CHAPTER ONE

THE EQUILIBRIUM OF BODIES FLOATING IN WATER

LEMMA

1. *The pressure which the water exercises on a submerged body at the individual points is normal to the surface of the body ; and the force, which any element of the submerged body sustains, is equal to the weight of the right cylinder of water, the base of which is equal to the element of the surface itself, truly the height is equal to the depth of the element below the surface of the water.*

DEMONSTRATION

Any particle of water is pressed downwards by the cylinder of water resting above, and the pressure is equal to the weight of this cylinder. Therefore any particle, pressed in this manner by so great a force, is trying to flow away in every direction, and the adjacent particles to be pressed on by this same force being exerted. Whereby if a body were submerged water, that will be pressed at the individual particles of its surface by just as great a force exerted by the particles of water, as by which the particles of water themselves are pressed, and that normal to the surface. With which the truth of the lemma agrees, but which is demonstrated more fully in hydrostatics. Q. E. D.

COROLLARY 1

2. Since the pressure of the water, which a submerged body experiences, shall be normal to the surface of the body at individual points, all the pressures acting together strive to compress the body, and so reducing it to the smallest volume.

COROLLARY 2

3. Therefore unless the surface of the body may have enough rigidity resisting the compression, a submerged body actually will be compressed by the water, and will be reduced to the minimum volume.

COROLLARIUM 3

4. Therefore since it will be agreed, however great a pressure the individual elements of the surface of the submerged body shall sustain, and likewise the directions of the pressure at the individual points shall be known ; and thus the force will be able to be determined, which the whole body endures from the individual pressures taken together.

SCHOLIUM

5. For the total pressure requiring to be determined, which a body submerged in water sustains, must begin with the individual elements. Indeed since the direction of the pressure of the water at the individual points shall be normal to the surface, these individual pressures will have to be resolved separately, before the mean direction of these and the equivalent force may be able to be assigned.

PROPOSITION 1

PROBLEM

6. *If a part AIB of the vertical plane of the surface EAIBF of the water* (Fig. 1) *shall be immersed, to find the force, which the part of the perimeter AIB sustains from the pressure of the water.*

SOLUTION

Therefore the surface of the water *AB* shall be a horizontal line and may be considered as the axis. The vertical applied lines *MP, mp* may be drawn to the axis from the element of the curve situated under the water at *Mm* and there may be called $AP = x$; $PM = y$, there will become:

$$Pp = Mn = dx,\ mn = dy \text{ and } Mm = \sqrt{(dx^2 + dy^2)} = ds.$$

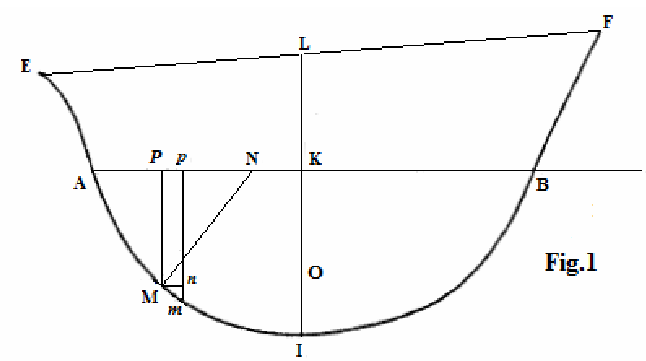

Fig.1

Now the pressure, which the element *Mm* sustains is equal to the water rectangle, of which the base is *Mm,* and the height is equal to *PM,* evidently equal to the depth of the element *Mm* below the surface of the water. This pressure therefore will be $= yds$, and its direction *MN* is normal to the surface at the element *Mm.* This pressure [force] may be resolved into two lateral parts, of which one shall act vertically upwards in the direction *MP,* the other truly shall be directed horizontally. Therefore on account of $MN : MP = ds : dx$, the force acting upwards on the element *Mm* $= ydx$; truly the force *Mm* pushing the element horizontally $= ydy$. Therefore on integrating, the force of the water propelling the whole arc *AM* horizontally $= \frac{y^2}{2}$; from which the force acting horizontally on the whole submerged curve *AMIB* shall become $= 0$, for with the point

M moved to *B* the applied line *y* will vanish. But the force, which pushes the whole arc *AM* upwards, will be $= \int ydx =$ area *AMP*.

Whereby the whole curve *AIB* will be urged upwards by a force, which is proportional to the area *AIB* ; and this force will be equal to the weight of the water present in the area *AIB* . Q.E.F.

COROLLARY 1

7. Therefore the force of the water acts to expel the figure *EIF* from the water, and indeed it will itself be expelled, unless it shall be retained in this position either by weight, or by some other external force.

COROLLARY 2

8. Since all the horizontal forces, by which all the submerged elements of the water curve are acted on, cancel each other out, it is evident the immersed figure *EIF* of water in no manner to be forced horizontally. Whereby also there is no need for a force being required to restrain the water.

COROLLARY 3

9. Therefore since the horizontal forces cancel each other out, and only the vertical forces shall remain, for any element of which *Mm* that force can be considered to be applied acting upwards, which is equal to the area of the element *PMmp*

SCHOLION

10. Before the forces may be able to be investigated, which the immersed body of water may sustain from the pressures of the water, it was necessary to begin from the surface, even if a case of this kind may not be given anywhere. But since more composite cases may be understood more clearly, if in the first place simpler cases may be subjected to an examination, and also it will be agreed to retain the same order here . On account of which I believe nobody to be offended by the uncalled for discussions, with which I have been compelled to use, while I have made mention of the weight, which the rectangle of water may have; indeed this ought to be indicated from the analogy with the following to be shown, where similar properties in these bodies will be discovered.

PROPOSITION 2

PROBLEM

11. *If a vertical plane figure of water were immersed, to find the mean direction of all the water pressures, and the equivalent force from all of these.*

SOLUTION

Since, after the individual water pressures exerted on the individual elements were resolved into vertical and horizontal components, all these horizontal pressures shall cancel each other mutually, only the vertical components from all the vertical pressures will be equivalent. Whereby only that is required, so that the mean direction of all these vertical forces and the equivalent force may be defined. But since all these forces shall have parallel directions, the equivalent force to be equal to all these forces taken together, and thus proportional to the area *AIB*. Again the direction also will be vertical, for example consider it to be *IL,* the distance *AK* of which may be found from some point *A*, by dividing the sum of all the moments relative to *A* by the sum of these forces, which is $=\int ydx$. But since the element *Mm* shall be urged upwards by the force $= ydx$, its moment with respect to the point $A = yxdx$. From which the sum of all the moments will be $\int yxdx$, if after the integration there may be put $x = AB$. Whereby the distance *AK* will be $=\frac{\int yxdx}{\int ydx}$, evidently with the position $x = AB$ after each integration. Therefore with the point *K* determined, the vertical *LI* drawn through that point will be the mean direction of all the water pressures, and the equivalent force itself will be $=\int ydx$, or the area *AIB*. Q.E.F.

COROLLARY I

12. If *O* were the centre of gravity of the area *AIB,* and from that the perpendicular *OK* may be drawn to the axis *AB*, there will be also $AK = \frac{\int yxdx}{\int ydx}$, as it is agreed from statics, whereby the right vertical line, drawn through the centre of gravity of the submerged part *AIB*, will be the mean direction of all the water pressure.

COROLLARY 2

13. Therefore a single force can be substituted in place of all the water pressures, forcing the figure vertically upwards in the direction *IL*, which is equal to the weight of water filling the area *AIB*.

SCHOLIUM

14. In which case, where the body of the immersed water is put as far as to the surface, concerning which we have elicited the pressure of the water, the same prevails also for these bodies, as will be shown soon ; evidently where all the horizontal forces cancel each other, and the mean direction shall be a vertical line passing through the centre of gravity of the volume of the immersed body, and so that the equivalent force is equal to the weight of water of the submerged part. Moreover even if these properties now satisfied shall be known, yet it is observed these provide a suitable way for the following requiring to be prepared on being elicited by this genuine analytical method.

PROPOSITION 3

PROBLEM

15. *If any body may have some part immersed in water, to determine the force which its part, submerged in the water, sustains from the pressures of the water.*

SOLUTION

The diagram shall represent the plane section of a flat body immersed in water, and thus *AEIFBL* shall be the horizontal section made on the surface of the water, by the surface of the body [at the base of the diagram] immersed vertically in the water, thus so that the part of the body situated below this section shall be established in the water (Fig 2). In this section some right line *AB* may be assumed for the axis and that two nearby ordinates may be drawn *LI, li,* which cut the other nearby parallel lines *EF, e f* at right angles to the axis ; from the points of intersection *Q, q, R, r* vertical lines are drawn downwards, cutting the element *Mm* on the surface of the body under the water, the area of which shall be *dS*.
Now there may be put

$$AP = x,\ PQ = y \text{ and } QM = z,$$

z will be the depth of the element *Mm* below the surface of the water. Whereby the pressure of the water, which the element *Mm* sustains, is equal to the weight of the cylinder of water, of which the base is *dS* and the height *z,* this weight may be expressed by

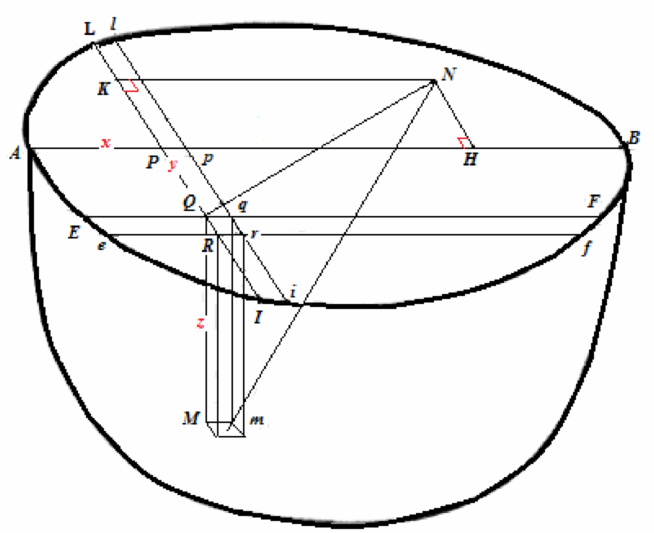

Fig. 2

zdS. Moreover the direction of this force, since it shall be normal to the surface *Mm*, shall be the normal *MN* drawn to the element, which produced shall cross the horizontal plane *ALB* at N, therefore *MN* will be the direction of water pressing on the element *Mm*; as the position of this normal is required to be found, the nature of the surface [where *z* is considered as a function of *x* and *y*] which shall be present under the water, may be expressed by this equation $dz = Pdx + Qdy \left[= \frac{\partial z}{dx}dx + \frac{\partial z}{dy}dy. \right]$ The normals *NH* and *NK* may be drawn from *N,* the one to the axis *AB* [or *x* axis], as well as the other to the applied line [or *y* axis] *LI* ; from the nature of the normal to the surface there will become $[KN] = PH = Pz$ and $QK = -Qz$ [*i.e.* we may consider an enlarged view of the *x* and *y* components of the normal to the elemental surface *Mn* projected onto the horizontal surface of the water, magnified by the depth *z*:]. The right line *QN* may be drawn, which, since it is horizontal, shall be normal to the vertical *QM,* and there will become

$$QN = z\sqrt{(P^2 + Q^2)} \text{ and } MN = z\sqrt{(1 + P^2 + Q^2)}.$$

Now the force pressing the element *Mm* in the direction *MN,* since the force is $= zdS$, may be resolved into two parts, of which one shall act upwards on the element *Mm* in the direction *MQ,* the other truly horizontally in a direction parallel to *QN* itself. Moreover, since there shall be

$$MN : MQ = \sqrt{(1 + P^2 + Q^2)} : 1,$$

the force acting upwards on the element *Mm* will become

$$= \frac{zdS}{\sqrt{\left(1 + P^2 + Q^2\right)}},$$

and the force acting on the element horizontally parallel to *QN* itself

$$= \frac{zdS\sqrt{\left(P^2 + Q^2\right)}}{\sqrt{\left(1 + P^2 + Q^2\right)}}.$$

But this latter force, since its direction is not constant, again may be resolved into two forces acting horizontally along the directions *QK* and *QF*, of which that one, which acts parallel to the applied line *LI*, will be

$$= \frac{-QzdS}{\sqrt{\left(1 + P^2 + Q^2\right)}},$$

and the other one, the direction of which is parallel to the axis *AB*

$$=\frac{PzdS}{\sqrt{\left(1+P^{2}+Q^{2}\right)}}.$$

But since the rectangle $Qr = dxdy$ shall be to the element of area $Mm = dS$ as QM to MN, the element *Mm* will become $= dxdy\sqrt{(1+P^2+Q^2)}$, with which value substituted in place of dS, the force acting upwards on the element $Mm = zdxdy =$ to the prism *RM*, or it is equal to the weight of the water, the volume of which is the prism *RM.* Therefore the whole surface of the body, which is under the water, shall be pushed upwards by a force which is equal to the sum of all the prisms, that is, which is equal to the weight of the water, the volume of which may be made equal to the part of the body present in the water. But the force, which shall act horizontally on the element *Mm* parallel to the applied line LI, will be $= -Qzdxdy$. Now if the abscissa $AP = x$ may be put constant, a horizontal force will be had, which shall act on all the elements placed under the strip *Li*, in the direction parallel to *LI* itself, on taking the integral of *Qzdxdy,* or on account of x being constant, this force will be $= -dx\int Qzdy.$ But if x were constant, there would become $dz = Qdy$, whereby this same force will become :

$$-dx\int zdz = -\frac{z^{2}dx}{2}.$$

Now with the point *Q* placed at *I,* where there is $z = 0$, $-\frac{z^2dx}{2}$ expresses the horizontal force parallel to *LI,* where the portion of the surface placed below the surface of the water, corresponding to the element *QqlI,* is pressed. Therefore with the point Q translated to *L,* where *z* again vanishes, this force also will vanish. Whereby the horizontal force, by which the part of the surface of the body put below the strip *LI* is pressed, shall be equal to zero. And consequently all the horizontal forces for the applied line *LI,* by which all the elements of the surface of the body put below the level of the water are pressed upon, cancel each other out. Then the horizontal force, the direction of which is parallel to the axis *AB*, by which the element *Mm* is pressed $= Pzdxdy$, now with $y = PQ$ taken to be constant, this horizontal force, by which a small part of the surface placed below the element *ER* is pressed

$$= dy\int Pzdx = \frac{z^{2}dy}{2},$$

on account of $Pdx = dz$ with *y* taken to be constant. Therefore with the point *Q* at *F,* where there shall be $z = 0$, the total horizontal force vanishes, where it is acting on the part of the surface of the body under the strip Ef. Therefore this horizontal force, where the whole surface of the body situated under the water also vanishes, the direction of which is parallel to the axis *AB*. Whereby the total pressure, which the body immersed in water endures from the pressures of the water, consists only on the vertical forces, by which the body is pushed upwards, and the sum of which is equal to the weight of the water, the volume of which is equal to the part of the body submerged. Q.E.I.

[Thus, Euler has given a mathematical argument validating Archimedes' Principle.]

COROLLARY 1

16. Since the horizontal forces of each kind, on which the individual elements of the submerged surface are acted on, altogether cancel each other out, the body will be forced upwards only by the water pressures, nor can a horizontal motion be impressed from these pressures.

COROLLARIUM 2

17. Since the force, by which a body of water is pressed upwards, shall be equal to the weight of the water, the volume of which shall be equal to the volume of the parts of the body submerged, it is evident the weight of the body striving to slip the body downwards is diminished by the upwards pressure of the water, as the force of gravity has the property of requiring the body to slip downwards. And indeed its proper weight to be diminished by the weight of the volume of water, which in turn is equal to the volume of the part of the body submerged.

COROLLARY 3

18. On account of which, if only a part of the body were immersed, so that only a volume of water equal to the weight of the body, then the struggle of the body trying to slip downwards vanishes, and the body will float on the water.

COROLLARY 4

19. It is observed from these also, that if a smaller part were immersed in the water than were required for floating, then the rest of the body shall be plunged deeper, until the volume of the object present in the water shall be equal to the weight of the body itself.

COROLLARY 5

20. On the other hand, if a greater part of the body than that required for floating were immersed in the water, then the force of the water pressing upwards will be greater than the weight of the body, and thus it will be raised upwards , until the volume of the submerged part shall be equal to the volume of the water which shall have a weight equal to the weight of the body.

SCHOLIUM 1

21. These, which have been brought forth, pertain chiefly to bodies for which the specific gravity is lighter than that of water. For if the body may exceed water with a greater specific gravity, then indeed the whole body submerged cannot occupy a volume of water of such a size that it will be of weight equal to the body itself. Therefore a body with a specific gravity heavier than water, even if the whole may be submerged in the water, will conserve the imbalance by slipping downwards, yet meanwhile this imbalance will be less than out of the water in the air. Clearly its weight will be diminished by the same weight of water of the same volume as the body has, and the ratio of this follows from the preceding, concerning these, which we have deduced for bodies lighter than water.

SCHOLIUM 2

22. Therefore so that the body may float on the water, it is required so that so great a part of this shall be immerged in the water, so that the volume of water of that part shall have the same weight as the body itself has. For if a greater or smaller part were to be submerged, the body either will rise or descent more, if it may be left to itself. Therefore this is the first and especially necessary requisite for floating, so that a body of a definite size may have a part immersed in the water; and this rule is accustomed to be enunciated by authors, so that they say, every floating body displaces as much water in its place, as shall be equal to the weight of the body itself. But this rule alone may not suffice for floating, for in addition another rule may be required, whereby the rule must be disposed where the body floats and remains at rest. Indeed though the horizontal forces may mutually cancel each other out, and on that account the body shall not be moved horizontally, yet from the vertical forces alone, from the weight of the body, and from the pressure arising from the water, even if they were equal to each other, motion can b produced, by which the state of rest may be disturbed, as will be shown further in the following.

PROPOSITION 4

PROBLEM

23. *If a body may be immersed, with some part out of the water, to determine the mean direction of all the water pressure exerted on the submerged part, and the equivalent force from all these parts.*

SOLUTION

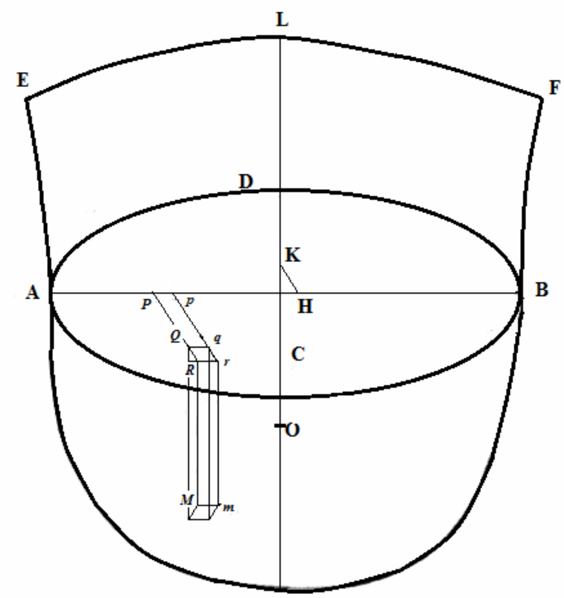

Fig. 3

Since all the horizontal forces, which arise from the resolution of the water pressure acting on the individual elements of the submerged part, shall mutually cancel each other out, only the vertical forces come into the consideration. The directions of which, since they shall be parallel to each other, also the vertical direction will be the mean direction of these and the equivalent force will be equal to the sum of all these vertical forces. On account of which the equivalent force from all the water pressures will be equal to the weight of the water, the volume of which is equal to the part of the body submerged. Moreover the direction of this, or the mean direction of all the water pressures will be the vertical right line *IK* (Fig.3), which occurs at the point *K* of the horizontal section of the body made with the surface *ABCD* of the water, and from the point *K* to the axis in the section for argument's sake assumed to be the perpendicular *KH* drawn. Now as before an element of the submerged part may be considered :

$$Mm = dS,$$

and thence the verticals to the plane section $ACBD$ may be drawn, and likewise the applied lines *QP, qp*. Then, as before, there may be put $AP = x$, $PQ = y$ and $QM = z$, and the nature of the submerged surface expressed by this equation $dz = Pdx + Qdy$; thus, so that there shall become $dS = dxdy\sqrt{(1+P^2+Q^2)}$. But the force, by which the element *Mm* is pressed vertically upwards, is equal to the weight of the volume of water $= zdxdy$, the moment of which with respect to the horizontal to the normal *AB,* and drawn through *A* is *zxdxdy*. Therefore on putting *x* constant and with the integral of *zdy* taken, thus so that it shall correspond to the whole ordinate drawn through the point *P*, $\int xdx\int zdy$ will give the sum of all the moments with respect to the horizontal through the point *A* drawn normally to *AB*. And this expression, if it may be divided by the volume of all the submerged parts, which is, which is $\int dx\int zdy$, will give rise to the distance *AH*. Moreover the distance *KH* will be had, if the sum of all the moments with respect to the axis *AB* may be divided by the volume of the submerged parts $\int dx \int zdy$. But the moment of the force acting vertically upwards on the element *Mm* is $yzdxdy$. Now with y placed constant and thus with the integral zdx taken, so that it may correspond to all the applied lines y, then $\int ydy \int zdx$ will give the sum of all the moments on this side of the axis *AB*, truly in a similar manner the sum of all the moments present beyond *AB* may be sought, and from this sum that sum may be subtracted, and the residue divided by $\int dx \int zdy$ will give the distance *HK*. Moreover with the point *K* known, the mean direction *IK* shall be found. Q.E.I.

COROLLARIUM 1

24. Since the force, by which the element *Mm* may be forced upwards, shall be proportional to the elementary prism *QRM*, it is evident the centre of gravity of the displaced water found in this manner, to pass through the right line *KI*, which shall be at *O*. Indeed by the same calculation, as we have used here, the centre of gravity is accustomed to be determined.

COROLLARY 2

25. Therefore the mean direction of all the water pressure [forces], which the submerged part of the body endures, is the vertical line which passes through the centre of gravity of the submerged part.

COROLLARY 3

26. Therefore if the body shall float on the water, in which case the volume of water of the submerged part shall weigh the same as the body, then the equivalent force from all the water pressures will be equal to the weight of the body.

COROLLARY 4

27. Therefore the body floating on the water is forced upwards by such a force, as great as the weight of the body. Truly the direction of this force acting upwards on the body is the vertical line passing through the centre of gravity of the submerged part.

SCHOLIUM

28. From the agreement that we have shown between the direction of the pressure of the water upwards and the right vertical line passing through the centre of gravity of the submerged part of the body, it is readily understood, this point *O* must be the centre of gravity of the body *AIB* considered as homogeneous. Whereby if some part of the submerged body were assembled from heterogeneous materials, yet with the merging together [of the up and down vertical forces] at the point *O* requiring to be investigated, this submerged part must be considered as homogeneous. For this reason so that so that I may avoid the ambiguity in the said centre of gravity, in that latter centre of gravity *O,* which must be sought from the consideration of a homogeneous body, I will call by the name centre of magnitude. Therefore the centre of the magnitude of the submerged part will be found, if the submerged part may be considered to be constructed only from homogeneous material, and its centre of gravity may be defined. And thus this centre of magnitude of the submerged part, also will be the centre of gravity of the water removed from its location, or of this water which had occupied the space *AIB*, before that body was immersed.

