Chapter 9

CONCERNING THE INITIAL GENERATION OF MOTION IN RIGID BODIES

THEOREM 6

616. If the effect of two forces acting together in generating the motion of a body is known, of which one acts through the centre of inertia, then the effect of the other acting separately also becomes known.

DEMONSTRATION

A pure progressive motion is generated by some force applied to the body at the centre of inertia, by which the individual elements of this are moved forwards along the direction of the force by equal small intervals, which, if the force acting is equal to V and the mass of the body is equal to M, in the small element of time dt, then these increments are equal to $\frac{Vgdt^2}{M}$. But if now the body, besides this force applied V at the centre of inertia, is acted on by some other force S and the effect of the two forces acting simultaneously is known, then the matter can thus be considered, as if the above body should be acted on by some force equal and opposite to V applied at the centre of gravity, by which the body thus is disturbed by the first effect, so that the whole body is carried backwards in the direction of this force through an interval equal to $\frac{Vgdt^2}{M}$, which effect joined with that above gives the effect of only the force S acting, which thus become known.

COROLLARY 1

617. Clearly the effect of the force S is equal to the effect produced by the two forces acting at the same time V and S, and on taking away this effect that is produced by the force V alone, that follows regarding the resolution of the motion which has been discussed above.

COROLLARY 2

618. Therefore if a rotational motion is impressed about some axis by the two forces *S* and *V* acting together, then by the single force *S* there is impressed on the body a mixed motion with the same rotational motion, and with a progressive motion, which is induced by a force equal and opposite to *V*.
[As the addition of *V* removes the forwards motion]

SCHOLIUM

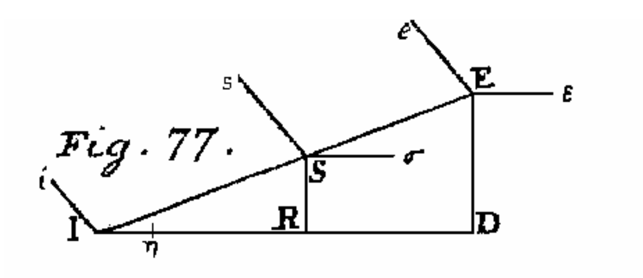
619. For the just principles of the adverse method [of subtracting forces] will be seen, because from the effect of the two forces acting at the same time, we can try to find the effect of one force acting alone. Now in problem 18, where we defined the forces by which the axis of rotation is not affected, we have seen that these forces can be reduced occasionally to one, but always to two forces. Whereby, as we wish the effect of only one force to prevail, so that this is to be effected, one of these two forces passes through the centre of oscillation of the body, and thus this theorem will be of the most use to us. From which it comes about, so that also any forces acting on the body are able to be reduced to two forces of this kind, as we now show.

THEOREM 7

620. However many forces there should be acting on a rigid body, and in whatever manner they should be applied, these can always be reduced to two forces, of which one passes through the centre of inertia of the body.

DEMONSTRATION

Let *I* be the centre of inertia of the body (Fig. 77), through which it pleases to draw some right line *ID*. Through the direction of any force acting [at some point S], there is drawn a plane normal to the line *ID* [coming out of the plane of the page], which cuts that line in the point *R*; and, unless the direction of this force is placed in that plane, then the force can be resolved into two forces *Ss* and $S\sigma$, of which one *Ss* shall be in the plane normal to *ID*, and the other $S\sigma$ now shall be parallel to the line *ID*.



[Note that often Euler represents a force by a line, one end of which is attached to the point of application on the body signified by a capital letter, while the other and usually free end is given an italic small letter; in addition, lines such as *Ss* are regarded as normal to the plane of the diagram/page/table, as indicated.]
But at a certain fixed point *D* there is put in place a plane normal to the line *ID*, and with the line *ISE* drawn, in place of the force *Ss*, the parallel forces *Ii* and *Ee* can be substituted at the points *I* and *E*, so that

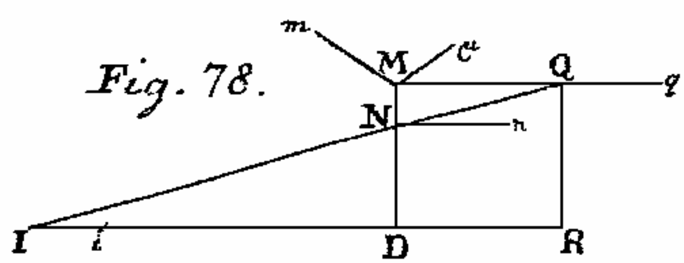
$$\text{force } Ii = \text{force } Ss \cdot \frac{DR}{ID}, \text{ and the force } Ee = \text{force } Ss \cdot \frac{IR}{ID};$$

in a similar manner, in place of the force $S\sigma$, the equivalent parallel forces $I\eta$ and $E\varepsilon$ are substituted, so that

$$\text{force } I\eta = \text{force } S\sigma \cdot \frac{DR}{ID}, \text{ and the force } E\varepsilon = \text{force } S\sigma \cdot \frac{IR}{ID}.$$

With such resolution put in place for all the forces acting, and from the individual forces, the two forces applied at the centre of inertia *I* are obtained, then also now the two forces *Ee* and $E\varepsilon$, the one placed in the plane normal to the axis *ID* at *D*, and the other normal to the plane or parallel to the axis *ID*.

Fig. 78.



[Now in general,] with all the forces which are applied to the centre of inertia *I* collected into one, equally all the forces *Ee* which are in the same plane, can be collected in one force, which shall be the force *Mm* (Fig. 78) ; and in a like manner all the forces $E\varepsilon$, which are parallel to each other, can be collected also into a single force, which shall be the force *Nn*, likewise parallel to the axis *ID*, just as that force *Mm* is turning in the plane *mMD* normal to the axis. In this manner, in place of all the forces acting, however many there should be, we obtain three forces, one applied to the centre *I* itself and two *Mm* and *Nn*, which three moreover can be reduced to two in this manner : The line *IN* is produced to *Q*, while the distance *QR* of this from the axis is made equal to the distance *DM* , drawn from *D* through *N* as far as crossing the force *Mm*, and then $ID : IR = DN : DM$. Then in place of the force *Nn* it is allowed to substitute the forces *Ii* and *Qq* parallel to this, so that

$$\text{force } Ii = \text{force } Nn \cdot \frac{MN}{DM}, \text{ and the force } Qq = \text{force } Nn \cdot \frac{DN}{DM}.$$

The first force meets with the rest of the applied forces in one point at the centre of inertia, now the latter Qq along its own direction can be considered to be applied at the point M and thus with the force Mm can become one force, which shall be the force $M\mu$, thus so that now all the forces acting are reduced to two, the one applied through the centre of inertia I, and the other the force $M\mu$.

COROLLARIUM 1

621. Because it is allowed to choose freely both the axis ID as well as the point D on that, the forces acting are able to be reduced to two forces of this kind in an infinite number of ways, of which one is applied through the centre of mass.

COROLLARY 2

622. But by one reduction of this kind, from the same principle, in place of the force $M\mu$ two other parallel forces can be substituted, of which one affects the centre of inertia I, and the other is applied now to some other line IM at some at some point, from which it is apparent that all forces can be reduced to being referred to the same line IM.

SCHOLIUM

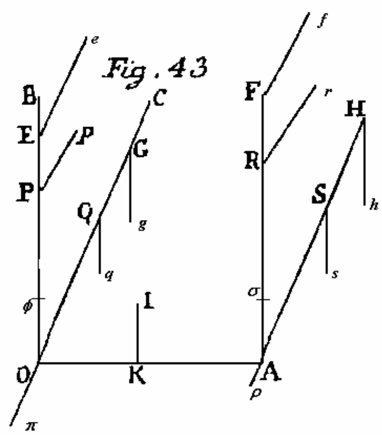
623. This theorem is of the greatest importance in the argument to be presented in this chapter, where our intention is to inquire into the generation of motion at first, when a free rigid body at rest is acted upon by some forces. For since these forces, however many there should be, also are able always to revert to two forces, one of which is applied to the centre of inertia, the effect of which is most easily determined; the whole business reduces to this, that the effect of a single force applied at the centre of inertia can be combined with some other force, and if the effect thus produced jointly can be designated, then the whole exercise [in determining the initial motion] will be accomplished. Therefore in the first place we shall consider how two forces of this kind should be prepared, so that from these a motion is impressed on a body about a given axis passing through the centre of inertia; for from this excellent idea it will be easy to pursue our intentions.

PROBLEM 57

624. To define two forces to be applied to a rigid body, of which the direction of one passes through the centre of inertia of the body, so that the body begins to turn about an axis through the centre of inertia of the body, due to the action of these.

SOLUTION

The centre of inertia lies at the point *O*, and let *OA* be the axis, about which the rotary motion must be produced ; and it is necessary that the forces acting are to be prepared thus, so that the axis experiences no force from these. Hence this problem is contained in the solution to problem 18 above (§390), where in the scholium §394 it is required to determine the forces generally shown, in order that for the end *O* all the forces are applied at the point *O* itself (Fig. 43). [Also, see the note at the end of § 357, Ch.3]Therefore putting the forces $Pp = 0$ and $Qq = 0$, from which on account of $KI = 0$ and $OK = 0$ the forces becoming



$$O\pi = \frac{\int xydM}{ab} \text{ and } O\varphi = \frac{\int xzdM}{ab}.$$

Then for the end *A*, in taking the forces $A\rho = 0$ and $A\sigma = 0$, the forces become :

$$Rr = \frac{\int xydM}{ab} \text{ and } Ss = \frac{\int xzdM}{ab},$$

thus in order that :

$$\text{the force } Rr = \text{force } O\pi \text{ and the force } Ss = \text{force } O\varphi;$$

as now it is required, that

$$AR\int xydM + AS\int xzdM = a\int rrdM.$$

Because it is possible to assume the plane *OAR* as it pleases, on account of the centre of inertia *I* put at O, then the integral becomes

$$\int xzdM = 0,$$

and hence only two forces remain satisfying the problem, one force

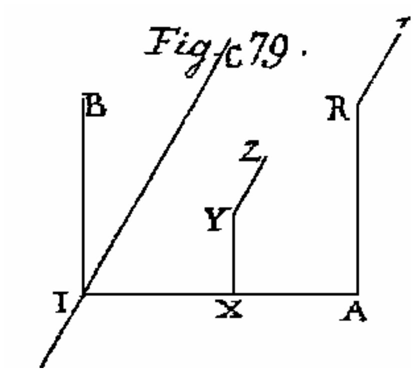
$$O\pi = \frac{\int xydM}{ab}$$

applied to the centre of inertia, and the other force

$$Rr = \frac{\int xydM}{ab}$$

applied at a distance from the axis

$$AR = \frac{a\int rrdM}{\int xydM}.$$



Hence the solution of the problem thus is reached: with a proposed axis of rotation *IA* of this kind, two directrices *IB* and *IC* are taken together (Fig. 79), so that for any element of the body *dM* with these parallel coordinates established $IX = x$, $XY = y$ and $YZ = z$ and on putting $XZ = \sqrt{(yy + zz)} = r$, making $\int xzdM = 0$. Moreover on taking the interval as it pleases $IA = a$, and

$$AR = \frac{a\int rrdM}{\int xydM} \text{ itself parallel to } IB,$$

then some force *Rr* itself parallel to *IC* and applied at the point *R* produces the proposed effect, but only if above an equal and opposite force to $I\pi$ is applied to the centre of inertia I; and with these in place, with the forces $Rr = I\pi = V$, since the moment with respect to the axis *IA* thus arising equal to

$$\frac{Va\int rrdM}{\int xydM},$$

in the element of time *dt* about the axis *IA*, the angle is generated :

$$d\omega = \frac{Vagdt^2}{\int xydM}.$$

COROLLARY 1

625. Since the interval $IA = a$, on which the distance *AR* depends, can be taken as you please, all the points *R* are found on the right line *IR* making an angle with the axis *IA*, the tangent of which is equal to $\frac{\int rrdM}{\int xydM}$, provided the plane *AIB* is thus taken so that it makes $\int xzdM = 0$.

COROLLARY 2

626. With this right line *IR* drawn some force applied to this line at some point and normal to the plane *AIB*, if at *I* an equal and opposite force to that above $I\pi$ can be applied, then the body begins to turn about the axis *IA*.

COROLLARY 3

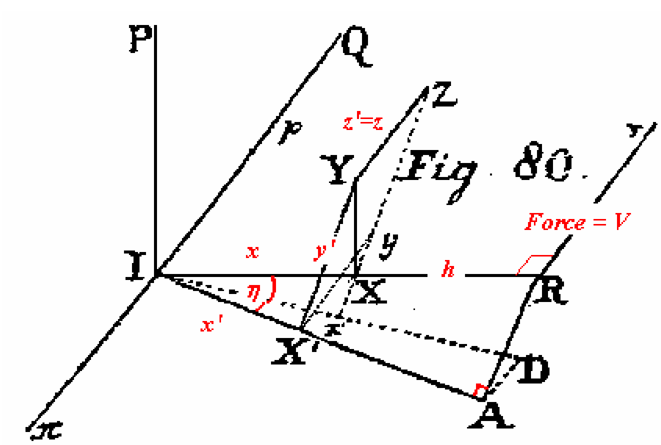
627. But for any proposed force *Rr*, if to that an equal and opposite force is applied at *I*, the body begins to turn about some axes passing through the centre of inertia, from which it is apparent only that it is placed in a plane through the centre of inertia *I* drawn normally to the direction of the force acting *Rr*.

