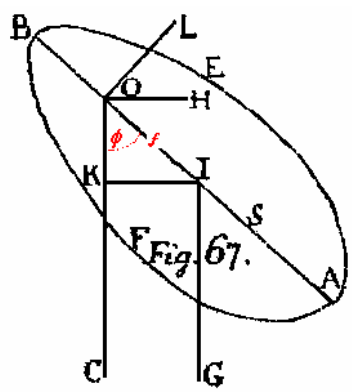
Chapter 7

CONCERNING THE OSCILLATORY MOTION OF HEAVY BODIES

PROBLEM 44

522. If a rigid body should be mobile about a fixed horizontal axis and the motion of this disturbed only by gravity, to determine the momentary change produced in the rotational motion.



SOLUTION

I assume this common property of gravity, by which the individual elements of mass are urged downwards proportionally along directions parallel to each other. Therefore in as much as the body is rigid, a single force equal to the weight of the body is equivalent to all these forces, the direction of which acts downwards passing through the centre of inertia. Whereby, if the mass of the body is taken as *M*, and the centre of inertia of this is at *I* (Fig. 67) and thus the right line *IG* is drawn vertically down, on account of the weight the body is acted on by a force in the direction *IG*, which is set up to be equal to the mass of the body, since we express the mass *M* itself by the weight of this body. Again, since the axis of gyration is horizontal, it is established normal to the plane passing through the centre of inertia *I*, which is vertical and is referred to as the plane of the diagram ; therefore the axis of gyration is considered as normal to this plane and crossing through the point *O*, thus with the right line *OI* drawn to *I* the distance is shown of the centre of inertia from the axis of gyration. With these premises put in place the body *AEBF* now is placed in the figure shown, and with *OC* drawn to the vertical the position of the body is noted by the angle *COI*. The interval $OI = f$ is put in place at the time t and the angle $CIO = \varphi$, then the force $IG = M$, the moment with respect to the axis of gyration is equal to $Mf \sin\varphi$, tending to diminish the angle *COI*, which is being substituted in problem 22 in place of the moment Vf. Now besides it is necessary to know the moment of inertia of the body about the axis of gyration *O*, indicated there by $\int rrdM$; hence finally the axis is considered passing through the centre of inertia *I* parallel to the axis of gyration, with respect to which the moment of inertia of the body is equal to *Mkk*, the moment of inertia of the same body about the axis of gyration *O* is equal to $M(ff + kk)$ on account of the separation of these axes *OI* = *f*. Hence, if the body thus is rotating, in order that the line *OI* falls on the vertical *OC*, the angular speed is equal to γ, since that is increased by the force acting, by § 408 the increase becomes

$$d\gamma = \frac{2g \cdot Mf\ sin\varphi}{M(ff+kk)}dt$$

or

$$d\gamma = \frac{2\,fgdt\,sin\varphi}{ff+kk};$$

but moreover with the line OI receding from the vertical OI with an angular speed equal to γ, it becomes

$$d\gamma = -\frac{2\,fgdt\,sin\varphi}{ff+kk}.$$

But in that case then $\gamma = \frac{-d\varphi}{dt}$, now from this $\gamma = \frac{d\varphi}{dt}$, on taking *dt* constant for each then

$$dd\varphi = -\frac{2\,fgdt^2\,sin\varphi}{ff+kk};$$

where the – sign is present, since the moment of the force acting tends to reduce the angle φ.

COROLLARIUM 1

523. If the body in place *AEBF* at this stage should have no motion, thus it will be urged by gravity towards the vertical line *OC*, in order that in the element of time *dt* it shall approach towards that by the angle

$$\frac{2\,fgdt^2\,sin\varphi}{ff+kk},$$

which is an infinitely small quantity of the second order.

COROLLARY 2

524. Therefore if the body were at rest, it may not remain at rest, unless $sin\varphi = 0$, that is unless the centre of inertia *I* has turned to the vertical *OC*. Whereby, if the body is suspended somehow in some manner, it is unable to remain at rest unless the line *OI* is vertical, which comes about if the centre of inertia should reach the lowest or highest location.

COROLLARY 3

525. But as often as the line *OI* should be oblique, the body is acted on by gravity to motion and, if now it should be moving, then the motion of this is disturbed either by being accelerated or decelerated, as the moving body either approaches *OC* or recedes from that.

COROLLARY 4

526. Also it is apparent, if the axis should pass through the centre of inertia *I*, in order that $OI = f = 0$, the moment of the weight vanishes and the rotational motion therefore plainly is not disturbed. Therefore in this case the body either remains at rest or will gyrate uniformly about the axis O.

SCHOLIUM

527. Here it is agreed to noted at once that the body likewise does not begin to be moved, if the whole mass of this should be gathered together in the centre of inertia *I*, as we have seen in the use in progressive motion. If indeed this total mass *M* of the body is actually gathered together in the centre of inertia *I*, the moment of inertia of this about the axis drawn through *I* vanishes and becomes *kk* = 0; and hence the motion therefore is being disturbed, in order that it becomes

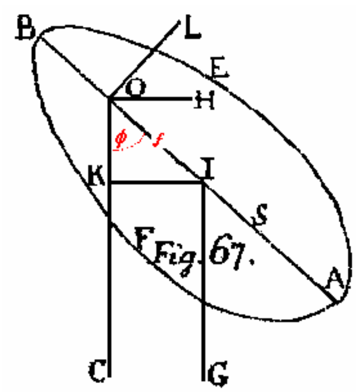
$$dd\varphi = \frac{-2gdt^2 \sin\varphi}{f},$$

which formula is greater than in the case proposed. Thus it is understood that the motion of on extended body, of such a kind as we contemplate here, is less disturbed by gravity than if the whole mass should be concentrated at the centre of inertia. Now below we will see a different point to be given on the line *OI* more removed from the axis *O*, at which if the whole mass of the body should be collected together, the same disturbed motion is to be allowed, which point deserves especially to be known in the rotational motion, since it is that point itself that is accustomed to be called the *centre of oscillation* and many instructions concerning the finding of this point occur here and there [in this tract].

PROBLEM 45

528. If the rigid body *AEBF* were mobile about a horizontal axis (Fig. 67) and the position and initial speed of this is given, to find the position and speed of this at any time.

SOLUTION



With everything to be used remaining as in the preceding problem, clearly the mass of the body is equal to *M*, the distance of the centre of inertia *I* from the axis of gyration *O* clearly is *OI* = *f* and the moment of inertia about the axis parallel to the axis of gyration and passing through *I* is equal to *Mkk*, the body may hold the place shown in the figure at the elapsed time *t* and let the angle $COI = \varphi$ arise with the vertical line CO ; and on taking the element *dt* constant we come upon this equation :

$$dd\varphi = \frac{-2\,fgdt^2 \sin\varphi}{ff+kk},$$

which multiplied by $2d\varphi$ and integrated offers

$$d\varphi^2 = \alpha dt^2 + \frac{4\,fgdt^2 \cos\varphi}{ff+kk},$$

from which the square of the speed is recognised :

$$\gamma\gamma = \alpha + \frac{4\,fg\cos\varphi}{ff+kk}.$$

Then for the sake of brevity, on putting

$$\frac{4\,fg}{ff+kk} = \lambda$$

on account of

$$d\varphi^2 = dt^2\left(\alpha + \lambda\cos\varphi\right)$$

there is found :

$$dt = \frac{d\varphi}{\sqrt{(\alpha+\lambda\cos\varphi)}}$$

and

$$t = \int \frac{d\varphi}{\sqrt{(\alpha+\lambda\cos\varphi)}},$$

where the constant α and the other, advanced from the final integration, must be defined from the given initial state.