LEMMA

29. *The body ABC* (Fig. 4), *to which the two forces Gg and Oo are the applied in parallel and opposite directions, cannot be in equilibrium, unless these forces shall be equal to each other, and the directions of these lie on the same right line.*

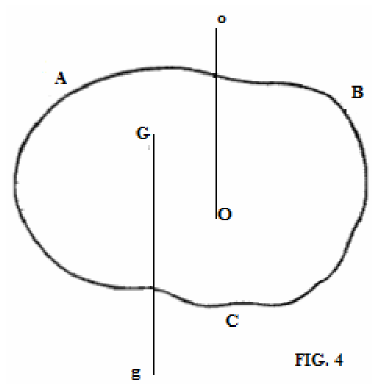

FIG. 4

DEMONSTRATION

If the forces were unequal, it is evident the body cannot in the least be in equilibrium : even if they may coincide, more strongly the body may move in their direction. But if the forces were equal, truly neither can they coincide, then these forces cause the body *ABC* to turn about itself about the region *BAC*. Whereby so that the body may remain at rest, it is necessary not only that the forces *Gg* and *Oo* shall be equal, but also so that the directions of these shall coincide. Q. E. D.

PROPOSITION 5

THEOREM

30. *A vessel, sitting freely on water, cannot be in a state of rest or equilibrium, unless the volume of the water of the submerged part may have a weight equal to the weight of the body itself, besides truly unless the centre of gravity of the whole body and the centre of the magnitude of the submerged part may lie on the same vertical right line.*

DEMONSTRATION

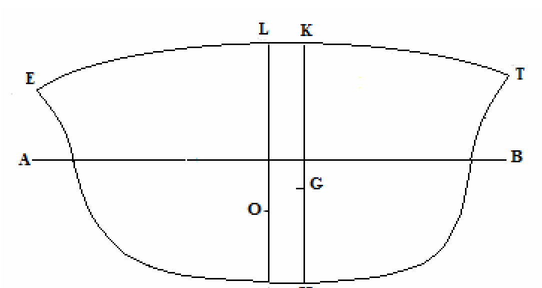

Fig. 5.

EAIBF (Fig. 5) shall be some resting water vessel and *AIB* its submerged part. Now since the body may be put in place to be free, no forces acting on the body need be considered, apart from the force of its weight and the pressures of the water, which its submerged part experiences. But the force of the weight of the body shall be equal to the weight itself, and its direction is a vertical right line passing through its centre of gravity. There may be put $P =$ to the total weight of the body, and *G* shall be its centre of gravity ; on account of this weight in place the body will be forced upwards in the direction *GH* by the force *P*. Whereupon the force from all the pressures of the water is equivalent to the force forcing the submerged part directly upwards in the direction *OL* passing through the centre of magnitude *O* of the submerged parts, which quantity is made equal to the weight of the volume of water occupied filled up by the submerged part; therefore with this weight *Q* in place on account of the pressures of the water, the body will be forced upwards in the direction *OL* by a force $= Q$. Therefore for this reason our water vessel to be resting entirely due to these two forces *P* and *Q* being applied, of which the one acts downwards in the direction *GH*, the other truly upwards in the direction *OL*. And thus by the lemma established the body cannot be in equilibrium, unless at the same time there shall be $Q = P$ and the right line *LI* shall fall on the right line *HK*. But there becomes $Q = P$, only if the part submerged in the water, for which volume, the weight of the water shall be equal to the weight of the body, then truly the lines *LI* and H*K* coincide, if the centre of gravity *G* of the whole body and the centre of magnitude *O* of the submerged part shall be placed on the same vertical line. Q. E. D.

COROLLARY 1

31. Therefore two forces are required for this, so that the body may be able to rest in equilibrium, if one or other shall be missing, the body cannot remain in a state of rest.

COROLLARY 2

32. Therefore as often as we see a vessel resting on water, then it is certain, from the volume of its submerged part, the weight of water displaced to be equal to the weight of the body itself. And besides the centre of the size of the displaced parts and the centre of gravity of the whole body to be situated on the same vertical line.

COROLLARY 3

33. Whereby if a vessel were floating in equilibrium, then the line joining the centre of gravity of the whole body and the centre of the magnitude of the submerged part , will be normal to the section of the water *AB*.

COROLLARY 4

34. Since the weight shall be agreed by experiment, by which a given volume of water shall press down, with the weight of any body given, it will be able to find the magnitude of the submerged part for the required equilibrium to be produced.

SCHOLIUM 1

35. So that I may avoid more circumlocutions in the following, in place of the whole description I may use more convenient terms. Thus the centre of gravity of the whole body I will simply only call the centre of gravity ; and the centre of the magnitude of the submerged part only the centre of magnitude, when these occur at no time can you call these by any other sense. Then also the section, which the body puts in place with the upper surface of the water, I will call more simply the water section. In a similar manner the vertical right line of the centre of gravity passing through the whole body will become the vertical right line passing through the whole body, and the vertical line passing through the centre of the magnitude of the whole submerged body will become... of the vertical centre of magnitude.

SCHOLIUM 2

36. When in the following with the aid of these rules we will determine the state of these bodies, in which they may be able to be settled in water, thus that must be understood, so that the body, if it is in a situation of this kind the definition of the water may be put in place most carefully, then at last it shall be rest in this situation. But if it may be removed or inclined a little from this position, then whether spontaneously it may restore itself, or truly if it may be received into another situation, is another question, which does not pertain here, but which I shall make clear in the following. But if the situation, in which the body may be imposed to be at rest in the water, thus if it were prepared, so that the body, if it may be inclined a little from that situation, itself may not be restored, but another situation may be sought, in which it may repose, then it is most difficult to bring

about, so that the body may persist in that state. Indeed even if the greatest skill may be arranged in that place, yet the lightest force either of the air or water at once may disturb it from there, thus so that it shall be most difficult, to prove cases of this kind by experiment. Just as there is no doubt, why a light branch placed vertically in water thus may not be returned, so that equilibrium may be obtained ; yet meanwhile put in place in this situation as if by itself at once will sink down, and will adopt a horizontal situation, in which it may repose.

PROPOSITION 6

THEOREM

37. *Every body* (Fig. 6), *which is generated from the rotation of a figure of some kind ACFB made about some axis AB, thus will be able to sit in water, so that its axis AB shall maintain a vertical situation, provided that the centre of gravity were put on the axis AB itself.*

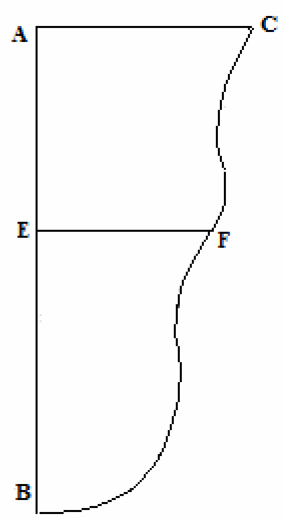

Fig. 6

DEMONSTRATION

The right line *EF* normal to the axis *AB* shall draw out the area *EBF,* which rotated around the axis will generate a solid, which shall be made with a volume equal to the weight of water which shall be equal to the whole weight of the body, and this solid to be the part required to be submerged in the water, so that the body may remain at rest, but only if the other part required may be found in this situation. But of this submerged part, which is generated from the area *EFB,* the centre of its magnitude falls on the axis *AB,* whereby since the centre of gravity is put to fall on the same axis also, thus the body will be able to rest on the water, so that the axis *AB* may maintain a vertical position. Q.E.D.

COROLLARY 1

38. It is understood also from this demonstration, that is solid body can float on water with the situation inverted, where *B* is put in place upwards and *A* truly downwards ; thus so that now two situations may be agreed, by which bodies of this kind can float on water.

COROLLARY 2

39. Hence it follows also a ball, the centre of gravity of which is placed at its centre, also can float placed in water in any situation. For any right line axis *AB* passing through the centre can maintain the position.

COROLLARY 3

40. Right cylinders pertain to this class of bodies, and likewise right cones both whole as well as truncated. Whereby these bodies also thus will be able to float in water, as the axes of these may maintain a vertical situation, but only if the centres of gravity of these may lie on the axes themselves.

SCHOLION

41. These round bodies have this property , that all the sections normal to the axis shall be circles, and the demonstration depends on this principle, because the axis *AB* shall pass through the centre of gravity of the individual sections. Whereby the same proposition shall prevail equally for bodies, the sections of which made normally to the axis shall be regular polygons of some kind, as well as for circles. On account of which also bodies of this kind thus will be able to float in water, as they shall maintain a vertical axes in place, but only if the centre of gravity of the whole body were put on this axis itself.

COROLLARY 4

42. Therefore if a body of this kind were made from a homogeneous material, then certainly the centres of gravity of these will lie on their axes. Whereby hence it will be allowed to conclude, all homogeneous bodies, which have figures of this kind, thus can float on water, so that the axes of these shall be vertical.

PROPOSITION 7

THEOREM

43. The cylindrical body *DEIH* (Fig. 7), *all the transverse sections of which DE, FG, HI are similar and equal to each other, can remain placed erect to the water, only if its centre of gravity were on the right line AB passing through the centres of gravity of all the sections.*

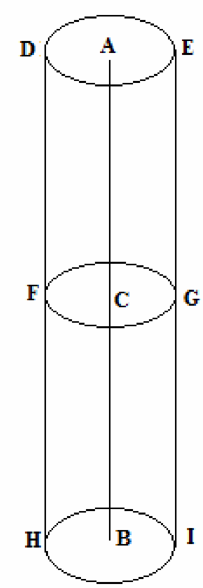

Fig. 7

DEMONSTRATION

So great a part *FGIH* shall be submerged, as great as the amount of water displaced, and that affair concerning the submerged part must be considered. The centre of the magnitude of this part submerged therefore will be situated on this line *BC*, certainly which passes through the centres of gravity of all the transverse sections or magnitudes. But the centre of gravity of the whole body is to be situated on the same line *AB*. Whereby since the right line *AB* shall be normal to the section *FG* of the water, the body in this situation will be able to float on the water. Q. E. D.

COROLLARY 1

44. Without myself being reminded, it is easy to understand also, the same body also placed inverted where *HI* points up and *DE* truly points down, to be able to stand on water ; thus so that two known cases shall be known, by which bodies of this kind can float on water.

COROLLARY 2

45. If the body *DEIH* were made from homogeneous material, the centre of gravity itself lies on the right line *AB*. Whereby bodies of this kind placed in water always will float erect, and that in two ways.

SCHOLIUM

46. All right prisms belong to bodies of this kind, besides common right cylinders, also whatever kind of bases they may have, either regular or irregular. Then also equally here all solids are considered which may be generated, if some plane figure may be drawn following straight lines normal, by its own motion always shall be moved parallel to the plane of the figure. And concerning all these, the theorem proposed prevails.

PROPOSITION 8

THEOREM

47. *The cylindrical body abdc* (Fig. 8), *such as we have considered in the preceding proposition, with bd situated horizontally, will be able to float on water, if its centre of gravity G may fall in the middle of its section ef . And indeed with the same in place, it will be able to float on water, where only the mean section ALBI is able to float placed vertical, if its centre of gravity were situated at G.*

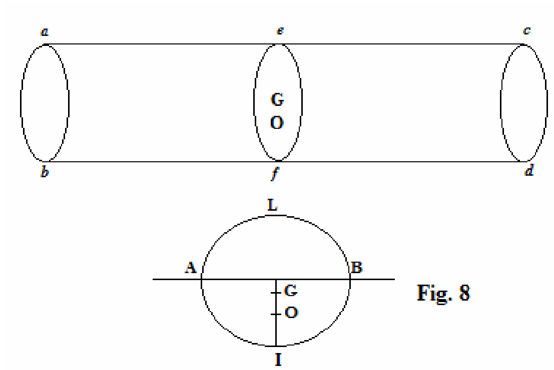

Fig. 8

DEMONSTRATION

Indeed here the body may float placed horizontally on water, and now just as great a part of this shall be immersed, as the amount required for equilibrium ; it is evident the centre of the magnitude *O* of the submerged part also must fall on the median section *ef*, and to be at the centre of magnitude of the submerged part itself. Therefore unless the centres *G* and *O* shall lie on the same vertical line, the body shall turn about that point until they acquire these centres. With which done, *LAIB* shall be the position of the mean section, *AB* the section of the water, *G* its centre of gravity, which we may put to agree with the centre of gravity of the whole, *O* the centre of the magnitude of the submerged part *AIB,* which itself agrees with the centre of magnitude for the body *ad.* Whereby if the right line *GO* were vertical, then not only the section *LI* but also the whole body will be able to float placed in water. Q.E.D.

COROLLARY 1

48. But so that only so great a part of the section *LAIB* may be immersed in water, if it is connected with the body, the same specific gravity with respect to the water must be attributed to this section, as the whole body has.

COROLLARY 2

49. If the body were homogeneous, then its centre of gravity not only will lie in the median section *ef* but also in addition will be situated at the centre of gravity of this mean section.

COROLLARY 3

50. Therefore so that the situation of cylindrical bodies of this kind may be determined, so that they are able to float horizontally on water, it will be sufficient to inquire, in what situation one section of that may be able to stand vertically.

SCHOLIUM

51. Therefore so that the situation may be able to be defined, so that cylindrical bodies of this kind may be able to lie horizontally on water, as far as it is with regard to the figure of cross-section. The problem therefore corresponds to this, so that for some given plane figure *LAIB,* the part *AIB* requiring to be immersed in water is determined by its specific gravity with respect to water, and the centre of gravity *G*, which indeed agrees with the quantity, so that the right line joining the centre of gravity *G* and the centre of the magnitude *O* of the part *AIB* shall be normal to the section of the water *AB*. On account of which, according to our design, it will be appropriate to consider some plane figures, and for which situations they shall be able to be investigated to be floating vertically in water. But we will consider especially cylinders bodies or homogeneous prisms, and hence on that account for the centre of gravity *G* of the figure *LAIB,* we may assume it to be the centre of gravity of this figure; thus so that the question for us is reduced to this: From the given figure *LAIB* by drawing the right line *AB*, the part *AIB* of a given magnitude can be removed, with this condition that the right line joining the centre of the whole figure to the parts cut off shall be perpendicular to the line *AB.* Therefore we shall begin with a triangle as the simplest figure and thence we may progress to quadrilaterals.

LEMMA

52. *If some line IPQ may be drawn through the centre of gravity G of the plane triangle ACB* (Fig. 9), *crossing the side BC produced at Q, there will become*

$$AC \cdot CQ - BC \cdot CP = 3CP \cdot CQ.$$

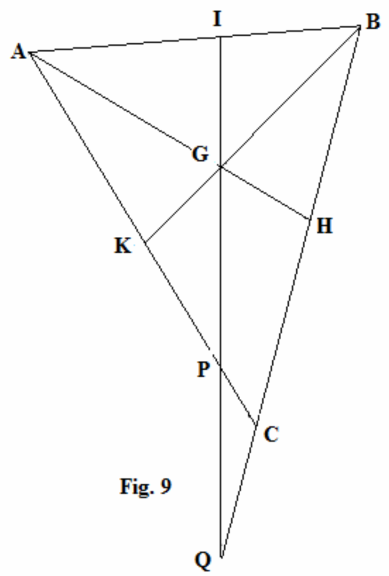

Fig. 9

DEMONSTRATION

The right lines *AH* and *BK* may be drawn from the angles *A* and *B* through the centre of gravity *G,* which, by the known property of the centre of gravity, will bisect the sides *BC* and *AC* and there will become

$AG : GH = 2:1$ and $BG : GK = 2:1$. Truly there is :

$$\sin API : \sin BQI = \frac{AG}{AP} : \frac{GH}{HQ} = \frac{GK}{KP} : \frac{BG}{BQ} = CQ : CP.$$

From which proportions there is elicited :

$$BC + 2CQ : AC - CP = BC + CQ : AC - 2CP = CQ : CP,$$

and hence

$$AC \cdot CQ - BC \cdot CP = 3CP \cdot CQ.$$

Q.E.D.

Note I:

With the sine rule applied to triangle *API* and to the associated triangle *APG*, we have:

$$\frac{\sin API}{\sin AIP} = \frac{\sin APG}{\sin AGP} = \frac{AG}{AP},$$

and likewise for triangle *QIB* and the associated triangle *QGB*:

$$\frac{\sin BQI}{\sin BIQ} = \frac{\sin BQG}{\sin BGQ} = \frac{GH}{QH},$$

and, where we note that as these angles are supplementary, $\sin AIP = \sin BIQ$; and the angle *API* can be replaced by the angle *APG*, and likewise the angle *BQI* by the angle *BQG*. Hence,

$$\frac{\sin API}{\sin AIP} : \frac{\sin BQI}{\sin BIQ} = \sin API : \sin BQI = \frac{AG}{AP} : \frac{GH}{QH} = \frac{CQ}{CP};$$

and the other ratio $\frac{GK}{KP} : \frac{BG}{BQ}$ can be treated in a similar manner, with the triangles KGP and PGQ. In this case we have the sine rule applied initially to triangle *KGP* and then to the associated triangle *APG*, giving :

$$\frac{\sin KPG}{\sin KGP} \cdot \frac{\sin GQB}{\sin QGB} = \frac{KG}{KP} \cdot \frac{GB}{BQ} = \frac{CQ}{CP},$$

as above.

Does $BC + 2CQ : AC - CP$ equal $\frac{AG}{AP} : \frac{GH}{HQ}$?

From Fig. 9 :

$$\frac{AG}{AP} : \frac{GH}{HQ} = \frac{AG \cdot HQ}{AP \cdot GH} = \frac{2HQ}{AP} = \frac{2HQ}{AC - PC} = \frac{BC + 2QC}{AC - PC} = \frac{CQ}{CP};$$

Does $BC + CQ : AC - 2CP$ equal $\frac{GK}{KP} : \frac{BG}{BQ}$?

From Fig. 9 :

$$\frac{GK}{KP} : \frac{BG}{BQ} = \frac{GK \cdot BQ}{KP \cdot BG} = \frac{BQ}{2KP} = \frac{BC + CQ}{2(KC - PC)} = \frac{BC + CQ}{AC - 2PC} = \frac{CQ}{CP}.$$

Again :

$$\frac{BC + 2CQ}{AC - CP} = \frac{BC + CQ}{AC - 2CP} = \frac{CQ}{CP}$$

gives :

$$BC \cdot AC - 2CP \cdot BC + 2CQ \cdot AC - 4CQ \cdot CP = AC \cdot BC + AC \cdot CQ - BC \cdot CP - CQ \cdot PC$$

hence :

$AC \cdot CQ - CP \cdot BC = 3CQ \cdot PC$. Voila!

COROLLARIUM 1

53. If the sine of the angle API may be put $= m$, and the sine of the angle $BQI = n$; there will become $m : n = CQ : CP$.
From which since there shall be :

$$CQ = \frac{AC \cdot CQ - BC \cdot CP}{3CP},$$

there will become

$$CQ = \frac{m \cdot AC - n \cdot BC}{3n}$$

and

$$CP = \frac{m \cdot AC - n \cdot BC}{3m}.$$

COROLLARY 2

54. If from A there may be a right line drawn parallel to GQ, then it shall cross BC produced, and that shall be $= 3GQ$. On account of which there shall become

$$CP : PQ = AC : 3GQ \text{ and } GQ = \frac{AC \cdot PQ}{3 \cdot CP}.$$

Therefore on putting $\sin ACB = k$ there will become

$$GQ = \frac{k \cdot AC}{3n}.$$

COROLLARY 3

55. If the right line QI were normal to AB, there will become

$$BC : AC = \cos API : \cos BQI.$$

Whereby if $\cos \text{ ang.} API = M$ and $\cos \text{ ang.} BQI = N$, there will become

$$BC : AC = M : N \text{ or } M \cdot AC = N \cdot BC.$$

PROPOSITION 9

PROBLEM

56. *For the proposed homogeneous triangle ACB* (Fig.10), *of which the specific gravity with respect to water shall be as p to q, to determine the cases, by which this triangle is able to float on water, so that the side AB may remain out of the water.*
[Some additions have been made to Fig. 10 and the calculations below set out in a more transparent manner, to make Euler's account more easily read.]

SOLUTION

The part sought being immersed in the water shall be aCb, so that the triangle may be able to float on the water, and the right line IPQ drawn through the centre of gravity both of the centre of gravity of the whole triangle ACB as well as of the submerged part aCb. Whereby so that it shall be able to float on the water in this situation, it will be required that the right line IQ shall be perpendicular to ab and in addition that the area of the triangle aCb shall be to the area ACB as $p:q$. Now there may be put $AC = a$, $BC = b$ and for the angle ACB, $\sin ACB = k$, and its $\cos ACB = K$. Again there shall be

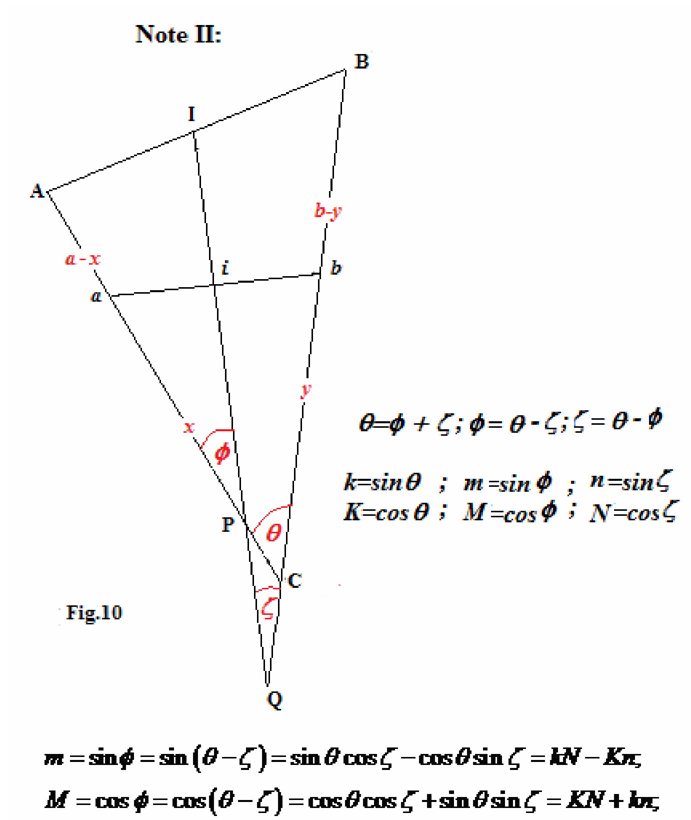

Fig.10

$aC = x,\ bC = y,\ \sin API = m,\ \cos API = M$,

likewise $\sin BQI = n$ and $\cos BQI = N$. With these in place there shall be $m = kN - Kn$ and $M = kn + KN$. Now since the right line QPI shall pass through the centre of gravity of the triangle ACB, there will become

$$CQ = \frac{ma - nb}{3n}.$$

Whereby since the same right line passes through the centre of gravity of the triangle aCb, there will become

$$CQ = \frac{mx - ny}{3n},$$

with which taken jointly there will be come $mx - ny = ma - nb$. Again, since the right line *PI* shall cross the right line *ab* normally, there will become $Mx = Ny$, from which equations there is elicited :

$$x = \frac{N(ma - nb)}{mN - Mn} \text{ and } y = \frac{M(ma - nb)}{mN - Mn}.$$

Truly since there is $m = kN - Kn$ and $M = kn + KN$, these values may be substituted into the equation

$$x = \frac{N(ma - nb)}{mN - Mn}$$

and on putting $t = \frac{n}{N}$ there will be produced:

$$x = \frac{ka - Kat - bt}{k - 2Kt - ktt} \quad \text{and} \quad y = \left(kt + K\right)x.$$

Hence truly there shall be found:

$$t = \frac{\frac{1}{2}\left(b + Ka\right) - Kx \pm \sqrt{\left(\frac{1}{4}\left(b + Ka\right)^2 - ax - Kbx + x^2\right)}}{kx}$$

and

$$y = \frac{1}{2}\left(b + Ka\right) - Kx \pm \sqrt{\left(\frac{1}{4}\left(b + Ka\right)^2 - ax - Kbx + x^2\right)}.$$

Finally the triangle *aCb* shall be to the triangle *ACB* as *xy* to *ab,* and therefore:

$$p : q = xy : ab \text{ and } y = \frac{pab}{qx},$$

with which value substituted in place of y in the above equation, there will be produced :

$$q^2x^4 - (a + Kb)q^2x^3 + (b + Ka)pqabx - a^2b^2p^2 = 0.$$

Therefore the value of x solved here from this equation will give the line *aC,* with which found if there may be assumed $bC = \frac{pab}{q \cdot aC}$, the section of the water *ab* will be found, and the part sought requiring to be submerged in the water. Q. E. I.

