eta=\tfrac{1}{2}\alpha+\beta,\quad \theta=\left(\alpha+\delta\right)\left(2\beta-\delta\right)$$

unde colligitur

$$ff-\left(\alpha+2\beta\right)f+\left(\alpha+\delta\right)\left(2\beta-\delta\right)=0.$$

Sit

$$f=\alpha+\delta;$$

erit

$$g=3\alpha+\delta \text{ et } \varepsilon=\alpha+\delta$$

hincque nascitur haec substitutio

$$\mathrm{X}=2\alpha n+2\alpha+2\delta-\frac{\left(3\alpha-2\beta+2\delta\right)\left(\alpha n+\alpha+\delta\right)}{\mathrm{X'}},$$

$$\mathrm{Y}=2\alpha n+4\alpha+2\delta-\frac{\left(3\alpha-2\beta+2\delta\right)\left(\alpha n+2\alpha+\delta\right)}{\mathrm{Y'}},$$

quae ducit ad sequentem aequationem

$$\left(\mathrm{X'}-\alpha n-3\alpha-\delta\right)\left(\mathrm{Y'}-\alpha n-2\alpha-\delta\right)$$
$$=\alpha\alpha nn+\alpha\left(2\alpha+2\beta\right)n+\left(2\alpha+\delta\right)\left(2\beta-\delta\right).$$

45. Tractetur haec aequatio simili modo secundum § 24 et ob valores

$$\lambda=-\alpha,\ \mu=-3\alpha-\delta,\ \nu=-2\alpha-\delta,$$
$$\zeta=\alpha,\ \eta=\alpha+\beta,\quad \theta=\left(2\alpha+\delta\right)\left(2\beta-\delta\right)$$

erit

$$ff-\left(2\alpha+2\beta\right)f+\left(2\alpha+\delta\right)\left(2\beta-\delta\right)=0,$$

unde sumatur

$$f=2\alpha+\delta,$$

fietque

$$g=5\alpha+\delta \text{ et } \varepsilon=2\alpha+\delta.$$

Habebitur ergo ista substitutio

$$\mathrm{X}' = 2\alpha n + 5\alpha + 2\delta - \frac{(5\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\mathrm{X}''},$$

$$\mathrm{Y}' = 2\alpha n + 7\alpha + 2\delta - \frac{(5\alpha-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{\mathrm{Y}''},$$

quae praebet hanc aequationem

$$\begin{aligned}&(\mathrm{X}''-\alpha n-5\alpha-\delta)(\mathrm{Y}''-\alpha n-3\alpha-\delta)\\&=\alpha\alpha nn+\alpha(3\alpha+2\beta)n+(3\alpha+\delta)(2\beta-\delta).\end{aligned}$$

46. Nunc igitur eodem modo erit

$$\begin{aligned}&\lambda=-\ \alpha,\ \mu=-5\alpha-\delta,\ \nu=-3\alpha-\delta,\\&\zeta=\alpha,\ \eta=\tfrac{3}{2}\alpha+\beta,\ \ \theta=(3\alpha+\delta)(2\beta-\delta),\end{aligned}$$

unde ob

$$f=3\alpha+\delta$$

colligitur

$$g=7\alpha+\delta \text{ et } \varepsilon=3\alpha+\delta .$$

Substitutio ergo

$$\mathrm{X}'' = 2\alpha n + 8\alpha + 2\delta - \frac{(7\alpha-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{\mathrm{X}'''},$$

$$\mathrm{Y}' = 2\alpha n + 10\alpha + 2\delta - \frac{(7\alpha-2\beta+2\delta)(\alpha n+4\alpha+\delta)}{\mathrm{Y}'''},$$

istam dabit aequationem

$$\begin{aligned}&(\mathrm{X}'''-\alpha n-7\alpha-\delta)(Y'''-\alpha n-4\alpha-\delta)\\&=\alpha\alpha nn+\alpha(4\alpha+2\beta)+(4\alpha+\delta)(2\beta-\delta).\end{aligned}$$

47. Cum lex progressionis hic sit satis manifesta, facile concluditur fore

$$xx = an + \delta - \cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+2\alpha+2\delta-\cfrac{(3\alpha n-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+5\alpha+2\delta-\cfrac{(5\alpha n-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{2\alpha n+8\alpha+2\delta-\cfrac{(7\alpha n-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{2\alpha n+11\alpha+2\delta-\text{etc.}}}}}$$

ubi notandum est ex aequatione proposita

$$xxyy=\alpha\alpha nn+2\alpha\beta n+\gamma$$

duplici modo dari δ, cum sit

$$\delta = \beta \pm \sqrt{(\beta\beta - \gamma)} ,$$

sicque binae eiusmodi series obtinentur, quarum altera proditura fuisset, si ubique pro f alteros valores assumsissemus.

ALIA RESOLUTIO BlNOS VALORES IPSIUS f ALTERNANDO

48. Sumamus in resolutione § 44

$$f = 2\beta - \alpha ,$$

ut sit

$$g = 2\alpha + 2\beta - \delta \text{ et } \varepsilon = 2\beta - \delta ;$$

erit substitutio

$$\mathrm{X} = 2\alpha n + \alpha + 2\beta - \frac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{\mathrm{X'}},$$

$$\mathrm{Y} = 2\alpha n + 3\alpha + 2\beta - \frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\mathrm{Y'}},$$

unde resultat haec aequatio

$$\begin{aligned}&(\mathrm{X'} - \alpha n - 2\alpha - 2\beta + \delta)(\mathrm{Y'} - \alpha n - \alpha - 2\beta + \delta)\\ &= \alpha\alpha nn + \alpha(2\alpha + 2\beta)n + (\alpha + \delta)(\alpha + 2\beta - \delta),\end{aligned}$$

quae ex superiori oritur, si ibi pro δ scribatur $-\alpha + 2\beta - \delta$, quo valore in sequentibus retento fiet

$$xx = an + \delta - \cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+4\beta-2\delta-\cfrac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{2\alpha n+6\alpha+4\beta-2\delta-\cfrac{(5\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{2\alpha n+9\alpha+4\beta-2\delta-\text{etc.}}}}}$$