COROLLARY 1

529. With the angle $COI = \varphi$ vanishing, let the angular speed

$$\gamma\gamma = \sqrt{\left(\alpha + \frac{4fg}{ff+kk}\right)}$$

be the greatest of all, moreover in the equalities for the prolongations of the line *OI* from the vertical *OC,* the speeds are equal; and unless the constant α is less than $\frac{4fg}{ff+kk}$, the body completes whole revolutions about the axis, since then the angular speed for the angle $\varphi = 180^{\circ}$ is real at this stage.

COROLLARY 2

530. But if $\alpha < \frac{4fg}{ff+kk}$, then the angle $COI = \varphi$ is unable to increase beyond a certain limit and the body, since pertaining to that, descends again and performs the motion of an oscillation; and on drawing *IK* horizontal, on account of $OK = f\cos\varphi$ the angle of displacement *COI* will correspond to the angular speed

$$\gamma = \sqrt{\left(\alpha + \frac{4g \cdot OK}{ff+kk}\right)}.$$

SCHOLIUM

531. Whether the body completes complete revolutions or it comes and go by oscillating, the determination of the motion demands the same calculation as the motion of a simple pendulum, in which an indefinitely small body is tied by a thread free of inertia, and rotating about the horizontal axis. Since we have now explained that motion in detail above, it would be superfluous the repeat the same calculation here ; thus it is sufficient for some case of the simple pendulum to be assigned to the motion of the body, since it is carried forwards by an equal angular motion. And here indeed only the length of this simple pendulum comes into the computation, as the motion of this depends only on the length of the pendulum, if indeed we attribute the same angular motion to each.

DEFINITION 9

532. For the rotational or oscillatory motion of any heavy body about a fixed horizontal axis, a *simple isochronous pendulum* is called upon, because when once it is displaced from the vertical by an equal amount, then it adopts the same angular speed, and hence continuously its position is given by a like angle.

EXPLANATION

533. If some body *AEBF* (Fig. 67) is put in place, since that is acted on by gravity alone and it rotates about the horizontal axis *O*, in the first place the centre of gravity *I* of this is to be considered, because if it is rotated from the vertical *OC* which indicates the natural position of the body at which it is at rest, then moreover the angle *COI* is called the *displacement from the natural position.* But if now there should be impressed a given motion to this body in a given displacement, a simple isochronous pendulum thus should be a comparison, in order that, if to that at an equal displacement there is impressed an equal angular motion, then the motion of this pendulum shall always correspond to the motion of the proposed body. Or as the whole calculation depends on the length of this pendulum, if that should be *OS,* and if it is considered to be suspended from the common axes *O*, then it will always accompany the motion of the body *AEBF* with its own motion, provided that once it has been given an equal motion. Likewise it is the case, that either this simple pendulum is taken to be fixed to the same axis or otherwise ; but since the displacements on both sides from the position of the vertical *OC* must always be the same and the displacement of the body from the position of the line *OI* has to be estimated, the simple pendulum is most conveniently considered to be suspended from the point *O*, in order that the position *OS* always falls on the line *OI* and the whole question of the determination of the point S is resolved.

COROLLARY 1

534. With the point *S* found on the line *OI* produced, the body likewise moves as if the whole mass were collected at this point *S*; then indeed on account of the extent of the body vanishing, there is obtained a simple pendulum of length *OS*.

COROLLARY 2

535. Therefore this point *S* must be sought on the line, which is drawn through the centre of inertia normal to the axis of gyration, even if here it is not necessary that a simple pendulum *OS* be put in place, suspended from the same point *O* on the axis.

SCHOLIUM

536. Since that pendulum carried with a similar motion, on account of the vanishing of the mass, is called a *simple pendulum*, in this manner any extended bodies mobile about fixed axes are usually called *compound pendulums*, thus so that here the question may be reduced to this, that for some proposed compound pendulum, since clearly it is a rigid body, a simple isochronous pendulum may be assigned, and indeed now we can easily resolve the question. Moreover it is to be observed that the thread, by which we understand the simple pendulum to be attached to the axis, not only put in place free of inertia, but also it must be considered as rigid, lest any bending is able to disturb the calculation.

PROBLEM 46

537. For some proposed heavy and rigid body *AEBF* mobile about a fixed horizontal axis *O* (Fig. 67), to define the simple isochronous pendulum *OS*.

SOLUTION

With the mass of the whole body put equal to *M* and the centre of inertia of this at *I*, hence there is drawn normal to the axis the right line $IO = f$, which now stands at an angle $COI = \varphi$ from the vertical *OC*; then let *Mkk* be the moment of inertia of the body about an axis drawn through *I* and parallel to the axis of rotation. With these in place, some initial motion should be impressed to the body, and in the elapsed time *t* the change in the motion is expressed by this formula :

$$dd\varphi = \frac{-2\,fgdt^2 \sin\varphi}{ff+kk}.$$

Now the length of the simple isochronous pendulum *OS* is put equal to *l*, since that stands apart by the same angle $COS = \varphi$ from the vertical, this change of the motion is allowed, so that it becomes :

$$dd\varphi = \frac{-2gdt^2 \sin\varphi}{l},$$

which indeed comes from the previous formula on putting $k = 0$ and $f = l$. Whereby, since the same change must arise in both cases, we obtain :

$$l = \frac{ff+kk}{f} \text{ or } l = f + \frac{kk}{f}.$$

COROLLARY 1

538. Hence the length of the simple isochronous pendulum *OS* exceeds the distance of the centre of inertia *I* from the axis of gyration *O* by the interval $IS = \frac{kk}{f}$. Now with the length *OS* = *l* known,

$$kk = f\left(l - f\right) = OI \cdot IS ,$$

thus in order that for the same body the rectangle $OI \cdot IS$ is constant.

COROLLARY 2

539. If the distance $OI = f$ is changed for the same body, it is apparent that as in the case *f* = 0 as $f = \infty$ that the simple isochronous pendulum *l* emerges as infinite ; moreover the shortest will be, if there is taken *f* = *k*, in which case $l = 2k$; in addition, it is always the case that $l > 2k$.

COROLLARY 3

540. With the length of the simple isochronous pendulum *l*, since likewise the small oscillations of the body and of this pendulum are isochronous, the time of each oscillation is equal to $\frac{\pi\sqrt{l}}{\sqrt{2g}}$ sec. (§ 215). Hence if there appears a length $l = \frac{2g}{\pi\pi}$, then the individual small oscillations are completed in a time of one second.