PROBLEM 58

628. If a rigid body at rest is acted on by some force, to determine at first the initial motion, which is generated in the body by that force about an axis in a plane placed normal to the direction of the force, if indeed that can occur.

SOLUTION

Let *I* be the centre of inertia of the body (Fig. 80), through which a plane is considered drawn normal to the direction of the force, which is referred to as the plane of the diagram, therefore the force acting *Rr* = *V* is understood to be set up normal to this, and the line *IR* is normal to that force, which is put as *IR* = *h*. There is applied to the above body at the centre of inertia a force $I\pi$ equal and opposite to that



force *V*, thus so that it is normal to the plane of the diagram from the opposite side. From these two forces acting simultaneously, the body begins to rotate about some axis passing through the centre of inertia, and from the preceding paragraph it is apparent that this axis is placed in the plane of the diagram, which therefore shall be the axis *IA*, for which the position is required: thus putting the angle $RIA = \eta$, so that with the normal *RA* drawn from R to that axis, then $RA = h\,sin\,\eta$ and $IA = h\,cos\,\eta$. Moreover since we do not yet know the position of this axis, we refer the individual elements of the body to the three directrices *IR*, *IP*, *IQ*, the first of which is given from the direction of the given force acting, the other *IP* in the plane of the diagram is normal to that, and the third *IQ* itself is set up normal to this plane. Hence the coordinates along these three directrices are $IX = x$, $XY = y$ and $YZ = z$. Then it is agreeable to obtain coordinates from the above formulas [referring to the axis *IA*], from *Y* to the axis of rotation *IA* there is drawn the normal *YX'*, and let these coordinates be :

$$IX' = x',\ X'Y = y' \text{ and } YZ = z' = z \text{ as before,}$$

of which the first two are determined from the preceding thus, so that

$$x' = x\,cos\,\eta - y\,sin\,\eta \quad \text{and} \quad y' = x\,sin\,\eta + y\,cos\,\eta.$$

But from these it is necessary that there is placed $\int x'\,zdM = 0$ and

tang *AIR* = tang $\eta = \frac{\int rrdM}{\int x'\,y'\,dM}$ (from § 625) on taking $rr = y'\,y' + zz$.

But now,

$$\int x'\,zdM = cos\,\eta \int xzdM - sin\,\eta \int yzdM\,,$$

$$\int rrdM = sin^2\,\eta \int xxdM + 2\,sin\,\eta\,cos\,\eta \int xydM + cos^2\,\eta \int yydM + \int zzdM\,,$$

$$\int x'\,y'\,dM = sin\,\eta\,cos\,\eta \int xxdM + \left(cos^2\,\eta - sin^2\,\eta\right) \int xydM - sin\,\eta\,cos\,\eta \int yydM\,.$$

We put in place these extended integrals for the whole body :

$$\int xxdM = A,\quad \int yydM = B,\quad \int zzdM = C,$$
$$\int xydM = D,\quad \int xzdM = E,\quad \int yzdM = F$$

and we have these equations :

$$E\,cos\,\eta - F\,sin\,\eta = 0$$

and

$$A\,sin^2\,\eta + D\left(cos^2\,\eta - sin^2\,\eta\right) \cdot \text{tang}\,\eta - B\,sin^2\,\eta$$

$$= A\,sin^2\,\eta + 2D\,sin\,\eta\,cos\,\eta + B\,cos^2\,\eta + C$$

or $D\ \text{tang}\,\eta + B + C = 0$; thus we obtain in two ways:

$$\text{tang}\,\eta = \tfrac{E}{F} \text{ and } \text{tang}\,\eta = \tfrac{-B-C}{D}\,,$$

which two values unless they are in agreement, the problem under the proposed condition, in which the axis of rotation is assumed in a plane in a direction normal to the direction of the force, cannot be resolved.

Hence we put the force to be applied thus to be, in order that it becomes :

$$\frac{E}{F}+\frac{B+C}{D}=0,$$

and the body begins to rotate about the axis *IA* in a plane placed normal to the direction of the force, so that

$$\text{tang } RIA=\frac{E}{F}=\frac{-B-C}{D}.$$

Then on account of the moment of the force $Vh\sin\eta$ and the moment of inertia about this axis, in the element of time *dt* it turns through the angle

$$d\omega=\frac{Vghdt^{2}\sin\eta}{A\sin^{2}\eta+B\cos^{2}\eta+2D\sin\eta\cos\eta+C}.$$

Which, since the effect of the two forces *Rr* and $I\pi$ jointly acting, in order that the effect of the force *Rr = V* alone, there is added to the above force the force *Ip = V* and there is impressed on the body besides the motion of gyration, a pure progressive motion along the direction *IQ* parallel to *Rr* itself, in which element of time *dt* there is completed the element of distance equal to $\frac{Vgdt^{2}}{M}$.

COROLLARY 1

629. Hence the solution of this problem is extended only to that case, in which the force acting *Rr = V* thus is applied to the body, in order that, from the integration formulas taken together, the condition is satisfied: $\frac{E}{F}+\frac{B+C}{D}=0.$

COROLLARY 2

630. But unless this property is had in place, the solution of the problem may be hidden at this stage and only this is agreed, that the rotation cannot be made about an axis, which is situated in a plane normal to the direction of the force acting.

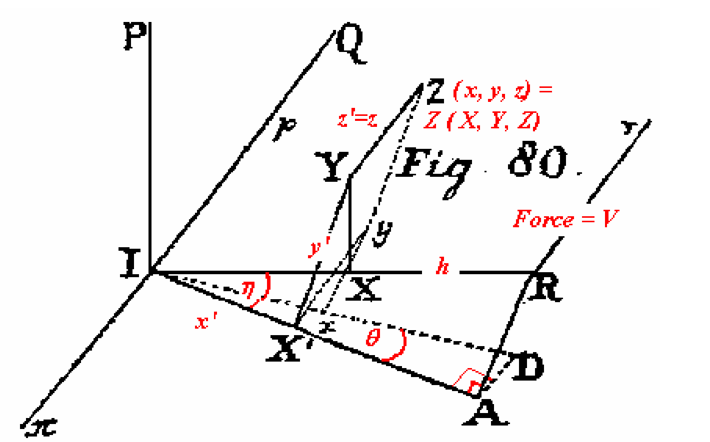
SCHOLIUM

631. Certainly it is considered wonderful because, when the preparation has been found for the forces that the above axes sustains, a complete desired solution is seen to be promised, yet now there is an infinite number of cases excluded, which our solution does not include. For since it is certain that a rotation about any other axis is not possible, unless clearly one which sustains no forces, then problem18 must supply the needs of the perfect solution, if indeed it should be a solution in any extension of the problem. The truth must be noted properly in this problem that no other forces are to be assumed, unless the directions of which can be found in planes normal to the axis, though still forces at an angle can be introduced, provided the forces parallel to the axis thus destroy each other. And in fact this comes about with the use in excluded cases, where the body must begin to make an initial rotation about an axis, which is inclined to the plane drawn normally through the centre of inertia to the force acting, as there then from the resolution of the force *Rr* there arises such a force parallel to the axis ; and certainly this must be taken into account, if we wish to resolve this problem in general.

PROBLEM 59

632. If a rigid body at rest is acted on by some force and likewise an equal and opposite force to that is applied to the centre of inertia, to define the axis about which it first begins to rotate.

SOLUTION



Let *I* be the centre of inertia of the body (Fig. 80), and in the diagram referred to as before, the plane through *I* is drawn normally to the direction of the force acting, which is *Rr* = *V*, in which there is put the distance *IR* = *h*. Then on taking the angle $RIA = \eta$ in this plane, in order that with the perpendicular *RA* drawn from *R* to *IA* then $IA = h\cos\eta$ and $RA = h\sin\eta$, with the normal *AD* drawn from *A* to the plane, and with *ID* drawn let the angle $AID = \vartheta$ and thus

$$AD = h\cos\eta \operatorname{tang}\vartheta \text{ and } ID = \frac{h\cos\eta}{\cos\vartheta};$$

the axis of rotation sought is thus the line *ID*, so that it is required to investigate both the angles η and ϑ. Hence the body must be expressed by three coordinates, of which one is taken on the axis itself *ID*. Therefore I assume a relation to be given between the coordinates *IX* = *x*, *XY* = *y* and *YZ* = *z*, of which the first [*x*] is taken on the line *IR* itself, the second [*y*] in the plane normal to the force and the third [*z*] taken parallel to the force *Rr*. From *Y* first to *IA* the perpendicular *YX'* is drawn, moreover in the plane *AID* normal to the diagram the perpendicular *X'y* is parallel to *YZ* and *yZ* is parallel to *X'Y*, then as we have seen before [Euler has the habit of using the same letter for a point on the diagram as for a length, as he does here with *x* and *y*. It should be noted in Fig. 80, that *yZ* is parallel to *X'Y*, and *YZ* is parallel to *X'y*, which is hence a vertical parallelogram, with *yZ* the upper horizontal side.]:

$$IX' = x\cos\eta - y\sin\eta,$$

$$X'y = YZ = z,$$

$$X'Y = yZ = x\sin\eta + y\cos\eta.$$

[This is an initiation into Euler Angles, in which rotations occur about 2 axes only here; the third axis is an axis of symmetry in this case and is not of concern. Thus, initially there is a rotation about *IQ* in the plane of the diagram through the angle η, corresponding to :

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

this is followed by a rotation about an axis normal to IX' through the angle ϑ :

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

from which we have :

$$\begin{pmatrix} \cos\eta\cos\vartheta & -\sin\eta\cos\vartheta & \sin\vartheta \\ -\cos\eta\sin\vartheta & \sin\eta\sin\vartheta & \cos\vartheta \\ \sin\eta & \cos\eta & 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ -\sin\vartheta & 0 & \cos\vartheta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

in which we note that one of the rotations is in the opposite sense to the other.]

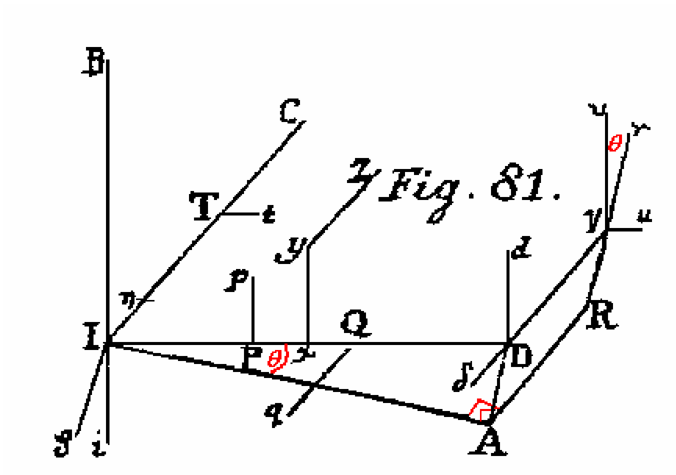
Then in the normal plane from *y* to *ID* the perpendicular *yx* is drawn, and there are new coordinates had, such as we desire, which are *Ix* = *X*, *xy* = *Y* and *yZ* = *Z*, and these are determined from the preceding thus

$$X = x\cos\eta\cos\vartheta - y\sin\eta\cos\vartheta + z\sin\vartheta,$$

$$Y = z\cos\vartheta - x\cos\eta\sin\vartheta + y\sin\eta\sin\vartheta,$$

$$Z = x\sin\eta + y\cos\eta.$$

[Clearly the point (*X*, *Y*, *Z*) is equivalent to (*x''*, *y''*, *z''*) above.]
These coordinates can now be projected onto the plane *IAD* of the diagram, shown



in figure 81, to which $AR = h\sin\eta$ now is normal to *AD*, and the force *Rr* is parallel to *AD*; *DV* is drawn parallel to *AR*, and the force is considered to be applied at the point *V*, so that *Vr* = *V*, and with *Vv* drawn parallel to *xy* and *Vu* parallel to *Ix*, from the angle $rVv = \vartheta$, the force *Vr* is resolved into these two : $Vv = V\cos\vartheta$ and $Vu = V\sin\vartheta$, which are applied to the opposite point *I*, the first of which is the force $Ii = V\cos\vartheta$, applied to *ID* now normal in the plane of the table, now the other acts along the axis *DI* . But the moment of the force $Vv = V\cos\vartheta$ about the axis *ID* is equal to
$V\cos\vartheta\cdot h\sin\eta = Vh\sin\eta\cos\vartheta$,
and on putting *YY* + *ZZ* = *RR* , the moment of inertia of the body about the axis $ID = \int RRdM$, from which in the time *dt* the body is turned through the angle

$$d\omega = \frac{Vghdt^2\sin\eta\cos\vartheta}{\int RRdM}.$$

Since the axis must experience no forces, the force $Vv = V\cos\vartheta$ is applied to the axis itself at *D*, in order that the force $Dd = V\cos\vartheta$; now the force $Vu = V\sin\vartheta$ is considered to be applied along its own direction to the perpendicular $IT = DV = h\sin\eta$, from which the first force arising for the axis acts along *ID* , which that above cancels; then now with the interval in place

$$ID = \frac{h\cos\eta}{\cos\vartheta} = a$$

two forces arise normal to the axis and to the plane of the diagram :

$$I\eta = D\delta = \frac{h\sin\eta}{a}V\sin\vartheta = V\ \text{tang}\,\eta\ \sin\vartheta\cos\vartheta$$