COROLLARY 1

57. Therefore so that the equation found may contain real positive roots, thus the triangle proposed will be able to float in just as many cases, so that the side *AB* shall remain out of the water, and only the angle C shall be immersed, provided there shall be $x< a$ and $y< b$.

COROLLARY 2

58. But if all 4 roots were real, then only three of these can be positive on account of the three alternative signs. Moreover the negative root proposed serves no useful purpose. Whereby no more than three cases can be given, by which the triangle can be floating on water in the prescribed manner.

COROLLARY 3

59. Besides, neither can x be greater than a nor can y be greater than b.
Whereby if it should eventuate that either x shall exceed a or y exceed b, both these cases will be useless.

COROLLARY 4

60. If the third side of the triangle *AB* may be put $= c$, and in this place the cosine K of the angle *ACB* may be introduced into the calculation, since

$$K = \frac{a^2 + b^2 - c^2}{2ab}$$

will be produced there :

$$q^2x^4 - \frac{q^3x^3}{2a}\left(3a^2 + b^2c^2\right) + \frac{apqx}{2}\left(a^2 + 3b^2 + c^2\right) - a^2b^2p^3 = 0.$$

COROLLARY 5

61. But two equations, in which x and y are present, the following are the most simple. Clearly the first is $qxy = pab$, and the second will be this:

$$y^2 - (b + Ka)y = x^2 - \left(a + Kb\right)x;$$

from which two equations the proposed problem is solved.

SCHOLION

62. Since the equation found shall have four dimensions and no more general dimensions will be considered to be allowed, thus so that scarcely any shall be deduced from that for use; we will be able to apply that for the cases of particular triangles, for which the equation shall be divisible, and the cases, from which floating can eventuate, themselves indeed can be assigned and set out.

EXAMPLE 1

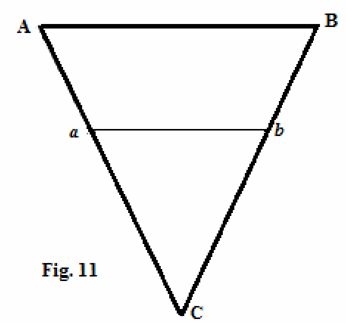

Fig. 11

63. The isosceles triangle *ACB* shall be proposed (Fig. 11), thus so that the sides *AC* and *BC,* which with the angle *C* placed underwater, shall be understood to be equal. Therefore there shall be put $BC = AC = a$, there will become $b = a$, and the equation found will changed into this :

$$q^2x^4 - (1+K)q^2ax^3 + (1+K)pqa^3x - a^4p^2 = 0,$$

which is resolved into these two equations by division :

$$\text{I. } qx^2 - pa^2 = 0 \text{ and II. } qx^2 - (1+K)qax + pa^2 = 0.$$

Of which that equation I gives

$$x = \pm\, a\sqrt{\frac{p}{q}},$$

but only the positive value has a place, since the sides AC and BC cannot be produced beyond C. Whereby from the first equation there will become :

$$aC = x = a\sqrt{\frac{p}{q}} \text{ and } bC = y = a\sqrt{\frac{p}{q}},$$

which therefore is the one case, where the isosceles triangle *ACB* can float on water; and the other part the isosceles triangle *aCb* likewise submerged, and the section of the water *ab* parallel to the base *AB,* and $AC : aC = \sqrt{q} : \sqrt{p}$. The two values are found for *x* from the other equation, of which one may be taken for *x*, the other will prevail for *y*, and thus so that either both or neither shall have a place. Therefore there will become

$$x=\frac{1}{2}(1+K)a\pm a\sqrt{\left(\frac{1}{4}(1+K)^2-\frac{p}{q}\right)},$$

and

$$y=\frac{1}{2}(1+K)a\mp a\sqrt{\left(\frac{1}{4}(1+K)^2-\frac{p}{q}\right)};$$

which values, lest they may become imaginary, shall be required so that:

$$1+K>2\sqrt{\frac{p}{q}} \text{ or } K>2\frac{p}{q}-1.$$

Besides, since both x and y must be smaller than a, it shall be required that

$$\frac{1}{2}(1+K)a\pm a\sqrt{\left(\tfrac{1}{4}(1+K)^2-\frac{p}{q}\right)}<1 \text{ or } \pm\sqrt{\left(\tfrac{1}{4}(1+K)^2-\frac{p}{q}\right)}<\frac{1}{2}(1-K),$$

from which there becomes

$$K=\frac{p}{(1-\alpha)q}-\alpha$$

with α denoting some positive number. On account of which where as well as the case designated, two remaining cases shall have a place, as it shall be required in these two equations

$$K=2\sqrt{\frac{p}{q}}-1+\beta \text{ and } K>\frac{p}{(1-\alpha)q}-\alpha$$

both α as well as β may obtain positive values. Therefore so that this may come about, these two remaining cases for which the triangle can float on water will arise:

$$\left.\begin{aligned} x&=\frac{1}{2}(1+K)a+a\sqrt{\left(\frac{1}{4}(1+K)^2-\frac{p}{q}\right)}\\ y&=\frac{1}{2}(1+K)a-a\sqrt{\left(\frac{1}{4}(1+K)^2-\frac{p}{q}\right)}\end{aligned}\right\}\textit{second case},$$

$$\left.\begin{aligned} x &= \frac{1}{2}(1+K)a - a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \\ y &= \frac{1}{2}(1+K)a + a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \end{aligned}\right\} \textit{third case.}$$

COROLLARY 1

64. If the triangle *ACB* were equilateral, the angle *C* will be 60°, and thus its cosine $K = \frac{1}{2}$. Whereby so that the two latter cases may find a place in the solution, initially it shall be required that

$$\frac{3}{2} > 2\sqrt{\frac{p}{q}} \text{ that is, so that } \frac{p}{q} < \frac{9}{16}.$$

Then also it will be required that there shall be

$$\frac{p}{q} = (1-\alpha)\left(\tfrac{1}{2}+\alpha\right) = \tfrac{1}{2} + \tfrac{1}{2}\alpha - \alpha^2,$$

with some positive number assumed for α. But so that $\frac{p}{q}$ shall be a positive fraction, it shall be required that $\alpha < 1$. Therefore it is required that there shall be

$$\frac{p}{q} > \frac{1}{2} \text{ and } \frac{p}{q} < \frac{9}{16}.$$

SCHOLIUM

65. With the other requirement, so that the two posterior cases may become possible, there shall be :

$$\frac{p}{q} < \frac{1}{4}(1+K)^2, \text{ and there may be put } \frac{p}{q} = \frac{1}{4}(1+K)^2 - \alpha^2,$$

and by the other requirement there must be $\alpha < \frac{1}{2}(1-K)$. Therefore there will become $\alpha^2 < \frac{1}{4}(1-K)^2$, and thus $\frac{p}{q} > K$.

On account of which for the given angle *ACB,* unless the ratio $\frac{p}{q}$ shall be contained within the limits $\frac{1}{4}(1+K)^2$ and K, besides the first situation required to be determined, no other is given, whereby the isosceles triangle with the vertex of the same turned downwards can float on the water. Truly there is always $\frac{1}{4}(1+K)^2 > K$, for indeed the difference is the square $\frac{1}{4}(1-K)^2$, whereby for any angle the triangles are able to show for *C*, which for the three ways with the angle with the angle C turned downwards, they are able to sit on the water.

COROLLARIUM 2

66. Moreover in the equiangular triangle, if there were $\frac{p}{q} > 1$ and $< \frac{9}{16}$, the two latter cases, for which the triangle can float on water, will become

$$x = \frac{3}{4}a \pm a\sqrt{\left(\frac{9}{16} - \frac{p}{q}\right)} \text{ and } x = \frac{3}{4}a \mp a\sqrt{\left(\frac{9}{16} - \frac{p}{q}\right)}.$$

COROLLARY 3

67. If the angle *C* were right, there will be $K = 0$, therefore triangles of this kind will be able to float on water in three ways, provided there shall be $\frac{p}{q} < \frac{1}{2}$. Truly then there will be:

$$\text{I. } x = a\sqrt{\tfrac{p}{q}};\ \text{II. } x = \tfrac{1}{2}a + a\sqrt{\left(\tfrac{1}{4} - \tfrac{p}{q}\right)} \text{ et III. } x = \tfrac{1}{2}a - a\sqrt{\left(\tfrac{1}{4} - \tfrac{p}{q}\right)}.$$

EXAMPLE 2

68. If the specific gravity of the triangle were prepared thus, so that there shall be $p:q = Kb:a$ the general equation will be changed into this :

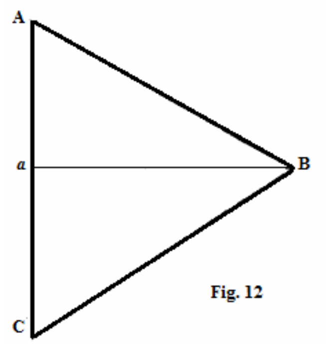

Fig. 12

$$x^4 - (a+Kb)x^3 + (b+Ka)Kb^2x - K^2b^4 = 0,$$

by which division it is resolved into these two equations:

$$\text{I. } x - Kb = 0; \text{ and II. } x^3 - ax^2 - Kabx + Kb^3 = 0.$$

The first of these equations gives $Ca = x = Kb$ and since there shall be $y = \dfrac{pab}{qx}$ there becomes $y = b$. Therefore the point b lies on B, and the right line Ba will be the section of the water (Fig. 12).

COROLLARY I

69. Since there shall be $Ca = Kb$, there becomes $\dfrac{Ca}{CB} = K =$ cosine of the angle C. Whereby the right line Ba will be perpendicular to the side AC. Therefore this perpendicular can always be the section of the water, if there were $p:q = Ca:CA$.

COROLLARY II

70. But this same circumstances for floating cannot happen, unless each of the angles A and C shall be acute, since otherwise the right line Ca will not fall within the triangle, which still is necessary.

SCHOLIUM

71. In a similar manner, if we were to assume $p:q = Ka:b$, the equation would be able to be divided by $x - a$, thus so that there would be produced $x = a$. Indeed this case does not differ from the previous one, except that the point B shall be transposed into A and vice versa. Therefore with these cases determined, by which a homogeneous triangle thus can float vertically on water, thus so that a single angle shall be immersed in the water, it remains that also we may examine these cases, for which triangles of this kind may be able to float with two angles immersed under the water; indeed entirely all the cases for these are contained in two ways, in which the triangle can float vertically.

PROPOSITION 10

PROBLEM

72. *For the proposed homogeneous triangle CAB* (Fig. 13), *the specific gravity of which shall be to the specific gravity of water as p to q, to determine the case, for which this triangle can be floating on water situated vertically, so that only the angle C may be above the surface of the water.*

SOLUTION

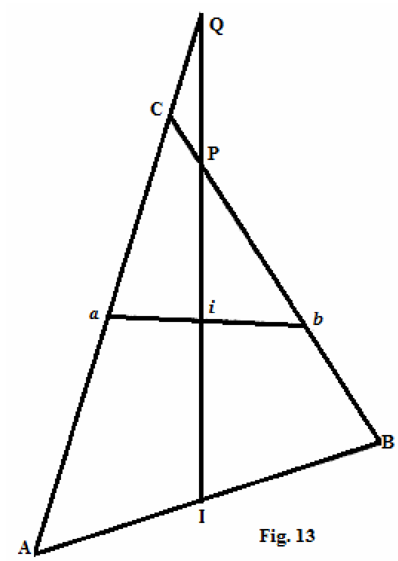

Fig. 13

aABb shall be the part required to be immersed in the water, which shall satisfy the question. The right line *IPQ* shall be drawn through the centre of gravity of the whole *ACB,* as well as of the immersed part *aABb,* which will have to be normal to the section of the water *ab.* But this same right line *IPQ* also will pass through the centre of gravity of the triangle *Cab* standing out above the water. On account of which here the question is reduced so that the position of the right line *ab* shall be found, which may be the part cut off, as it were, from the whole triangle *aABb*, which shall be to the whole triangle as *p* to *q;* and so that the right line joining the centre of gravity of the whole triangle and of the triangle *Cab* shall be normal to the section of the water *ab.* This latter condition agrees completely with the preceding problem, however the first condition disagrees with that, because here the area of the exposed triangle *Cab* itself shall be had to the area of the whole triangle *CAB* as; since there this ratio shall be as *p* to *q.* On account of which the solution of the first problem will be adapted to this problem only by writing $q-p$ in place of p. [Essentially by rearranging the basic equation for upthrust: weight of triangle.]

Therefore, there may be put as before :

$AC=a,\ BC=b,\ \cos\text{ang.}ACB=K,$ and $Ca=x$ as well as $Cb=y$,

as well as *Ob*= *y,* there will become :

$$q^2x^4-(a+Kb)q^2x^3(b+Ka)(q-p)qabx-(q-p)^2a^2b^2=0.$$

From which equation the value of x arising will give the corresponding value itself:

$$y=\frac{(q-b)ab}{qx}.$$

Q. E. I.

COROLLARY 1

73. Which equation found therefore will contain the real and positive roots for *x,* which themselves are smaller than a and for which the corresponding values of y are smaller than *b,* with just as many ways the proposed triangle thus will be able to float on water, so that the angles *A* and *B* may be present under the water.

COROLLARY 2

74. Moreover if all the roots of the proposed equation shall be real, of these only three are able to be positive. Whereby it cannot happen, that a triangle shall be able to float on water in more than three ways, so that only the angle *C* shall appear above the water.

COROLLARY 3

75. If the third side *AB* may be put $= c$, and here the cosine *K* of the angle *ACB* may be introduced, then the following equation will arise:

$$q^2x^4 - \frac{q^2x^3}{2a}\left(3a^2 + b^2 - c^2\right) + \frac{(q-p)qax}{2}\left(a^2 + 3b^2c^2\right) - \left(q-p\right)^2 a^2b^2 = 0.$$

COROLLARY 4

76. If there were $p : q = 1 : 2$, this equation agrees with the equation of the above proposition. Whereby in this case the same right line *ab* will be able to be the section of the water, both if the angle *C* alone were out of the water, as well as if it were submerged in the water.

EXAMPLE

77. An isosceles triangle *ACB* may be put in place, evidently the side $BC = AC$ or $b = a$, and it produces this same equation

$$q^2x^4 - (1+K)q^2ax^3 + (1+K)\left(q-p\right)qa^3x - \left(q-p\right)^2 a^4 = 0,$$

which by division will be changed into these two equations

$$\text{I. } qx^2(q-p)a^2 = 0 \text{ et II. } qx^2(1+K)qax + (q-p)a^2 = 0.$$

The first equation of which gives $x = a\sqrt{\frac{q-p}{q}}$, to which there corresponds also: $y = a\sqrt{\frac{q-p}{q}}$, therefore in this case also an isosceles triangle will be appearing above the water, and the section of the water *ab* will be parallel to the base *AB*. The other equation gives these two quantities for x :

$$x=\tfrac{1}{2}(1+K)a\pm a\sqrt{\left(\tfrac{1}{4}(1+K)^2-\tfrac{(q-p)}{q}\right)},$$

to which there will correspond respectively

$$y=\tfrac{1}{2}(1+K)a\mp a\sqrt{\left(\tfrac{1}{4}(1+K)^2-\tfrac{(q-p)}{q}\right)}.$$

Therefore besides the first case found another cannot be given, unless there shall be

$$\tfrac{q-p}{q}<\tfrac{1}{4}(1+K)^2,$$

but even if there were $\tfrac{q-p}{q}<\tfrac{1}{4}(1+K)^2$, this alone does not suffice for the realisation of the latter cases; for in addition it is required that both x as well as y shall themselves be less than a, that which will have a place if there were $\tfrac{q-p}{q}>K$.
On account of which, so that the latter cases may become real, it will be required that the ratio $\dfrac{p}{q}$ shall be contained between these limits $1-K$ and $1-\tfrac{1}{4}(1+K)^2$.

SCHOLIUM

78. Therefore all the cases can be defined from these two propositions, by which a given homogeneous triangle placed vertically can float on water. Likewise it can be noted the number of cases can be great enough, provided several roots of the equation found are real and satisfying the question. But if all the roots were useful, then it will be able for a triangle to float on water in 18 different ways ; for indeed there are three cases, by which the individual angles are immersed in water, with just as many the individual vertices emerge from the water. But it scarcely can happen, that all the eighteen cases shall become real, because if the specific gravity of the triangle may have a required property for one given angle, the same no further can be satisfied for the remaining angles. thus the equilateral triangle, if indeed its specific gravity to water may be contained between the two ratios 8:16 and 9:16, in which case it permits the most roots requiring to be satisfied, it can float on water in as much as 12 different ways.

LEMMA

79. *If the perpendicular GQ may be drawn from the centre of gravity G of the triangle ABC to the side AB* (Fig. 14), *the segment of the side AB will be*

$$AQ = \frac{3\cdot AB^2 + AC^2 - BC^2}{6\cdot AB}.$$

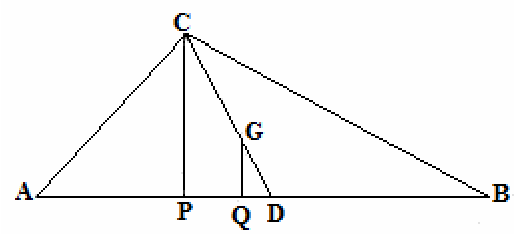

Fig. 14

DEMONSTRATION

The right line *CD* is drawn from the angle *C* through the centre of gravity G to the line *AB*, from the nature of the centre of gravity there will be $AD = BD$, and $GD = \frac{1}{3}CD$. Again the perpendicular *CP* may be sent from *C* to *AB*, there will become:

$$AP = \frac{AB^2 + AC^2 - BC^2}{2\cdot AB},$$

and hence

$$DP = \frac{AB}{2} - AP = \frac{BC^2 - AC^2}{2AB}.$$

But since *CP* shall be parallel to *GQ* itself, there will be $PQ = \frac{2}{3}DP$, from which there becomes

$$PQ = \frac{BC^2 - AC^2}{3\cdot AB}.$$

On account of which

$$AQ = AP + PQ = \frac{3AB^2 + AC^2 - BC^2}{6\cdot AB}.$$

Q.E.D.

COROLLARY 1

80. Therefore the perpendicular *GQ* sent from the centre of gravity *G* itself to the line *AB* will be the third part of the perpendicular *CP*, thus so that there will be $GQ = \frac{1}{3}CP$.

COROLLARY 2

81. If the angle *ACB* were right, there will become $AB^2 = AC^2 + BC^2$, therefore in this case there becomes

$$AQ = \frac{2AC^2 + BC^2}{3AB} = \tfrac{1}{3}AB + \frac{AC^2}{3AB}.$$

And since there shall be

$$CP = \frac{AC \cdot BC}{AB} \text{ there will be } GQ = \frac{AC \cdot BC}{3 \cdot AB}$$

COROLLARY 3

82. If the triangle *ACB* were isosceles, thus so that there shall be $AC = BC$, there will become $AQ = \tfrac{1}{2}AB$. And in this case there will become $GQ = \tfrac{1}{3}\sqrt{AC^2 - \tfrac{1}{4}AB^2}$.

LEMMA

83. *If the right line EF shall cut the triangle EDF from the right angled parallelogram ABDC* (Fig. 15)*, and from the centre of gravity of the rectangle G of the whole rectangle AD, the perpendicular GH shall be sent to the right line EF, to find the point H.*

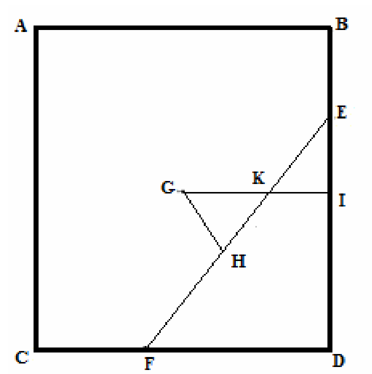

Fig. 15

SOLUTION

From the centre of gravity *G* of the rectangle the perpendicular *GI* shall be sent to the line *BD* cutting the right line *EF* at *K;* there will be $DI = \tfrac{1}{2}BD$, and $EI = ED - \tfrac{1}{2}BD$, likewise $GI = \tfrac{1}{2}AB$. Now on account of the similar triangles *EIK, EDF*, there will become

$$IK = \frac{DF \cdot EI}{DE} = DF - \frac{DF \cdot BD}{2DE},$$

and

$$EK = \frac{EF \cdot EI}{DE} = EF - \frac{EF \cdot BD}{2 \cdot DE}.$$

Thereupon there will be had

$$GK = \tfrac{1}{2}AB - KI = \tfrac{1}{2}AB - DF + \frac{DF \cdot BD}{2DE},$$

and on account of the similar triangles *GHK* and *EDF* this proportionality will be had :

$$EF : DF = GK : KH,$$

from which it follows:

$$KH = \frac{AB \cdot DF}{2EF} - \frac{DF^2}{EF} + \frac{DF^2 \cdot BD}{2DE \cdot EF}.$$

On account of which there will be found:

$$EH = EF - \frac{BD \cdot DE}{2EF} + \frac{AB \cdot DF}{2EF} - \frac{DF^2}{EF},$$

or

$$EH = \frac{2EF^2 + AB \cdot DF - BD \cdot DE}{2EF}.$$

Q.E.I.

