CHAPTER FIVE

CONCERNING THE RESISTANCE

WHICH PLANE FIGURES EXPERIENCE MOVING IN WATER

PROPOSITION 49

PROBLEM

465. *If a plane figure may be moved straight into water with a given speed, to define the resistance, or the diminution of the motion which it will experience, while it traverses a given distance.*

SOLUTION

A plane figure is said to be moved straight in water, when its direction is perpendicular to the plane surface itself. The right line *AB* therefore will represent a plane surface, the area of which shall be $=aa$, to be moved along the direction of the normal *CO* to the same surface (Fig. 72). The weight or mass of the body, which this plane surface has, shall be $=M$, which encounters the water at *AEFB* , and its speed must correspond to a height *v*, by which it is progressing initially along the right line

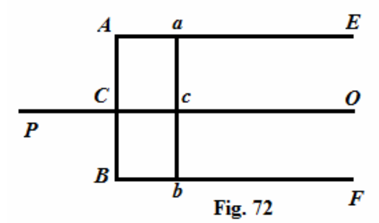


Fig. 72

CO into the water, and by which it shall go on itself progressing, except that it shall be reduced by the resistance. Now with the force of the resistance requiring to be defined, the body may be considered to be progressing for a brief instant of time, thus so that the plane surface *AB* may arrive at *ab* passing through the small absolute distance $Aa=Bb=dx$; and the remaining speed, on passing through this small space, must correspond to [a body falling from rest to] a height $v-dv$. But while the body is progressing through this small volume *Cc*, the water *ABba* will be displaced from its location by the interaction, thus so that meanwhile the colliding body may displace the mass of water *ABba,* the volume of which will be a^2dx, and its mass or weight may be expressed properly by ma^2dx, with *m* denoting the specific gravity of water. Therefore the body *M* with its speed $\sqrt{v}$ travels straight into the resting mass mass of water ma^2dx , from which the direction of the force in this collision is understood to be normal to the surface *AB* involved, and to be passing through the centre of gravity of this surface *C*, thus so that likewise the centre of gravity of the mass of water *ABba* shall be situated on the right line *Cc* ; therefore the body *M* will be acted on in this interaction by a certain force *CP,* the direction of which will be opposite to the direction of the motion *CO.* Therefore it will be required to address the rules requiring to be defined for the diminution of the motion, and indeed these are the rules, which are considered for perfectly soft bodies [*i.e.* completely inelastic collisions], since water at least in this case, may be considered from experiments to be freed from all elasticity. And thus since before the collision the quantity of motion present $= M\sqrt{v}$, truly after the collision, since the mass of water *ABba* will be moving with the same speed as the body *M,* evidently it must correspond to the height $v - dv$, the quantity of the motion:

$$\left(M + ma^2dx\right)\sqrt{(v-dv)} = \left(M + ma^2dx\right)\left(\sqrt{v} - \frac{dv}{2\sqrt{v}}\right) = \left[M\sqrt{v} - \frac{Mdv}{2\sqrt{v}} + ma^2dx\sqrt{v} - ma^2dx\frac{dv}{2\sqrt{v}}\right];$$

these two quantities will be required to be equal to each other, from which there becomes :

$$\frac{Mdv}{2\sqrt{v}} = ma^2dx\sqrt{v}, \text{ or } Mdv = 2ma^2vdx.$$

[The first term in the square brackets corresponds to the initial momentum, the last term is second order to be ignored, and the middle two terms must be equal and opposite to preserve linear momentum in the collision. We may interpret this last equation as the work done by the mass M falling through a height *dv* to be equal to the work done by the incremental mass of water $2ma^2dx$ falling through the distance *v*.]

Now so great a force *p* may be put in place, so that the body shall be acted on by the force in the direction *CP*, yet meanwhile the body is moved through the distance $Cc = dx$, so that the same motion may be able to be become smaller, there shall become

$$dv = \frac{pdx}{M};$$

[Again, the work done by *M* falling *dv* is put equal to the retarding force *p* acting horizontally on *M* through the increment *dx*.]
and thus, from above, $p = 2ma^2v$; from which it is understood the resistance of the water on the surface a^2 to be moving with a speed corresponding to the height *v,* the motion to be directly equivalent to a weight of water of volume $2a^2v$; or equal to the weight of a cylinder, the base of which shall be equal to the surface a^2 moving into the water ; truly the height shall be made equal to twice the height corresponding to the speed of the body. Again therefore water is effective through resistance, and if the body *M* may be acted on by so great a force, as we may assign in the direction *CP,* to be impinging directly normally on the surface of the body in the water, and to be passing through the centre of gravity *C* of its surface. Q. E. I.

[Thus, in a roundabout manner, Euler implicitely involves the conservation of energy principle by equating the work done by a weight falling from a height to the change it experiences in the square of a speed, clearly a universal property, before such quantities had been understood properly, or interactions defined in terms of energy changes.]

COROLLARY 1

466. Therefore the resistance is reduced to the force, which the body endowed with a plane surface experiences when travelling directly into water, of which both the direction as well as the quantity is expressed by the weight given.

COROLLARY 2

467. Therefore the mean direction of the resistance, which the plane surface encounters from the direct water motion, is normal to the surface itself and passes through its centre of gravity.

COROLLARY 3

468. Moreover the magnitude of the resistance maintains a ratio composed from the surface area itself, and from the square of the speed; and on this account, for the same surface, the resistances are in the ratio of the speeds squared.

COROLLARY 4

469. If a volume of water $=V$ may be put equal to the weight M of the body itself, [whereby $m=\frac{M}{V}$] there will become $V:M=2a^2v$: weight of the cylinder of water, of which the area of the base is *aa* and the height *2v*. On account of which the resistance, which the plane surface a^2 must endure with a speed due to the height *v* in passing straight into water, is equivalent to the weight of a cylinder of water of magnitude $\frac{2Ma^2v}{V}$.

COROLLARY 5

470. Therefore the body at rest will experience the same force, on the plane surface of which water strikes with a speed due to the height *v*, therefore so that the effect arising from the collision of the bodies will depend to a great extent on the respective speed, which in each case is the same.

COROLLARY 6

471. Therefore this proposition prevails equally for the motion of a body a resting water, and is required to be determined in rivers, if indeed the surface experiencing the resistance were plane, and that pointing directly into the water, or the water were to strike the same directly.

SCHOLIUM 1

472. Even now much is discussed between authors, who have written about the resistance of water, whether the resistance shall be the same for a double cylinder of water, of which the equal base shall be the source of the resistance of the surface requiring to be removed at once, and the height equal to the height due to the speed, just as we have indeed found here, or only with a simple cylinder. But here we have elicitd a double cylinder of this kind requiring to be expressed for the resistance of the water, since we have put the water particles to be perfectly soft and without anything from the side, which indeed the trials suggest. But if perfect elasticity may be attributed to the water, certainly another account of the resistance will arise. If indeed the rules, which are applied to the collision of elastic bodies, may be called into help, then indeed the quadruple of the cylinder mentioned will arise, and the resistance will be found $= 4ma^2v$. But since this here from this consideration a greater speed will be communicated to the water, than the body itself retains, thus the water must spring from the body, so that a vacuum will be left between the body and the water. Whereby on account of the weight of the water, where it will be unable to press its parts together, the rules of close contact, which are adapted for elastic bodies, here will be unable to be used; but the general principle, on which these rules depend will be required to be used, and which is consistant with the conservation of the *vis viva* [lit. *living force,* a concept introduced by Leibniz, and which may be considered to be an early form of the conservation of momentum principle in the interaction between free bodies]. Therefore it is required to be established on account of the close contact, the body *M* and the water *ABba* certainly to acquire the same speed. Truly with this in place, which before the present interaction $= Mv$, indeed after the interaction the *vis viva* $= \left(M + ma^2dx\right)(v - dv)$, with these equated there will become $Mdv = ma^2vdx$; from which an equivalent force of the resistance will arise equal to the weight ma^2v, that is to a cylinder of water of base a^2 and of height *v.* But whichever account of the resistance may be had, the calculation remains the same, indeed it differs only by the coefficient of this cylinder of water, which in that case is 2, truly for this case is 1. On account of which there is not much we can do about that controversy, since, whichever case shall prevail, the proportions shall remain the same, to which we shall attend especially; indeed in each case the direction of the resistance is normal to the plane surface travelling directly into the water, and passes through the centre of gravity of its surface, and besides in each case proportional to the area of the surface and to the square of the speed taken together. Moreover experiments, which have been put in place concerned with the resistance of bodies in moving water are seen to be in favour of the simple cylinder, which agrees wonderfully well with the argument aimed at the conservation of living forces. Also it is readily apparent the resistance must be smaller than in the solution we have found; indeed there, since the body is accompanied by the water after the collision, the forces following must be weaker than we have assumed.

SCHOLIUM 2

473. Evidently the experiments, which Newton set out for globes dropped in water, were seen to show the resistance clearly enough only by a simple cylinder of water, of which the height evidently is equal to the simple height requiring to generate the speed to be established. Moreover truly because the water, besides this resistance, which proceeds from striking the sides, has another resistance arising from the tenacity of the particles [*i.e.* the viscosity], is to be defined by experiment not without a little difficulty, so great shall be the resistance alone arising from the sides being struck. Therefore whatever it shall be, with the experiments of the latter hypothesis, by which the resistance along a simple cylinder of water is set forth, they shall be agreed on well enough, we will adopt that hypothesis here, and the resistance, which the plane surface experiences striking straight into the water, we will measure by the weight of a cylinder of water, of which the base shall be equal to the area of the surface, with the height truly of the speed due to the height; thus in the case of corollary 4 the resistance will be equal to $\frac{Ma^2v}{V}$ itself required to be considered. Truly the same hypothesis of the resistance can be confirmed by the following argument, but not indeed being proved clearly. The vessel *ACDB* shall be filled completely with water (Fig. 73), the height of which shall be $AC = v$, the hole *DE* shall be bored through this vessel at the side below, of which the area shall be $= a^2$, water must flow throught this hole with a speed corresponding to the height *v,* now the vessel of the outflow *Ed* may be placed directly opposite to the plane of the bolt *de* , equal to the size of the opening, and here the bolt will sustain the same force from the outflowing water, as if it may strike against water at rest, itself with a speed corresponding to a height *v*. Moreover it is seen to be agreed, the bolt at *de* to be subjected to the same pressure, as if it were present at *DE*, truly in this case the bolt will stop up the opening entirely and the outflow will be completely stopped up; but now the pressure will appear equal to the weight of the cylinder of water, of which the base is equal to the surface of the bolt a^2, truly with a height of the altitude $AC = v$, from which it follows the resistance of the plane surface in the water is required to be estimated from the water of a simple cylinder, the height of which shall be equal to the speed corresponding to the altitude. Also experiments confirm this reasoning well enough, for if some greater bolt may be used, than the opening *DE,* a greater resistance of the water may be perceived, yet it may be seen for bolts of this size required to be used, certainly water may flow out from its sides, and it will exercise a greater pressure, than if the bolt were only equal to the opening; on account of which there is no doubt why the bolt should not have an surface greater than the amplitude of the opening, so that the assigned pressure shall be perceived. Again, so that the same hypothesis may be confirmed, if the elasticity of the water may granted, which cannot be removed completely, and especially, if the conservation of the *living forces* may be put in place, the

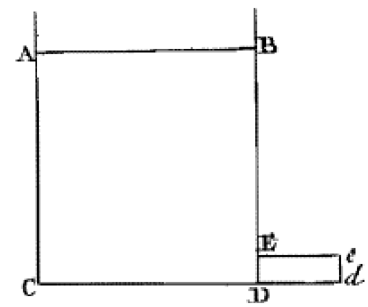


Fig. 73

greatest use of which is considered everywhere, thus only there we will have received less doubt, and on that the whole doctrine of resistance to be constructed there; and therefore even more, since they may favour especially these experiments.

PROPOSITION 50

474. *If a plane surface may be moved obliquely in water, to determine the resistance, by which the motion of the surface will be retarded by the water.*

SOLUTION

Because the plane surface is said to be moved obliquely, when the direction of the motion is put in place for some oblique angle itself, *AB* shall represent the plane surface, of which the area shall be $=a^2$ (Fig. 74); which may be moved in water in the direction *MC,* which shall constitute with the plane surface *AB* the angle *ACM,* of which the sine shall be $=n$*;* with the whole sine put $=1$; truly the speed with which the surface is moved shall be due to the the altitude *v.* Now as before the surface *AB* shall be considered to be moved in the water through the small space $Cc=dx$, and with this completed to arrive at *ab,* meanwhile by necessity an interaction will be had with the mass of water *ABba,* the volume of which is $=na^2dx$. Therefore a small part of the surface opposes the motion, as if it were moving directly into the water, and that in the ratio of the sine of the angle of incidence to the whole sine; and on that account the resistance from this chapter, which is allowed in the direction of the motion, is required to be diminished in the ratio of the sine of the angle of incidence *ACM* to the whole sine. Then in whatever oblique manner the individual particles of water may strike the surface, yet the impulse will be normal to the surface *AB*, thus so that the resistance will exert a force on the surface *AB*, the direction of which will be the normal *CP,* and which will pass through the centre of gravity *G.* But since all the impacts of this surface with the individual particles of water are oblique, they will be less effective than if they were direct, and in the ratio of the sine of the angle of incidence *AGM* to the whole sine. Therefore since for this oblique impulse, on account of the twofold causes, the ratio of the sine of the angle of incidence to the whole sine must be diminished twice, and the total resistance will be had, while the surface *AB* is moved obliquely in the water, to the resistance, which the same surface moved with the same speed directly shall experience, so that the square of the sine of the angle of incidence *MCA* to the square of the whole sine from this is as a^2 to 1. Whereby since the resistive force in the case of the right motion shall be $=ma^2v$, or to the weight of the cylinder of water, of which the base is $=a^2$ and the height equal to the altitude due to the speed, the resistive force for the present case will be $=n^2ma^2v$, that is to the weight of the cylinder of water having the base equal to the surface

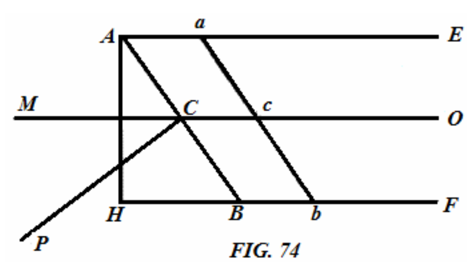


FIG. 74

itself and the height equal to the altitude due to the speed, multiplied by the square of the sine of the angle of incidence *MCA* with the whole sine put $=1$. Q. E. I.

COROLLARY 1

475. Therefore the resistance, which the same plane experiences under diverse angles moving with the same speed in the water, is in the square ratio of the sine of the angle which the plane makes with the direction of the motion.

COROLLARY 2

476. Therefore if the resistive force were known, which the plane sufferd in the direct motion in the water, likewise the resistance will be known, which the same plane will experience striking the water obliquely in some way.

COROLLARY 3

477. Therefore in whatever direction the plane surface may be moving in the water, the direction of the resistance is always the same, normal to the plane of the surface, and passing through the centre of gravity of the surface itself.

COROLLARY 4

478. Again the resistance, which the same plane may experience in the motion thrugh the water, under various angles and with different speeds, is in the ratio composed from the squares of the speeds, and from the square of the sine of the angle with which it strikes the water.

COROLLARY 5

479. Moreover the resistances, which different planes experience by moving in the water, maintain the ratio composed from the simple areas, the squares of the speeds, and the squares of the sines of the angles, with which they meet in water.

SCHOLIUM 1

480. These problems provide the basic framework for determining the resistance, which bodies of any kind of shape experience moving in water. Indeed the resistance will depend on the anterior surface of the body which travels through the water, certainly which contends alone with the particles of water, moreover the posterior part of the body experience no resistance from the water, since there that part does not dash against the water. Though indeed even the posterior part may be seen to be affected by the water, while the place for the water, which the body has left behind after itself, being occupied, in part is forced back after the forwards motion and adds to the accelerated motion, yet this effect is scarcely noticeable, and

for this reason it does not deserve to be considered here; to which it approaches, so that the theory of water not only shall be raised to that degree of perfection, so that the effect of water arising from the back part of floating bodies may be able to be defined. Therefore with this consideration overlooked, if the anterior surface of a body floating on water were or may be agreed to be either a plane or to depend on several planes, the absolute resistance will be able to be defined with the aid of these two problems. Truly besides, these problems attend to the resistance of some bodies with predetermine surfaces required to be assigned; for a surface of some kind was required to be prepared, that were accustomed to be considered to be prepared from innumerable planes, and from the rules of statics the total resistance, which emerges from the resistances of the individual parts, will be able to be defined by integration, with which understood, both the mean direction of all the resistances, as well as the equivalent force itself, will be allowed to be determined.

SCHOLIUM 2

481. Therefore since now it shall be proposed to investigate the resistance, by which any floating water bodies endure indefinitely, so that the same may be resolved conveniently and clearly, it will be required to follow a certain order. Therefore at first in this chapter I will consider only plane figures floating in water either horizontally or vertically, and I will deteremine the force of the resistance of each direction, then indeed it will allow for these bodies themselves to move easily. Truly these shapes, which we may put to float horizontally on water, we will assume to be provided with an axis or diameter, so that the ships, which we consider mainly here, partake of a plane diameter, which shall pass vertically through the keel, from which the individual horizontal sections of the keel will be provided. But here a huge distinction arises in the resistance, whether a surface of this kind may be moved in the water following the direction of its diameter, or obliquely ? Indeed if it may be moved following the direction of its diameter, it is evident the mean direction of the resistance on account of the similarity from each side of the diameter to be brought about, to be with the diameter itself in place, thus so that in this case only the size of the resistive force must be investigated; but if a surface of this kind may not be progressing in the water along the direction of its diameter, then both the mean direction as well as the magnitude of the resistance itself will have to be found separately, which investigation therefore will be had of greater difficulty. Then in the following chapter in a similar manner we are going to turn our attention to investigate the resistance of bodies floating on water; indeed we will consider only bodies of this kind, which shall be provided with a vertical plane to the diameter, so that the condition of the ship may be observed in the first place, in which again with the handling required to be attended to, especially with regard to the direction of the motion, whether that shall be made along the didrection of the diameter of the water section, or according to some oblique diameter; indeed for the first case the mean direction of the resistance is given at once, truly the latter is not going to be found without a great deal of effort . But we will bring to each treatment of this problem, from which it may be clear, since the figure of the ship shall be the most suitable with regard to the resistance; since then both the minumum resistance, as well as the most suitable direction for the resistance, will be chosen. But before we undertake all these things required to established, this is the most suitable place for examining the effect of the rudder by

turning the ship about a vertical axis ; since the rudder is accustomed to be provided with a plane surface, the force of which, as it endures being driven against the water, therefore will be able to be defined at once from this proposition.

PROPOSITION 51

PROBLEM

482. *If the ship may be progressing in some direction, and the rudder may be turned to a given angle, to find the force, which the rudder will have requiring to rotate the ship about the vertical axis passing through the centre of gravity.*

SOLUTION

Since the mean direction of the force of the water, into which the rudder has intruded, passes through the centre of gravity of the plane surface of the rudder, and is normal to that, the horizontal section *ARBm* of the ship passing through the centre of gravity *C* of the rudder *AD* may be considered (Fig. 75). Moreover it is evident here not all of the rudder, but only its part, which has immersed the centre of gravity in the water ought to be taken. And thus in the figure *A* will represent the stern, *B* the prow, *AB* the keel of the ship, and *AD* the natural place holding the rudder. Moreove *G* shall be the point of the vertical axis drawn through the centre of gravity of the ship, in which it passes through the horizontal plane *ARBm*; truly *GM* shall be the direction of the course or motion of the ship, thus in order that the angle *BGM* may denote the declination of the course of the ship from the course direction, which is considered to be along the direction of the keel *GB*, truly the inclination of the rudder shall be at the angle *DAd,* thus so that the position *Ad* may be obtained, so that it will be struck in the water along the direction *cm* parallel to the direction of the course *GM*. Now the sine of the angle $BGM = m$, the cosine $= \mu$; truly the sine of the angle $DAd = n$, the cosine $= \nu$, with the whole sine always $=1$. Again the area or surface of the rudder receiving the force of the water s $= a^2$,

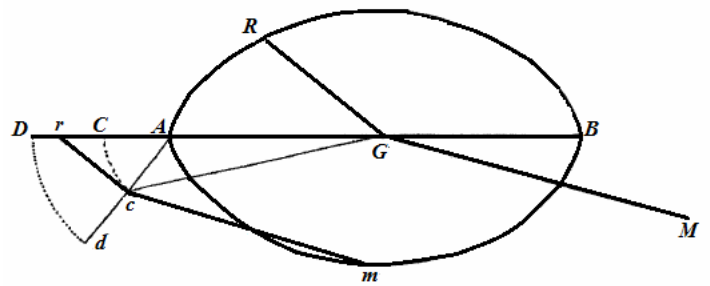


FIG. 75

$$AC = Ac = b,\ AG = f\ ;$$

and the speed, with which the ship may be moved, shall be due to the height *v.* And finally the weight of the boat shall be $= M$, the volume of the submerged part of the boat *V,* and the moment of inertia of the ship with respect to the vertical axis $= Mk^2$. From these premises *Acm* will be the angle within which the rudder Ad interacts with the water, which since it shall be $= DAd + BGM$, its sine will be $= mv + np$; hence therefore the resistive force which the rudder experiences will be

$$= \frac{(mv + n\mu)^2 Ma^2 v}{V},$$

the direction of which cr will be transmitted through the centre of gravity c of the rudder, and it will be normal to Ad. Therefore the moment of this force for the ship turning about the vertical axis will be

$$= (mv + n\mu)^2 \frac{Ma^2 v}{V}.Gr\sin Arc.$$

Truly since the angle Acr is right, the sine of $Arc = v$, and $Ar = \frac{b}{v}$; from which there becomes $Gr = f + \frac{b}{v}$. So that the moment of the force of the rudder for the ship requiring to be rotated about G will be

$$= \frac{(mv + n\mu)^2 Ma^2 v(b + vf)}{V}.$$

Which divided by the moment of inertia of the ship with respect to the vertical axis Mk^2, will give the force of gyration of the ship about the same vertical axis

$$= \frac{(mv + n\mu)^2 a^2 v(b + vf)}{Vk^2};$$

to which force the momentary angular acceleration of the ship is proportional, which is impressed around the vertical axis of the ship through the centre of gravity. Q. E. I.

COROLLARY 1

483. Therefore for the same ship, so that the greater were the expression,

$$(mv + n\mu)^2 (b + vf)$$

thus the greater will be the effect of the rudder requiring to rotate the ship; from which the angle DAd will be able to be defined, so that the effect of the rudder will be a maximum.

COROLLARY 2

484. Therefore if the cosine of the angle *DAd* or v were put $= x$, there will become $n = \sqrt{1-xx}$, and the formula,

$$\left(mx+\mu\sqrt{1-xx}\right)^2 (b+fx),$$

or the square of its radius,

$$\left(mx+\mu\sqrt{1-xx}\right)\sqrt{b+fx}$$

will become a maximum since x will be determined from this equation:

$$2\left(m-\frac{\mu x}{V(1-xx)}\right)(b+fx)+f\left(mx+\mu\sqrt{1-xx}\right)=0,$$

which changes into this :

$$m(2b+3fx)\sqrt{1-xx} = \mu\left(3fxx+2bx-f\right).$$

COROLLARIUM 3

485. If therefore the course of the ship were directed straight ahead, so that the angle *BGM* shall vanish, there will become $m=0$, and $\mu=1$, and the gyratory force $=\dfrac{n^2a^2v\left(b+vf\right)}{Vk^2}$; therefore the maximum effect of the rudder will be preserved, if there were $3fxx+2bx-f=0$, that is, if the cosine of the angle *DAd*

$$x=\frac{-b\pm\sqrt{bb+3ff}}{3f}.$$

COROLLARY 4

486. Therefore if b were so small, so that it may vanish besides f, the cosine of the angle *DAd* $=\dfrac{1}{\sqrt{3}}$, so that it is responsible for the maximum effect of the rudder, that is the angle *DAd* will be $54°, 44'$.

COROLLARY 5

487. If the course of the ship were inclined in the direction by the angle *BGM,* moreover the rudder will be left in its natural state $-AD$, yet it will establish an effect on the rotation of the ship, of which the gyratory force will become $\frac{m^2a^2v(b+f)}{Vk^2}$.

COROLLARY 6

488. Moreover the force of the rudder affects the motion of the ship itself, which change is found, if the force of the resistance $\frac{(mv+n\mu)^3 Ma^2v}{V}$ may be considered to be applied to the centre of gravity in the direction parallel to *GR*; evidently it will retard the motion of the ship in its direction by the force $\frac{(mv+n\mu)^3 Ma^2v}{V}$; but the force $\frac{(\mu v-mn)(mv+n\mu)^2 Ma^2v}{V}$ will disturb the ship from its rectilinear path.

COROLLARY 7

489. In addition the rudder itself in the situation *Ad* will attempt to be rotating about A along the direction *dD,* which attempt will be expressed by the moment

$$\frac{(mv+n\mu)^2 Ma^2bv}{V};$$

therefore so great a force must be applied to the rudder for the rudder to be continued in its position *Ad*.

COROLLARY 8

490. Therefore if the ship may be carried in its oblique course, thus there will be a need for the rudder to remainin its natural state *AD*, which force will be expressed by the moment $\frac{Mn^2a^2bv}{V}$.

COROLLARY 9

491. Finally it is evident all these forces exerted by the rudder, with all else being equal, to increase in the square ratio of the speed, with which the ship is progressing.

SCHOLIUM

492. Therefore in this proposition not only have we defined by how great a force the ship' s rudder may be acting around the vertical axis drawn through the centre of gravity, but also we have determined in the corollaries, by how great an amount both the speed of the ship as well as the direction of the course may be affected by the rudder. Moreover, also, we have assigned the force, which the ship's captain must use for the rudder, in a given situation with so much requiring to be preserved, clearly this force is required of the master, so that its moment with respect of the axis which is present around the moveable rudder, may be made equal to the moment found, by which the rudder may be extended from the situation *Ad* towards *AD*. Truly also it is understood, unless the plane *ARBm* shall pass through the centre of gravity *G* of the ship, the force of the rudder also exercises itself on the ship around the horizontal axes with moments of the components of the force around both the length as well as the width of the ship, but which inclination barely deserves a mention, since it shall be small, and then only will eventuate, when the rudder is adjusted. On account of which with the rudder dealt with, we may return to that proposition itself, and indeed in the first place, we will investigate how great a resistance plane figures floating in water may experience.

PROPOSITION 52

PROBLEM

493. *The plane figure MAN* (Fig. 76) *with the given diameter AP shall float on water along the direction AC of the diameter AP itself with a given speed, to find the resistance, which this figure shall experiance from the water.*
[In what follows we may assume the plane figure to be of unit height.]

SOLUTION

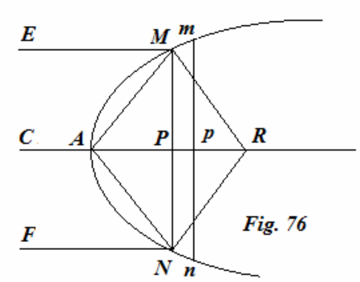


Fig. 76

First it is evident, since the figure is progressing along the direction of the axis *AC* in in the water, on account of everything being similar on both sides, the mean direction of the resistance must lie on the diameter *AP* itself, thus so that there will be a need to determine this quantity. On this account the speed must be put to correspond to the height *v*, with which the figure is progressing in the water along the direction *AC* and the two ordinates *MPN, mpn* be drawn to the diameter *AP* equal and similar to the elements *Mm, Nn* requiring to be cut off on both sides of the curve, it is required to investigate how much resistance which elements may contribute. There may be put

$$AP = x,\ PM = PN = y$$

there will become:

$$Pp = dx, \text{ and } Mm = Nn = \sqrt{dx^2 + dy^2} = ds.$$

Now the sine of the angle, by which the elements *Mm* and *Nn* are pushed against by the water, is equal to $= \frac{dy}{ds}$. But if these elements may be struck directly or normally in the water, the force of the resistance will become $= vds$, that is, to the weight of a small cylinder of water of base *ds* and of height *v*. Therefore in the present case the force, which each element experiences, will be equal to $= \frac{vdy^2}{ds}$; and the direction of each force is normal to the element itself, and thus the normals *MR* and *NR* will intersect. If now these two forces may be resolved in pairs, of which one will have the direction of the applied lines, the other parallel to the axis *AP,* these former will mutually cancel out, truly the latter will add together, and will have the mean incident direction *AP*. On account of which resistance of the elements of the figure *Mm, Nn,* in the direction *AP,* the force of the resistance $= \frac{2vdy^3}{ds^2}$; from which the resistance of the whole curve *MAN* will be acquired

$$= 2\int \frac{vdy^3}{ds^2} = 2v\int \frac{dy^3}{ds^2} = \left[2v\int \left(\frac{dy}{ds} \right)^2 dy \right],$$

on account of the constant *v*, and the direction of this force in place will be along the diameter *AP.* Q.E.I.

COROLLARY 1

494. Therefore the direction of the resistance, which a figure of this kind experiences moved in water along the diameter *AC*, will be directly opposite to the direction of the motion, and on this account the motion will be retarded so much by the resistance, the direction truly will not be affected, if indeed the centre of gravity of the figure were situated on the diameter *AP* .

COROLLARY 2

495. Therefore the resistance from *A* and on progressing as far as *M* and *N*, so that there may become $dy = 0$, that is so that the tangents to the curve shall become parallel to the axis *AP*. On account of which if the curve were indefinite, the resistance must be considered only as far as the parts of the branches *AM* and *AN* , which lie between *A* and the place where $dy = 0$.

COROLLARY 3

496. If only with the ordinate *MPN* directly in the water, that is the shape may be moving with the same speed along the direction *AP*, then the resistance it will experience will become

$=2vy$; and where the resistance of the ordinate *MPN* itself will be had to the resistance of the curve *MAN* as y to $\int\frac{dy^3}{ds^2}$.

COROLLARY 4

497. Since there is $dy<ds$ everywhere, there will become

$$\frac{dy^3}{ds^2}<dy \text{ and thus } \int\frac{dy^3}{ds^2}<y\ ;$$

on account of which the resistance which the curve *MAN* experiences always to be smaller than the resistance, which the ordinate *MPN* alone may indicate.

COROLLARY 5

498. Therefore the resistance of the figure *MAN* will be smaller, when it shall differ more from the right transversal *MPN*; or where everywhere the element of the transversal of the applied line dy is smaller with respect to the element of the curve.

COROLLARY 6

499. But since there shall become

$$\frac{dy^3}{ds^2}=dy-\frac{dx^2dy}{ds^2} \text{ on account of } ds^2=dx^2+dy^2,$$

the resistance, which the portion of the curve *MAN* maintains,

$$=2vy-2v\int\frac{dx^2dy}{ds^2}\ .$$

Therefore the excess of the resistance of the ordinate *MN* over the resistance of the arc *MAN* will become

$$=2v\int\frac{dx^2dy}{ds^2}\ .$$

COROLLARY 7

500. Therefore if only the figure *MAN* may be put in place instead of the curve *MAN,* in which case nowhere shall there be $dy=0$, and with the ends *MN* terminated with the same

speed along the direction *PC,* then it shall be moving the following along opposite to *PR*, the resistance in the first case to the resistance in the second case will be as $y-\int\frac{dx^2dy}{ds^2}$ to *y*. Therefore in the first case the resistance for the curve *MAN* is put in place*,* truly in the second case the resistance for the line *MN* .

SCHOLIUM

501. Since the formula, by which the resistance of the arc *MAN* is expressed, $2v\int\frac{dy^3}{ds^2}$, generally cannot be integrated, the resistance itself cannot be represented by finite quantities, and it is evident, the resistance to depend on the mutual position of the individual elements. On account of which it will be expedient to descend to more specific examples, and to consider given curves, for which the value of $\int\frac{dy^3}{ds^2}$ itself may be assigned, indeed it will be able to collect from these, curves of this kind which may experience greater or smaller resistance. Clearly in this way the mind of the reader will be prepared for curves, which either amongst all of which, or between these alone, which may be endowed with a certain property, on understanding that these may partake of a minimum resistance ; I am going to investigate curves of this kind in the following examples.

EXAMPLE 1

502. The figure or at least its anterior part *EA F* (Fig. 77), which experiences the resistance, shall be an isosceles triangle, which must be progressing forwards in the water along the direction of the diameter *AB* towards the vertex *A*, with a speed due to the height *v.* The direction of the resistance lies on the right line *AB,* and truly its magnitude thus will be defined from the general solution. On putting $AB=a,\ BE=BF=b,$ and as before $AP=x,\ PM=PN=y,$ there will become $a:b=x:y,$ from which there shall be $y=\frac{bx}{a};\ dy=\frac{bdx}{a};$ and $ds=\frac{dx}{a}\sqrt{\left(a^2+b^2\right)}.$

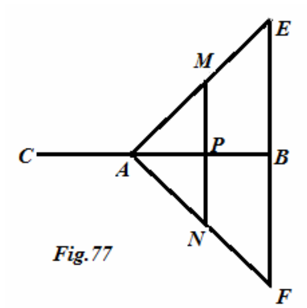


Fig.77

From these there will become :

$$\frac{dy^3}{ds^2}=\frac{b^3dx}{a\left(a^2+b^2\right)},$$

and on integrating,

$$\int\frac{dy^3}{ds^2}=\frac{b^3x}{a\left(a^2+b^2\right)},$$

on account of which the resistance, which the part *MAN* will be experiencing, will be $= \frac{b^3xv}{a(a^2+b^2)}$, and on putting $x=a$, the desired resistance will be had, which the whole triangle *EAF* will endure $= \frac{2b^3v}{a^2+b^2}$.

COROLLARY 1

503. Therefore the resistance, which the angle *EAF* experiences will be to the resistance of the base *EF* moved in water with the same speed and in the same direction as *bb* to a^2+b^2, that is as BE^2 to AE^2.

COROLLARY 2

504. Therefore it is evident the resistance of the triangle *EAF* in water with the vertex *A* foremost, truly itself to have to the resistance which the same triangle turned with the base *EF* foremost, to be in the square ratio of the sine of the angle *BAE* to the whole sine.

COROLLARY 3

505. Therefore where the angle *EAF* were smaller or more acute with the base *EF* remaining the same, there the resistance experienced will be smaller, as will become evident with the leading vertex *A* turned around in the water.

COROLLARY 4

506. Therefore in the above figure with the same base *EF* put in place, which shall experience the minimum resistance, the triangle will be infinitely great, of which the vertex *A* will go off to infinity: clearly the resistance of such a triangle is zero.

EXEMPLUM 2

507. The curve *MAN* (Fig. 76) shall be a parabola, of which any ordinate shall be either continuous, or composed from the equal roots *AM* and *AN* of the same parabola, which must be moved in water in the direction *AC* with a speed corresponding to the height *v*, thus so that there shall become

$$y = \frac{x^m}{a^{m-1}}.$$

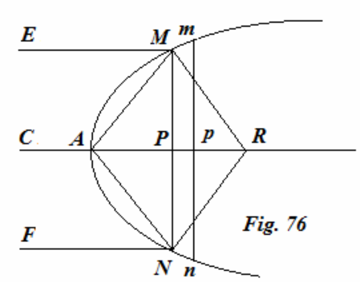


Fig. 76

Therefore the direction of the resistance, which the part *MAN* will experience, certainly lies in the direction of the diameter *AP,* truly thus will be defined from the equation

$$y=\frac{x^m}{a^{m-1}}.$$

Since there shall be

$$dy=\frac{mx^{m-1}dx}{a^{m-1}},$$

there will become

$$ds^2=dx^2\left(1+\frac{m^2x^{2m-2}}{a^{2m-2}}\right)=\frac{dx^2\left(a^{2m-2}+m^2x^{2m-2}\right)}{a^{2m-2}}$$

and

$$\frac{dy^3}{ds^2}=\frac{m^3x^{3m-3}dx}{a^{m-1}\left(a^{2m-2}+m^2x^{2m-2}\right)}=dy-\frac{ma^{m-1}x^{m-1}dx}{a^{2m-2}+m^2x^{2m-2}}.$$

Hence therefore there will become:

$$\int\frac{dy^3}{ds^2}=y-ma^{m-1}\int\frac{x^{m-1}dx}{a^{2m-2}+m^2x^{2m-2}}=y-\int\frac{a^{\frac{2m-2}{m}}dy}{a^{\frac{2m-2}{m}}+m^2y^{\frac{2m-2}{m}}},$$

which expression multiplied by $2v$ will give the magnitude of the resistance. Therefore for the Appollonian parabola for which there is $m=\frac{1}{2}$, the resistance will be

$$=2vy-2v\int\frac{4yydy}{aa+4yy}=2v\int\frac{aady}{aa+4yy}=av\ \text{Arc.t.}\frac{2y}{a}.$$

Therefore the whole parabola continued to infinity experiences a finite resistance which will be $=\frac{\pi}{2}av,$ with $1:\pi$ denoting the ratio of the diameter to the periphery. Truly for the remaining cases no exceedingly concise formulas of this kind will be found, on account of which with these dismissed we will progress to other curves requiring to be considered.

EXAMPLE 3

508. If the anterior part receiving the resistance were the arc of the circle *MAN* to be moved horizontally along the direction *CA*, the radius of which becomes $AC=a$, the direction of the resistance on account of the similar parts on each side fall on the diameter *AC.* Moreover on putting

$$AP=x \text{ and } PM=PN=y;$$

there will become

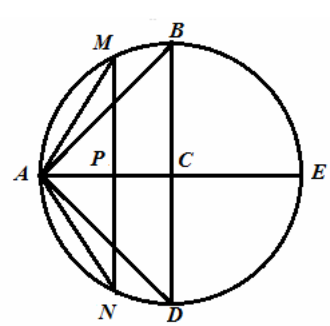


Fig. 78

$$x = a - \sqrt{a^2 - y^2} \text{ and } ds = \frac{ady}{\sqrt{a^2 - y^2}}$$

from which there becomes

$$\frac{dy^3}{ds^2} = dy - \frac{y^2 dy}{aa}, \text{ and } \int \frac{dy^3}{ds^2} = y - \frac{y^3}{3a^2}.$$

Therefore the resistance which the arc *MAN* will experience in the direction AC will be

$$= 2v\left(y - \frac{y^3}{3aa} \right).$$

Therefore if the anterior part of the figure were the whole semicircle *BAD,* the resistance which it will experience will be $\frac{4av}{3}$. On account of which, if the whole figure were the circle *ABED,* in whatever region of this may be moved forward in the water, the resistance of the semicircle always will be the subject of the resistance, and on account of all the similar parts the direction of the resistance will bisect the anterior semicircle and will pass through the centre, and from that the resistance will be always $= \frac{4av}{3}$.