Now in addition the forces are obtained $Ii = Dd = V\cos\vartheta$, which must be cancelled by the elementary forces. But from problem 16, the elementary forces here in place provide the two forces *Pp* and *Qq* applied at the two points *P* and *Q*, in order that the force

$$IP = \frac{\int XZdM}{\int ZdM}, \quad \text{and the force } Pp = \frac{Vh\sin\eta\cos\vartheta\cdot\int ZdM}{\int RRdM},$$
$$IQ = \frac{\int XYdM}{\int YdM}, \quad \text{and the force } Qq = \frac{Vh\sin\eta\cos\vartheta\cdot\int YdM}{\int RRdM}.$$

But since the motion here starts in the contrary sense, and we have assumed here, these forces must be put in place equivalent to the preceding forces; and since on account of the centre of inertia I then $\int YdM = 0$ and $\int ZdM = 0$, are satisfied by moments, in order that it must be the case that
$$Pp\cdot IP = Dd\cdot ID \text{ and } Qq\cdot IQ = D\delta\cdot ID,$$
and thus we have these two equations :

$$\frac{Vh\sin\eta\cos\vartheta\int XZdM}{\int RRdM} = Vh\cos\eta$$
and
$$\frac{Vh\sin\eta\cos\vartheta\int XYdM}{\int RRdM} = Vh\sin\eta\sin\vartheta$$
or
$$\sin\eta\cos\vartheta\int XZdM = \cos\eta\int RRdM$$
and
$$\cos\vartheta\int XYdM = \sin\vartheta\int RRdM.$$
Therefore now with the integrals in place, from the principal coordinates x, y and z there are produced :

$$\int xxdM = A, \quad \int yydM = B, \quad \int zzdM = C,$$
$$\int xydM = D, \quad \int xzdM = E, \quad \int yzdM = F,$$

and since $RR = YY + ZZ$ then there is

$$\int RRdM = A\left(\sin^2\eta + \cos^2\eta\sin^2\vartheta\right) + B\left(\cos^2\eta + \sin^2\eta\sin^2\vartheta\right) + C\cos^2\vartheta$$
$$+2D\sin\eta\cos\eta\cos^2\vartheta - 2E\cos\eta\sin\vartheta\cos\vartheta + 2F\sin\eta\sin\vartheta\cos\vartheta,$$

$$\int XYdM = -A\cos^2\eta\sin\vartheta\cos\vartheta - B\sin^2\eta\sin\vartheta\cos\vartheta + C\sin\vartheta\cos\vartheta$$
$$+2D\sin\eta\cos\eta\sin\vartheta\cos\vartheta + E\cos\eta\left(\cos^2\vartheta-\sin^2\vartheta\right)$$
$$-F\sin\eta\left(\cos^2\vartheta-\sin^2\vartheta\right),$$

$$\int XZdM = A\sin\eta\cos\eta\cos\vartheta - B\sin\eta\cos\eta\cos\vartheta + D\cos\vartheta\left(\cos^2\eta-\sin^2\eta\right)$$
$$+E\sin\eta\sin\vartheta + F\cos\eta\sin\vartheta,$$

with which values substituted, the two equations found are put into these forms :

$$\text{I.}\quad -A\cos\eta\sin^2\vartheta - B\cos\eta - C\cos\eta\cos^2\vartheta - D\sin\eta\cos^2\vartheta$$
$$+E\left(1+\cos^2\eta\right)\sin\vartheta\cos\vartheta - F\sin\eta\cos\eta\sin\vartheta\cos\vartheta = 0,$$
$$\text{II.}\quad -A\sin\vartheta - B\sin\vartheta + E\cos\eta\cos\vartheta - F\sin\eta\cos\vartheta = 0,$$

of which the latter produces

$$\text{tang }\vartheta = \frac{E\cos\eta - F\sin\eta}{A+B}.$$

But $\text{II}\cdot\cos\eta\sin\vartheta - \text{I}$ produces

$$B\cos\eta\cos^2\vartheta + C\cos\eta\cos^2\vartheta + D\sin\eta\cos\vartheta - E\sin\vartheta\cos\vartheta = 0,$$

thus gathering together,

$$\text{tang }\vartheta = \frac{(B+C)\cos\eta + D\sin\eta}{E};$$

and hence finally :

$$\text{tang }\eta = \frac{EE-(A+B)(B+C)}{(A+B)D+EF},$$

thus both the angles $RIA = \eta$ and $AID = \vartheta$, and thus the axis of rotation *ID* becomes known.

COROLLARY 1

633. Therefore for some proposed force $Rr = V$, to this likewise an equal and opposite force is applied at the centre of inertia *I*, if through *I* the plane *PIR* is drawn normal to the direction of the force and with the line of the perpendicular *IQ*, from these three directrices *IR*, *IP*, *IQ* for some element of the body *dM* in place at *Z* the parallel coordinates are taken $IX = x$, $XY = y$ and $YZ = z$, and hence from the nature of the body the following six values can be gathered together in the following :

$$\int xxdM = A, \quad \int yydM = B, \quad \int zzdM = C,$$
$$\int xydM = D, \quad \int xzdM = E, \quad \int yzdM = F,$$

COROLLARY 2

634. From these found in the plane *RIP* in the direction of the normal force from the opposite of the positive coordinates $XY = y$ or in the negative region there is taken the angle $RIA = \eta$, in order that

$$\text{tang } \eta = \frac{EE-(A+B)(B+C)}{(A+B)D+EF},$$

from which found above that plane in the region of the positive coordinates $YZ = z$ there arises the angle $AID = \vartheta$, in order that

$$\text{tang } \vartheta = \frac{E\cos\eta - F\sin\eta}{A+B}$$

or

$$\text{tang } \vartheta = \frac{(A+B)\cos\eta + D\sin\eta}{E},$$

and the line *ID* is the axis of rotation.

COROLLARY 3

635. On putting the distance $IR = h$ then the moment of the force acting about this axis *ID* is equal to $Vh\sin\eta\cos\vartheta$ and the moment of inertia = $\int RRdM$, because that is also equal to

$$\text{tang } \eta\cos\vartheta\int XZdM = \cot\vartheta\int XYdM,$$

the value of this from the preceding is easily elicited : hence in the element of time *dt* the angle turned through becomes

$$d\omega = \frac{Vghdt^2\sin\eta\cos\vartheta}{\int RRdM}.$$

SCHOLIUM

636. Thus behold our general problem, about which the greater part of this chapter revolves, fully solved ; thus indeed the previous case immediately follows, clearly in which the angle $\vartheta = 0$; for then there becomes from the first formula

$$\text{tang}\,\eta = \frac{E}{F}$$

and from the second

$$\text{tang}\,\eta = \tfrac{-B-C}{D},$$

which values unless they agree, that case cannot be put in place. Moreover in turn, if

$$DE + (B+C)F = 0,$$

on account of

$$B + C = \tfrac{-DE}{F}$$

there is made

$$\text{tang}\,\eta = \tfrac{E}{F}$$

and

$$\text{tang}\,\vartheta = 0.$$

Moreover I observe from these, which have been handled for the principal axes, the there is the relation

$$\int XYdM = \frac{-d\cdot\int RRdM}{2d\vartheta} \text{ on taking only } \vartheta \text{ to be variable}$$

and

$$\int XZdM = \frac{-d\cdot\int RRdM}{2d\eta\cos\vartheta} \text{ on taking only } \eta \text{ to be variable.}$$

With which values put in place two principal conditions are postulated :

$$\frac{\sin\eta\cdot d\cdot\int RRdM}{2d\eta} = \cos\eta\int RRdM \quad\text{and}\quad -\frac{\cos\vartheta\cdot d\cdot\int RRdM}{2d\vartheta} = \sin\vartheta\cdot\int RRdM,$$

in the first of which only η, and in the second only ϑ is variable. Therefore the integral of each is the same

$$\int RRdM = \alpha\sin^2\eta\cos^2\vartheta,$$

thus in turn I conclude that the angles η and ϑ thus are required to be defined, in order that the quantity

$$\frac{\sin^2\eta\cos^2\vartheta}{\int RRdM}$$

becomes a minimum, because hence the same two equations arise to be solved. But the same formula arises, if $d\omega^2\int RRdM$ or $\int dM\cdot RRd\omega^2$ is given a minimum, in which since $Rd\omega$ denotes the speed of the element dM and therefore $dM\cdot RRd\omega^2$ the *vim vivam* [the *vis viva* or living force in the acc. case], as it is called, hence we gather together this conspicuous theorem :

THEOREM 8

637. If a rigid body at rest is acted on by some force and from that above there is applied an equal and opposite force at the centre of inertia, and to that body about an axis of the same kind passing through the centre of inertia there is impressed a rotational motion, in order that the whole body thus is given the minimum amount of vim vivam [double the rotational kinetic energy in modern terms],which is the sum of all the elements by the squares of their speeds acquired multiplied together.

DEMONSTRATION

For whatever axis is taken passing through the centre of inertia, with respect to this the proposed force V shall have a certain moment, which shall be Vf, then also now with respect to this axis the body is given a certain moment of inertia, which shall be equal to $\int RRdM$, and each depending on the position of the axis ; hence moreover in an element of time dt there is generated about this axis the angle

$$d\omega = \frac{Vfgdt^2}{\int RRdM}$$

and the infinitely small angular velocity is given by

$$\gamma = \frac{2Vfgdt}{\int RRdM},$$

hence the speed of the element dM at a distance R from the axis shall be equal to $R\gamma$ and thus the vis viva $= R^2\gamma^2 dM$. Hence the total vis viva acquired in the element of time dt is equal to :

$$\gamma\gamma \int R^2 dM = \frac{4VVffggdt^2}{\int RRdM},$$

which as Vg and dt are constant will be $\frac{ff}{\int RRdM}$ is reduced to a minimum, and from this condition the position of the axis can be determined. Hence moreover the determination of the same axis results, as we have found before, thus so that from this principal of the minimum vis viva the same solution can be elicited.

SCHOLIUM

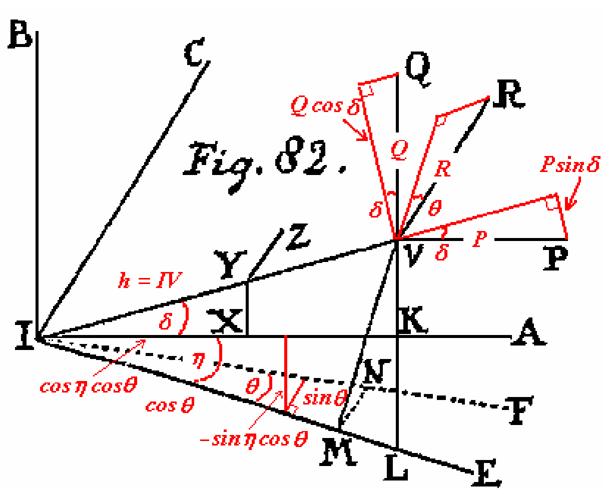
638. In order that the use of the solution found before could be retained, up to this stage this has not been difficult, because for each force acting the character of the body must be recalled according to its own particular coordinates. A remedy to that disadvantage is produced by these, which we have shown from the principal axes of the body, about which if the moment of inertia should once be found, then the moments of inertia about any other axes can be found easily.

And even for the present circumstances it suffices for the relation of the body to the coordinates from the principal axes to be put in place, since hence the relation to any other three coordinates can be derived. On account of which I can resolve the above problem thus, so that I can assume the force to be given acting about some principal axis ; and I can reach for the solution itself from the principal now established, that generates the minimum vis viva.

PROBLEM 60

639. For the given principal axes of a rigid body and the moments of inertia about these, if that is acted on by some force and likewise there is applied another equal and opposite force to the centre of inertia, to define the axis, about which the body begins to rotate.