49. At aequatio modo eruta cum § 25 collata dat

$$\begin{aligned}&\lambda = -\alpha,\ \mu = -2\alpha - 2\beta + \delta,\ \nu = -\alpha - 2\beta + \delta,\\ &\xi = \alpha,\ \eta = \alpha + \beta,\ \ \theta = (\alpha + \delta)(\alpha + 2\beta - \delta),\end{aligned}$$

et

$$ff - 2(a + \beta)f + (\alpha + \delta)(\alpha + 2\beta - \delta) = 0..$$

Si hic sumeremus

$$f = \alpha + 2\beta - \delta,$$

haberemus formuIam modo inventam. Sit ergo

$$f=\alpha+\delta\,;$$

erit

$$g=4\alpha+\delta \text{ et } \varepsilon=\alpha+\delta,$$

ideoque

$$\text{X}'=2\alpha n+3\alpha+2\beta-\frac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{\text{X}''},$$

$$\text{Y}'=2\alpha n+5\alpha+2\beta-\frac{(3\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{\text{Y}''},$$

hincque nascitur aequatio

$$\begin{aligned}&(\text{X}'''-\alpha n-5\alpha-2\beta+\delta)(\text{Y}'''-\alpha n-2\alpha-2\beta+\delta)\\&=\alpha\alpha nn+\alpha(4\alpha+2\beta)n+(2\alpha+\delta)(2\alpha+2\beta-\delta).\end{aligned}$$

quae ex praecedente [§ 45] oritur, si ibi pro δ scribatur $-\alpha+\delta$*;* sicque erit

$$xx=an+\delta-\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+6\alpha+2\delta-\text{etc.}}}}}$$

50. Verum illa aequatio altero modo resoluta ob valores

$$\begin{aligned}&\lambda=-\alpha,\ \mu=-4\alpha-\delta,\ \nu=-2\alpha-\delta,\\&\xi=\alpha,\ \eta=\tfrac{3}{2}\alpha+\beta,\ \ \theta=(2\alpha+\delta)(\alpha+2\beta-\delta),\end{aligned}$$

dat

$$ff-(3\alpha+2\beta)f+(2\alpha+\delta)(\alpha+2\beta-\delta)=0\,;$$

unde nunc sumamus

$$f=\alpha+2\beta-\delta\,,$$

ut sit

$$g=5\alpha+2\beta-\delta \text{ et } \varepsilon=\alpha+2\beta-\delta\,,$$

prodibitque haec substitutio

$$\text{X}''=2\alpha n+5\alpha+2\beta-\frac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{\text{X}'''},$$

$$\text{Y}''=2\alpha n+7\alpha+2\beta-\frac{(3\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{\text{Y}'''},$$

quae ducit ad hanc aequationem

$$\left(\mathrm{X}''' - \alpha n - 5\alpha - 2\beta + \delta\right)\left(\mathrm{Y}'' - \alpha n - 2\alpha - 2\beta + \delta\right)$$
$$= \alpha\alpha nn + \alpha\left(4\alpha + 2\beta\right)n + \left(2\alpha + \delta\right)\left(2\alpha + 2\beta - \delta\right).$$

51. Cum lex progressionis hinc iam colligi possit, habebimus

$$xx = an + \delta - \cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+2\beta-\cfrac{(\alpha+2\beta-2\delta)(\alpha n+2\beta-\delta)}{2\alpha n+3\alpha+2\beta-\cfrac{(3\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{2\alpha n+5\alpha+2\beta-\cfrac{(3\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{2\alpha n+7\alpha+2\beta-\text{etc.}}}}}$$

quae fractio continua ob satis concinnam progressionis legem est notatu digna.

RESOLUTIO AEQUATIONIS $xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$ PER § 28

52. Posito

$$\delta = \beta + \sqrt{\left(\beta\beta - \gamma\right)}$$

ob

$$\lambda = 0,\quad \mu = 0,\quad \nu = 0,\quad \xi = \alpha,\quad \eta = \beta,\quad \theta = \gamma$$

erit

$$g = \delta,\ f = -\alpha + \delta \ \text{ et } \ \varepsilon = \delta,$$

unde substitutio

$$xx = \alpha n - \alpha + \delta + \frac{(-\alpha-2\beta+2\delta)(\alpha n-\alpha+\delta)}{\mathrm{X}},$$

$$yy = \alpha n + \delta + \frac{(-\alpha-2\beta+2\delta)(\alpha n+\delta)}{\mathrm{Y}}$$

dabit hanc aequationem ex § 27

$$\left(\mathrm{X} + \alpha n - \alpha + \delta\right)\left(\mathrm{Y} + \alpha n + \delta\right)$$
$$= \alpha\alpha nn + \alpha\left(2\beta - \alpha\right)n + \left(\delta - \alpha\right)\left(2\beta - \delta\right).$$