SCHOLIUM

541. Hence an easy method is put together by which the moment of inertia of each body can be defined practically. For with the body suspended from a horizontal axis, around which is able to rotate freely, with every attention taken in the first place, the distance of the centre of inertia *I* from the axis of rotation *O* is defined, surely by
OI = *f*, since that can be done practically; then the body is set in motion to perform small oscillations, and with several counted in a given time, the time of a single oscillation is found, which is equal to τ sec., hence there is obtained $l = \frac{2g\tau\tau}{\pi\pi}$; from which there is found $kk = f\left(l - f\right)$, and the weight of the body *M* multiplied by *kk* gives the moment of inertia about the axis passing through the centre of inertia and parallel to the axis of gyration. Also this experiment can be performed several times, while the body is suspended from various axes

successively, which are still parallel to each other, from which we may return a more reliable value for the true value of kk. Hence in turn, it is possible also to investigate the length of the individual simple pendulum [with a period] of oscillation of one second, seeing that neither is it possible for simple pendulums to be used in experiments, nor is it possible for the distance fallen g in an absolute time of one second to be determined accurately enough by experiment. Hence for a suspended body, it is necessary to know the quantities f and kk accurately, and that time is found from the length of the simple pendulum $l = f + \frac{kk}{f}$; then, if the time of one small oscillation τ is observed, there is obtained $g = \frac{\pi\pi l}{2\tau\tau}$ and hence the length of the simple pendulum with a period of one second is found from $\frac{2g}{\pi\pi} = \frac{l}{\tau\tau}$.

[We note that Euler's period for a simple pendulum $\tau = \pi\sqrt{\frac{l}{2g_e}}$ is half the modern period $T = 2\pi\sqrt{\frac{l}{g}}$, though in the *Mechanica* he had given the time for an oscillation as that of a complete to and fro return swing; presumably at this time in the 1760's, others had not followed his lead, and so he reverted to the old usage of Huygens and Newton and others of the time. We also note again that Euler's constant that is now called g_e to prevent confusion, is the distance fallen in a time of one second; thus $2\,g_e$ is numerically equal to the acceleration of gravity g. With these thoughts in mind we can recover the modern formula, to which Euler's result is of course equivalent.]

DEFINITION 9

542. The centre of oscillation in a compound pendulum is the point at which if the whole mass of the body were to be gathered, the same oscillatory motion would be produced. Moreover it is taken on a right line which passes through the centre of inertia of the body normal to the axis of rotation.

COROLLARY 1

543. Therefore the distance of the centre of oscillation from the axis of rotation is equal to the length of a simple isochronous pendulum : and the interval $IS = \frac{kk}{f}$ is more distant from the axis of rotation O than the centre of inertia.

COROLLARY 2

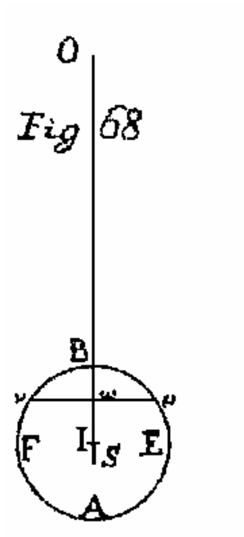
544. Therefore, as regards finding the centre of oscillation *S,* it is necessary to know the moment of inertia about the axis passing through the centre of inertia *I* and parallel to the axis of rotation, since if it is equal to *Mkk*, then it must be divided by *Mf*, that is by the product of the mass of the body *M* by the distance of the axis of rotation from the centre of inertia $OI = f$, and then the quotient $\frac{Mkk}{Mf}$ shows the distance of the centre of oscillation from the centre of inertia.

SCHOLIUM

545. In this manner, the investigation of the motion of compound pendulums is accustomed to lead to an investigation of the centre of oscillation, although for that it is sufficient to know the length of the simple isochronous pendulum, nor by any other account does it have forces acting on it, so that this pendulum is considered to be applied from the same axis of suspension and indeed drawn along the line through the centre of inertia normal to the axis of suspension. Now this is the most convenient way of considering the matter, and if the body should hang in the resting position, so that the line drawn through the centre of inertia normally to the axis likewise is vertical, then the centre of oscillation on the same line is lower than the centre of inertia is placed; nor indeed is it necessary here, that the body is considered as in motion. Thus the line *OI* is considered to fall on the vertical *OC*, on which the centre of oscillation *S* is placed lower than the centre of inertia *I*, that here actually maintains the name of the centre of gravity, thus so that the interval $IS = \frac{Mkk}{Mf} = \frac{kk}{f}$. Whereby the calculation of the centre of oscillation is easily brought about by the calculation, that we have treated above for finding the moment of inertia.

EXAMPLE

546. The experiments mentioned before are accustomed to be set up with a sphere made from a homogeneous material (Fig. 68), which hanging with the aid of a thread *OB* start to make small oscillations, where indeed a thread so thin has to be taken, so that the mass of this can be taken as nothing besides the mass of the sphere. Therefore let the radius of the sphere be $BI = b$ and the distance of the point of suspension *O* from the centre of the sphere *I*, which likewise is the centre of inertia or of gravity, clearly then $OI = f$, so that as we found above, $kk = \frac{2}{5}bb$. Whereby the centre of oscillation is at *S*, in order that $IS = \frac{2bb}{5f}$, or the oscillations agree with the oscillations of a simple pendulum, the length of which is equal to $f + \frac{2bb}{5f}$. Hence, in order that this pendulum oscillates in a time of one second, it is necessary that



$$f + \frac{2bb}{5f} = \frac{2g}{\pi\pi}$$

or

$$ff = \frac{2gf}{\pi\pi} - \frac{2}{5}bb,$$

thus

$$f = \frac{g}{\pi\pi} \pm \sqrt{\left(\frac{g}{\pi^4} - \frac{2}{5}bb\right)},$$

thus, so that there is a twofold value for *f*, which likewise taken give $\frac{2g}{\pi\pi}$. Both these values become equal, if only a sphere is taken, so that

$$bb = \frac{5gg}{2\pi^4} \text{ and } b = \frac{g}{\pi\pi}\sqrt{\frac{5}{2}};$$

that is, the radius of the sphere in Rhenish feet must be equal to 2,50317, and then the distance $OI = f$ becomes 1,583144 ft., thus so that the point of suspension or the axis of rotation must be taken within the sphere. Moreover since then

$$f = \frac{g}{\pi\pi} = b\sqrt{\frac{5}{2}}$$

or $f = k$, it is apparent in this case that the sphere oscillates the fastest. Clearly if $I\omega = b\sqrt{\frac{5}{2}}$ is put in place, with the horizontal $\mu\nu$ drawn, which refers to the axis of rotation, then $\cos B\mu = \sqrt{\frac{5}{2}}$ and thus the arc $B\mu = 50^{\circ}\ 46'$. But if the sphere were very small, as it is accustomed to happen, in order that one second is produced, then there must be taken

$$OI = \frac{2g}{\pi\pi} - \frac{\pi\pi bb}{5g};$$

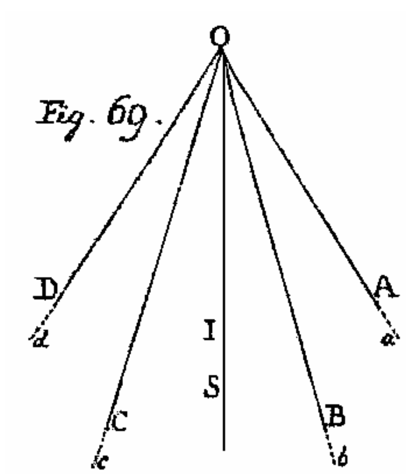
whereby, so that a sphere hanging from the point B performs this, the radius of this must be

$$b = \frac{\left(\sqrt{65}-5\right)g}{2\pi\pi} = 0{,}155136g \text{ nearly.}$$

PROBLEM 47

547. If a rigid body mobile about a horizontal axis should be constructed from several parts, the centres of inertia and the moments of inertia of which are known, to define the centre of oscillation of the whole body.