SOLUTION

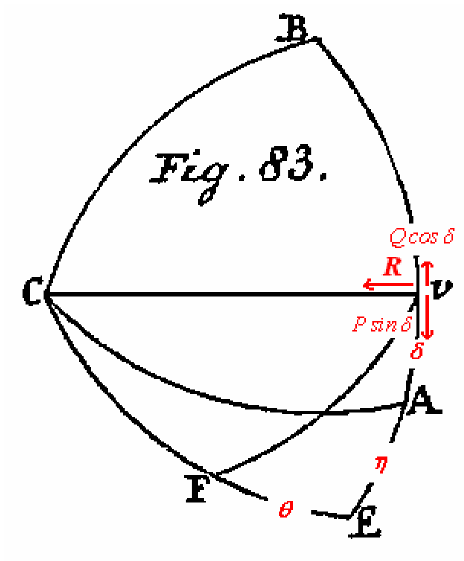


Let I be the centre of inertia of the body and the right lines IA, IB, IC the three principal axes of this (Fig. 82), about which the moments of inertia are Maa, Mbb, Mcc. Now the body is acted on by some force, the passage of this through the plane AIB is known from two of the principal axes, which shall be at the point V, distant from I by the interval $IV = h$ with the angle present $AIV = \delta$; moreover the force itself, as if it should be applied to this point, can be resolved into three forces parallel to the axes, which are the forces $VP = P$, the force $VQ = Q$ and the force $VR = R$, to which therefore three equal and opposite forces have been understood to be applied at the point I. Therefore from these the body begins to turn about some axis passing through the centre of inertia I, which shall be IF inclined to the plane BIA at the angle $FIE = \vartheta$, with the angle present $AIE = \eta$, which two angles it is required to find. Now in the first place the moment of inertia is sought about the axis IF, which, since we have $\cos AIF = \cos\eta\cos\vartheta$, $\cos BIF = -\sin\eta\cos\vartheta$, $\cos CIF = \sin\vartheta$, then from above, this becomes

$$M\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right).$$

Then the moments of the forces *P, Q, R* about this axis *IF* are to be found; but from before it is apparent with *VM* drawn normal to *IE* , in order that $VM = h\sin(\delta+\eta)$, the moment of the force *VR* = *R* is equal to

$$R\cdot VM\cdot\cos\vartheta = Rh\sin(\delta+\eta)\cos\vartheta.$$



[Note that the components of the forces generating moments about the axis *IF* must be perpendicular to *IF*. This task is much simplified by considering the forces to act on the surface of a sphere as follows :] Now, in order that the moments of the remaining forces are able be found easily, the points *V, A, B, C, E, F* may be considered on a spherical surface, the centre of which is at the point *I* (Fig. 83). Hence the arcs *AB, AC* and *BC* are quadrants, and $AV = \delta, AE = \eta, EF = \vartheta$; and the forces *P, Q, R* applied at *V* are resolved in pairs, of which the first of each is normal to the surface of the sphere, and the second of each is tangential to the spherical surface, where the former passing through the centre bear no moments, and thus it is sufficient to consider the latter, which are :

the force along $VA = P\sin AV$,

the force along $VB = Q\sin BV$,

and

the force along $VC = R\sin CV = R$

on account of the quadrant *CV*. These forces are again resolved along the direction *VF* and in the other at right angles to that, where the first placed with the axis *IF* in the same plane bear no moment, but the other forces will be

$$P\sin AV\sin AVF - Q\sin BV\sin BVF - R\sin CVF,$$

the direction of which, since it shall be normal to the plane *IFV*, is also in the plane normal to the axis *IF* , thus, since the distance from the axis is equal to $h\sin FV$, since $AV = \delta$ and $\sin BVF = \sin AVF$, the moment sought is equal to

$$h\left(\left(P\sin\delta - Q\cos\delta\right)\sin AVF\ \sin FV - R\cos AVF\cdot\sin FV\right)$$

but [from the sine rule for spherical triangles :]

$$\sin AVF\cdot\sin FV = \sin FE = \sin\vartheta,$$

and if the moment is given

$$Ph\sin\delta\sin\vartheta - Qh\cos\delta\sin\vartheta - Rh\cos AVF\cdot\sin FV,$$

then from the sphere there is :

$$\cos AVF\cdot\sin FV = \sin\left(\delta+\eta\right)\cos\vartheta,$$

thus, so that the moment sought is equal to

$$Ph\sin\delta\sin\vartheta - Qh\cos\delta\sin\vartheta - Rh\sin\left(\delta+\vartheta\right)\cos\vartheta,$$

from which the angle arising in the element of time *dt* becomes :

$$d\omega = \frac{ghdt^2\left(P\sin\delta\sin\vartheta - Q\cos\delta\sin\vartheta - R\sin\left(\delta+\eta\right)\cos\vartheta\right)}{M\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right)}.$$

On account of which this expression must return a minimum

$$\frac{\left(\left(P\sin\delta - Q\cos\delta\right)\sin\vartheta - R\sin\left(\delta+\eta\right)\cos\vartheta\right)^2}{aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta},$$

[naively, we may consider this expression as the square of the torque applied divided by the moment of inertia :

torque $= \tau = Fr = I\alpha;$ hence $\tau^2 = \left(I\alpha\right)^2 / I;$ This last quantity is proportional to the *vis viva* or the kinetic energy, on taking the angular velocity arising proportional to the angular acceleration α, which is now minimised]
in the first place we place only ϑ to be variable, and it becomes :

$$\begin{aligned}&2\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right)\\ &\qquad\left(\left(P\sin\delta - Q\cos\delta\right)\cos\vartheta + R\sin\left(\delta+\eta\right)\sin\vartheta\right) =\\ &2\left(-aa\cos^2\eta\sin\vartheta\cos\vartheta - bb\sin^2\eta\sin\vartheta\cos\vartheta + cc\sin\vartheta\cos\vartheta\right)\\ &\left(\left(P\sin\delta - Q\cos\delta\right)\sin\vartheta - R\sin\left(\delta+\eta\right)\cos\vartheta\right),\end{aligned}$$

which is reduced to this form :

$$(P\sin\delta - Q\cos\delta)\left(aa\cos^2\eta + bb\sin^2\eta\right)\cos\vartheta + Rcc\sin(\delta+\eta)\sin\vartheta = 0,$$

hence there arises :

$$\text{tang }\vartheta = \frac{(Q\cos\delta - P\sin\delta)\left(aa\cos^2\eta + bb\sin^2\eta\right)}{Rcc\sin(\delta+\eta)} = 0.$$

Now on taking η to be variable, we obtain :

$$2\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right)\left(-R\sin(\delta+\eta)\cos\vartheta\right) =$$
$$2\left(-aa\sin\eta\cos\eta\cos^2\vartheta + bb\sin\eta\cos\eta\cos^2\vartheta\right)$$
$$\left((P\sin\delta - Q\cos\delta)\sin\vartheta - R\sin(\delta+\eta)\cos\vartheta\right),$$

which is reduced to this form :

$$R\cos\vartheta\left(aa\cos\delta\cos\eta\cos^2\vartheta - bb\sin\delta\sin\eta\cos^2\vartheta + cc\cos(\delta+\eta)\sin^2\vartheta\right) =$$
$$(Q\cos\delta - P\sin\delta)(bb-aa)\sin\eta\cos\eta\sin\vartheta\cos^2\vartheta,$$

where in place of $Q\cos\delta - P\sin\delta$ there is put

$$\frac{Rcc\sin(\delta+\eta)\text{tang }\vartheta}{aa\cos^2\eta + bb\sin^2\eta},$$

with which reduction accomplished, this equation is arrived at :

$$\cos^2\vartheta(aa\cos\delta\cos\eta - bb\sin\delta\sin\eta)\left(aa\cos^2\eta + bb\sin^2\eta\right)$$
$$+cc\sin^2\vartheta(aa\cos\delta\cos\eta - bb\sin\delta\sin\eta) = 0,$$

which divided by $aa\cos\delta\cos\eta - bb\sin\delta\sin\eta$ produces

$$\cos^2\vartheta\left(aa\cos^2\eta + bb\sin^2\eta\right) + cc\sin^2\vartheta = 0$$

an impossible equation on account of all the positive parts. Whereby with the divisor used we obtain the determination of the angle η, clearly

$$\text{tang }\eta = \frac{aa\cos\delta}{bb\sin\delta};$$

from which again it is deduced :

$$\text{tang }\vartheta = \frac{(Q\cos\delta - P\sin\delta)aabb}{Rcc\sqrt{\left(a^4\cos^2\delta + b^4\sin^2\delta\right)}}$$

or, lest doubt remains in ambiguity of the sign of the root,

$$\text{tang } \vartheta = \frac{(Q\cos\delta - P\sin\delta)aa\cos\eta}{Rcc\sin\delta} = \frac{(Q\cos\delta - P\sin\delta)bb\sin\eta}{Rcc\cos\delta}.$$

Now with this axis found, if again on substituting the above value $Q\cos\delta - P\sin\delta$ the moment of the forces acting is deduced about this axis, equal to

$$\frac{Rh\sin(\delta+\eta)\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right)}{\left(aa\cos^2\eta + bb\sin^2\eta\right)\cos\vartheta},$$

thus the element of the angle $d\omega$ that is generated in the time dt about the axis will be :

$$d\omega = \frac{Rghdt^2\sin(\delta+\eta)}{M\cos\vartheta\left(aa\cos^2\eta + bb\sin^2\eta\right)} = \frac{Rghdt^2\sqrt{\left(a^4\cos^2\delta + b^4\sin^2\delta\right)}}{Maabb\cos\vartheta},$$

if in $\sin(\delta+\eta)$ in place of the angle η the value found is substituted.

COROLLARY 1

640. Therefore from the point *V* (Fig. 82), in which direction the force acting in the plane *AIB* acts, at once there is found in the same plane the line *IE*, to which the axis of rotation *IF* hangs over; for on putting the angle $AIV = \delta$ then

$$\text{tang } AIE = \text{tang } \eta = \frac{aa\cos\delta}{bb\sin\delta}$$

which does not depend on the direction of this force.

COROLLARY 2

641. Whereby, if the force acting passes through the principal axis *IA*, then the angle *AIE* becomes right and the axis of rotation *IF* will be in a plane normal to *IA*. But on account of $\delta = 0$ and $\eta = 90^{\circ}$ then there is

$$\text{tang } EIF = \text{tang } \vartheta = \frac{Qbb}{Rcc}.$$

COROLLARY 3

642. If the moments of inertia about the axes *IA* and *IB* were equal, then

$$\text{tang } \eta = \cot\delta = \text{tang}\left(90^{\circ} - \delta\right)$$

and thus the angle *VIE* is right; therefore in this case the axis of rotation *IF* is normal to the line *IV* and $aa = bb$ makes

$$\text{tang } \vartheta = \frac{(Q\cos\delta - P\sin\delta)aa}{Rcc}.$$

COROLLARY 4

643. If the force acting, which is equal to *V* and the direction of this crosses the plane *AIB* at the point *V*, from the resolution of this there arises the forces *P, Q, R*, only that drive the body, induces the motion assigned to that body about the axis found *IF* , now in addition a progressive motion begins along its own direction, from which in the element of time *dt* it completes the increment of length equal to $\frac{Vgdt^2}{M}$.

SCHOLIUM

644. In the solution of this problem it is reasonably pleasing to see, how the calculation, which initially is seen to be not a little intricate, immediately leads to greater simplicity as if it were induced freely, and in which an excellent judgement of the truth can be discerned. Indeed most such useful calculations are taken, that we turn to a happy outcome while in the investigation of the truth, as by straying away from the path of truth we are accustomed to slide into inextricable calculations. And indeed the principle of the minimum, which I have used here, provides an elegant solution, which avoids a great many more intricies, if we had wished to find that from the first principles of mechanics. Hence now the problem by which the present chapter is completed, can be handled by this method.

PROBLEM 61

645. If a rigid body at rest is acted on by some forces, to define the first motion of the elements, which will be produced in that body.

SOLUTION

From theorem VII all the forces acting, however many there should be, are reduced to two, of which the one is applied to the centre of inertia, and the other is directed beyond this centre ; the first of these shall be equal to *S*, and the second equal to *V*. From these two forces found, only the force *V* need be considered, a force equal and opposite to that at the centre is considered to be applied, in order that the centre of inertia even now may be kept at rest. Hence it may be discerned, where the direction of this force *V* passes through some plane between two principle axes of the body, and from the preceding problem there as an axis of rotation is sought [Euler uses the word 'gyration' consistently rather than 'rotation', as we have done in the translation of this work; generally the word *gyration* appears in texts referring to an axis, but then it is not used in phrases such as *gyrational velocity*, etc;] about which the body begins to turn, as an infinitely small angle is produced in the first element of time. But then the above progressive motion may be impressed on the body, to which there is to be found the force, with the other force *V*, acting along its own direction and considered to be applied to the centre of inertia of the body also, so that thus the body is acted on in conjunction with the first force S; and because with each applied to the centre of inertia, thus a pure motion arises, which if combined with the rotation found before, the total effect from the proposed forces is produced.

COROLLARY 1

646. If the force *V* vanishes, that is, if as single force *S* is given applied to the centre of inertia, which is equivalent to all the forces acting, then, as we have seen above now, only an progressive motion is impressed on the body.

COROLLARY 2

647. But if the force *S* should be equal to the force *V*, but in the opposite direction, which happens, if all the forces acting have been prepared, so that they cancel each other out applied along their directions through the centre of inertia, then the centre of inertia remains at rest and only the rotational motion is generated.

COROLLARY 3

648. In all the remaining cases a mixed motion is generated in the body, the one progressive, and the other about some about some axis passing through the centre of inertia of the body; each of which can be considered and determined separately.

SCHOLIUM

649. The same effect is produced by these forces acting, even if the body is [already] turning in a motion, now on account of this motion the mixing together becomes more difficult to know. For if now the body is rotating about a different axis and now it is made to move, not only the angular speed, but also the axis of rotation itself changes, thus so that now it begins to rotate about another axis passing through the centre of inertia. And with this variation of the axis the greatest perturbation of the motion is put in place, to the explanation of which in the first place it is convenient to determine accurately the momentary perturbation of this kind, which argument we present in the following chapter.

CAPUT IX

DE PRIMA MOTUS GENERATIONE IN CORPORIBUS RIGIDIS

THEOREMA 6

616. Si cognitus fuerit effectus duarum virium iunctim agentium in motu generando, quarum altera ipsi centro inertiae sit applicata, alterius etiam seorsim agentis effectus innotescet.

DEMONSTRATIO

A vi corpori in ipso centro inertiae applicata generatur motus progressivus purus, quo singula eius elementa secundum directionem vis per aequalia spatiola promoventur, quae, si vis sollicitans sit = V et massa corporis = M, tempusculo dt sunt = $\frac{Vgdt^2}{M}$. Quodsi iam corpus praeter vim hanc V centro inertiae applicatam sollicitetur ab alia vi quacunque S effectusque harum duarum virium simul agentium fuerit cognitas, res ita concipiatur, quasi corpus insuper a vi contraria ipsi V aequali et cento inertiae applicata sollicitaretur, qua prior effectus ita turbabitur, ut totum corpus motu progressivo secundum directionem huius vis retro feratur per spatiolum $= \frac{Vgdt^2}{M}$, qui effectus cum illo coniunctus dabit effectum solius vis S sollicitantis, qui propterea innotescet.