COROLLARY 1

509. Therefore the resistance of the semicircle *BAD* itself will be had to the resistance of the diameter *BD* of the same speed moved directly as $\frac{4av}{3}$ to $2av$, that is as 2 : 3.

COROLLARY 2

510. But if the same isosceles triangle *BAD* may be considered to be floating on the water with the same speed and in the same direction, its resistance will be $= av$. Therefore the resistance of the triangle *BAD,* it twice as small as the resistance of the diameter *BD* ; and the resistances of the circle *BAD,* of the triangle *BAD* and of the diameter *BD* will have resistances between each other as these numbers 4: 3: 6.

COROLLARY 3

511. But if moreover the resistance for an indefinite arc smaller than the semicircle *MAN* alone may be experienced, its resistance will be to the resistance of the chord *MN* striking against the water with the same speed, as $y - \frac{y^3}{3aa}$ to y, or as $3aa - yy$ to $3aa$.

COROLLARY 4

512. If in addition the isosceles triangle *MAN* shall be moving in the same direction and with the same speed, its resistance will be $= \frac{y^3 v}{aa - a\sqrt{aa-yy}}$. On account of which the resistances of the segment *MAN,* of the triangle *MAN* and of the chord *MN* will be between themselves as

$$\frac{6aa-2yy}{3aa} : \frac{y^2}{a^2 - a\sqrt{aa-yy}} : 2$$

or so that

$$6aa - 2yy : 3a^2 + 3a\sqrt{aa-yy} : 6aa.$$

COROLLARY 5

513. Therefore the maximum of these three resistances is that experienced by the chord *MN*. Truly the resistance of the segment *MAN* becomes equal to the resistance of the triangle *MAN,* if there shall become $y = \frac{a\sqrt{3}}{2}$, that is, if the arc *AM* shall be 60°. Therefore the resistance of the segment *MAN* will be greater than the resistance of the triangle *MAN* if the arc *AM* shall exceed 60°, truly less if the arc *AM* were smaller than 60°.

EXEMPLUM 4

514. Now the anterior elliptic arc *MAN* (Fig. 79), shall be part of the plane figure floating on the water, or part of the ellipse *ABED,* of which one semi-axis shall be $AG = a$, the other $BC = b$, there will be put in place

$$AP = x,\ PM = y,\ CP = a - x,$$

and

$$a^2 = (a-x)^2 + \frac{aayy}{bb},$$

from which there becomes

$$x = a - \frac{a}{b}\sqrt{(bb-yy)}.$$

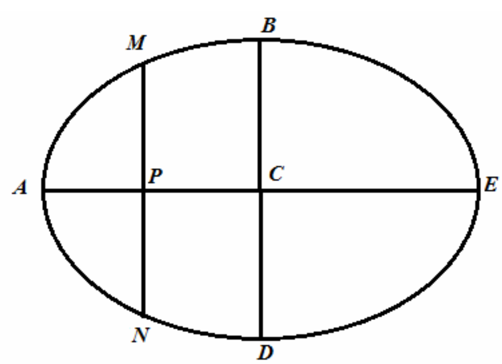


Fig. 79

Hence therefore there will become

$$dx = \frac{aydy}{b\sqrt{(b^2 - y^2)}} \quad \text{and} \quad ds^2 = \frac{dy^2\left(b^4 + \left(a^2 - b^2\right)yy\right)}{bb\left(bb - yy\right)},$$

$$\left[i.e.\ ds^2 = dx^2 + dy^2 = \frac{a^2y^2dy^2 + b^2\left(b^2 - y^2\right)dy^2}{b^2\left(b^2 - y^2\right)} = \frac{\left(a^2 - b^2\right)y^2dy^2 + b^4dy^2dy^2}{b^2\left(b^2 - y^2\right)} \right]$$

on account of which there will become

$$\frac{dy^3}{ds^2} = \frac{bbdy(bb - yy)}{b^4 + \left(aa - bb\right)yy} = -\frac{bbdy}{aa - bb} + \frac{aab^4dy}{\left(aa - bb\right)\left(b^4 + \left(aa - bb\right)yy\right)}.$$

For the integration of this formula requiring to be resolved two cases are required to be considered, the one if $a > b$, the other if $a < b$. With the first indeed from the case with the resistance from the square of the circle, with the latter depending on logarithms . Therefore the first more acute vertex A of the ellipse may be moved in the direction of the major axis EA in the water, there will become

$$\int \frac{dy^3}{ds^2} = -\frac{bby}{aa - bb} + \frac{aab^4dy}{\left(aa - bb\right)^{\frac{3}{2}}} \text{Atang.} \frac{y\sqrt{\left(aa - bb\right)}}{bb},$$

and thus the resistance which the arc MAN will experience will be

$$= \frac{2a^2b^4v}{\left(aa - bb\right)^{\frac{3}{2}}} \text{Atang.} \frac{y\sqrt{\left(aa - bb\right)}}{bb} - \frac{2b^2vy}{aa - bb}.$$

Whereby if the integral of the semi-ellipse BAD may constitute the anterior part of the figure, the resistance will be

$$= \frac{2a^2b^2dy}{\left(aa - bb\right)^{\frac{3}{2}}} \text{Atang.} \frac{\sqrt{\left(aa - bb\right)}}{b} - \frac{2b^3v}{aa - bb}.$$

But if now $a < b$, or A the more obtuse vertex of the ellipse, there will become

$$\int \frac{dy^3}{ds^2} = \frac{bby}{bb - aa} + \frac{aabb}{2\left(aa - bb\right)^{\frac{3}{2}}} l\left(\frac{bb + y\sqrt{\left(aa - bb\right)}}{bb - y\sqrt{\left(aa - bb\right)}}\right);$$

therefore in this case the arc *MAN* will produce the resistance

$$=\frac{bbvy}{bb-aa}+\frac{aabbv}{(aa-bb)^{\frac{3}{2}}}l\left(\frac{bb+y\sqrt{(aa-bb)}}{bb-y\sqrt{(aa-bb)}}\right).$$

But if moreover *b* shall be greater or less than *a,* the integral will be given by the infinite series

$$\int\frac{dy^3}{ds^2}=y-\frac{aay^3}{3b^4}+\frac{aa(aa-bb)y^5}{5b^8}$$
$$-\frac{aa(aa-bb)^2y^7}{7b^{12}}+\frac{aa(aa-bb)^3y^9}{9b^{16}}-\frac{aa(aa-bb)^4y^{11}}{11b^{20}}+\text{etc.}$$

from which the resistance which the arc *MAN* will endure

$$=2vy\left(1-\frac{aay^2}{3b^4}+\frac{aa(aa-bb)y^4}{5b^8}-\frac{aa(aa-bb)^2y^6}{7b^{12}}+\text{etc.}\right).$$

Therefore if the whole central of the ellipse *BAD* may endure the resistance, the resistance will become

$$=2bv\left(1-\frac{a^2}{3bb}+\frac{a^2(aa-bb)}{5b^4}-\frac{a^2(aa-bb)^2}{7b^6}+\frac{a^2(aa-bb)^3}{9b^8}-\text{etc.}\right).$$

COROLLARY 1

515. Therefore if the same ellipse shall be moving first along the direction of the axis *EA,* then along the direction of the axis *DB* ; the resistance in the first case will be

$$=2bv\left(1-\frac{a^2}{3bb}+\frac{a^2(a^2-b^2)}{5b^4}-\frac{a^2(a^2-b^2)^2}{7b^6}+\text{etc.}\right),$$

truly the resistance in the second case will become

$$=2av\left(1-\frac{bb}{3aa}-\frac{bb(aa-bb)}{5a^4}-\frac{bb(aa-bb)^2}{7a^6}-\text{etc.}\right).$$

COROLLARY 2

516. Therefore the resistance of an ellipse progressing along the major axis *EA* to the resistance of the same body progressing along the minor axis *DB* with the same speed is itself had as

$$b-\frac{a^2}{3b}+\frac{a^2\left(a^2-b^2\right)}{5b^3}-\text{etc. to } a-\frac{bb}{3a}-\frac{bb\left(aa-bb\right)}{5a^3}-\text{etc.}$$

where *a* is the semi-major, *b* truly the semi-minor axis.

COROLLARY 3

517. Therefore if the difference of the axioms shall be very small, on putting $a=b+d$ with *d* denoting an extremely small magnitude, the resistance of the semi-ellipse *BAD* to the resistance of the semi-ellipse *ABE* shall be as

$$\frac{2b}{3}-\frac{4d}{15}+\frac{2dd}{21b} \quad \text{to} \quad \frac{2b}{3}+\frac{14d}{15}+\frac{2dd}{21b},$$

which is as 1 to

$$1+\frac{9d}{5b}+\frac{18dd}{25bb};$$

which agrees especially with this ratio, 1 to $\left(1+\frac{d}{b}\right)^{\frac{9}{5}}$, or $b^{\frac{9}{5}}:a^{\frac{9}{5}}$.

COROLLARY 4

518. But truly the ratio, which the semi-ellipse *BAD* maintains to the resistance of the semi-ellipse *ABE* is as

$$2a^2b^2\text{Atang.}\frac{\sqrt{\left(aa-bb\right)}}{b}-2b^3\sqrt{\left(aa-bb\right)}$$

to

$$2a^3\sqrt{\left(aa-bb\right)}-a^2b^2l\left(\frac{a+\sqrt{\left(aa-bb\right)}}{a-\sqrt{\left(aa-bb\right)}}\right)$$

therefore this ratio, if indeed *b* may not differ much from *a*, will approach closely to this ratio $b^{\frac{9}{5}}:a^{\frac{9}{5}}$.

EXAMPLE 5

519. The figure floating on the water advancing in the direction CA shall be had composed from the two equal circular segments ABE and ADE (Fig. 80), on putting

$$AC = a,\ BC = CD = b,$$

the radius of the circle, of which the segments are taken, will be

$$\frac{aa+bb}{2b}.$$

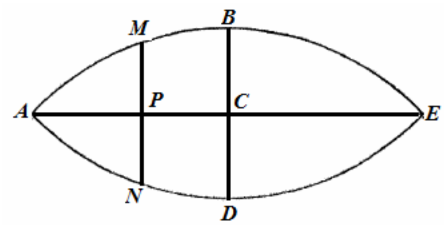


Fig. 80

Whereby on putting

$$AP = x,\ PM = PN = y$$

there will be had

$$x = a - \sqrt{\left(a^2 - \frac{\left(a^2-b^2\right)}{b}y - yy\right)}$$

and

$$dx = \frac{\left(aa-bb\right)dy + 2bydy}{2\sqrt{(a^2b^2-(a^2-b^2)by-bbyy)}},$$

from which there becomes

$$ds^2 = \frac{\left(a^2+b^2\right)^2 dy^2}{4a^2b^2-4(a^2-b^2)by-4bbyy},$$

On account of which there will be obtained :

$$\frac{dy^3}{ds^2} = \frac{4a^2b^2dy-4(a^2-b^2)bydy-4bbyydy}{\left(a^2+b^2\right)^2},$$

and hence

$$\int\frac{dy^3}{ds^2} = \frac{4a^2b^2y-2(a^2-b^2)byy-\frac{4}{3}bby^3}{\left(a^2+b^2\right)^2};$$

which expression multiplied by *2v* will give the resistance, which the portion *MAN* experiences. On account of which if there may be put $y = b$ the resistance will arise, which the whole anterior part *BAD* experiences $= \dfrac{4b^3v(3a^2+b^2)}{3\left(a^2+b^2\right)^2}$. But the resistance, which the same figure experiences, if it may be moved in the direction *CB* with the same speed, will be from example 3

$$=\frac{2av(3a^4+2a^2b^2+3b^4)}{3\left(a^2+b^2\right)^2}.$$

COROLLARY 1

520. The resistance which is experienced by the right line *BD* moved in the direction *CA*, is

$$=2bv=\frac{6bv(a^2+b^2)^2}{3\left(a^2+b^2\right)^2};$$

from which the resistance of the figure *BAD* moved in the direction *CA* to the resistance of the right line *BD* moved in the same direction will be as

$$6aabb+2b^4 \text{ to } 3a^4+6aabb+3b^4,$$

from which the resistance of the right line *BD* is much greater than the resistance of the figure *BAD*.

COROLLARY 2

521. But the resistance, which the figure *ABED* moved along the direction *CA* will experience, will itself have the ratio to the resistance of the same figure moved with the same speed in the direction *DB* as

$$6a^2b^3+2b^5 \text{ to } 3a^5+6a^3b^2+3b^4;$$

therefore if $a > b$ the former will always be smaller than the latter.

SCHOLIUM

522. From these considerations it is understood well enough that no finite figure may be given, which may experience the minimum resistance among all other curves containing the same terms. For, whatever figure may be assigned experiencing the minimum resistance, at once another curve will be able to be shown, which will experience a smaller resistance, such a curve or only a part of that may be required to be extended into that region in which the motion occurs. On account of this property, not even an example of a plane figure sought may be able to be proposed, which moving forwards horizontally in the water may experience the minimum resistance ; indeed no minimum solution may itself be declared in finite terms. Moreover from which it may be evident, whatever finite figures may be preferred in the account of the resistance, to which it is required to attend, other conditions likewise may be present, from

which the curve sought may be considered to be finite. But questions of this kind can be formed, so that either all the figures shall have the same area, or amongst all the curves with the same surrounding perimeter, one may be determined there, which may experience the minimum resistance along the given direction of the motion in the water. Truly, it will be convenient to establish the following lemma for solving questions of this kind, whereby a method is produced for solving all the problems of this kind.

LEMMA

523. *To find a curve, which may make use of some maximum or minimum property, either among all whole curves, or between these only, which shall be provided either with a certain single property, or equally with several properties.*

SOLUTION

Both that property, which must be a maximum or minimum in the curve sought, as well as these properties, where in curves some choice is required to be made from which the indefinite integral formulas may be expressed ; and from these formulas, without any discrimination in place, a certain maximum or minimum property may be contained, or common properties, the nature of the curve sought will be defined in the following manner. The integrals of the individual formulas proposed may be reduced to the orthogonal ordinates x and y, so that in these no other quantities will be present besides x and y, with the differentials of these both of the first as well as of higher orders. Moreover with dx put constant there shall become

$$dy = pdx,\ dp = qdx,\ dq = rdx,$$

etc. , from which substitutions some proposed integral formula will be reduced to a form of this kind $\int Zdx$, in which Z will be a quantity composed from the finite quantities $x, y, p, q, r,$ etc. Whereby if this quantity Z may be differentiated, its differential will have such a form, so that there shall become

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

From this differential the following quantity may be formed

$$V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.}$$

and the values V of this kind may be deduced from the individual integral formulas proposed, which must be in common, either to be a maximum or minimum, or from all the curves from which the question is required to be defined, must be in common. From these finally the individual values of V found may be multiplied respectively by some constant quantities, and

the sum of these products may be put $=0$, which equation will express the nature of the curve sought.
Therefore with these restored in place of $p, q, r,$ etc. evidently the values of the assumed

$$p=\frac{dy}{dx},\ q=\frac{ddy}{dx^2},\ r=\frac{d^3y}{dx^3}\ \text{etc.}$$

so that an equation may be obtained for the curve sought containing only the two variables x and y together with their differentials, in which dx shall be constant. Q. E. I.

COROLLARY 1

524. Therefore if the area of the curve $\int ydx$ must be either a maximum or a minimum, or all the curves of the same area may be put in place, from which those sought to be defined of the same area are to be put in place, there will become

$$Z=y,\ \text{and}\ dZ=dy,$$

from which the value of V corresponding to the formula $\int ydx$ will be $=1$.

$$\left[\begin{array}{l} dZ=Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.}\ =1.dy \\ \& \\ V=N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\text{etc.}=N=1. \end{array}\right]$$

COROLLARY 2

525. If a curve either of the maximum or minimum length may be desired, or all the curves may be put in place from these sought, which must be of the same length, this same property will be expressed by this formula $\int\sqrt{\left(dx^2+dy^2\right)}$, with the aid of which substitution is reduced to this $\left[\int Zdx\right]=\int dx\sqrt{(1+pp)}$; therefore there will become:

$$Z=\sqrt{(1+pp)}\ \text{and}\ dZ=\frac{pdp}{\sqrt{(1+pp)}},$$

from which there will become:

$$\left[dZ = Mdx + Ndy + Pdp + \text{etc.} \ = \frac{pdp}{\sqrt{(1+pp)}}\right]$$

$$M = N = 0,\ P = \frac{p}{\sqrt{(1+pp)}},\ Q = 0,\ \text{etc.}$$

and thus the corresponding value V of the formula $\int dx\sqrt{(1+pp)}$ will be

$$\left[V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.} = -\frac{d}{dx}\left(\frac{p}{\sqrt{(1+pp)}}\right) = \left(-\frac{dp}{dx}\cdot\frac{1}{\sqrt{(1+pp)}} + \frac{1}{2}\frac{2p^2}{(1+pp)^{\frac{3}{2}}}\cdot\frac{dp}{dx}\right)\right]$$

$$= \frac{-dp}{(1+pp)^{\frac{3}{2}}dx}.$$

COROLLARY 3

526. If a curve of this kind may be sought, which is floating horizontally along the direction of the axis, in which the abscissa of x may be taken, the resistance must be taken to be the minimum experienced, then that same formula $\int\frac{dy^3}{ds^2}$ will be required to be taken to be a minimum, truly this formula on account of

$$dy = pdx \ \text{ and } \ ds^2 = dx^2(1+pp),$$

will be changed into this: $\int\frac{p^3dx}{1+pp}$. Therefore since there shall be

$$[dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}]$$

$$Z = \frac{p^3}{1+pp},\ \text{there will become } N = 0,\ \left[\frac{dZ}{dp} = \frac{3p^2}{1+pp} - \frac{p^3\cdot 2p}{(1+pp)^2}\right] = \frac{3pp+p^4}{(1+pp)^2},$$

and the corresponding value of V will be $= -\frac{dP}{dx}$.

COROLLARY 4

527. If therefore among all the curves that one may be desired, which shall experience the maximum or minimum resistance, which will be had from the single formula $\int \frac{dy^3}{ds^2}$; on account if which the corresponding value of V must be $=0$. Therefore there will be had

$$dP = 0, \text{ and } P = \frac{3pp + p^4}{(1+pp)^2} = m,$$

by which equation the nature of the line sought will be expressed.

COROLLARY 5

528. Therefore since from this equation p may become constant, there shall be

$$p = k, \; dy = kdx, \text{ and there will become } y = kx + c$$

from which there becomes

$$k = \frac{y-c}{x} = p.$$

Which value substituted into the equation found will give an algebraic equation between x and y, hence

$$(y-c)^4 + 3xx(y-c)^2 = m\left(x^2 + (y-c)^2\right)^2,$$

which indeed is for a right line or for several connected right lines.

COROLLARY 6

529. Whereby on putting $x = 0$ likewise there may become $y = 0$, or there must become either $c = 0$ or $m = 1$. But if there shall be $m = 1$ then $p = 1$ and $y = x$; but if there may be put $c = 0$ there will be had

$$y^4 + 3xxy^2 = m\left(x^2 + y^2\right)^2,$$

and hence

$$y = \frac{\pm x\sqrt{2m}}{\sqrt{\left(3 - 2m \pm \sqrt{(9-8m)}\right)}},$$

which equation includes four right lines.

SCHOLIUM

530. This lemma extends the most widely, since not only from these problems, by which one curve may be desired, which will deserve to be resolved, from which all the curves can be produced generally, and which may possess a certain property of the maximum or minimum; but also may be adapted for these problems, not from all the possible curves available from which, but only from these which may be equally endowed with one or more properties of some kind of the maxima or minima, of which it may be wished to make use. Therefore the use of this lemma extends much further than to isoperimetric problems, as indeed has been treated at this point, so that the method is extended to all kinds of curves whether of the same length, or some other certain property equally possible requiring that to be defined, which may entertain the use of some maximum or minimum property. For besides which the method will consider only a single use, which may be agreed on for all curves, that also in the account of the integral formulas which must be common to the maxima or minima of all curves, for a huge restriction is objectionable; for its use may cease, and at once may be introduced either into another or into the same integral formula of the second order or into another either of the same or higher order of this same kind, while the method treated by this lemma may be extended to differentials of any orders. But if this same arc of the curve may be contained by this curve itself or the integrals of other forms in the quantity Z itself, the lemma brings forth no greater superiority, but must be connected up by another method, which, since the use of this in the following does not occur, here we have passed over.

PROPOSITION 53

PROBLEM

531. *Among all the curves AM, with the axis AP and with the applied line PM maintaining that same area AM, which may form the figure AMN around the axis AP on both sides, in allowing the minimum or maximum resistance, if indeed it may be progressing in the direction of the diameter PA* (Fig. 76).

SOLUTION

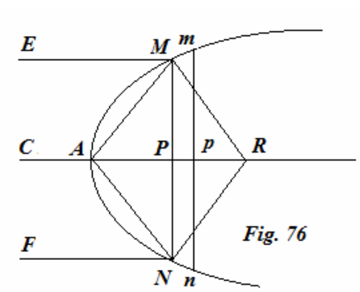


Fig. 76

With the abscissa put $AP = x$, the applied line $PM = y$, here the question corresponds, so that between all the curves, in which $\int ydx$ maintains the same value, that will be determined in which

$$\int \frac{dy^3}{ds^2} \text{ or } \int \frac{p^3dx}{1+pp}$$

on putting $dy = pdx$, shall be a maximum or minimum. Therefore here the value $V = 1$ will correspond to the first formula $\int ydx$; truly for the latter

$$\int \frac{p^3 dx}{1+pp} \text{ there is } V = -\frac{dP}{dx}$$

with there being put

$$P = \frac{3pp + p^4}{(1+pp)^2} \text{ and } \int Pdp = \frac{p^3}{1+pp}.$$

Therefore for the curve sought this equation will be obtained :

$$dP = \frac{dx}{a} \text{ and } P = \frac{x+b}{a} = \frac{3pp + p^4}{(1+pp)^2}.$$

But from the same equation multiplied by *p*,

$$pdP = \frac{pdx}{a} = \frac{dy}{a};$$

on integrating there arises

$$Pp - \frac{p^3}{1+pp} = \frac{y+c}{c} = \frac{2p^3}{(1+pp)^2},$$

from which there becomes

$$x = \frac{3app + ap^4}{(1+pp)^2} - b \text{ and } y = \frac{2ap^3}{(1+pp)^2} - c$$

from which formulas the curve sought is constructed without difficulty. Moreover the area will be

$$\int ydx = \frac{a^2P^2p}{2} - \frac{a^2Pp^3}{1+pp} + \frac{a^2p^5}{2(1+pp)^2} - cx + 2a^2\int \frac{p^4dp}{(1+pp)^4}$$
$$= \frac{2a^2p^5}{(1+pp)^4} + 2a^2\int \frac{p^4dp}{(1+pp)^4} - cx;$$

or truly the resistance is

$$\int \frac{dy^3}{ds^2} = \frac{aPp^2}{1+pp} - a\int P^2dp = \frac{2ap^5}{(1+pp)^3} - 4a\int \frac{p^4dp}{(1+pp)^4}.$$

From these equations both that curve, which has the maximum resistance, as well as that with the minimum resistance which is experienced, will be contained. Q. E. I.

COROLLARY 1

532. If there may be put $b=0$ and $c=0$, the curve will remain the same; for another such axis may be taken parallel to the first, and another initial abscissa. And thus for this axis, if the abscissa shall be

$$x=\frac{3app+2ap^4}{(1+pp)^2},$$

the applied line will be

$$y=\frac{2ap^3}{(1+pp)^2}.$$

COROLLARY 2

533. Therefore if there may be taken $p=0$, then there will become both $x=0$ as well as $y=0$; therefore initially the curve lies on the axis of the abscissas, and on account of $\frac{dy}{dz}=0$, the curves will be tangential to the axis at this location.

COROLLARY 3

534. If there may be put $p=\infty$, there will become $x=a$ and $y=0$, on account of which in this place where $x=a$, the curves will meet the axis again, but truly the tangent will be normal to the axis.

COROLLARY 4

535. Then it is evident both the abscissa as well as the applied line to be able to increase only as far as to certain limits ; indeed a maximum value will be obtained for both *x* and *y* on putting $p=\sqrt{3}$, and in this case there becomes

$$x=\tfrac{9}{8}a \text{ and } y=\frac{3\sqrt{3}}{8}a.$$

COROLLARY 5

536. Finally p may have either a positive or negative value, the x abscissa remains the same, but y will obtain a negative value on taking p negative, from which it is understood the axis in which x may be taken to be the abscissa, likewise to be the diameter of the curve found.

SCHOLIUM 1

537. Since on taking

$$x = \frac{3ap^2 + ap^4}{(1+pp)^2} \text{ there shall become } y = \frac{2ap^3}{(1+pp)^2},$$

the curve will be algebraic, and easily described by infinitely many points. For the

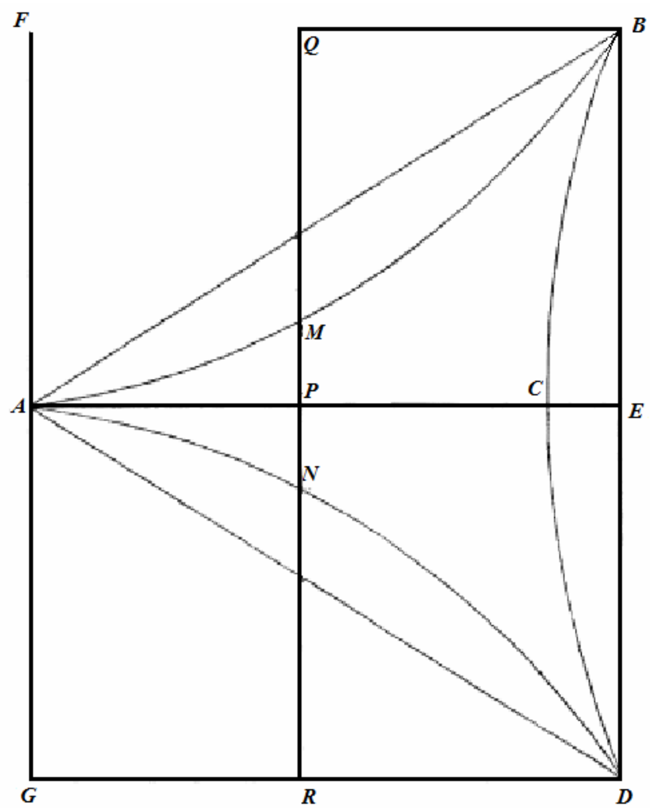


Fig. 81

direction of the axis AC may be taken, along which the figure shall be moving parallel in the water, and the construction found will provide the triangular curve *AMBCDNA* (Fig. 81) having the three cusps *A, B, D* providing the angles of the equilateral triangle *ABD*, and the three parts comprised between the cusps *AMB, AND* and *BCD* will be equal and similar to each other. Moreover there will become

$$AC = a,\ AE = \tfrac{9}{8}a \text{ and } BE = DE = \frac{3\sqrt{3}}{8}a$$

truly tangents at *B* and *D* making an angle of 30 degrees with the right line *BD*. Therefore since this curve follows the direction of the axis *AC* with the motion between all the others of this same capacity both maximum as well as minimum, the resistance in water may be experienced, the part *BMAND* will be allowed to be experiencing the minimum resistance, truly the area *BCD* the maximum. Whereby if the curve may be desired, which may experience the same minimum resistance between all the parts containing the same area, for that either the arc *AMB* or *AND* or a part of each will be required to be taken. Moreover for ships it will be most convenient to take half of each anterior part, that is the figure *DNAG* or *BMAF*, thus so that *D* may fall on the prow, and the right line *DG* to be the keel of the ship; if indeed the figure *DNA* may be disposed to each part of the axis *DG* which the figure will have in water following the direction *GD* among all the other areas containing the same area *DNAG*, and it will be experiences the minimum resistance passing through the points *D* and *A*; and this same curve will have the area *DNAG* among all the others drawn through *A* and *D* and experiencing that same maximum resistance. So that moreover the nature of this curve may be examined for ships especially adapted with regard of the axis *DG*, there shall be $DR = t$, $NR = u$, and since there shall become

$$t = \tfrac{9}{8}a - x \text{ and } u = \frac{3\sqrt{3}}{8}a - y,$$

there will become

$$DR = t = \frac{(3-pp)^2 a}{8(1+pp)^2},$$

and

$$NR = u = \frac{\left(p-\sqrt{3}\right)^2\left(3p^2\sqrt{3}+2p+\sqrt{3}\right)a}{8(1+pp)^2}.$$

Truly also this first equation to be conserved, which is

$$DR = t = \frac{(3-pp)^2 a}{8(1+pp)^2},$$

$$AP = GR = x = \frac{3app + ap^4}{(1+pp)^3}, \text{ and } PN = y = \frac{2ap^3}{(1+pp)^2};$$

from which there will become

$$dx = \frac{2apdp(3-pp)}{(1+pp)^3}, \text{ and } PN = y = \frac{2appdp(3-pp)}{(1+pp)^3},$$

and

$$\sqrt{\left(dx^2+dy^2\right)}=ds=\frac{2apdp\left(3-pp\right)}{\left(1+pp\right)^{\frac{5}{2}}},$$

from which the arc itself will become:

$$AN=s=\frac{2ap\left(3pp-1\right)}{\left(1+pp\right)^{\frac{3}{2}}}+\frac{2a}{8},$$

thus so that the curve found shall be rectifiable. But the resistance which the part *AN* may experience will become as

$$\int\frac{dy^3}{ds^2}=\frac{\frac{1}{2}ap+\frac{7}{6}ap^3+2ap^5}{(1+pp)^3}-\frac{a}{2}\int\frac{dp}{1+pp}.$$

Therefore on putting $p=\sqrt{3}$ the whole curve will become $AND=\frac{4}{3}a,$ truly its sub tangent which itself will be had from the arc *AND* will be to the sub tangent *AD* as 16 to $9\sqrt{3}$. Truly the resistance which the whole curve *DNA* will experience will be as

$$\frac{11a\sqrt{3}}{32}-\frac{\pi a}{6}$$

with π denoting the periphery of the circle, of which the diameter is 1. Therefore the resistance of the whole curve will itself be had to the resistance of the chord *AD* as 4 : 9 approx. Besides the curve *AND* has the tangent at *A* parallel to the axis *GD*, and the tangent at *D* makes an angle of 60 degrees with the axis; truly at *A* and *D* the radius of osculation is infinitely small. Moreover in this manner it may be seen easily that this curve itself *AND* shall have the radius of oscillation with respect to the sub tangent AD, from the part *BCD*, where there is

$$CE=\tfrac{1}{8}a,\ BE=DE=\frac{3\sqrt{3}}{8}a,$$

and the radius of osculation at the mean point *C* is $=2a$ *;* from which the construction is easily prepared.

COROLLARY 6

538. Therefore if the figure *AND* may be granted to the anterior part of the ship, with *D* being the prow and *DG* the keel, the ship progressing in the direction *GD* not only will experience the minimum resistance but in addition if it may be moved thus, so that not only shall it be moving forwards thus, so that the chord *AD* shall become normal to direction of the course, then it will experience the maximum resistance, since the curve *AND* is congruent to *BCD.*

COROLLARY 7

539. Therefore this figure commends itself for the optimum form requiring to be attributed to ships ; for not only is it required to be progressing in the direction of the keel with the

minimum resistance offered, but also so that the resistance may become exceptionally great in any oblique course.

COROLLARY 8

540. Truly the resistance which the figure *AND* may experience if it may be moving in the water in the direction normal to the chord *AD*, and will be to normal to that chord and

$$=\frac{11a\sqrt{3}}{16}+\frac{\pi\alpha}{6}.$$

Truly if the figure may be moving in the direction along *GD*, and from each part of the axis *DG* similar to it the resistance will be

$$=\frac{11a\sqrt{3}}{16}-\frac{\pi\alpha}{6}.$$

COROLLARY 9

541. Therefore if the individual horizontal sections of the anterior parts of the ship will have a figure of this kind, so that the halves of these all shall be equal or similar to the figure *DNA,* then it will have the best adapted figure of the ship requiring to overcome the resistance of the water, and likewise it will comprise the greatest space, the accounting of which is required to be had especially in ships.

SCHOLIUM 2

542. But since here these shapes are brought forth only for plane figures floating in water, they may be extended to plane figures floating horizontally, not to solid figures, such as ships, unless they may be adapted with the greatest degree of caution. Thus the plane figures experiencing the minimum resistance among all the spaces of equal capacity, which here finds a place in solids not found unless all the horizontal sections of the floating body shall be equal to each other; and on that account if these will be able to be confirmed by experiment, asserting it will be agreed to use the same thickness everywhere, so that the law of resistance as plane figures or of vanishing thickness will hold; clearly in this case the sides asserting the resistance which maintain the same vertical position are required to be excepted, and thus they meet the water particles below the surface at the same angles as any horizontal sections. But if the submerged figures may maintain an oblique situation to the water, or if the resisting sides were not to oppose the water vertically but inclined to the horizontal, then the angle of incidence differs from that angle, under which it crosses only a horizontal section of the water ; and on that account in bodies of this kind, the force of the resistance shall be known, which the individual horizontal sections experience, yet the resistance of the whole thence will be unable to be defined. On account of which nothing amiss will be derived from the conclusions for the resistance of the body, and it is indicated to suspend the deliberations : so that in the following

we shall be determining the resistance which any bodies floating in water are going to endure indefinitely.

PROPOSITION 54

PROBLEM

543. *If some plane figure BCA* (Fig. 82) *situated vertically in the water may be moved along the horizontal direction MD with a given speed, to determine both the magnitude as well as the mean direction of the resistance which it may encounter.*

SOLUTION

AC may be taken for the vertical axis, in which the abscissa shall be $CP = x = MQ,$ the applied line $PM = y = CQ$; and the arc $AM = s$*;* the sine of the angle, where the point *M* of the curve *AMB* meets the water $= \frac{dx}{ds}$, from which the element of the resistance, which the element *ds* will experience, the force will be $= \frac{vdx^2}{ds}$, with *v* denoting the height corresponding to the speed with which the figure will be moved, the direction of which force will be *MN* normal to the curve at *M.* Now this force may be resolved into two sides, the direction of one will be of the horizontal *MP,* of the other the vertical *MQ,* and the direction of the horizontal force will be

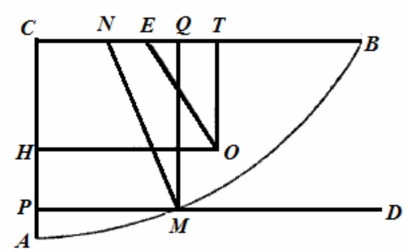


Fig. 82

$$MP = -\frac{vdx^3}{ds^2},$$

and of the vertical force

$$MQ = +\frac{vdx^2dy}{ds^2}.$$

Hence the sum of all the horizontal forces which the arc *AM* experiences

$$= -v\int \frac{dx^3}{ds^2},$$

and the sum of all the vertical forces

$$= +v\int \frac{dx^2dy}{ds^2};$$

thus with these integrals taken so that they vanish on making s or $y=0$. Whereby if there may be put $x=0$, then the forces will be produced which the whole curve *AMB* experiences under the water. Moreover the direction of the total horizontal force

$$=-v\int\frac{dx^3}{ds^2}$$

shall be along *OH*; the direction of the total vertical force

$$=+v\int\frac{dx^2dy}{ds^2}$$

shall be along *OI*; with the moments summed with respect to the point *C*,

$$-CH\cdot v\int\frac{dx^3}{ds^2}=-v\int\frac{xdx^3}{ds^2};$$

and

$$CI\cdot v\int\frac{dx^2dy}{ds^2}=v\int\frac{ydx^3dy}{ds^2}.$$

Hence therefore there will be obtained:

$$CH=\frac{\int xdx^3:ds^2}{\int dx^3:ds^2}\text{ and }CI=\frac{\int ydx^2dy:ds^2}{\int dx^3dy:ds^2};$$

thus with all the integrals taken so that they may vanish on putting s or $y=0$ and then on making $x=0$. Therefore the effect of the total resistance consists in this, so that the figure may be forced back in the direction of the horizontal line *OH* by a force

$$=-v\int\frac{dx^3}{ds^2},$$

and likewise again it shall be forced in the direction *OI* by the force

$$=v\int\frac{dx^2dy}{ds^2}.$$

Therefore the direction of the whole resistance lies on *OK* with there being

$$IK:OI=-\int\frac{dx^3}{ds^2}:\int\frac{dx^2dy}{ds^2},$$

from which there will become

$$IK = \frac{-OI\int dx^3 : ds^2}{\int dx^2 dy : ds^2} = \frac{-\int x dx^3 : ds^2}{\int dx^2 dy : ds^2};$$

and thus

$$CK = \frac{\int (xdx + ydy) dx^2 : ds^2}{\int dx^2 dy : ds^2}.$$

Truly the tangent of the angle *OKB* will be

$$= \frac{\int dx^2 dy : ds^2}{-\int dx^3 : ds^2},$$

from which the position of the mean direction of the resistance *OK* is known. Truly the resistive force itself will be

$$= v\sqrt{\left(\int \frac{dx^3}{ds^2}\right)^2 + \left(\int \frac{dx^2 dy}{ds^2}\right)^2}.$$

Q.E.I.

COROLLARY 1

544. Therefore the resistance in the figure *BCA* will produce at twofold effect, of which the one consists in a slowing down of the motion of the figure, and arises from the horizontal force $= -v\int \frac{dx^3}{ds^2}$, of which the direction is *OH.*

COROLLARY 2

545. Moreover the other force arises from the resistance $= v\int \frac{dx^2 dy}{ds^2}$, the direction of which does not affect the vertical motion of the figure along *OI*, but raised that from the water as if it were made lighter.

COROLLARY 3

546. Therefore unless the vertical force $v\int \frac{dx^2 dy}{ds^2}$ may vanish or may become, while the figure may be moved or will emerge more from the water, and likewise if it may be made

lighter, and therefore it will be raised more from the water, so that it may be moved faster in the water; clearly the decrease of the weight is as the square of the speed.

COROLLARY 4

547. But unless there shall be everywhere either $dy = 0$, which happens when the line *BMA* either will become a vertical right line, or *dy* may become negative somewhere, the force will always maintain a positive value with that same vertical figure rising from the water.

COROLLARY 5

548. Then this vertical force, since its direction lies on the prow, the figure itself thus will be inclined, so that the prow may be raised; truly the stern may be lowered, unless the horizontal force *OH* may be placed deeper than the centre of gravity, and thus may bring about the opposite effect.