SOLUTION



The horizontal axis of rotation is considered normal to the plane of the figures at the point *O* (Fig. 69), and let *A, B, C, D* be the centres of inertia of the parts, from which the body is composed, of which the masses of the parts shall be *A, B, C, D*, and the moments of inertia about the parallel axes of rotation, and passing through each centre of inertia shall be $Aa^2, Bb^2, Cc^2, Dd^2;$ moreover, the centres of inertia shall be distant from the axes of rotation by the intervals *AO, BO, CO, DO*; likewise indeed, either these intervals point towards the same point of the axis *O* or to different points, because the moments of the weights as well as the moments of inertia only depend on the distances from the axis, nor any difference of the points *O* is conferred by that. Therefore in the first place the centre of inertia of the whole body *I* is defined and the mass of this shall be equal to $M = A + B + C + D$, which is to be placed on such a line *OI*, in order that

$$A\cdot OA\cdot sin\,AOI + B\cdot OB\cdot sin\,BOI = C\cdot OC\cdot sin\,COI + D\cdot OD\cdot sin\,DOI;$$

then also :

$$M\cdot OI = A\cdot AO\cdot cos\,AOI + B\cdot BO\cdot sin\,BOI + C\cdot CO\cdot cos\,COI + D\cdot DO\cdot cos\,DOI;$$

which quantity must be written in the above formula $IS = \frac{Mkk}{Mf}$ in place of Mf.
But the moment if inertia of the whole body about the axis of rotation $M(ff + kk)$, which is put together from the parts, becomes :

$$A\left(AO^2 + aa\right) + B\left(BO^2 + bb\right) + C\left(CO^2 + cc\right) + D\left(DO^2 + dd\right).$$

Whereby, since

$$OS = \frac{M(ff + kk)}{Mf},$$

becomes

$$OS = \frac{A\left(AO^2+aa\right)+B\left(BO^2+bb\right)+C\left(CO^2+cc\right)+D\left(DO^2+dd\right)}{A\cdot AO\cdot cos\,AOI+B\cdot BO\cdot cos\,BOI+C\cdot CO\cdot cos\,COI+D\cdot DO\cdot cos\,DOI}.$$

COROLLARY 1

548. If the individual parts are considered separately and the centres of oscillation of these are put in place at the points a, b, c, d, on account of

$$Oa = \frac{A\left(AO^2+aa\right)}{A\cdot AO}$$

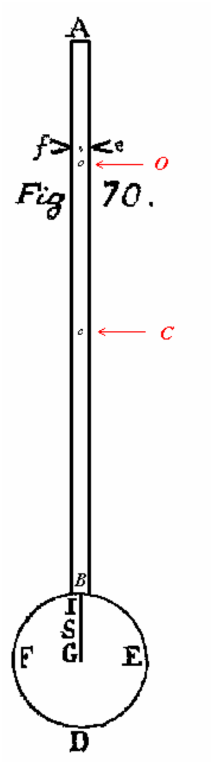
then it is found that :

$$OS = \frac{A\cdot OA\cdot Oa+B\cdot OB\cdot Ob+C\cdot OC\cdot Oc+D\cdot OD\cdot Od}{A\cdot AO\cdot cos\,AOI+B\cdot BO\cdot cos\,BOI+C\cdot CO\cdot cos\,COI+D\cdot DO\cdot cos\,DOI}.$$

COROLLARY 2

549. Moreover with the centre of inertia or gravity found of the whole body *I*, in place of the denominator it is possible to put in place $M\cdot OI$; but from the preceding, the static centre of gravity of the whole body from the given centres of gravity of the parts is easily put together.

EXAMPLE



550. Let a pendulum be composed from the right cylinder *ACB* and from that attached sphere *BEDF*, which shall be free to move about the horizontal axis *eOf* (Fig. 70), and the centre of oscillation *S* of this pendulum is sought. Moreover the rod and the sphere are made from a uniform material and the length of the rod *AB* is put equal to *a*, the weight equal to *A*, and the distance of the extremity *B* from the axis of rotation *O*, *BO* is equal to *b*, while the radius of the base of this cylinder is equal to *c* ; let the centre of inertia of this be at *C*, so that it becomes $AC = BC = \frac{1}{2}a$ and $OC = b - \frac{1}{2}a$, now the moment of inertia about the axis drawn through *C* and parallel to the axis of rotation is equal to $A\left(\frac{1}{12}aa + \frac{1}{4}cc\right)$. Again let the mass of the attached sphere be equal to *E*, the radius *BG* = *e*, then the centre of inertia of this is at *G*, and the moment of inertia = $\frac{2}{5}Eee$. Now let *I* be the centre of inertia of the whole body, then

$$(A+E)\cdot OI = A\left(b-\tfrac{1}{2}a\right) + E\left(e+b\right) = Mf;$$

then the moment of inertia about the axis of gyration is equal to

$$A\left(\frac{1}{12}aa + \frac{1}{4}cc + \left(b-\tfrac{1}{2}a\right)^2\right) + E\left(\tfrac{2}{5}ee + (b+e)^2\right),$$

that must be substituted in place of $M\left(ff+kk\right)$. Thus the centre of oscillation is at S, so that it shall be

$$OS = \frac{A\left(\frac{1}{3}aa - ab + bb + \frac{1}{4}cc\right) + E\left(bb + 2be + \frac{7}{5}ee\right)}{A\left(b-\frac{1}{2}a\right) + E\left(b+e\right)},$$

hence on account of $OG = b + e$ there becomes

$$GS = \frac{A\left(be + \frac{1}{2}ab - \frac{1}{2}ae - \frac{1}{3}aa - \frac{1}{4}cc\right) - E\cdot\frac{2}{5}ee}{A\left(b-\frac{1}{2}a\right) + E\left(b+e\right)}.$$

COROLLARY 1

551. If the axis or rotation *O* is taken at the top of the rod *A*, so that there arises $b = a$, then

$$OS = \frac{A\left(\tfrac{1}{3}aa+\tfrac{1}{4}cc\right)+E\left(aa+2ae+\tfrac{7}{5}ee\right)}{A\cdot\tfrac{1}{2}a+E\left(a+e\right)}$$

and

$$GS = \frac{A\left(\tfrac{1}{2}ae+\tfrac{1}{6}aa-\tfrac{1}{4}cc\right)-E\cdot\tfrac{2}{5}ee}{A\cdot\tfrac{1}{2}a+E\left(a+e\right)},$$

if indeed we take the point *S* to fall beyond *G*.

COROLLARY 2

552. If for the sake of example,
$E=30A, a=b=3$ ft., $e=\frac{1}{12}$ ft., and $c=\frac{1}{500}$ ft., thus so that *cc* may be safely neglected, then

$$OG = 3\frac{1}{12} = 3{,}0833$$

and

$$OS = \frac{3+285\frac{7}{24}}{1\frac{1}{2}+92\frac{1}{2}} = \frac{288\frac{7}{24}}{94} = 3{,}0669,$$

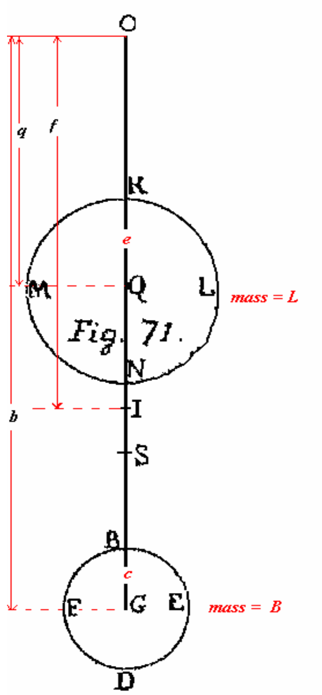
and in this case the point *S* falls above *G*; but if the mass of the rod should vanish, it will be given by OS = 3,0842, and thus *S* will fall below *G*.