Sumtis autem X et Y negativis, ut sit ex § 28

$$xx=\alpha n-\alpha+\delta+\frac{(\alpha+2\beta-2\delta)(\alpha n-\alpha+\delta)}{\mathrm{X}},$$

$$yy=\alpha n+\delta+\frac{(\alpha+2\beta-2\delta)(\alpha n+\delta)}{\mathrm{Y}},$$

habebitur

$$\begin{gathered}(\mathrm{X}+\alpha n+\alpha-\delta)(\mathrm{Y}-\alpha n-\delta)\\ =\alpha\alpha nn-\alpha(2\beta-\alpha)n-(\delta-\alpha)(2\beta-\delta).\end{gathered}$$

53. Haec aequatio porro secundum easdem formulas tractata praebet

$$\begin{gathered}\lambda=-\alpha,\ \mu=\alpha-\delta,\ \nu=-\delta,\\ \xi=\alpha,\ \eta=-\tfrac{1}{2}\alpha+\beta,\ \ \theta=(\delta-\alpha)(2\beta-\delta),\end{gathered}$$

unde fit

$$gg-(2\beta-\alpha)g+(\delta-\alpha)(2\beta-\delta)=0,$$

ergo vel

$$g=\delta-\alpha \text{ vel } g=2\beta-\delta$$

et

$$f=-a+g \ \text{ atque } \ e=g.$$

Quare substitutio erit

$$\mathrm{X}=2\alpha n-2\alpha+g+\delta+\frac{(2\beta-2g)(\alpha n-\alpha+g)}{\mathrm{X}'},$$

$$\mathrm{Y}=2\alpha n+g+\delta+\frac{(2\beta-2g)(\alpha n+g)}{\mathrm{Y}'},$$

quae ducit ad hanc aequationem

$$\begin{gathered}(\mathrm{X}'-\alpha n+\alpha-g)(\mathrm{Y}'-\alpha n+\alpha-g)\\ =\alpha\alpha nn+\alpha(2\beta-2\alpha)n+(g-\alpha)(2\beta-\alpha-g).\end{gathered}$$

54. Retineamus hanc litteram g geminum valorem involventem et sequentes per g', g'' indicemus. Cum ergo hic sit

$$\begin{gathered}\lambda=-\alpha,\ \mu=\alpha-g,\ \nu=\alpha-g,\\ \zeta=\alpha,\ \eta=-\alpha+\beta,\ \ \theta=(g-\alpha)(2\beta-\alpha-g),\end{gathered}$$

erit vel

$$g'=g-\alpha \ \text{ vel } \ g'=2\beta-\alpha-g$$

hincque

$$f=-2\alpha+g' \ \text{ et } \ \varepsilon=g'$$

ideoque

$$X' = 2\alpha n - 3\alpha + g + g' + \frac{(2\beta - 2g')(\alpha n - \alpha + g')}{X''},$$

$$Y' = 2\alpha n - \alpha + g + g' + \frac{(2\beta - 2g')(\alpha n + g')}{Y''},$$

unde prodit haec aequatio

$$\begin{aligned}&(X'' - \alpha n + \alpha - g')(Y'' - \alpha n + 2\alpha - g')\\ &= \alpha\alpha nn + \alpha(2\beta - 3\alpha)n + (g' - \alpha)(2\beta - \alpha - g').\end{aligned}$$

55. Nunc igitur porro erit

$$\begin{aligned}&\lambda = -\alpha,\ \mu = \alpha - g',\ \nu = 2\alpha - g',\\ &\zeta = \alpha,\ \eta = \beta - \tfrac{3}{2}\alpha,\ \ \theta = (g' - \alpha)(2\beta - \alpha - g')\end{aligned}$$

hincque vel

$$g'' = g' - \alpha \ \text{ vel } \ g'' = 2\beta - 2\alpha - g'$$

et

$$f = -3\alpha + g'' \ \text{ atque } \ \varepsilon = g''.$$

Quare substitutio

$$X'' = 2\alpha n - 4\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n - \alpha + g'')}{X'''},$$

$$Y'' = 2\alpha n - 2\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n + g'')}{Y'''}$$

dabit hanc aequationem

$$\begin{aligned}&(X''' - \alpha n + \alpha - g'')\ (Y''' - \alpha n + 3\alpha - g'')\\ &= \alpha\alpha nn + \alpha(2\beta - 4\alpha)n + (g'' - \alpha)(2\beta - 3\alpha - g'').\end{aligned}$$

56. Iam pro huius aequationis resolutione est

$$\begin{aligned}&\lambda = -\alpha,\ \mu = \alpha - g'',\ \nu = 3\alpha - g'',\\ &\xi = \alpha,\ \eta = \beta - 2\alpha,\ \ \theta = (g'' - \alpha)(2\beta - 3\alpha - g''),\end{aligned}$$

unde vel

$$\begin{aligned}&g''' = g'' - \alpha \ \text{ vel } \ g''' = 2\beta - 3\alpha - g'',\\ &f = -4\alpha + g''' \ \text{ atque } \ \varepsilon = g'''.\end{aligned}$$

Quare ex substitutione

$$X''' = 2\alpha n - 5\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n - \alpha + g''')}{X''''},$$

$$Y'' = 2\alpha n - 3\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n + g''')}{Y''''}$$

orietur haec aequatio

$$\begin{aligned}&(X'''' - \alpha n + \alpha - g''')(Y'''' - \alpha n + 4\alpha - g''')\\ &= \alpha\alpha nn + \alpha(2\beta - 5\alpha)n + (g''' - \alpha)(3\beta - 4\alpha - g''').\end{aligned}$$