COROLLARY 6

549. Moreover the horizontal force *OH,* by which the motion of the figure may be retarded, there will be smaller, where the figure were more pointed towards *B*. And if amongst all the figures enclosing the same area *BAG* that one may be sought, which will be retarded minimally by the water, that figure itself will be found, which has been found in the previous proposition. Likewise indeed the resistance itself will be had whether the figure *BMA* may be put in place to be moved horizontally or vertically.

COROLLARY 7

550. Moreover the figure will be raised maximally from the water, or the vertical force *OI* will be a maximum, if the curved line *BMA* may become straight, when the angle to the horizontal *BC* may be set up as $54^{\circ}, 44'$, or its cosine is $\frac{1}{\sqrt{3}}$.

SCHOLIUM 1

551. This proposition serves mainly for defining the resistance, which the keel of a ship endures perpetually while progressing through the water, indeed from which not only is it understood how much the motion of the ship is retarded by the resistance of the keel, but also how much the ship itself will be raised as if rendered lighter by the resistance of the water. Moreover if in addition the body moving in the water were prepared thus, so that all the vertical sections in the direction of the motion shall be similar and equal to each other, then from this proposition the resistance also can be found, thus so that if a cylinder of water lying horizontally were moving, so that its axis shall be normal to the direction of the motion then the curve *AMB* will be the arc of a circle, and hence the resistance will become known. Then truly this same proposition will have a great use in the following, where we are going to investigate, how great a resistance a plane horizontal figure may experience, which is moving

along at an oblique angle, for in this case the resistance of each of the half figures is required to be investigated, and from each mean direction the whole resistance is going to be concluded. For in our case likewise the resistance itself will be had whether the figure *BMAC* in place shall be progressing in the water either vertically or horizontally.

EXAMPLE 1

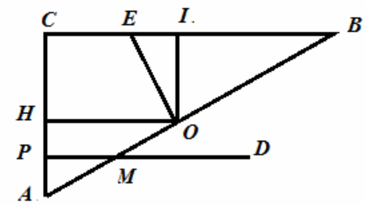


Fig. 83

552. *BAC* (Fig. 83) shall be a plane triangle figure which is moving in water along the direction *MD* with a speed corresponding to the height v ; of which any resistance may be determined easily from proposition 50, moreover we are going to investigate this case with the aid of the illustrious formula found here. Thus there shall become

$BC = a, AC = b,$ on account of $CP = x,\ PM = y$ there will become $b - x : y = b : a$ and there will become

$$y = a - \frac{ax}{b},\ dy = -\frac{adx}{b} \text{ and } ds = \frac{dx\sqrt{a^2+b^2}}{b}.$$

Hence the force of the horizontal resistance acting along the direction *OH* will become

$$= -v\int \frac{dx^3}{ds^2} = -\frac{b^2 v}{a^2+b^2}\int dx,$$

which integrated thus, so that it shall vanish on putting $y = 0$ or $x = b$ will become $= \frac{b^2 v(b-x)}{a^2+b^2}$, therefore on putting $x = 0$, the total horizontal force $= \frac{b^2 v}{a^2+b^2}$. Truly the vertical force or the direction of which along *OI* is:

$$= v\int \frac{dx^2 dy}{ds^2} = \frac{-abv}{a^2+b^2}\int dx = \frac{abv(b-x)}{a^2+b^2},$$

from which the whole vertical force produced $= \frac{ab^2 v}{a^2+b^2}$. Truly for the position of the lines *OH* and *OI* requiring to be found these integrals now are known :

$$\int \frac{dx^3}{ds^2} = \frac{-b^3}{a^2+b^2} \quad \text{and} \quad \int \frac{dx^2 dy}{ds^2} = \frac{ab^2}{a^2+b^2},$$

on account of which these are required to be known:

$$\int \frac{xdx^3}{ds^2} \text{ and } \int \frac{ydx^2dy}{ds^2}.$$

Truly there is:

$$\int \frac{xdx^3}{ds^2} = \frac{b^2}{a^2+b^2}\int xdx = \frac{-b^4}{2\left(a^2+b^2\right)}$$

and

$$\int \frac{ydx^2dy}{ds^2} = \frac{-a^2}{a^2+b^2}\int \left(b-x\right)dx = \frac{aabb}{2\left(a^2+b^2\right)};$$

with the integration used understood thus to be absolute, so that the integrals may vanish on putting $x=b$ and then on making $x=0$. Hence therefore there will become

$$CH = \frac{b}{2} = \tfrac{1}{2}AC \text{ and } \mathrm{CI} = \frac{a}{2} = \tfrac{1}{2}BC;$$

therefore the point *O* falls on the middle of the line *AB.* Moreover since with *OK* put for the mean direction of the resistance , there becomes

$$IK : OI = b : a = AC : BC,$$

from which it is seen by analogy the mean direction of the resistance *OK* to be normal to the right line *AB;* finally the strength of the resistance *OK* itself is $= \dfrac{bbv}{\sqrt{\left(a^2+b^2\right)}}$; which indeed all follow at once from proposition 50.

COROLLARIUM

553. Therefore since there shall be $IK : OI = b : a$ there will become

$$IK = \frac{bb}{2a} \text{ and } CK = \frac{aa-bb}{2a};$$

truly the angle which the mean direction of the resistance *OK* makes with the horizontal *BC* is $=$ ang. *CAB* , or of which the tangent is $= \dfrac{a}{b}$.

EXAMPLE 2

554. The figure *BCA* (Fig. 84) shall be half the segment from the circle or of the circular arc *BMA*, of which the tangent at *A* shall be horizontal, and of which therefore the centre falls at the point E of the vertical right line *AGE*.
There may be put, $BC = a$, $AC = b$ and the radius $AE = c$, there will become

$$c^2 = a^2 + (c-b)^2 \quad \text{or} \quad c = \frac{aa+bb}{2b}.$$

Now put

$CP = x$, and $PM = y$, there will become $EP = c - b + x$,
and hence from the nature of the circle

$$c^2 = y^2 + (c-b+x)^2 \text{ or } y = \sqrt{c^2 - (c-b+x)^2}$$

from which there becomes $y = 0$, if there becomes $x = b$. Moreover again there will become

$$dy = \frac{-(c-b+x)dx}{\sqrt{c^2-(c-b+x)^2}} \quad \text{and} \quad ds = \frac{cdx}{\sqrt{c^2-(c-b+x)^2}};$$

and hence

$$\frac{dx^2}{ds^2} = 1 - \frac{(c-b+x)^2}{cc}.$$

Therefore the following integrals may be sought with this condition that they may vanish on putting $y = 0$ or $x = b$, and with these in place after the integration there may be put $x = 0$ or $y = a$.

Moreover there will be found

$$\int \frac{dx^3}{ds^2} = \frac{-bb(3c-b)}{3cc}, \quad \int \frac{dx^2dy}{ds^2} = \int \frac{yydy}{cc} = \frac{a^3}{3cc},$$

and

$$\int \frac{xdx^3}{ds^2} = \int xdx - \int \frac{xdx(c-b+x)^2}{cc} = \frac{-b^3(4c-b)}{12cc} \text{ and } \int \frac{ydx^2dy}{ds^2} = \frac{a^4}{4cc}.$$

From these there is found:

$$CH = \frac{b(4c-b)}{4(3c-b)} \text{ and } CI = \frac{3a}{4}.$$

Truly besides the horizontal force in the direction *OH*

$$=\frac{bbv(3c-b)}{3cc}$$

and the vertical force in the direction *OI*

$$=\frac{a^3v}{3cc},$$

whereby the mean direction of the whole resistance will be *OK,* with there being

$$IK=\frac{b^3\left(4c-b\right)}{4a^3} \quad \text{or} \quad CK=\frac{3a^3-4bc+b^4}{4a^3},$$

and the tangent of the angle *OKI* will become $=\frac{a^3}{bb\left(3c\ -b\right)}$. Therefore there will become the equivalent force of the whole resistance, acting in the direction *OK,* which is

$$=\frac{v}{3cc}\sqrt{a^6+b^4\left(3c-b\right)^2}.$$

COROLLARY 1

555. Since there shall be $c=\frac{aa+bb}{2b}$, there will become $aa=2bc-bb$ and thus

$$CK=\frac{12bbcc-16b^3c+4b^4}{4a^3}=\frac{bb(c-b)\left(3c-b\right)}{a^3}.$$

Therefore if *OK* may be produced, then that will concur with *AC* in the centre of the circle *E* itself, indeed there becomes :

$$\frac{CE}{CK}=\frac{a^3}{bb(3c-b)}=\text{tang. ang.}OKI=\frac{OI}{KI}.$$

COROLLARY 2

556. Therefore the mean direction *OE* of the resistance passes through the centre of the circle *E* itself, and since it will constitute the angle *AEO* with the right line *AE*, its tangent will be

$$=\frac{bb(3c-b)}{a^3}=\frac{(3c-b)\sqrt{b}}{(2c-b)\sqrt{(2c-b)}}, \quad \text{truly the sine } =\frac{(3c-b)\sqrt{b}}{c\sqrt{(8c-3b)}}.$$

COROLLARY 3

557. Truly the magnitude of the whole resistance, which itself acts in the direction *OE* is

$$=\frac{v}{3cc}\sqrt{8b^3c^3-3b^4cc}=\frac{bv}{3c}\sqrt{b(8c-3b)};$$

clearly is equal to the weight of the cylinder of water the height of which is *v,* of which the base truly

$$=\frac{b}{3c}\sqrt{b(8c-3b)},$$

multiplied by the width or thickness of the figure, if it has any such a dimension.

COROLLARY 4

558. Since the direction of the resistance, which the individual elements endure, is normal to the curve, that will pass through the centre *E*, from which at once it follows the mean direction of the resistance which is experienced by the arc *BMA*, must pass through the centre *E*.

COROLLARY 5

559. If the arc *AMB* may be equal to the quadrant, there will become $a=b=c$, and the tangent of the angle *AEO* will be $=2$; truly the magnitude of the equivalent resistive force will be $=\frac{c\sqrt{5}}{3}$.

COROLLARY 6

560. But if the arc *AMB* may be changed into a semicircle so that there may become $b=2c$, the angle *AEO* will become right or the direction of the mean resistance will be horizontal, moreover the strength of the total resistance will be produced $=\frac{4c}{3}$, provided now as before it will be allowed to consider §509.

SCHOLIUM 2

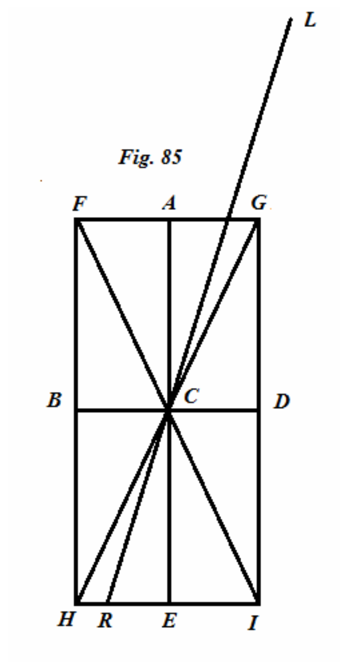
561. Therefore since for a figure suitable equipped with two similar and equal parts, if it may be moved horizontally along the direction of the diameter, as we will have determined the resistance for a plane figure placed vertically in water, we will return to the establishment of plane figures placed floating horizontally in water, and we will also define the resistance, when not only may it be moving directly along the diameter but also obliquely. For this investigation

is much more difficult than the preceding, since the resistance, which the figure experiences from the unequal interactions with the water from each part of the diameter shall be dissimilar; and on that account both the mean direction of the resistance as well as the magnitude of the resistance will be required to be determined. For it is easily understood in oblique motion of this kind the mean direction of the resistance does not coincide with the diameter, but to cut the diameter somewhere, and from that some certain angle to be established; therefore we will call that point the centre of resistance in which the diameter and the mean direction of the resistance intersect each other the closest ; certainly the effect of this resistance is known by necessity to be the greatest, in some figure requiring to be turned about the vertical axis. Moreover we will begin this treatment with the simplest figures, and indeed initially we will consider rectangular parallelogram, whereby it is understood, both how much the mean direction of the resistance as well as that resistance itself for the various oblique courses will be changed.

PROPOSITION 55

PROBLEM

562. *If the right angled parallelogram FGIH* (Fig. 85) *may be moving along some oblique direction CL in water, to find both the direction and magnitude of the resistance which it will experience.*



SOLUTION

The width of the rectangle *FGIH* shall be $FG = HI = a$, with the length $FH = GI = b$, and likewise the normal *AE* may be drawn through the centre of the figure *C* with the normal *BD* moving across, so that there shall become $AC = \frac{1}{2}b$, and $BC = CD = \frac{1}{2}a$. Moreover the sine of the oblique angle *ACL* of the direction of the course shall be $= m$, truly the cosine $= n$, with the whole sine put $= 1$, thus so that there shall become $m^2 + n^2 = 1$; but the speed with which the rectangle is progressing in the direction *CL*, shall correspond to the height *v*. Now while this figure may be advancing, the two sides *FG* and *GI* will be exposed to resistance, and the sine of the angle with which the side *FG* may strike the water will be $= n$; truly that of the angle by which the side *GI* intrudes into the water is $= m$. Therefore the resistance which the side *FG* will endure will be $= n^2av$, and its direction will fall on the axis *ACE;* truly the resistance which the side *GI* endures will be $= m^2bv$, and its direction will be along the right line *DB*. Therefore the resistance will be equivalent to the two forces applied at the point *C* of which the one is n^2av and has the direction *CE* ; the other truly is m^2bv in the direction *CB*. Therefore the mean resistance lies in the direction of the right line *CR* with the tangent of the angle $RCE = \frac{m^2 b}{n^2 a}$, and the magnitude of its resistance acting in the direction *CR* is $v\sqrt{n^4 a^2 + m^4 b^2}$. Q.E.I.

COROLLARY 1

563. Therefore in whatever direction the rectangular parallelogram may be progressing, the mean direction always will pass through its midpoint *C,* or the centre of the resistance falls on the centre of the figure *C.*

COROLLARIUM 2

564. Therefore if likewise the centre of gravity of the rectangle shall lie at the centre of the figure *C*, then the resistance will be without any force requiring to rotate the figure about the centre of gravity, and the whole resistance is expended towards altering the motion itself.

COROLLARIUM 3

565. If the tangent of the angle *LCA* may be put $= v$ there will become $\frac{m}{n} = v$, [Recall that v is normally reserved for the height fallen from rest to generate the square of the speed.] and the tangent of the angle *RCE* will become $\frac{v^2 b}{a}$. On account of which the direction of the resistance will be directly opposite to the motion itself, if there were either $v = 0$, that is if the figure shall be progressing along the direction *CA*, or $v = \frac{a}{b}$, that is if the figure shall be progressing along the diagonal *HCG*.

COROLLARY 4

566. The angle *ACL* shall be $< ACG$ or $v = \frac{a}{\alpha b}$, with α denoting a number greater than unity, the tangent of the angle *ECR* $v = \frac{a}{\alpha^2 b}$, from which it follows the angle *ECR* to become smaller than the angle *LCA*. Truly on the other hand, if the angle $ACL > ACG$, then the angle *ECR* also will be greater than the angle *ACL.*

COROLLARIUM 5

567. Moreover the tangent of the difference of the angles *ACL* and *ECR* $=\frac{v^2b-va}{a+v^3b}$, from which the difference of the angles will become a maximum if *v* may be taken from this equation

$$bbv^4-2abv^3-2abv+aa=0.$$

SCHOLIUM

568. The roots of this equation $b^2v^4-2abv^3-2abv+a^2=0,$ can be found in the following manner, where the common biquadratic equations are accustomed to be reduced to cubics, but here it happens to be more convenient, that a pure cubic equation may be produced. Indeed if $a=kb$, there will be had $v^4-2kv^3-2kv+kk=0$, of which the factors, these equations $v^2-\alpha v+\beta=0$ and $v^2-\delta v+\varepsilon=0;$ may be put in place and there will become :

$$\alpha+\delta=2k,\ \beta+\varepsilon+\alpha\delta=0,\ \alpha\varepsilon+\beta\delta=2k \text{ and } \beta\varepsilon=k^2.$$

Let $\alpha\delta=2h;$ there will become

$$\alpha-\delta=2\sqrt{k^2-2h},$$

and

$$\alpha=k+2\sqrt{k^2-2h} \text{ et } \delta=k-2\sqrt{k^2-2h}.$$

But since again there becomes

$$\beta+\varepsilon=-2h,\ \text{and } \beta\varepsilon=k^2,$$

will become

$$\beta-\varepsilon=2\sqrt{h^2-k^2},\ \text{et } \beta=-h+\sqrt{h^2-k^2} \text{ ac } \varepsilon=-h-\sqrt{h^2-k^2}.$$

Since finally there shall be $\alpha\varepsilon+\beta\delta=2k$, there will become $k+kh=-\sqrt{\left(k^2-2h\right)\left(h^2-k^2\right)}$, from which from the assumed squares there becomes

$$k^2=-k^4-2h^3,\ \text{ or } h=\frac{\sqrt[3]{k^2\left(k^2+1\right)}}{\sqrt[3]{2}}=-\frac{\sqrt[3]{4k^2\left(k^2+1\right)}}{2}.$$

Moreover from the given *h*, α, β, δ, and ε, will be given from the above equations, and thence there will be either

$$v=\frac{\alpha}{2}\pm\sqrt{\left(\frac{\alpha^2}{4}-\beta\right)}\ \text{ or }\ v=\frac{\delta}{2}\pm\sqrt{\left(\frac{\delta^2}{4}-\varepsilon\right)}.$$

Moreover, with these substitutions made, there will be found either

$$v=\frac{k+\sqrt{\left(k^2+\sqrt[3]{4k^2\left(k^2+1\right)}\right)}\pm\sqrt{\left(2k^2-\sqrt[3]{4kk\left(k^2+1\right)}+2k\sqrt{\left(k^2+\sqrt[3]{4k^2\left(k^2+1\right)}\right)}\right)-2\sqrt{\left(\sqrt[3]{16k^4\left(k^2+1\right)^2}-4k^2\right)}}}{2}$$

or

$$v=\frac{k-\sqrt{\left(k^2+\sqrt[3]{4k^2\left(k^2+1\right)}\right)}\pm\sqrt{\left(2k^2-\sqrt[3]{4kk\left(k^2+1\right)}-2k\sqrt{\left(k^2+\sqrt[3]{4k^2\left(k^2+1\right)}\right)}\right)-2\sqrt{\left(\sqrt[3]{16k^4\left(k^2+1\right)^2}-4k^2\right)}}}{2},$$

which therefore are the four roots of this biquadratic equation

$$v^4-2kv^3-2kv+kk=0.$$

Therefore on putting $\frac{a}{b}$ in place of k the oblique course will be had from that, since the direction of the resistance agrees minimally with these ; moreover that property of the ship depends on this difference of the signs , by which the course also can be directed into the wind, for which reason also we have agreed that same discrepancy to have escaped attention, and requiring especially to be investigated more carefully.

PROPOSITION 56

PROBLEM

569. *The rhombus ABED* (Fig. 86) *floating horizontally on water shall be moving obliquely along the direction CL yet thus so that only the two anterior lateral sides AB and AD may sustain resistance: to define both the magnitude and direction of the resistance.*

SOLUTION

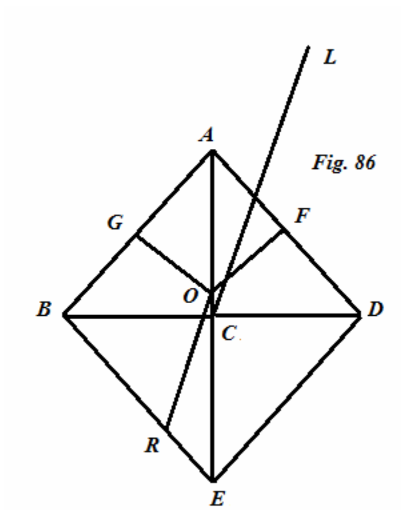


Fig. 86

Any two sides of the rhombus may be put to be $AB = AD = a$, the radius $AC = b$, and half of the other diagonal $BC = CD = c$, so that there shall be $a^2 = b^2 + c^2$. Truly the direction of the motion *CL* shall make the angle *ACL* with the axis *CA* of which the sine shall be $= m$, truly the cosine $= n$ with the whole sine $= 1$; finally the speed with which the rhombus may be moving in this direction shall correspond to the height *v*. Now since the sine of the angle *CAD* shall be $= \frac{c}{a}$ and the cosine $= \frac{b}{a}$; the sine of the angle under which the side *AD* is impinged by the water $= \frac{nc + mb}{a}$, truly the sine of the angle under which the side *AB* is driven against the water $= \frac{nc - mb}{a}$. Hence the resistance, which the side *AD* will experience will become $= \frac{(nc + mb)^2 v}{a}$, and its direction will be the right line *FO*, which *AD* stands on normally at its midpoint *F*. In a similar manner the resistance *AB* will be $= \frac{(nc + mb)^2 v}{a}$, and its direction will be the right line *GO* at the mid point *G* of the right line *AB* normal to *AB*. Therefore the centre of the resistance will be the point *O*, with there being $AO = \frac{aa}{2b}$. Each force of the resistance may be resolved into two sides, which shall be parallel to the diagonals *AE* and *BD*, and the resistance acting along *AE* will become

$$\frac{cv(nc+mb)^2 + cv(nc-mb)^2}{a^2} = \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2};$$

truly the force of the resistance acting along the direction parallel to *DB* itself

$$=\frac{bv(nc+mb)^2-bv(nc-mb)^2}{a^2}=\frac{4mnb^2cv}{a^2}.$$

From these the mean direction of the total resistance *OR* is found, which since the angle *ROE* will be made with the axis *AE* of which the tangent is $=\frac{2mnb^2}{n^2c^2+m^2b^2}$, truly the magnitude of the resistance produced

$$=\frac{2cv}{a^2}\left(n^4c^4+2m^2n^2b^2c^2+m^4b^4+4m^2n^2b^4\right).$$

Q.E.I.

COROLLARY 1

570. Also in this figure it is agreed *O* is the centre of the resistance, in order that the course *CL* may be inclined to the axis *CA*, provided the angle *ACL* shall not exceed the angle *CAD* ; that is provided there shall be $\frac{m}{n}<\frac{c}{b}$.

COROLLARY 2

571. Therefore if the centre of gravity of the figure likewise shall lie at the point *O,* then the resistance will not rotate the figure, but only will affect the progress of its motion, with that either being slowed down or by changing the course.

COROLLARY 3

572. Moreover, the angle *EOR* will be greater than the angle *ACL,* if there were

$$\frac{2mnb^2}{n^2c^2+m^2b^2}>\frac{m}{n}$$

that is, if there were

$$2nb^2-n^2c^2>m^2b^2,\ \text{ seu }\ \frac{m}{n}<\frac{\sqrt{2b^2-c^2}}{b}.$$

But since there is $\frac{m}{n}<\frac{c}{b}$, it is clear, if there were $b>c$ or $AC>CD$ then the angle *EOR* always to be greater than the angle *ACL.*

COROLLARY 4

573. If there may become $\frac{m}{n}=\frac{c}{b}$, in which case only the side *AD* will be exposed to the resistance, than the tangent of the angle *EOR* will be $=\frac{b}{c}$. Evidently in this case the angle *EOR* will be the complement of the angle *ACL* for the right line which indeed is apparent from these at once.

PROPOSITION 57

PROBLEM

574. *If the figure floating on water AE* (Fig. 87) *were composed from the right angled parallelogram HKNM, and from the two equal isosceles triangles HAK and MEN put in place on the two opposite sides HK et MN, and this figure may be moved forwards in the direction CL obliquely to the diameter AE, to find both the magnitude and the direction of the resistance.*

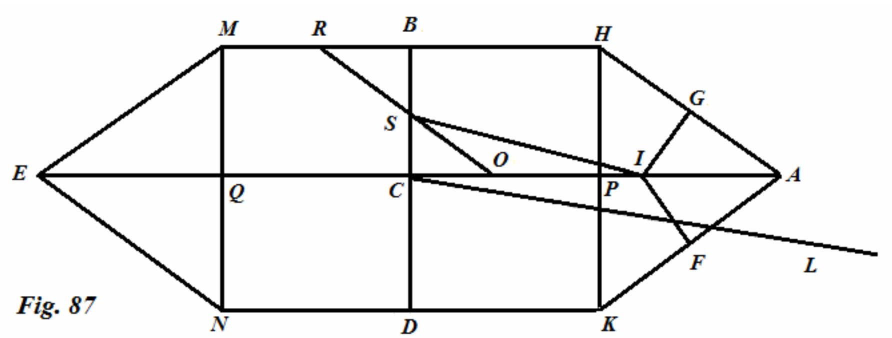


Fig. 87

SOLUTION

Initially in the triangle *HAK,* the side may be put

$$AK=AH=a,\ AP=b,\ HP=PK=c,$$

thus so that there shall become $a^2=b^2+c^2$. Then the length *MH* or *KN* of the rectangle *HMNK* shall be $=2f$, or from the transverse diameter *BD* drawn there shall become $KD=DN=f$; moreover from the obliqueness of the angle *ACL* of the course the sine shall be $=m$, the cosine $=n=\sqrt{1-mm}$; which angle shall be smaller than the angle *CEN,* so that only the three sides *HA, AK* and *KN* shall be exposed to the resistance, finally the speed with which this figure is progressing in the direction *CL* shall correspond to the height *v.* Now in the first place the resistance may be considered, which the sides of the triangle *HA* and *AK* experience, of which the mean direction *IS,* by the preceding proposition, passes through the point *I,* with there being $AI = \frac{aa}{2b}$, and which will constitute the angle *CIS* with the axis *AE*, the tangent of which is $= \frac{2mnb^2}{n^2c^2 + m^2b^2}$, truly from that the force of the resistance will be

$$= \frac{2cv}{a^2}\sqrt{\left(n^2c^2 + m^2b^2\right)^2 + 4m^2n^2b^4};$$

or which will render the same resistance will be equivalent to the two forces applied at *I*, of which the one acts towards *IC* and is $= \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2}$, the other has the direction normal to this, and is $= \frac{4mnb^2cv}{a^2}$. With these established we may inquire into the resistance of the side *KN,* which strikes the water under the angle of which the sine is $= m$, therefore its resistance is $= 2m^2 fv$, the direction of which is normal to *KN* and lies on the same *DB*. Therefore this resistance, if it may be compared with the former, which the sides of the triangle *HA, AK* experience, will present the centre of the resistance at *O* so that there shall become

$$CO : IO = \frac{4mnb^2cv}{a^2} : 2m^2 fv = 2nb^2c : ma^2 f;$$

from which there becomes

$$CI : CO = 2nb^2c + ma^2 f : 2nb^2c.$$

Truly there is

$$CI = f + b - \frac{a^2}{2b} = \frac{2bf + 2bb - aa}{2b};$$

and thus

$$CO = \frac{nbc\left(2bf + 2bb - aa\right)}{2nb^2c + ma^2 f}.$$

Therefore the total resistance is reduced to two forces applied at the point *O*, of which one acts in the direction *OC* and is

$$= \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2},$$

truly the other which will be

$$= 2m^2 fv + \frac{4mnb^2cv}{aa},$$

is normal to that direction. Hence the total resistance in the mean direction is the right line *OR,* which with the keel constitutes the angle *EOR*, of which the tangent is

$$=\frac{m^2a^2f+2mnb^2c}{n^2c^3+m^2b^2c};$$

and the magnitude of the resistance will be

$$=\frac{2v}{a^2}\sqrt{\left(n^2c^3+m^2b^2c\right)^2+\left(m^2a^2f+2mnb^2c\right)^2}.$$

Q.E.I.

COROLLARY 1

575. Therefore in this figure the position of the centre of resistance *O* is not fixed, but depends on the obliquity of the course unless there shall become

$$f+b=\frac{a^2}{2b},$$

in which case it falls on C. For if the angle *ACL* vanishes, then the point *O* lies on the point *I* itself, and so that as the obliquity of the course *ACL* may become greater, there the point *O* approaches closer to *C*.

COROLLARY 2

576. Therefore the figure on an oblique course cannot avoid being rotated in this manner by the resistance about the centre of gravity, unless in that case where *O* falls on *C*. Therefore with this being prevented, there is need for new forces.

COROLLARY 3

577. Moreover with the course maintaining the same angle *ACL* of obliqueness, the angle *EOR* which the mean direction of the resistance makes with the axis or keel *AE*, therefore will be greater, so that the rectangular parallelogram *HMNK* were longer.

COROLLARY 4

578. Therefore the greater the angle *EOR* will exceed the angle *ACL,* so that the greater this magnitude $m^2na^2f+2mn^2b^2c$ surpassed this quantity $mn^2c^3+m^3b^2c$. Therefore the greater is this same excess, whereby the longer will be the middle part of the figure or of the rectangular parallelogram.

SCHOLIUM

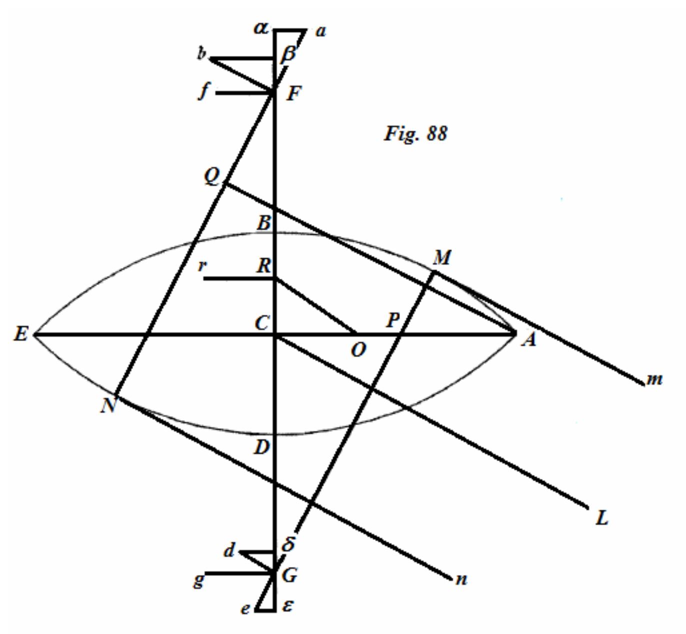
579. From these cases it is seen to be clear enough, to be agreed to consider the evidence, concerning the resistance which any figure experiences advancing obliquely in water. Evidently it shall be required for ships which are propelled by the wind, that the forwards motion shall be from that region from which the wind comes, and it may be able to establish,

how great this same property may be obtained from the difference of the angles, which the direction of forwards motion makes with the mean direction of the resistance of the keel, and it will be enabled to compare how great this same difference would become, were the ship better adapted for the desired outcome. Moreover, it is understood from these properties introduced by the difference of these angles, that the resistance of the sides of the ship thus becomes greater with respect to the resistance which is experienced by the prow moving directly forwards. On account of which, it will be agreed in the first place, ships to be constructed thus, so that the motion along the keel shall be considered to offer the minimum resistance, if the ship may become exceedingly long, and its sides shall be required to have almost a plane shape; then truly so that, if the course may undertake a little obliquity, the resistance will be increased maximally. Therefore on this account also for the case of the prow of the ship, which thus is requiring to be shaped skillfully in the direction of the course, so that only the minimum resistance may be experienced, indeed as great as the remaining circumstances permit. But all these matters will be explained more fully in the following chapter and in the other book. Moreover so that the position of the centre of the resistance may be reached, so that these may be able to be indicated more easily, it has been considered to present the following propositions.

PROPOSITION 58

PROBLEM

580. *If the figure ABED* (Fig. 88) *may depend on two equal and similar circular segments, set out on either side of the common chord AE; and with that figure may be moved in water obliquely along the direction CL, to determine both the direction and magnitude of the resistance.*



SOLUTION

F shall be the centre of the arc *ADE,* and *G* the centre of the arc *ABE,* and there shall be put

$$FD = GB = c,\ AC = EC = a,\ BC = CD = b,$$

there will be

$$FC = GC = c - b,$$

and from the nature of the circle $a^2 - (c-b)^2 = c^2$ from which there shall become $2bc = a^2 + b^2$. Now the sine of the oblique angle *ACL* of the course $= m$, and the cosine $= n$, and the tangents *Mm* and *Nn* shall be drawn parallel to the direction of the course *CL,* and the radii *MG* and *FN,* the angles *MGB, NFD* will be equal to the angle *ACL,* therefore of these the sine $=m$, and the cosine $=n$. And thus since the sine of the arcs *BM* and *DN* shall be $=m$ and the cosines $=n$, truly the sine of the arcs *AB* and *AD* shall be $=\frac{a}{c}$ and the cosine $=\frac{\sqrt{c^2-a^2}}{c}$, truly,

$$\text{the sine of the arc } AM = \frac{an-m\sqrt{c^2-a^2}}{c} \text{ and sim. the cosine } = \frac{am+n\sqrt{c^2-a^2}}{c};$$

truly,

$$\text{sine } ADN = \frac{an+m\sqrt{c^2-a^2}}{c} \text{ and cosine } ADN = \frac{n\sqrt{c^2-a^2}-ma}{c}.$$

Whereby if the perpendicular *APQ* may be drawn from *A* to the radii *GM* and *FN,* which are parallel to each other, there will become

$$AP = na - m\sqrt{c^2-a^2} = na - m(c-b)$$

and

$$GP = ma + n\sqrt{c^2-a^2} = ma + n(c-b).$$

In a similar manner there will become

$$AQ = na + m(c-b), \text{ and } FQ = n(c-b) - ma.$$

But these arcs *AM* and *ADN* are the parts of the same figure, which alone experience the resistance; therefore the speed shall be required to be defined according to the resistance experienced by each arc, by which the figure is progressing corresponding to the height *v*. Moreover the mean direction of the resistance of the arc *AM* passes through the centre of the arc *G,* and is reduced to the two forces *Gd, Ge,* of which that one *Gd* is normal to *GM*, truly this one *Ge* acts in the direction *MG*; moreover from § 554

$$\text{the force } Gd = \frac{MP^2(3MG-MP)v}{3MG^2} \text{ and the force } Ge = \frac{AP^3\cdot v}{3MG^2}.$$

But there is

$$MP = c - ma - n(c-b) \text{ and } MG = c;$$

and thus

$$3MG - MP = 2c + ma + n(c-b)$$

with their being

$$AP = na - m(c-b).$$

In a similar manner the resistance, which acts on the arc *ADN,* is reduced to the two forces *Fb* and *Fa* similarly applied at the point *F*, so that *bF* shall be perpendicular to *NF* and *a* shall be lace on the right line *NF* produced; and the forces will become

$$Fb = \frac{NQ^2\left(3NF - NQ\right)v}{3NF^2} \text{ and } Fa = \frac{AQ^3 \cdot v}{3NF^2},$$

with there being

$$NQ = c + ma - n(c - b),$$

and

$$NF = c,\ 3NF - NQ = 2c - ma + n\left(c - b\right),$$

and

$$AQ = na + m(c - b).$$

These forces may be resolved into forces in pairs, of which the first of each pair shall lie on *FG*, the second of each shall be normal to *FG*, which is done easily, while the sine of the angle $eG\varepsilon$ shall be $= m,$ and the cosine of the angle $aF\alpha$ shall be $= n.$ Truly the force is found

$$Gg = \frac{nv \cdot MP^2(3MG - MP) + mv \cdot AP^3}{3MG^2}$$

and the force

$$G\varepsilon = \frac{nv \cdot AP^3 - mv \cdot MP^2(3MG - MP)}{3MG^2}.$$

In the same manner, the force

$$Ff = \frac{nv \cdot MQ^2(3NF - NQ) - mv \cdot AQ^3}{3MG^2}$$

and the force

$$F\alpha = \frac{nv \cdot AQ^3 + mv \cdot NQ^2(3NF - NQ)}{3MG^2}$$

There may be put

and

$$MP = c - ma - nf,\ NQ = c + ma - nf,$$

$$3MG - MP = 2c + ma + nf,\ 3NF - NQ = 2c - ma + nf,$$

and

$$AP = na - mf, \text{ and } AQ = na + mf.$$

Hence the force will become

$$Gg=\frac{v}{3cc}\left(2nc^3-3n^2c^2f-2mna^3+\left(nn-mm\right)f^3\right)$$

and the force

$$G\varepsilon=\frac{v}{3cc}\left(\left(n^2-m^2\right)a^3+2mnf^3-2mc^3+3m^2ac^2\right).$$

And similarly the force

$$Ff=\frac{v}{3cc}\left(2nc^3-3n^2c^2f+2mna^3+\left(nn-mm\right)f^3\right)$$

and the force

$$F\alpha=\frac{v}{3cc}\left(\left(n^2-m^2\right)a^3-2mnf^3+2mc^3+3m^2ac^2\right).$$

Now the mean direction *OR* of the whole resistance will be

$$CR=\frac{2mna^3f}{2nc^3-3n^2c^2f+\left(nn-mm\right)f^3};$$

and the two forces will be equivalent to the forces *Rr* and *RB* applied at the point *R*, and the forces will become

$$Rr=\frac{v}{3cc}\left(2nc^3-3nnc^2f+2mna^3+\left(nn-mm\right)f^3\right)$$

and

$$RB=\frac{4mv}{3cc}\left(c^3-nf^3\right),$$

from which there will become

$$CO=\frac{na^3f}{c^3-nf^3}.$$

Therefore the tangent of the angle *ROC* which the mean direction of the resistance makes with the axis *AE* ,will be

$$=\frac{2mc^3-2mnf^3}{2nc^3-3n^2c^2f+\left(nn-mm\right)f^3},$$

and the magnitude of this resistance will be

$$\frac{2v}{3cc}\sqrt{4c^6-12n^3c^5f+4n\left(nn-3mm\right)c^3f^3+9n^4c^4ff-6nn\left(nn-mm\right)c^2f^4+f^6}.$$

Q.E.I.

COROLLARY 1

581. Therefore the position of the centre of the resistance *O* is variable, and will depend on the obliquity of the course or the angle *ACL.* Whereby the greater the angle *ACL* shall become*,* thus the closer the point *O* will approach to *C*.

COROLLARY 2

582. If the angle *ACL* shall be infinitely small, the point *O* will be maximally removed from *C* ; indeed the distance will become

$$OC=\frac{a^3 f}{c^3-f^3}=\frac{af(c+f)}{cc+cf+ff}$$

on account of $aa=cc-ff$. But if there may become $n=\frac{f}{c}$*,* in which case the point *M* falls on *A*, the distance will become the minimum

$$OC=\frac{a^3 ff}{c^4-f^4}=\frac{aff}{cc+ff}.$$

COROLLARY 3

583. Therefore the interval, through which the centre of the resistance *O* wanders, while the point *M* is moved from *B* as far as to *A*, is

$$=\frac{af(c+f)}{cc+cf+ff}-\frac{aff}{cc+ff}=\frac{ac^3 f}{(cc+ff)(cc+cf+ff)}=a\left(\frac{f}{c}-\frac{ff}{cc}-\frac{f^3}{c^3}+\frac{2f^4}{c^4}\right)$$

approximately ; therefore less than $\frac{af}{c}$.