SCHOLIUM

553. This final case thus worth noting, since a common thread, if it should be very thin and light with respect of the sphere, clearly indeed is seen here to convey hardly anything to the centre of oscillation, but here it is the case that although the sphere is thirty time heavier than the thread, an account of this cannot be neglected without a significant error. Indeed we may put this pendulum oscillations to be completed in one second, and the length of the isochronous simple pendulum hence is required to be determined. Therefore this equation with the mass of the thread ignored gives a length equal to 3,0842 ft., yet actually with the thread is shall only be 3,0669 ped., thus so that an error of 0,0173 ped. = $2\frac{1}{2}$ lin. is committed, clearly the smallest to be tolerated. But if, with the remaining dimensions kept in place, the thread at this stage should be lighter and $E = 60A$, then $OS = \frac{3+570\frac{7}{12}}{1\frac{1}{2}+185} = 3{,}0755$, in place of this if there is taken 3,0842, then the error committed is equal to 0,0087 ft $= \frac{5}{4}$ lin. [1lin.= 12th part of an inch; there is an error in Euler's arithmetic in calculating *OS*, which has been corrected here as in the *O.O.*]

PROBLEM 48

554. If the pendulum should be established from the narrowest rod *OB* free of inertia, yet rigid, and with the sphere *BDEF* (Fig. 71), to find the place where another sphere must be attached to the same rod, so that the most frequent oscillations are made.



SOLUTION

Since the axis of rotation shall be at *O*, let the distance $OG = b$ and the radius of the sphere to be fixed below be $BG = c$, and the mass of this globe is equal to B; then let the mass of the other sphere to be attached be equal to L and the radius $QK = e$, for the position of this is sought at the distance $OQ = q$. With these in place, let the common centre of inertia be *I*, then $(B+L)OI = Bb + Lq = Mf$, then the moment of inertia of the whole pendulum about the axis of rotation is now equal to :

$$B\left(\frac{2}{5}cc+bb\right)+L\left(\frac{2}{5}ee+qq\right)=M\left(ff+kk\right).$$

Whereby, if the centre of oscillation should be set up at *S*, then [the length of the simple pendulum : see § 543, etc]

$$OS=\frac{B\left(\frac{2}{5}cc+bb\right)+L\left(\frac{2}{5}ee+qq\right)}{(Bb+Lq)},$$

which must be the shortest length, so that the oscillations become the most prompt. Hence this equation arises [on differentiating the expression for the period and equating to zero]

$$2BLbq - BL\left(\tfrac{2}{5}cc + bb\right) - \tfrac{2}{5}LLee + LLqq = 0$$

or

$$Lq = -Bb + \sqrt{\left(BBbb + BLbb + \tfrac{2}{5}BLcc + \tfrac{2}{5}LLee\right)},$$

hence the distance $OQ = q$ becomes known, from which again the length of the simple isochronous pendulum can be gathered:

$$OS = \tfrac{2}{L}\sqrt{\left(BBbb + BLbb + \tfrac{2}{5}BLcc + \tfrac{2}{5}LLee\right)} - \tfrac{2Bb}{L} = 2q.$$

Hence, if both spheres were constructed from the same material, on account of $B{:}L = c^3{:}e^3$ then

$$OS = \frac{2\sqrt{\left(c^6bb + c^3e^3 + \frac{2}{5}c^5e^3 + \frac{2}{5}e^8\right)} - 2c^3b}{e^3}$$

and

$$OQ = q = \frac{\sqrt{\left(c^6bb + c^3e^3bb + \frac{2}{5}c^5e^3 + \frac{2}{5}e^8\right)} - c^3b}{e^3}.$$

COROLLARY 1

555. If the diameters of the spheres were small, so that *cc* and *ee* are able to be ignored besides *bb*, then the distance $OQ = q$ thus can be taken so that it becomes :

$$OQ = \frac{\sqrt{B(B+L)} - B}{L}b,$$

and the length of the simplest isochronous pendulum will be

$$2\cdot OQ = 2b\cdot\frac{\sqrt{B(B+L)} - B}{L}.$$

COROLLARY 2

556. Clearly, if the other sphere *KLMN* is disregarded, then the required length is

$$OS = b + \frac{2cc}{5b},$$

which is a greater length than with that sphere connected, if it should be that

$$b + \frac{2cc}{5b} > 2e\sqrt{\frac{2}{5}}.$$

From which, unless

$$e > \frac{\sqrt{5}}{2\sqrt{2}}\left(b + \frac{2cc}{5b}\right),$$

the oscillations are able to be returned more promptly with the other sphere attached.

COROLLARY 3

557. But if

$$e = \frac{\sqrt{5}}{2\sqrt{2}}\left(b + \frac{2cc}{5b}\right),$$

however large the mass of this sphere L should be, for the most rapid oscillations obtained, it is necessary to take

$$OQ = q = \tfrac{1}{2}b + \frac{cc}{5b},$$

and then the length of the simple isochronous pendulum will be equal to $\frac{1}{2}b + \frac{cc}{5b}$, for all, and if the sphere $KLMN$ should be removed.

COROLLARY 4

558. If both of the spheres should be equal, so that $L = B$ and $e = c$, the most rapid oscillations arise on taking

$$OQ = q = \sqrt{\left(2bb + \tfrac{4}{5}cc\right)} - b;$$

and if it is allowed to ignore cc in comparison with bb,

$$OQ = OG\left(\sqrt{2} - 1\right);$$

and hence the length of the isochronous simple pendulum is equal to :

$$2 \cdot OG\left(\sqrt{2} - 1\right) = 0{,}828427\ OG.$$

COROLLARY 5

559. If both spheres consist of the same material, it is possible to define the radius e of the sphere $KLMN$, in order that with that duly joined the oscillations are made the most rapidly; it follows that e must be sought from this equation :

$$16e^{10} - 48c^5e^5 - 600bbc^6ee + 9c^6\left(5bb + 2cc\right)^2 - 120bbc^3e^5 = 0.$$

SCHOLIUM

560. Moreover it is apparent, in which the radius *e* of the sphere *KLMN* is less with its mass remaining at *L*, from this to give rise to a smaller distance $OQ = q$, and therefore the oscillations from this to be more rapid. But now with the radius *e* remaining fixed the oscillations become swifter, if the mass *L* of the attached sphere should be as large as possible ; for if $L = 0$, then

$$OS = b + \frac{2cc}{5b},$$

which is the maximum value, since indeed by attaching the other sphere, the oscillations are made more rapidly [Opera Omnia note: Euler has omitted the case $e < \frac{5bb+2cc}{2b\sqrt{10}}$.] But now, if it should be the case that

$$5bb + 2cc = 2be\sqrt{10} \text{ or } e = \frac{5bb+2cc}{2b\sqrt{10}},$$

however large the mass *L* of the sphere should be, with that duly attached the oscillations remain of the same duration, and if this sphere now should be made larger, the oscillations henceforth are produced slower. But if both spheres were made from materials with equal weight, then the size to be attached, so that the motion of the oscillations should become the most rapid, must be determined from an equation of the tenth order; now if the axis passes through the centre *G* of the sphere *BCDF*, so that $b = 0$, from which there is produced $e = c\sqrt[5]{\frac{3}{2}}$ for the radius of the attached sphere, and for the position of this :