57. His igitur colligendis ex aequatione proposita

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

posito

$$\delta = \beta + \sqrt{(\beta\beta - \gamma)},$$

si pro litteris g, g', g'', g''' etc. sequentes valores, gemini assumantur

$$g = \begin{Bmatrix}\delta - \alpha\\ 2\beta - \delta\end{Bmatrix},\ g' = \begin{Bmatrix}g - \alpha\\ 2\beta - \alpha - g\end{Bmatrix},\ g'' = \begin{Bmatrix}g' - \alpha\\ 2\beta - 2\alpha - g'\end{Bmatrix},\ g''' = \begin{Bmatrix}g'' - \alpha\\ 2\beta - 3\alpha - g''\end{Bmatrix}\text{etc.,}$$

colligetur pro xx sequens valor

$$xx = \alpha n - \alpha + \delta + \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{2\alpha n - 2\alpha + \delta + g + \cfrac{(2\beta - 2g)(\alpha n - \alpha + g)}{2\alpha n - 3\alpha + g + g' + \cfrac{(2\beta - 2g')(\alpha n - \alpha + g')}{2\alpha n - 4\alpha + g' + g'' + \text{etc.}}}}$$

qui ergo ob geminos valores singularum $\delta,\ g,\ g',\ g'',\ g'''$ etc. in infinitum variari potest.

58. Si eadem aequatio simili modo retentis valoribus omnibus ambiguis secundum § 25 resolvatur ac sumto

$$\delta = \beta \pm \sqrt{(\beta\beta - \gamma)},$$

ponatur

$$f = \begin{Bmatrix}\alpha + \delta\\ 2\beta - \delta\end{Bmatrix},\ f' = \begin{Bmatrix}\alpha + f\\ \alpha + 2\beta - f\end{Bmatrix},\ f'' = \begin{Bmatrix}\alpha + f'\\ 2\alpha + 2\beta - f'\end{Bmatrix},\ f''' = \begin{Bmatrix}\alpha + f''\\ 3\alpha + 2\beta - f''\end{Bmatrix}\text{etc.,}$$

erit

$$xx=\alpha n+\delta-\cfrac{(\alpha-2\beta+2\delta)(\alpha n+\delta)}{2\alpha n+\alpha+\delta+f-\cfrac{(\alpha-2\beta+2f)(\alpha n+f)}{2\alpha n+2\alpha+f+f'+\cfrac{(\alpha-2\beta+2f')(\alpha n+f')}{2\alpha n+3\alpha+f'+f''-\cfrac{(\alpha-2\beta+2f'')(\alpha n+f'')}{2\alpha n+4\alpha+f'+f''-\text{etc.}}}}}.$$

59. Simili modo aequationem propositam secundum § 24 tractando si posito

$$\delta=\beta\pm\sqrt{(\beta\beta-\gamma)},$$

statuatur ut ante

$$f=\begin{Bmatrix}\alpha+\delta\\2\beta-\delta\end{Bmatrix},\ f'=\begin{Bmatrix}\alpha+f\\\alpha+2\beta-f\end{Bmatrix},\ f''=\begin{Bmatrix}\alpha+f'\\2\alpha+2\beta-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}\alpha+f''\\3\alpha+2\beta-f''\end{Bmatrix}\text{etc.},$$

erit

$$xx=\alpha n+\delta-\cfrac{(2\beta-\alpha-2\delta)(\alpha n+\delta)}{-\alpha-\delta+f-\cfrac{(2\beta+\alpha-2f)(\alpha n+f)}{-f+f'-\cfrac{(2\beta+\alpha-2f')(\alpha n+f')}{-\alpha-f'+f''-\cfrac{(2\beta+3\alpha-2f'')(\alpha n+f'')}{-f'+f''-\cfrac{(2\beta+3\alpha-2f''')(\alpha n+f''')}{-\alpha-f'''+f'''-\cfrac{(2\beta+5\alpha-2f'''')(\alpha n+f'''')}{-f''''+f''''-\text{etc.}}}}}}}$$

60. Porro ex § 27 si post

$$\delta=\beta\pm\sqrt{(\beta\beta-\gamma)},$$

statuatur

$$g=\begin{Bmatrix}\delta-\alpha\\2\beta-\delta\end{Bmatrix},\ g'=\begin{Bmatrix}g-\alpha\\2\beta-\alpha-g\end{Bmatrix},\ g''=\begin{Bmatrix}g'-\alpha\\2\beta-2\alpha-g'\end{Bmatrix},\ g'''=\begin{Bmatrix}g''-\alpha\\2\beta-3\alpha-g''\end{Bmatrix}\text{etc.},$$

erit

$$xx=\alpha n-\alpha+\delta-\cfrac{(2\beta-2\delta+\alpha)(\alpha n+\alpha+\delta)}{-g-\delta-\cfrac{(2\beta-2g)(\alpha n-\alpha+g)}{g'-g+\alpha-\cfrac{(2\beta-2g'-2\alpha)(\alpha n-\alpha+g')}{g''-g'-\cfrac{(2\beta-2g''-2\alpha)(\alpha n-\alpha+g'')}{g'''-g''+\alpha-\cfrac{(2\beta-2g'''-4\alpha)(\alpha n-\alpha+g''')}{g''''+g'''-\cfrac{(2\beta-2g''''-4\alpha)(\alpha n-\alpha+g'''')}{g'''''-g''''+\alpha-\text{etc.}}}}}}}$$