COROLLARY 4

584. If the segments *ABE* and *ADE* may be changed into semicircles, then there will become $f=0$, therefore in this case the centre of resistance *O* falls on the same point *C*. Whereby the greater were *I,* that is where these segments were smaller, thus the more distant will be the centre of resistance *O* from *C*.

COROLLARY 5

585. So that the difference of the angles *COR* and *ACL* may be perceived more distinctly, we may put the angle *ACL* to be infinitely small, in which case there shall become $m =$ infinitely small number and $n = 1$, and the tangent of the angle *ACL* $= m$. Therefore the tangent of the angle *COR* will be

$$= \frac{2m(c^3 - f^3)}{2c^3 - 3ccf + f^3} = \frac{2m(c^2 + cf + ff)}{2c^2 - cf - ff} = \frac{2m(c^2 + cf + f^2)}{\left(c - f\right)(2c + f)};$$

from which the angle *ACL* itself will be had to the angle *COR* as $2cc - cf - ff$ to $2cc + 2cf + 2ff$.

COROLLARY 6

586. Therefore if the obliquity of the course or the angle *ACL* were exceedingly small, then the angle *COR* will be greater than the angle *ACL,* unless there shall be $f = 0$, in which case the figure will be changed into the whole circle. Indeed if the figure is a whole circle the angles *ACL* and *COR* are equal always, and the points *O* and *C* coincide.

COROLLARY 7

587. If the obliquity may become greatest or the arc *AM* may vanish, so that only the arc of the resistance *ADE* shall be set out, then there will become

$$m = \frac{a}{c} \text{ and } n = \frac{f}{c},$$

and the tangent of the angle *COR* will be

$$= \frac{a(c^4 - f^4)}{f(cc - ff)^2} = \frac{a(cc + ff)}{f(cc - ff)}.$$

Therefore the tangent of the angle *ACL* will itself be had to the tangent of the angle *COR* as $cc - ff$ to $cc + ff$.

COROLLARY 8

588. From these it is understood where the greater were f with respect to c, or where the segments *ABE* and *ADE* shall be smaller, therefore any greater obliquity the angle *COR* to exceed the angle *ACL.*

COROLLARY 9

589. If the angle *ACL* vanished, then on account of $m = 0$ and $n = 1$, the force of the total resistance produced

$$= \frac{2v(2c^3 - 3ccf + f^3)}{3cc} = \frac{2v(c-f)(2c^2 - cf - ff)}{3cc} = \frac{2v(c-f)^2\left(2c + f\right)}{3cc},$$

but if the obliquity may become a maximum or

$$m = \frac{a}{c} \text{ and } n = \frac{f}{c}$$

then the total resistance produced $= \frac{4a^3v\sqrt{cc+3ff}}{3c^3}$.

SCHOLIUM

590. Thus here I have considered chiefly the figure composed from the two circular segments, which shall be suitable enough for the examination of the resistance of ships. Indeed whatever horizontal sections of ships may not be in complete agreement with that same figure, yet if the preceding cases likewise may be taken into consideration, it will be able to assign both the position of the centre of the resistance as well as the mean direction of the resistance without difficulty for any oblique course. Indeed it has been shown well enough, where the greater figure was cone-shaped, there the centre of the resistance to be situated towards the prow with the remaining parts. Also the celebrated Johan Bernoulli has subjected this same figure to an examination in the tract entitled : *Manoeuvre des Vaisseaux*, and has investigated in a singular manner in place for the centre of resistance, thus only for the case where the obliquity of the course is a minimum, or the angle *ACL* is infinitely small; moreover in this case it is agreed the centre of resistance to be put in place at that point, where the mean direction of the resistance which the arc *AB* or *AD* alone experiences intersects the axis *AE* in the direction of the course. But this same point does not agree with our point *O,* when the angle *ACL* vanishes. For following the Bernoulli method the interval is found

$$CO = \frac{af(2c+f)}{(c+f)^2},$$

since actually there shall become

$$CO = \frac{af(c+f)}{cc+cf+ff}.$$

From which it is understood the centre of resistance, when the obliquity of the course is very small, cannot be defined from the resistance which each part of the curve experiences in the direction of the course, but actually it is required to introduce the course of the obliquity into the consideration, just as has been done by us in this proposition. But if other figures besides the circular ones were proposed, then the resistance in the oblique course scarcely neither can nor cannot be determined, on account of the exceedingly complex nature of the calculation; on account of which I have refrained from being led into investigations of this kind. Yet I will attempt in this case only, to define the position of the centre of the resistance and the mean direction, where the course differs minimally from the direct course, certainly which case may be subjected to be investigated more easily, and it can be freed in a certain way from a tedious calculation.

PROPOSITION 59

PROBLEM

591. *If the figure floating on water may be composed from the two parts AMBE and ANDE with equal and similar parts put in place on each side of the axis AE* (Fig. 89), *and that may be moved in the direction CL which shall constitute the infinitely small angle ACL with the axis AC; to determine the mean direction of the resistance OR, and the magnitude of the same resistance.*

SOLUTION

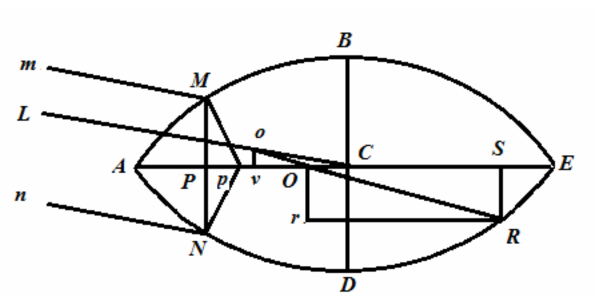


Fig. 89

Since the obliquity of the course is put to be infinitely small and the same part of each figure *AMB* and *AND* will experience the same resistance, which if the course were direct, it would be the exposition of the resistance; since not only must these parts become infinitely small while the arc *AMB* be increased, while the arc *AND* be diminished, but also they are thrust upon the water at an infinitely small angle, thus so that with care it may be allowed to ignore the resistance of these. Therefore with the ordinate *MPN* drawn, there shall be $AP = x$, $PM = PN = y$, and the arc $AM = AN = s$, truly the sine of the angle *ACL* may be put r $= m$, and the cosine $= n$, m will become infinitely small and therefore $n = 1$, moreover the speed with which the figure is progressing shall be due to the height *v*. Now the lines *mM* and *nN* may be drawn parallel to *LO* itself, which will represent the direction, by which the points *M* and *N* may be driven against in the water; moreover the sine of the angle *AMm* will be

$$= \frac{ndy + mdx}{ds} = \frac{dy + mdx}{ds},$$

but the sine of the angle *ANn*

$$=\frac{ndy-mdx}{ds}=\frac{dy-mdx}{ds}.$$

Therefore the resistance, which the element *ds* experiences at *M* will become

$$=\frac{v(dy^2+2mdxdy)}{ds},$$

the direction of which will be normal to the curve *Mp.* Truly the resistance which the element *ds* will experience at N will become

$$=\frac{v(dy^2-2mdxdy)}{ds},$$

in the direction of the normal *Np.* Therefore the element *ds* at *M* will be urged in the direction *MP* by the force

$$=\frac{vdxdy(dy+2mdx)}{ds^2},$$

moreover, in the direction parallel to the axis *AC* by the force

$$=\frac{vdy^2(dy+2mdx)}{ds^2}$$

In the same way the element *ds* at *N* will be acted on in the direction *NP* by the force

$$=\frac{vdxdy^2(dy-2mdx)}{ds^2};$$

and in a direction parallel to the axis *AC* by the force

$$=\frac{vdy^2(dy-2mdx)}{ds^2}$$

Therefore the sum of the forces by which both the elements jointly will be acted on in the direction *AC* is $=\frac{2vdy^3}{ds^2}$; but the excess, by which they are disturbed in the direction *MN,* $=\frac{4mvdx^2dy}{ds^2}$. Now *oO* shall be the mean direction of the resistance, and with the perpendicular *ov* drawn from *o* to *AC,* with the integrals taken as far as to *B* and *D*

$$ov = \frac{4mv\int \frac{ydy^2dx}{ds^2}}{2v\int \frac{dy^3}{ds^2}} = \frac{2m\int ydy^2dx : ds^2}{\int dy^3 : ds^2}$$

and

$$Av = \frac{4mv\int \frac{xdx^2dy}{ds^2}}{4mv\int \frac{dx^2dy}{ds^2}} = \frac{\int xdx^2dy : ds^2}{\int dx^2dy : ds^2}.$$

Again, if *oO* is the mean direction of the resistance, there will be

$$ov : vO = 2m\int \frac{dx^2dy}{ds^2} : \int \frac{dy^3}{ds^2},$$

from which there becomes

$$vO = \frac{\int ydy^2dx : ds^2 :}{\int dx^2dy : ds^2,}$$

and

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{\int dx^2dy : ds^2}$$

which expression determines the position of the centre of the resistance *O*. Therefore the total resistance is reduced to two forces acting at the point *O*, of which one is $= 2v\int \frac{dy^3}{ds^2}$, acting in the direction *Ov,* the other truly is $= 4mv\int \frac{dx^2dy}{ds^2}$, of which the direction is *ov* normal to *AE*. Finally from these, the mean direction *OR* will make the angle *EOR* with the axis *AE,* of which the tangent is $= \frac{2m\int dx^2dy : ds^2}{\int dy^3 : ds^2}$.

Q.E.I.

COROLLARY 1

592. Hence it is clear the locus of the centre of the resistance *O* generally to be different from that, which was found before indicated in the following manner (§ 590), indeed through that there is produced

$$AO = \frac{\int (xdx + ydy)dy^2 : ds^2}{\int dy^2dx : ds^2},$$

since however there shall be actually

$$AO = \frac{\int (xdx + ydy)dy^2 : ds^2}{\int dx^2 dy : ds^2}.$$

COROLLARY 2

593. Therefore the equally small angle *ROE*, and the ratio it will have to the angle *ACL* to be held as :

$$2\int \frac{dx^2 dy}{ds} \text{ to } \int \frac{dy^3}{ds^2},$$

which ratio therefore will be finite. For infinitely small angles are as the tangents or sines of these.

COROLLARY 3

594. Therefore the force of the resistance, which acts along the direction of the axis *AE* is equal to that resistance, which may be experienced by the same figure if it may be moved in the direction of the course directly along the axis *CA*.

SCHOLIUM 1

595.It is understood at once from the solution, from which condition all the integrals which occur shall be going to be accepted. Evidently in the first place all the integrals are required to be put in place thus, so that all the integrals may vanish on putting either x or $y = 0$. Then for the maximum width of the figure it is required to consider, which if it is *BD,* there must be put $x = AC$ or $y = BC$, since that part of the figure alone experiences the resistance which is situated between the prow *A* and the maximum width of the figure *BD.*

COROLLARY 4

596. Since the resistance along the direction *AE* shall be as $\int \frac{dy^3}{ds^2}$, and the tangent of the angle $ROE = \frac{2m\int dx^2 dy : ds^2}{\int dy^3 : ds^2}$, it is understood, so that there resistance shall be less on being moved directly forwards, but there to be greater the more the angle *ROE* shall be overcoming the angle *ACL.*

COROLLARY 5

597. But the integral $\int \frac{dx^2 dy}{ds^2}$ depends on the integration $\int \frac{dy^3}{ds^2}$: indeed since there shall be $dx^2 + dy^2 = ds^2$ there will become

$$\int \frac{dx^2 dy}{ds^2} + \int \frac{dy^3}{ds^2} = y,$$

and thus

$$\int \frac{dx^2 dy}{ds^2} = y - \int \frac{dy^3}{ds^2},$$

from which the tangent of the angle *EOR*

$$= \frac{2my}{\int dy^3 : ds^2} - 2m = \frac{2m \cdot BC}{\int dy^3 : ds^2} - 2m.$$

COROLLARY 6

598. Therefore among all the figures passing through the points *A* and *B*, that of the given obliquity *ACL* will produce the maximum angle *EOR*, which will experience the minimum resistance in the direct course.

COROLLARY 7

599. Then so that it pertains to the position of the centre of resistance *O*, since there shall be

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{\int dx^2 dy : ds^2}$$

there will become

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{BC - \int dy^3 : ds^2}.$$

So that the resistance of the figure is smaller in the direct course, thus the nearer the centre of the resistance *O* will be placed to the prow *A*, with the numerator remaining

$$\int \frac{(xdx + ydy)dxdy}{ds^2}.$$

EXAMPLE 1

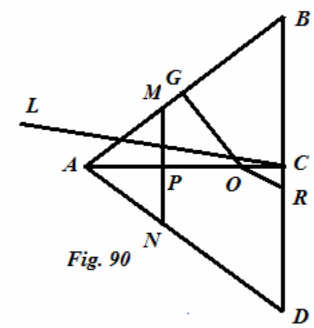


Fig. 90

600. The anterior part of the figure shall be receiving the isosceles triangle resistance *BAD* (Fig. 90), in which there shall be

$$AC = a,\ BC = CD = b,$$

and

$$AB = AD = \ c = \sqrt{a^2 + b^2}.$$

Truly the direction of the course shall be *CL,* and the sine of the angle *ACL* which is infinitely small, shall be $= m$;and the speed due to the height *v*. Now on putting

$$AP = x,\ PM = PN = y,\ \text{there will be}\ y = \frac{bx}{a},$$

and

$$dy = \frac{bdx}{a},\ \text{and}\ ds = \frac{cdx}{a}.$$

Now *O* shall be the centre of the resistance, and *OR* the mean direction of the resistance, there will be on account of

$$\int \frac{dx^2dy}{ds^2} = \int \frac{abdx}{cc} = \frac{a^2b}{cc},\quad \int \frac{dy^3}{ds^2} = \int \frac{b^3dx}{acc} = \frac{b^3}{cc},$$

$$\int \frac{xdx^2dy}{ds^2} = \int \frac{abxdx}{cc} = \frac{a^3b}{2cc},$$

and

$$\int \frac{ydxdy^2}{ds^2} = \int \frac{b^3xdx}{acc} = \frac{ab^3}{2cc},$$

the distance

$$AO = \frac{a^3b + ab^3}{2a^2b} = \frac{cc}{2a},$$

from which it is apparent the centre of the resistance point *O* to lie on the same axis *AC*, at which the right line *GO,* which is normal to *AB* and bisects that, meets the right line *AC* , just as now is agreed from the preceding. Then the tangent of the angle *COR* is $= \frac{2ma^2}{b^2}$, thus so that the angle *ACL* itself may be had to the angle *COR* as b^2 to $2a^2$; therefore just as there will become

$$2a^2 > b^2 \text{ or } \frac{BC}{AC} < \sqrt{2},$$

of if the angle *BAC* were less than 54^o, 45', the whole angle *COR* will exceed the angle *ACL*. Finally the force of the resistance acting in the direction *CO* is $=\frac{2b^3v}{cc}$; and the force which will be acting in the direction of the normal to *OC* will be $=\frac{4ma^2bv}{cc}$.

EXAMPLE 2

601. The figure shall correspond to the two equal and similar segments of circles *ABE, ADE* and there shall be $AC = a$, $BC = CD = b$, and the radius of the circle from which these segments have been taken chosen shall be $= c$. Now for the sake of brevity there may be put $c - b = f$, so that there shall become $cc = a^2 + ff$. Again *CL* shall be the direction of the course and the sine of the angle $ACL = m$, which is put infinitely small, and the height $= v$ must correspond to the speed. Now since there shall be $AP = x$, $PM = PN = y$, from the nature of the circle there shall be

$$x = a - \sqrt{c^2 - (f+y)^2},\ dx = \frac{(f+y)dy}{\sqrt{c^2-(f+y)^2}} \text{ and } ds = \frac{cdy}{\sqrt{c^2-(f+y)^2}},$$

from which the following integrals will be found:

$$\int \frac{dy^3}{ds^2} = \int \frac{ccdy - (f+y)^2 dy}{cc} = b + \frac{f^3}{3cc} - \frac{c}{3} = \frac{2c^3 - 3ccf + f^3}{3cc} = \frac{(c-f)^2(2c+f)}{3cc}$$

and

$$\int \frac{dx^2dy}{ds^2} = c - f - \frac{(c-f)^2(2c+f)}{3cc} = \frac{(c-f)(cc+cf+ff)}{3cc} = \frac{c^3-f^3}{3cc}.$$

From which there becomes

$$\int \frac{xdx^2dy}{ds^2} = \frac{a(c^3-f^3)}{3cc} - \int \frac{(f+y)^2 dy\sqrt{cc-(f+y)^2}}{cc},$$

and

$$\int \frac{ydxdy^2}{ds^2} = \int \frac{(f+y)ydy}{cc}\sqrt{cc-(f+y)^2};$$

from which there will become:

$$\int\frac{(xdx+ydy)dxdy}{ds^2}=\frac{a(c^3-f^3)}{3cc}-\frac{f}{cc}\int(f+y)dy\sqrt{cc-(f+y)^2}$$
$$=\frac{a(c^3-f^3)}{3cc}-\frac{a^3f}{cc}=\frac{ab}{3}.$$

From these there becomes

$$AO=\frac{abcc}{c^3-f^3}=\frac{acc}{cc+cf+ff}$$

and

$$CO=\frac{af(c+f)}{cc+cf+ff}$$

as above (§ 582). Moreover the tangent of the angle *COR,* which the mean direction of the resistance *OR* makes with AC, $=\frac{2m(cc+cf+f)}{(c-f)(2c+f)}$, as above (§585).

EXAMPLE 3

602. The figure of the ellipse *ABED* shall be floating on water (Fig. 91) , of which the semi axis *AC* shall be $=a$, the other $BC=CD=b$, and *CL* shall be the direction of the course, differing from the axis *AC* by an infinitely small amount from the direction*,* thus so that the sine *m* of the angle *ACL* shall be infinitely small ; truly the height corresponding to the speed, by which this figure may be moved forwards, must be *v*. Now with the abscissa put to be $AP=x$, with the applied lines $PM=PN=y$, there will become

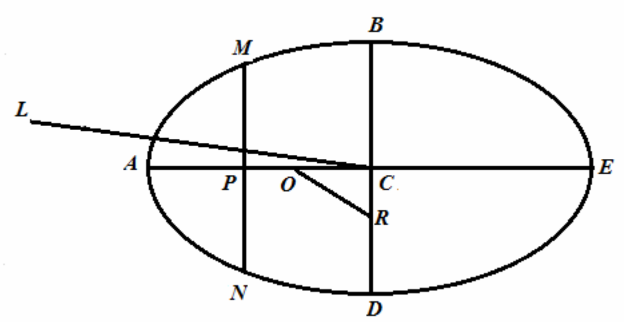


Fig. 91

$$y=\frac{b}{a}\sqrt{2ax-xx},\text{ or } x=a-\frac{a}{b}\sqrt{bb-yy};$$

hence there shall become thus:

$$dx=\frac{aydy}{b\sqrt{bb-yy}}\text{ and } ds^2=\frac{dy^2\left(b^4+\left(a^2-b^2\right)yy\right)}{bb(bb-yy)}.$$

Whereby the integrals, with the aid of which thus there will become :

$$\int\frac{dx^2dy}{ds^2}=\int\frac{a^2y^2dy}{b^4+\left(a^2-b^2\right)yy}=\frac{aab}{aa-bb}-\frac{a^2b^4}{aa-bb}\int\frac{dy}{b^4+\left(a^2-b^2\right)yy}$$

where two cases are required to be considered, for which if there were $a>b$ or $a<b;$ indeed if $a>b$ or $AC>BC$ there will become

$$\int\frac{dx^2dy}{ds^2}=\frac{a^2b}{aa-bb}-\frac{a^2b^2}{\left(aa-bb\right)^{\frac{3}{2}}}\text{Atang}.\frac{\sqrt{aa-bb}}{b},$$

but if $a<b$ there will become

$$\int\frac{dx^2dy}{ds^2}=\frac{a^2b^2}{2\left(bb-aa\right)^{\frac{3}{2}}}l\frac{b+\sqrt{bb-aa}}{b-\sqrt{bb-aa}}-\frac{a^2b}{bb-aa}$$
$$=\frac{a^2b^2}{2\left(bb-aa\right)^{\frac{3}{2}}}l\frac{b+\sqrt{bb-aa}}{a}-\frac{a^2b}{bb-aa}.$$

But since there shall be

$$\int\frac{dy^3}{ds^2}=b-\int\frac{dx^2dy}{ds^2};$$

there will be in that case, where there is $a>b,$

$$\int\frac{dy^3}{ds^2}=\frac{-b^3}{aa-bb}-\frac{a^2b^2}{\left(aa-bb\right)^{\frac{3}{2}}}\text{Atang}.\frac{\sqrt{aa-bb}}{b},$$

truly in the case where there is $a<b$ there will be

$$\int\frac{dy^3}{ds^2}=\frac{b^3}{bb-aa}-\frac{a^2b^2}{\left(aa-bb\right)^{\frac{3}{2}}}l\frac{b+\sqrt{aa-bb}}{a}.$$

From which then, since there shall be

$$\frac{xdx^2dy}{ds^2}=\frac{a^3y^2dy-\frac{a^3}{b}y^2dy\sqrt{bb-yy}}{b^4+\left(aa-bb\right)yy},$$

and

$$\frac{ydxdy^2}{ds^2}=\frac{abyydy\sqrt{bb-yy}}{b^4+(aa-bb)yy},$$

there will become

$$\int\frac{(xdx+ydy)dxdy}{ds^2}=a\int\frac{a^2y^2dy}{b^4+(aa-bb)yy}-\frac{a}{b}\int\frac{(aa-bb)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy}.$$

Truly there becomes:

$$\int\frac{(aa-bb)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy}=\frac{\pi}{4}\cdot\frac{bb(a-b)^2}{(aa-bb)}$$

with $\pi:1$ denoting the ratio of the periphery to the diameter in a circle. On account of which if $a>b$ the integral will become

$$\int\frac{(xdx+ydy)dxdy}{ds^2}=\frac{a^3b}{aa-bb}-\frac{a^3b^2}{(aa-bb)^{\frac{3}{2}}}\text{Atang.}\frac{\sqrt{aa-bb}}{b}-\frac{\pi ab(a-b)^2}{4(aa-bb)},$$

but if $a<b$ there will become

$$\int\frac{(xdx+ydy)dxdy}{ds^2}=\frac{a^3b^2}{(aa-bb)^{\frac{3}{2}}}l\frac{b+\sqrt{bb-aa}}{a}-\frac{a^3b}{bb-aa}+\frac{\pi ab(a-b)^2}{4(aa-bb)}.$$

And thus the centre of the resistance will be at O so that there shall be

$$AO=a-\frac{\pi ab(a-b)^2}{4a^2b-\frac{4a^2b^2}{\sqrt{aa-bb}}\text{Atang.}\frac{\sqrt{aa-bb}}{b}};$$

and thus

$$CO=\frac{\pi(a-b)^2}{4a-\frac{4ab}{\sqrt{a^2-b^2}}\text{Atang.}\frac{\sqrt{a^2-b^2}}{b}},$$

in the case where $a>b$. But in the case where $a<b$, there will become

$$CO=\frac{-\pi\left(a-b\right)^2}{\frac{4ab}{\sqrt{bb-aa}}l\frac{b+\sqrt{bb-aa}}{a}-4a}=\frac{\pi\left(b-a\right)^2\sqrt{bb-aa}}{4abl\frac{b+\sqrt{bb-aa}}{a}-4a\sqrt{bb-aa}}.$$

Finally the tangent of the angle *COR* will be

$$=\frac{2ma^2\sqrt{aa-bb}-2ma^2b\cdot\text{Atang.}\frac{\sqrt{aa-bb}}{b}}{-bb\sqrt{aa-bb}+a^2b\cdot\text{Atang.}\frac{\sqrt{aa-bb}}{b}}=\frac{-2ma^2\sqrt{bb-aa}+2ma^2b\cdot l\frac{b+\sqrt{bb-aa}}{a}}{bb\sqrt{bb-aa}-a^2b\cdot l\frac{b+\sqrt{bb-aa}}{a}},$$

of which expression, that one will prevail if $a>b$, truly this one if $a<b$. Therefore the mean direction of the resistance *OR* becomes known when the elliptic figure proposed is considered to move along the direction *CL*.

COROLLARY 1

603. If the integrations, which depend both on the quadrature of the circle as well as the hyperbola, may be resolved into series, there will become

$$CO=\frac{\pi(a-b)^2}{4a\left(\frac{aa-bb}{3bb}-\frac{\left(aa-bb\right)^2}{5b^4}+\frac{\left(aa-bb\right)^3}{7b^6}-\text{etc.}\right)}$$

and the tangent of the angle *COR*

$$=\frac{2m\left(\frac{aa-bb}{3bb}-\frac{\left(aa-bb\right)^2}{5b^4}+\frac{\left(aa-bb\right)^3}{7b^6}-\text{etc.}\right)}{-\frac{bb}{aa}+1-\frac{\left(aa-bb\right)}{3bb}+\frac{\left(aa-bb\right)^2}{5b^4}-\frac{\left(aa-bb\right)^3}{7b^6}+\text{etc.}}$$

which formulae prevail equally whither there shall be $a>b$ or $a<b$.

COROLLARY 2

604. Since there shall be

$$-\frac{bb}{aa}+1=\frac{aa-bb}{aa},$$

the tangent of the angle *COR* will become

$$=\frac{2m\left(\frac{1}{3bb}-\frac{(aa-bb)}{5b^4}+\frac{(aa-bb)^2}{7b^6}-\text{etc.}\right)}{\frac{1}{aa}-\frac{1}{3bb}+\frac{aa-bb}{5b^4}-\frac{(aa-bb)^2}{7b^6}+\text{etc.}}$$

and in a similar manner the interval will become:

$$CO=\frac{\pi(a-b)}{4a(a+b)\left(\frac{1}{3bb}-\frac{(aa-bb)}{5b^4}+\frac{(aa-bb)^2}{7b^6}-\text{etc.}\right)}$$

COROLLARY 3

605. If the ellipse may approximate to becoming a circle, so that there shall be nearly $b=a$, with there being $b=a-dw$, on account of the terms vanishing there will become $\frac{3\pi dw}{8}=CO$; and in this case also the tangent of the angle COR becomes $=m$, or the angle COR is equal to the angle ACL.

SCHOLIUM 2

606. The integration of the differential formula $\frac{(a^2-b^2)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy}$, which occurs in this example is noteworthy, because there the integral in that case, where there is put $y=b$, is expressed by a finite and simple form, against all expectations. For if an indefinite integral may be desired, then an especially long and intricate expression shall be found, from which also it shall be with great difficulty for the case $y=b$ to be shown. Therefore in this integration I have made use of a special way only, where at once for that single case, where there is $y=b$, an integral will arise, the basis of which consists in this: so that there shall become:

$$\int y^{m+2}dy(bb-yy)^{\frac{n}{2}}=\frac{(m+1)bb}{m+n+3}\int y^m dy(bb-yy)^{\frac{n}{2}},$$

in that case, where there is put $y=b$. Hence therefore there is found :

$$\int(\alpha+\beta y^2+\gamma y^4+\delta y^6+\varepsilon y^8+\text{etc.})y^m dy(bb-yy)^{\frac{n}{2}}$$

$$=\left(\alpha+\frac{\beta(m+1)bb}{m+n+3}+\frac{\gamma(m+1)(m+3)b^4}{(m+n+3)(m+n+5)}+\frac{\delta(m+1)(m+3)(m+5)b^6}{(m+n+3)(m+n+5)(m+n+7)}+\text{etc}\right)\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}.$$

Now since there shall be

$$\frac{1}{cc+yy}=\frac{1}{cc}-\frac{yy}{c^4}+\frac{y^4}{c^6}-\frac{y^6}{c^8}+\text{etc.}$$

there will become

$$\int\frac{y^{m+2}dy\left(bb-yy\right)^{\frac{n}{2}}}{cc+yy}=\left(\frac{1}{cc}-\frac{(m+1)b^2}{(m+n+3)c^4}+\frac{(m+1)(m+3)b^4}{(m+n+3)(m+n+5)c^6}\right.$$
$$\left.-\frac{(m+1)(m+3)(m+5)b^6}{(m+n+3)(m+n+5)(m+n+7)c^8}+\text{etc.}\right)\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}$$

indeed in that case were there shall become $y=b$. But that same series, though it may progress to infinity, yet indeed on putting $\frac{b}{c}=z$, can be reduced to a finite form, the sum of this series is

$$=\frac{(m+n+1)c^{m-1}(bb+cc)^{\frac{n}{2}}}{b^{m+n+1}}\int\frac{z^{m+n}dz}{\left(1+zz\right)^{\frac{n+2}{2}}},$$

thus with this integral taken so that it shall vanish on putting $z=0$. Therefore on putting

$$\int\frac{z^{m+n}dz}{\left(1+zz\right)^{\frac{n+2}{2}}}=C$$

there will become

$$\int\frac{y^{m+2}dy\left(bb-yy\right)^{\frac{n}{2}}}{cc+yy}=\frac{(m+n+1)Cc^{m-1}(bb+cc)^{\frac{n}{2}}}{b^{m+n+1}}\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}.$$

But when *m* is an even number and *n* odd, then $\int y^m dy(bb-yy)^{\frac{n}{2}}$ can be reduced further to $\int\frac{dy}{\sqrt{bb-yy}}$, of which integral in the case where $y=b$ becomes $\frac{\pi}{2}$, with π denoting the periphery of the circle of which the diameter is $=1$. Moreover, there will be

$$\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}$$

$$=\frac{\pi}{2}\cdot\frac{1\cdot3\cdot5...(m-1)}{2\cdot4\cdot6...m}\cdot\frac{1}{m+2}\cdot\frac{3}{m+4}\cdot\frac{5}{m+6}\cdot.....\cdot\frac{n}{m+n+1}b^{m+n+1};$$

from which finally there is had:

$$\int\frac{y^m dy(bb-yy)^{\frac{n}{2}}}{cc+yy}=\frac{\pi}{2}\cdot\frac{1\cdot3\cdot5...(m-1)}{2\cdot4\cdot6...m}$$
$$\cdot\frac{1}{m+2}\cdot\frac{3}{m+4}\cdot\frac{5}{m+6}\cdot.....\frac{n}{m+n+1}\cdot(m+n+1)c^{m-1}(bb+cc)^{\frac{n}{2}}C$$

with there being

$$C=\int\frac{z^{m+n}dz}{(1+zz)^{\frac{n+2}{2}}},\text{ and } z=\frac{b}{c},$$

thus so that on account of the even number *m*, *n* truly will be odd, and C shall be an algebraic quantity. Therefore from these for our case of the applied line, where on putting

$$\frac{b^4}{aa-bb}=cc,$$

our formula changes into this

$\int\frac{yydy\sqrt{bb-yy}}{cc+yy}$, from which there becomes $m=2$ and $n=1$.

Therefore there will be

$$C=\int\frac{z^3dz}{(1+zz)^{\frac{3}{2}}}=\frac{2+zz}{\sqrt{1+zz}}-2=\frac{2cc+bb}{c\sqrt{bb+cc}}-2,$$

and thus with the desired integral

$$=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot4\left(2cc+bb-2c\sqrt{bb+cc}\right)=\frac{\pi}{4}\left(\sqrt{bb+cc}-c\right)^2=\frac{\pi bb(a-b)^2}{4(aa-bb)},$$

just as we have established above.

PROPOSITION 60

PROBLEM

607. If the plane rectilinear figure *ABCDEF* (Fig. 92) where inscribed in a circle, while it shall be moved along some direction *OL* in water, the mean direction of the resistance will always pass through the centre of the circle *O*.

DEMONSTRATION

Because the direction of the resistance, which, which some line of the figure experiences from the water, is normal to the side itself at its midpoint, and any side shall be a chord of the circumscribed circle ; the direction of the resistance of any side will pass through the centre of the circumscribed circle *O*. Therefore however many sides of the figure may be taken, the direction of the individual resistance will pass through the centre of the circle *O* ; and on that account the mean direction of all the individual resistances by necessity shall pass through the same centre *O*. Therefore the total resistance, which the proposed figure shall experience moved along some direction, passes through the centre of the circumscribed circle *O*. Q.E.D.

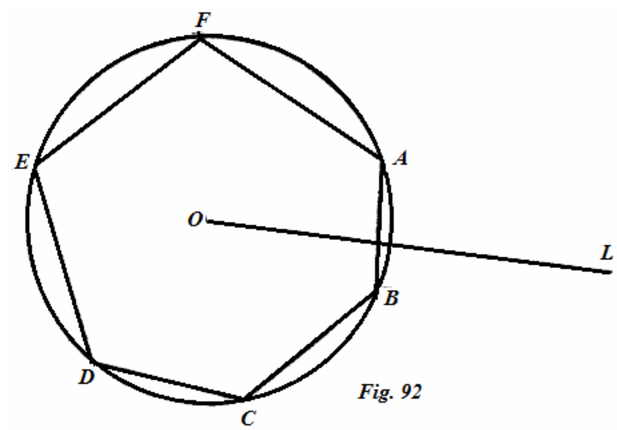


Fig. 92

COROLLARY 1

608. Therefore if the centre of gravity of this figure likewise were placed at the centre of gravity of the circumscribed figure, then the resistance will have no force for rotating the figure, in whatever direction the figure also maybe progressing.

COROLLARY 2

609. Also it is understood, if only the anterior part of the figure were inscribed in the circle, no oblique course shall be so great, that the posterior parts of the figure may be taken as parts of the resistance; while equally the mean direction of the resistance to be going to pass through the centre of the circle of the circumscribed prow *O*.

COROLLARY 3

610. Therefore if the diameter of this figure were provided, the diameter through the centre of the circumscribed circle in this case will have a fixed centre of resistance placed at the centre of the circumscribed circle itself.

SCHOLIUM

611. This is a conspicuous property of rectilinear figures inscribed in a circle, so that for these a constant location may be obtained for the centre of resistance, however great the obliquity of the shall be force, while in other figures the position of the centre of resistance for various oblique courses will be changed to such an extend, and it is observed that same property of figures being inscribed to be appropriate for the circle, thus so that it may not agree in other figures. Moreover it would be superfluous to consider several more plane figures, floating in water, since from these introduced it shall be easy to judge how to form the resistance of any oblate figure. Hence on this account with the resistance disregarded, which only of lines, either right or curved, just as the boundaries of plane figures may experience in water, we shall move on to the following chapter and in that to solid figures, which will pertain particularly to our principles, to be going to investigate the resistance any body may experience floating in water, which resistance must be derived from the surface of the body immersed in the water and to be derived from the water striking that surface. Evidently in a similar manner, which we have used up to this stage, every surface is considered to be constructed from innumerable planes, the individual resistances of which may be experienced by the surfaces themselves and to be proportional as well to the square of the angle of incidence. Thus so that if a plane of which the area shall be $= aa$ may strike underwater at an angle of which the sine is m, with the velocity due to the height v, then the force of the resistance will be equivalent to the weight of a cylinder of water of which the base is a^2 and the height $= m^2 v$, truly the direction of the resistance will be normal to that same plane surface, and passes through its centre of gravity, just as had been shown satisfactorily at the start of this chapter. But I shall consider bodies of this kind only, which enjoy a vertical plane diameter, which may be separated into two equal and similar pieces, for bodies of this kind alone deserve to be considered according to our principles. Moreover we may put the direction of the course, that is, which shall be placed in the plane of the diameter itself ; with which considered together the investigation of resistance shall become easier, since the direction of the resistance will be provided at once, certainly which lies equally in the diagonal plane. Therefore so much remains, so that the magnitude of the resistance, and of that itself established, which it has in the diagonal plane, may be defined. Indeed in the first place for this case we will set out the most general proposition, which in the following thus shall be able to progress easily to several kinds of figures.

CAPUT QUINTUM

DE RESISTENTIA, QUAM FIGURAE PLANAE IN AQUA MOTAE PATIUNTUR

PROPOSITIO 49

PROBLEMA

465. *Si figura plana data celeritate in aquae directe moveatur, definire resistentiam seu motus diminutionem, quam patietur, dum datum spatium percurrit.*

SOLUTIO

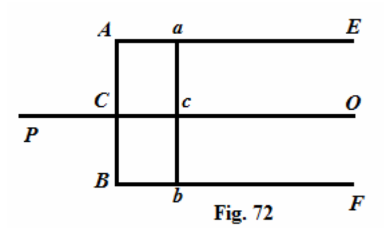


Fig. 72

Figura plana in aqua directe moveri dicitur, quando eius directio ad ipsam superficiem planam est perpendicularis. Repraesentet igitur recta *AB* superficiem planam, cuius area sit $= aa$, in aqua motam in directione *CO* ad ipsam superficiem normali (Fig. 72). Sit pondus seu massa corporis, quod hanc superficiem planam habet, quae in aquam *AEFB* incurrit $= M$, eiusque celeritas, qua in recta *CO* progreditur, et reipsa progredi pergeret, nisi resistentia adesset, debita altitudini *v*. Iam ad vim resistentiae definiendam concipiatur corpus momento temporis progredi, ita ut superficies plana *AB* perveniat in *ab* absoluto spatiolo $Aa = Bb = dx$; sitque celeritas quam peracto hoc spatiolo retinebit debita altitudini $v - dv$. Dum autem corpus per spatiolum *Cc* progreditur, aquam *ABba* de loco suo pellet per conflictum, ita ut corpus interea collisionem transigat cum mole aquea *ABba,* cuius volumen erit a^2dx, eiusque massa seu pondus propterea exprimatur per ma^2dx, denotante *m* aquae gravitatem specificam. Incurrit igitur corpus *M* celeritate sua $\sqrt{v}$ in molem aqueam ma^2dx quiescentem directe, ex quo perspicuum est directionem vis, quam corpus in hoc conflictu sentiet, fore normalem ad superficiem incurrentem *AB,* atque transituram esse per centrum gravitatis *C* superficiei ipsius, eo quod in recta *Cc* simul centrum gravitatis molis aquae *ABba* situm erit; urgebitur ergo corpus *M* in hoc conflictu vi quadam *CP,* cuius directio directe erit contraria directioni motus *CO.* Ad diminutionem motus igitur definiendam regulas communicationis motus in subsidium vocari oportet, et quidem eas, quae ad corpora perfecte mollia spectant, cum aquam hoc saltem casu omni elasticitate carere experimenta satis declarent. Cum itaque ante conflictum motus quantitas adsit $= M\sqrt{v}$, post conflictum vero, quoniam moles aquea *ABba* eadem celeritate movebitur qua corpus *M,* debita scilicet altitudini $v - dv$, erit motus quantitas

$$\left(M + ma^2dx\right)\sqrt{(v-dv)} = \left(M + ma^2dx\right)\left(\sqrt{v} - \frac{dv}{2\sqrt{v}}\right) \; ;$$

has duas motus quantitates inter se aequales esse oportet, unde oritur

$$\frac{Mdv}{2\sqrt{v}} = ma^2dx\sqrt{v}, \text{ seu } Mdv = 2ma^2vdx.$$

Ponatur nunc potentia *p* tanta, ut corpus in directione *CP* sollicitando, interea dum corpus per spatium $Cc = dx$ movetur, eandem motus diminutionem producere posset, foret

$$dv = \frac{pdx}{M};$$

ideoque $p = 2ma^2v$ *;* ex quo perspicitur aquae resistentiam in superficiem a^2 celeritate debita altitudini *v* directe motam aequivalere ponderi voluminis aquae $2a^2v$ *;* seu aequalem esse ponderi cylindri aquei, cuius basis aequalis sit superficiei incurrenti in aquam a^2 *;* altitudo vero adaequet duplam altitudinem celeritati corporis debitam. Idem ergo aqua per resistentiam efficit, ac si corpus *M* sollicitaretur a potentia tanta, quantam assignavimus in directione *CP,* ad superficiem corporis in aquam directe impingentem normali, et per eius ipsius superficiei centrum gravitatis *C* transeunte. Q. E. I.