COROLLARY

84. From the similitude of the same triangles *GHK* and *EDF* there follows also to become $EF : ED = GK : GH$. Therefore from this ratio the distance is found from the centre of gravity *G* to the right line *EF* itself:

$$\frac{AB \cdot DE}{2EF} + \frac{BD \cdot DF}{2EF} - \frac{DE \cdot DF}{EF} = \frac{AB \cdot DE + BD \cdot DF - 2DE \cdot DF}{2EF}.$$

PROPOSITION 11

PROBLEM

85. *For the proposed homogeneous rectangular parallelogram ABDC* (Fig. 15), *of which the specific gravity shall be to water as p to q, to find the case, for which this rectangular parallelogram thus is able to float on water, so that only the angle D may be immersed..*

SOLUTION

EF shall be the section of the water sought, and *EDF* the part required to be immersed in the water, there will be $\frac{1}{2} DE \cdot DF : AB \cdot BD = p : q$; and the right line joining the centres of gravity of the whole rectangle and of the part *EDF*, must be normal to the section of the water *EF*. Whereby the normals from the centre of gravity of the rectangle, and from the centre of gravity of the triangle *EDF* will have to lie on the same right line *EF*. Whereby by § 83 and § 81 there will be

$$\frac{2DE^2 + AB \cdot DF - BD \cdot DE}{2EF} = \frac{2DE^2 + DF^2}{3EF}.$$

Now on putting

$$AB = CD = a,\ AC = BD = b,\ DF = x \text{ and } DE = y,$$

there will become

$$qyx = 2pab \text{ or } y = \frac{2pab}{qx},$$

and

$$2y^2 + 3ax - 3by = 2x^2,$$

which, with its value $\frac{2pab}{qx}$ substituted in place of y, will be changed into this:

$$2qqx^4 - qqax^3 + 6pqab^2x - 8p^2a^2b^2 = 0.$$

From which equation the value of x itself may be found, and therefore the value of y itself. Q. E. I.

COROLLARY 1

86. This equation at most can have three positive real roots, but of these only those sought that satisfy the question, which for the x values give smaller values for a, and likewise the values of y correspond to smaller values of b .

COROLLARY 2

87. Since the triangle EDF cannot exceed half of the rectangle, it is evident the ratio q to p cannot be smaller than half, thus so that there must be either

$$\frac{q}{p} = 2 \text{ or } \frac{q}{p} > 2.$$

EXAMPLE

88. If the rectangle may be changed into a square, so that there shall be $b = a$, this equation shall be had:

$$2q^2x^4 - 3qqax^3 + 6pqa^3x - 8ppa^4 = 0,$$

which on division is resolved into these two parts

$$\text{I. } qx^2 - 2pa^2 = 0 \text{ et II. } 2qx^2 - 3qax + 4pa^2 = 0.$$

That first equation of these gives :

$$x = a\sqrt{\frac{2p}{q}},$$

to which there corresponds

$$y = a\sqrt{\frac{2p}{q}},$$

in which case therefore x and y are equal, and the section of the water *EF* becomes parallel to the diagonal *BC.*
A twofold value of this is elicited from the other equation :

$$x = \tfrac{3a}{4} \pm a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)},$$

to which some twofold value of y will correspond, evidently

$$y = \tfrac{3a}{4} \mp a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)}.$$

Therefore lest these two values may become imaginary, there shall be required to be

$$\tfrac{2p}{q} < \tfrac{9}{16} \text{ or } \tfrac{p}{q} < \tfrac{9}{32}.$$

Then lest it shall not exceed the length *a,* there shall be required to be

$$a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)} < \tfrac{a}{4} \text{ or } \frac{p}{q} > \frac{1}{4}.$$

Whereby if $\frac{p}{q}$ may be contained between these limits $\frac{8}{32}$ and $\frac{9}{32}$ three cases are given, from which the square thus is able to float in the water, so that only the angle *D* shall be immersed in the water.

SCHOLIUM.

89. The same solution changed a little also will satisfy this problem, where the cases are sought, from which the rectangular parallelogram thus may be able to float on water, so that the three angles *B, A, C* may be submerged in the water, and only the angle *D* may stand out. For in this case equally as before the position of the right line *EF* must be found, to which the right line joining the centres of gravity of the rectangle *AD* and of the triangle *EDF* shall be normal. Only this same problem differs from the preceding, since here the area *ABEFC* must have the ratio p to q to the whole rectangle. Therefore since in

this case the ratio of the area of the triangle *EDF* to the whole rectangle must be as $q-p$ to *q,* the preceding solution must be adapted to this by putting $q-p$ in place of p. Therefore with the same denominations remaining as before, there will become

$$2qqx^4-3qqax^3+6(q-p)qab^2x-8(q-p)^2a^2b^2=0,$$

from which value of x found there will correspond the value

$$y=\frac{2(q-p)ab}{qx}.$$

If the rectangle may be changed into a square, the same will prevail, which have been found in the example, but only if there may be put $q-p$ in place of p. Thus the square will be able to float in water in three different ways, so that only the angle *D* may emerge above the water, if $\frac{p}{q}$ shall be contained between these limits $\frac{8}{32}$ and $\frac{9}{32}$.

LEMMA

90. *From the given trapezium ABDC* (Fig.16), *in which the sides AC and BD shall be parallel to each other, and normal to the base CD, to find the point H for which the right line GH drawn through the centre of the trapezium G falls normally on the right line AB.*

SOLUTION

The right line *AB* then may be produced to the base CD*,* until they meet at *I* and from each of the triangles *ACI* and *BDI* with the centres of gravity may be considered dropped perpendicularly on *AI*. Now from statics it shall be going to be agreed the moment of the trapezium with respect to the point *I* to be equal to the difference of the moments of the triangles. Moreover the moment of the figure with respect to *I* is obtained, if the area is multiplied by the distance of the point, at which the perpendicular from its centre of gravity dropped on *AI* falls, from the point I. Thus the moment of the trapezium will be $= ABDC\cdot IH$, and by § 81 there will become

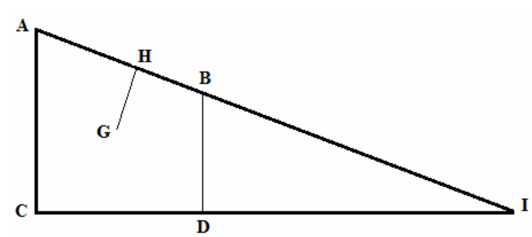

Fig. 16

$$\text{moment of the triangle } ACI=\frac{ACI(2CI^2+AC^2)}{3AI}$$

and

moment of the triangle $BDI = \frac{BDI(2DI^2 + BD^2)}{3BI}$.

On account of which this equation will had :

$$ABDC \cdot IH = \frac{ACI(2CI^2 + AC^2)}{3AI} - \frac{BDI(2DI^2 + BD^2)}{3BI}$$

from which the point H will be determined. Therefore there may be put

$$AB = a,\ AC = b,\ BD = c \text{ and } CD = \sqrt{\left(a^2 - (b-c)^2\right)} = d,$$

and there will become:

$$AI = \frac{ab}{b-c},\ BI = \frac{ac}{b-c},\ CI = \frac{bd}{b-c} \text{ and } DI = \frac{cd}{b-c}.$$

From these there will be produced:

$$ABDC = \frac{(b+c)d}{2},\ ACI = \frac{bbd}{2(b-c)} \text{ and } BDI = \frac{ccd}{2(b-c)}.$$

Now there may be put $AH = x$, there will become $HI = \frac{ab}{b-c} - x$, with which values substituted at last there will be obtained

$$AH = x = \frac{a^2(b+2c) + b^3 - c^3}{3a(b+c)}.$$

Q.E.I.

COROLLARY 1

91. In a similar manner it will be able for the perpendicular GH itself to be determined with the aid of the moments, moreover the calculation finally leads to :

$$GH = \frac{d\left(b^3 - c^3\right)}{3a\left(bb - cc\right)} = \frac{d\left(bb + bc + cc\right)}{3a\left(b+c\right)}.$$

COROLLARY 2

92. If there were $AC = BD$ or $c = b$, in which case the trapezium is changed into a rectangular parallelogram, and there will be produced :

$$AH = \tfrac{a}{2} \text{ et } GH = \tfrac{bd}{2a} = \tfrac{b}{2},$$

on account of $d = a$ in this case.

PROPOSITION 12

PROBLEM

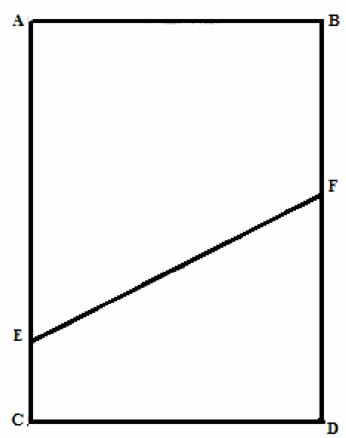

Fig. 17

93. *To determine the cases, for which the homogeneous rectangular parallelogram ABDC, the specific gravity of which shall be to water as p to q, thus shall be able to float on water, so that the two angles C and D shall be under water, truly the remaining A and B to be present above the water* (Fig. 17).

SOLUTION

EF shall be the section of the water, which satisfies the question; initially there shall be required to be

$\frac{CE + DF}{2} : AC = p : q$. Thence it is required, that the right line joining the centres of gravity of the whole rectangle and of the part *ECDF* shall be normal to *EF*, that which will come about, if the perpendiculars dropped from the centres of gravity of each if the trapeziums *ECDF* and *EABF* shall coincide with *EF*. Therefore by the above lemma, this equation will be obtained:

$$\frac{EF^2(CE + 2DF) + CE^3 - DF^3}{3EF\left(CE + DF\right)} = \frac{EF^2(AE + 2BF) + AE^3 - BF^3}{3EF(AE + BF)}.$$

Therefore there may be put:

$$AB = CD = a,\ AC = BD = b,\ CE = x,\ DF = y,$$

there will become:

$$AE = b - x,\ BF = b - y, \text{ and } EF^2 = a^2 + x^2 - 2xy + y^2.$$

Now, on account of the first condition there becomes

$$\frac{x+y}{2}:b=p:q \text{ or } x+y=\frac{2bp}{q},$$

truly the other condition supplies this equation :

$$\frac{\left(a^2+(x-y)^2\right)(x+2y)+x^3-y^3}{x+y}=\frac{\left(a^2+(x-y)^2\right)(3b-x-2y)+(b-x)^3-(b-y)^2}{2b-x-y}$$

Which equation is reduced by division to these two equations:

$$x=y \text{ and } a^2+2\left(y^2+yx+xx\right)=3b(y+x).$$

But since there shall be

$$x+y=\frac{2bp}{q},$$

the first equation shall give

$$x=y=\frac{bp}{q},$$

truly the latter:

$$2qqxx-4pqbx+8p^2b^2-6pqb^2+q^2a^2=0,$$

from which equation the value of x is found :

$$x=\frac{pb}{q}\pm\sqrt{\left(\frac{3pb^2}{q}-\frac{3p^2b^2}{q^2}-\frac{a^2}{2}\right)};$$

and there will become:

$$y=\frac{pb}{q}\mp\sqrt{\left(\frac{3pb^2}{q}-\frac{3p^2b^2}{q^2}-\frac{a^2}{2}\right)}$$

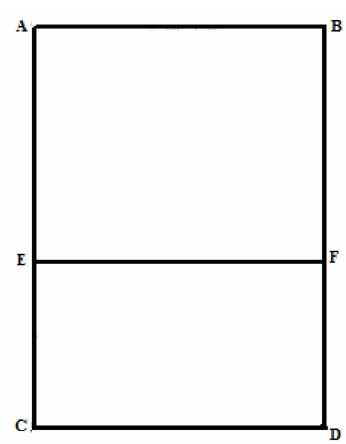

Fig. 18

COROLLARIUM 1

94. The first case, where $x = y = \frac{pb}{q}$, always has a place, provided it will have been a rectangle with a specific gravity lighter than water, or $p < q$. But then the section of the water will be *EF* parallel to the sides *AB* and *CD* (Fig. 18).

COROLLARY 2

95. Therefore this solution shows only the four cases, by which a homogeneous rectangle can float on water. For from that any side can be held horizontally on water, such as *CD*.

COROLLARY 3

96. But the two remaining cases are not always able to have a place (Fig. 17). Indeed so that they may be able to be present it is required, so that in the first place they shall be real, then secondly so that they shall be positive, and in the third place so that they shall be less than the side *b*.

COROLLARY 4

97. Truly these values found for *x* and *y*, where it is required that they shall be real, so that there shall be:

$$a^2 < \frac{6pb^2}{q} - \frac{6p^2b^2}{q^2},$$

that is, so that there shall be

$$\frac{a}{b} < \sqrt{\left(\frac{6pq - 6pp}{qq}\right)}.$$

Secondly, so that they shall be positive it is necessary that there shall be

$$a^2 > \frac{6pqb^2 - 8p^2b^2}{q^2} \quad \text{or} \quad \frac{a}{b} > \sqrt{\frac{6pq - 8pp}{qq}}.$$

Truly in the third place where it is required that they shall be less than the side *b,* so that there shall be :

$$a^2 > \frac{10pb^2}{q} - \frac{8p^2b^2}{q^2} \quad \text{or} \quad \frac{a}{b} > \sqrt{\left(\frac{10pq - 8pp - 2qq}{qq}\right)}.$$

COROLLARY 5

98. If the rectangle were twice as light as water, so that there shall be $\frac{p}{q} = \frac{1}{2}$; so that the latter cases shall become real, there will be required to be

$$\frac{a}{b} < \frac{\sqrt{6}}{2} \text{ and } \frac{a}{b} > 1 \text{ and in the third case } \frac{a}{b} > 1.$$

And thus it will be required that $\frac{a}{b}$ shall be contained between the limits 1 and $\frac{\sqrt{6}}{2}$.

COROLLARY 6

99. If there were $p : q = 3:4$, so that by the latter ratio the rectangle shall be able to float on the water rectangle, it will be required that there shall be

$$\text{in the first place, } \frac{a}{b} < \sqrt{\frac{9}{8}}, \text{ according to the second } \frac{a}{b} > 0 \text{ and by the third } \frac{a}{b} > 1.$$

Whereby it is necessary that $\frac{a^2}{b^2}$ shall be contained between the limits $\frac{8}{8}$ et $\frac{9}{8}$.

COROLLARY 7

100. If there were $p : q = 1:4$, since the latter ways of requiring the body to float become useless, it shall be necessary that

$$\frac{a}{b} < \sqrt{\frac{9}{8}}, \text{ and } \frac{a}{b} > 1, \text{ but not } \frac{a}{b} > 0.$$

Whereby the limits will be 1 and $\sqrt{\frac{9}{8}}$ as above.

SCHOLIUM

101. It is no wonder that the limits shall become the same in these two latter cases. For if *EF* is the section of the water for the submerged part *EGDF,* also it will be able to be the section of the water for the submerged part *EABF* ; and in this case it will have a place if the specific gravity of the rectangle to the water were as $q-p$ to q. And if in the given limits for $\frac{a}{b}$ there may be put $q-p$ in place of p, the first limit will not be changed at all, truly the second and the third are interchanged, thus so that they must always produce the same limits for each case.

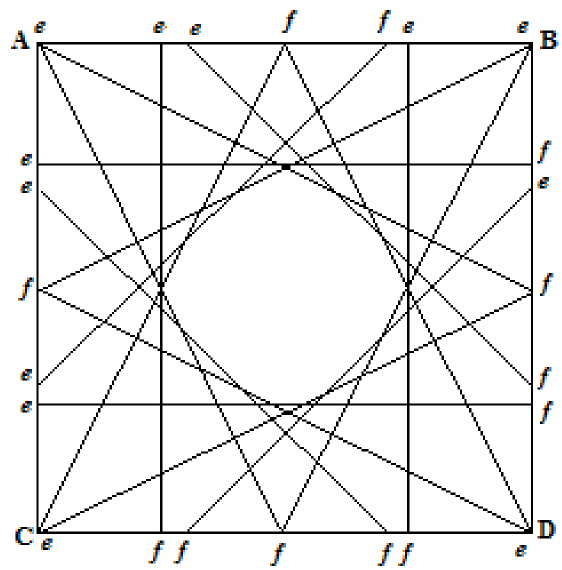

Fig. 19

EXAMPLE

102. The rectangle may be changed into the square *ABDC* (Fig. 19), and the number of ways its specific gravity must be prepared shall be sought, so that the corresponding number of ways the square may be able to float in water shall be found. Therefore since there is $b=a$, in the first place there becomes

$$x=y=\frac{pa}{q}$$

then

$$x=\frac{p}{q}a\pm a\sqrt{\left(\frac{3p}{q}-\frac{3pp}{qq}-\frac{1}{2}\right)},$$

and

$$y=\frac{p}{q}a\mp a\sqrt{\left(\frac{3p}{q}-\frac{3pp}{qq}-\frac{1}{2}\right)},$$

which latter expressions, so that they shall become real and useful, it is required, so that in the first place $\frac{q}{p}$ shall be contained between the limits $3+\sqrt{3}$ and $3-\sqrt{3}$. Then it is required that there shall be either

$$\frac{q}{p} > 4 \text{ or } \frac{q}{p} < 2.$$

Truly in the third case it is necessary that there shall be either

$$\frac{q}{p} > 2 \text{ vel } \frac{q}{p} < \frac{4}{3}.$$

Whereby these cases will become real, either if $\frac{q}{p}$ lies between these limits $3+\sqrt{3}$ and 4 or may be contained between these $3-\sqrt{3}$ and $\frac{4}{3}$. Hence also in the first case the square can float on water, so that a single angle shall be under the water, truly the other so that three angles may be present under the water (§ 88, § 89), and this only in a single way, for two angle to remain with the rest exposed require that $\frac{q}{p}$ either shall be contained between the limits 4 and $\frac{32}{9}$, or between the limits $\frac{4}{3}$ and $\frac{32}{23}$, which limits therefore are excluded from these. Whereby if $\frac{q}{p}$ either shall be contained between the limits $3+\sqrt{3}$ and $\frac{32}{9}$ or between the limits $3-\sqrt{3}$ and $\frac{32}{23}$, then the square both by this, as well as by the preceding proposition 16 will be able to float on water in different ways. Therefore if there shall be $p:q=1:4$, there will be sixteen sections of the water, by which the square can float on water, just as they are expressed by *ef* in the figure, which same sections also prevail for the square, of which the specific gravity to water is as 3 to 4. Moreover unless the specific gravities may be contained between these said ratios, then the square will be able to float on water only in eight ways.

SCHOLIUM

103. These matters which we have related here, concerned with the divers ways, by which either homogeneous triangles or rectangular parallelograms are able to float on water, also pertain to homogeneous prisms, of which the sections are either triangles or rectangles, as we have shown above. Evidently there will be just as many ways for prisms of this kind to be able to float on water while the axes of these remain in place horizontally, as we have assigned both for triangle as well as rectangles in these propositions. But certainly these are not all the cases, by which prisms are able to float on water, for besides these they can float on water in many other ways, while the axis of these shall be either vertical or horizontal, or inclined at some angle to the horizontal, just as it will easily be allowed to bring together. But to pursue these further would be against our intentions, as we may linger too long over these.

DEFINITION

104. I call a plane the diametrical plane, by which a body is divided into two similar and equal parts, thus so that all the sections of the body, which arise from the planes normal to the diametrical plane, shall be divided by this diametrical plane into two similar and equal parts.

COROLLARY

105. Therefore the provided centre of the magnitude of the body lies on the same diametrical plane itself, since from each part of that the body itself shall be similar and equal.

SCHOLIUM

106. Bodies which have a diametrical plane, are merited before others, since they may be subjected to examination; indeed all bodies, which are used to be floating in water, are prepared thus, so that they are permitted to have a diametrical plane. Thus in all ships a vertical plane passing through the keel will have this property, as ships may be divided into two similar and equal parts. Moreover so that all ships may be built in this manner according to this plan, since there shall be no account given, whereby one part may have a different figure to the other. On account of which, for the most suitable figures of ships requiring to be determined, it suffices for the one part to be assigned about the diametrical plane, so that the other part must be put in place similar and equal to that part.

PROPOSITION 13

PROBLEM

107. *A body, which is provided with a diametrical plane, thus always can float on water, so that the diametrical plane shall be vertical, only if the centre of gravity of the body may fall on the diametrical plane.*

DEMONSTRATION

If a body of this kind thus may be placed on the water, so that the diametric plane may maintain the vertical position, the section of the water will be normal to the vertical plane, and thus the submerged part is divided into two similar and equal parts. On account of which the centre of the magnitude of the submerged parts falls on the diametric plane, in which also the centre of gravity of the whole body is put to fall. Consequently the body thus will be going to be inclined, so that the diametrical plane shall remain vertical, if it can be done, so that the right line joining the centres of gravity and of the magnitude shall become vertical, in which case the body therefore will float on the water. Q. E. D.

SCHOLIUM 1

108. A situation of this kind, where the diametric plane is vertical, we see ships of all kinds to be seated firmly on the water, unless they may be inclined from this situation by alien forces. And then both from their construction as well as for ships being loaded as that is the main source of the encumbrance, so that the centre of gravity may fall in the diametrical plane, so that ships may be able to sail according to this situation described.

SCHOLIUM 2

109. Therefore the principles have been set out in this chapter from which the state of equilibrium of bodies floating on water is maintained, and likewise a method has been set out, with the aid of which it will be allowed to determine the principles by which, for the state of any proposed bodies, by which they are able to float on water. Moreover we have considered bodies to be entirely free, which are not disturbed by any external forces, but are established in equilibrium with only the force of their weight as well as the pressure of the water being present; but in the following we will progress towards the determination both for the motion as well as the equilibrium of the bodies, which either are not free or disturbed by external forces. But now we go on to the second chapter, in which we will investigate the motion, whereby a body may be set up not in equilibrium, itself to be restored to a state of equilibrium. Evidently in this chapter we will consider bodies imposed thus on water, so that either no part ought to be immersed in water, or the right line joining the centres of gravity and of the magnitudes under water shall not be vertical, and we will enquire, how they may be taken back to a state of equilibrium. But since a body of water may not be able to be imposed on thus, so that from the vestiges it may be arranged to be returned to its state of equilibrium; cases of this kind are going to be examined here, by which a body sent off from its state of equilibrium by an external force, and with that force ceasing, it may be returned to the state of equilibrium.

CAPUT PRIMUM

DE AEQUILIBRIO CORPORUM AQUAE INSIDENTIUM

LEMMA

1. *Pressio quam aqua in corpus submersum exercet in singulis punctis est normalis ad corporis superficiem; et vis, quam quodlibet superficiei submersae ele. mentum sustinet, aequalis est ponderi cylindri aquei recti, cujus basis aequalis est ipsi superficiei elemento, altitudo vero aequalis profunditati elementi infra supremam aquae superficiem.*

DEMONSTRATIO

Quaelibet aquae particula deorsum premitur a cylindrulo aqueo superincumbente, et pressio aequatur ponderi huius cylindri. Quaevis ergo particula hoc modo pressa tanta vi quaquaversum diffluere conatur, hocque ipso conatu particulas adiacentes eadem vi premit. Quare si corpus fuerit aquae submersum, id in singulis suae superficiei punctis a particulis aquae tanta vi premetur, quanta ipsae particulae premuntur, idque normaliter in superficiem. Unde veritas lemmatis constat, quae autem plenius in hydrostatica evincitur. Q. E. D.