COROLLARIUM 1

617. Effectus nempe vis S aequalis est effectui a binis viribus V et S simul agentibus producto, demendo hunc effectum, quem sola vis V produceret, secundum ea, quae supra de resolutione motus sunt praecepta.

COROLLARIUM 2

618. Si ergo a duabus viribus V et S simul urgentibus corpori imprimatur motus gyratorius circa quempiam axem, a vi sola S corpori imprimetur motus mixtus ex eodem gyratorio et progressio, qui ipsi a vi ipsi V aequali et contraria induceretur.

SCHOLION

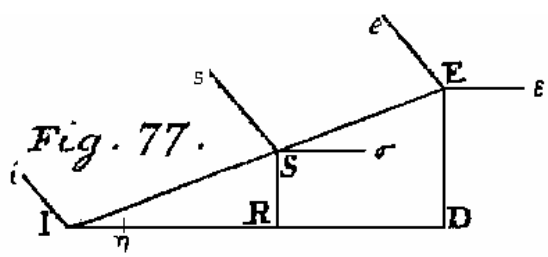
619. Legibus iustae methodi adversari videbitur, quod ex effectu duarum virium simul agentium in effectum unius vis inquirere conemur. Verum in problemate 18, ubi vires definivimus, a quibus axis gyrationis non afficiatur, vidimus has vires rarissime ad unicam, semper autem ad duas reduci posse. quarum ergo effectus in corpus quiescens assignari poterit. Quare, ut unius tantum vis effectum definire valeamus, efficiendum est, ut illarum binarum virium altera per ipsum corporis centrum inertiae transeat, sicque hoc Theorema amplissimum nobis praestabit usum. Quo accedit, ut etiam vires quaecunque corpus sollicitantes ad duas huiusmodi vires reduci queant, quemadmodum iam docebimus.

THEOREMA 7

620. Quotcunque fuerint vires corpus rigidum sollicitantes et quomodocunque fuerint applicatae, eae semper ad duas reduci possunt, quarum altera per ipsum centrum inertiae corporis transeat.

DEMONSTRATIO

Sit *I* centrum inertia corporis (Fig. 77), per quod pro lubitu ducatur recta quaecunque *ID*. Per directionem cuiuslibet vis sollicitantis ducatur planum ad rectam *ID* normale, quod eam secet in puncto *R*; ac, nisi directio huius vis in isto plano sit sita, ea resolvatur in duas vires *Ss* et $S\sigma$, quarum illa sit in plano ad *ID* normali, altera vero $S\sigma$ ipsi rectae *ID* sit parallela.

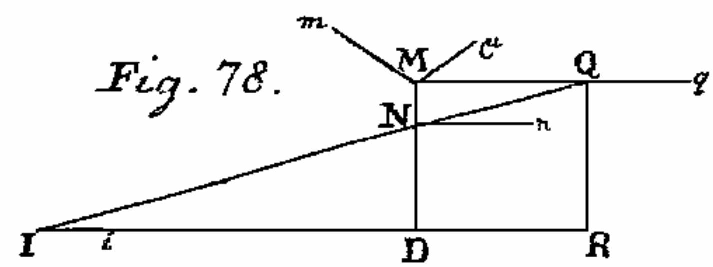


Ad certum punctum fixum *D* statuatur planum ad rectam *ID* normale, ac ducta recta *ISE* loco vis *Ss* in punctis *I* et *E* substitui poterunt vires *Ii* et *Ee* ipsi parallelae, ut sit

$$\text{vis } Ii = \text{vi } Ss \cdot \frac{DR}{ID} \quad \text{et} \quad \text{vis } Ee = \text{vi } Ss \cdot \frac{IR}{ID};$$

simili modo loco vis $S\sigma$ substituantur vires $I\eta$ et $E\varepsilon$ ipsi aequivalentes parallelae, ut sit

$$\text{vis } I\eta = \text{vi } S\sigma \cdot \frac{DR}{ID} \quad \text{et} \quad \text{vis } E\varepsilon = \text{vi } S\sigma \cdot \frac{IR}{ID}.$$



Talis resolutio in omnibus viribus corpus sollicitantibus instituatur, atque ex singulis obtinebuntur binae vires ipsi centro inertiae *I* applicatae, tum vero etiam binae vires Ee et $E\varepsilon$, illa in plano ad axem *ID* in *D* normali sita, haec vero ad istud planum normalis seu axi *ID* parallela. Omnibus viribus, quae centro inertiae *I* applicantur, in unam collectis omnes vires *Ee*, quia in eodem sunt plano, pariter in unam colligi poterunt, quae sit vis *Mm* (Fig. 78) ; similique modo omnes vires $E\varepsilon$, quia sunt inter se parallelae, etiam in unam colligi possunt, quae sit vis *Nn*, axi *ID* itidem parallela, quemadmodum illa *Mm* in plano *mMD* ad axem normali versatur. Hoc modo loco omnium virium sollicitantium, quotcunque fuerint, nanciscimur tres vires, unam ipsi centro inertia *I* applicatam et binas *Mm* et *Nn*, quae tres autem porro ad duas reducentur hoc modo : Producatur recta *IN* in *Q*, donec eius ab axe distantia *QR* aequalis fiat distantiae *DM* ex *D* per *N* ad occursum vis *Mm* usque ductae, eritque $ID : IR = DN : DM$. Tum loco vis *Nn* substituere licebit vires *Ii* et *Qq* ipsi parallelas, ut sit

$$\text{vis } Ii = \text{vi } Nn \cdot \frac{MN}{DM} \quad \text{et} \quad \text{vis } Qq = \text{vi } Nn \cdot \frac{DN}{DM}.$$

Prior cum reliquis centro inertiae applicatis in unam coalescit, posterior vero *Qq* secundum suam directionem in ipso puncto *M* applicata concipi potest sicque cum vi *Mm* pariter uniri potest, quae sit vis $M\mu$, ita ut nunc omnes vires sollicitantes reductae sint ad duas, alteram centro inertiae *I* applicatam, alteram vero istam vim $M\mu$.

COROLLARIUM 1

621. Quoniam tam axem *ID* quam in eo punctum *D* pro lubitu assumere licet, vires sollicitantes infinitis modis ad huiusmodi binas vires, quarum altera ipsi centro inertiae sit applicata, reduci possunt.

COROLLARIUM 2

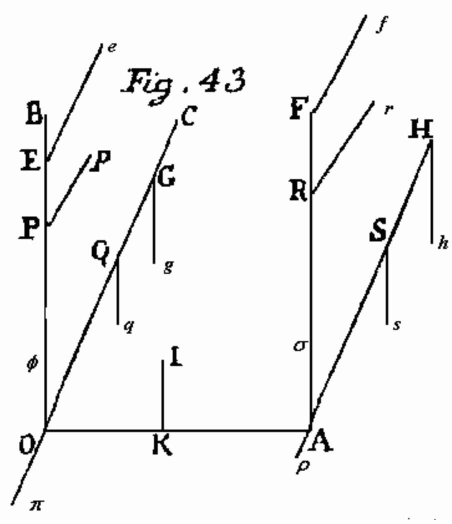
622. Facta autem una huiusmodi reductione per eadem principia loco vis $M\mu$ duae aliae ipsi parallelae substituti possunt, quarum altera centrum inertiae I afficiat, altera vero in puncto quovis alio rectae IM sit applicata, unde patet omnes reductiones ad eundem rectam IM referri.

SCHOLION

623. Theorema hoc maximi est momenti in argumento huius capitis evolvendo, ubi propositum nobis est in primam motus generationem inquirere, quando corpus rigidum quiescens et liberum a viribus quibuscunque sollicitatur. Cum enim hac vires, quotcunque etiam fuerint, semper ad binas revocari queant, quarum altera ipsi centro inertiae sit applicata, huiusque effectus sit determinatu facillimus, totum negotium eo redit, ut effectus ab unica quaecunque centro inertiae applicata combinari poterit, ac si effectus inde coniunctim productus assignari poterit, totum negotium erit confectum. Primum ergo dispiciamus, quomodo duae huiusmodi vires comparatae esse debeant, ut ab iis corpori motus circa datum axem per eius centrum inertiae transeuntem imprimatur; hoc enim praestito facile erit institutum nostrum prosequi.

PROBLEMA 57

624. Definire duas vires corpori rigido applicandas, quarum alterius directio per centrum inertiae transeat, ut corpus ab iis sollicitatum circa datum axem per eius centrum inertiae transeuntem converti incipiat.



SOLUTIO

Incidat centrum inertiae in punctum O sitque OA axis, circa quem motus gyratorius generari debeat; ac necesse est, ut vires sollicitantes ita sunt comparatae, ut axis ab illis nihil patiatur. Hoc ergo problema continetur i problemate 18 supra§ 390 soluto, ubi in scholio § 394 vires generaliter exhibitas ita determinari oportet, ut pro termino O omnes vires ipsi puncto O sint applicatae (Fig. 43). Ponatur ergo vires $Pp = 0$ et $Qq = 0$, unde ob $KI = 0$ aeque ac $OK = 0$ fiet vis

$$O\pi = \frac{\int xydM}{ab} \text{ et vis } O\varphi = \frac{\int xzdM}{ab}.$$

Deinde pro termino A sumantur vires $A\rho = 0$ et $A\sigma = 0$ fientque vires

$$Rr = \frac{\int xydM}{ab} \text{ et } Ss = \frac{\int xzdM}{ab},$$

ita ut sit

$$\text{vis } Rr = \text{vi } O\pi \text{ et vis } Ss = \text{vi } O\varphi;$$

tum vero requiritur, ut sit

$$AR\int xydM + AS\int xzdM = a\int rrdM.$$

Quoniam planum OAR ob centrum inertiae I in O positum pro lubitu assumi potest, id ita assumi poterit, ut fiat

$$\int xzdM = 0,$$

hincque duae tantum supersunt vires problemati satisfacientes, altera vis

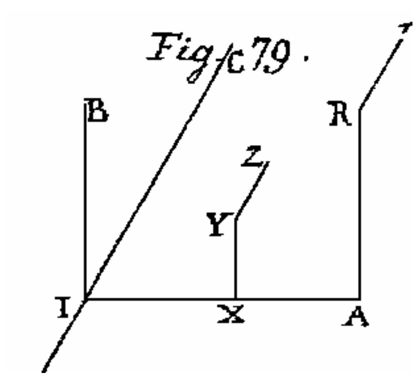
$$O\pi = \frac{\int xydM}{ab}$$

ipsi centro inertiae applicata, altera vis

$$Rr = \frac{\int xydM}{ab}$$

in distantia ab axe

$$AR = \frac{a\int rrdM}{\int xydM}$$



applicanda. Hinc solutionem problematis ita complectemur: cum axe gyrationis proposito IA eiusmodi binae directrices IB et IC coniungatur (Fig. 79), ut constitutis pro quovis corporis elemento dM coordinatis illis parallelis $IX = x$, $XY = y$ et $YZ = z$ positaque $XZ = \sqrt{(yy + zz)} = r$ fiat $\int xzdM = 0$. Tum sumto intervallo pro lubitu $IA = a$ et ipsi IB parallela,

$$AR = \frac{a\int rrdM}{\int xydM},$$

quacunque vis Rr ipsi IC parallela et in puncto R applicata effectum propositum producet, si modo insuper centro inertiae I vis illi aequalis et contrara $I\pi$ applicetur; et positis his viribus $Rr = I\pi = V$, cum momentum respectu axis IA inde natum sit =

$$AR = \frac{Va\int rrdM}{\int xydM},$$

tempusculo dt circa axem IA generalitur angulus

$$d\omega = \frac{Vagdt^2}{\int xydM}.$$

COROLLARIUM 1

625. Cum intervallum $IA = a$, a quo distantia AR pendet, pro lubitu accipi possit, omnia puncta R reperiuntur in linea recta IR faciente cum axe IA angulum, cuius tangens = $\frac{\int rrdM}{\int xydM}$, dummo planum AIB ita sit sumtum, ut fiat $\int xzdM = 0$.

COROLLARIUM 2

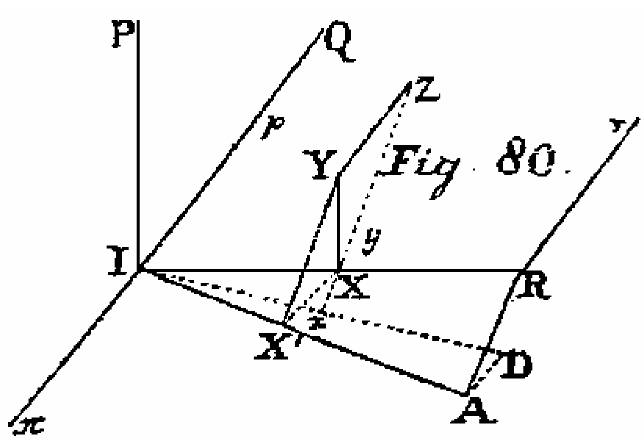
626. Ducta hac recta IR quaelibet vis huic rectae in quovis puncto applicata et ad planum AIB normalis, si in I vis illi aequalis et contraria $I\pi$ insuper applicetur, corpus circa axem IA converti incipiet.