COROLLARIUM 1

466. Reducta igitur est resistentia, quam corpus plana superficie praeditum directe in aquam incurrens patitur, ad potentiam, cuius tum directio tum quantitas pondere expressa datur.

COROLLARIUM 2

467. Media igitur directio resistentiae, quam superficies plana in aqua directe mota patitur, est normalis ad ipsam superficiem et per eius centrum gravitatis transit.

COROLLARIUM 3

468. Quantitas autem resistentiae tenet rationem compositam ex ipsa superficie et quadrato celeritatis; et hancobrem pro eadem superficie resistentiae sunt in duplicata ratione celeritatum.

COROLLARIUM 4

469. Si aquae volumen pondere ipsius corporis *M* pondus adaequantis ponatur $= V$, erit $V : M = 2a^2v$ ad pondus cylindri aquei, cuius basis est *aa* et altitudo *2v.* Quocirca resistentia, quam superficies plana a^2 celeritate altitititudini *v* debita in aquam directe occurrens patitur, aequivalet ponderi $\dfrac{2Ma^2v}{V}$.

COROLLARIUM 5

470. Eandem ergo vim corpus quiescens sentiet, in cuius superficiem planam aqua celeritate altitudini v debita impingit, ideo quod effectus ex collisione corporum ortus tantum a celeritate respectiva pendet, quae utroque casu est eadem.

COROLLARIUM 6

471. Haec ergo propositio aeque valet ad motum corporum in aqua quiescenti, ac in fluviis determinandum, siquidem superficies resistentiam patiens fuerit plana, atque ea directe in aquam, vel aqua directe in ipsam impingat.

SCHOLION 1

472. Multum etiamnum inter Auctores, qui de aquae resistentia scripserunt, disputatur, utrum resistentia aequivaleat duplo cylindro aqueo, cuius basis aequalis sit superficiei resistentiam directe excipienti, et altitudo aequalis altitudini celeritati debitae, prout hic quidem invenimus, an simplo tantum cylindro. Elicuimus hic autem duplum eiusmodi cylindri ad resistentiam aquae exprimendam, quia posuimus aquae particulas perfecte molles et omni elatere carentes, quod quidem experimenta suadent. At si aquae perfecta elasticitas tribuatur, utique alia resistentiae ratio prodiret. Si enim regulae, quae in collisione corporum elasticorum locum habent, in subsidium vocentur, tum adeo quadruplum memorati cylindri prodiret, resistentiaque reperietur $= 4ma^2v$. Sed cum hac consideratione aquae maior celeritas communicetur, quam ipsum corpus retinet, aqua a corpore ita resilire deberet, ut vacuum inter corpus et aquam relinqueretur. Quod cum ob aquae pondus, quo eius partes inter se comprimantur evenire nequeat, regulae communicationis, quae corporibus elasticis sunt accommodatae, locum hic invenire non poterunt; sed principium generale, quo illae regulae nituntur, et quod in conservatione virium vivarum consistit, erit adhibendum. Ob aquae compressionem igitur utique est statuendum, corpus M et aquam $ABba$ eandem acquirere celeritatem. Hoc vero posito, quia ante conflictum vis viva adest $= Mv$, post conflictum vero vis viva est $= \left(M + ma^2dx\right)(v - dv)$, his aequatis fiet $Mdv = ma^2vdx$; unde potentia aequivalens resistentiae orietur = ponderi ma^2v, hoc est cylindro aqueo basis a^2 et altitudini v. Quaecuncunque autem resistentiae ratio locum habeat, calculus manet idem, differt enim tantum coefficiente istius cylindri aquei, qui illo casu est 2 hoc vero 1. Quamobrem istam controversiam non multum curabimus, cum, utervis casus valeat, proportiones maneant eadem, ad quas praecipue attendemus; utroque enim casu directio resistentiae est normalis ad superficiem planam directe in aquam incurrentem, atque per ipsius superficiei centrum gravitatis transit, estque praeterea utroque casu proportionalis areae superficiei et quadrato celeritatis coniunctim. Experimenta autem, quae circa resistentiam corporum in aqua motorum sunt instituta pro simplici cylindro pugnare videntur, id quod cum argumento ex conservatione virium vivarum petito mirifice congruit. Facile etiam

patet resistentiam minorem esse debere, quam in solutione invenimus; ibi enim, quia aqua post collisionem corpus comitatur, impulsus sequentes debiliores esse debent quam assumsimus.

SCHOLION 2

473. Experimenta scilicet, quae NEWTONUS cum globis in aqua delapsis instituit, satis clare evincere videntur resistentiam tantum per simplicem cylindrum aqueum, cuius altitudo scilicet aequetur simplici altitudini celeritatem generanti, esse expondendam. Praeterea vero quia aqua praeter hanc resistentiam, quae ab allisione proficiscitur, aliam habet resistentiam a tenacitate particularum oriundam, haud parum difficile est definire per experimenta, quanta sit resistentia a sola allisione orta. Quidquid igitur sit, cum experimenta posteriori hypothesi, qua resistentia per simplum cylindrum aqueum exponitur, satis sint consentanea, eam hypothesin hic adoptabimus, et resistentiam, quam superficies plana in aquam directe impingens patitur, mensurabimus pondere cylindri aquei, cuius basis aequetur areae superficiei, altitudo vero ipsi altitudini celeritati debitae; ita in casu corollarii 4 resistentia aequalis erit pondenda ipsi $\frac{Ma^2v}{V}$. Eadem vero resistentiae hypothesis confirmari potest sequenti argumento non quidem apodictico. Sit vas amplissimum aqua repletum *ACDB* (Fig. 73), cuius altitudo $AC = v$, pertusum sit hoc vas infra ad latus foramine *DE* cuius area sit $= a^2$, effluet aqua per hoc foramen celeritate debita altitudini *v,* iam venae aquae effluentis *Ed* opponatur directe obex planus *de* ipsi foramini amplitudine aequalis, atque hic obex ab effluente aqua eandem vim sustinebit, ac si ipse celeritate altitudini *v* debita directe contra aquam quiescentem impingeret. Consentaneum autem videtur, obicem in *de* eandem pressionem esse passurum, ac si in *DE* esset collocatus, hoc vero casu obex omnino obturabit foramen effluxumque penitus impediet; nunc autem pressionem patietur aequalem ponderi cylindri aquei, cuius basis aequatur ipsi superficiei obicis a^2, altitudo vero altitudini $AC = v$, ex quo sequitur resistentiam superficiei planae in aqua directe motae aestimandam esse ex simplici cylindro aqueo, cuius altitudo altitudini celeritati debitae aequalis sit. Experimenta etiam hoc ratiocinium satis confirmant, nam quanquam si obex maior adhibeatur, quam est foramen *DE,* resistentia aquae maior sentiatur, tamen hoc magnitudini obicis tribuendum videtur, quippe ad cuius latera aqua defluit, maioremque pressionem exercet, quam si obex orificium tantum aequaret; quamobrem non dubitandum est quin obex superficiem maiorem non habens quam est amplituda foraminis, assignatam pressionem sit sensurus. Quoniam porro eadem hypothesis confirmatur, si aquae elasticitas, quae omnino adimi non potest, tribuatur, et praecipue, si conservatio virium vivarum statuatur, cuius usus ubique summus conspicitur, eo minus dubitabimus eam solam recipere, eique totam resistentiae doctrinam superstruere; idque eo magis, cum illi experimenta maxime faveant.

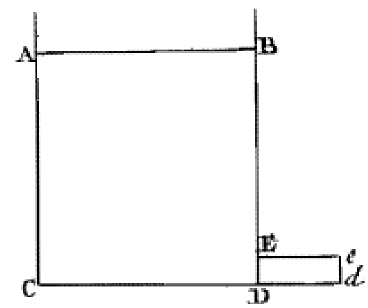


Fig. 73

PROPOSITIO 50

474. *Si superficies plana in aqua oblique moveatur, determinare resistentiam, qua motus superficiei ab aqua retardabitur.*

SOLUTIO

Quia superficies plana oblique moveri dicitur, quando directio motus ad ipsam angulum constituit obliquum, repraesentet *AB* superficiem planam, cuius area sit $=a^2$ (Fig. 74); quae moveatur in aqua directione *MC,* quae cum plano superficiei *AB* angulum constituat *ACM* cuius sinus sit $=n$; posito sinu toto $=1$; celeritas vero qua superficies movetur debita sit altitudini *v.* Concipiatur iam ut ante superficies *AB* in aqua promoveri per spatiolum $Cc=dx$, atque hoc absoluto pervenire in *ab,* interea conflictum habuerit necesse est cum mole aquae *ABba,* cuius volumen est $=na^2dx$. Minor igitur aquae portio motui superficiei obstat, quam si directe in aqua moveretur, idque in ratione sinus anguli incidentiae ad sinum totum; et hancobrem ex hoc capite resistentia, quam pateretur in motu directo, diminuenda est in ratione sinus anguli incidentiae *ACM* ad sinum totum. Deinde quanquam superficies in singulas aquae particulas oblique impingit, tamen impulsus directio erit ad superficiem *AB* normalis, ita ut resistentia in superficiem *AB* vim exerat, cuius directio ad eam erit normalis *CP,* atque per ipsius superficiei centrum gravitatis *G* transibit. At quoniam omnes conflictus huius superficiei cum singulis aquae

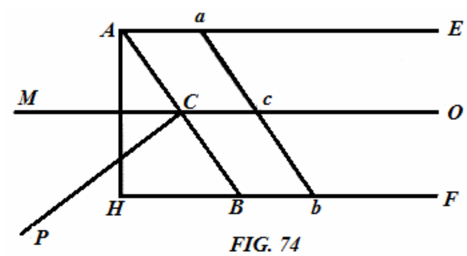


FIG. 74

particulis sunt obliqui, minus erunt efficaces, quam si essent directi, idque in ratione sinus anguli incidentiae *AGM* ad sinum totum. Cum igitur in ista impulsione obliqua resistentia ob duplicem causam bis debeat diminui in ratione sinus anguli incidentiae ad sinum totum, se habebit resistentia, dum superficies *AB* in aqua oblique movetur, ad resistentiam, quam eadem superficies eadem celeritate directe mota pateretur, ut
quadratum sinus anguli incidentiae *MCA* ad quadratum sinus totius hoc est ut a^2 ad 1. Quare cum vis resistentiae in casu motus recti sit $=ma^2dx$, seu ponderi cylindri aquei, cuius basis est $=a^2$ et altitudo aequalis altitudini debitae celeritati, erit vis resistentiae pro praesenti casu $=n^2ma^2v$, hoc est ponderi cylindri aquei basin habentis aequalem ipsi superficiei et altitudinem aequalem altitudini celeritati debitae, multiplicato per quadratum sinus anguli incidentiae *MCA* posito sinu toto $=1$. Q. E. I.

COROLLARY 1

475. Resistentia igitur, quam idem planum sub diversis angulis in aqua motum eadem celeritate patitur, est in duplicata ratione sinus anguliquem planum cum directione motus constituit.

COROLLARIUM 2

476. Si igitur cognita fuerit vis resistentiae, quam planum in aqua directe motum suffert, simul innotescet resistentia, quam idem planum utcunque oblique in aquam impingens patietur.

COROLLARIUM 3

477. In quacunque igitur directione superficies plana in aqua moveatur, directio resistentiae semper est eadem, est normalis ad planum superficiei, atque per centrum gravitatis ipsius superficiei transit.

COROLLARIUM 4

478. Resistentia porro, quam idem planum sub variis angulis diversisque celeritatibus in aqua motum patitur, est in ratione composita ex duplicata celeritatum, et duplicata sinus anguli quo in aquam impingit.

COROLLARIUM 5

479. Resistentiae autem, quas diversa plana in aqua mota sufferunt, rationem tenent compositam ex simplici arearum, duplicata celeritatum et duplicata sinuum angulorum, quibus in aquam incurrunt.

SCHOLION 1

480. Inserviunt haec problemata instar basis ad resistentiam determinandam, quam corpora cuiuscunque figurae in aqua mota patiuntur. Pendet enim resistentia a corporis superficie anteriore qua in aquam incurrit, quippe quae sola cum particulis aquae conflictatur, pars autem corporis posterior ab aqua nullam patitur resistentiam, eo quod ea ad aquam non allidit. Quamquam enim etiam pars posterior ab aqua affici videatur, dum aqua locum, quem corpus post se reliquit, occupans, in partem posticam impetum facit ac motum accelerat, tamen iste effectus vix est sensibilla, et hancobrem hic considerari non meretur; ad quod accedit, quod theoria aquae nondum sit ad eum perfectionis gradum evecta, ut aquae effectus in posticam corporis natantis partem definiri queat. Hac igitur consideratione praetermissa, si corporis aquae innatantis anterior superficies vel plana fuerit vel ex planis pluribus constet, ope duorum horum problematum resistentia absolute poterit definiri. Praeterea vero inserviunt haec problemata ad resistentiam corporum quacunque superficie praeditorum assignandam; quomodocunque enim superficies fuerit comparata, ea more solito tanquam ex innumerabilibus planis composita considerari, atque ex regulis staticis resistentia totalis, quae ex resistentiis singulorum elementorum emergit, per integrationem definiri poterit, quo pacto tam directionem mediam omnium resistentiarum, quam ipsam potentiam aequivalentem determinare licebit.

SCHOLION 2

481. Cum igitur nunc propositum sit resistentiam indagare, quam corpora quaecunque aquae innatantia perpetiuntur, quo tota ista tractatio commode et dilucide absolvatur, certum ordinem sequi oportebit. Primum igitur hoc capite figuras tantum planas aquae tum horizontaliter tum verticaliter innatantes considerabo, atque vim resistentiae eiusque directionem determinabo, inde enim ad ipsa corpora facilius transire licebit. Eas vero figuras, quas aquae horizontaliter innatare ponemus, axe seu diametro praeditas assumemus, quia naves, ad quas hic potissimum respicimus, plano diametrali, quod verticaliter per spinam transeat, gaudent, ex quo singulae sectiones horizontales diametro spinae navis parallela erunt praeditae. Hic autem in resistantia ingens oritur discrimen, utrum eiusmodi superficies secundum diametri suae directionem in aqua moveatur, an oblique ? si enim secundum directionem diametri moveatur, manifestum est mediam directionem resistentiae ob similem ex utraque diametri parte effectum, esse in ipsa diametro positam, ita ut hoc casu tantum quantitas vis resistentiae investigari debeat; sin autem eiusmodi superficies non secundum diametri suae directionem in aqua progrediatur, tum seorsim tam mediam directionem, quam ipsam quantitatem resistentiae inveniri oportet, quae investigatio propterea plus habebit difficultatis. Deinceps in capite sequente simili modo in resistentia corporum ipsorum aquae innatantium investiganda versabimur; eiusmodi enim corpora tantum contemplabimur, quae praedita sint plano diametrali verticali, quo navium conditio imprimis spectetur, in qua tractatione iterum praecipue ad directionem motus erit attendendum, utrum is fiat secundum diametrum sectionis aquae, an ad diametrum oblique; priore enim casu media directio resistentiae sponte datur, posteriore vero haud exiguo labore demum est investiganda. In utraque autem tractatione eiusmodi problemata afferemus, ex quibus pateat, quaenam navium figura ratione resistentiae sit aptissima; quae tum ex minima resistentia tum ex idonea resistentiae directione desumentur. Antequam autem haec omnia evolvenda suscipiamus, hic locus maxime est idoneus ad effectum gubernaculi in nave circa axem verticalem convertenda inquirendum; quoniam gubernaculum superficie plana solet esse praeditum, cuius ideo vis, quam contra aquam impingens patitur, ex ista propositione facile definiri potest.

PROPOSITIO 51

PROBLEMA

482. *Si navis in directione quacunque progrediatur, atque gubernaculum ad datum angulum convertatur, invenire vim; quam gubernaculum habebit ad navim circa axem verticalem per centrum gravitatis transeuntem convertendam.*

SOLUTIO

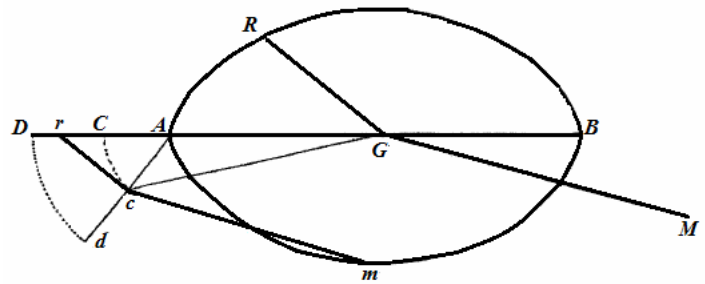


FIG. 75

Quoniam media directio vis aquae, in quam gubernaculum irruit, per centrum gravitatis superficiei planae gubernaculi transit, ad eamque est normalis, concipiatur sectio navis horizontalis *ARBm* per gubernaculi *AD* centrum gravitatis *C* transiens (Fig. 75). Manifestum autem est hic non totius gubernaculi, sed eius tantum partis, quae aquae est immersa centrum gravitatis sumi debere. Repraesentabit itaque in figura *A* puppim, *B* proram, *AB* spinam navis, *AD* gubernaculum situm tenens naturalem. Sit autem *G* punctum axis verticalis navis per eius centrum gravitatis ducti, in quo per planum horizontale *ARBm* transit; *GM* vero sit directio cursus seu motus navis, ita ut angulus *BGM* denotet declinationem cursus navis a cursu directio, qui secundum directionem spinae *GB* fieri censetur, gubernaculum vero inclinatum sit ad angulum *DAd,* ita ut situm *Ad* obtineat, quo secundum directionem *cm* directioni cursus *GM* parallelam in aquam impingit. Sit nunc anguli *BGM* sinus $= m$, cosinus $= \mu$; anguli vero *DAd* sinus $= n$, cosinus $= v$, existente semper sinu toto $= 1$. Sit porro area vel superficies gubernaculi vim aquae excipiens $= a^2$,

$$AC = Ac = b,\ AG = f\ ;$$

celeritasque, qua navis movetur, debita sit altitudini *v.* Denique sit pondus navis $= M$, volumen partis submersae navis *V,* et momentum inertiae navis respectu axis verticalis $= Mk^2$. His praemissis erit *Acm* angulus sub quo gubernaculum *Ad* ad aquam allidit, qui cum sit $= DAd + BGM$, erit sinus eius $= mv + np$; hinc igitur vis resistentiae, quam gubernaculum sentiet erit

$$= \frac{(mv + n\mu)^2\, Ma^2 v}{V},$$

cuius directio *cr* transibit per gubernaculi centrum gravitatis *c,* eritque ad *Ad* normalis. Momentum ergo huius vis ad navem circa axem verticalem circumvertendam erit

$$= (mv + n\mu)^2 \frac{Ma^2 v}{V}.Gr\sin Arc.$$

Est vero ob angulum *Acr* rectum, sinus $Arc = v$, et $Ar = \dfrac{b}{v}$; unde fit

$Gr = f + \frac{b}{v}$. Quo circa momentum vis gubernaculi ad navem circa G convertendam erit

$$= \frac{(mv + n\mu)^2 Ma^2 v(b + vf)}{V}.$$

Quod divisum per momentum inertiae navis respectu axis verticalis Mk^2, dabit vim gyratoriam navis circa eundem axem verticalem

$$= \frac{(mv + n\mu)^2 a^2 v(b + vf)}{Vk^2};$$

cui vi acceleratio momentanea motus angularis, qui navi circa axem verticalem per centrum gravitatis ductum imprimitur, est proportionalis. Q. E. I.

COROLLARIUM 1

483. Pro eadem ergo navi, quo maior fuerit expressio

$$(mv + n\mu)^2 (b + vf)$$

eo maior erit effectus gubernaculi ad navem convertendam; ex quo angulus DAd definiri poterit, quo effectus gubernaculi fit maximus.

COROLLARIUM 2

484. Si igitur anguli DAd cosinus seu v ponatur $= x$, erit $n = \sqrt{(1 - xx)}$, atque formula seu eius radix quadrata

$$\left(mx + \mu\sqrt{(1 - xx)}\right)^2 (b + fx)$$

fiet maximum, cum x determinabitur ex hac aequatione:

$$2\left(m - \frac{\mu x}{V(1 - xx)}\right)(b + fx) + f\left(mx + \mu\sqrt{(1 - xx)}\right) = 0,$$

quae transit in hanc

$$m(2b + 3fx)\sqrt{(1 - xx)} = \mu(3fxx + 2bx - f).$$

COROLLARIUM 3

485. Si ergo navis cursu directo progrediatur, ut angulus *BGM* evanescat, erit $m=0$, et $\mu=1$, atque vis gyratoria $=\frac{n^2a^2v\left(b+vf\right)}{Vk^2}$; maximum igitur gubernaculum praestabit effectum, si fuerit $3fxx+2b\,x-f=0$, hoc est, si fuerit anguli *DAd* cosinus

$$x=\frac{-b\pm\sqrt{(bb+3ff)}}{3f}.$$

COROLLARIUM 4

486. Si igitur *b* tam fuerit parvum, ut prae *f* evanescat, erit anguli *DAd,* quo maximum effectum praestat gubernaculum, cosinus $=\frac{1}{\sqrt{3}}$; hoc est angulus *DAd* erit 54°, 44'.

COROLLARIUM 5

487. Si navis cursus a directo declinet angulo *BGM,* gubernaculum autem in situ naturali $-AD$ relinquatur, praestabit tamen gubernaculum effectum ad navem convertendam, cuius vis gyratoria erit $\frac{m^2a^2v\left(b+f\right)}{Vk^2}$.

COROLLARIUM 6

488. Afficit autem praeterea vis gubernaculi ipsum navis motum, quae mutatio reperitur, si vis resistentiae $\frac{\left(mv+n\mu\right)^3Ma^2v}{V}$; concipiatur in directione parallela *GR* centro gravitatis applicata; retardabitur scilicet motus navis in directione sua a potentia $\frac{\left(mv+n\mu\right)^3Ma^2v}{V}$; at a semita rectilinea deturbabitur potentia $\frac{\left(\mu v-mn\right)\left(mv+n\mu\right)^2Ma^2v}{V}$.

COROLLARIUM 7

489. Habebit insuper gubernaculum in situ *Ad* conatum sese circa *A* convertendi secundum plagam *dD,* qui conatus exprimetur momento

$$\frac{\left(mv+n\mu\right)^2Ma^2bv}{V};$$

tanta igitur vis a gubernatore adhiberi debet ad gubernaculum in situ *Ad* continendum.

COROLLARIUM 8

490. Si igitur navis cursu obliquo feratur, vi adeo opus erit ad gubernaculum in situ naturali *AD* conservandum, quae vis exprimetur momento $\frac{Mn^2a^2bv}{V}$.

COROLLARIUM 9

491. Manifestum denique est omnes has vires a gubernacula exertas ceteris paribus crescere in duplicata ratione celeritatum, quibus navis progrediatur.

SCHOLION

492. In hac igitur propositione non solum definivimus quanta vi gubernaculum navem circa axem verticalem per centrum gravitatis ductum circumagat, sed etiam quantum tam ipsius navis celeritatem, quam cursus directionem afficiat, in corollariis determinavimus. Praeterea etiam vim assignavimus, quam nauclerus adhibere debet ad gubernaculum, in dato situ conservandum tanta, scilicet haec naucleri vis requiritur, ut eius momentum respectu axis circa quem gubernaculum mobile existit, adaequet momentum inventum, quo gubernaculum ex situ *Ad* versus *AD* tendit. Intelligitur vero etiam, nisi planum *ARBm* per navis centrum gravitatis transeat, vim gubernaculi etiam se exerere ad navem circa axem horizontalem tam longitudinalem quam latitudinalem inclinandam, quae inclinatio autem attendi vix meretur, cum sit exigua, atque tum solum eveniat, quando gubernaculum usurpatur. Quamobrem misso gubernacula ad ipsum propositum revertamur, ac primo quidem, quantam resistentiam figurae planae aquae innatantes patiantur investigemus.

PROPOSITIO 52

PROBLEMA

493. *Innatet aquae figura plana MAN* (Fig. 76) *diametro AP praedita secundum directionem AC ipsius diametri AP data cum celeritate, invenire resistentiam, quam haec figura ab aqua patietur.*

SOLUTIO

Primum perspicuum est, quia figura secundum directionem axis *AC* in aqua progreditur, ob utrinque omnia similia mediam directionem resistentiae in ipsam diametrum *AP* incidere debere, ita ut tantum opus sit eius quantitatem determinare. Hancobrem ponatur celeritas, qua figura in aqua secundum directionem *AC* progreditur, debita altitudini

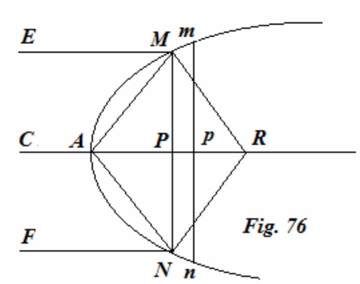


Fig. 76

v ac ducantur ad diametrum *AP* duae ordinatae orthogonales *MPN, mpn,* utrinque aequalia et similia curvae elementa
Mm, Nn abscindentia, quae elementa quantam resistentiam excipiant
est indagandum. Ponatur

$$AP = x,\ PM = PN = y$$

erit

$$Pp = dx, \text{ et } Mm = Nn = \sqrt{\left(dx^2 + dy^2\right)} = ds.$$

Iam anguli, quo elementa *Mm* et *Nn* in aquam directe seu illidunt, sinus est $= \frac{dy}{ds}$. Si autem elementa haec in aquam directe seu normaliter impingerent, foret vis resistentiae $= vds$ hoc est ponderi cylindruli aquei basis *ds* et altitudinis *v.* Praesenti igitur casu vis, quam utrumque elementum patitur, erit $= \frac{vdy^2}{ds}$; cuius utriusque vis directio est normalis ad ipsa elementa, ideoque in normales *MR* et *NR* incidet. Si nunc hae duae vires resolvantur in binas, quarum alterae directiones habeant in applicatis, alterae parallelas axi *AP,* illae se mutuo destruent, hae vero conspirabunt, habebuntque mediam directionem in *AP* incidentem. Quamobrem ob resistentiam elementorum *Mm, Nn,* figurae in directione *AP* resistetur vi $= \frac{2vdy^3}{ds^2}$; ex quo tota curva *MAN* resistentiam
pariet

$$= 2\int \frac{vdy^3}{ds^2} = 2v\int \frac{dy^3}{ds^2},$$

ob *v* constantem, huiusque vis directio sita erit in ipsa diametro *AP.* Q.E.I.

COROLLARIUM 1

494. Directio resistentiae ergo, quam eiusmodi figura secundum diametrum
AC in aqua promota sentit, directe contraria erit directioni motus, et hancobrem motus tantum a resistentia retardabitur, directio vero non afficietur, siquidem figurae centrum gravitatis in diametro *AP* fuerit situm.

COROLLARIUM 2

495. Resistentia ergo ab *A* ad *M* et *N* progrediendo eousque crescit, quoad fiat $dy = 0$, hoc est quoad curvae tangentes axi *AP* fiant parallelae. Quamobrem si curva fuerit indefinita, resistentia ex iis tantum ramorum *AM* et *AN* portionibus aestimari debet, qui inter *A* et loca ubi est $dy = 0$ interiacent.

COROLLARIUM 3

496. Si sola ordinata *MPN* in aqua directe, hoc est secundum directionem *AP* eadem celeritate moveretur, tum resistentia quam sentiret, foret $= 2vy$; ex quo resistentia ordinatae *MPN* se habebit ad resistentiam curvae *MAN* ut y ad $\int \frac{dy^3}{ds^2}$.

COROLLARIUM 4

497. Quoniam ubique est $dy < ds$, erit

$$\frac{dy^3}{ds^2} < dy \text{ ideoque } \int \frac{dy^3}{ds^2} < y \ ;$$

quamobrem resistentia, quam curva *MAN* patitur semper minor erit quam resistentia, quam sola ordinata *MPN* sentiret.

COROLLARIUM 5

498. Eo minor ergo erit figurae *MAN* resistentia, quo magis discrepat a recta transversali *MPN*; sive quo minus ubique est elementum applicatae *dy* respectu elementi curvae.

COROLLARIUM 6

499. Cum autem sit

$$\frac{dy^3}{ds^2} = dy - \frac{dx^2 dy}{ds^2} \text{ ob } ds^2 = dx^2 + dy^2,$$

erit resistentia, quam curvae portio *MAN* perpetitur

$$= 2vy - 2v\int \frac{dx^2 dy}{ds^2} .$$

Excessus ergo resistentiae ordinatae *M N* super resistentiam arcus *MAN* erit

$$= 2v\int \frac{dx^2 dy}{ds^2} .$$

COROLLARIUM 7

500. Si ergo sola figura *MAN* curva *MAN* in qua nusquam sit $dy = 0$ et recta *MN* terminata eadem celeritate tum secundum directionem *PC,* tum secundum contrariam *PR* moveatur, erit resistentia in priore casu ad resistentiam in posteriore ut $y - \int \frac{dx^2 dy}{ds^2}$ ad *y.* Priore scilicet casu curva *MAN,* posteriore vero recta *MN* resistentiae exponitur.

SCHOLION

501. Quoniam formula, qua resistentia arcus *MAN* exprimitur $2v\int \frac{dy^3}{ds^2}$ generaliter integrari nequit, ipsa resistentia quantitatibus finitis exhiberi non potest, ac manifestum est, resistentiam a mutua positione singulorum elementorum pendere. Quamobrem expediet ad specialiora descendere, atque datas curvas considerare, pro quibus valor ipsius $\int \frac{dy^3}{ds^2}$ assignari queat, ex his enim facilius colligi poterit, cuiusmodi curvae minorem maioremve resistentiam patiantur. Hoc scilicet modo animus lectoris praeparabitur ad curvas, quae vel inter omnes, vel inter eas, quae certa quadam proprietate sint praeditae, minimam patiantur resistentiam, cognoscendas; cuiusmodi curvas deinceps sum investigaturus.

EXEMPLUM 1

502. Sit figura seu eius saltem pars anterior *EA F* (Fig. 77), quae resistentiam sentit, triangulum isosceles, quod secundum directionem diametri *AB* cuspide *A* antiorsum versa in aqua progrediatur celeritate debita altitudini *v.* Incidet directio resistentiae in rectam *AB,* eius vero quantitas ita ex generali solutione definietur. Positis $AB = a,\ BE = BF = b,$ et ut ante $AP = x,\ PM = PN = y,$ erit $a : b = x : y,$ unde fit

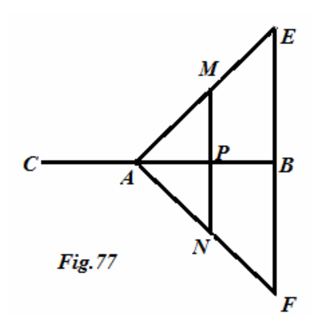


Fig.77

$$y = \frac{bx}{a};\ dy = \frac{bdx}{a};\ \text{et}\ ds = \frac{dx}{a}\sqrt{\left(a^2 + b^2\right)}.$$

Ex his fient

$$\frac{dy^3}{ds^2} = \frac{b^3 dx}{a\left(a^2 + b^2\right)},$$

atque integrando

$$\int \frac{dy^3}{ds^2} = \frac{b^3 x}{a\left(a^2+b^2\right)},$$

quamobrem resistentia, quam pars *MAN* patietur, erit $= \dfrac{b^3 xv}{a\left(a^2+b^2\right)}$ atque posito $x = a$,

habebitur resistentia desiderata, quam triangulum totum *EAF* suffert $= \dfrac{2b^3 v}{a^2+b^2}$.

COROLLARIUM 1

503. Resistentia ergo, quam sentit angulus *EAF* erit ad resistentiam basis *EF* eadem celeritate et in eadem directione in aqua motae ut *bb* ad a^2+b^2, hoc est ut BE^2 ad AE^2.

COROLLARIUM 2

504. Perspicuum igitur est resistentiam trianguli *EAF* in aqua vertice *A* antiorsum vero se habere ad resistentiam, quam idem triangulum base *EF* antiorsum versa patitur in duplicata ratione sinus anguli *BAE* ad sinum totum.

COROLLARIUM 3

505. Quo minor igitur seu acutior fuerit angulus *EAF* manente basi *EF* eadem, eo minor erit resistentia, quam triangulum vertice *A* antiorsum verso in aqua sentiet.

COROLLARIUM 4

506. Figura plana igitur super eadem basi *EF* constituta, quae minimam patietur resistentiam, erit triangulum infinite magnum, cuius vertex *A* in infinitum abit: talis quippe trianguli resistentia est nulla.

EXEMPLUM 2

507. Sit curva *MAN* (Fig. 76) parabola, cuiusvis ordinis, sive continua, sive ex aequalibus eiusdem parabolae ramis *AM* et *AN* composita, quae celeritate altitudini *v* debita in directione *AC* moveatur in aqua, ita ut sit

$$y = \frac{x^m}{a^{m-1}}.$$

Resistentiae directio, quam portio *MAN* patietur, incidet utique in directionem diametri *AP*, quantitas vero ex aequatione

$$y=\frac{x^m}{a^{m-1}}$$

ita definietur. Cum sit

$$dy=\frac{mx^{m-1}dx}{a^{m-1}},$$

erit

$$ds^2=dx^2\left(1+\frac{m^2x^{2m-2}}{a^{2m-2}}\right)=\frac{dx^2\left(a^{2m-2}+m^2x^{2m-2}\right)}{a^{2m-2}}$$

atque

$$\frac{dy^3}{ds^2}=\frac{m^3x^{3m-3}dx}{a^{m-1}\left(a^{2m-2}+m^2x^{2m-2}\right)}=dy-\frac{ma^{m-1}x^{m-1}dx}{a^{2m-2}+m^2x^{2m-2}}.$$

Hinc igitur oritur

$$\int\frac{dy^3}{ds^2}=y-ma^{m-1}\int\frac{x^{m-1}dx}{a^{2m-2}+m^2x^{2m-2}}=y-\int\frac{a^{\frac{2m-2}{m}}dy}{a^{\frac{2m-2}{m}}+m^2y^{\frac{2m-2}{m}}},$$

quae expressio ducta in *2v* dabit quantitatem resistentiae. Pro parabola ergo Appolloniana quae est $m=\frac{1}{2}$, erit resistentia

$$=2vy-2v\int\frac{4yydy}{aa+4yy}=2v\int\frac{aady}{aa+4yy}=av\ \text{Arc.t.}\frac{2y}{a}.$$

Tota ergo parabola in infinitum continuata resistentiam patitur finitam quae erit $=\frac{\pi}{2}av,$ denotante $1:\pi$ rationem diametri ad peripheriam. Pro reliquis vero casibus non admodum concinnae formulae reperiuntur, quamobrem his missis ad alias curvas considerandas pergemus.

EXEMPLUM 3

508. Si curvae in aqua horizontaliter promotae secundum directionem *CA* pars anterior resistentiam excipiens fuerit arcus circuli *MAN*, cuius radius *AC* fit $=a$, resistentiae directio ob partes utrinque similes cadet in diametrum *AC*. Positis autem

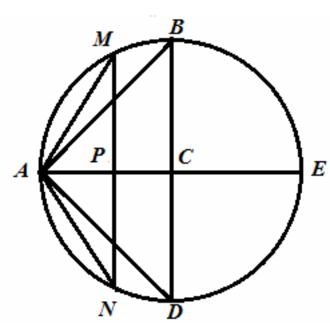


Fig. 78

$$AP=x\ \text{ et }\ PM=PN=y\ ;$$

erit

$$x=a-\sqrt{a^2-y^2}\ \text{ et }\ ds=\frac{ady}{\sqrt{a^2-y^2}}$$

unde fit

$$\frac{dy^3}{ds^2} = dy - \frac{y^2 dy}{aa}, \text{ atque } \int \frac{dy^3}{ds^2} = y - \frac{y^3}{3aa}.$$

Resistentia igitur quam patietur arcus *MAN* in directione *AC* erit

$$= 2v\left(y - \frac{y^3}{3aa} \right).$$

Si ergo pars figurae anterior fuerit semi circulus integer *BAD,* erit resistentia quam patietur $\frac{4av}{3}$. Quamobrem si integra figura fuerit circulus *ABED,* in quamcunque plagam is in aqua promoveatur, semicirculus semper resistentiae erit obnoxius, atque ob omnes partes similes directio resistentiae
semicirculum anteriorem bisecabit et per centrum transibit, ipsaque resistentia perpetuo erit $= \frac{4av}{3}$.

COROLLARIUM 1

509. Resistentia ergo semicirculi *BAD* se habebit ad resistentiam diametri *BD* eadem celeritate directe motae ut $\frac{4av}{3}$ ad $2av$ hoc est ut 2 : 3.

COROLLARIUM 2

510. Si autem concipiatur triangulum isosceles *BAD* eadem celeritate et in eadem directione aquae innatare, erit eius resistentia $= av$. Duplo igitur minor est resistentia trianguli *BAD,* quam resistentia diametri *BD* ; atque semicirculi *BAD* trianguli *BAD* et diametri *BD* resistentiae se inter se habebunt ut isti numeri 4: 3: 6.