61. Possent autem permiscendis his reductionibus innumerabiles aliae fractiones continuae elici, quae omnes valorem ipsius *xx* exprimerent; verum his quaternis formis generalibus, quibus prima § 33 exhibita addi potest, acquiescamus easque ad casum quempiam determinatum accommodemus. Sit scilicet $xxyy=nn$ seu quaeratur eiusmodi series

$$a,\ b,\ c,\ d,\ e,\ f\ \text{etc.},$$

ut sit

$$ab=1,\ bc=2,\ cd=3,\ de=4,\ ef=5\ \text{etc.},\ \ldots .\ xy=n,$$

atque iam notavimus (§ 12) fore

$$aa=\tfrac{2}{\pi},\ \ bb=\tfrac{\pi}{2},\ \ cc=\tfrac{2}{1}\cdot\tfrac{2}{\pi},\ \ dd=\tfrac{3}{2}\cdot\tfrac{\pi}{2},\ \ ee=\tfrac{2\cdot 4}{1\cdot 3}\cdot\tfrac{2}{\pi}\ \ \text{etc.}$$

Deinde vero ex § 6 colligitur

$$xx=n\cdot\frac{\int z^{n}dz:\sqrt{(1-zz)}}{\int z^{n-1}dz:\sqrt{(1-zz)}}$$

seu per productum infinitum

$$xx=n\cdot\frac{n(n+2)}{(n+1)^{2}}\cdot\frac{(n+2)(n+4)}{(n+3)^{2}}\cdot\frac{(n+4)(n+6)}{(n+5)^{2}}\cdot\frac{(n+6)(n+8)}{(n+7)^{2}}\cdot\text{etc.}$$

Hunc igitur valorem ipsius *xx,* quemadmodum per fractiones continuas exprimi possit, videamus.

62. Cum igitur pro aequatione

$$xxyy=nn$$

sit $\alpha=1,\ \beta=0,\ \gamma=0$, erit secundum § 33

$$\mathrm{N}=n-\tfrac{1}{2}\ \ \text{et}\ \ \mathrm{B}=0,$$

unde fit

$$xx=n-\frac{1}{2}+\cfrac{1:4}{2n-1+\cfrac{9:4}{2n-1+\cfrac{25:4}{2n-1+\cfrac{49:4}{2n-1+\cfrac{1}{1+\text{etc.}}}}}}$$

sive

$$2xx=2n-1+\cfrac{1}{2(2n-1)+\cfrac{9}{2(2n-1)+\cfrac{25}{2(2n-1)+\cfrac{49}{2(2n-1)+\cfrac{81}{2(2n-1)+\text{etc.}}}}}}$$

63. Porro ex §59 ob

$$\beta=0,\ \ \gamma=0,\ \text{et}\ \ \delta=0,$$

si sumatur

$$f=\begin{Bmatrix}1\\0\end{Bmatrix},\ f'=\begin{Bmatrix}1+f\\1-f\end{Bmatrix},\ f''=\begin{Bmatrix}1+f'\\2-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}1+f''\\3-f''\end{Bmatrix},\ f''''=\begin{Bmatrix}1+f'''\\4-f'''\end{Bmatrix},$$

erit

$$xx=n+\cfrac{n}{f-1-\cfrac{(1-2f)(n+f)}{f'-f-\cfrac{(1-2f')(n+f')}{f''-f'-1-\cfrac{(3-2f'')(n+f'')}{f'''-f''+\cfrac{(3-2f''')(n+f''')}{f''''-f'''-1-\cfrac{(5-2f'''')(n+f'''')}{f'''''-f''''-\cfrac{(5-2f''''')(n+f''''')}{f''''''-f'''''-\text{etc.}}}}}}}}$$

vel ex § 58 sub iisdem denominationibus

$$xx=n-\cfrac{n}{2n+1+f-\cfrac{(1+2f)(n+f)}{2n+2+f+f'-\cfrac{(1+2f')(n+f')}{2n+3+f'+f''-\cfrac{(1+2f'')(n+f'')}{2n+4+f''+f'''-\cfrac{(1+2f''')(n+f''')}{2n+5+f'''+f''''-\text{etc.}}}}}}$$

64. Deinde posito

$$g=\begin{Bmatrix}-1\\0\end{Bmatrix},\ g'=\begin{Bmatrix}g-1\\-1-g\end{Bmatrix},\ g''=\begin{Bmatrix}g'-1\\-2-g'\end{Bmatrix},\ g'''=\begin{Bmatrix}g''-1\\-3-g''\end{Bmatrix}\ \text{etc.},$$

erit ex § 60

$$xx=n-1-\cfrac{n-1}{g+\cfrac{2g(n-1+g)}{g'-g+1+\cfrac{2(1+g')(n-1+g')}{g''-g'+1+\cfrac{2(1+g'')(n-1+g'')}{g'''-g''+1+\cfrac{2(1+g''')(n-1+g''')}{g''''-g'''+\text{etc.}}}}}}.$$

atque ex §57

$$xx=n-1-\cfrac{n-1}{2n-2+g-\cfrac{2g(n-1+g)}{2n-3+g+g'-\cfrac{2g'(n-1+g')}{2n-4+g'+g''-\cfrac{2g''(n-1+g'')}{2n-5+g''+g'''-\text{etc.}}}}}$$