COROLLARIUM 1

2. Cum pressio, quam corpus aquae submersum patitur, sit in singulis punctis ad superficiem corporis normalis, omnes pressiones coniunctim tendent ad corpus comprimendum et in minus spatium reducendum.

COROLLARIUM 2

3. Nisi ergo superficies corporis satis habeat firmitatis compressioni resistendi, corpus aquae submersum revera comprimetur et in minus spatium redigetur.

COROLLARIUM 3

4. Cum igitur constet, quantam pressionem singula superficiei corporis submersi elementa sustineant, simulque pressionum in singulis punctis directiones sint cognitae; determinari poterit vis, quam totum corpus a singulis aquae pressionibus coniunctis suffert.

SCHOLION

5. Ad pressionem totalem, quam corpus aquae submersum sustinet, determinandam, a singulis elementis incipi debet. Cum enim directio pressionum aquae in singulis punctis sit normalis ad superficiem, hae singulae pressiones seorsim resolvi debebunt, antequam earum media directio et potentia aequivalens assignari queat.

PROPOSITIO 1

PROBLEMA

6. *Si superficiei planae verticalis EAIBF* (Fig. 1) *portio AIB aquae sit immersa, invenire vim, quam portio perimetri AIB a pressione aquae sustinet.*

SOLUTIO

Sit *AB* superficies aquae ideoque linea recta horizontalis, quae instar axis consideretur. Ex elemento curvae sub aqua sitae *Mm* ducantur ad axem applicatae verticales *MP, mp* vocenturque $AP = x$; $PM = y$, erit

$$Pp = Mn = dx,\ mn = dy \text{ et } Mm = \sqrt{(dx^2 + dy^2)} = ds.$$

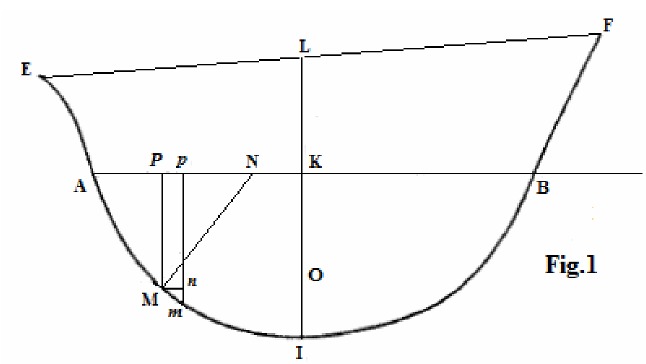

Fig.1

Iam pressio, quam elementum *Mm* sustinet aequatur rectangulo aqueo, cuius basis est *Mm,* et altitudo aequalis *PM,* profunditati scilicet elementi *Mm* infra aquae superficiem. Haec igitur pressio erit $= yds$, eiusque directio est normalis *MN* in elementum *Mm.* Resolvatur haec pressio in duas laterales, quarum altera tendat verticaliter sursum in directione *MP,* altera vero horizontaliter sit directa. Erit ergo ob $MN : MP = ds : dx$ vis elementum *Mm* sursum premens $= ydx$; vis vero elementum *Mm* horizontaliter pellens $= ydy$. Integrando igitur erit vis aquae totum arcum *AM* horizontaliter propellens $= \frac{y^2}{2}$; unde vis totam curvam submersam *AMIB* horizontaliter urgens fiet $= 0$, puncto enim *M* in *B* translata evanescet applicata *y.* Vis autem, quae totum arcum *AM* sursum premit, erit $= \int ydx =$ areae *AMP.* Quare tota curva *AIB* sursum urgebitur vi, quae areae *AIB* est proportionalis; atque ista vis aequalis erit ponderi aquae aream *AIB* occupantis. Q.E.F.

COROLLARIUM 1

7. Vis igitur aquae tendet figuram *EIF* verticaliter sursum ex aqua expellere, et reipsa expellet, nisi vel gravitate vel alia vi externa in hoc situ retineatur.

COROLLARIUM 2

8. Quia omnes vires horizontales, quibus elementa omnia curvae aquae submersa urgentur, sese destruunt, apparet figuram *EIF* aquae immersam neutiquam horizontaliter urgeri. Quare etiam nulla opus est vi ad motum horizontalem cohibendum.

COROLLARIUM 3

9. Cum igitur vires horizontales sese destruant, et solae verticales supersint, cuivis elemento *Mm* concipi potest vis id verticaliter sursum pellens applicata, quae aequalis est areae elemento *PMmp.*

SCHOLION

10. Antequam vires, quas corpora aquae immersa a pressionibus aquae sustineant, investigare queant, necesse erat a superficiebus ordiri, etiamsi huiusmodi casus in mundo non detur. Sed cum casus magis compositi facilius enodentur, si prius simpliciores examini subiiciantur, eundem ordinem etiam hic retinere convenit. Quamobrem neminem offendi arbitror locutionibus impropriis, quibus uti coactus sum, dum ponderis, quod rectangulum aqueum habeat, mentionem feci; hoc enim ad analogiam cum sequentibus declarandam indicare oportuit, ubi similes proprietates in ipsis corporibus detegentur.

PROPOSITIO 2

PROBLEMA

11. *Si figura plana verticalis aquae fuerit immersa, invenire mediam directionem omnium pressionum aquae, et potentiam iis omnibus aequivalentem.*

SOLUTIO

Cum, postquam singulae pressiones aquae in elementa exertae resolutae fuerint in verticales et horizontales, hae omnes pressiones horizontales se mutuo destruant, solae verticales omnibus aquae pressionibus aequivalebunt. Quare hoc tantum requiritur, ut harum virium verticalium media directio et potentia aequivalens definiatur. Sed cum hae omnes potentiae directiones habeant parallelas, erit potentia aequivalens iis omnibus simul sumtis aequalis ideoque proportionalis area *AIB.* Directio porro erit quoque verticalis puta *IL* cuius distantia *AK* a puncto *A* invenietur, dividendo summam omnium momentorum ad quodpiam punctum fixum *A* relatorum per ipsarum potentiarum summam, quae est $= \int ydx.$ Sed cum elementum *Mm* sursum urgeatur vi $= ydx$, erit eius momentum respectu puncti $A = yxdx$. Unde summa omnium momentorum erit $\int yxdx$, si post integrationem ponatur $x = AB$. Quare distantia *AK* erit $= \frac{\int yxdx}{\int ydx}$, posito scilicet post utramque integrationem $x = AB$. Puncto ergo *K* determinato, erit verticalis per id ducta *LI* media directio omnium aquae pressionum, atque potentia ipsa aequivalens erit $= \int ydx.$ seu areae *AIB*. Q.E.F.

COROLLARIUM I

12. Si *O* fuerit centrum gravitatis area *AIB,* et ex eo perpendicularis *OK* ad axem *AB* ducatur, erit etiam $AK = \frac{\int yxdx}{\int ydx}$, uti ex staticis constat, quare recta verticalis, per centrum gravitatis partis submersae *AIB* ducta, erit media directio omnium aquae pressionum.

COROLLARIUM 2

13. Loco omnium ergo aquae pressionum substitui potest unica vis, figuram in directione *IL* verticaliter sursum pellens, quae aequalis est ponderi aquae aream *AIB* implentis.

SCHOLION

14. Quae in casu, quo corpus aquae immersum ponitur tantum superficiei, circa pressionem aquae elicuimus, eadem quoque pro ipsis corporibus valent; uti mox ostendetur; scilicet quod potentiae horizontales omnes sese destruant et media directio sit linea verticalis per centrum gravitatis voluminis sub aquae mersi transiens, atque quod potentia aequivalens aequetur ponderi aquae parti submersae volumine aequali. Ceterum etiamsi hae proprietates iam satis sint cognitae, tamen eas methodo hac genuina analytice eruere ad viam ad sequentia praeparandam idoneum visum est.

PROPOSITIO 3

PROBLEMA

15. *Si corpus quodcunque ex aliqua parte aquae immergatur, determinare vim, quam eius pars aquae submersa a pressionibus aquae sustinet.*

SOLUTIO

Repraesentet planum tabulae sectionem corporis mersi verticalem, sitque *AEIFBL* sectio corporis horizontalis in aquae superficie facta, ita ut corporis

pars infra hanc sectionem sita in aqua versetur (Fig 2). In hac sectione summatur recta quaecunque *AB* pro axe ad eumque ducantur duae ordinatae proximae *LI, li,* quas secent aliae rectae proximae axi parallelae *EF, ef*; ex punctis intersectionum *Q, q, R, r* deorsum ducantur verticales, abscidentes in superficie corporis sub aqua elementum *Mm,* cuius areola sit *dS.*
Iam ponatur

$$AP = x,\ PQ = y \text{ et } QM\ \ z,$$

erit *z* profunditas elementi *Mm* infra superficiem aquae. Quare pressio aquae, quam elementum *Mm* sustinet, aequatur ponderi cylindri aquae, cuius basis est *dS* et altitudo *z,* exponatur hoc pondus per *zdS.* Directio autem huius vis cum sit normalis ad superficiem, ducatur ad elementum *Mm* normalis *MN,* quae producta plano horizontali *ALB* occurrat inN, erit ergo *MN* directio vis aquae elementum *Mm* prementis, ad cuius positionem inveniendam exprimat haec aequatio $dz = Pdx + Qdy$ naturam superficiei, quae sub aqua versatur. Ex *N* tam ad axem *AB,* quam ad applicatam *LI* ducantur normales *NH* et *NK,* erit ex indole normalium $PH = Pz$ et $QK = Qz$. Ducatur recta *QN,* quae, quia est horizontalis, normalis erit ad verticale: *QM,* eritque

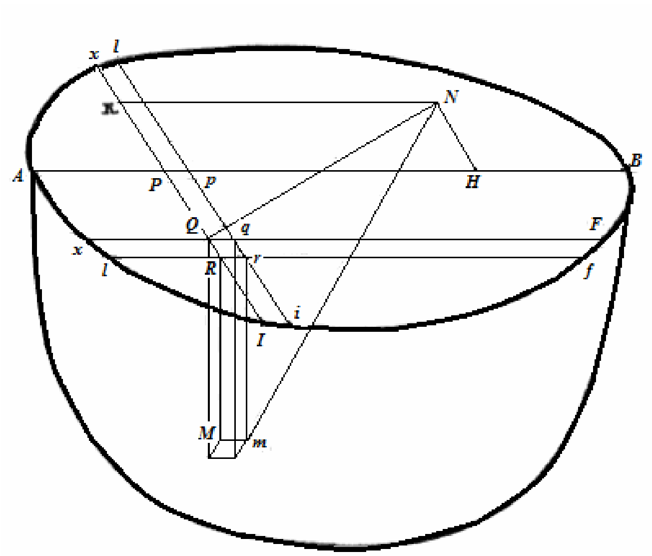

Fig. 2

$$QN = z\sqrt{(P^2 + Q^2)} \ \text{ et } \ MN = z\sqrt{(1 + P^2 + Q^2)}.$$

Iam vis premens elementum *Mm* in directione *MN,* quae vis est $= zdS$, resolvatur in binas, quarum altera elementum *Mm* sursum urgeat in directione *MQ,* altera vero horizontaliter in directione parallela ipsi *QN.* Cum autem sit

$$MN : MQ = \sqrt{(1 + P^2 + Q^2)} : 1,$$

erit vis verticalis elementum *Mm* sursum pellens

$$= \frac{zdS}{\sqrt{\left(1 + P^2 + Q^2\right)}},$$

et vis horizontaliter iuxta parallelam ipsi *QN* urgens

$$=\frac{zdS\sqrt{\left(P^2+Q^2\right)}}{\sqrt{\left(1+P^2+Q^2\right)}}.$$

Haec autem vis, quia eius directio non est constans, resolvatur denuo in duas secundum horizontales *QK* et *QF* sollicitantes, quarum illa, quae secundum parallelam applicatae *LI* agit, erit

$$=\frac{-QzdS}{\sqrt{\left(1+P^2+Q^2\right)}},$$

et altera, cuius directio axi *AB* est parallela

$$=\frac{PzdS}{\sqrt{\left(1+P^2+Q^2\right)}}.$$

Sed cum sit rectangulum $Qr=dxdy$ ad areolam $Mm=dS$ ut *QM* ad *MN,* erit elementum $Mm=dxdy\sqrt{(1+P^2+Q^2)}$, quo valore loco *dS* substituto prodibit vis elementum *Mm* sursum urgens $=zdxdy=$ prismati *RM*, seu aequalis est ponderi aquae, cuius volumen est prisma *RM.* Tota ergo corporis superficies, quae est sub aqua, premetur sursum a vi, quae aequalis est summae omnium prismatum, hoc est, quae aequalis est ponderi aquae, cuius volumen adaequat partem corporis in aqua versantem. Vis autem, quae elementum *Mm* horizontaliter secundum parallelam applicatae *LI* sollicitat, erit $=-Qzdxdy$. Si nunc ponatur abscissa $AP=x$ constans, habebitur vis horizontalis, quae omnia elementa sub fascia *Li* posita sollicitat, in directione parallela ipsi *LI* sumendo integrale ipsius *Qzdxdy,* seu ob *x* constans erit haec vis $=-dx\int Qzdy$. At si *x* est constans, erit $dz=Qdy$, quare ista vis erit

$$-dx\int zdz=-\frac{z^2dx}{2}.$$

Posito nunc puncto *Q* in *I,* ubi est $z=0$, exprimet $-\frac{z^2dx}{2}$ vim horizontalem parallelam ipsi *LI,* qua portiuncula superficiei sub aqua positae respondens elemento *QqiI* urgetur. Translato ergo puncto Q in *L,* quo *z* iterum evanescit, haec vis quoque evanescet. Quare vis horizontalis, qua portio superficiei corporis sub fascia *LI* positae urgetur, aequalis fit nihilo. Et consequenter omnes vires horizontales parallelae applicatis *LI,* quibus omnia superficiei corporis sub aquae positae elementa urgentur, sese destruunt. Deinde vis horizontalis, cuius directio axi *AB* est parallela, qua elementum *Mm* urgetur est $=Pzdxdy$, sumto nunc $y=PQ$ constante, erit vis ista horizontalis, qua superficiei portiuncula sub elemento *ER* posita urgetur

$$= dy\int Pzdz = \frac{z^2dy}{2}$$

propter $Pdx = dz$ sumto y constante. Translato ergo puncto Q in F, ubi fit $z = 0$, tota vis horizontalis, qua portio superficiei corporis sub fascia Ef sita sollicitatur, evanescet. Ergo etiam haec vis horizontalis, cuius directio parallela est axi AB, qua tota corporis superficies sub aqua sita urgetur, evanescit. Quocirca tota pressio, quam corpus in aqua immersum a pressionibus aquae sustinet, consistit in solis viribus verticalibus, quibus corpus sursum urgetur, quarumque summa aequatur ponderi aquae, cuius volumen aequale est parti corporis submersae. Q.E.I.

COROLLARIUM 1

16. Quia vires horizontales utriusque generis, quibus singula superficiei sub aqua mersae elementa sollicitantur, coniunctim sese destruunt, corpus a pressionibus aquae tantum sursum pellitur, neque ipsi ab his pressionibus motus horizontalis imprimi potest.

COROLLARIUM 2

17. Cum vis, qua corpus aquae immersum sursum pellitur, aequalis sit ponderi aquae, cuius volumen aequatur volumini partis corporis submersae, manifestum est nisum corporis, quam vi propriae gravitatis habet deorsum labendi, a pressionibus aquae diminui. Et quidem eius proprium pondus diminuetur pondere voluminis aquae, quod aequale est volumini partis corporis submersae.

COROLLARIUM 3

18. Quamobrem si tanta pars corporis aquae fuerit immersa, ut tantum aquae volumen aequiponderet ipsi corpori, tum nisus deorsum labendi corporis evanescet, corpusque aquae innatabit.

COROLLARIUM 4

19. Ex his quoque perspicitur, si minor pars, quam ad natandum requiritur, aquae fuerit immersa, tum corpus sibi relictum profundius immergi, donec in aqua volumen occupet, cuius pondus aequale sit ipsi corporis ponderi.

COROLLARIUM 5

20. Contra si maior pars, quam ad natandum requiritur, aquae immergatur, tum vis aquae corpus sursum pellens maior erit, quam corporis gravitas, ideoque sursum elevabitur, donec parti submersae aequale volumen aquae aequiponderans sit ipsi corpori.

SCHOLION 1

21. Haec, quae sunt allata, pertinent potissimum ad corpora, quae gravitate specifica leviora sunt quam aqua. Nam si corpus gravitate specifica aquam superet, tum ne totum quidem corpus aquae submersum tantae aquae quantitatis locum occupare potest, quae ipsi corpori aequiponderaret. Corpus igitur gravitate specifica gravius quam aqua, etiam totum aquae submersum, conservabit nisum deorsum labendi, interim tamen hic nisus minor erit quam extra aquam in aere. Diminuetur scilicet eius pondus pondere aquae eiusdem voluminis, quod corpus habet, eiusquae ratio ex praecedentibus aeque fluit, quam eorum, quae de corporibus levioribus aqua deduximus.

SCHOLION 2

22. Quo igitur corpus aquae innatet, oportet ut tanta pars eius aquae immergatur, ut volumen aquae illi parti aequale idem habeat pondus, quod ipsum corpus habet. Nam si maior vel minor pars esset submersa, corpus vel ascenderet vel descenderet magis, si sibi ipsi esset relictum. Hoc igitur est primum atque maxime necessarium requisitum ad natandum, ut corpus definitae magnitudinis partem aquae habeat immersam; haecque regula ab auctoribus ita enunciari solet, ut dicant, omne corpus natans tantum aquae de suo loco depellere, quantum pondere ipsum corpus adaequet. Hoc autem solum ad natandum non sufficit, sed insuper alius canon requiritur, secundum quem corpus, quo natet et in quiete persistat, dispositum esse debet. Quanquam enim vires horizontales se mutuo destruant, et hancobrem corpus horizontaliter non promoveatur, tamen a solis viribus verticalibus, ex gravitate corporis, et pressionibus aquae ortis, etiamsi inter se fuerint aequales, motus potest generari, quo quies turbatur, uti in sequentibus fusius ostendetur.

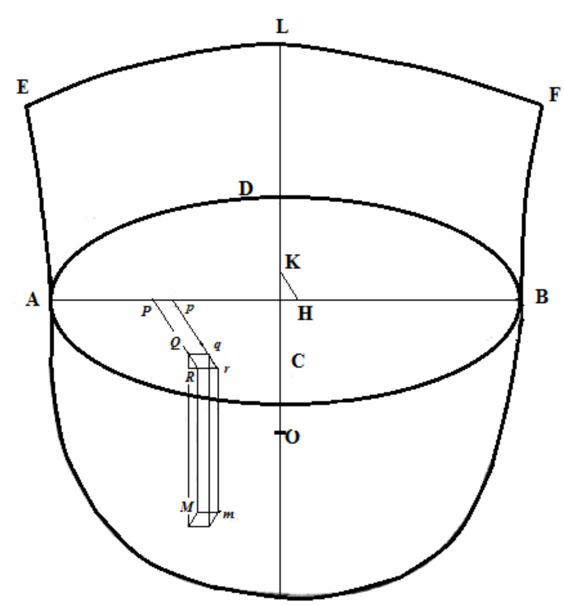

Fig. 3

PROPOSITIO 4

PROBLEMA

23. *Si corpus quodcunque ex parte aquae immergatur, determinare mediam directionem omnium pressionum aquae in partem submersam exertarum, atque potentiam omnibus illis pressionibus aequivalentem.*

SOLUTIO

Cum vires horizontales omnes, quae ex resolutione pressionum aquae in singula partis submersae elementa oriuntur, se mutuo destruant, solae vires verticales in considerationem veniunt. Quarum directiones, cum sint inter se parallelae, erit quoque media earum directio verticalis atque potentia aequivalens aequabitur summae omnium virium verticalium. Quamobrem potentia aequivalens omnibus aquae pressionibus aequalis erit ponderi aquae, cuius volumen aequale est parti corporis submersae. Sit autem eius directio, seu media directio omnium pressionum aquae recta verticalis *IK* (Fig.3), quae sectioni corporis horizontali :& *ABCD* cum superficie aquae factae occurrat in puncto *K,* et ex puncto *K* ad axem in sectione pro lubitu assumtum ducatur perpendicularis *KH.* Iam consideretur ut ante elementum partis submersae

$$Mm = dS,$$

indeque verticales ad planum sectionis *ACBD* ducantur, itemque applicatae *QP, qp.* Tum ponatur ut ante $AP = x,\ PQ = y$ et $QM = z$, exprimatque haec aequatio $dz = Pdx + Qdy$ naturam superficiei submersae; ita ut sit $dS = dxdy\sqrt{(1+P^2+Q^2)}$. Vis autem, qua elementum *Mm* verticaliter sursum urgetur, aequatur ponderi aquae voluminis $= zdxdy$, cuius momentum respectu horizontalis ad *AB* normalis et per *A* ducta est *zxdxdy.* Posito igitur *x* constante et sumto integrali ipsius *zdy,* ita ut toti ordinatae per punctum *P* ductae respondeat, dabit $\int xdx\int zdy$ summam omnium momentorum respectu horizontalis per punctum *A* ad *AB* normaliter ductae. Haecque expressio si dividatur per volumen partis submersae, quod est $\int dx\int zdy$, prodibit distantia *AH.* Distantia autem *KH* habebitur, si summa omnium momentorum respectu axis *AB* dividatur per volumen partis submersae $\int dx\int zdy$. Momentum autem vis elementum *Mm* verticaliter sursum pellentis est *yzdxdy.* Posito nunc *y* constante et sumto integrali *zdx* ita, ut omnibus applicatis *y* respondeat, tum dabit $\int ydy\int zdx$ summam omnium momentorum cis axem *AB,* simili vero modo quaeratur summa omnium momentorum ultra *AB* existentium, et ab hac summa illa summa subtrahatur residuumque per $\int dx\int zdy$ divisum dabit distantiam *HK.* Cognito autem puncto *K* innotescit media directio quaesita *IK.* Q.E.I.

COROLLARIUM 1

24. Cum vis, qua elementum *Mm* sursum urgetur, proportionalis sit prismati elementari *QRM,* manifestum est rectam *KI* hoc modo inventam per centrum gravitatis voluminis aquae submersi, quod sit in *O,* transire. Eodem enim calculo, quo hic usi sumus, centrum gravitatis solet determinari.

COROLLARIUM 2

25. Media ergo directio omnium aquae pressionum, quas corporis pars submersa patitur, est linea verticalis, quae per centrum gravitatis partis submersae transit.

COROLLARIUM 3

26. Si ergo corpus aquae innatet, quo casu volumen aquae parti submersae aequale aequiponderat ipsi corpori, tum potentia omnibus pressionibus aquae aequivalens aequalis erit ponderi ipsius corporis.

COROLLARIUM 4

27. Corpus igitur aquae innatans ab aqua tanta vi sursum pellitur, quantum est corporis pondus. Huius vero vis corpus sursum pellentis directio est linea verticalis per centrum gravitatis partis submersae transiens.