COROLLARIUM 3

511. Sin autem arcus indefinitus semicirculo minor *MAN* solus resistentiam patiatur, erit resistentia ipsius ad resistentiam chordae *MN* eadem celeritate contra aquam directe impingentis ut $y - \frac{y^3}{3aa}$ ad y seu ut $3aa - yy$ ad $3aa$.

COROLLARIUM 4

512. Si insuper triangulum isosceles *MAN* in eadem directione et eadem celeritate moveatur, erit eius resistentia $= \frac{y^3 v}{aa - a\sqrt{aa - yy}}$. Quamobrem segmenti *MAN,* trianguli *MAN* et chordae *MN* resistentiae erunt inter se ut

$$\frac{6aa-2yy}{3aa}:\frac{y^2}{a^2-a\sqrt{aa-yy}}:2$$

seu ut

$$6aa-2yy:3a^2+3a\sqrt{aa-yy}:6aa.$$

COROLLARIUM 5

513. Harum trium ergo resistentiarum maxima est, quam chorda *MN* patitur. Resistentia vero segmenti *MAN* aequalis fit resistentiae trianguli *MAN,* si sit $y=\frac{a\sqrt{3}}{2}$, hoc est si arcus *AM* sit 60 graduum. Resistentiae igitur segmenti *MAN* erit maior quam resistentia trianguli *MAN* si arcus *AM* excedat 60 gradus, minor vero si arcus *AM* fuerit 60° minor.

EXEMPLUM 4

514. Sit nunc figurae planae aquae innatantis pars anterior arcus ellipticus *MAN* (Fig. 79) seu portio ellipsis *ABED,* cuius alter semiaxis $AG=a$, alter $BC=b$, erit positis

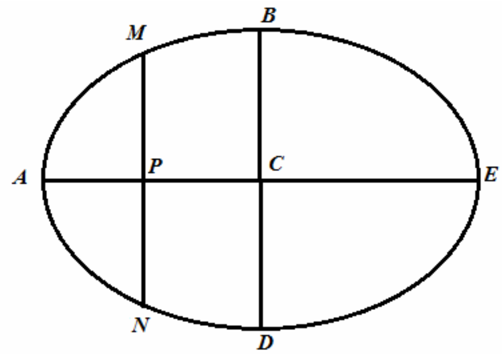


Fig. 79

$$AP=x,\ PM=y,\ CP=a-x,$$

et

$$a^2=(a-x)^2+\frac{aayy}{bb},$$

unde fit

$$x=a-\frac{a}{b}\sqrt{(bb-yy)}.$$

Hinc igitur erit

$$dx=\frac{aydy}{b\sqrt{(b^2-y^2)}} \quad \text{atque} \quad ds^2=\frac{dy^2\left(b^4+\left(a^2-b^2\right)yy\right)}{bb(bb-yy)},$$

quamobrem habebitur

$$\frac{dy^3}{ds^2}=\frac{bbdy(bb-yy)}{b^4+(aa-bb)yy}=-\frac{bbdy}{aa-bb}+\frac{aab^4dy}{(aa-bb)\left(b^4+(aa-bb)yy\right)}.$$

Ad integrationem huius formulae absolvendam duo considerandi sunt casus, alter si $a > b$, alter si $a < b$. Priore enim casu resistentia a quadratura circuli, posteriore a logarithmis pendebit. Moveatur igitur primum ellipsis vertex acutior *A* in directione axis maioris *EA* in aqua, erit

$$\int \frac{dy^3}{ds^2} = -\frac{bby}{aa-bb} + \frac{aab^4 dy}{(aa-bb)^{\frac{3}{2}}} \text{Atang.} \frac{y\sqrt{(aa-bb)}}{bb},$$

ideoque resistentia quam arcus *MAN* patietur erit

$$= \frac{2a^2b^4v}{(aa-bb)^{\frac{3}{2}}} \text{Atang.} \frac{y\sqrt{(aa-bb)}}{bb} - \frac{2b^2vy}{aa-bb}.$$

Quare si integra semi ellipsis *BAD* constituat partem figurae anteriorem, erit resistentia

$$= \frac{2a^2b^2dy}{(aa-bb)^{\frac{3}{2}}} \text{Atang.} \frac{\sqrt{(aa-bb)}}{b} - \frac{2b^3y}{aa-bb}.$$

Sit autem nunc $a < b$ seu *A* ellipsis vertex obtusior, erit

$$\int \frac{dy^3}{ds^2} = \frac{bby}{bb-aa} + \frac{aabb}{2(aa-bb)^{\frac{3}{2}}} l \left(\frac{bb + y\sqrt{(aa-bb)}}{bb - y\sqrt{(aa-bb)}} \right);$$

hoc igitur casu resistentia arcus *MAN* prodibit

$$= \frac{bbvy}{bb-aa} + \frac{aabbv}{(aa-bb)^{\frac{3}{2}}} l \left(\frac{bb + y\sqrt{(aa-bb)}}{bb - y\sqrt{(aa-bb)}} \right).$$

Sive autem *b* sit maius sive minus quam *a* per seriem infinitam erit

$$\int \frac{dy^3}{ds^2} = y - \frac{aay^3}{3b^4} + \frac{aa(aa-bb)y^5}{5b^8}$$
$$- \frac{aa(aa-bb)^2 y^7}{7b^{12}} + \frac{aa(aa-bb)^3 y^9}{9b^{16}} - \frac{aa(aa-bb)^4 y^{11}}{11b^{20}} + \text{etc.}$$

unde fit resistentia quam arcus *MAN* patietur

$$=2vy\left(1-\frac{aay^2}{3b^4}+\frac{aa(aa-bb)y^4}{5b^8}-\frac{aa(aa-bb)^2y^6}{7b^{12}}+\text{etc.}\right).$$

Si igitur ellipsis tota medietas *BAD* resistentiam patiatur, erit resistentia

$$=2by\left(1-\frac{a^2}{3bb}+\frac{a^2(aa-bb)}{5b^4}-\frac{a^2(aa-bb)^2}{7b^6}+\frac{a^2(aa-bb)^3}{9b^8}-\text{etc.}\right).$$

COROLLARIUM 1

515. Si igitur eadem ellipsis tum secundum directionem axis *EA,* tum secundum directionem axis *DB* in aqua moveatur; erit resistentia in priori casu

$$=2bv\left(1-\frac{a^2}{3bb}+\frac{a^2(a^2-b^2)}{5b^4}-\frac{a^2(a^2-b^2)^2}{7b^6}+\text{etc.}\right),$$

resistentia vero in posteriore

$$=2av\left(1-\frac{bb}{3aa}-\frac{bb(aa-bb)}{5a^4}-\frac{bb(aa-bb)^2}{7a^6}-\text{etc.}\right).$$

COROLLARIUM 2

516. Resistentia ergo ellipsis secundum axem maiorem *EA* progredientis se habet ad resistentiam eiusdem ellipsis secundum axem minorem *DB* eadem celeritate promotae ut

$$b-\frac{a^2}{3b}+\frac{a^2(a^2-b^2)}{5b^3}-\text{etc. ad } a-\frac{bb}{3a}-\frac{bb(aa-bb)}{5a^3}-\text{etc.}$$

ubi a est semiaxis maior, b vero semiaxis minor.

COROLLARIUM 3

517. Si ergo differentia axium sit valde parva puta $a=b+d$ denotante d quantitatem vehementer parvam, erit resistentia semissis ellipsis *BAD* ad resistentiam semissis *ABE* ut

$$\frac{2b}{3}-\frac{4d}{15}+\frac{2dd}{21b},\ \text{ad}\ \frac{2b}{3}+\frac{14d}{15}+\frac{2dd}{21b},$$

hoc est ut 1 ad

$$1+\frac{9d}{5b}+\frac{18dd}{25bb};$$

quae congruit maxime cum hac ratione 1 ad

$$\left(1+\frac{d}{b}\right)^{\frac{9}{5}} \quad \text{seu} \quad b^{\frac{9}{5}}:a^{\frac{9}{5}}.$$

COROLLARIUM 4

518. Vera autem ratio, quam resistentia semissis *BAD* tenet ad resistentiam semissis *ABE* est ut

$$2a^2b^2\text{Atang.}\frac{\sqrt{(aa-bb)}}{b}-2b^3\sqrt{(aa-bb)}$$

ad

$$2a^3\sqrt{(aa-bb)}-a^2b^2 1\left(\frac{a+\sqrt{(aa-bb)}}{a-\sqrt{(aa-bb)}}\right)$$

quae igitur ratio, siquidem b ab a non multum discrepet, proxime accedet ad hanc $b^{\frac{9}{5}}:a^{\frac{9}{5}}$.

EXEMPLUM 5

519. Sit figura aquae in directione *CA* innatans composita ex duobus habebitur segmentis circularibus aequalibus *ABE* et *ADE* (Fig. 80), erit positis

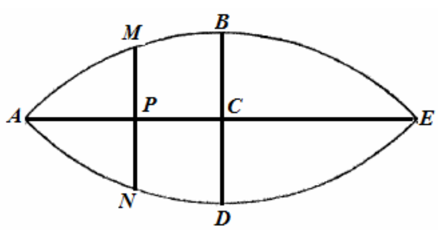


Fig. 80

$$AC=a,\ BC=CD=b,$$

radius circuli, cuius segmenta sunt sumta $\frac{aa+bb}{2b}$. Quare positis

$$AP=x,\ PM=PN=y$$

habebitur

$$x=a-\sqrt{\left(a^2-\frac{(a^2-b^2)}{b}y-yy\right)}$$

atque

$$dx = \frac{(aa - bb)dy + 2bydy}{2\sqrt{(a^2b^2 - (a^2 - b^2)by - bbyy)}},$$

unde erit

$$ds^2 = \frac{(a^2 + b^2)^2 dy^2}{4a^2b^2 - 4(a^2 - b^2)by - 4bbyy},$$

Quamobrem obtinebitur

$$\frac{dy^3}{ds^2} = \frac{4a^2b^2dy - 4(a^2 - b^2)bydy - 4bbyydy}{(a^2 + b^2)^2},$$

hincque

$$\int \frac{dy^3}{ds^2} = \frac{4a^2b^2y - 2(a^2 - b^2)byy - \frac{4}{3}bby^3}{(a^2 + b^2)^2};$$

quae expressio ducta in *2v* dabit resistentiam, quam portio *MAN* patitur. Quocirca si ponatur $y = b$ proveniet resistentia, quam integra pars anterior *BAD* sufferet $= \frac{4b^3v(3a^2 + b^2)}{3(a^2 + b^2)^2}$.

Resistentia autem, quam sentiet eadem figura, si in directione *CB* eadem celeritate moveatur, erit ex exemplo 3

$$= \frac{2av(3a^4 + 2a^2b^2 + 3b^4)}{3(a^2 + b^2)^2}.$$

COROLLARIUM 1

520. Resistentia quam patitur recta *BD* in directione *CA* mota, est

$$= 2bv = \frac{6bv(a^2 + b^2)^2}{3(a^2 + b^2)^2};$$

unde erit resistentia figurae *BAD* in directione *CA* motae ad resistentiam rectae *BD* in eadem directione motae ut

$$6aabb + 2b^4 \text{ ad } 3a^4 + 6aabb + 3b^4,$$

unde resistentia rectae *BD* multo maior est quam resistentia figurae *BAD*.

COROLLARIUM 2

521. Resistentia autem, quam patietur figura *ABED* mota secundum directionem *CA* se habebit ad resistentiam eiusdem figurae motae eadem celeritate in directione *DB* ut

$$6a^2b^2 + 2b^5 \text{ ad } 3a^5 + 6a^3b^2 + 3b^4;$$

si ergo $a > b$ resistentia prior semper est minor quam posterior.

SCHOLION

522. His consideratis satis intelligitur nullam dari figuram finitam, quae minimam patiatur resistentiam inter omnes alias iisdem terminis contentas. Quaecunque enim assignetur figura minimam patiens resistentiam, statim alia exhiberi posset, quae minorem resistentiam sentiret, tantum datam curvam vel eius tantum portionem versus eam regionem in quam fit motus elongando. Hanc ob rem nequidem quaestio foret instar problematis proponere inveniendam figuram planam, quae in aqua horizontaliter promota minimam sentiret resistentiam; ipsa enim solutio nullum dari minimum in finitis declararet. Quo autem appareat, quaenam figurae finitae reliquis ratione resistentiae sint praeferendae ad alias conditiones simul est attendendum, quibus curva quaesita cogatur esse finita. Eiusmodi autem quaestiones formari possunt, ut vel inter omnes figuras eandem aream habentes, vel inter omnes eadem perimetro cinctas ea determinetur, quae secundum datam directionem in aqua mota minimam patiatur resistentiam. Ad solvendas vero istius modi quaestiones conveniet lemma sequens praemittere, quo methodus omnia huius generis problemata solvendi continetur.

LEMMA

523. *Invenire curvam, quae maximi minimive proprietate quapiam gaudeat, vel inter omnes omnino curvas, vel inter eas tantum, quae una quadam sive pluribus proprietatibus aequaliter sint praeditae.*

SOLUTIO

Tam ea proprietas, quae in curva quaesita maxima minimave esse debet, quam eae proprietates, quae in curvas, e quibus electio est facienda, formulis integralibus indefinitis exprimentur; ex iisque formulis nullo discrimine habito, quaenam proprietatem maximam minimamve contineat, aut proprietates communes, natura curvae quaesitae sequenti modo definietur. Singulae formulae integrales propositae ad ordinatas orthogonales x et y reducantur, ut in illis aliae quantitates non insint praeter x et y, cum ipsorum differentalibus tam primi quam altiorum graduum. Posito autem dx constante fiat

$$dy = pdx,\ dp = qdx,\ dq = rdx,$$

etc. quibus substitutionibus quaeque formula proposita integralis reducetur ad huiusmodi formam $\int Zdx$, in qua Z erit quantitas composita ex finitis quantitatibus *x, y, p, q, r,* etc. Quare si ista quantitas *Z* differentietur, eius differentiale talem habebit formam, ut sit

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

Ex hoc differentiali formetur sequens quantitas

$$V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.}$$

atque eiusmodi valores *V* eliciantur ex singulis formulis integralibus propositis, quae vel maximum minimumve esse, vel omnibus curvis ex quibus quaesita est definienda, communes esse debent. Hi denique singuli valores *V* inventi multiplicentur per constantes quantitates quascunque respective, eorumque productorum summa ponatur $= 0$, quae aequatio naturam curvae quaesitae exprimet.
Hoc igitur facto restituantur loco *p, q, r,* etc. assumti valores scilicet

$$p = \frac{dy}{dx},\ q = \frac{ddy}{dx^2},\ r = \frac{d^3y}{dx^3}\ \text{etc.}$$

ut obtineatur aequatio pro curva quaesita solas binas variabiles *x* et *y* continens cum suis differentialibus, in qua sit *dx* constans. Q. E. I.

COROLLARIUM 1

524. Si igitur area curvae $\int ydx$ vel maxima minimave esse debeat, vel omnes curvae, ex quibus quaesita est definienda eiusdem areae ponuntur, erit

$$Z = y, \text{et } dZ = dy,$$

unde formulae $\int ydx$ valor ipsi *V* respondens erit $= 1$.

COROLLARIUM 2

525. Si vel curva maximae minimaeve longitudinis desideretur, vel omnes curvae, ex quibus quaesita debet inveniri eiusdem longitudinis ponantur, exprimetur ista proprietas hac formula $\int\sqrt{\left(dx^2 + dx^2\right)}$ quae ope substitutionis reducitur ad hanc $\int dx\sqrt{(1+pp)}$ erit ergo

$$Z = \sqrt{(1+pp)} \text{ et } dZ = \frac{pdp}{\sqrt{(1+pp)}},$$

unde erit

$$N=0,\ P=\frac{p}{\sqrt{(1+pp)}},\ Q=0,\ \text{etc.}$$

ideoque valor ipsius V formulae $\int dx\sqrt{(1+pp)}$ respondens erit

$$\frac{-dp}{(1+pp)^{\frac{3}{2}}dx}.$$

COROLLARIUM 3

526. Si eiusmodi curva quaeratur, quae aquae horizontaliter innatans secundum directionem axis, in quo abscissae x capiuntur, minimam pati debeat resistentiam, tum ista formula $\int\frac{dy^3}{ds^2}$ minimam esse oportebit, haec vero formula ob

$$dy=pdx \ \text{ et } \ ds^2=dx^2(1+pp)$$

abit in hanc $\int\frac{p^3dx}{1+pp}$. Cum igitur sit

$$Z=\frac{p^3}{1+pp},\ \text{erit}\ N=0,\ \ P=\frac{3pp+p^4}{(1+pp)^2},$$

atque valor ipsius V respondens erit $=-\frac{dP}{dx}$.

COROLLARIUM 4

527. Si igitur inter omnes omnino curvas ea desideretur, quae maximam minimamve resistentiam patiatur, unica habebitur formula $\int\frac{dy^3}{ds^2}$; cuius propterea valor ipsius V respondens debet esse $=0$. Habebitur ergo

$$dP=0,\ \ \text{et}\ \ P=\frac{3pp+p^4}{(1+pp)^2}=m$$

qua aequatione natura lineae quaesitae exprimetur.

COROLLARIUM 5

528. Cum igitur ex hac aequatione fiat p constans, sit

$$p = k \text{ erit } dy = kdx \text{ et } y = kx + c$$

unde fiet

$$k = \frac{y-c}{x} = p.$$

Qui valor in aequatione inventa substitutus dabit aequationem algebraicam inter x et y hanc

$$(y-c)^4 + 3xx(y-c)^2 = m\left(x^2 + (y-c)^2\right)^2,$$

quae quidem est pro linea recta seu pluribus rectis connexis.

COROLLARIUM 6

529. Quo posito $x = 0$ fiat simul $y = 0$, debebit esse vel $c = 0$ vel $m = 1$. At si sit $m = 1$ fiet $p = 1$ et $y = x$; sin autem ponatur $c = 0$ habebitur

$$y^4 + 3xxy^2 = m\left(x^2 + y^2\right)^2,$$

hincque

$$y = \frac{\pm x\sqrt{2m}}{\sqrt{\left(3 - 2m \pm \sqrt{(9-8m)}\right)}},$$

quae aequatio quatuor lineas rectas complectitur.

SCHOLION

530. Lemma hoc latissime patet, cum non solum iis problematis, quibus ex omnibus omnino curvis una, quae maximi minimive proprietate quapiam gaudeat, desideratur, resolvendis inserviat; sed etiam ad ea problemata sit accommodatum, quibus non ex omnibus curvis possibilibus, sed ex iis tantum, quae una pluribusve quibuscunque proprietatibus aequaliter sint praeditae una maximi minimive proprietate gaudens desideratur. Multo amplior igitur extat huius lemmatis usus, quam problematis Isoperimetrici, prout id quidem adhuc est tractatum, quo methodus traditur ex omnibus curvis vel eiusdem longitudinis vel aliam quandam proprietatem aequaliter possidentibus eam definiendi, quae ali qua maximi minimive proprietate gaudeat. N am praeterquam
quod methodus haec usitata unicam tantum spectat proprietatem, quae in omnes curvas competat, ea quoque ratione ipsarum formularum integralium quae vel maximae minimaeve vel omnibus curvis communes esse debent, ingenti restrictioni est obnoxia; cessat enim eius usus, statim atque in alteram sive in utramque formulam integralem differentialia secundi altiorisve cuiusdam gradus ingrediuntur, dum methodus hoc lemmate tradita ad cuiusvis gradus differentialia extenditur. At si ipse curvae arcus vel aliae formulae integrales in ipsa quantitate

Z contineatur, lemma allatum nullum amplius praestat usum, sed cum alia methodo coniungi debet, quam, quia eius usus in sequentibus non occurit, hic praetermisimus.

PROPOSITIO 53

PROBLEMA

531. *Inter omnes curvas AM cum axe AP et applicata PM eandem aream comprehendentes invenire eam AM, quae circa axem AP utrinque disposita formet figuram AM N in aqua minimam maximamve patientem resistentiam, si quidem in directione diametri PA progrediatur* (Fig. 76).

SOLUTIO

Positis abscissa $AP = x$, applicata $PM = y$, quaestio huc redit, ut inter omnes curvas, in quibus $\int ydx$ eundem obtinet valorem, ea determinetur in qua

$$\int \frac{dy^3}{ds^2} \text{ seu } \int \frac{p^3dx}{1+pp}$$

posito $dy = pdx$, sit maximum vel minimum. Priori igitur formulae $\int ydx$ respondet hic valor $V = 1$; posteriori vero

$$\int \frac{p^3dx}{1+pp} \text{ est } V = -\frac{dP}{dx}$$

existente

$$P = \frac{3pp + p^4}{(1+pp)^2} \text{ et } \int Pdp = \frac{p^3}{1+pp} .$$

Pro curva ergo quaesita obtinebitur ista aequatio

$$dP = \frac{dx}{a} \text{ atque } P = \frac{x+b}{a} = \frac{3pp+p^4}{(1+pp)^2}.$$

At ex eadem aequatione differentiali per p multiplicata

$$pdP = \frac{pdx}{a} = \frac{dy}{a}$$

oritur integrando

$$Pp - \frac{p^3}{1+pp} = \frac{y+c}{c} = \frac{2p^3}{(1+pp)^2},$$

unde fiet

$$x=\frac{3app+ap^4}{(1+pp)^2}-b \text{ et } y=\frac{2ap^3}{(1+pp)^2}-c$$

ex quibus formulis curva quaesita non difficulter construitur. Erit autem area

$$\int ydx=\frac{a^2P^2p}{2}-\frac{a^2Pp^2}{1+pp}+\frac{a^2p^5}{2(1+pp)^2}-cx+2a^2\int\frac{p^4dp}{(1+pp)^4}$$
$$=\frac{2a^2p^5}{(1+pp)^4}+2a^2\int\frac{p^4dp}{(1+pp)^4}-cx;$$

resistentia vero est seu

$$\int\frac{dy^3}{ds^2}=\frac{aPp^2}{1+pp}-a\int P^2dp=\frac{2ap^5}{(1+pp)^3}-4a\int\frac{p^4dp}{(1+pp)^4}.$$

His autem aequationibus tam ea curva, quae maximam, quam quae minimam patitur resistentiam continetur. Q. E. I.

COROLLARIUM 1

532. Si ponatur $b=0$ et $c=0$, curva manebit eadem; alius enim tantum axis priori parallelus accipitur, aliudque initium abscissarum. Pro hoc itaque axi si sumatur abscissa

$$x=\frac{3app+2ap^4}{(1+pp)^2},$$

erit applicata

$$y=\frac{2ap^3}{(1+pp)^2}.$$

COROLLARIUM 2

533. Si ergo sumatur $p=0$, tum fiet tam $x=0$ quam $y=0$; in initio igitur abscissarum incidet curva in axem, atque ob $\frac{dy}{dz}=0$, curva hoc loco ab axe tangetur.

COROLLARIUM 3

534. Si ponatur $p=\infty$, fiet $x=a$ et $y=0$ hancobrem eo loco ubi est $x=a$, curva iterum in axem incidet, hic vero tangens curvae erit normalis ad axem.

COROLLARIUM 4

535. Deinde perspicuum est tam abscissam quam applicatam usque ad certos tandem terminos crescere posse; obtinebit enim tam x quam y maximum valorem ponendo $p=\sqrt{3}$, hocque casu fit

$$x=\tfrac{9}{8}a \text{ et } y=\frac{3\sqrt{3}}{8}a.$$

COROLLARIUM 5

536. Denique sive p affirmativum sive negativum habeat valorem, abscissa x manet eadem, at y negativum obtinet valorem sumto p negativo, ex quo intelligitur axem in quo abscissae x capiuntur, simul esse diametrum curvae inventae.

SCHOLION 1

537. Cum sumto

$$x=\frac{3ap^2+ap^4}{(1+pp)^2} \text{ sit } y=\frac{2ap^3}{(1+pp)^2},$$

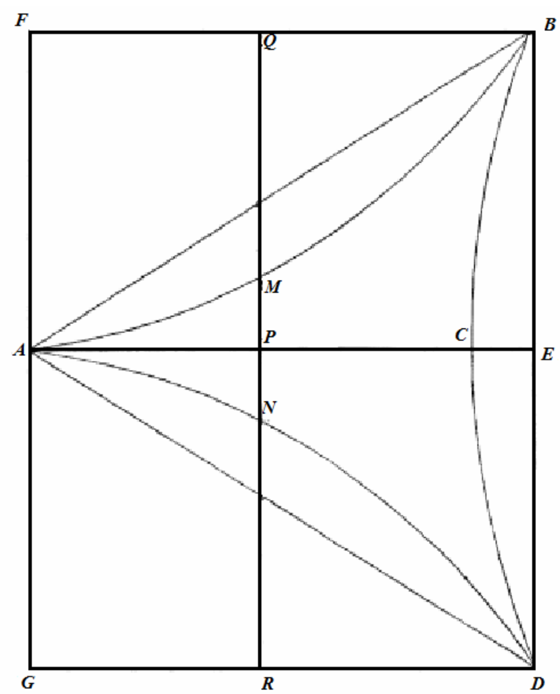


Fig. 81

curva erit algebraica, atque per infinita puncta descriptu facilis. Sumatur enim axis AC directioni, secundum quam figura in aqua movetur parallelus, atque constructio inventa praebebit curvam triangularem $AMBCDNA$ (Fig. 81) tres habentem cuspides A, B, D ad angulos trianguli aequilateri *ABD* dispositas, ac tres portiones inter cuspides comprehensae, *AMB, AND* et *BCD* erunt inter se aequales et similes. Erit autem

$$AC = a,\ AE = \tfrac{9}{8}a \text{ et } BE = DE = \frac{3\sqrt{3}}{8}a$$

tangentes vero in *B* et *D* cum recta *BD* constituant angulum 30 graduum. Cum igitur haec curva secundum directionem axis *AC* mota inter omnes alias eiusdem capacitatis tam maximam quam minimam, in aqua patiatur resistentiam, intelligere licet portionem *BMAND* minimam esse passuram resistentiam aream vero *BCD* maximam. Quare si curva desideretur, quae inter omnes eandem aream continentes minimam patiatur resistentiam, pro ea vel arcus *AMB* seu *AND* vel portio quaecunque erit accipienda. Pro navibus autem commodissimum erit utrique semissi partis anterioris accipere figuram *DNAG* seu *BMAF* ita ut *D* cadat in proram, et recta *DG* in spinam navis; si enim figura *DNA* ad utramque partem axis *DG* disponatur habebitur figura quae in aqua secundum directionem *G D* inter omnes alias eandem aream *DNAG*continentes, et per puncta *D etA* transeuntes minimam patietur resistentiam; atque haec eadem curva inter omnes alias per *A* et *D* ductas et eandem resistentiam patientes maximam habebit aream *DNAG.* Quo autem natura huius curvae navibus maxime accommodata respectu axis *DG* inspiciatur, sit $DR = t,\ NR = u$, atque cum sit

$$t = \tfrac{9}{8}a - x \text{ et } u = \frac{3\sqrt{3}}{8}a - y,$$

erit

$$DR = t = \frac{(3 - pp)^2 a}{8(1 + pp)^2},$$

atque

$$NR = u = \frac{\left(p - \sqrt{3}\right)^2 \left(3p^2\sqrt{3} + 2p + \sqrt{3}\right)a}{8(1 + pp)^2}.$$

Potest vero etiam aequatio prima non incongrue conservari, qua est

$$DR = t = \frac{(3 - pp)^2 a}{8(1 + pp)^2},$$

$$AP = GR = x = \frac{3app + ap^4}{(1 + pp)^3}, \text{ et } PN = y = \frac{2ap^3}{(1 + pp)^2};$$

ex qua erit

$$dx = \frac{2apdp(3 - pp)}{(1 + pp)^3}, \text{ et } PN = y = \frac{2appdp(3 - pp)}{(1 + pp)^3},$$

atque

$$\sqrt{\left(dx^2+dy^2\right)}=ds=\frac{2apdp\left(3-pp\right)}{\left(1+pp\right)^{\frac{5}{2}}},$$

unde fiet ipse arcus

$$AN=s=\frac{2ap\left(3pp-1\right)}{\left(1+pp\right)^{\frac{3}{2}}}+\frac{2a}{8},$$

ita ut curva inventa sit rectificabilis. Resistentia autem quam patietur pars AN erit ut

$$\int\frac{dy^3}{ds^2}=\frac{\frac{1}{2}ap+\frac{7}{6}ap^3+2ap^5}{(1+pp)^3}-\frac{a}{2}\int\frac{dp}{1+pp}.$$

Posito igitur $p=\sqrt{3}$ proveniet tota curva $AND=\frac{4}{3}a,$ eius vero subtensa unde arcus AND se habebit ad subtensam AD ut 16 ad $9\sqrt{3}$. Resistentia vero quam patietur tota curva DNA erit ut

$$\frac{11a\sqrt{3}}{32}-\frac{\pi a}{6}$$

denotante π peripheriam circuli, cuius diameter est 1. Resistentia curvae ergo se habet ad resistentiam chordae AD ut 4 : 9 proxime. Haec curva AND praeterea in A habet tangentem axi GD parallelam, atque D curva cum axe facit angulum 60 graduum; in A et D vero radius osculi est infinite parvus.

Quomodo autem se haec curva AND respectu subtensae AD habeat, ex parte BCD facilius perspicitur, ubi est

$$CE=\tfrac{1}{8}a,\ BE=DE=\frac{3\sqrt{3}}{8}a,$$

atque radius osculi in puncto medio C est $=2a$; unde constructio practica facile concinnatur.

COROLLARIUM 6

538. Si igitur parti navis anteriori tribuatur figura AND, existente D prora et DG spina, navis in directione GD progrediens non solum minimam patietur resistentiam sed insuper si ita moveatur, ut chorda AD ad cursus directionem fiat normalis, tunc maximam patietur resistentiam, quia curva AND congruit cum BCD.

COROLLARIUM 7

539. Hoc igitur ipso haec figura se commendat ad navibus optimam formam tribuendam; nam non solum requiritur ut navis in directione spinae progrediens minimam offendat resistentiam, sed etiam ut in cursu obliquo resistentia fiat vehementer magna.

COROLLARIUM 8

540. Resistentia vero quam sentiet figura *AND* si in directione ad chordam *AD* normali in aqua moveatur, erit ad hanc chordam normalis atque

$$=\frac{11a\sqrt{3}}{16}+\frac{\pi\alpha}{6}.$$

Si vero figura secundum directionem *GD* moveatur, atque ex utraque parte axis *DG* sui sit similis erit resistentia

$$=\frac{11a\sqrt{3}}{16}-\frac{\pi\alpha}{6}.$$

COROLLARIUM 9

541. Si ergo partis anterioris navis aquae submersae singulae sectiones horizontales habuerint eiusmodi figuram, ut earum semisses omnes aequales sint vel similes figurae *DNA,* tum navis aptissimam habebit figuram ad aquae resistentiam superandam, atque simul comprehendet maximum spatium, cuius ratio in navibus praecipue est habenda.

SCHOLION 2

542. Quae autem hic sunt allata proprie tantum ad figuras planas aquae horizontaliter innatantes extenduntur, neque ad corpora solida, ac naves nisi cum summa cautione possunt accommodari. Ita figura plana minimam patiens resistentiam inter omnes aequicapaces, quae hic est inventa in solidis locum non invenit nisi omnes corporis natantis sectiones horizontales sint inter se aequales; et hancobrem si haec per experimenta confirmare lubuerit, asseres ubique eiusdem crassitudinis adhibere convenit, qui eandem resistentiae legem tenebunt ac figurae planae seu crassitiei evanescentis; hoc scilicet casu latera assertum resistentiam excipientia situm tenent verticalem, ideoque sub iisdem angulis in aquae particulas incurrunt, sub quibus quaelibet sectiones horizontales. At si figura submersa obliquum teneat situm ad aquam, seu si latera resistentia aquae opposita non fuerint verticalia sed ad horizontem inclinata, tum angulus incidentiae differt, ab illo angulo, sub quo sola sectio horizontalis aquae occurrit; et hancobrem in eiusmodi corporibus, quam vis resistentiae, quas singulae sectiones horizontales sentiunt, sint cognitae, tamen resistentia totalis exinde definiri nequit. Quo circa ne ex hic traditis vitiosae deriventur conclusiones pro resistentia corporum, consultum est iudicium suspendere: quoad in sequentibus resistentiam, quam quaecunque corpora in aqua perpetiuntur, simus determinaturi.

PROPOSITIO 54

PROBLEMA

543. *Si figura quaecunque plana BCA* (Fig. 82) *situ verticali in aqua secundum directionem horizontalem MD promoveatur data cum celeritate, determinare resistentiae, quam offendet tam quantitatem quam directionem mediam.*

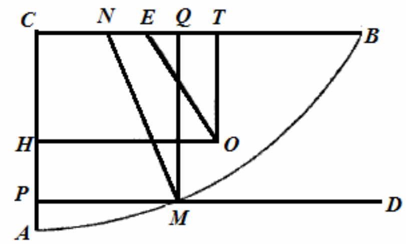


Fig. 82

SOLUTIO

Sumatur verticalis *AC* pro axe, in quo sit abscissa $CP = x = MQ$, applicata $PM = y = CQ$; atque arcus $AM = s$; erit sinus anguli, quo curvae *AMB* punctum *M* in aquam incurrit $= \frac{dx}{ds}$, ex quo resistentiae, quam elementum *ds* patietur, vis erit $= \frac{vdx^2}{ds}$, denotante *v* altitudinem celeritati qua figura promovetur, debitam, cuius vis directio erit *M N* normalis ad curvam in *M.* Resolvatur nunc haec vis in binas laterales, quarum alterius directio sit horizontalis *MP,* alterius verticalis *MQ,* eritque vis horizontalis

$$MP = -\frac{vdx^3}{ds^2},$$

et vis verticalis

$$MQ = +\frac{vdx^2dy}{ds^2}.$$

Hinc erit summa omnium virium horizontalium quas arcus *AM* patitur

$$= -v\int \frac{dx^3}{ds^2},$$

et summa virium verticalium

$$= +v\int \frac{dx^2dy}{ds^2};$$

ita sumtis his integralibus ut evanescant facto *s* vel $y = 0$. Quare si ponatur $x = 0$, tum prodibunt vires quas tota curva *AMB* ab aqua patitur. Sit autem vis totalis horizontalis

$$= -v\int \frac{dx^3}{ds^2},$$

directio *OH,* vis totalis vero verticalis

$$= +v\int \frac{dx^2 dy}{ds^2}$$

directio *OI*, erit sumendis momentis respectu puncti *C,*

$$-CH \cdot v\int \frac{dx^3}{ds^2} = -v\int \frac{xdx^3}{ds^2};$$

atque

$$CI \cdot v\int \frac{dx^2 dy}{ds^2} = v\int \frac{ydx^3 dy}{ds^2}.$$

Hinc igitur obtinetur

$$CH = \frac{\int xdx^3 : ds^2}{\int dx^3 : ds^2};\ \text{et } CI = \frac{\int ydx^2 dy : ds^2}{\int dx^3 dy : ds^2};$$

omnibus integralibus ita sumtis ut evanescant posito s seu $y = 0$ tumque facto $x = 0$. Effectus igitur resistentiae totalis in hoc consistit, ut figura retro urgeatur in directione horizontali *OH* a vi

$$= -v\int \frac{dx^3}{ds^2},$$

simulque sursum urgeatur in directione *OI* a vi

$$= v\int \frac{dx^2 dy}{ds^2}.$$

Media ergo directio totius resistentiae cadet in *OK* existente

$$IK : OI = -\int \frac{dx^3}{ds^2} : \int \frac{dx^2 dy}{ds^2},$$

unde erit

$$IK = \frac{-OI\int dx^3 : ds^2}{\int dx^2 dy : ds^2} = \frac{-\int xdx^3 : ds^2}{\int dx^2 dy : ds^2};$$

ideoque

$$CK = \frac{\int (xdx + ydy)dx^2 : ds^2}{\int dx^2 dy : ds^2}.$$

Anguli vero *OKB* tangens erit

$$= \frac{\int dx^2 dy : ds^2}{-\int dx^3 : ds^2},$$

ex quibus positio mediae directionis resistentiae *OK* cognoscitur. Ipsa vero resistentiae vis erit

$$= v\sqrt{\left(\int \frac{dx^3}{ds^2}\right)^2 + \left(\int \frac{dx^2 dy}{ds^2}\right)^2}.$$

Q.E.I.

COROLLARIUM 1

544. Duplicem igitur resistentia in figuram *BCA* exerit effectum, quorum alter consistit in motu figurae retardando, atque oritur a vi horizontali $= -v\int \frac{dx^3}{ds^2}$, cuius directio est *OH*.

COROLLARIUM 2

545. Altera autem vis ex resistentia orta $= v\int \frac{dx^2 dy}{ds^2}$, cuius directio est verticalis secundum *OI* motum figurae non afficit, sed eam ex aqua elevat et quasi leviorem facit.

COROLLARIUM 3

546. Nisi igitur vis verticalis $v\int \frac{dx^2 dy}{ds^2}$ evanescat vel negativa fiat, figura dum movetur ex aqua magis emerget, perinde ac si levior esset facta; eoque magis elevabitur ex aqua, quo celerius in aqua progreditur; decrementum scilicet gravitatis est ut quadratum celeritatis.

COROLLARIUM 4

547. Nisi autem ubique sit vel $dy = 0$, quod evenit quando linea *BMA* abit in rectam verticalem, vel usquam fiat *dy* negativum, vis ista verticalis figuram ex aqua elevans semper tenebit valorem affirmativum.

COROLLARIUM 5

548. Deinde haec vis verticalis, quia eius directio in proram cadit, figuram etiam ita inclinabit, ut prora elevetur; puppis vero deprimatur, nisi vis horizontalis *OH* profundius sit sita quam centrum gravitatis, ideoque inclinationem contrariam efficiat.

COROLLARIUM 6

549. Vis autem horizontalis *OH,* qua motus figurae retardatur, eo erit minor, quo magis figura versus *B* fuerit cuspidata. Atque si inter omnes figuras eandem aream *BAG* comprehendentes ea quaeratur, quae ab aqua quam minime retardetur, ea ipsa reperietur figura, quae in propositione praecedente est inventa. Perinde enim se habet resistentia sive figura *BMA* horizontali situ promoveatur sive verticali.

COROLLARIUM 7

550. Maxime autem figura ex aqua elevabitur, seu vis verticalis *OI* erit maxima, si linea curva *BMA* abeat in rectam, quae angulum cum horizontali *BC* constituat $54^{\circ}, 44'$ seu cuius cosinus est $\frac{1}{\sqrt{3}}$.