COROLLARIUM 3

627. Proposita autem quacunque vi Rr, cui aequalis in I contrarie sit applicata, corpus circa quempiam axem per centrum inertiae transeuntem verti incipiet, de quo tantum patet, quod situs sit in plano per centrum inertiae I ad directionem vis sollicitantis Rr normaliter ducto.

PROBLEMA 58

628. Si corpus rigidum quiescens a vi quacunque sollicitetur, determinare primum initium motus, qua ab ea vi in corpore generabitur circa axem in plano ad directionem vis normali situm, siquidem fieri queat.



SOLUTIO

Sit I centrum inertiae corporis (Fig. 80), per quod ductum concipiatur planum ad directionem vis normale, quod ipso plano tabulae referatur, cui ergo vis sollicitans $Rr = V$ normaliter insistere est intelligenda, et recta IR ad eam sit normalis, quae ponatur $IR = h$. Applicetur corpori insuper in centro inertiae vis $I\pi$ illi aequalis et contraria, ita ut ex opposito in planum tabulae sit normalis. Ab his duabus viribus simul agentibus corpus circa quempiam axem per centrum inertiae transeuntem converti incipiet atque ex pargrapho praecendete patet hunc axem in plano tabulae fore situm, qui propterea sit IA, pro cuius positione quaeri debet angulus $RIA = \eta$ ita, ut ducta ex R ad eum normalis RA sit $RA = h\sin\eta$ et $IA = h\cos\eta$. Quoniam autem positionem huius axis nondum novimus, referamus singula corporis elementa ad ternas directrices IR, IP, IQ, quarum prima ex directione vis sollicitatis datur, altera IP in plano tabulae ad eam sit normalis ac tertia IQ ipsi huic plano normaliter insistat. Sint ergo coordinatae secundum has ternas directrices $IX = x$, $XY = y$ et $YZ = z$. Deinde ad coordinatas superioribus formulis consentaneas obtinendas ex Y ad axem gyrationis IA ducatur normalis YX', sintque istae coordinatae :

$$IX' = x',\ X'Y = y' \text{ et } YZ = z' = z \text{ ut ante},$$

quarum priores per praecedentes ita determinentur, ut sit

$$x' = x\cos\eta - y\sin\eta \quad \text{et} \quad y' = x\sin\eta + y\cos\eta.$$

Ex his autem necesse est fiat $\int x'zdM = 0$ et

tang AIR = tang $\eta = \dfrac{\int rrdM}{\int x'y'dM}$ (§ 625) existente $rr = y'y' + zz$.

At vero erit

$$\int x'zdM = \cos\eta\int xzdM - \sin\eta\int yzdM,$$
$$\int rrdM = \sin^2\eta\int xxdM + 2\sin\eta\cos\eta\int xydM + \cos^2\eta\int yydM + \int zzdM,$$
$$\int x'y'dM = \sin\eta\cos\eta\int xxdM + \left(\cos^2\eta - \sin^2\eta\right)\int xydM - \sin\eta\cos\eta\int yydM.$$

Ponamus haec integralia per totum corpus extensa :

$$\int xxdM = A,\quad \int yydM = B,\quad \int zzdM = C,$$
$$\int xydM = D,\quad \int xzdM = E,\quad \int yzdM = F$$

atque habebimus has aequationes :

$$E\cos\eta - F\sin\eta = 0$$

et

$$A\sin^2\eta + D\left(\cos^2\eta - \sin^2\eta\right)\cdot \text{tang}\,\eta - B\sin^2\eta$$

$$= A\sin^2\eta + 2D\sin\eta\cos\eta + B\cos^2\eta + C$$

seu $D\ \text{tang}\,\eta + B + C = 0$; unde duplici modo nanciscimur :

$$\text{tang}\,\eta = \frac{E}{F} \text{ et } \text{tang}\,\eta = \frac{-B-C}{D},$$

qui bini valores nisi consentiant, problema sub conditione proposita, qua axis gyrationis in plano ad directionem vis normali assumitur, resolvi nequit.

Ponamus ergo vim ita esse applicatam, ut fiat

$$\frac{E}{F} + \frac{B+C}{D} = 0,$$

atque corpus gyrari incipiet circa axem *IA* in plano ad directionem vis normali situm, ut sit

$$\text{tang}\, RIA = \frac{E}{F} = \frac{-B-C}{D}.$$

Tum ob momentum vis $= Vh\sin\eta$ et momentum inertiae respectu huius axis tempusculo *dt* vertetur per angulum

$$d\omega = \frac{Vghdt^2 \sin\eta}{A\sin^2\eta + B\cos^2\eta + 2D\sin\eta\cos\eta + C}.$$

Qui cum sit effectus binarum virium *Rr* et $I\pi$ iunctim agentium, ut prodeat effectus solius vis $Rr = V$, addatur insuper vis $Ip = V$ et corpori praeter motum gyatorium imprimetur motus progressivus purus secundum directionem *IQ* ipsi *Rr* parallelam, quo tempusculo dt conficietur spatiolum = $\frac{Vgdt^2}{M}$.

COROLLARIUM 1

629. Solutio ergo huius problematis ad eos tantum casus extenditur, quibus vis sollicitans $Rr = V$ corpori ita est applicata, ut collectis formulis integralibus expositis fiat $\frac{E}{F} + \frac{B+C}{D} = 0.$

COROLLARIUM 2

630. Nisi autem haec proprietas locum habeat, solutio problematis adhunc latet hocque tantum constat gyrationem non fieri circa axem, qui situs sit in plano ad directionem vis sollicitantis normali.

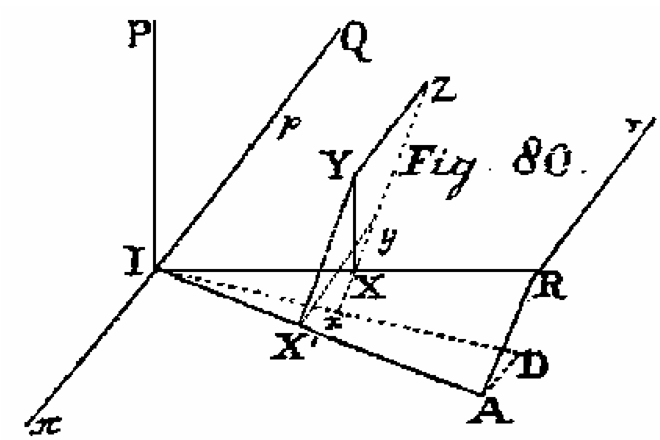
SCHOLION

631. Mirum utique videbitur, quod, cum praeparatio ex viribus, quas axis supra sustinere inventus est, petita solutionem completam polliceri sit visa, nunc tamen infiniti casus excludantur, quos nostra solutio non complectatur. Cum enim certum sit conversionem circa nullum alium axem fieri posse, nisi qui nullas plane vires sustineat, problema 18 perfectam solutionem suppeditare deberet, siquidem ipsum in omni extensione fuisset solutum. Verum probe notandum est in hoc problemate nullas alias vires esse assumtas, nisi, quarum directiones reperiantur in planis ad axem normalibus, cum tamen vires etiam obliquae induci potuissent, dummodo vires axi parallelae inde natae se destruerent. Atque hoc revera usu venit in casibus exclusis, ubi corpus initium gyrationis capere debet circa axem, qui est inclinatus ad planum per centrum inertiae ad vim sollicitantem normaliter ductum, quoniam tum ex resolutione vis *Rr* nascitur vis tali axi parallela; cuius utique ratio debet, si hoc problema in genere resolvere velimus.

PROBLEMA 59

632. Si corpus rigidum quiescens a vi quacunque sollicitetur eique simul in centro inertiae vis aequalis et contraria applicata fuerit, definire axem, circa quem primum gyrari incipiet.

SOLUTIO



Sit *I* centrum inertiae corporis (Fig. 80), ac tabula referat ut ante planum per *I* ad directionem vis sollicitantis, quae sit *Rr* = *V*, normaliter ductum, in quo ponatur distantia *IR* = *h*. Tum sumto in hoc plano angulo $RIA = \eta$, ut ducto ex *R* in *IA* perpendiculo *RA* sit $IA = h\cos\eta$ et $RA = h\sin\eta$, ducatur in *A* ad planum normalis *AD*, sitque ducta *ID* angulus $AID = \vartheta$ ideoque

$$AD = h\cos\eta \operatorname{tang}\vartheta \text{ et } ID = \frac{h\cos\eta}{\cos\vartheta};$$

quae linea *ID* sit axis gyrationis quaestus ita, ut ambos angulos η et ϑ investigare oporteat. Corpus ergo exprimi debet per ternas coordinatas, quarum una in ipso axe *ID* capiatur. Dari igitur assumo relationem inter coordinatas *IX* = *x*, *XY* = *y* et *YZ* = *z*, quarum prima in ipsa recta *IR*, secunda in plano ad vim normali ac tertia ipsi vi *Rr* parallela capiatur. Ex *Y* primo ad *IA* perpendicularis *YX'* ducatur, in plano autem ad tabulam normali *AID* perpendicularis *X'y* ipsi *YZ* et *yZ* ipsi *X'Y* parallela, erit, ut ante videmus :

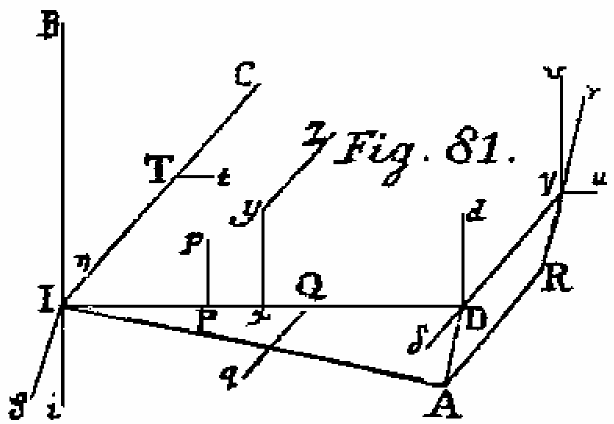
$$IX' = x\cos\eta - y\sin\eta,$$

$$X'y = YZ = z,$$

$$X'Y = yZ = x\sin\eta + y\cos\eta.$$

Tum in plano normali ex y ad ID ducatur perpendicularis yx, et habebuntur novae coordinatae, quales desideramus, quae sint $Ix = X$, $xy = Y$ et $yZ = Z$, atque ita per praecedentes determinatur

$$X = x\cos\eta\cos\vartheta - y\sin\eta\cos\vartheta + z\sin\vartheta,$$
$$Y = z\cos\vartheta - x\cos\eta\sin\vartheta + y\sin\eta\sin\vartheta,$$
$$Z = x\sin\eta + y\cos\eta.$$



Hae iam coordiniatae plano IAD in planum tabulae proiecto in figura 81 repraesententur, ad quod iam $AR = h\sin\eta$ erit normalis et vis Rr ipsi AD parallela; ducatur DV ipsi AR parallela, et vis in puncto V applicata concipiatur, ut sit vis $Vr = V$, ductisque Vv ipsi xy et Vu ipsi Ix parallela, ab angulum $rVv = \vartheta$, vis Vr resolvitur in binas has : $Vv = V\cos\vartheta$ et $Vu = V\sin\vartheta$, quae contrarie puncto I applicentur, quarum prior sit vis $Ii = V\cos\vartheta$, ad ID iam in plano tabulae normalis, altera vero axis secundum DI urgebitur. Momentum autem vis $Vv = V\cos\vartheta$ respectu axis ID est $= V\cos\vartheta\cdot h\sin\eta = Vh\sin\eta\cos\vartheta$, et posito $YY + ZZ = RR$ momentum inertiae corporis respectu axis $ID = \int RRdM$, unde tempusculo dt conversio fiet per angulum

$$d\omega = \frac{Vghdt^2 \sin\eta\cos\vartheta}{\int RRdM}.$$

Cum axis nullas vires sentire debeat, vis $Vv = V\cos\vartheta$ ipsi axi in D applicetur, ut sit vis $Dd = V\cos\vartheta$; vis vero $Vu = V\sin\vartheta$ in sua directione perpendiculo $IT = DV = h\sin\eta$ applicata concipiatur, ex qua pro axe primo vis nascitur secundum ID urgens, quae superiorem illam destruit; tum vero posito intervallo

$$ID = \frac{h\cos\eta}{\cos\vartheta} = a$$

inde oriuntur binae vires ad axem et planum tabulae normales

$$I\eta = D\delta = \frac{h\sin\eta}{a}V\sin\vartheta = V\ \mathrm{tang}\,\eta\ \sin\vartheta\cos\vartheta$$

Praeterea vero habentur vires $Ii = Dd = V\cos\vartheta$, quae per vires elementares destrui debent. At ex problemate 16 vires elementares huc accomodatae praebent binas vires Pp et Qq in punctis P et Q applicandae, ut sit

$$IP = \frac{\int XZdM}{\int ZdM}, \quad \text{vis } Pp = \frac{Vh\sin\eta\cos\vartheta\cdot\int ZdM}{\int RRdM},$$
$$IQ = \frac{\int XYdM}{\int YdM}, \quad \text{vis } Qq = \frac{Vh\sin\eta\cos\vartheta\cdot\int YdM}{\int RRdM}.$$