SCHOLION

28. Ex convenientia, quam demonstravimus inter mediam directionem pressionum aquae et rectam verticalem per centrum gravitatis partis submersae, transeuntem, facile intelligitur, hoc punctum *O* esse debere centrum gravitatis corporis *AIB* tanquam homogenei considerati. Quare utcunque corporis pars submersa fuerit ex heterogenea materia, conflata tamen in puncto *O* investigando haec pars submersa tanquam homogenea considerari debet. Hanc ob causam ut ambiguitatem in voce centri gravitatis evitem, in posterum istud gravitatis centrum *O,* quod ex consideratione corporis homogenei quaeri debet, centri magnitudinis nomine appellabo. Centrum igitur magnitudinis partis submersae invenietur, si pars submersa tanquam ex materia homogenea constans consideretur, eiusque centrum gravitatis definiatur. Hoc itaque centrum magnitudinis partis submersae, quoque erit centrum gravitatis aquae de suo loco depulsae, vel eius aquae, quae ante quam corpus immergebatur, spatium *AIB* occupaverat.

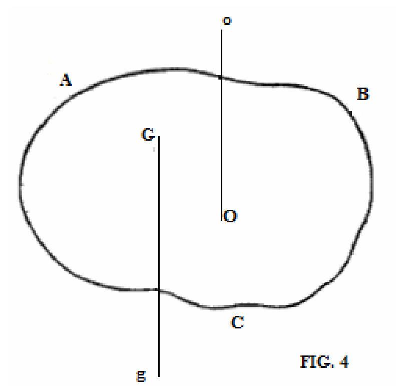

FIG. 4

LEMMA

29. *Corpus ABC* (Fig. 4), *cui duae potentiae Gg et Co in directionibus parallelis et contrariis sunt applicatae, in aequilibrio esse non potest, nisi illae potentiae sint inter se aequales, earumque directiones in eandem rectam incidant.*

DEMONSTRATIO

Si potentiae fuerint inaequales, manifestum est corpus in aequilibrio esse minime posse: nam etiamsi coinciderent, fortior corpus in sua directione promoveret. At si potentiae fuerint aequales, neque vero coincidant, tum eae corpus *ABC* secundum plagam *BAC* circa se ipsum convertent. Quare quo corpus quiescat, necesse est ut potentiae *Gg* et C*o* non solum sint aequales, sed etiam ut earum directiones coincidant. Q. E. D.

PROPOSITIO 5

THEOREMA

30. *Corpus aquae libere insidens inquiete seu aequilibrio esse nequit, nisi tum pars submersa volumine adaequet pondus aquae ipsi corpori aequale, tum vero nisi totius corporis centrum gravitatis atque centrum magnitudinis partis submersae in eandem rectam verticalem incidant.*

DEMONSTRATIO

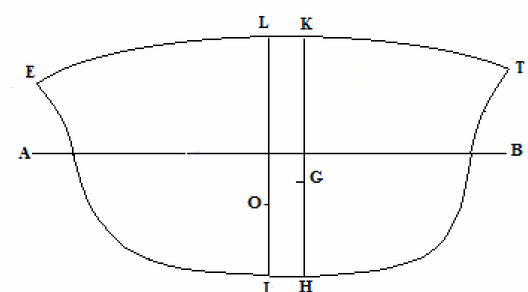

Fig. 5.

Sit *EAIBF* (Fig. 5) corpus aquae utcunque insidens et *AIB* eius pars submersa. Cum iam corpus ponatur liberum, nullas sentiet sollicitationes ad motum, praeter suam vim gravitatis et pressiones aquae, quas eius pars submersa patitur. At vis corporis gravitatis aequatur ponderi ipsius, eiusque directio est recta verticalis transiens per ipsius centrum gravitatis. Ponatur $P =$ ponderi totius corporis, et sit *G* eius centrum gravitatis; quibus positis corpus propter gravitatem deorsum urgebitur in directione *GH* a vi *P*. Deinde omnibus aquae pressionibus aequivalet vis directe sursum pellens in directione *OL* per centrum magnitudinis *O* partis submersae transeunte, quae quantitate adaequat pondus aquae spatium a parte submersa occupatum implentis; posito ergo hoc pondere *Q* propter pressiones aquae corpus in directione *OL* sursum urgebitur vi $= Q$. Corpori igitur nostro aquae hac ratione insidenti omnino duae vires *P* et *Q* sunt applicatae, quarum altera corpus in directione *GH* deorsum, altera vero in directione *OL* sursum sollicitat. Per lemma itaque praemissum corpus in aequilibrio esse nequit, nisi simul sit $Q = P$ et recta *LI* in rectam *HK* incidat. Fit autem $Q = P$, si tanta pars aquae submergatur, quae volumine adaequet pondus aquae ipsius corporis ponderi aequale, deinde vero lineae *LI* et H*K* coincident, si centrum gravitatis *G* totius corporis et centrum magnitudinis *O* partis submersae in eadem recta verticali sint sita. Q. E. D.

COROLLARIUM 1

31. Duo ergo requiruntur ad hoc, ut corpus aquae insidens possit esse in aequilibrio, quorum si alterutrum desit, corpus in quiete persistere nequit.

COROLLARIUM 2

32. Quoties ergo corpus aquae insidere videmus, tum certum est, partem eius submersam volumine aequalem esse ponderi aquae ipsius corporis ponderi aequali. Atque praeterea centrum magnitudinis partis submersae et centrum gravitatis totius corporis in eadem recta verticali esse sita.

COROLLARIUM 3

33. Quare si corpus aquae innatans fuerit in aequilibrio, tum recta iungens centrum gravitatis totius corporis et centrum magnitudinis partis submersae, erit normalis in sectionem aquae AB.

COROLLARIUM 4

34. Cum per experimenta constet pondus, quo datum aquae volumen gravitat, ex dato corporis cuiusvis pondere inveniri poterit quantitas partis submersae ad aequilibrium producendum requisita.

SCHOLION 1

35. Quo in sequentibus plures circumlocutiones evitem, loco integrarum descriptionum terminis convenientibus utar. Ita centrum gravitatis totius corporis vocabo tantum simpliciter centrum gravitatis; atque centrum magnitudinis partis submersae tantum centrum magnitudinis, cum hae voces nunquam alio sensu occurrant. Deinde etiam sectionem, quam corpus cum suprema aquae superficie constituit, vocabo simpliciter sectionem aquae. Simili modo verticalis centri gravitatis nobis erit recta verticalis per centrum gravitatis totius corporis transiens, atque verticalis centri magnitudinis debitabit rectam verticalem per centrum magnitudinis partis submersae transeuntem.

SCHOLION 2

36. Quando in sequentibus ope harum regularum eos corporum situs determinabimus, in quibus aquae insidere possint, id ita intelligi debet, ut corpus, si in eiusmodi situ definito aquae accuratissime collocetur, tum demum in hoc situ sit quieturum. At si tantillum ex hoc situ removeatur vel inclinetur, utrum tum sponte sese restituat, an vero in alium situm se recipiat? alia quaestio est, quae huc nondum pertinet, sed quam in sequentibus enodabo. Si autem situs, in quo corpus aquae impositum quiescere potest, ita fuerit comparatus, ut corpus, si tantillum ex eo situ declinetur, se non restituat, sed alium situm quaerat, in quo acquiescat, tum difficillimum est efficere, ut corpus in eo situ persistat. Etsi enim summa sollertia in eum situm collocetur, tamen levissima vi vel aeris vel aquae statim ex eo deturbabitur, adeo ut difficillimum

sit, huiusmodi casus per experimenta comprobare. Veluti nullum est dubium, quin baculus levis aquae verticaliter ita immitti queat, ut aequilibrium obtineatur; interim tamen in hoc situ constitutus quasi sua sponte procumbet, situmque horizontalem affectabit, in quo acquiescat.

PROPOSITIO 6

THEOREMA

37. *Omne corpus* (Fig. 6), *quod generatur ex rotatione cuiuscunque figurae ACFB circa axem quempiam AB, facta, ita aquae insidere poterit, ut eius axis AB teneat situm verticalem, dummodo centrum gravitatis fuerit positum in ipso axe AB.*

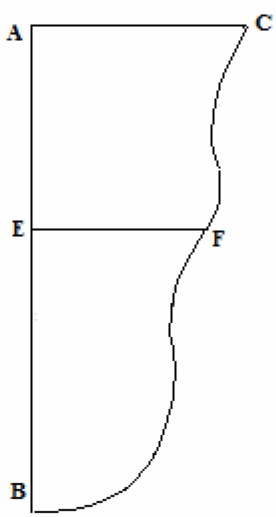

Fig. 6

DEMONSTRATIO

Recta *EF* ad axem *AB* normali abscindatur area *EBF,* quae circa axem conversa generet solidum, quod volumine adaequet pondus aquae totius corporis ponderi aequale, poteritque hoc solidum esse pars aquae immergenda, quo corpus in quiete permaneat, si modo alterum requisitum in hoc situ locum inveniat. At huius partis submersae, quae generatur ab area *EFB,* centrum magnitudinis cadit in axem *AB,* quare cum in eundem quoque centrum gravitatis cadere ponatur, poterit corpus ita aquae insidere, ut axis *AB* teneat situm verticalem. Q.E.D.

COROLLARIUM 1

38. Ex hac demonstratione etiam intelligitur, hoc corpus quoque inverso situ, quo *B* sursum *A* vero deorsum dirigitur, aquae insidere posse; adeo ut iam duo constent situa, quibus huiusmodi corpora aquae insidere possunt.

COROLLARIUM 2

39. Hinc etiam sequitur globum, cuius centrum gravitatis in ipso centro est positum, quocunque situ aquae insidere posse. Recta enim quaelibet per centrum transiens axis *AB* locum sustinere potest.

COROLLARIUM 3

40. Ad hanc corporum classem pertinent cylindri recti itemque coni recti tam integri quam truncati. Quare etiam haec corpora aquae ita innatare poterunt, ut eorum axes teneant situm verticalem, si modo eorum centra gravitatis in ipsos axes incidant.

SCHOLION

41. Corpora haec rotunda hanc habent proprietatem, ut omnes sectiones ad axem normales sint circuli, et demonstratio hoc nititur principio, quod axis *AB* per singularum sectionum transversalium centra gravitatis transeat. Quare eadem propositio aeque valebit pro corporibus, quorum sectiones ad axem normaliter factae sint polygona regularia quaecunque, ac pro circulis. Quamobrem etiam huiusmodi corpora aquae ita innatare poterunt, ut axes situm verticalem teneant, si modo centrum gravitatis totius corporis in hoc ipso axe fuerit positum.

COROLLARIUM 4

42. Si ergo huiusmodi corpora ex materia homogenea fuerint fabricata, tum utique eorum centra gravitatis in axes suos incident. Quare hinc concludere licet, omnia corpora homogenea, quae eiusmodi habent figuras, aquae ita insidere posse, ut eorum axes sint verticales.

PROPOSITIO 7

THEOREMA

43. *Corpus cylindricum DEIH* (Fig. 7), *cuius omnes sectiones transversales DE, FG, HI sunt inter se aequales et similes, situ erecto verticali aquae insistere potest, si modo eius centrum gravitatis fuerit in recta AB per omnium sectionum centra gravitatis transeunte.*

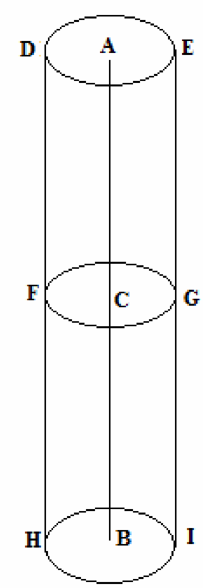

Fig. 7

DEMONSTRATIO

Ponatur portio *FGIH* tanta, quanta aquae immergi debet, eaque re ipsa aquae concipiatur submersa. Erit ergo huius partis submersae centrum magnitudinis in recta *BC* situm, quippe quae transit per omnium sectionum transversalium centra gravitatis seu magnitudinis. In eadem autem recta *AB* situm esse ponitur centrum gravitatis totius corporis. Quare cum recta *AB* ad sectionem aquae *FG* sit normalis, corpus in hoc situ aquae insidere poterit. Q. E. D.

COROLLARIUM 1

44. Me non monente facile etiam intelligitur, idem corpus situ quoque inverso quo *HI* sursum *DE* vero deorsum vergit, aquae insistere posse; ita ut consequenter duo situs sint cogniti, quibus huiusmodi corpora aquae insidere possunt.

COROLLARIUM 2

45. Si corpus *DEIH* fuerit ex materia homogenea factum, centrum gravitatis perse in rectam *AB* incidit. Quare huiusmodi corpora semper aquae situ erecto insidere poterunt, idque duplici modo.

SCHOLION

46. Ad hoc corporum genus pertinent praeter cylindros vulgares rectos omnia prismata recta, quascunque etiam habeant bases sive regulares sive inregulares. Deinde etiam pariter huc referuntur omnia solida quae generantur, si figura quaecunque plana secundum ductum lineae rectae ad planum figurae perpendicularis motu sibi semper parallelo moveatur. Atque de his omnibus valet Theorema propositum.

PROPOSITIO 8

THEOREMA

47. *Corpus cylindricum abdc* (Fig. 8), *quale in praecedente propositione consideravimus, situ horizontali bd aquae insidere poterit, si eius centrum gravitatis G in eius sectionem mediam ef incidat. Et quidem eodem situ aquae insidere poterit, quo sectio media sola ALBI aquae situ verticali innatare potest, si eius centrum gravitatis in G fuerit situm.*

DEMONSTRATIO

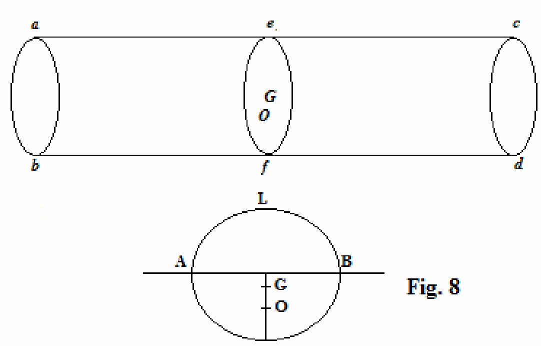

Fig. 8

Insideat enim hoc corpus aquae situ horizontali, sitque tanta eius pars iam immersa, quanta ad aequilibrium requiritur; manifestum est centrum magnitudinis *O* partis submersae quoque in sectionem mediam *ef* cadere debere, esseque in ipso centro magnitudinis partis ipsius sectionis mediae submersae. Nisi ergo haec centra *G* et C in eandem rectam verticalem incidant, convertatur corpus eousque donec illa centra hoc requisitum acquirant. Quo facto sit *LAIB* situs sectionis mediae, *AB* sectio aquae, *G* eius centrum gravitatis, quod congruere ponimus cum totius centro gravitatia, *O* centrum magnitudinis partis submersae *AIB,* quod per se convenit cum centro magnitudinis in ipso corpore *ad.* Quare si recta *GO* fuerit verticalis, tam sola sectio *LI* quam totum corpus hoc situ aquae insidere poterit. Q.E.D.

COROLLARIUM 1

48. Quo autem sectionis *LAIB* solius tanta pars aquae immergatur, quanta immergi debet, si est cum corpore coniunctum, ipsi huic sectioni eadem gravitas specifica respectu aquae tribui debet, quam habet integrum corpus.

COROLLARIUM 2

49. Si corpus fuerit homogeneum, tum eius centrum gravitatis non solum in sectionem mediam *ef* incidet sed insuper in ipsius sectionis mediae centro gravitatis erit situm.

COROLLARIUM 3

50. Ut igitur huiusmodi corporum cylindricorum situs, quo horizontaliter aquae insidere possunt, determinetur, sufficiet inquirere, in quonam situ una eius sectio aquae verticaliter insistere possit.

SCHOLION

51. Quo igitur definiri queat situs, quo huiusmodi corpora cylindrica aquae horizontaliter incubare possint, tantum ad figuram sectionum transversalium respiciendum est. Problema ergo huc redit, ut data quacunque figura plana *LAIB,* eius gravitate specifica respectu aquae et centro gravitatis *G* determinetur pars *AIB* aquae immergenda, quae quidem quantitate constat, ut recta iungens centrum gravitatis *G* et centrum magnitudinis *O* partis *AIB* sit in sectionem aquae *AB* normalis. Quamobrem ad nostrum institutum conveniet aliquot huiusmodi figurarum planarum considerare, et quibusnam sitibus aquae verticaliter innatare queant investigare. Praecipue autem ad corpora cylindrica seu prismatica homogenea respiciemus, et hanc ob causam pro centro gravitatis *G* figurae *LAIB* sumemus ipsius figurae centrum gravitatia; ita ut nobis quaestio huc reducatur: A data figura *LAIB* ducendo rectam *AB* partem *AIB* datae magnitudinis abscindere, hac conditione ut recta iungens centra totius figurae et partis abscissae perpendicularis sit ad rectam *AB.* Incipiamus igitur a triangulo tanquam figura simplicissima indeque ad quadrilatera progrediemur.

LEMMA

52. *Si per trianguli ACB* (Fig. 9) *centrum gravitatis G ducatur recta quaecunque IPQ lateri BC producto in Q occurrens, erit* $AC\cdot CQ-BC\cdot CP=3CP\cdot CQ.$

DEMONSTRATIO

Ex angulis *A* et *B* per centrum gravitatis *G* ducantur rectae *AH* et *BK,* quae per notam centri gravitatis proprietatem bisecabunt latera *BC* et *AC* eritque $AG:GH=2:1$ et $BG:GK=2:1$. Est vero

$$\sin API:\sin BQI=\frac{AG}{AP}:\frac{GH}{HQ}=\frac{GK}{BG}:\frac{BG}{BQ}=CQ:CP.$$

Ex quibus proportionibus eruitur

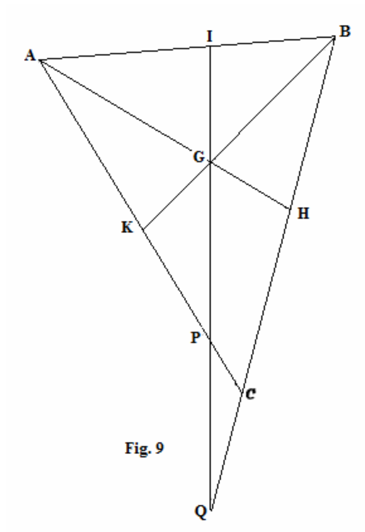

Fig. 9

$$BC+2CQ:AC-CP=BC+CQ:AC\;-2CP=CQ:CP\,,$$

atque hinc

$$AC\cdot CQ-BC\cdot CP=3CP\cdot CQ.$$

Q.E.D.

COROLLARIUM 1

53. Si sinus anguli *API* ponatur $=m$, et sinus anguli $BQI=n$; erit $m:n=CQ:CP$. Unde cum sit

$$CQ=\frac{AC\cdot CQ-BC\cdot CP}{3CP},$$

erit

$$CQ=\frac{m\cdot AC-n\cdot BC}{3n}$$

et

$$CP=\frac{m\cdot AC-n\cdot BC}{3m}.$$

COROLLARIUM 2

54. Si ex *A* ducatur ipsi *GQ* parallela, donec ipsi *BC* productae occurrat, erit ea $=3GQ$. Quocirca erit

$$CP:PQ=AC:3GQ \text{ atque } GQ=\frac{AC\cdot PQ}{3\cdot CP}.$$

Posito ergo $\sin ACB=k$ erit

$$GQ=\frac{k\cdot AC}{3n}.$$

COROLLARIUM 3

55. Si recta *QI* fuerit normalis in *AB,* erit

$$BC : AC = \cos API : \cos BQI.$$

Quare si $\cos$ ang.API ponatur M et $\cos$ ang.$BQI = N$, erit

$$BC : AC = M : N \text{ seu } M \cdot AC = N \cdot BC.$$

PROPOSITIO 9

PROBLEMA

56. *Proposito triangulo homogeneo ACB* (Fig.10), *cuius ad aquam gravitas specifica sit ut p ad q, casus determinare, quibus hoc triangulum aquae ita innatare potest, ut latus AB maneat aquam situm.*

SOLUTIO

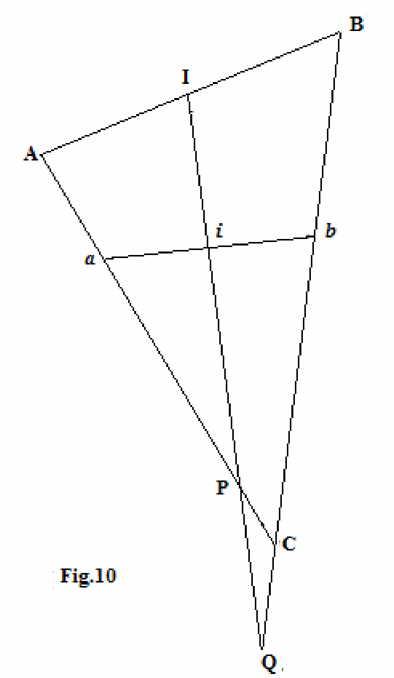

Fig.10

Sit aCb pars quaesita aquae immergenda, quo triangulum aquae insidere queat, sitque recta IPQ ducta per centra gravitatis tum totius trianguli ACB tum partis submersae aCb. Quo ergo triangulum in hoc situ aquae insidere queat, oportet ut recta IQ sit in ab perpendicularis atque praeterea ut area trianguli aCb sit ad aream ACB ut $p:q$. Ponatur nunc $AC = a,\ BC = b$ atque sin anguli $ACB = k$, eiusque $\cos = K$. Porro sit $aC = x,\ bC = y,\ sinAPI = m$, eius $\cos = M$, item $\sin BQI = n$ et $\cos = N$. His positis erit $m = kN - Kn$ et $M = kn + KN$. Cum nunc recta QPI transeat per centrum gravitatis trianguli ACB, erit

$$CQ = \frac{ma - nb}{3n}.$$

Deinde quia eadem recta per centrum gravitatis trianguli aCb transit, erit

$$CQ = \frac{mx - ny}{3n},$$

quibus coniunctis erit $mx - ny = ma - nb$. Porro cum recta PI normaliter occurrat rectae ab, erit $Mx = Ny$ ex quibus aequationibus elicitur

$$x = \frac{N(ma - nb)}{mN - Mn} \text{ et } y = \frac{M(ma - nb)}{mN - Mn}.$$

Quoniam vero est $m = kN - Kn$ et $M = kn + KN$, substituantur hi valores in aequatione

$$x = \frac{N(ma - nb)}{mN - Mn}$$

et posito $t = \frac{n}{N}$ prodibit

$$x = \frac{ka - Kat - bt}{k - 2Kt - ktt} \quad \text{atque} \quad y = (kt + K)x.$$

Hinc vero invenitur

$$t = \frac{\frac{1}{2}(b + Ka) - Kx \pm \sqrt{\left(\frac{1}{4}(b + Ka)^2 - ax - Kbx + x^2\right)}}{kx}$$

atque

$$y = \frac{1}{2}(b + Ka) - Kx \pm \sqrt{\left(\frac{1}{4}(b + Ka)^2 - ax - Kbx + x^2\right)}.$$

Denique est triang. *aCb* ad triang. *ACB* ut *xy* ad *ab,* erit ergo

$$p : q = xy : ab \text{ et } y = \frac{pab}{qx},$$

quo valore loco y in superiore aequatione substituto prodibit

$$q^2x^4 - (a + Kb)q^2x^3 + (b + Ka)pqabx - a^2b^2p^2 = 0.$$

Valor igitur ipsius x ex hac aequatione erutus dabit latus *aC,* quo invento si sumatur $bC = \frac{pab}{q \cdot aC}$, habebitur sectio aquae *ab,* atque pars aquae immergenda quaesita. Q. E. I.