SCHOLION 1

551. Haec propositio potissimum inservit ad resistentiam definiendam, quam spina navis in aqua progredientis perpetitur, ex ea enim intelligitur non solum quantum motus navis a spinae resistentia retardetur, sed etiam quantum ipsa navis a resistentia aquae elevetur et quasi levior reddatur. Si autem praeterea corpus in aqua motum ita fuerit comparatum, ut omnes sectiones verticales in directione motus factae sint inter se similes et aequales, tum ex hac propositione quoque resistentia colligi potest, ita si cylindrus aquae horizontaliter incubans ita moveatur, ut eius axis ad directionem motus sit normalis tum curva *AM B* erit arcus circuli, atque hinc resistentia innotescet. Deinde vero eadem haec propositio magnam habebit utilitatem in sequentibus, ubi sumus investigaturi, quantam resistentiam figura plana horizontalis quae in aqua secundum directionem obliquam progreditur, patiatur, hoc enim casu resistentia utriusque semissis figurae seorsim est investiganda, et ex utraque media directio totius resistentiae concludenda. Nostro enim casu perinde se habet resistentia sive figura *BMAC* in situ verticali sive horizontali in aqua progrediatur.

EXEMPLUM 1

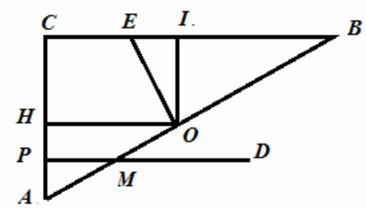


Fig. 83

552. Sit figura plana triangulum *BAC* (Fig. 83) quod in aqua secundum directionem *MD* progrediatur celeritate altitudini *v* debita; cuius resistentia quam Fig.83 quam facile ex propositione 50 determinatur, tamen eam ope formularum hic inventarum illustrationis causa sumus investigaturi. Sit itaque

$$BC = a, AC = b, \text{ ob } CP = x,\ PM = y \text{ erit } b - x : y = b : a$$

ideoque

$$y = a - \frac{ax}{b},\ dy = -\frac{adx}{b} \text{ et } ds = \frac{dx\sqrt{a^2+b^2}}{b}.$$

Hinc erit vis horizontalis resistentiae secundum directionem *OH* agens

$$= -v\int \frac{dx^3}{ds^2} = -\frac{b^2 v}{a^2+b^2}\int dx,$$

quae ita integrata, ut evanescat posito $y = 0$ seu $x = b$ erit $= \frac{b^2 v(b-x)}{a^2+b^2}$, posito ergo $x = 0$, erit vis horizontalis totalis $= \frac{b^2 v}{a^2+b^2}$. Vis vero verticalis seu cuius directio est *OI* est

$$= v\int \frac{dx^2 dy}{ds^2} = \frac{-abv}{a^2+b^2}\int dx = \frac{abv(b-x)}{a^2+b^2},$$

unde prodit vis verticalis totalis $= \frac{ab^2 v}{a^2+b^2}$. Ad positionem vero rectarum *OH* et *OI* inveniendam iam cognita sunt

$$= \int \frac{dx^3}{ds^2} = \frac{-b^3}{a^2+b^2} \quad \text{et} \quad \int \frac{dx^2 dy}{ds^2} = \frac{ab^2}{a^2+b^2},$$

quamobrem quaerenda sunt

$$\int \frac{xdx^3}{ds^2} \quad \text{et} \quad \int \frac{ydx^2 dy}{ds^2}.$$

Est vero

$$\int \frac{xdx^3}{ds^2} = \frac{b^2}{a^2+b^2}\int xdx = \frac{-b^4}{2\left(a^2+b^2\right)}$$

et

$$\int \frac{ydx^2dy}{ds^2} = \frac{-a^2}{a^2+b^2}\int (b-x)\,dx = \frac{aabb}{2\left(a^2+b^2\right)};$$

integratione uti est praeceptum ita absoluta ut integralia evanescant posito $x=b$ tumque facto . Hinc igitur erit

$$CH = \frac{b}{2} = \tfrac{1}{2}AC \text{ et } \mathrm{CI} = \frac{a}{2} = \tfrac{1}{2}BC;$$

punctum igitur *O* cadit in ipsum medium rectae *AB.* Quoniam autem posita *OK* media directione resistentiae, est

$$IK : OI = b : a = AC : BC,$$

ex qua analogia perspicitur mediam resistentiae directionem *OK* esse normalem ad rectam *AB;* vis denique ipsa resistentiae *OK* est $= \dfrac{bbv}{\sqrt{\left(a^2+b^2\right)}}$; quae quidem omnia ex propositione 50 sponte consequuntur.

COROLLARIUM

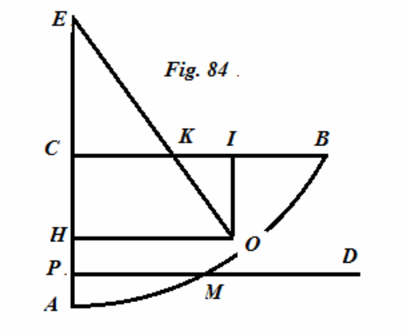
553. Cum igitur sit $IK : OI = b : a$ erit

$$IK = \frac{bb}{2a} \text{ et } CK = \frac{aa-bb}{2a};$$

angulus vero quem media directio resistentiae *OK* cum horizontali *BC* constituit est $=$ ang. CAB, eiusve tangens est $= \dfrac{a}{b}$.

EXEMPLUM 2

554. Sit figura *BCA* (Fig. 84) semisegmentum circulare seu *BMA* arcus circuli, cuius tangens in *A* sit horizontalis, cuius ideo centrum cadet in *E* punctum rectae verticalis *AGE*.



Ponatur, $BC = a$, $AC = b$ et radius $AE = c$, erit

$$c^2 = a^2 + (c-b)^2 \quad \text{seu} \quad c = \frac{aa + bb}{2b}.$$

Posito nunc $CP = x$, et $PM = y$, erit $EP = c - b + x$, indeque ex natura circuli

$$c^2 = y^2 + (c-b+x)^2 \text{ seu } y = \sqrt{c^2 - (c-b+x)^2}$$

ex qua fit $y = 0$, si est $x = b$. Erit autem porro

$$dy = \frac{-(c-b+x)dx}{\sqrt{c^2-(c-b+x)^2}} \quad \text{atque} \quad ds = \frac{cdx}{\sqrt{c^2-(c-b+x)^2}};$$

hincque

$$\frac{dx^2}{ds^2} = 1 - \frac{(c-b+x)^2}{cc}.$$

Quaerantur igitur sequentia integralia hac conditione ut evanescant posito $y = 0$ seu $x = b$, in iisque ponatur post integrationem $x = 0$ seu $y = a$.
Reperietur autem

$$\int \frac{dx^3}{ds^2} = \frac{-bb(3c-b)}{3cc}, \quad \int \frac{dx^2 dy}{ds^2} = \int \frac{yydy}{cc} = \frac{a^3}{3cc},$$

atque

$$\int \frac{xdx^3}{ds^2} = \int xdx - \int \frac{xdx(c-b+x)^2}{cc} = \frac{-b^3(4c-b)}{12cc} \quad \text{et} \quad \int \frac{ydx^2dy}{ds^2} = \frac{a^4}{4cc}.$$

Ex his invenitur

$$CH = \frac{b(4c-b)}{4(3c-b)} \text{ et } CI = \frac{3a}{4}.$$

Praeterea vero erit vis horizontalis in directione *OH*

$$=\frac{bbv(3c-b)}{3cc}$$

et vis verticalis in directione *OI*

$$=\frac{a^3v}{3cc},$$

quare media directio totius resistentiae erit *OK* existente

$$IK=\frac{b^3(4c-b)}{4a^3} \quad \text{seu} \quad CK=\frac{3a^3-4bc+b^4}{4a^3},$$

angulique *OKI* tangens erit $=\frac{a^3}{bb(3c\ \ -b)}$. Universa igitur resistentia aequipollebit vi in directione *OK* urgenti quae est

$$=\frac{v}{3cc}\sqrt{a^6+b^4(3c-b)^2}.$$

COROLLARIUM 1

555. Cum sit $c=\frac{aa+bb}{2b}$, erit $aa=2bc-bb$ ideoque

$$CK=\frac{12bbcc-16b^3c+4b^4}{4a^3}=\frac{bb(c-b)(3c-b)}{a^3}.$$

Si ergo *OK* producatur concurret ea cum *AC* in ipso centro circuli *E,* est enim

$$\frac{CE}{CK}=\frac{a^3}{bb(3c-b)}=\text{tang. ang.}OKI=\frac{OI}{KI}.$$

COROLLARIUM 2

556. Resistentiae igitur media directio *OE* per ipsum circuli centrum *E* transit, atque cum recta *AE* angulum *AEO* constituet, cuius tangens erit

$$=\frac{bb(3c-b)}{a^3}=\frac{(3c-b)\sqrt{b}}{(2c-b)\sqrt{(2c-b)}}, \quad \text{sinus vero}=\frac{(3c-b)\sqrt{b}}{c\sqrt{(8c-3b)}}.$$

COROLLARIUM 3

557. Quantitas vero totius resistentiae, quae se in directione *OE* exerit est

$$=\frac{v}{3cc}\sqrt{8b^3c^3-3b^4cc}=\frac{bv}{3c}\sqrt{b(8c-3b)};$$

aequatur scilicet ponderi cylindri aquei cuius altitudo est *v,* basis vero

$$=\frac{b}{3c}\sqrt{b(8c-3b)}$$

ducto in latitudinem seu crassitiem figurae si quam habet.

COROLLARIUM 4

558. Quoniam directio resistentiae, quam singula elementa patiuntur, est ad curvam normalis, ea per centrum *E* transibit, unde sponte sequitur mediam directionem resistentiae quam arcus *BMA* patitur, per centrum *E* transire debere.

COROLLARIUM 5

559. Si arcus *AMB* quadranti aequetur, fiet $a=b=c$, atque anguli *AEO* tangens erit $=2$; potentiae vero resistentiae aequivalentis quantitas erit $=\frac{c\sqrt{5}}{3}$.

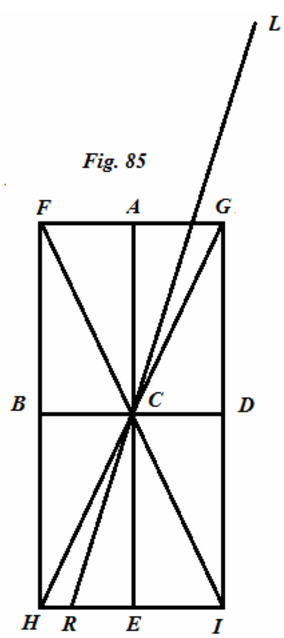
COROLLARIUM 6

560. At si arcus *AMB* abeat in semicirculum ut fiat $b=2c$, angulus *AEO* fiet rectus seu media resistentiae directio erit horizontalis, vis autem totius resistentiae prodibit $=\frac{4c}{3}$, prout iam ex ante § 509 traditis colligere licet.

SCHOLION 2

561. Cum igitur tam pro figura plana duabus partibus similibus et aequalibus gaudente, si secundum directionem diametri horizontaliter promoveatur, quam pro figura plana in aqua verticaliter promota resistentiam determinaverimus, ex instituto revertemur ad figuras planas aquae in situ horizontali innatantes, atque resistentiam etiam definiemus, cum non directe secundum diametrum sed oblique promoventur. Haec enim investigatio multo difficilior est quam praecedens, cum resistentia, quam figura ex utraque diametri parte ob dissimilem allisionem ad aquam patitur sit dissimilis; et hancobrem tam directionem mediam resistentiae quam ipsam resistentiae quantitatem determinari oportet. Facile enim intelligitur in istiusmodi motu obliquo directionem mediam

Fig. 85



resistentiae non in diametrum incidere, sed diametrum alicubi secare, cum eaque certum quendam angulum constituere; ubi illud punctum in quo diameter et media directio resistentiae se intersecant brevitatis ergo centrum resistentiae appellabimus; quippe cuius cognitio ad effectum resistentiae, in figura circum axem verticalem convertenda summe est necessaria. Incipiemus autem hanc tractationem a figuris simplicioribus, et primo quidem rectangulum parallelogrammum consideremus, quo perspiciatur, quantum tam media directio resistentiae quam ipsa resistentia pro varia cursus obliquitate immutetur.

PROPOSITIO 55

PROBLEMA

562. *Si parallelogrammum rectangulum FGIH* (Fig. 85) *in aqua secundum directionem quamcunque obliquam CL promoveatur, invenire resistentiae quam patietur tum directionem tum quantitatem.*

SOLUTIO

Sit rectanguli *FGIH* latitudo $FG = HI = a$, longitudo $FH = GI = b$, atque ducatur axis *AE* itemque per centrum figurae *C* transversa normalis *BD,* ut sit $AC = \frac{1}{2}b$, et $BC = CD = \frac{1}{2}a$. Anguli autem obliquitatis cursus *ACL* sinus sit $= m$, cosinus vero $= n$, posito sinu toto $= 1$, ita ut futurum sit $m^2 + n^2 = 1$; celeritas autem qua rectangulum in directione *CL* progreditur, debita sit altitudini *v.* Iam dum haec figura promovetur, latera bina *FG* et *GI* erunt resistentiae exposita, atque anguli quo latus *F G* in aquam impingit sinus erit $= n$; anguli vero quo latus *GI* in aquam irruit sinus est $= m$. Resistentia igitur quam latus *FG* patietur erit $= n^2av$, eiusque directio incidet in axem *ACE;* resistentia vero quam patietur latus *G I* erit $= m^2bv$, eiusque directio erit recta *DB.* Resistentia ergo aequivalet duabus viribus in puncto *C* applicatis quarum altera est n^2av et directionem habet *CE* alterius vero m^2bv directio est *CB.* Media consequenter resistentiae directio incidet in rectam *CR* existente anguli *RCE* tangente $= \frac{m^2b}{n^2a}$, ipsiusque resistentiae in directione *CR* urgentis quantitas est $v\sqrt{n^4a^2 + m^4b^2}$. Q.E.I.

COROLLARIUM 1

563. In quacunque igitur directione parallelogrammum rectangulum progrediatur, media directio resistentiae perpetuo transibit per eius punctum medium *C,* seu centrum resistentiae incidet in centrum figurae *C.*

COROLLARIUM 2

564. Si igitur simul centrum gravitatis rectanguli in centrum figurae *C* incidat, tum resistentia omni carebit vi figuram circa centrum gravitatis convertendi, atque tota resistentia impendetur ad motum ipsum alterandum.

COROLLARIUM 3

565. Si anguli *LCA* tangens ponatur $= v$ erit $\frac{m}{n} = v$, atque anguli *RCE* tangens erit $\frac{v^2 b}{a}$. Hancobrem resistentiae directio directe contraria erit ipsi motui, si fuerit vel $v = 0$, hoc est si figura secundum directionem *CA* progrediatur, vel $v = \frac{a}{b}$ hoc est si figura secundum diagonalem *HCG* progrediatur.

COROLLARIUM 4

566. Sit angulus $ACL < ACG$ seu $v = \frac{a}{\alpha b}$ denotante α numerum unitate maiorem, erit anguli *ECR* tangens $v = \frac{a}{\alpha^2 b}$, unde sequitur angulum *ECR* fore minorem angulo *LCA* . Contra vero si fuerit angulus $ACL > ACG$, tum angulus *ECR* quoque maior erit quam angulus *ACL*.

COROLLARIUM 5

567. Differentiae autem angulorum *ACL* et *ECR* tangens est $= \frac{v^2 b - va}{a + v^3 b}$, unde differentia horum angulorum prodibit maxima si capiatur *v* ex hac aequatione

$$bbv^4 - 2abv^3 - 2abv + aa = 0.$$

SCHOLION

568. Radices huius aequationis $b^2 v^4 - 2abv^3 - 2abv + a^2 = 0$, eo modo reperiri possunt, quo vulgo aequationes biquadratae ad cubicas reduci solent sed hic commode accidit, ut cubica aequatio prodeat pura. Sit enim $a = kb$, habebitur $v^4 - 2kv^3 - 2kv + kk = 0$, cuius factores ponantur hae aequationes $v^2 - \alpha v + \beta = 0$ et $v^2 - \delta v + \varepsilon = 0$; eritque

$$\alpha + \delta = 2k,\ \beta + \varepsilon + \alpha\delta = 0,\ \alpha\varepsilon + \beta\delta = 2k \text{ et } \beta\varepsilon = k^2.$$

Sit $\alpha\delta = 2h$; erit

$$\alpha - \delta = 2\sqrt{k^2 - 2h},$$

atque

$$\alpha = k + 2\sqrt{k^2 - 2h} \text{ et } \delta = k - 2\sqrt{k^2 - 2h}.$$

Quia autem porro est

$$\beta + \varepsilon = -2h, \text{ et } \beta\varepsilon = k^2,$$

erit

$$\beta - \varepsilon = 2\sqrt{h^2 - k^2}, \text{ et } \beta = -h + \sqrt{h^2 - k^2} \text{ ac } \varepsilon = -h - \sqrt{h^2 - k^2}.$$

Cum denique sit $\alpha\varepsilon + \beta\delta = 2k$, erit $k + kh = -\sqrt{\left(k^2 - 2h\right)\left(h^2 - k^2\right)}$, unde sumendis quadratis oritur

$$k^2 = -k^4 - 2h^3, \text{ seu } h = \frac{\sqrt[3]{k^2\left(k^2+1\right)}}{\sqrt[3]{2}} = -\frac{\sqrt[3]{4k^2\left(k^2+1\right)}}{2}.$$

Dato autem h dabuntur α, β, δ, et ε, per superiores aequationes, indeque erit vel

$$v = \frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha^2}{4} - \beta\right)} \text{ vel } v = \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta^2}{4} - \varepsilon\right)}.$$

Factis autem substitutionibus reperitur vel

$$v = \frac{k + \sqrt{\left(k^2 + \sqrt[3]{4k^2\left(k^2+1\right)}\right)} \pm \sqrt{\left(2k^2 - \sqrt[3]{4kk\left(k^2+1\right)} + 2k\sqrt{\left(k^2 + \sqrt[3]{4k^2\left(k^2+1\right)}\right)}\right) - 2\sqrt{\left(\sqrt[3]{16k^4\left(k^2+1\right)^2} - 4k^2\right)}}}{2}$$

vel

$$v = \frac{k - \sqrt{\left(k^2 + \sqrt[3]{4k^2\left(k^2+1\right)}\right)} \pm \sqrt{\left(2k^2 - \sqrt[3]{4kk\left(k^2+1\right)} - 2k\sqrt{\left(k^2 + \sqrt[3]{4k^2\left(k^2+1\right)}\right)}\right) - 2\sqrt{\left(\sqrt[3]{16k^4\left(k^2+1\right)^2} - 4k^2\right)}}}{2}$$

quae ergo sunt omnes quatuor radices huius aequationis biquadratae

$$v^4 - 2kv^3 - 2kv + kk = 0.$$

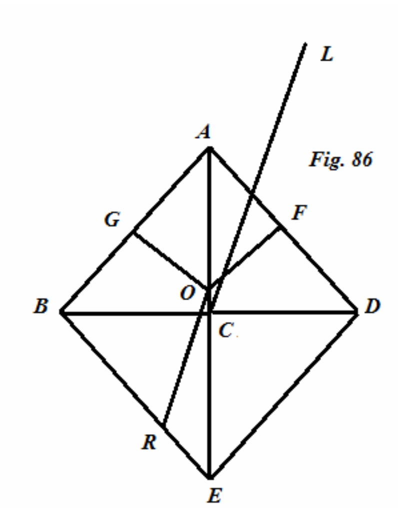
Posito igitur $\frac{a}{b}$ loco k habebuntur illi cursus obliqui, cum quibus directio resistentiae minime congruit; ab hoc autem discrimine pendet insignis illa navium proprietas, qua cursus etiam versus ventum dirigi potest, quam ob causam etiam istam discrepantiam quo casu maxima evadit diligentius investigandam censuimus.

PROPOSITIO 56

PROBLEMA

569. *Moveatur rhombus ABED* (Fig. 86) *aquae horizontaliter insidens oblique secundum directionem CL ita tamen ut sola bina latera anteriora AB et AD resistentiam sustineant: definire resistentiae tam directionem quam magnitudinem.*

SOLUTIO



Ponatur rhombi latus quodvis $AB = AD = a$, semidiameter $AC = b$, et alterius diagonalis semissis $BC = CD = c$, ut sit $a^2 = b^2 + c^2$. Directio vero motus CL cum axe CA angulum constituat ACL cuius sinus sit $= m$, cosinus vero $= n$ posito sinu toto $= 1$; celeritas denique qua rhombus in hac directione promovetur sit debita altitudini v. Cum iam sit anguli CAD sinus $= \frac{c}{a}$ et cosinus $= \frac{b}{a}$ erit anguli sub quo latus AD in aquam impingit sinus $= \frac{nc + mb}{a}$, anguli vero sub quo latus AB in aquam impingit sinus $= \frac{nc - mb}{a}$. Hinc resistentia, quam latus AD patietur erit $= \frac{(nc + mb)^2 v}{a}$, eiusque directio erit recta FO, quae ad AD in eius puncto medio F normaliter insisit. Simili modo resisentia lateris AB erit $= \frac{(nc + mb)^2 v}{a}$, eiusque directio erit recta *GO* in puncto medio *G* rectae *AB* ad *AB* normalis. Centrum igitur resistentiae erit punctum *O,* existente $AO = \frac{aa}{2b}$. Resolvatur utraque resistentiae vis in duas laterales, quae diagonalibus *AE* et *BD* sint parallelae, erit resistentia secundum *AE* urgens

$$\frac{cv(nc+mb)^2 + cv(nc-mb)^2}{a^2} = \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2}$$

resistentiae vero vis secundum directionem ipsi *DB* parallelam agens

$$\frac{bv(nc+mb)^2 - bv(nc-mb)^2}{a^2} = \frac{4mnb^2cv}{a^2}.$$

Ex his media totius resistentiae directio reperitur *OR,* quae cum axe *AE* angulum constituet *ROE* cuius tangens est $= \frac{2mnb^2}{n^2c^2 + m^2b^2}$, ipsius vero resistentiae quantitas prodit

$$= \frac{2cv}{a^2}\left(n^4c^4 + 2m^2n^2b^2c^2 + m^4b^4 + 4m^2n^2b^4\right).$$

Q.E.I.

COROLLARIUM 1

570. In hac ergo figura quoque constans est centrum resistentiae *O,* utcunque cursus *CL* ab axe *CA* declinet, dummodo angulus *ACL* non superet angulum *CAD* ; hoc est dummodo sit $\frac{m}{n} < \frac{c}{b}$.

COROLLARIUM 2

571. Si ergo figurae centrum gravitatis simul incidat in punctum *O,* tum resistentia figuram non convertet, sed tantum eius motum progressivum afficiet, eum vel retardando vel cursum inflectendo.

COROLLARIUM 3

572. Angulus autem *EOR* maior erit quam angulus *ACL* si fuerit

$$\frac{2mnb^2}{n^2c^2 + m^2b^2} > \frac{m}{n}$$

hoc est si fuerit

$$2nb^2 - n^2c^2 > m^2b^2, \text{ seu } \frac{m}{n} < \frac{\sqrt{2b^2 - c^2}}{b}.$$

Quia autem est $\frac{m}{n} < \frac{c}{b}$, perspicuum est, si fuerit $b > c$ seu $AC > CD$ tum angulum *EOR* semper maiorem fore angulo *ACL*.

COROLLARIUM 4

573. Si fiat $\frac{m}{n} = \frac{c}{b}$, quo casu solum latus *AD* resistentiae erit expositum, tum anguli *EOR* tangens erit $= \frac{b}{c}$. Scilicet angulus *EOR* hoc casu complementum erit anguli *ACL* ad rectum quod quidem ex se sponte patet.

PROPOSITIO 57

PROBLEMA

574. *Si figura aquae innatans AE* (Fig. 87) *fuerit composita ex parallelogrammo rectangulo HKNM et duobus triangulis isoscelibus aequalibus HAK et MEN super lateribus oppositis HK et MN constitutis, haecque figura in directione CL ad diametrum AE obliqua promoveatur, invenire resistentiae tum directionem tum magnitudinem.*

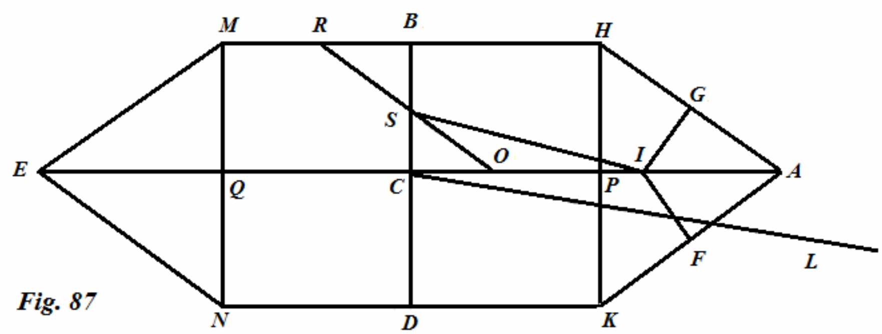


Fig. 87

SOLUTIO

Ponantur primo in triangulo *HAK,* latus

$$AK = AH = a, \; AP = b, \; HP = PK = c,$$

ita ut sit $a^2 = b^2 + c^2$. Deinde rectanguli *HMNK* longitudo *MH* seu *KN* sit $= 2f$, vel ducta diametro transversali *BD* sit $KD = DN = f$; anguli autem obliquitatis cursus *ACL* sinus sit $= m$, cosinus $= n = \sqrt{1 - mm}$; qui angulus minor sit quam angulus *CEN,* quo tria latera *HA, A K* et *KN* solum sint resistentiae exposita, celeritas denique qua haec figura in directione *CL* progreditur debita sit altitudini *v.* Consideretur nunc primum resistentia, quam sola trianguli latera *HA* et *AK* patiuntur cuius media directio *IS* per propositionem antecedentem transit per punctum *I,* existente $AI = \frac{aa}{2b}$, atque cum axe *AE* angulum *CIS* constituet, cuius tangens est $= \frac{2mnb^2}{n^2c^2 + m^2b^2}$, ipsa vero resistentiae vis erit

$$= \frac{2cv}{a^2}\sqrt{\left(n^2c^2 + m^2b^2\right)^2 + 4m^2n^2b^4};$$

seu quod eodem redit resistentia aequivalebit duabus viribus in *I* applicatis, quarum altera urget versus *IC* estque $= \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2}$, altera directionem habet ad hanc normalem, estque $= \frac{4mnb^2cv}{a^2}$. His evolutis inquiramus in resistentiam lateris *KN,* quod in aquam sub angulo cuius sinus est $= m$ impingit, eius igitur resistentia est $= 2m^2 fv$ cuius directio ad *KN* est normalis
atque in ipsam *DB* incidit. Haec ergo resistentia si cum priore, quam latera trianguli *HA, AK* sufferunt coniungatur, praebebit centrum resistentiae in *O* ut sit

$$CO : IO = \frac{4mnb^2cv}{a^2} : 2m^2 fv = 2nb^2bf : ma^2 f;$$

unde fit

$$CI : CO = 2nb^2c + ma^2 f : 2nb^2c.$$

Est vero

$$CI = f + b - \frac{a^2}{2b} = \frac{2bf + 2bb - aa}{2b};$$

ideoque

$$CO = \frac{nbc\left(2bf + 2bb - aa\right)}{2nb^2c + ma^2 f}.$$

Tota ergo resistentia ad duas vires in puncto *O* applicatas reducitur, quarum altera urget in directione *OC* estque

$$= \frac{2cv\left(n^2c^2 + m^2b^2\right)}{a^2},$$

alterius vero quae erit

$$= 2m^2 fv + \frac{4mnb^2cv}{aa}$$

directio ad illam est normalis. Hinc totius resistentiae media directio est recta *OR,* quae cum axe angulum *EOR* constituit, cuius tangens

$$\frac{m^2a^2 f + 2mnb^2c}{n^2c^3 + m^2b^2c};$$

atque ipsius resistentiae quantitas erit

$$=\frac{2v}{a^2}\sqrt{\left(n^2c^3+m^2b^2c\right)^2+\left(m^2a^2f+2mnb^2c\right)^2}.$$

Q.E.I.

COROLLARIUM 1

575. In hac igitur figura situs centri resistentiae O non est fixus, sed pendet ab obliquitate cursus nisi sit

$$f+b=\frac{a^2}{2b},$$

quo casu in C incidit. Nam si angulus ACL evanescit, tum punctum O incidet in ipsum punctum I, atque quo maior fit obliquitas cursus ACL, eo propius punctum O ad C accedit.

COROLLARIUM 2

576. In huius modi igitur figura evitari nequit quin in cursu obliquo figura a resistentia circa gravitatis centrum quandoque convertatur, nisi eo casu quo cadit O in C. Ad hanc ergo conversionem impediendam opus erit novis viribus.

COROLLARIUM 3

577. Manente autem eodem angulo ACL obliquitatis cursus, angulus EOR quem media directio resistentiae cum axe seu spina AE constituit, eo erit maior, quo longius fuerit parallelogrammum rectangulum $HMNK$.

COROLLARIUM 4

578. Eo magis autem angulus EOR excedet angulum ACL, quo magis haec quantitas $m^2na^2f+2mn^2b^2c$ superat hanc $mn^2c^3+m^3b^2c$. Eo maior autem est iste excessus, quo longior fuerit figurae pars media seu parallelogrammum rectangulum.

SCHOLION

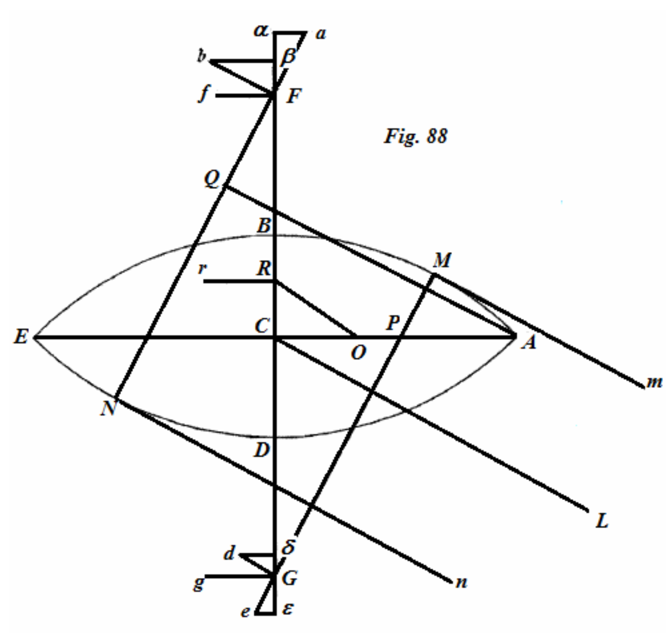
579. Ex his casibus satis clare perspicitur, quomodo de resistentia quam quaecunque figura in aqua oblique promota patitur, iudicium ferri conveniat. Scilicet cum in navibus quae vento propelluntur requiratur, ut cursus quam maxime in eam plagam institui queat, unde ventus venit, quantum ista proprietas obtineatur ex differentia angulorum, quos directio cursus et media
directio resistentiae cum spina constituit, colligere licebit, quo maior enim fuerit ista differentia, eo aptior erit navis ad istum scopum consequendum. Ex allatis autem intelligitur horum angulorum differentiam eo fore maiorem quo maior fuerit resistentia laterum navis respectu resistentiae quam prora directe promota sentit. Hancobrem primo naves ita construi convenit, ut secundum spinam directe promotae minimam sentiant resistentiam; tum vero ut, si

cursus tantillum suscipiatur obliquus, resistentia maxime augeatur; qui posterior scopus obtinetur, si navis fiat vehementer longa, eiusque latera figuram fere planam sint habitura. Hanc itaque etiam ob causam pars navium anterior, quae in cursu directo sola resistentiam patitur ita est conformanda, ut minimam
patiatur resistentiam, quantum quidem id reliquae circumstantiae permittunt. Sed haec omnia tum in sequenti capite tum vero in altero libro uberius exponentur. Quod autem ad situm centri resistentiae attinet, quo de eo facilius indicari queat, sequentes propositiones afferre est visum.

PROPOSITIO 58

PROBLEMA

580. *Si figura ABED* (Fig. 88) *ex duobus aequalibus similibusque segmentis circularibus constet, super communi chorda AE utrinque dispositis; eaque figura in aqua promoveatur oblique secundum directionem CL, determinare resistentiae tum directionem tum magnitudinem.*



SOLUTIO

Sit F centrum arcus $ADE,$ et G centrum arcus $ABE,$ ponaturque

$$FD = GB = c,\ AC = EC = a,\ BC = CD = b,$$

erit

$$FC = GC = c - b,$$

atque ex natura circuli $a^2-\left(c-b\right)^2=c^2$ unde fit $2bc=a^2+b^2$. Sit iam anguli obliquitatis cursus *ACL* sinus $=m$, cosinusque $=n$, ac ducantur tangentes *Mm* et *Nn* parallelae directioni cursus *CL,* radiique *MG* et *FN,* erunt anguli *MGB, NFD* aequales angulo *ACL,* eorumque propterea sinus
$=m$, cosinusque $=n$. Cum itaque arcuum *BM* et *DN* sinus sit $=m$ et cosinus $=n$, arcuum vero *AB* et *AD* sinus sit $=\frac{a}{c}$ et cosinus $=\frac{\sqrt{c^2-a^2}}{c}$, erit arcus *AM*

$$\text{sinus}=\frac{an-m\sqrt{c^2-a^2}}{c}\ \text{ et cosinus}=\frac{am+n\sqrt{c^2-a^2}}{c};$$

arcus vero *ADN*

$$\text{sinus}=\frac{an+m\sqrt{c^2-a^2}}{c}\ \text{ et cosinus}=\frac{n\sqrt{c^2-a^2}-ma}{c}.$$

Quare si ex *A* ad radios *GM* et *FN,* qui inter se sunt paralleli ducatur perpendicularis *APQ* erit

$$AP=na-m\sqrt{c^2-a^2}=na-m\left(c-b\right)$$

et

$$GP=ma+n\sqrt{c^2-a^2}=ma+n\left(c-b\right).$$

Simili modo erit

$$AQ=na+m(c-b),\ \text{et}\ FQ=n(c-b)-ma.$$

Sunt autem arcus *AM* et *ADN* eae figurae partes, quae solae resistentiam patiuntur; ad resistentiam igitur utriusque arcus definiendam sit celeritas, qua figura progreditur debita altitudini *v.* Resistentiae autem arcus *AM* media directio transit per centrum arcus *G,* reduciturque ad duas potentias *Gd, Ge,* quorum illa *Gd* ad *GM* est normalis, haec vero *Ge* cum *MG* in directum iacet;
est autem ex § 554

$$\text{vis } Gd=\frac{MP^2\left(3MG-MP\right)v}{3MG^2}\ \text{ et vis } Ge=\frac{AP^3\cdot v}{3MG^2}.$$

At est

$$MP=c-ma-n(c-b)\ \text{ et }\ MG=c\ ;$$

ideoque

$$3MG-MP=2c+ma+n\left(c-b\right)$$

existente

$$AP=na-m(c-b).$$

Simili modo resistentia, quam patitur arcus *ADN,* reducetur ad duas vires *Fb* et *Fa* in puncto *F* similiter applicatas, ut sit *bF* ad *NF* perpendicularis et *a* situm sit in recta *NF* producta; eritque vis

$$Fb = \frac{NQ^2\left(3NF - NQ\right)v}{3NF^2} \text{ et vis } Fa = \frac{AQ^3 \cdot v}{3NF^2},$$

existente

$$NQ = c + ma - n(c - b),$$

et

$$NF = c,\ 3NF - NQ = 2c - ma + n\left(c - b\right),$$

atque

$$AQ = na + m(c - b).$$

Resolvantur hae vires in binas quarum alterae in *FG* incidant, alterae ad *FG* sint normales, quod facile fit, dum angulorum $eG\varepsilon$ et $aF\alpha$ sinus sit $= m$, cosinusque $= n$. Reperitur vero vis

$$Gg = \frac{nv \cdot MP^2(3MG - MP) + mv \cdot AP^3}{3MG^2}$$

et vis

$$G\varepsilon = \frac{nv \cdot AP^3 - mv \cdot MP^2(3MG - MP)}{3MG^2}.$$

Pari ratione est vis

$$Ff = \frac{nv \cdot MQ^2(3NF - NQ) - mv \cdot AQ^3}{3MG^2}$$

et vis

$$F\alpha = \frac{nv \cdot AQ^3 + mv \cdot NQ^2(3NF - NQ)}{3MG^2}$$

Ponatur

$$FC = GC = c - b = f;\ \text{ erit } a^2 + f^2 = c^2,$$

atque

$$MP = c - ma - nf,\ NQ = c + ma - nf,$$

$$3MG - MP = 2c + ma + nf,\ 3NF - NQ = 2c - ma + nf,$$

atque

$$AP = na - mf,\ \text{ ac } AQ = na + mf.$$

Hinc erit vis

$$Gg = \frac{v}{3cc}\left(2nc^3 - 3n^2c^2f - 2mna^3 + \left(nn - mm\right)f^3\right)$$

et vis

$$G\varepsilon = \frac{v}{3cc}\left(\left(n^2 - m^2\right)a^3 + 2mnf^3 - 2mc^3 + 3m^2ac^2\right).$$

Similiterque vis

$$Ff = \frac{v}{3cc}\left(2nc^3 - 3n^2c^2f + 2mna^3 + \left(nn - mm\right)f^3\right)$$

et vis

$$F\alpha = \frac{v}{3cc}\left(\left(n^2 - m^2\right)a^3 - 2mnf^3 + 2mc^3 + 3m^2ac^2\right).$$

Sit nunc *OR* media directio totius resistentiae, erit

$$CR = \frac{2mna^3f}{2nc^3 - 3n^2c^2f + \left(nn - mm\right)f^3};$$

viribusque superioribus aequivalebunt duae vires *Rr* et *RB* in puncto *R* applicatae, eritque vis

$$Rr = \frac{v}{3cc}\left(2nc^3 - 3nnc^2f + 2mna^3 + \left(nn - mm\right)f^3\right)$$

et vis

$$RB = \frac{4mv}{3cc}\left(c^3 - nf^3\right),$$

unde proveniet

$$CO = \frac{na^3f}{c^3 - nf^3}.$$

Anguli igitur *ROC* quem media directio resistentiae cum axe *AE* constituit, tangens erit

$$\frac{2mc^3 - 2mnf^3}{2nc^3 - 3n^2c^2f + \left(nn - mm\right)f^3},$$

atque ipsius resistentiae quantitas erit

$$\frac{2v}{3cc}\sqrt{4c^6 - 12n^3c^5f + 4n\left(nn - 3mm\right)c^3f^3 + 9n^4c^4ff - 6nn\left(nn - mm\right)c^2f^4 + f^6}.$$

Q.E.I.