Cum autem motus hic in contrarium sensum incipiat, atque ibi assumisus, hae vires praecedentibus aequivalentes statui debent; et quia ob I centrum inertiae sit $\int YdM = 0$ et $\int ZdM = 0$, omnia ad momenta revocantur, ut esse debeat

$$Pp\cdot IP = Dd\cdot ID \text{ et } Qq\cdot IQ = D\delta\cdot ID,$$

sicque habebimus has duas aequationes :

$$\frac{Vh\sin\eta\cos\vartheta\int XZdM}{\int RRdM} = Vh\cos\eta$$

et

$$\frac{Vh\sin\eta\cos\vartheta\int XYdM}{\int RRdM} = Vh\sin\eta\sin\vartheta$$

sive

$$\sin\eta\cos\vartheta\int XZdM = \cos\eta\int RRdM$$

et

$$\cos\vartheta\int XYdM = \sin\vartheta\int RRdM.$$

Nunc igitur positis integralibus ex coordinatis principalibus x, y et z natis :

$$\int xxdM = A, \quad \int yydM = B, \quad \int zzdM = C,$$
$$\int xydM = D, \quad \int xzdM = E, \quad \int yzdM = F,$$

ob $RR = YY + ZZ$ erit

$$\int RRdM = A\left(\sin^2\eta + \cos^2\eta\sin^2\vartheta\right) + B\left(\cos^2\eta + \sin^2\eta\sin^2\vartheta\right) + C\cos^2\vartheta$$
$$+ 2D\sin\eta\cos\eta\cos^2\vartheta - 2E\cos\eta\sin\vartheta\cos\vartheta + 2F\sin\eta\sin\vartheta\cos\vartheta,$$

$$\int XYdM = -A\cos^2\eta\sin\vartheta\cos\vartheta - B\sin^2\eta\sin\vartheta\cos\vartheta + C\sin\vartheta\cos\vartheta$$
$$+2D\sin\eta\cos\eta\sin\vartheta\cos\vartheta + E\cos\eta\left(\cos^2\vartheta - \sin^2\vartheta\right)$$
$$-F\sin\eta\left(\cos^2\vartheta - \sin^2\vartheta\right),$$

$$\int XZdM = A\sin\eta\cos\eta\cos\vartheta - B\sin\eta\cos\eta\cos\vartheta + D\cos\vartheta\left(\cos^2\eta - \sin^2\eta\right)$$
$$+E\sin\eta\sin\vartheta + F\cos\eta\sin\vartheta,$$

quibus valoribus substitutis binae aequationes inventae induent has formas :

$$\text{I.}\quad -A\cos\eta\sin^2\vartheta - B\cos\eta - C\cos\eta\cos^2\vartheta - D\sin\eta\cos^2\vartheta$$
$$+E\left(1+\cos^2\eta\right)\sin\vartheta\cos\vartheta - F\sin\eta\cos\eta\sin\vartheta\cos\vartheta = 0,$$
$$\text{II.}\quad -A\sin\vartheta - B\sin\vartheta + E\cos\eta\cos\vartheta - F\sin\eta\cos\vartheta = 0,$$

quarum posterior praebet

$$\text{tang }\vartheta = \frac{E\cos\eta - F\sin\eta}{A+B}.$$

At $\text{II}\cdot\cos\eta\sin\vartheta - \text{I}$ praebet

$$B\cos\eta\cos^2\vartheta + C\cos\eta\cos^2\vartheta + D\sin\eta\cos\vartheta - E\sin\vartheta\cos\vartheta = 0,$$

unde colligitur

$$\text{tang }\vartheta = \frac{(B+C)\cos\eta + D\sin\eta}{E};$$

hincque tandem

$$\text{tang }\eta = \frac{EE-(A+B)(B+C)}{(A+B)D+EF},$$

unde ambo anguli $RIA = \eta$ et $AID = \vartheta$ atque adeo axis gyrationis *ID* innotescit.

COROLLARIUM 1

633. Proposita ergo vi quacunque *Rr = V*, cui simul aequalis in ipso centro inertiae *I* contrarie sit applicata, si per *I* planum ad directionem vis normale ducatur *PIR* atque ad id recta *IQ* perpendicularis, his ternis directricibus *IR, IP, IQ* pro quovis elemento corporis *dM* in *Z* sito parallelae capiantur coordinatae *IX = x, XY = y* et *YZ = z,* hincque ex indole corporis colligi debent sequentibus sex valores :

$$\int xxdM = A,\quad \int yydM = B,\quad \int zzdM = C,$$
$$\int xydM = D,\quad \int xzdM = E,\quad \int yzdM = F,$$

COROLLARIUM 2

634. His inventis in plano *RIP* ad directionem vis normali ex opposito coordinatarum *XY* = *y* positivarum seu in regione negativarum capiatur angulus $RIA = \eta$, ut sint

$$\text{tang } \eta = \frac{EE-(A+B)(B+C)}{(A+B)D+EF},$$

quo invento super illo plano in regione coordinatarum *YZ* = *z* positavarum erigatur $AID = \vartheta$, ut sit

$$\text{tang } \vartheta = \frac{E\cos\eta - F\sin\eta}{A+B}$$

seu

$$\text{tang } \vartheta = \frac{(A+B)\cos\eta + D\sin\eta}{E},$$

eritque recta *ID* axis gyrationis.

COROLLARIUM 3

635. Posita distantia *IR* = *h* erit respectu huius axis *ID* momentum vis sollicitantis = $Vh\sin\eta\cos\vartheta$ et momentum inertiae = $\int RRdM$, quod etiam est =

$$\text{tang } \eta\cos\vartheta\int XZdM = \cot\vartheta\int XYdM,$$

cuius valor ex praecedentibus facile eruitur: inde elemento temporis *dt* conversio fit per angulum

$$d\omega = \frac{Vghdt^2 \sin\eta\cos\vartheta}{\int RRdM}.$$

SCHOLION

636. En ergo problema nostrum generale, in quo summa hius capitis versatur, perfecte solutum; unde quidem casus ante tractatus sponte fluit, quippe quo est angulus $\vartheta = 0$; nam tum fit ex formula priori

$$\text{tang}\,\eta = \frac{E}{F}$$

et ex posteriore

$$\text{tang}\,\eta = \frac{-B-C}{D},$$

qui valores nisi conveniant, casus ille locum habere nequit. Vicisim autem, si fuerit

$$DE + (B+C)F = 0,$$

ob

$$B+C=\frac{-DE}{F}$$

fit

$$\operatorname{tang}\eta=\frac{E}{F}$$

et

$$\operatorname{tang}\vartheta=0.$$

Caeterum hic observo ex iis, quae supra de axibus principalibus sunt tradita, esse

$$\int XYdM=\frac{-d\cdot\int RRdM}{2d\vartheta} \text{ sumto tantum } \vartheta \text{ variabili}$$

et

$$\int XZdM=\frac{-d\cdot\int RRdM}{2d\eta\cos\vartheta} \text{ sumto tantum } \eta \text{ variabili.}$$

Quibus valoribus substitutis binae conditiones principales postulant

$$\frac{\sin\eta\cdot d\cdot\int RRdM}{2d\eta}=\cos\eta\int RRdM \quad \text{et} \quad -\frac{\cos\vartheta\cdot d\cdot\int RRdM}{2d\vartheta}=\sin\vartheta\cdot\int RRdM,$$

in quarum priore tantum η, in posteriore tantum ϑ est variabile. Utriusque igitur idem est integrale

$$\int RRdM=\alpha\sin^2\eta\cos^2\vartheta,$$

unde vicissim concludo angulos η et ϑ ita definire oportet, ut quantitas

$$\frac{\sin^2\eta\cos^2\vartheta}{\int RRdM}$$

fiat minimum, quoniam hinc eaedem binae aequationes resolvendae proveniunt. Eadem autem formula oritur, si $d\omega^2\int RRdM$ seu $\int dM\cdot RRd\omega^2$ reddatur minimum, in qua cum $Rd\omega$ denotet celeritatem elementi dM ideoque $dM\cdot RRd\omega^2$ eius vim vivam, uti vocatur, hinc colligimus istud insigne Theorema :

THEOREMA 8

637. Si corpus rigidum quiescens sollicitetur a vi quacunque eique insuper in centro inertiae applicata sit vis aequalis et contraria, ei circa eiusmodi axem per centrum inertiae transeuntem primo instanti motus gyratiorius imprimetur, ut totum corpus inde mimimam adipiscatur vim vivam, quae est aggregatum omnium elementorum per quadrata celeritatum suarum acquisitarum multiplicatorum.

DEMONSTRATIO

Quicunque enim axis per centrum inertiae transiens accipatur, eius respectu vis proposita V certum obtinebit momentum, quod sit Vf, tum vero etiam corpus eius respectu certum obtinebit momentum inertiae, quod sit = $\int RRdM$, utrumque a situ axis assumti pendens; hinc autem tempuscolo dt generabitur circa hunc axem angulus

$$d\omega = \frac{Vfgdt^2}{\int RRdM}$$

et celeritas angularis infinite parva

$$\gamma = \frac{2Vfgdt}{\int RRdM},$$

unde elementi dM ab axe intervallo = R distantia celeritas fit = $R\gamma$ ideoque vis viva = $R^2\gamma^2 dM$. Totius ergo corporis vis viva tempusculo infinite parvo dt acquisita erit =

$$\gamma\gamma\int R^2 dM = \frac{4VVffggdt^2}{\int RRdM},$$

quae ob Vg et dt constans erit minimum, si $\frac{ff}{\int RRdM}$ reddatur minimum, atque ex hac conditione positio axis determinetur. Hinc autem eadem axis determinatio resultat, quam ante invenimus, ita ut ex hoc principio minimi eadem solutio erui potuisset.

SCHOLION

638. Quod ad usum solutionis ante inventae attinet, hoc adhuc nimis est molestum, quod pro unaquaque vi sollicitante indoles corporis ad peculiares coordinatas revocari debeat. Cui incommodo remedium affertur per ea, quae supra de axibus principalibus cuisque corporis docuimus, quorum respectu si momenta inertiae semel fuerint inventa, facillime inde respectu omnium aliorum axium colligi possunt. Atque etiam pro praesenti instituto sufficit relationem corporis ad coordinatas axibus principalibus parallelas posse, quoniam et hinc relatio ad quasvis alias ternas coordinatas derivari potest. Quamobrem problema superius ita resolvam, ut vim sollicitantem respectu axium principalium dari assumam; ac solutionem ipsam ex principio iam stabilito, quod minima vis viva generetur, petam.

PROBLEMA 60

639. Datis corporis rigidi axibus principalibus eorumque respectu momentis inertiae, si id a vi quacunque sollicetur simulque ipsi in centro inertiae applicata sit alia vis illi aequalis et contrara, definire axem, circa quem corpus primum gyrari incipiet.

SOLUTIO

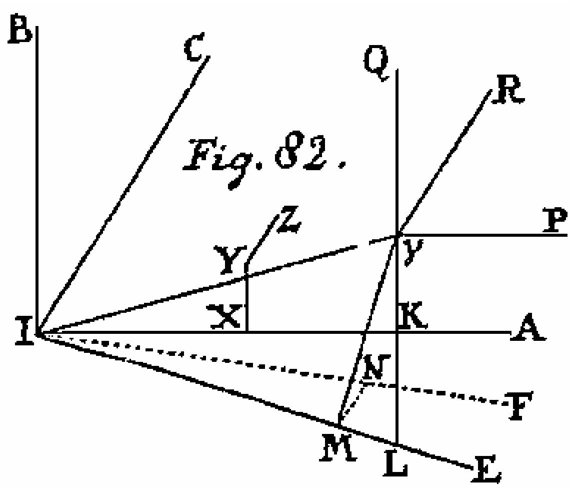


Fig. 82.

Sit I centrum inertiae corporis et rectae IA, IB, IC eius tres axes principales (Fig. 82), quorum respectu momenta inertiae sint Maa, Mbb, Mcc. Iam a quacunque vi corpus sollicitetur, notetur eius transitus per planum binis axibus principalibus interceptum AIB, qui fit in puncto V, ab I distante intervallo $IV = h$ existente angulo $AIV = \delta$; ipsa autem vis, quasi huic puncto esset applicata, resolvetur in ternas axibus parallelas, quae sint vis $VP = P$, vis $VQ = Q$ et vis $VR = R$, quibus igitur aequales et contrariae in puncto I applicatae sunt intelligendae. Ab his ergo corpus circa quempiam axem per centrum inertiae I transeuntem verti incipiet, qui sit IF ad planum BIA inclinatus angulo $FIE = \vartheta$, existente angulo $AIE = \eta$, quos binos angulos investigari oportet. Iam primo respectu huius axis IF quaeratur momentum inertiae, quod, cum sit $\cos AIF = \cos\eta\cos\vartheta,\quad \cos BIF = -\sin\eta\cos\vartheta,\quad \cos CIF = \sin\vartheta,$ erit per superiora

$$M\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right).$$

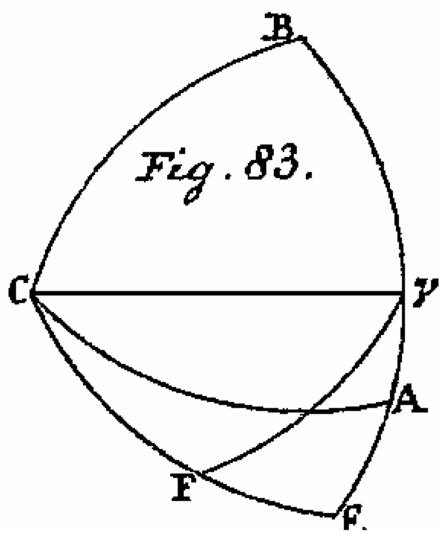


Fig. 83.