65. Generaliter ergo pro serie *a, b, c, d* etc., in qua sit

$$ab=p,\ bc=p+q,\ cd=p+2q,\ de=p+3q\ \text{etc.},\ \ldots\ xy=p+nq,$$

ex superioribus [§ 6] constat esse

$$xx=(p+nq)\cdot\frac{(p+nq)(p+(n+2)q)}{(p+(n+1)q)(p+(n+1)q)}\cdot\frac{(p+(n+2)q)(p+(n+4)q)}{(p+(n+3)q)(p+(n+3)q)}\cdot\frac{(p+(n+4)q)(p+(n+6)q)}{(p+(n+5)q)(p+(n+5)q)}\cdot\text{etc.},$$

et per formulas integrales

$$xx=(p+nq)\cdot\frac{\int z^{p+(n+1)q-1}dz:\sqrt{(1-z^{2q})}}{\int z^{p+nq-1}dz:\sqrt{(1-z^{2q})}}$$

posito $z=1$. Iam ob

$$xxyy=qqnn+2pqn+pp$$

habebimus $a=q,\ \beta=p$ et $\gamma=pp,$ hinc $\delta=p.$ Quare ex § 33 erit

$$\mathrm{N}=nq-\tfrac{q}{2}+p \text{ et } \mathrm{B}=0$$

ideoque

$$xx=p+q\left(n-\frac{1}{2}\right)+\cfrac{\frac{1}{4}qq}{2p+q(2n-1)+\cfrac{\frac{9}{4}qq}{2p+q(2n-1)+\cfrac{\frac{25}{4}qq}{2p+q(2n-1)+\cfrac{\frac{49}{4}qq}{2p+q(2n-1)+\text{etc.}}}}}$$

66. At per reliquas formulas, si ponamus primo

$$f=\begin{Bmatrix}q+p\\ p\end{Bmatrix},\ f'=\begin{Bmatrix}q+f\\ q+2p-f\end{Bmatrix},\ f''=\begin{Bmatrix}1+f'\\ 2q+2p-f'\end{Bmatrix},\ f'''=\begin{Bmatrix}q+f''\\ 3q+2p-f''\end{Bmatrix}, \text{ etc.}$$

habebimus ex § 59

$$xx=qn+p+\cfrac{q(qn+p)}{f-p-q-\cfrac{(q+2p-2f)(qn+f)}{f'-f-\cfrac{(q+2p-2f')(qn+f')}{f''-f'-q-\cfrac{(3q+2p-2f'')(qn+f'')}{f'''-f''-\cfrac{(3q+2p-2f''')(qn+f''')}{f''''-f'''-q-\text{etc.}}}}}}$$

et ex §58

$$xx=qn+p-\cfrac{q(qn+p)}{2qn+q+p+f-\cfrac{(q-2p+2f)(qn+f)}{2qn+2q+f+f'-\cfrac{(q-2f+2f')(qn+f')}{2qn+3q+f+f'-\text{etc.}}}}$$

ubi ex tribus numeris datis *p, q, n* bini quicunque negativi assumi possunt, quandoquidem aequatio resolvenda hinc nullam mutationem patitur.

67. Deinde si ponamus

$$g=\begin{Bmatrix}p-q\\ p\end{Bmatrix},\ g'=\begin{Bmatrix}g-q\\ 2p-q-g\end{Bmatrix},\ g''=\begin{Bmatrix}g'-q\\ 2p-2q-g'\end{Bmatrix},\ g'''=\begin{Bmatrix}g''-q\\ 2p-3q-g''\end{Bmatrix}, \text{ etc.}$$

erit per § 60

$$xx=qn-q+p-\cfrac{q(qn-q+p)}{g-p-\cfrac{2(p-g)(qn-q+g)}{g'-g+q-\cfrac{2(p-g'-q)(qn-q-g')}{g''-g'-\cfrac{2(p-g''-q)(qn-q-g'')}{g'''-g''+q-\text{etc.}}}}}$$

per §57

$$xx = qn - q + p - \cfrac{q(qn-q+p)}{2qn-2q+p+g+\cfrac{2(p-g)(qn-q+g)}{2qn-3q+g+g'+\cfrac{2(p-g')(qn-q+g)}{2qn-4q+g'+g''+\text{etc.}}}}$$

68. Verum de his expressionibus in infinitum excurrentibus tenendum est eas saepenumero seriebus divergentibus aequivalere, ita ut, quo ulterius earum valores colligamus, eo magis a veritate aberremus; quod incommodum tamen in expressione prima non usu venit. Quamobrem his casibus de iis eadem locum habent, quae de serierum divergentium natura iam annotavi, scilicet eas spectandas esse tanquam formulas infinitas ex evolutione cuiuspiam formulae finitae natas, quae nihilominus pro earum summa sit habenda, etiamsi, ubicunque in collectione partium substiterimus, verum nunquam attingamus.

69. Examinemus etiam seriem

$$a,\ b,\ c,\ d\ \text{ etc.},$$

in qua sit

$$ab = \tfrac{\beta}{\gamma},\ \ bc = \tfrac{\alpha+\beta}{\alpha+\gamma},\ \ cd = \tfrac{2\alpha+\beta}{2\alpha+\gamma},\ de = \tfrac{3\alpha+\beta}{3\alpha+\gamma}\ \text{ etc.}$$

hincque in genere

$$xy = \tfrac{\alpha n+\beta}{\alpha n+\gamma},$$

ac habebitur

$$x = \tfrac{\alpha n+\beta}{\alpha n+\gamma}\cdot\tfrac{\alpha(n+1)+\gamma}{\alpha(n+1)+\beta}\cdot\tfrac{\alpha(n+2)+\beta}{\alpha(n+2)+\gamma}\cdot\tfrac{\alpha(n+3)+\gamma}{\alpha(n+3)+\beta}\cdot\text{etc.}$$

sive

$$x = \frac{\int z^{\alpha n+\gamma-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\alpha n+\beta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}.$$

Cum nunc, si esset $n=\infty$, foret $x = y = 1$, valores x et y continuo magis ad unitatem accedent; quare ponatur

$$x = 1+\tfrac{\mathrm{A}}{\mathrm{X}}\ \text{ et }\ y = 1+\tfrac{\mathrm{A}}{\mathrm{Y}}$$

fietque

$$(\mathrm{X+A})(\mathrm{Y+A}) = \tfrac{\alpha n+\beta}{\alpha n+\gamma}\mathrm{XY}$$

seu

$$(\beta-\gamma)\mathrm{XY} - \mathrm{A}(\alpha n+\gamma)\mathrm{X} - \mathrm{A}(\alpha n+\gamma)\mathrm{Y} = \mathrm{AA}(\alpha n+\gamma)$$