COROLLARIUM 1

581. Locus igitur centri resistentiae *O* est variabilis, pendetque ab obliquitate cursus seu angulo *ACL.* Quo maior enim fit angulus *ACL,* eo propius punctum *O* ad *C* accedit.

COROLLARIUM 2

582. Si angulus *ACL* sit infinite parvus, punctum *O* a *C* maxime erit remotum; erit enim distantia

$$OC = \frac{a^3 f}{c^3 - f^3} = \frac{af(c+f)}{cc+cf+ff}$$

propter $aa = cc - ff$. At si fiat $n = \frac{f}{c}$, quo casu punctum *M* in *A* cadit, erit distantia minima

$$OC = \frac{a^3 ff}{c^4 - f^4} = \frac{aff}{cc+ff}.$$

COROLLARIUM 3

583. Intervallum igitur, per quod centrum resistentiae *O* vagatur, dum punctum *M* a *B* usque ad *A* pro movetur, est

$$= \frac{af(c+f)}{cc+cf+ff} - \frac{aff}{cc+ff} = \frac{ac^3 f}{(cc+ff)(cc+cf+ff)} = a\left(\frac{f}{c} - \frac{ff}{cc} - \frac{f^3}{c^3} + \frac{2f^4}{c^4}\right)$$

proxime; minus igitur est quam $\frac{af}{c}$.

COROLLARIUM 4

584. Si segmenta *ABE* et *ADE* abeant in semicirculos, tum fiet $f = 0$, hoc igitur casu centrum resistentiae *O* in ipsum punctum *C* cadit. Quo maior autem fuerit *I,* hoc est quo minora fuerint segmenta illa, eo magis centrum resistentiae *O* a *C* distat.

COROLLARIUM 5

585. Ut differentia angulorum *COR* et *ACL* distinctius percipiatur, ponamus angulum *ACL* esse infinite parvum, quo casu fit $m =$ infinite parvo et $n = 1$, angulique *ACL* tangens $= m$. Anguli ergo *COR* tangens erit

$$= \frac{2m(c^3 - f^3)}{2c^3 - 3ccf + f^3} = \frac{2m(c^2 + cf + ff)}{2c^2 - cf - ff} = \frac{2m(c^2 + cf + f^2)}{(c-f)(2c+f)};$$

unde se habebit angulus *ACL* ad angulum *COR* ut *2cc- cf- If* ad
2cc + 2cf + 2ff.

COROLLARIUM 6

586. Si ergo obliquitas cursus seu angulus *ACL* fuerit vehementer exiguus, tum angulus *COR* maior erit angulo *ACL,* nisi sit $f=0$, quo casu figura in integrum circulum abit. Semper enim si figura est integer circulus anguli *ACL* et *COR* sunt aequales, atque puncta *O* et *C* coincidunt.

COROLLARIUM 7

587. Si obliquitas fiat maxima seu arcus *AM* evanescat, ut solus arcus *ADE* resistentiae exponatur, tum fiet

$$m=\frac{a}{c} \text{ et } n=\frac{f}{c},$$

atque anguli *COR* tangens erit

$$=\frac{a(c^4-f^4)}{f(cc-ff)^2}=\frac{a(cc+ff)}{f(cc-ff)}.$$

Anguli igitur *ACL* tangens se habebit ad anguli *COR* tangentem ut $cc-ff$ ad $cc+ff$.

COROLLARIUM 8

588. Ex his intelligitur quo maior fuerit *f* respectu *c,* seu quo minora sint segmenta *ABE* et *ADE,* eo magis pro quavis obliquitate excedere angulum *COR* angulum *ACL.*

COROLLARIUM 9

589. Si angulus *ACL* evanescit, tum ob $m=0$ et $n=1$, prodit totius resistentiae vis

$$=\frac{2v(2c^3-3ccf+f^3)}{3cc}=\frac{2v(c-f)(2c^2-cf-ff)}{3cc}=\frac{2v(c-f)^2(2c+f)}{3cc},$$

at si obliquitas fiat maxima seu

$$m=\frac{a}{c} \text{ et } n=\frac{f}{c}$$

tum prodit tota resistentia $=\dfrac{4a^3v\sqrt{cc+3ff}}{3c^3}$.

SCHOLION

590. Hanc figuram ex duobus segmentis circularibus compositam ideo potissimum hic sum contemplatus, quod ad cognitionem resistentiae navium satis sit idonea. Quamvis enim sectiones horizontales navium non admodum congruant cum ista figura, tamen si praecedentes casus simul in considerationem ducantur, non difficile erit pro quavis cursus obliquitate tam centri

resistentiae locum, quam mediam resistentiae directionem aestimatione assignare. Satis enim manifestum est, quo magis figura fuerit cuspidata, eo propius centrum resistentiae versus proram esse situm ceteris paribus. Eandem hanc etiam figuram Celeb. Ioh. BERNOULLII in tractatu cui titulus est: Manoeuvre des Vaisseaux, examini subiecit, atque peculiari modo in locum centri resistentiae inquisivit, eo tantum casu quo obliquitas cursus est quam minima, seu angulus *ACL* infinite parvus; censet autem hoc casu centrum resistentiae in eo puncto fore constitutum, ubi media directio resistentiae quam arcus *AB* vel *AD* solus in cursu directo patitur, axem *AE* intersecat. At istud punctum non congruit cum nostro puncto *O,* quando angulus *ACL* evanescit. Secundum methodum enim Bernoullianam reperitur intervallum

$$CO = \frac{af(2c+f)}{(c+f)^2},$$

cum revera sit

$$CO = \frac{af(c+f)}{cc+cf+ff}.$$

Ex quo intelligitur centrum resistentiae, cum obliquitas cursus est infinite parva, ex resistentia quam utraque curvae pars in cursu directo patitur, definiri non posse, sed revera cursum obliquum in considerationem duci oportere, quemadmodum in hac propositione a nobis est factum. Sed si aliae figurae praeter circulares fuerint propositae, tum resistentia in cursu obliquo vix ac ne vix quidem potest determinari, ob calculum nimis prolixum; quocirca eiusmodi investigationibus supersedendum esse duxi. Tentabo autem tantum eo casu, quo cursus obliquus minime a directo differt, locum centri resistentiae et mediam resistentiae directionem definire, quippe qui casus facilius examini subiicitur, et a taediosis calculis quodammodo liberari potest.

PROPOSITIO 59

PROBLEMA

591. *Si figura aquae innatans constet ex duabus partibus AMBE et ANDE aequalibus et similibus utrinque ad axem AE dispositis* (Fig. 89), *eaque moveatur in directione CL quae cum axe AC constituat angulum ACL infinite parvum; determinare mediam directionem resistentiae OR, ipsamque resistentiae quantitatem.*

SOLUTIO

Quia cursus obliquitas ponitur infinite parva eadem utrinque figurae portio *AMB* et *AND* resistentiam patietur, quae si cursus foret directus, resistentiae esset exposita; cum non solum eae partes quibus tum arcus *AMB* augeri, tum arcus *AND* diminui deberet, fiunt infinite parvae, sed etiam sub angulo infinite parvo in aquam impingunt, ita ut

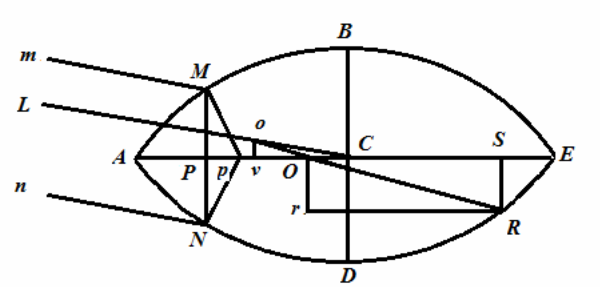


Fig. 89

earum resistentiam tuto negligere liceat. Ducta igitur ordinata *MPN,* sit $AP = x$, $PM = PN = y$, et arcus $AM = AN = s$, anguli vero ACL sinus ponatur $= m$, cosinusque $= n$, erit m infinite parvum et propterea $n = 1$, celeritas autem qua figura
progreditur debita sit altitudini v. Ducantur iam ipsi LO parallelae mM et nN, quae directionem repraesentabunt, qua puncta M et N in aquam impingunt; erit autem anguli AMm sinus

$$=\frac{ndy+mdx}{ds}=\frac{dy+mdx}{ds},$$

anguli autem ANn sinus

$$=\frac{ndy-mdx}{ds}=\frac{dy-mdx}{ds}.$$

Resistentia ergo, quam elementum ds in M sufferet erit

$$=\frac{v(dy^2+2mdxdy)}{ds},$$

eiusque directio erit normalis ad curvam Mp. Resistentia vero quam elementum ds in N sufferet erit

$$=\frac{v(dy^2-2mdxdy)}{ds},$$

in directione normalis Np. Elementum igitur ds in M urgebitur in directione MP vi

$$=\frac{vdxdy(dy+2mdx)}{ds^2}$$

at in directione axi AC parallela vi

$$=\frac{vdy^2(dy+2mdx)}{ds^2}$$

Simili modo elementum ds in N urgebitur, in directione NP vi

$$=\frac{vdxdy^2(dy-2mdx)}{ds^2}$$

et in directione axi AC parallela vi

$$=\frac{vdy^2(dy-2mdx)}{ds^2}$$

Summa ergo virium qua ambo elementa coniunctim in directione AC urgentur est $=\frac{2vdy^3}{ds^2}$; at excessus, quo in directione MN sollicitantur $=\frac{4mvdx^2dy}{ds^2}$. Sit nunc oO media resistentiae directio, ductoque ex o ad AC perpendiculo ov, erit integralibus usque ad B et D sumtis

$$ov = \frac{4mv\int \frac{ydy^2dx}{ds^2}}{2v\int \frac{dy^3}{ds^2}} = \frac{2m\int ydy^2dx : ds^2}{\int dy^3 : ds^2}$$

atque

$$Av = \frac{4mv\int \frac{xdx^2dy}{ds^2}}{4mv\int \frac{dx^2dy}{ds^2}} = \frac{\int xdx^2dy : ds^2}{\int dx^2dy : ds^2}.$$

Porro si *oO* est media directio resistentiae, erit

$$ov : vO = 2m\int \frac{dx^2dy}{ds^2} : \int \frac{dy^3}{ds^2},$$

unde fit

$$vO = \frac{\int ydy^2dx : ds^2 :}{\int dx^2dy : ds^2,}$$

atque

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{\int dx^2dy : ds^2}$$

quae expressio determinat locum centri resistentiae *O*. Tota igitur resistentia reducitur ad duas vires in puncto *O* applicatas, quarum altera est $= 2v\int \frac{dy^3}{ds^2}$, agens in directione *Ov,* altera vero est

$= 4mv\int \frac{dx^2dy}{ds^2}$, cuius directio est *ov* ad *AE* normalis. Ipsa denique media directio *OR* cum

axe *AE* angulum constituet *EOR* cuius tangens est $= \frac{2m\int dx^2dy : ds^2}{\int dy^3 : ds^2}$.

Q.E.I.

COROLLARIUM 1

592. Hinc patet locum centri resistentiae *O* omnino esse diversum ab eo, qui secundum modum ante indicatum (§ 590) reperitur, per eum enim prodit

$$AO = \frac{\int (xdx + ydy)dy^2 : ds^2}{\int dy^2dx : ds^2},$$

cum tamen revera sit

$$AO = \frac{\int (xdx + ydy)dy^2 : ds^2}{\int dx^2 dy : ds^2}.$$

COROLLARIUM 2

593. Angulus igitur *ROE* pariter infinite parvus, rationemque habebit ad angulum *ACL* uti se tenet,

$$2\int \frac{dx^2 dy}{ds} \text{ ad } \int \frac{dy^3}{ds^2},$$

quae ergo ratio erit finita. Anguli enim infinite parvi sunt ut eorum tangentes vel sinus.

COROLLARIUM 3

594. Resistentiae ergo vis, quae agit secundum directionem axis *AE* aequalis est illi resistentiae, quam pateretur eadem figura si cursu directo secundum directionem axis *CA* moveretur.

SCHOLION 1

595. Ex solutione sponte intelligitur, qua conditione omnia integralia, quae occurrunt, sint accipienda. Scilicet primo omnes integrationes ita sunt instituendae, ut omnia integralia evanescant posito vel x vel $y = 0$. Deinde ad maximam figurae latitudinem est respiciendum, quae si est BD, poni debet $x = AC$ vel $y = BC$, quoniam ea pars figurae solum resistentiam patitur quae sita est inter proram *A* et maximam figurae latitudinem *BD*.

COROLLARIUM 4

596. Cum resistentia secundum directionem *AE* sit ut $\int \frac{dy^3}{ds^2}$, atque anguli *ROE* tangens $= \frac{2m\int dx^2 dy : ds^2}{\int dy^3 : ds^2}$, intelligitur, quo figurae directe promotae minor fuerit resistentia, eo magis angulum *ROE* esse superaturum angulum *ACL*.

COROLLARIUM 5

597. Pendet autem integratio $\int \frac{dx^2 dy}{ds^2}$ ab integratione $\int \frac{dy^3}{ds^2}$: cum enim sit $dx^2 + dy^2 = ds^2$ erit

$$\int \frac{dx^2dy}{ds^2} + \int \frac{dy^3}{ds^2} = y,$$

ideoque

$$\int \frac{dx^2dy}{ds^2} = y - \int \frac{dy^3}{ds^2},$$

unde fit anguli *EOR* tangens

$$= \frac{2my}{\int dy^3 : ds^2} - 2m = \frac{2m \cdot BC}{\int dy^3 : ds^2} - 2m.$$

COROLLARIUM 6

598. Inter omnes igitur figuras per puncta *A* et *B* transeuntes, ea pro data obliquitate *ACL* maximum angulum *EOR* producet, quae in cursu directo minimam patitur resistentiam.

COROLLARIUM 7

599. Deinde quod ad locum centri resistentiae *O* attinet, cum sit

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{\int dx^2dy : ds^2}$$

erit

$$AO = \frac{\int (xdx + ydy)dxdy : ds^2}{BC - \int dy^3 : ds^2}.$$

Quo minor ergo est resistentia figurae in cursu directo, eo propius centrum resistentiae *O* ad proram *A* erit situm, manente numeratore $\int \frac{(xdx + ydy)dxdy}{ds^2}$.

EXEMPLUM 1

600. Sit pars figurae anterior resistentiam excipiens triangulum isosceles *BAD* (Fig. 90), in quo sit

$AC = a$, $BC = CD = b$,
et
$AB = AD = c = \sqrt{a^2 + b^2}$.
Directio vero cursus sit *CL*, angulique *ACL* qui est infinite parvus, sinus $= m$; et celeritas debita altitudini *v*. Iam posito

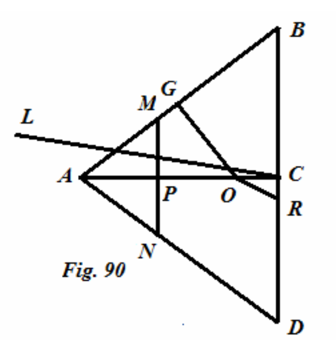


Fig. 90

$$AP = x,\ PM = PN = y,\ \text{erit}\ y = \frac{bx}{a},$$

et

$$dy = \frac{bdx}{a},\ \text{atque}\ ds = \frac{cdx}{a}.$$

Sit nunc O centrum resistentiae, et OR media directio resistentiae, erit ob

$$\int \frac{dx^2 dy}{ds^2} = \int \frac{abdx}{cc} = \frac{a^2 b}{cc},\quad \int \frac{dy^3}{ds^2} = \int \frac{b^3 dx}{acc} = \frac{b^3}{cc},$$
$$\int \frac{xdx^2 dy}{ds^2} = \int \frac{abxdx}{cc} = \frac{a^3 b}{2cc},$$

atque

$$\int \frac{ydxdy^2}{ds^2} = \int \frac{b^3 xdx}{acc} = \frac{ab^3}{2cc},$$

distantia

$$AO = \frac{a^3 b + ab^3}{2a^2 b} = \frac{cc}{2a},$$

unde patet centrum resistentiae O in idem axis AC punctum incidere, in quo recta GO, quae ad AB est normalis eamque bisecat, rectae AC occurrit, prouti ex praecedentibus iam constat. Deinde anguli COR tangens est $= \frac{2ma^2}{b^2}$, ita ut se habeat angulus ACL ad angulum COR uti b^2 ad $2a^2$; quoties igitur fuerit

$$2a^2 > b^2 \quad \text{seu} \quad \frac{BC}{AC} < \sqrt{2},$$

sive angulus BAC minor quam $54^{\circ}, 45'$, toties angulus COR excedet angulum ACL. Vis denique resistentiae agens in directione CO est $= \frac{2b^3 v}{cc}$; atque vis qua in directione ad OC normali sollicitabitur erit $= \frac{4ma^2 bv}{cc}$.

EXEMPLUM 2

601. Constet figura ex duobus segmentis circularibus ABE, ADE aequalibus et similibus sitque $AC = a,\ BC = CD = b,$ atque radius circuli ex quo haec segmenta sunt desumta sit $= c$. Ponatur autem brevitatis causa $c - b = f$, ut sit $cc = a^2 + ff$. Porro sit CL directio cursus angulique ACL, qui ponitur infinite parvus, sinus $= m$, et celeritati altitudo debita $= v$. Iam cum sit
$AP = x,\ PM = PN = y,$ erit ex natura circuli

$$x = a - \sqrt{c^2 - (f+y)^2}, \quad dx = \frac{(f+y)dy}{\sqrt{c^2 - (f+y)^2}} \quad \text{et} \quad ds = \frac{cdy}{\sqrt{c^2 - (f+y)^2}},$$

unde sequentia integralia reperientur

$$\int \frac{dy^3}{ds^2} = \int \frac{ccdy - (f+y)^2 dy}{cc} = b + \frac{f^3}{3cc} - \frac{c}{3} = \frac{2c^3 - 3ccf + f^3}{3cc} = \frac{(c-f)^2(2c+f)}{3cc}$$

atque

$$\int \frac{dx^2 dy}{ds^2} = c - f - \frac{(c-f)^2(2c+f)}{3cc} = \frac{(c-f)(cc+cf+ff)}{3cc} = \frac{c^3 - f^3}{3cc}.$$

Deinde est

$$\int \frac{xdx^2 dy}{ds^2} = \frac{a(c^3 - f^3)}{3cc} - \int \frac{(f+y)^2 dy\sqrt{cc-(f+y)^2}}{cc},$$

atque

$$\int \frac{ydxdy^2}{ds^2} = \int \frac{(f+y)ydy}{cc}\sqrt{cc-(f+y)^2};$$

unde erit

$$\int \frac{(xdx+ydy)dxdy}{ds^2} = \frac{a(c^3-f^3)}{3cc} - \frac{f}{cc}\int (f+y)dy\sqrt{cc-(f+y)^2}$$

$$= \frac{a(c^3-f^3)}{3cc} - \frac{a^3 f}{cc} = \frac{ab}{3}.$$

Ex his oritur

$$AO = \frac{abcc}{c^3 - f^3} = \frac{acc}{cc+cf+ff}$$

atque

$$CO = \frac{af(c+f)}{cc+cf+ff}$$

ut supra(§ 582). Anguli autem *COR,* quem media directio resistentiae *OR* cum axe *AC* constituit tangens est $= \dfrac{2m(cc+cf+f)}{(c-f)(2c+f)}$ uti supra (§ 585).

EXEMPLUM 3

602. Sit figura aquae insidens ellipsis *ABED* (Fig. 91) cuius semiaxis AC sit $= a$, alter $BC = CD = b$ atque *CL* cursus directio infinite parum dissidens ab axe *AC,* ita ut anguli *ACL*

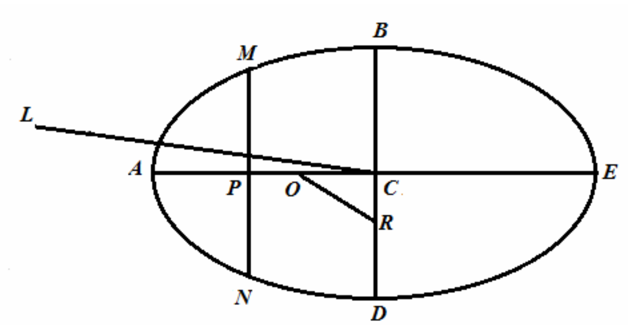


Fig. 91

sinus *m* sit infinite parvus; altitudo vero celeritati, qua haec figura promovetur, debita sit *v*. Iam positis abscissa
$AP = x$, applicatis $PM = PN = y$,
erit
$y = \frac{b}{a}\sqrt{2ax - xx}$, sive $x = a - \frac{a}{b}\sqrt{bb - yy}$;
hinc igitur fit

$$dx = \frac{aydy}{b\sqrt{bb - yy}} \quad \text{et} \quad ds^2 = \frac{dy^2\left(b^4 + \left(a^2 - b^2\right)yy\right)}{bb\left(bb - yy\right)}.$$

Quare integralia, quibus opus est ita se habebunt

$$\int \frac{dx^2 dy}{ds^2} = \int \frac{a^2 y^2 dy}{b^4 + \left(a^2 - b^2\right)yy} = \frac{aab}{aa - bb} - \frac{a^2 b^4}{aa - bb}\int \frac{dy}{b^4 + \left(a^2 - b^2\right)yy}$$

ubi duo casus, sunt considerandi, prout fuerit $a > b$ vel $a < b$; si enim $a > b$ seu $AC > BC$ provenit

$$\int \frac{dx^2 dy}{ds^2} = \frac{a^2 b}{aa - bb} - \frac{a^2 b^2}{\left(aa - bb\right)^{\frac{3}{2}}} \text{Atang.} \frac{\sqrt{aa - bb}}{b},$$

at si $a < b$ erit

$$\int \frac{dx^2 dy}{ds^2} = \frac{a^2 b^2}{2\left(bb - aa\right)^{\frac{3}{2}}} l \frac{b + \sqrt{bb - aa}}{b - \sqrt{bb - aa}} - \frac{a^2 b}{bb - aa}$$
$$= \frac{a^2 b^2}{2\left(bb - aa\right)^{\frac{3}{2}}} l \frac{b + \sqrt{bb - aa}}{a} - \frac{a^2 b}{bb - aa}.$$

Quia autem est

$$\int \frac{dy^3}{ds^2} = b - \int \frac{dx^2 dy}{ds^2}$$

erit eo casu, quo est $a > b$,

$$\int \frac{dy^3}{ds^2} = \frac{-b^3}{aa - bb} - \frac{a^2 b^2}{\left(aa - bb\right)^{\frac{3}{2}}} \text{Atang.} \frac{\sqrt{aa - bb}}{b},$$

casu vero quo est $a < b$ erit

$$\int \frac{dy^3}{ds^2} = \frac{b^3}{bb-aa} - \frac{a^2b^2}{(aa-bb)^{\frac{3}{2}}} l \frac{b+\sqrt{aa-bb}}{a}.$$

Deinde cum sit

$$\frac{xdx^2dy}{ds^2} = \frac{a^3y^2dy - \frac{a^3}{b}y^2dy\sqrt{bb-yy}}{b^4+(aa-bb)yy},$$

atque

$$\frac{ydxdy^2}{ds^2} = \frac{abyydy\sqrt{bb-yy}}{b^4+(aa-bb)yy},$$

erit

$$\int \frac{(xdx+ydy)dxdy}{ds^2} = a\int \frac{a^2y^2dy}{b^4+(aa-bb)yy} - \frac{a}{b}\int \frac{(aa-bb)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy}.$$

Est vero

$$\int \frac{(aa-bb)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy} = \frac{\pi}{4}\cdot\frac{bb(a-b)^2}{(aa-bb)}$$

denotante $\pi:1$ rationem peripheriae ad diametrum in circulo. Quamobrem si $a > b$ erit

$$= \frac{a^3b}{aa-bb} - \frac{a^3b^2}{(aa-bb)^{\frac{3}{2}}}\text{Atang.}\frac{\sqrt{aa-bb}}{b} - \frac{\pi ab(a-b)^2}{4(aa-bb)},$$

at si $a < b$ erit

$$\int \frac{(xdx+ydy)dxdy}{ds^2} = \frac{a^3b^2}{(aa-bb)^{\frac{3}{2}}} l \frac{b+\sqrt{bb-aa}}{a} - \frac{a^3b}{bb-aa} + \frac{\pi ab(a-b)^2}{4(aa-bb)}.$$

Centrum itaque resistentiae situm erit in O ut sit

$$AO = a - \frac{\pi ab(a-b)^2}{4a^2b - \frac{4a^2b^2}{\sqrt{aa-bb}}\text{Atang.}\frac{\sqrt{aa-bb}}{b}};$$

ideoque

$$CO = a - \frac{\pi\left(a-b\right)^2}{4a - \frac{4ab}{\sqrt{a^2-b^2}}\text{Atang.}\frac{\sqrt{a^2-b^2}}{b}},$$

casu quo est $a > b$. At casu quo est $a < b$ erit

$$CO = \frac{-\pi\left(a-b\right)^2}{\frac{4ab}{\sqrt{bb-aa}}l\frac{b+\sqrt{bb-aa}}{a} - 4a} = \frac{\pi\left(b-a\right)^2\sqrt{bb-aa}}{4abl\frac{b+\sqrt{bb-aa}}{a} - 4a\sqrt{bb-aa}}.$$

Anguli denique *COR* tangens erit

$$= \frac{2ma^2\sqrt{aa-bb} - 2ma^2b\text{Atang.}\frac{\sqrt{aa-bb}}{b}}{-bb\sqrt{aa-bb} + a^2b\text{Atang.}\frac{\sqrt{aa-bb}}{b}} = \frac{-2ma^2\sqrt{bb-aa} + 2ma^2bl\frac{b+\sqrt{bb-aa}}{a}}{bb\sqrt{bb-aa} - a^2bl\frac{b+\sqrt{bb-aa}}{a}},$$

quarum expressionum, illa valet si $a > b$, haec vero si $a < b$. Innotescit igitur media directio resistentiae *OR* quam figura proposita elliptica secundum directionem *CL* promota sentit.

COROLLARIUM 1

603. Si integrationes, quae tum a quadratura circuli tum hyperbolae pendent, per series absolvantur, erit

$$CO = \frac{\pi(a-b)^2}{4a\left(\frac{aa-bb}{3bb} - \frac{\left(aa-bb\right)^2}{5b^4} + \frac{\left(aa-bb\right)^3}{7b^6} - \text{etc.}\right)}$$

atque anguli *COR* tangens

$$= \frac{2m\left(\frac{aa-bb}{3bb} - \frac{\left(aa-bb\right)^2}{5b^4} + \frac{\left(aa-bb\right)^3}{7b^6} - \text{etc.}\right)}{-\frac{bb}{aa} + 1 - \frac{\left(aa-bb\right)}{3bb} + \frac{\left(aa-bb\right)^2}{5b^4} - \frac{\left(aa-bb\right)^3}{7b^6} + \text{etc.}}$$

quae formulae aeque valent sive sit $a > b$ sive $a < b$.

COROLLARIUM 2

604. Cum sit

$$-\frac{bb}{aa}+1=\frac{aa-bb}{aa},$$

erit anguli *COR* tangens

$$=\frac{2m\left(\frac{1}{3bb}-\frac{(aa-bb)}{5b^4}+\frac{(aa-bb)^2}{7b^6}-\text{etc.}\right)}{\frac{1}{aa}-\frac{1}{3bb}+\frac{aa-bb}{5b^4}-\frac{(aa-bb)^2}{7b^6}+\text{etc.}}$$

similique modo fiet intervallum

$$CO=\frac{\pi(a-b)}{4a(a+b)\left(\frac{1}{3bb}-\frac{(aa-bb)}{5b^4}+\frac{(aa-bb)^2}{7b^6}-\text{etc.}\right)}$$

COROLLARIUM 3

605. Si ellipsis proxime ad circulum accedat ita ut prope sit $b=a$, existente $b=a-dw$, erit ob terminos evanescentes $\frac{3\pi dw}{8}=CO$; atque hoc casu etiam fit anguli *COR* tangens $=m$, seu angulus *COR* aequalis erit angulo *ACL*.

SCHOLION 2

606. Integratio formulae differentialis $\frac{(a^2-b^2)yydy\sqrt{bb-yy}}{b^4+(aa-bb)yy}$, quae in hoc exemplo occurrit notatu est digna, eo quod integrale eo casu, quo ponitur $y=b$, contra omnem expectationem finite et tam simplici forma exprimitur. Si enim integrale indefinitum desideraretur, tum maxime prolixa et intricata expressio inveniretur, ex qua etiam difficillimum foret integrale pro casu $y=b$ exhibere. Peculiari igitur in hac integratione usus sum modo, quo statim pro eo solum casu, quo est $y=b$, integrale prodit, cuius fundamentum in hoc consistit: quod sit

$$\int y^{m+2}dy(bb-yy)^{\frac{n}{2}}=\frac{(m+1)bb}{m+n+3}\int y^{m+2}dy(bb-yy)^{\frac{n}{2}},$$

eo casu, quo ponitur $y=b$. Hinc igitur reperitur

$$\int(\alpha+\beta y^2+\gamma y^4+\delta y^6+\varepsilon y^8+\text{etc.})y^m dy(bb-yy)^{\frac{n}{2}}$$

$$=\left(\alpha+\frac{\beta(m+1)bb}{m+n+3}+\frac{\gamma(m+1)(m+3)b^4}{(m+n+3)(m+n+5)}+\frac{\delta(m+1)(m+3)(m+5)b^6}{(m+n+3)(m+n+5)(m+n+7)}+\text{etc}\right)\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}.$$

Cum nunc sit

$$\frac{1}{cc+yy}=\frac{1}{cc}-\frac{yy}{c^4}+\frac{y^4}{c^6}-\frac{y^6}{c^8}+\text{etc.}$$

erit

$$\int\frac{y^{m+2}dy\left(bb-yy\right)^{\frac{n}{2}}}{cc+yy}=\left(\frac{1}{cc}-\frac{(m+1)b^2}{(m+n+3)c^4}+\frac{(m+1)(m+3)b^4}{(m+n+3)(m+n+5)c^6}\right.$$
$$\left.-\frac{(m+1)(m+3)(m+5)b^6}{(m+n+3)(m+n+5)(m+n+7)c^6}+\text{etc.}\right)\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}$$

eo quidem casu quo fit $y=b$. Series autem ista, quanquam in infinitum progreditur, tamen ad formam finitam potest reduci posito enim $\frac{b}{c}=z$, summa illius seriei est

$$=\frac{(m+n+1)c^{m-1}(bb+cc)^{\frac{n}{2}}}{b^{m+n+1}}\int\frac{z^{m+n}dz}{(1+zz)^{\frac{n+2}{2}}}$$

integrali hoc ita sumto ut evanescat posito $z=0$. Posito igitur

$$\int\frac{z^{m+n}dz}{(1+zz)^{\frac{n+2}{2}}}=C$$

erit

$$\int\frac{y^{m+2}dy\left(bb-yy\right)^{\frac{n}{2}}}{cc+yy}=\frac{(m+n+1)Cc^{m-1}(bb+cc)^{\frac{n}{2}}}{b^{m+n+1}}\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}.$$

Quando autem *m* est numerus par et *n* impar, tum $\int y^m dy(bb-yy)^{\frac{n}{2}}$ ulterius reduci potest ad $\int\frac{dy}{\sqrt{bb-yy}}$, cuius integrale casu quo $y=b$ fit $\frac{\pi}{2}$ denotante π peripheriam circuli cuius diameter est $=1$. Erit autem

$$\int y^m dy\left(bb-yy\right)^{\frac{n}{2}}$$

$$=\frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\ldots(m-1)}{2\cdot 4\cdot 6\ldots m}\cdot\frac{1}{m+2}\cdot\frac{3}{m+4}\cdot\frac{5}{m+6}\cdot\ldots\ldots\frac{n}{m+n+1}b^{m+n+1}$$

unde denique habetur

$$\int\frac{y^m dy\left(bb-yy\right)^{\frac{n}{2}}}{cc+yy}=\frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\ldots(m-1)}{2\cdot 4\cdot 6\ldots m}$$
$$\cdot\frac{1}{m+2}\cdot\frac{3}{m+4}\cdot\frac{5}{m+6}\cdot\ldots\ldots\frac{n}{m+n+1}\cdot(m+n+1)c^{m-1}\left(bb+cc\right)^{\frac{n}{2}}C$$

existente

$$C=\int\frac{z^{m+n}dz}{\left(1+zz\right)^{\frac{n+2}{2}}},\ \text{ et }\ z=\frac{b}{c},$$

ita ut ob m numerum parem n vero imparem C sit quantitas algebraica. His igitur ad nostrum casum applicatis, quo posito

$$\frac{b^4}{aa-bb}=cc,$$

formula nostra transit in hanc

$\int\frac{yydy\sqrt{bb-yy}}{cc+yy}$, unde fit $m=2$ et $n=1$.

Erit ergo

$$C=\int\frac{z^3dz}{\left(1+zz\right)^{\frac{3}{2}}}=\frac{2+zz}{\sqrt{1+zz}}-2=\frac{2cc+bb}{c\sqrt{bb+cc}}-2,$$

ideoque integrale desideratum

$$=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot 4\left(2cc+bb-2c\sqrt{bb+cc}\right)=\frac{\pi}{4}\left(\sqrt{bb+cc}-c\right)^2=\frac{\pi bb\left(a-b\right)^2}{4\left(aa-bb\right)},$$

prouti supra posuimus.

PROPOSITIO 60

PROBLEMA

607. Si figura plana rectilinea ABCDEF (Fig. 92) fuerit circulo inscriptibilis, tum secundum quamcunque directionem OL in aqua moveatur, media directio resistentiae perpetuo per centrum circuli O transibit.

DEMONSTRATIO

Quoniam resistentiae directio, quam latus quodcunque figurae ab aqua suffert, ad ipsum latus in suo puncto medio est normalis, atque quodlibet latus sit chorda circuli circumscripti; directio resistentiae cuiusvis lateris per centrum circuli circumscripti *O* transibit. Quotcunque igitur latera figurae resistentiam excipiant, singulorum directio resistentiae per centrum *O* transibit; et hancobrem harum singularum resistentiarum media directio per idem centrum *O* transeat necesse est. Resistentia ergo totalis, quam figura proposita secundum quamcunque directionem promota patitur, per centrum circuli circumscripti *O* transit. Q.E.D.

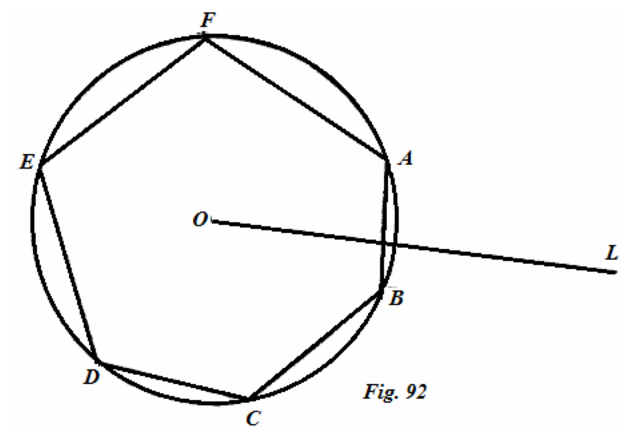


Fig. 92

COROLLARIUM 1

608. Si igitur huius figurae centrum gravitatis simul in centro circuli circumscripti fuerit situm, tum resistentia nullam habebit vim ad figuram convertendam, in quacunque directione etiam figura progrediatur.

COROLLARIUM 2

609. Intelligitur etiam, si modo anterior figurae pars circulo fuerit inscriptilis, neque cursus obliquitas sit tanta, ut posteriores figurae partes resistentiam excipiant; tum pariter resistentiae mediam directionem per centrum circuli prorae circumscripti *O* esse transituram.

COROLLARIUM 3

610. Si ergo huiusmodi figura diametro fuerit praedita, diameter per centrum circuli circumscripti transibit hocque casu centrum resistentiae fixum habebit situm in ipso centro circuli circumscripti.

SCHOLION

611. Insignis haec est proprietas figurarum rectilinearum circulo inscriptibilium, quod in iis centrum resistentiae constantem obtineat situm, quantum vis cursus sit obliquus, dum in aliis figuris situs centri resistentiae pro varia cursus obliquitate tantopere mutetur, videturque ista proprietas propria figurarum circulo inscriptibilium, ita ut in alias figuras non competat. Superfluum autem foret plures alias figuras planas, aquae innatantes considerare, cum ex allatis
facile sit iudicium de resistentia cuiuscunque figurae oblatae formare. Hancobrem missa resistentia, quam tantum lineae sive rectae sive curvae tanquam termini figurarum planarum in aqua patiuntur, progrediamur ad caput sequens in eoque ad figuras solidas, quae proprie ad institutum nostrum pertinent, investigaturi quantam resistentiam quodcunque corpus in aqua promotum sufferat, quae resistentia ex superficie corporis aquae submersa et in aquam impingente derivari debet. Simili scilicet modo, quo hactenus sumus usi, superficies omnis constare concipitur ex innumeris planis, quorum singula resistentiam patiuatur ipsis superficiebus et quadrato anguli incidentiae coniunctim proportionalem. Ita si superficies plana cuius area sit $=aa$ in aquam impingat sub angulo cuius sinus est m, velocitate debita altitudini v, tum vis resistentiae aequivalebit ponderi cylindri aquei cuius basis est a^2 et altitudo $=m^2v$, directio vero resistentiae erit ad ipsam superficiem planam normalis, atque per eius centrum gravitatis transit, prout in initio huius capitis satis est ostensum. Eiusmodi autem corpora tantum considerabo, quae plano diametrali verticali gaudeant, quo in duo frusta aequalia et similia dispescantur, huius modi enim corpora pro nostro instituto tantum considerari merentur. Praeterea cursus directionem ponemus directam, hoc est, quae in ipso plano diametrali sit sita; qua adiunctione inquisitio resistentiae fit facilior, cum directio resistentiae sponte se praebeat, quippe quae pariter in planum
diagonale incidit. Tantum igitur superest, ut quantitas resistentiae, et ipsa eius, quam plano diagonali habet, positio definiatur. Primum quidem pro hoc casu propositionem maxime generalem praemittemus, quo deinceps eo facilius ad quasvis figurarum species progredi liceat.