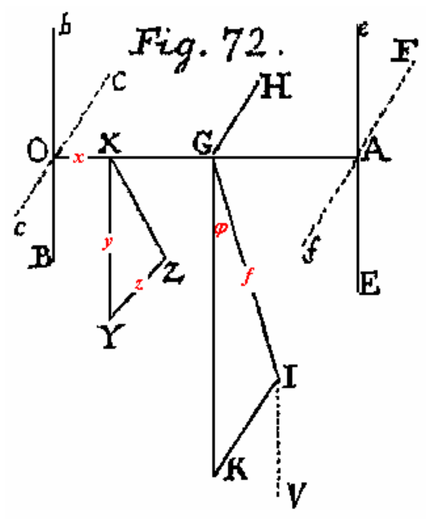
$$OQ = q = \sqrt{\left(\frac{2}{5}\frac{c^5}{e^3} + \frac{2}{5}e^2\right)} = c\sqrt[10]{\frac{8}{27}}$$

and the length of the simple isochronous pendulum is equal to $c\sqrt[10]{\frac{8}{27}}$. Therefore from the axis of rotation passing through the centre of the first sphere, the other sphere thus must be transported, so that the distance between the centres is $OQ = c\sqrt[10]{\frac{8}{27}}$, which is less than the radius of this sphere, $e = c\sqrt[5]{\frac{3}{2}}$. Moreover several questions of this kind about the motion of the oscillations can be proposed, which from the principles established here can be solved without difficulty. But it will be of great interest to investigate the size of the forces sustained by the axis during the rotational motion.

PROBLEM 49

561. While a heavy rigid body is rotating about a fixed horizontal axis *OA*, to define the forces, which the axis will sustain at any time at the two given points *O* and *A*.

SOLUTION



The diagram represents a plane with the vertical passing through the axis of rotation *OA* (Fig. 72), and now the centre of inertia of the body is rotating beyond this plane at *I*, thus perpendiculars *IK* and *IG* are drawn both to the vertical plane and to the axis, then the angle $IGK = \varphi$ is the departure of the body from the natural position, and on putting the distance $IG = f$ then $KI = f \sin\varphi$ and $GK = f \cos\varphi$. Moreover the mass of the body is equal to *M*, which since likewise this expresses the weight, the force acting is equal to *M* in the vertical direction *IV*, the moment of this is equal to $Mf \sin\varphi$ tending to decrease the angle *IGK*. Then some element of the body *dM* is to be considered at *Z* , from which with the perpendiculars drawn *ZY, ZX* to the vertical plane and to the axis, the coordinates are be called

$$OX = x, \quad XY = y \quad \text{and} \quad YZ = z,$$

and hence

$$OG = \frac{\int xdM}{M}, \quad GK = \frac{\int ydM}{M}, \quad \text{and} \quad KI = \frac{\int zdM}{M};$$

on putting the distance

$$\text{XZ}=\sqrt{(yy+zz)} = r$$

then $\int rrdM$ expresses the moment of inertia of the body about the axis *OA*, which is equal to Mkk; and then on putting the distance of the end points of the axis $OA = a$, and through both there are drawn the lines *BOb*, *COc* and *EAe*, *FAf* with themselves parallel to *KG* and *KI*, [so giving the directions of the *y* and *z* components of the reactive forces]. With these drawn prepared according to problem 23, in the first place I observe that no force is present along the axis in the same plane; but since here the moment of the force $Mf\ sin\varphi$ turns in the opposite sense, and we have assumed it becomes $Vf = -Mf\ sin\varphi$.

Now therefore on account of the force *IV* = *M*, which is considered to be applied along the direction of the axis *GK* at *G*, the axis sustains these forces at the points *O* and *A* :

along *OB* the force $= \frac{M \cdot AG}{a}$, along *AE* the force $= \frac{M \cdot OG}{a}$.

With which these are to be joined, which arise from opposing the elementary forces applied, which are

for the end O: along *Ob* the force $= \frac{f\ sin\varphi \cdot \int (a-x)zdM}{akk}$; along *Oc* the force $= \frac{f\ sin\varphi \cdot \int (a-x)ydM}{akk}$

for the end A: along *Ae* the force $= \frac{f\ sin\varphi \cdot \int xzdM}{akk}$; along. *Ae* the force $= \frac{f\ sin\varphi \cdot \int xydM}{akk}$

and the axis sustain these forces on account of the action of the weight of the body, and now on account of the motion, with which it now rotates; if the angular speed is called γ, then the axis sustains these forces at the points O and A :

at the end *O*: along *OB* the force $= \frac{\gamma\gamma \int (a-x)zdM}{2ag}$; along *OC* the force $= \frac{\gamma\gamma \int (a-x)ydM}{2ag}$

at the end *A*: along *AE* the force $= \frac{\gamma\gamma \int xzdM}{2ag}$; along *AE* the force $= \frac{\gamma\gamma \int xydM}{2ag}$

[It may be of interest to indicate more fully here the source of these forces. For the first set, applied to an element of mass *dM* at *Z* (*x*, *y*, *z*), we note that in general this element has an angular acceleration α equal to that of the rigid body, produced by a linear force $y\alpha dM$ acting normally to the *xz* plane and normal to the axis, and similarly by a linear force $z\alpha dM$ for the yz plane. These forces are 'shared out' according to the principle of moments about the pivots, and α is given by the torque $Mf\ sin\varphi$ divided by the moment of inertia *I*. The latter forces are of course the centripetal forces, treated in an equivalent manner.]

COROLLARY 1

562. If the distance of the supports *O* and *A* from *G* are called *OG* = *b* and *AG* = *c*, so that $a = b + c$, now there is put *GX* = *u*, then $x = b - u$ and $a - x = c + u$; thus

$$\int(a-x)zdM = \int(c+u)zdM = Mc\cdot KI + \int uzdM,$$
$$\int(a-x)ydM = \int(c+u)ydM = Mc\cdot GK + \int uydM,$$
$$\int xzdM = \int(b-u)zdM = Mb\cdot KI - \int uzdM,$$
$$\int xydM = \int(b-u)ydM = Mb\cdot GK - \int uydM.$$

COROLLARY 2

563. With these values introduced, the axis at the point *O* sustains these forces : first along the direction *OB*, the force

$$\frac{Mc}{a} - \frac{Mcf\ sin\varphi\cdot KI}{akk} - \frac{f\ sin\varphi\cdot\int uzdM}{akk} + \frac{\gamma\gamma\cdot Mc\cdot GK}{2ag} + \frac{\gamma\gamma\cdot\int uydM}{2ag},$$

then along the direction, *OC* the force

$$\frac{Mcf\ sin\varphi\cdot GK}{akk} + \frac{f\ sin\varphi\cdot\int uydM}{akk} + \frac{\gamma\gamma\cdot Mc\cdot KI}{2ag} + \frac{\gamma\gamma\cdot\int uzdM}{2ag}.$$