Sit

$$\mathrm{A} = \beta-\gamma$$

seu

$$x = 1+\tfrac{\beta-\gamma}{\mathrm{X}}\ \text{ et }\ y = 1+\tfrac{\beta-\gamma}{\mathrm{Y}}$$

habebiturque

$$\mathrm{XY} - (\alpha n+\gamma)\mathrm{X} - (\alpha n+\gamma)\mathrm{Y} = (\beta-\gamma)(\alpha n+\gamma)$$

sive

$$(\mathrm{X}-\alpha n-\gamma)(\mathrm{Y}-\alpha n-\gamma)=(\alpha n+\beta)(\alpha n+\gamma).$$

70. Ex hac iam aequatione valores X et Y per formulas supra datas infinitis modis exhiberi possunt, ex quibus § 19 maxime convergentem suppeditat.
Cum autem sit

$$\lambda=-\alpha,\ \mu=-\gamma,\ \nu=-\gamma,\ \xi=\alpha,\ \eta=\tfrac{\beta+\gamma}{2}\ \text{et}\ \theta=\beta\gamma,$$

fiet

$$\mathrm{X}=2\alpha n-\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma+\frac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{X}'}$$

$$\mathrm{Y}=2\alpha n+\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma+\frac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{Y}'}$$

hincque emergit ista nova aequatio

$$\left(\mathrm{X}'-\alpha n-\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)\left(\mathrm{Y}'-\alpha n+\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\right)\gamma=(\alpha n+\beta)(\alpha n+\gamma).$$

71. Quodsi haec aequatio denuo simili modo evolvatur, ob

$$\lambda=-\alpha,\ \mu=-\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,\ \nu=\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,$$
$$\xi=\alpha,\ \eta=\tfrac{\beta+\gamma}{2}\ \text{et}\ \theta=\beta\gamma,$$

ut sit $\eta\eta-\theta=\frac{1}{4}(\beta-\gamma)^2$, orietur haec substitutio

$$\mathrm{X}'=2\alpha n-\alpha+\beta+\gamma+\frac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{X}''},$$

$$\mathrm{Y}'=2\alpha n+\alpha+\beta+\gamma+\frac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{\mathrm{Y}''},$$

quae praebet istam aequationem

$$\left(\mathrm{X}''-\alpha n-2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma\right)\left(\mathrm{Y}''-\alpha n+2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\right)\gamma=(\alpha n+\beta)(\alpha n+\gamma).$$

72. Iam cum hic sit

$$\lambda=-\alpha,\ \mu=-2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,\ \nu=2\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma,$$
$$\xi=\alpha,\ \eta=\tfrac{\beta+\gamma}{2}\ \text{et}\ \theta=\beta\gamma,$$

orietur haec substitutio

$$\mathrm{X}'' = 2\alpha n - \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{\mathrm{X}'''},$$

$$\mathrm{Y}'' = 2\alpha n + \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{\mathrm{Y}'''}$$

hincque ista aequalitas

$$\left(\mathrm{X}''' - \alpha n - 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma\right)\left(\mathrm{Y}''' - \alpha n + 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma\right) = (\alpha n + \beta)(\alpha n + \gamma).$$

73. Hoc modo progrediendo consequemur tandem aequationis propositae hanc resolutionem

$$xy = \frac{\alpha n+\beta}{\alpha n+\gamma}$$

$$x = 1 + \cfrac{\beta-\gamma}{2\alpha n-\alpha+\frac{1}{2}\beta+\frac{3}{2}\gamma+\cfrac{\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\cfrac{4\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\cfrac{9\alpha\alpha-\frac{1}{4}(\beta-\gamma)^2}{2\alpha n-\alpha+\beta+\gamma+\text{etc.}}}}}$$

ita ut sit

$$x = \frac{\int z^{\alpha n+\gamma-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\alpha n+\beta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}.$$

74. Percurramus quaedam exempla sitque prima

$$\alpha = 1,\ \beta = 2 \text{ et } \gamma = 0;$$

erit

$$x = \frac{\int z^{n-1}dz:\sqrt{(1-zz)}}{\int z^{n+1}dz:\sqrt{(1-zz)}} = \frac{n+1}{n}$$

ideoque

$$x = 1 + \cfrac{2}{2n+\cfrac{1-1}{2n+1+\text{etc.}}} = 1 + \frac{1}{n},$$

uti patet. Atque in genere, si $\alpha = 1$, quoties $\beta - \gamma$ est numerus par, valor ipsius x rationaliter exprimitur.

75. Manente $\alpha = 1$ sit

$$\beta = 1 \text{ et } \gamma = 0;$$

erit

$$x = \frac{\int z^{n-1}dz:\sqrt{(1-zz)}}{\int z^{n}dz:\sqrt{(1-zz)}},$$

ex aequatione

$$xy = \frac{n+1}{n},$$

at per fractionem continuam

$$x = 1 + \cfrac{1}{2n-\frac{1}{2}+\cfrac{1-\frac{1}{4}}{2n+\cfrac{4-\frac{1}{4}}{2n+\cfrac{9-\frac{1}{4}}{2n+\text{etc.}}}}} = 1 + \cfrac{2}{4n-1+\cfrac{1\cdot 3}{4n+\cfrac{3\cdot 5}{4n+\cfrac{5\cdot 7}{4n+\text{etc.}}}}}$$

Sumto autem

$$\beta = 0 \text{ et } \gamma = 1$$

prodit valor reciprocus

$$\frac{1}{x} = 1 - \cfrac{1}{2n+\frac{1}{2}+\cfrac{1-\frac{1}{4}}{2n+\cfrac{4-\frac{1}{4}}{4n+\text{etc.}}}} = 1 - \cfrac{2}{4n+1+\cfrac{1\cdot 3}{4n+\cfrac{3\cdot 5}{4n+\cfrac{5\cdot 7}{4n+\text{etc.}}}}}$$

cuius consensus cum praecedente facile perspicitur.