COROLLARIUM 1

57. Quot igitur aequatio inventa continet radices reales et affirmativas, tot casibus triangulum propositum aquae ita insidere poterit, ut latus *AB* extra aquam maneat, solusque angulus C immergatur, dummodo sit $x < a$ et $y < b$.

COROLLARIUM 2

58. Si autem omnes 4 radices fuerint reales, tum earum tres tantum possunt esse affirmativae ob ternas tantum signorum alternationes. Radix autem negativa proposito non inservit. Quare non dari possunt plures tribus casus, quibus triangulum praescripto modo aquae insistere potest.

COROLLARIUM 3

59. Praeterea neque x maius esse potest quam a neque y maius quam b. Quare si eveniat ut vel x excedat a vel y excedat b, tum hi quoque casus erunt inutiles.

COROLLARIUM 4

60. Si tertium latus trianguli AB ponatur $= c$, hocque loco cosinus Kang. ACB in computum introducatur, propter

$$K = \frac{a^2 + b^2 - c^2}{2ab}$$

prodibit

$$q^2x^4 - \frac{q^3x^3}{2a}\left(3a^2 + b^2c^2\right) + \frac{apqx}{2}\left(a^2 + 3b^2 + c^2\right) - a^2b^2p^3 = 0.$$

COROLLARIUM 5

61. Duae autem aequationes, in quibus insunt x et y, simplicissimae sunt sequentes. Prima scilicet est $qxy = pab$, atque altera erit ista

$$y^2 - (b + Ka)y = x^2 - \left(a + Kb\right)x;$$

quibus duabus aequationibus problema propositum solvitur.

SCHOLION

62. Cum aequatio inventa habeat quatuor dimensiones neque generaliter concepta divisionem admittat, ita ut vix quicquam ex illa ad usum deduci queat; eam ad casus particulares triangulorum accommodabimus, pro quibus aequatio fit divisibilis, atque casus, quibus natatio evenire potest, reipsa assignari et repraesentari possunt.

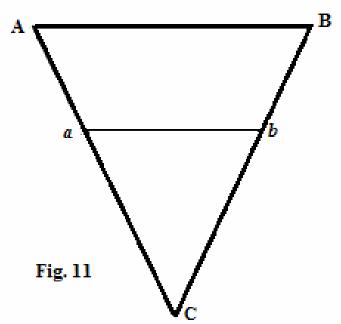

Fig. 11

EXEMPLUM 1

63. Sit triangulum propositum ACB (Fig. 11) isosceles, ita ut latera AC et BC, quae angulum C sub aqua situm, comprehendunt sint aequalia. Ponatur ergo $BC = AC = a$, erit $b = a$, atque aequatio inventa abibit in hanc

$$q^2x^4 - (1 + K)q^2ax^3 + (1 + K)pqa^3x - a^4p^2 = 0,$$

quae divisione resolvitur in has duas

$$\text{I. } qx^2 - pa^2 = 0 \text{ et II. } qx^2(1+K)qax + pa^2 = 0.$$

Quarum illa aequatio dat

$$x = \pm\, a\sqrt{\frac{p}{q}},$$

solus autem valor affirmativus habet locum, quia latera AC et BC non ultra C producta ponuntur. Quare ex prima aequatione erit

$$aC = x = a\sqrt{\frac{p}{q}} \text{ et } bC = y = a\sqrt{\frac{p}{q}},$$

qui ergo est unus casus, quo triangulum isosceles, ACB aquae insidere potest; eritque pars submersa aCb itidem triangulum isosceles, et sectio aquae ab parallela basi $AB,$ atque $AC : aC = \sqrt{q} : \sqrt{p}.$ Ex altera aequatione pro x duo obtinentur valores, quorum alter si pro x capiatur, alter pro y valebit, adeo ut vel neuter vel uterque sit locum habiturus. Erit vero

$$x = \frac{1}{2}(1+K)a \pm a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)},$$

atque

$$y = \frac{1}{2}(1+K)a \mp a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)};$$

qui valores ne fiant imaginarii, oportet ut sit

$$1+K > 2\sqrt{\frac{p}{q}} \text{ seu } K > 2\frac{p}{q} - 1.$$

Praeterea quia tam x quam y minores esse debent quam $a,$ oportet sit

$$\frac{1}{2}(1+K)a \pm a\sqrt{\left(\tfrac{1}{4}(1+K)^2 - \frac{p}{q}\right)} < 1 \text{ seu } \pm\sqrt{\left(\tfrac{1}{4}(1+K)^2 - \frac{p}{q}\right)} < \frac{1}{2}(1-K),$$

unde erit

$$K = \frac{p}{(1-\alpha)q} - \alpha$$

denotante α quemcunque numerum affirmativum. Quamobrem quo praeter casum assignatum adhuc duo reliqui locum habeant, oportet ut in his duabus aequationibus

$$K = 2\sqrt{\frac{p}{q}} - 1 + \beta \text{ et } K > \frac{p}{(1-\alpha)q} - \alpha$$

tam α quam β obtineant valores affirmativos. Quod si ergo evenerit, prodibunt duo reliqui casus quibus triangulum aquae insidere potest

$$\left.\begin{aligned} x &= \frac{1}{2}(1+K)a + a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \\ y &= \frac{1}{2}(1+K)a - a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \end{aligned}\right\} \textit{casus secundus},$$

$$\left.\begin{aligned} x &= \frac{1}{2}(1+K)a - a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \\ y &= \frac{1}{2}(1+K)a + a\sqrt{\left(\frac{1}{4}(1+K)^2 - \frac{p}{q}\right)} \end{aligned}\right\} \textit{casus tertius}.$$

COROLLARIUM 1

64. Si triangulum ACB fuerit aequilaterum, erit angulus C 60°, ideoque eius cosinus $K = \frac{1}{2}$. Quare quo duo posteriores casus locum inveniant, oportet sit primo ut

$$\frac{3}{2} > 2\sqrt{\frac{p}{q}} \text{ hoc est ut } \frac{p}{q} < \frac{9}{16}.$$

Deinde necesse est quoque ut sit

$$\frac{p}{q} = (1-\alpha)\left(\tfrac{1}{2}+\alpha\right) = \tfrac{1}{2} + \tfrac{1}{2}\alpha - \alpha^2,$$

sumto pro α numero affirmativo quocunque. Quod autem $\frac{p}{q}$ sit fractio affirmativa oportet sit $\alpha < 1$. Requiritur ergo ut sit $\frac{p}{q} > \frac{1}{2}$ et $\frac{p}{q} < \frac{9}{16}$.

SCHOLION

65. Cum alterum requisitum, ne duo posteriores casus fiant impossibiles, sit

$$\frac{p}{q} < \frac{1}{4}(1+K)^2, \text{ ponatur } \frac{p}{q} = \frac{1}{4}(1+K)^2 - \alpha^2,$$

et per alterum requisitum debebit esse $\alpha < \frac{1}{2}(1-K)$. Erit ergo

$\alpha^2 < \frac{1}{4}(1-K)^2$ ideoque $\frac{p}{q} > K$.

Quocirca dato angula *ACB,* nisi ratio $\frac{p}{q}$ contineatur intra limites $\frac{1}{4}(1+K)^2$ et *K,* praeter situm primo determinatum alius non datur, quo triangulum isosceles vertice deorsum verso aquae innatare potest. Est vero semper $\frac{1}{4}(1+K)^2 > K$, differentia enim est quadratum $\frac{1}{4}(1-K)^2$, quare pro quovis angulo *C* triangula exhiberi possunt, quae tribus modis angulo *C* deorsum verso aquae insidere possunt.

COROLLARIUM 2

66. In triangulo autem aequilatero, si fuerit $\frac{p}{q} > 1$ et $< \frac{9}{16}$, duo casus posteriores, quibus triangulum aquae insidere potest, erunt

$$x = \frac{3}{4}a \pm a\sqrt{\left(\frac{9}{16} - \frac{p}{q}\right)} \text{ et } x = \frac{3}{4}a \mp a\sqrt{\left(\frac{9}{16} - \frac{p}{q}\right)}.$$

COROLLARIUM 3

67. Si angulus *C* fuerit rectus, erit $K = 0$, huiusmodi igitur triangula triplici modo aquae innatare poterunt, dummodo sit $\frac{p}{q} < \frac{1}{2}$. Tum vero erit

$$\text{I. } x = a\sqrt{\tfrac{p}{q}};\ \text{II. } x = \tfrac{1}{2}a + a\sqrt{\left(\tfrac{1}{4} - \tfrac{p}{q}\right)} \text{ et III. } x = \tfrac{1}{2}a - a\sqrt{\left(\tfrac{1}{4} - \tfrac{p}{q}\right)}.$$

EXEMPLUM 2

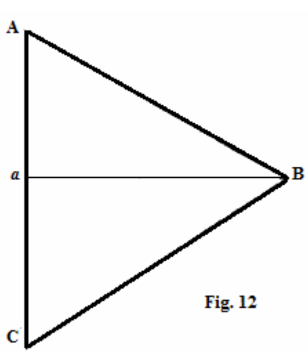

Fig. 12

68. Si gravitas specifica trianguli ita fuerit comparata ut sit $p : q = Kb : a$ abibit aequatio generalis in hanc:

$$x^4 - (a+Kb)x^3 + (b+Ka)Kb^2x - K^2b^4 = 0,$$

quae divisione resolvitur in has duas

$$\text{I. } x - Kb = 0; \text{ et II. } x^3 - ax^2 - Kabx + Kb^3 = 0.$$

Harum aequationum prima dat $Ca = x = Kb$ et cum sit $y = \frac{pab}{qx}$ fiet $y = b$. Incidet ergo punctum *b* in *B,* et recta *Ba* erit sectio aquae (Fig. 12).

COROLLARIUM I

69. Cum sit $Ca = Kb$, erit $\frac{Ca}{CB} = K = \text{cosinuiang}.AC$. Quare recta *Ba* erit perpendicularis in latus *AC.* Haec ergo perpendicularis semper potest esse sectio aquae, si fuerit $p : q = Ca : CA$.

COROLLARIUM II

70. Ista autem innatatio locum habere non potest, nisi uterque angulus *A* et C sit acutus, alioquin enim recta *Ca* non intra triangulum caderet, quod tamen est necesse.

SCHOLION

71. Simili modo si assumsissemus $p : q = Ka : b$, aequatio dividi potuisset per $x - a$, ita ut prodiisset $x = a$. Hic vero casus a priore non differt, nisi quod punctum *B* in *A* et vicissim sit translatum. Determinatis ergo casibus, quibus triangulum homogeneum aquae verticaliter ita innatare potest, ut unicus angulus aquae immergatur, superest ut etiam in eos casus inquiramus, quibus huiusmodi triangula cum duobus angulis sub aqua submersis innatare queant; his enim in duobus modis continentur omnes omnino casus, quibus triangulum aquae verticaliter innatare potest.

PROPOSITIO 10

PROBLEMA

72. *Proposito triangulo homogeneo CAB* (Fig. 13), *cuius gravitas specifica sit ad aquam ut p ad q, determinare casus, quibus hoc triangulum situ verticali aquae ita insidere potest, ut solus angulus C supra aquam emineat.*

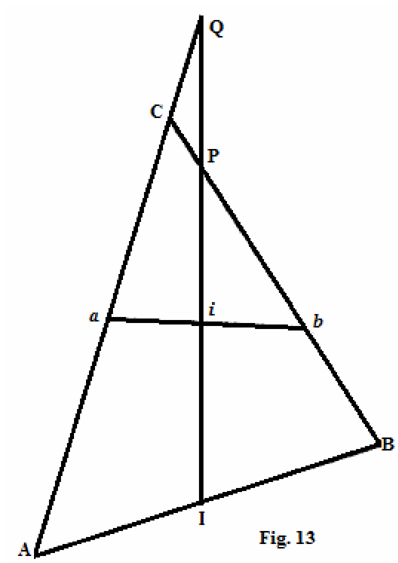

Fig. 13

SOLUTIO

Sit *aABb* pars aquae immergenda, quae quaesito satisfaciat. Per centra gravitatis tam totius trianguli *ACB,* quam partis submersae *aABb* ducatur recta *IPQ,* quae debebit esse normalis in sectionem aquae *ab.* At ista recta *IPQ* quoque transibit per centrum gravitatis trianguli *Cab* supra aquam eminentis. Quamobrem quaestio huc reducitur ut inveniatur positio rectae *ab,* quae a toto triangulo partem *aABb* abscindat, quae sit ad totum ut *p* ad *q;* atque ut recta iungens centra gravitatis totius trianguli et trianguli *Cab* sit normalis ad sectionem aquae *ab.* Haec ergo posterior conditio prorsus convenit cum praecedente problemate, prior vero in hoc discrepat, quod hic area trianguli *Cab* se habere debeat ad aream trianguli totius *CAB* ut *q-p ad q;* cum haec ratio ibi esset ut *p* ad *q.* Quamobrem solutio prioris problematis ad hoc accomodabitur, si tantum *qp* loco *p* scribatur. Ponatur ergo ut ante $AC = a,\ BC = b,\ \text{cosang.}ACB = K,$ et $Ca = x$ atque $Cb = y,$ atque *Ob*= *y,* erit

$$q^2x^4 - \left(a + Kb\right)q^2x^3(b + Ka)\left(q - p\right)qabx - \left(q - p\right)^2 a^2b^2 = 0.$$

Ex qua aequatione valor ipsius *x* erutus dabit respondentem valorem ipsius

$$y = \frac{\left(q - b\right)ab}{qx}.\ \text{Q. E. I.}$$

COROLLARIUM 1

73. Quod igitur aequatio inventa continebit radices reales et affirmativas pro *x,* quae sunt ipso *a* minores et quibus respondent valores ipsius *y* minores quam *b,* tot modis triangulum propositum aquae ita insidere poterit, ut anguli *A* et *B* sub aqua versentur.

COROLLARIUM 2

74. Sin autem aequationis propositae omnes radices sunt reales, earum tres tantum esse possunt affirmativae. Quare fieri non potest, ut triangulum pluribus quam tribus modis aquae ita innatare queat, ut solus angulus *C* extra aquam emineat.

COROLLARIUM 3

75. Si tertium latus *AB* ponatur $= c$, hocque loco cosinus *K* anguli *ACB* introducatur, tum sequens orietur aequatio

$$q^2x^4 - \frac{q^2x^3}{2a}\left(3a^2 + b^2 - c^2\right) + \frac{(q-p)qax}{2}\left(a^2 + 3b^2c^2\right) - \left(q - p\right)^2 a^2b^2 = 0.$$

COROLLARIUM 4

76. Si fuerit $p : q = 1 : 2$, haec aequatio congruit cum aequatione superioris propositionis. Quare hoc casu eadem recta *ab* poterit esse sectio aquae,
tam si angulus *C* fuerit solus extra aquam, quam si fuerit aquae submersus.

EXEMPLUM

77. Ponatur triangulum propositum *ACB* isosceles, latus scilicet

$BC = AC$ seu $b = a$, atque prodibit ista aequatio

$$q^2x^4-(1+K)q^2ax^3+(1+K)(q-p)qa^3x-(q-p)^2a^4=0,$$

quae per divisionem abit in has duas aequationes

$$\text{I. } qx^2(q-p)a^2=0 \text{ et II. } qx^2(1+K)qax+(q-p)a^2=0.$$

Harum aequationum prior dat $x=a\sqrt{\frac{q-p}{q}}$,

cui quoque respondet $y=a\sqrt{\frac{q-p}{q}}$,

hoc ergo casu triangulum extra aquam eminens erit quoque isosceles, et sectio aquae *ab* parallela basi *AB*. Altera aequatio has duas pro *x* dat quantitates

$$x=\tfrac{1}{2}(1+K)a\pm a\sqrt{\left(\tfrac{1}{4}(1+K)^2-\tfrac{(q-p)}{q}\right)},$$

quibus respective respondet

$$y=\tfrac{1}{2}(1+K)a\mp a\sqrt{\left(\tfrac{1}{4}(1+K)^2-\tfrac{(q-p)}{q}\right)}.$$

Praeter primum ergo situm inventum alius non dari potest, nisi sit

$$\tfrac{q-p}{q}<\tfrac{1}{4}(1+K)^2,$$

at etiamsi fuerit $\frac{q-p}{q}<\frac{1}{4}(1+K)^2$, hoc solum non sufficit ad realitatem posterium casuum; insuper enim requiritur ut tam *x* quam *y* sint ipso *a* minores, id quod locum habebit si fuerit $\frac{q-p}{q}>K$.

Quamobrem, quo casus posteriores fiant reales, oportet ut ratio inter hos *q* limites $1-K$ et $1-\frac{1}{4}(1+K)^2$ contineatur.

SCHOLION

78. Ex his igitur duabus propositionibus omnes casus definiri possunt, quibus datum triangulum homogeneum aquae situ verticali innatare potest. Perspicitur autem simul numerum casuum satis magnum esse posse, prout plures radices aequationum inventarum sunt reales et quaesito satisfacientes. Omnes autem radices si fuerint utiles, tum triangulum

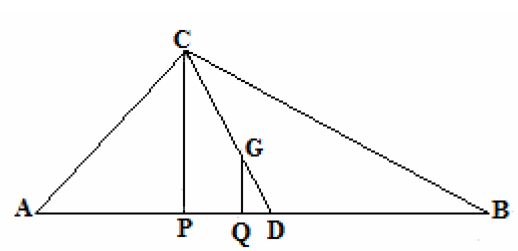

Fig. 14

18 diversis modis aquae innatare poterit; tres enim sunt casus, quibus singuli anguli aquae immerguntur, totidemque quibus singuli extra aquam eminent. Vix autem evenire potest, ut omnes octodecim casus fiant reales, quia si gravitas specifica trianguli ad aquam pro uno angulo requisitam habet proprietatem, eadem pro reliquis angulis non amplius satisfacere potest. Ita triangulum aequilaterum, si quidem eius gravitas specifica ad aquam contineatur inter rationes 8:16 et 9:16, quo casu plurimas admittit radices satisfacientes, 12 tantum diversis modis aquae insidere potest.

LEMMA

79. *Si ex trianguli ABC centro gravitatis G in latus AB perpendicularis GQ ducatur* (Fig. 14), *erit lateris AB segmentum*

$$AQ = \frac{3\cdot AB^2 + AC^2 - BC^2}{6\cdot AB}.$$

DEMONSTRATIO

Ex angulo C per centrum gravitatis G ad latus AB ducatur recta CD, erit ex nota centri gravitatis natura $AD = BD$, atque $GD = \frac{1}{3}CD$. Demittatur porro ex C *ad AB* perpendiculum CP, erit

$$AP = \frac{AB^2 + AC^2 - BC^2}{2\cdot AB},$$

hincque

$$DP = \frac{AB}{2} - AP = \frac{BC^2 - AC^2}{2AB}.$$

Cum autem CP *sit* parallela ipsi GQ erit $PQ = \frac{2}{3}DP$ unde fiet

$$PQ = \frac{BC^2 - AC^2}{3\cdot AB}.$$

Quamobrem erit

$$AQ = AP + PQ = \frac{3AB^2 + AC^2 - BC^2}{6\cdot AB}.$$

Q.E.D.

COROLLARIUM 1

80. Ipsum ergo perpendiculum GQ ex centro gravitatis trianguli G in latus AB demissum erit tertia pars perpendicularis CP, ita ut sit $GQ = \frac{1}{3}CP$.

COROLLARIUM 2

81. Si angulus ACB fuerit rectus, erit $AB^2 = AC^2 + BC^2$, hoc ergo casu fiet

$$AQ = \frac{2AC^2 + BC^2}{3AB} = \frac{1}{3}AB + \frac{AC^2}{3AB}.$$

Atque cum sit

$$CP = \frac{AC \cdot BC}{AB} \text{ erit } GQ = \frac{AC \cdot BC}{3 \cdot AB}$$

COROLLARIUM 3

82. Si triangulum *ACB* fuerit isosceles, ita ut sit $AC = BC$ erit $AQ = \frac{1}{2}AB$. Atque hoc casu erit $GQ = \frac{1}{3}\sqrt{AC^2 - \frac{1}{4}AB^2}$.

LEMMA

83. *Si a parallelogrammo rectangulo ABDC* (Fig. 15) *recta EF abscindatur triangulum EDF, atque* Fig. 15 *ex centro gravitatis G rectanguli totius AD in rectam EF perpendiculum GH demittatur, invenire punctum H.*

SOLUTIO

Ex centro gravitatis *G* rectanguli in latus *BD* demittatur perpendicularis *GI* secans rectam *EF* in *K* erit $DI = \frac{1}{2}BD$, et $EI = ED - \frac{1}{2}BD$, itemque $GI = \frac{1}{2}AB$. Iam ob triangula *EIK, EDF* similla, erit

$$IK = \frac{DF \cdot EI}{DE} = DF - \frac{DF \cdot BD}{2DE},$$

atque

$$EK = \frac{EF \cdot EI}{DE} = EF - \frac{EF \cdot BD}{2 \cdot DE}.$$

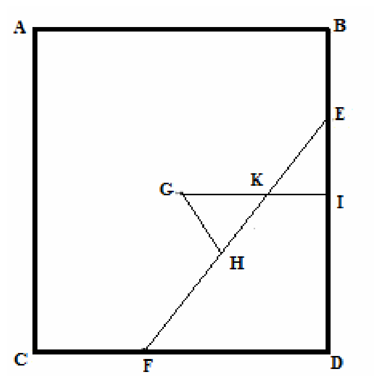

Fig. 15

Deinde habebitur

$$GK = \frac{1}{2}AB - KI = \frac{1}{2}AB - DF + \frac{DF \cdot BD}{2DE},$$

atque propter triangula *GHK* et *EDF* similla ista analogia

$$EF : DF = GK : KH,$$

ex qua sequitur

$$KH = \frac{AB \cdot DF}{2EF} - \frac{DF^2}{EF} + \frac{DF^2 \cdot BD}{2DE \cdot EF}.$$

Quocirca reperietur

$$EH = EF - \frac{BD \cdot DE}{2EF} + \frac{AB \cdot DF}{2EF} - \frac{DF^2}{EF},$$

seu

$$EH = \frac{2EF^2 + AB \cdot DF - BD \cdot DE}{2EF}.$$

Q.E.I.