Moreover, now at the point *A* these become :

firstly along the direction *AE* the force

$$\frac{Mb}{a} - \frac{Mbf\ sin\varphi\cdot KI}{akk} + \frac{f\ sin\varphi\cdot\int uzdM}{akk} + \frac{\gamma\gamma\cdot Mb\cdot GK}{2ag} + \frac{\gamma\gamma\cdot\int uydM}{2ag},$$

then along the direction *AF* the force

$$\frac{Mbf\ sin\varphi\cdot GK}{akk} - \frac{f\ sin\varphi\cdot\int uydM}{akk} + \frac{\gamma\gamma\cdot Mb\cdot KI}{2ag} - \frac{\gamma\gamma\cdot\int uzdM}{2ag}.$$

COROLLARY 3

564. If the body should be put together thus, so that it is divided in the plane *IGK* into two equal and similar parts, and let $GO = GA = \frac{1}{2}a$, on account of

$$\int uzdM = 0 \quad \text{and} \quad \int uydM = 0$$

the axis sustains these forces at the point *O*

$$\text{along } OB \text{ the force} = \tfrac{1}{2}M - \frac{Mf\ sin\varphi \cdot KI}{2kk} + \frac{\gamma\gamma \cdot M \cdot GK}{4g},$$

$$\text{along } OC \text{ the force} = \frac{Mf\ sin\varphi \cdot GK}{2kk} + \frac{\gamma\gamma \cdot M \cdot KI}{4g},$$

but at the point *A* it will sustain these forces

$$\text{along } AE \text{ the force} = \tfrac{1}{2}M - \frac{Mf\ sin\varphi \cdot KI}{2kk} + \frac{\gamma\gamma \cdot M \cdot GK}{4g},$$

$$\text{along } AF \text{ the force} = \frac{Mf\ sin\varphi \cdot GK}{2kk} + \frac{\gamma\gamma \cdot M \cdot KI}{4g};$$

therefore in this case the forces do not depend on the magnitude of the distance *OA* = *a*.

COROLLARY 4

565. Therefore in the case in which $\int uydM = 0$ and $\int uzdM = 0$, nothing stands in the way by which the distance *OA* = *a* is taken to vanish and the axis can be retained as a single point *G*; this certainly sustains the two forces :

$$\text{the one along } GK = M - \frac{Mff\ sin^2\varphi}{kk} + \frac{M\gamma\gamma f\ cos\varphi}{2g},$$

$$\text{and the other along } GH = \frac{Mff\ sin\varphi\, cos\varphi}{kk} + \frac{M\gamma\gamma f\ sin\varphi}{2g}$$

with *GH* itself being present parallel to *KI*.

SCHOLIUM

566. Bodies, which commonly are accustomed to be put into oscillatory motion, thus have been composed so that, they can be cut into two equal and similar parts by the plane which is drawn normally through their centre of mass and axes of gyration; from these therefore the place is had, that the axis can retain as single point. Clearly, if figure 67 represents a plane drawn with the vertical through the centre of inertia *I* of such a body and normal to the axis of rotation, which for the figure is considered to be placed normally at *O*, with the vertical line *OC* present and *OH* in a plane to the horizontal, the axis at the point *O* in indicating the manner it will sustain the forces. Clearly the angle *COI* is itself put equal to φ, the distance *OI = f*, the mass of the body is equal to *M*, the moment of inertia of this about the axis of gyration is equal to Mkk and the angular speed at that position is equal to γ, unless the angle *COI* tends towards a maximum or a minimum, then the axis *O* sustains the two forces,

$$\text{one along } OC \ = M - \frac{Mff\ sin^2\varphi}{kk} + \frac{M\gamma\gamma\, f\ cos\varphi}{2g},$$

$$\text{and the other along } OH = \frac{Mff\ sin\varphi cos\varphi}{kk} + \frac{M\gamma\gamma\, f\ sin\varphi}{2g}.$$

Therefore at first with the force acting downwards and the support sustaining that force; now on account of the other force in that direction, at which the centre of inertia is rotating, the body attempts to proceed horizontally on the support, which effect it is agreed to be prevented by the bar. When the centre of inertia is wandering off in the opposite direction, this horizontal force is directed in the opposite direction. Moreover both forces depend on two parts, of which the one is due to the action of the weight, and the other due to the rotary motion itself, and with *OL* drawn normal to *OI* these parts may be reduced to just a few, so that the axis is acted on at the point *O* by these forces :

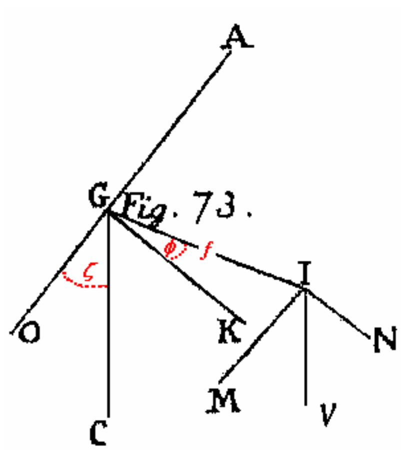
along OC by a force $= M$, along OL by a force $= \frac{Mff\ sin\varphi}{kk}$, along OI by a force $= \frac{Mf\gamma\gamma}{2g}$.

If there should not be $\int uydM = 0$ and $\int uzdM = 0$, then in addition the axis sustains these above forces § 563 at the points *O* and *A* of figure 72, which involve these integral formulas, since the remaining parts cannot be reduced to one point.

PROBLEM 50

567. If the axis *OA*, about which the heavy rigid body is free to move, should not be horizontal, to define the rotational motion as well as the forces which the axis sustains.

SOLUTION



Through the axis *OA* there is considered a plane drawn in the vertical, in which there is the vertical line *GC* (Fig. 73), and the angle $OGC = \zeta$, and the complement of this $90^{\text{o}} - \zeta$ gives the inclination of the axis *OA* to the horizontal. Now the centre of inertia *I* of the body may be found beyond this vertical plane, from which with *IG* = *f* drawn normal to the axis, and from *G* in a vertical plane and equally normal to the axis, *GK* is drawn normally in the plane *IGK* itself normal to the [initial] vertical plane, and there is put in place the angle $IGK = \varphi$, measuring the departure of the body from its natural position; for the *GI* moves in the plane *IGK*. The mass of the body and the same weight of this is put equal to *M* and the moment of inertia about the axis *OA* = *Mkk*, which likewise is gathered as if the axis were horizontal ; for the inclination is only observed for the force acting. But the effect of the weight thus responds to this, so that the body is acted on at the point *I* by a force equal to *M* in the vertical direction IV; in order that this is may be resolved there are drawn *IM* and *IN* parallel to *GO* and *GK*, and the lines *IM*, *IV* and *IN* are in a vertical plane and the angle $MIV = \zeta$. Hence from the force *IV* = *M* there are two forces arising, the one $IM = M\cos\zeta$ and the other along $IN = M\sin\zeta$. As before since it is parallel to the axis, clearly nothing is added to the motion, but all is expended on the axis, as we have shown above. Therefore only the force $IN = M\sin\zeta$ remains for the motion, the direction of which is parallel to *GK*, the moment arises equal to $Mf\sin\zeta$, the direction of this is parallel to GK, and there arises the moment equal to $Mf\sin\zeta\sin\varphi$ tending to minimise the angle *IGK*, and for the motion to be defined, the above formulas prevail for the horizontal axis found, except, as in place of the moments of the force acting, which before was equal to $Mf\ sin\zeta$, here must be written as $Mf\ sin\zeta\ sin\varphi$, or since *M* denotes the weight of the body, in place of this there must be written $M\ sin\zeta$, but since the moment of inertia is present, the rest need not be changed. Whereby the motion is similar to the motion of a simple pendulum about a horizontal axis, the length of which is equal to $\frac{Mkk}{Mf\ sin\zeta} = \frac{kk}{f\ sin\zeta}$, so that the motion itself is perfectly determined. But for the restraining forces, which the axis meanwhile sustain, if placed as it pleases at the given points *O* and *A*, in the first place on account of the force $IM = M\ cos\zeta$ the axis is urged now along its own direction *AO* in addition by so large a force, $\frac{GI}{OA}\cdot M\ cos\zeta$ at both *O* and *A*, at the point *A* clearly acting along the direction parallel to *GI*, but now at *O* along the opposite direction. Then besides these forces at the points *O* and *A* from the same forces acting, which we have determined in the preceding problem, by observing only this, that for *M* it is necessary to write $M\ sin\zeta$ and $f\ sin\zeta\ sin\varphi$ in place of $f\ sin\varphi$. Certainly, if our inclined axis is in figure 72 *OA* and everything remains as in the preceding problem, then the axis besides these forces, is sustained in addition to these forces by the force arising $IM = M\ cos\zeta$. In the first place at the point *O* along the direction *OB* the force :