76. Quodsi iam pro *n* successive numeri 1, 2, 3, 4 etc. substituantur, reperientur sequentes fractiones continuae

$$\frac{\pi}{2} = 1 + \cfrac{2}{3+\cfrac{1\cdot 3}{4+\cfrac{3\cdot 5}{4+\cfrac{5\cdot 7}{4+\cfrac{7\cdot 9}{4+\text{etc.}}}}}}$$

$$\frac{2}{1}\cdot\frac{2}{\pi} = 1 + \cfrac{2}{7+\cfrac{1\cdot 3}{8+\cfrac{3\cdot 5}{8+\cfrac{5\cdot 7}{8+\cfrac{7\cdot 9}{8+\text{etc.}}}}}}$$

77. Hinc etiam vicissim huiusmodi fractionum continuarum valores investigari poterunt. Sit enim proposita haec fractio

$$\frac{\pi}{2} = 1 + \cfrac{2}{3+\cfrac{1\cdot 3}{4+\cfrac{3\cdot 5}{4+\cfrac{5\cdot 7}{4+\cfrac{7\cdot 9}{4+\text{etc.}}}}}}$$

$$\frac{2}{1}\cdot\frac{2}{\pi}=1+\cfrac{2}{7+\cfrac{1\cdot 3}{8+\cfrac{3\cdot 5}{8+\cfrac{5\cdot 7}{8+\cfrac{7\cdot 9}{8+\text{etc.}}}}}}$$

$$\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2}=1+\cfrac{2}{11+\cfrac{1\cdot 3}{12+\cfrac{3\cdot 5}{12+\cfrac{5\cdot 7}{12+\cfrac{7\cdot 9}{12+\text{etc.}}}}}}$$

$$\frac{2\cdot 2\cdot 4}{1\cdot 3\cdot 3}\cdot\frac{2}{\pi}=1+\cfrac{2}{15+\cfrac{1\cdot 3}{16+\cfrac{3\cdot 5}{16+\cfrac{5\cdot 7}{16+\cfrac{7\cdot 9}{16+\text{etc.}}}}}}$$

77. Hinc etiam vicissim huiusmodi fractionum continuarum valores investigari poterunt. Sit enim proposita haec fractio

$$s=\cfrac{\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{4\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{9\alpha\alpha-\delta\delta}{m-\alpha+\cfrac{16\alpha\alpha-\delta\delta}{m-\alpha+\text{etc.}}}}}$$

erit [§ 73]

$$\beta-\gamma=2\delta \text{ et } 2\alpha n+\beta+\gamma=m$$

unde

$$\beta=2\delta+\gamma \text{ et } 2\alpha n=m-2\delta-2\gamma$$

sicque

$$x=1+\frac{2\delta}{m-\alpha-\delta+s}=\frac{m-\alpha+\delta+s}{m-\alpha-\delta+s}$$

ergo

$$s=\frac{2\delta}{x-1}-m+\alpha+\delta.$$

Verum est

$$x=\frac{\int z^{\frac{1}{2}m-\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}}{\int z^{\frac{1}{2}m+\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}},$$

unde, si ponatur

$$\int z^{\frac{1}{2}m-\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}=\mathrm{P} \text{ et } \int z^{\frac{1}{2}m+\delta-1}dz:\sqrt{\left(1-z^{2\alpha}\right)}=Q,$$

erit

$$s=\frac{(m-\alpha+\delta)Q-(m-\alpha-\delta)P}{P-Q}.$$

78. Hinc si ponatur

$$\delta=\varepsilon\sqrt{-1},$$

ut sit

$$s=\cfrac{\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{4\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{9\alpha\alpha+\varepsilon\varepsilon}{m+\cfrac{16\alpha\alpha+\varepsilon\varepsilon}{m+\text{etc.}}}}}$$

ob

$$z^{-\delta}=z^{-\varepsilon\sqrt{-1}}=e^{-\varepsilon\sqrt{-1}\cdot lz}=\cos.\varepsilon lz-\sqrt{-1}\cdot\sin.\varepsilon lz$$

et

$$z^{\delta}=\cos.\varepsilon lz-\sqrt{-1}\cdot\sin.\varepsilon lz,$$

statuatur

$$\int\frac{z^{\frac{1}{2}m-1}dz\cos.\varepsilon lz}{\sqrt{(1-z^{2\alpha})}}=R$$

et

$$\int\frac{z^{\frac{1}{2}m-1}dz\sin.\varepsilon lz}{\sqrt{(1-z^{2\alpha})}}=S,$$

erit

$$P=R-S\sqrt{-1}\ \text{ et }\ Q=R+S\sqrt{-1}$$

ideoque

$$s=\frac{2(m-\alpha)S\sqrt{-1}+2\varepsilon R\sqrt{-1}}{-2S\sqrt{-1}}=\alpha-m-\varepsilon\frac{R}{S},$$

ubi notandum est integralia R et S ita sumi debere, ut posito $z=0$ evanescant, tum vero poni $z=1$.