COROLLARIUM

84. Ex eorundem triangulorum *GHK* et *EDF* similitudine sequitur etiam fore $EF : ED = GK : GH$. Ex hac ergo analogia reperitur ipsa centri gravitatis *G* a recta *EF* distantia

$$\frac{AB \cdot DE}{2EF} + \frac{BD \cdot DF}{2EF} - \frac{DE \cdot DF}{EF} = \frac{AB \cdot DE + BD \cdot DF - 2DE \cdot DF}{2EF}.$$

PROPOSITIO 11

PROBLEMA

85. *Proposito parallelogrammo rectangulo homogeneo ABDC* (Fig. 15), *cuius gravitas specifica sit ad aquam ut p ad q, invenire casus, quibus hoc parallelogrammum rectangulum aquae ita insidere potest, ut solus angulus D immergatur.*

SOLUTIO

Sit *EF* sectio aquae quaesita, et *EDF* portio aquae immergenda, erit $\frac{1}{2} DE \cdot DF : AB \cdot BD = p : q$; atque recta jungens centra gravitatis totius rectanguli, et partis *EDF* normalis esse debet in sectionem aquae *EF.* Quare normales ex centro gravitatis rectanguli, et centro gravitatis trianguli *EDF* in idem rectae *EF* punctum incidere debebunt. Quare per § 83 et § 81 erit

$$\frac{2DE^2 + AB \cdot DF - BD \cdot DE}{2EF} = \frac{2DE^2 + DF^2}{3EF}.$$

Positis nunc

$$AB = CD = a,\ AC = BD = b,\ DF = x \text{ et } DE = y,$$

erit

$$qyx = 2pab \text{ seu } y = \frac{2pab}{qx},$$

atque

$$2y^2 + 3ax - 3by = 2x^2,$$

quae si loco y eius valor $\frac{2pab}{qx}$ substituatur abit in hanc

$$2qqx^4 - qqax^3 + 6pqab^2x - 8p^2a^2b^2 = 0.$$

Ex qua aequatione invenietur valor ipsius x, ex eoque valor ipsius y. Q. E. I.

COROLLARIUM 1

86. Aequatio haec ad summum tres habere potest radices affirmativas reales, quarum autem ea tantum quaesito satisfaciunt, quae pro x valores dant ipso a minores, et quibus simul valores ipsius y respondent ipso b minores.

COROLLARIUM 2

87. Cum triangulum EDF dimidium rectanguli excedere nequeat, perspicuum est rationem q ad p minorem dupla esse non posse, ita ut esse debeat vel

$$\frac{q}{p} = 2 \text{ vel } \frac{q}{p} > 2.$$

EXEMPLUM

88. Si rectangulum abeat in quadratum, ut sit $b = a$, habebitur ista aequatio

$$2q^2x^4 - 3qqax^3 + 6pqa^3x - 8ppa^4 = 0,$$

quae divisione resolvitur in has duas

$$\text{I. } qx^2 - 2pa^2 = 0 \text{ et II. } 2qx^2 - 3qax + 4pa^2 = 0.$$

Harum aequationum illa dat

$$x = a\sqrt{\frac{2p}{q}},$$

cui respondet

$$y = a\sqrt{\frac{2p}{q}},$$

quo ergo casu x et y sunt aequales, et sectio aquae EF fit parallela diagonali BC. Ex altera aequatione eruitur duplex valor ipsius

$$x = \tfrac{3a}{4} \pm a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)},$$

cui quoque iste duplex valor ipsius y respondet, scilicet

$$y = \tfrac{3a}{4} \mp a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)}.$$

Ne igitur hi valores fiant imaginarii, oportet sit

$$\tfrac{2p}{q} < \tfrac{9}{16} \text{ seu } \tfrac{p}{q} < \tfrac{9}{32}.$$

Deinde ne excedant latus a, oportet sit

$$a\sqrt{\left(\tfrac{9}{16} - \tfrac{2p}{q}\right)} < \frac{a}{4} \text{ seu } \frac{p}{q} > \frac{1}{4}.$$

Quare si $\frac{p}{q}$ contineatur inter hos limites $\frac{8}{32}$ et $\frac{9}{32}$ tres casus dantur, quibus quadratum aquae ita insidere potest, ut solus angulus D aquae immergatur.

SCHOLION

89. Eadem solutio parumper immutata etiam huic problemati satisfaciet, quo quaeruntur casus, quibus parallelogrammum rectangulum aquae ita insidere queat, ut tres anguli *B, A, C* sub aquam mergantur, solusque angulus *D* emineat. Hoc enim casu pariter ut ante positio rectae *EF* inveniri debet, in quam recta iungens centra gravitatis rectanguli *AD* et trianguli *EDF* sit normalis. Hoc tantum differt istud problema a praecedente, quod hic area *ABEFC* ad totum rectangulum rationem habere debeat p ad q. Cum ergo hoc casu ratio areae trianguli *EDF* ad rectangulum totum esse debeat ut $q-p$ ad q, solutio praecedens ad hunc casum accommodabitur ponendo $q-p$ loco p. Manentibus igitur ceteris denominationibus ut ante, erit

$$2qqx^4 - 3qqax^3 + 6(q-p)qab^2x - 8\left(q-p\right)^2 a^2b^2 = 0,$$

ex qua valori ipsius x invento respondet valor ipsius

$$y = \tfrac{2(q-p)ab}{qx}.$$

Si rectangulum abeat in quadratum, eadem valebunt, quae in exemplo sunt inventa, si modo loco p ponatur $q-p$. Ita quadratum tribus diversis modis aquae ita poterit innatare, ut solus angulus D extra aquam emineat, si $\frac{p}{q}$ contineatur intra hos limites $\frac{8}{32}$ et $\frac{9}{32}$.

LEMMA

90. *Dato trapezio ABDC* (Fig.l6), *in quo latera AC et BD sint inter se parallela, atque ad basin CD normalia, invenire punctum H in quo recta G H per centrum gravitatis trapezii G ducta in latus AB normaliter incidit.*

SOLUTlO

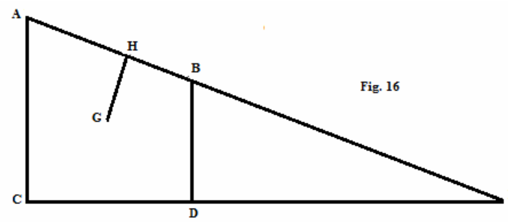

Fig. 16

Producantur tum latus *AB* tum basis CD, donec concurrant in *I* et ex utriusque trianguli *ACI* et *BDI* centris gravitatis concipiantur perpendicula in *AI* demissa. Iam ex staticis A constat fore momentum trapezii respectu puncti *I* aequale differentiae momentorum triangulorum. Momentum autem figurae respectu *I* obtinetur, si *c* area multiplicetur in distantiam puncti, in quo perpendiculum ex eius centro gravitatis in *AI* demissum in *AI* incidit, a puncto I. lta momentum trapezii erit $= ABDC\cdot IH$, atque per § 81 erit

$$\text{momentum trianguli } ACI=\frac{ACI(2CI^2+AC^2)}{3AI}$$

atque

$$\text{momentum trianguli } BDI=\frac{BDI(2DI^2+BD^2)}{3BI}.$$

Quamobrem habebitur ista aequatio

$$ABDC\cdot IH=\frac{ACI(2CI^2+AC^2)}{3AI}-\frac{BDI(2DI^2+BD^2)}{3BI}$$

ex qua punctum *H* determinabitur. Ponatur ergo

$$AB=a,\ AC=b,\ BD=c\ \text{ et }\ CD=\sqrt{\left(a^2-(b-c)^2\right)}=d,$$

eritque

$$AI=\frac{ab}{b-c},\ BI=\frac{ac}{b-c},\ CI=\frac{bd}{b-c}\ \text{ atque }\ DI=\frac{cd}{b-c}.$$

Ex his prodibit

$$ABDC=\frac{(b+c)d}{2},\ ACI=\frac{bbd}{2(b-c)}\ \text{et}\ BDI=\frac{ccd}{2(b-c)}.$$

Iam ponatur $AH=x$, erit $HI=\frac{ab}{b-c}-x$, quibus valoribus substitutis obtinebitur tandem

$$AH=x=\frac{a^2\left(b+2c\right)+b^3-c^3}{3a\left(b+c\right)}.$$

Q.E.I.

COROLLARIUM 1

91. Simili modo ipsum perpendiculum *GH* ope momentorum poterit determinari, prodibit autem calcula ad finem perducto

$$GH=\frac{d\left(b^3-c^3\right)}{3a\left(bb-cc\right)}=\frac{d\left(bb+bc+cc\right)}{3a\left(b+c\right)}.$$

COROLLARIUM 2

92. Si fuerit $AC=BD$ seu $c=b$, quo casu trapezium mutatur in parallelogrammum rectangulum, prodibit

$$AH=\tfrac{a}{2}\ \text{et}\ GH=\tfrac{bd}{2a}=\tfrac{b}{2}$$

propter hoc casu $d=a$.

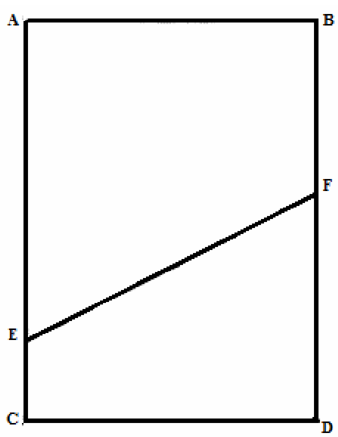

Fig. 17

PROPOSITIO 12

PROBLEMA

93. *Determinare casus, quibus parallelogrammum rectangulum ABDC homogeneum, cuius gravitas specifica ad aquam sit ut p ad q, aquae ita insidere potest, ut duo anguli C et D sub aqua, reliqui vero A et B extra aquam versentur* (Fig. 17).

SOLUTIO

Sit *EF* sectio aquae, quae quaestioni satisfaciat, oportet primo ut sit

$\frac{CE+DF}{2}:AC=p:q.$ Deinde requiritur, ut recta iungens centra gravitatis totius rectanguli et portionis *ECDF* in *EF* sit normalis, id quod eveniet, si perpendicula ex utriusque trapezii *ECDF* et *EABF* centris gravitatis in *EF* demissa coincidant. Per lemma igitur praemissum obtinebitur ista aequatio

$$\frac{EF^2(CE+2DF)+CE^3-DF^3}{3EF\left(CE+DF\right)}=\frac{EF^2(AE+2BF)+AE^3-BF^3}{3EF(AE+BF)}.$$

Ponantur ergo

$$AB=CD=a,\ AC=BD=b,\ CE=x,\ DF=y,$$

erit

$$AE=b-x,\ BF=b-y,\ \text{et } EF^2=a^2+x^2-2xy+y^2.$$

Iam propter primam conditionem fit

$$\frac{x+y}{2}:b=p:q \text{ seu } x+y=\frac{2bp}{q},$$

altera vero conditio hanc suppeditat aequationem

$$\frac{\left(a^2+\left(x-y\right)^2\right)\left(x+2y\right)+x^3-y^3}{x+y}=\frac{\left(a^2+\left(x-y\right)^2\right)\left(3b-x-2y\right)+(b-x)^3-\left(b-y\right)^2}{2b-x-y}$$

Quae aequatio per divisionem reducitur ad has duas

$$x=y \text{ et } a^2+2\left(y^2+yx+xx\right)=3b(y+x).$$

Cum autem sit

$$x+y=\frac{2bp}{q}$$

dabit prior aequatio

$$x=y=\frac{bp}{q}$$

posterior vero

$$2qqxx-4pqbx+8p^2b^2-6pqb^2+q^2a^2=0,$$

ex qua aequatione eruitur valor ipsius

$$x = \frac{pb}{q} \pm \sqrt{\left(\frac{3pb^2}{q} - \frac{3p^2b^2}{q^2} - \frac{a^2}{2}\right)}$$

eritque

$$y = \frac{pb}{q} \mp \sqrt{\left(\frac{3pb^2}{q} - \frac{3p^2b^2}{q^2} - \frac{a^2}{2}\right)}$$

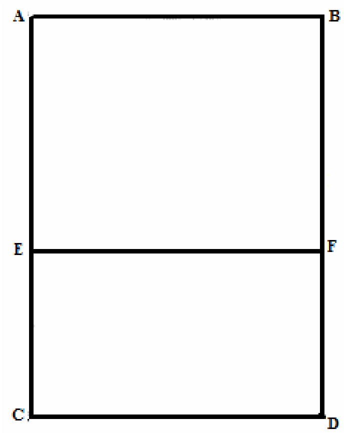

Fig. 18

COROLLARIUM 1

94. Primus ergo casus, quo $x = y = \frac{pb}{q}$ semper locum habet, dummodo fuerit rectangulum gravitate specifice levius quam aqua seu $p < q$. Tum autem sectio Fig. 18 aquae erit *EF* parallela lateribus *AB* et *CD* (Fig. 18).

COROLLARIUM 2

95. Haec igitur sola solutio quatuor exhibet casus, quibus rectangulum homogeneum aquae innatare potest. Quodlibet enim latus per eam in aqua situm horizontalem prout *CD* tenere poterit.

COROLLARIUM 3

96. Duo autem reliqui casus (Fig. 17) non semper locum habere possunt. Quo enim existere queant requiritur, ut primo sint reales, deinde ut sint affirmativi
et tertio ut sint minor quam latus *b*.

COROLLARIUM 4

97. Hi vero pro *x* et *y* inventi valores, quo sint reales requiritur ut sit

$$a^2 < \frac{6pb^2}{q} - \frac{6p^2b^2}{q^2},$$

hoc est, ut sit

$$\frac{a}{b} < \sqrt{\left(\frac{6pq - 6pp}{qq}\right)}.$$

Deinde quo sint affirmativi necesse est ut sit

$$a^2 > \frac{6pqb^2 - 8p^2b^2}{q^2} \quad \text{seu} \quad \frac{a}{b} > \sqrt{\frac{6pq - 8pp}{qq}}.$$

Tertio vero quo sint latere b minores oportet ut sit

$$a^2 > \frac{10pb^2}{q} - \frac{8p^2b^2}{q^2} \quad \text{seu} \quad \frac{a}{b} > \sqrt{\left(\frac{10pq - 8pp - 2qq}{qq}\right)}.$$

COROLLARIUM 5

98. Si fuerit rectangulum duplo levius quam aqua, ut sit $\frac{p}{q} = \frac{1}{2}$; quo posteriores casus fiant reales, oportet sit

$$\frac{a}{b} < \frac{\sqrt{6}}{2} \text{ atque } \frac{a}{b} > 1 \text{ et tertio } \frac{a}{b} > 1.$$

Requiritur, itaque ut $\frac{a}{b}$ contineatur inter limites 1 et $\frac{\sqrt{6}}{2}$.

COROLLARIUM 6

99. Si fuerit $p : q = 3:4$, quo rectangulum posteriore ratione aquae insidere queat, oportet ut sit

$$\text{primo } \frac{a}{b} < \sqrt{\frac{9}{8}}, \text{ secundo } \frac{a}{b} > 0 \text{ et tertio } \frac{a}{b} > 1.$$

Quare necesse est ut $\frac{a^2}{b^2}$ contineatur intra limites $\frac{8}{8}$ et $\frac{9}{8}$.

COROLLARIUM 7

100. Si fuerit $p : q = 1 : 4$, ne posteriores aquae innatandi modi fiant inutiles, necesse est ut sit

$$\frac{a}{b} < \sqrt{\frac{9}{8}}, \text{ atque } \frac{a}{b} > 1 \text{ nec non } \frac{a}{b} > 0.$$

Quare limites erunt 1 et $\sqrt{\frac{9}{8}}$ ut supra.

SCHOLION

101. Quod in his duobus casibus posterioribus limites sint iidem, mirandum non est. Nam si *EF* est sectio aquae pro parte submersa *EGDF,* quoque sectio aquae esse poterit pro parte submersa *EABF ;* hicque casus locum habet si gravitas specifica rectanguli ad aquam fuerit ut $q-p$ ad q. Atque si in limitibus pro $\frac{a}{b}$ datis ponatur $q-p$ loco p, primus omnino non mutatur, secundus vero et tertius commutantur, adeo ut semper pro utroque casu iidem limites prodire debeant.

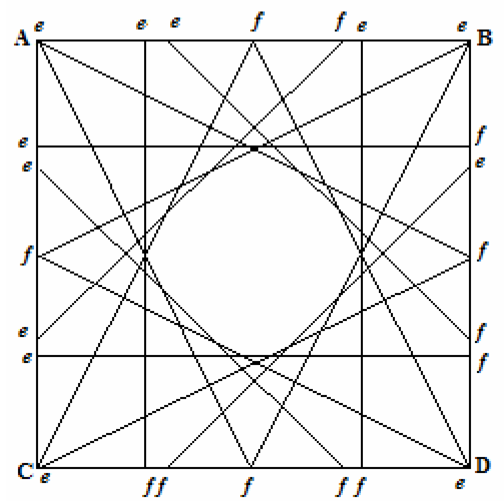

Fig. 19

EXEMPLUM

102. Abeat rectangulum in quadratum *ABDC* (Fig. 19) et quaeratur, quo modo eius gravitas specifica comparata esse debeat, ut quadratum plurimis diversis modis aquae insidere queat. Quia ergo est $b=a$, fiet primo

$x=y=\frac{pa}{q}$ deinde

$$x=\frac{p}{q}a\pm a\sqrt{\left(\frac{3p}{q}-\frac{3pp}{qq}-\tfrac{1}{2}\right)}$$

atque

$$y=\frac{p}{q}a\mp a\sqrt{\left(\frac{3p}{q}-\frac{3pp}{qq}-\tfrac{1}{2}\right)},$$

quae posteriores expressiones, quo fiant reales et utiles, oportet ut primo $\frac{q}{p}$ contineatur intra limites $3+\sqrt{3}$ et $3-\sqrt{3}$. Deinde opus est ut sit vel

$$\frac{q}{p}>4 \text{ vel } \frac{q}{p}<2.$$

Tertio vero necesse est ut sit vel

$$\frac{q}{p}>2 \text{ vel } \frac{q}{p}<\frac{4}{3}.$$

.

Quare casus hi fient reales si $\frac{q}{p}$ vel intra hos limites $3+\sqrt{3}$ et 4 vel intra hos

$3-\sqrt{3}$ et $\frac{4}{3}$ contineatur. Deinceps etiam hoc quadratum priore casu aquae insidere potest, ut unicus angulus sub aqua, altero vero ut tres sub aqua existant (§ 88, § 89), hocque uno saltem modo, duo enim reliqui ibi expositi requirunt ut $\frac{q}{p}$ vel contineatur intra limites 4 et $\frac{32}{9}$, vel intra limites $\frac{4}{3}$ et $\frac{32}{23}$ qui ergo limites ab illis excluduntur. Quare si $\frac{q}{p}$ contineatur vel intra limites $3+\sqrt{3}$ et $\frac{32}{9}$ vel intra limites $3-\sqrt{3}$ et $\frac{32}{23}$, tum quadratum tam per hanc, quam praecendentem propositionem 16 diversis modis aquae insidere poterit. Sit ergo $p:q=1:4$, erunt sedecim sectiones aquae, quibus quadratum aquae insidere potest, prout in figura sunt expressae per *ef,* quae eaedem sectiones quoque valent pro quadrato, cuius gravitas specifica ad aquam est ut 3 ad 4.
Nisi autem gravitas specifica intra dictas rationes contineatur, tum octo tantum modis quadratum aquae insidere poterit.

SCHOLION

103. Quae igitur hic de diversis modis, quibus vel triangula vel parallelogramma rectangula homogenea aquae insidere possunt, tradidimus, ea quoque pertinent ad prismata homogenea, quarum sectiones sunt vel triangula vel rectangula, uti supra monstravimus. Tot scilicet modis huiusmodi prismata, dum eorum axes manent in situ horizontali, aquae insidere poterunt, quot tum pro triangulis tum rectangulis in his propositionibus assignavimus. Hi autem utique non sunt omnes casus, quibus prismata aquae innatare possunt, nam praeter hos pluribus quoque aliis modis aquae insidere poterunt, dum eorum axes vel sint verticales vel ad horizontem oblique positi, quemadmodum ex allatis facile colligere licet. His autem ulterius persequendis contra nostrum esset institutum, ut immoraremur.

DEFINITIO

104. Planum diametrale voco planum, quo corpus in duas partes similes et aequales dividitur, ita ut omnes sectiones corporis, quae fiunt planis ad planum diametrale normalibus, ab hoc plano diametrali dividantur in duas partes similes et aequales.

COROLLARIUM

105. Corporis igitur plano diametrali praediti centrum magnitudinis incidit in ipsum planum diametrale, cum ex utraque eius parte corpus sit sibi simile et aequale.

SCHOLION

106. Corpora, quae habent planum diametrale, prae ceteris merentur, quae examini subiiciantur; omnia enim corpora, quae ad natandum in aqua adhibentur, ita sunt

comparata, ut planum diametrale admittant. Sic in omnibus navibus planum verticale per spinam transiens habebit hanc proprietatem, ut naves in duas partes similes et aequales dividat. Quod autem omnes naves hoc modo fabricentur ratio in promtu est, cum nulla sit ratio, cur ex una parte aliam habeant figuram, quam ex altera. Quamobrem ad figuram navium aptissimam determinandam sufficit alteram medietatem circa planum diametrale assignasse, quippe cui altera pars similis et aequalis constitui debet.

PROPOSITIO 13

PROBLEMA

107. *Corpus, quod plano diametrali est praeditum, aquae semper ita innatare potest, ut planum diametrale sit verticale, si modo corporis centrum gravitatis in planum diametrale cadat.*

DEMONSTRATIO

Sit corpus huiusmodi aquae ita impositum, ut planum diametrale teneat situm verticalem, erit sectio aquae normalis ad planum verticale, ideoque pars aquae submersa a plano diametrali in duas partes similes et aequales dividetur. Quamobrem huius partis submersae centrum magnitudinis cadet in planum diametrale, in quod etiam centrum gravitatis totius corporis cadere ponitur. Consequenter corpus ita inclinando, ut planum diametrale maneat verticale, effici potest, ut recta iungens centra gravitatis et magnitudinis fiat verticalis, quo ergo casu corpus aquae innatabit. Q. E. D.

SCHOLION 1

108. Huiusmodi situ, quo planum diametrale est verticale, videmus omnis generis naves aquae insidere, nisi a viribus alienis ex hoc situ declinentur. Atque tum in construendis tum onerandis navibus in id potissimum est incumbendum, ut centrum gravitatis in planum diametrale cadat, quo naves hoc situ descripto aquae innatare queant.

SCHOLION 2

109. Exposita igitur sunt in hoc capite principia quibus status aequilibrii corporum aquae innatantium nititur, atque simul methodus est tradita, cuius ope pro quibuscunque corporibus propositis situs, quibus aquae insidere possunt, determinare licet. Corpora autem hic sumus contemplati omnino libera, quae a nullis viribus externis sunt sollicitata, sed a solis viribus tum gravitatis tum pressionum aquae in aequilibrium constituuntur; in sequentibus autem tam ad motum quam aequilibrium corporum, quae vel non sunt libera vela viribus externis sollicitantur determinandum progrediemur. Nunc autem ad caput secundum pergimus, in quo investigabimus motum, quo corpus non in aequilibrio constitutum, sese in situm aequilibrii restituit. In hoc scilicet capite considerabimus corpora aquae ita imposita, ut vel non debita pars aquae sit immersa, vel recta iungens

centra gravitatis et magnitudinis non sit verticalis, et inquiremus, quomodo ex hoc situ in statum aequilibrii se recipiant. Cum autem corpus aquae non ita imponi queat, quin e vestigio in statum aequilibrii collocetur, eiusmodi casus hic examinandi venient, quibus corpus ab externa vi ex aequilibrio est depulsum, et cessante ea vi in statum aequilibrii revertitur.