Deinde momenta virium P, Q, R respectu axis huius IF sunt investiganda; ex antecedentibus autem patet ducta VM ad IE normali, ut $VM = h\sin(\delta+\eta)$, fore vis $VR = R$ momentum = $R\cdot VM\cdot\cos\vartheta = Rh\sin(\delta+\eta)\cos\vartheta$.

Verum, quo reliquarum virium momenta facilius inveniri queant, puncta V, A, B, C, E, F in superficie sphaerica considerentur, cuius centrum sit in I (Fig. 83). Erunt ergo arcus AB, AC et BC quadrantes,

$AV = \delta, AE = \eta, EF = \vartheta$; et vires *P, Q, R* in *V* applicatae resolvantur in binas, quarum alterae sint in superficiem sphaericam normales, alterae superficiem sphaericam tangant, ubi priores per centrum transeuntes nulla praebent momenta, unde solas posteriores considerasse sufficit, quae erunt :

$$\text{vis sec. } VA = P\,sin\,AV,$$
$$\text{vis sec. } VB = Q\,sin\,BV,$$

et

$$\text{vis sec. } VC = R\,sin\,CV = R$$

ob *CV* quadrantem. Hae vires porro resolvantur secundum directionem *VF* et aliam ad eam normalem, ubi priores cum axe *IF* in eodem plano sitae nullum praebent momentum, alterae autem vires erunt

$$P\,sin\,AV\;sin\,AVF - Q\,sin\,BV\;sin\,BVF - R\,sin\,CVF,$$

quarum directio, cum sit ad planum *IFV* normalis, erit etiam in plano ad axem IF normali, unde, cum distantia ab axe sit =
$h\,sin\,FV$, ob $AV = \delta$ et $sin\,BVF = sin\,AVF$ erit momentum quaesitum =

$$h\big(\big(P\,sin\,\delta - Q\,cos\,\delta\big)sin\,AVF\;sin\,FV - R\,cos\,AVF\cdot sin\,FV\big)$$

at

$$sin\,AVF\cdot sin\,FV = sin\,FE = sin\,\vartheta,$$

sique momentum habebitur

$$Ph\,sin\,\delta\,sin\,\vartheta - Qh\,cos\,\delta\,sin\,\vartheta - Rh\,cos\,AVF\cdot sin\,FV,$$

at ex sphaericis est

$$cos\,AVF\cdot sin\,FV = sin\big(\delta+\eta\big)cos\,\vartheta,$$

ita ut momentum quaesitum sit =

$$Ph\,sin\,\delta\,sin\,\vartheta - Qh\,cos\,\delta\,sin\,\vartheta - Rh\,sin\big(\delta+\vartheta\big)cos\,\vartheta,$$

ex quo angulus tempusculo *dt* genitus fit

$$d\omega = \frac{ghdt^2\big(P\,sin\,\delta\,sin\,\vartheta - Q\,cos\,\delta\,sin\,\vartheta - R\,sin\big(\delta+\eta\big)cos\,\vartheta\big)}{M\big(aa\,cos^2\,\eta\,cos^2\,\vartheta + bb\,sin^2\,\eta\,cos^2\,\vartheta + cc\,sin^2\,\vartheta\big)}.$$

Quocirca minimum reddi debet haec expressio

$$\frac{\left(\left(P\sin\delta-Q\cos\delta\right)\sin\vartheta-R\sin\left(\delta+\eta\right)\cos\vartheta\right)^2}{aa\cos^2\eta\cos^2\vartheta+bb\sin^2\eta\cos^2\vartheta+cc\sin^2\vartheta},$$

statuamus primo ϑ tantum variabile, et fiet :

$$\begin{aligned}&2\left(aa\cos^2\eta\cos^2\vartheta+bb\sin^2\eta\cos^2\vartheta+cc\sin^2\vartheta\right)\\&\qquad\left(\left(P\sin\delta-Q\cos\delta\right)\cos\vartheta+R\sin\left(\delta+\eta\right)\sin\vartheta\right)=\\&2\left(-aa\cos^2\eta\sin\vartheta\cos\vartheta-bb\sin^2\eta\sin\vartheta\cos\vartheta+cc\sin\vartheta\cos\vartheta\right)\\&\left(\left(P\sin\delta-Q\cos\delta\right)\sin\vartheta-R\sin\left(\delta+\eta\right)\cos\vartheta\right),\end{aligned}$$

quae reducitur ad hanc formam

$$\left(P\sin\delta-Q\cos\delta\right)\left(aa\cos^2\eta+bb\sin^2\eta\right)\cos\vartheta+Rcc\sin\left(\delta+\eta\right)\sin\vartheta=0,$$

unde oritur

$$\text{tang }\vartheta=\frac{\left(Q\cos\delta-P\sin\delta\right)\left(aa\cos^2\eta+bb\sin^2\eta\right)}{Rcc\sin\left(\delta+\eta\right)}=0.$$

Nunc sumto η variabili obtinebimus :

$$\begin{aligned}&2\left(aa\cos^2\eta\cos^2\vartheta+bb\sin^2\eta\cos^2\vartheta+cc\sin^2\vartheta\right)\left(-R\sin\left(\delta+\eta\right)\cos\vartheta\right)=\\&2\left(-aa\sin\eta\cos\eta\cos^2\vartheta+bb\sin\eta\cos\eta\cos^2\vartheta\right)\\&\qquad\left(\left(P\sin\delta-Q\cos\delta\right)\sin\vartheta-R\sin\left(\delta+\eta\right)\cos\vartheta\right),\end{aligned}$$

qui reducitur ad hanc formam

$$\begin{aligned}&R\cos\vartheta\left(aa\cos\delta\cos\eta\cos^2\vartheta-bb\sin\delta\sin\eta\cos^2\vartheta+cc\cos\left(\delta+\eta\right)\sin^2\vartheta\right)=\\&\qquad\left(Q\cos\delta-P\sin\delta\right)\left(bb-aa\right)\sin\eta\cos\eta\sin\vartheta\cos^2\vartheta,\end{aligned}$$

ubi si loco $Q\cos\delta-P\sin\delta$ ponatur

$$=\frac{Rcc\sin\left(\delta+\eta\right)\text{tang }\vartheta}{aa\cos^2\eta+bb\sin^2\eta},$$

facta reductione pervenitur ad hanc aequationem

$$\cos^2\vartheta(aa\cos\delta\cos\eta - bb\sin\delta\sin\eta)\left(aa\cos^2\eta + bb\sin^2\eta\right)$$
$$+cc\sin^2\vartheta(aa\cos\delta\cos\eta - bb\sin\delta\sin\eta) = 0,$$

quae per $aa\cos\delta\cos\eta - bb\sin\delta\sin\eta$ divisa praebet

$$\cos^2\vartheta\left(aa\cos^2\eta + bb\sin^2\eta\right) + cc\sin^2\vartheta = 0$$

aequationem impossibilem ob omnes partes positivas. Quare divisore utentes nanciscimur determinationem anguli η, scilicet

$$\text{tang }\eta = \frac{aa\cos\delta}{bb\sin\delta};$$

ex quo porro colligitur

$$\text{tang }\vartheta = \frac{(Q\cos\delta - P\sin\delta)aabb}{Rcc\sqrt{\left(a^4\cos^2\delta + b^4\sin^2\delta\right)}}$$

vel, ne ambiguitas signi radicalis dubium relinquat,

$$\text{tang }\vartheta = \frac{(Q\cos\delta - P\sin\delta)aa\cos\eta}{Rcc\sin\delta} = \frac{(Q\cos\delta - P\sin\delta)bb\sin\eta}{Rcc\cos\delta}.$$

Hoc iam axe invento, si porro $Q\cos\delta - P\sin\delta$ valor superior substitutu colligitur momentum virium sollicitantium respectu istius axis =

$$= \frac{Rh\sin(\delta+\eta)\left(aa\cos^2\eta\cos^2\vartheta + bb\sin^2\eta\cos^2\vartheta + cc\sin^2\vartheta\right)}{\left(aa\cos^2\eta + bb\sin^2\eta\right)\cos\vartheta},$$

unde angulus elementaris $d\omega$ tempusculo dt circa axem genitus erit

$$d\omega = \frac{Rghdt^2\sin(\delta+\eta)}{M\cos\vartheta\left(aa\cos^2\eta + bb\sin^2\eta\right)} = \frac{Rghdt^2\sqrt{\left(a^4\cos^2\delta + b^4\sin^2\delta\right)}}{Maabb\cos\vartheta},$$

si in $\sin(\delta+\eta)$ loco anguli η valor repertus substituatur.

COROLLARIUM 1

640. Ex puncto ergo V (Fig. 82), in quo directio vis sollicitantis planum AIB traiicit, statim invenitur in eodem plano recta IE, cui axis gyrationis IF imminet; posito enim angulo $AIV = \delta$ erit

$$\text{tang } AIE = \text{tang } \eta = \frac{aa\cos\delta}{bb\sin\delta}$$

neque a directione ipsius vis pendet.

COROLLARIUM 2

641. Quare, si vis sollicitans per axem principalem IA transeat, angulus AIE fit rectus gyrationis IF in plano ad axem IA normali. At ob $\delta = 0$ et $\eta = 90^{\circ}$ erit

$$\text{tang } EIF = \text{tang } \vartheta = \frac{Qbb}{Rcc}.$$

COROLLARIUM 3

642. Si momenta inertiae respectu axium IA et IB fuerint aequalia, erit

$$\text{tang } \eta = cot\,\delta = \text{tang}\left(90^{\circ} - \delta\right)$$

ideoque angulus VIE rectus; hoc igitur casu axis gyrationis IF erit ad rectam IV normalis et $aa = bb$ fiet

$$\text{tang } \vartheta = \frac{\left(Q\cos\delta - P\sin\delta\right)aa}{Rcc}.$$

COROLLARIUM 4

643. Si vis sollicitans, quae sit = V et cuius directio planum AIB in puncto V traiicit, ex cuius resolutione nascuntur vires P, Q, R, sola in corpus agat, ea corpori motum assignatum circa axem inventum IF inducet, praeterea vero ipsi motum progressivum secundum suam directionem imprimet, qua tempusculo dt conficiet spatiolum = $\frac{Vgdt^2}{M}$

SCHOLION

644. In solutione huius problematis iucundum sane erat perspicere, quomodo calculus, qui initio non parum intricatus videbatur, continuo ad maiorem simplicitatem quasi sponte fuerit perductus, in quo eximium veritatis criterium cernitur. Plerumque enim talis calculi commoditas deprehenditur, dum in veritatis investigatione felici successu versamur, cum contra a veritatis tramite aberrantes in calculos inextricabiles illabi solemus. Ac principium quidem minimi, quo hic sum usus, elegantem suppeditavit solutionem, quae multo intricatior evasisset, si eam ut ante ex primis mechanicae principiis petere voluissemus. Nunc ergo problema, quo praesens caput absolvitur, in genere pertractare licebit.

PROBLEMA 61

645. Si corpus rigidum quiescens a viribus quibuscunque sollicitetur, definire primum motum elementarum, qui in eo generabitur.

SOLUTIO

Ex Theoremate VII omnes vires sollicitantes, quotcunque fuerint, reducantur ad binas, quarum altera ipsi centro inertiae sit applicata, altera vero extra hoc centrum directa; harum prior sit = S, posterior = V. His duabus viribus inventis primo sola vis V consideretur, cui aequalis in ipso centro inertiae contrarie applicata concipatur, ut centrum inertiae etiamnunc in quiete conservetur. Dispiciatur ergo, ubi directio illius vis V per planum aliquod intra binos axes principales corporis transeat, et ex problemate praecedente quaeratur tam axis gyrationis, circa quem corpus primum converti incipiet, quam angulus infinite parvus primo tempusculo productus. Tum autem corpori insuper motus progressivus imprimetur, ad quem inveniendum vis illa altera V secundum suam directionem ipsam quoque centro inertiae applicata concipiatur, ita ut coniunctim cum vi priore S iam corpus sollicet; et quia utraque centro inertiae est applicata, inde orietur motus progressivus purus, qui si cum gyratorio ante invento combinetur, habebitur totus effectus a viribus propositis productus.

COROLLARIUM 1

646. Si vis V evanescat, hoc est, si unica detur vis S centro inertiae applicata, quae omnibus viribus sollicitantibus aequivaleat, tum, ut supra iam vidimus, corpori solus motus progressivus imprimitur.

COROLLARIUM 2

647. Sin autem vis *S* aequalis sit vi *V*, sed directionem habeat oppositam, quod fit, si vires sollicitantes ita fuerint comparatae, ut omnes quaeque in sua directione centro inertiae applicatae se mutuo destruerent, tum centrum inertiae in quiete perseverabit solusque motus gyratorius generabitur.

COROLLARIUM 3

648. Reliquis casibus omnibus in corpore motus mixtus generabitur, alter progressivus, alter circa certum quendam axem per centrum inertiae transeuntem; quorum utrumque seorsim considerare ac determinare licet.

SCHOLION

649. Idem effectus producetur ab his viribus sollicitantibus, etiam si corpus in motu versetur, verum ob huius motus admixtionem difficilius cognosci poterit. Si enim corpus iam circa alium axem gyretur ac nunc incitatur, non solum celeritas angularis, sed etiam ipse axis gyrationis mutabitur, ita ut nunc circa alium axem per centrum inertiae transeuntem gyrari incipiat. Atque in hac axis variatione maxima motus perturbatio est sita, ad quam explicandam primo conveniet huiusmodi perturbationem momentaneam accurate determinari, quod argumentum in sequente capite evolvamus.