$$\frac{Mc\, sin\zeta}{a} - \frac{Mcff\ sin\zeta\ sin^2\varphi}{akk} - \frac{f\ sin\zeta\ sin\varphi\cdot\int uzdM}{akk} + \frac{Mcf\gamma\gamma\ cos\varphi}{2ag} + \frac{\gamma\gamma\int uydM}{2ag}$$

and along the direction *OC* the force

$$\frac{Mcff\ sin\zeta\ sin\varphi\, cos\varphi}{akk} + \frac{f\ sin\zeta\ sin\varphi\cdot\int uydM}{akk} + \frac{Mcf\gamma\gamma\ sin\varphi}{2ag} + \frac{\gamma\gamma\int uzdM}{2ag}.$$

Then at the point *A* along the direction *AE* the force

$$\frac{Mb\, sin\zeta}{a} - \frac{Mbff\ sin\zeta\ sin^2\varphi}{akk} + \frac{f\ sin\zeta\ sin\varphi\cdot\int uzdM}{akk} + \frac{Mbf\gamma\gamma\ cos\varphi}{2ag} - \frac{\gamma\gamma\int uydM}{2ag}$$

and along the direction *AF* the force

$$\frac{Mbff\ sin\zeta\ sin\varphi\, cos\varphi}{akk} - \frac{f\ sin\zeta\ sin\varphi\cdot\int uydM}{akk} + \frac{Mbf\gamma\gamma\ sin\varphi}{2ag} - \frac{\gamma\gamma\int uzdM}{2ag},$$

where *OA* = *a*, *OG* = *b*, *AG* = *c* and the angular speed = γ, with the integrations taken as we have described here.

COROLLARY 1

568. Since the length of the simple pendulum is equal to $\frac{kk}{f\ sin\zeta}$, the body completes it own slower oscillations about the inclined, than if the axis were horizontal, and if the oscillations should be the shortest, the time of one will be equal to $\pi\sqrt{\frac{kk}{2\,fg\ sin\zeta}}$ sec.

COROLLARY 2

569. If the axis has been inclined, also the force sustained along its own direction *AO* (Fig. 73), which is $M\ cos\zeta$, and all the other forces are normal to the axes and can be recalled for the two given points *O* and *A*.

COROLLARY 3

570. If the body is bisected by the plane *IGK* into two equal and similar parts, the values of the integrals $\int uydM$ and $\int uzdM$ vanish and all the forces besides these, which arise from the force *IM*, are able to be reduced to one point *G* , as above.

SCHOLIUM

571. These are the problems, which were considered to be shown concerning the rotational motion of rigid bodies about a fixed axis, where indeed the determination of the motion itself has been reduced, so that it can be handled with no more difficulty than the motion of a body about a fixed axis, if the moment of inertia were investigated in this way. But the forces, which the axis sustain during the motion, are much more troublesome to examine and generally calculate, since from the shape of the body the values of the two integrals $\int xydM$ and $\int xzdM$ must be elicited. Now this investigation is of the greatest importance, if we wish to progress to the motion of bodies about axes which are not fixed; where in the first place indeed it is agreed to examine these more carefully, in which the axis remains at rest spontaneously, even if from the outside it is not retained. Therefore from some proposed rigid body it is to be inquired, whether axes of some kind can be given in that, if about which the body undertakes rotational motion, thus the axis itself should sustain no forces; then also it can be considered, by which forces the body about such an axis must be acted on by such forces, so that hence also no forces arise parting from the axis.

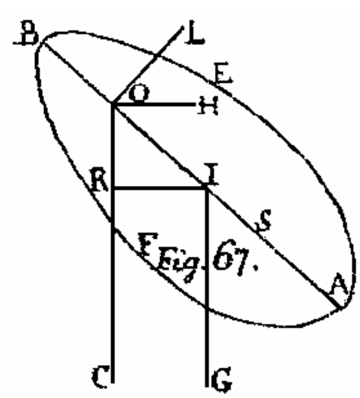
CAPUT VII

DE MOTU OSCILLATORIO CORPORUM GRAVIUM

PROBLEMA 44

522. Si corpus rigidum fuerit mobile circa axem horizontalem fixum eiusque motus a sola gravitate turbetur, determinare mutationem momentaneam in motu gyratorio productam.

SOLUTIO



Communem hic gravitatis hypothesin assumo, qua singula corporis elementa massis proportionaliter deorsum urgentur secundum directiones inter se parallelas. Quatenus ergo corpus est rigidum, his omnibus viribus aequivalet una vis ponderi corporis aequalis, cuius directio deorsum tendens per eius centrum inertiae transit. Quare, si corporis massa dicatur = *M* eiusque centrum inertiae sit in *I* (Fig. 67) indeque deorsum ducatur recta verticalis *IG*, ob gravitatem corpus sollicitabitur in directione *IG* a vi, quae ipsa massae *M* aequalis est statuenda, quandoquidem ipsam massam *M* per pondus huius corporis exprimimus. Porro, cum axis gyrationis sit horizontalis, ad eum normaliter constituatur planum per centrum inertiae *I* transiens, quod erit verticale et ipso plano tabulae referetur; axis igitur gyrationis ad hoc planum normalis per punctum *O* traiectus concipiatur, unde gyrationis ad hoc planum normalis per punctum *O* traiectus concipiatur, unde ad *I* ducta recta *OI* exhibet distantiam centri inertiae *I* ab axe gyrationis. His praemissis teneat nunc corpus *AEBF* situm in figura repraesentatum, ductaque verticali *OC* ex angulo *COI* situs corporis innotescit. Ponatur intervallum $OI = f$ et ad tempus = t angulus $CIO = \varphi$, erit vis $IG = M$, momentum respectu axis gyrationis = $Mf\ sin\varphi$, tendens ad angulum *COI* minuendum, quae in problemate 22 loco momenti Vf est substituenda. Praeterea vero necesse est nosse momentum inertiae corporis respectu axis gyrationis *O*, ibi per $\int rrdM$ indicatum; hunc in finem concipiatur axis per ipsum centrum inertiae *I* transiens axi gyrationis parallelus, cuius respectu sit momentum inertiae corporis = *Mkk*, erit eiusdem momentum inertiae respectu axis gyrationis *O* = $M\left(ff+kk\right)$ ob intervallum horum axium *OI* = *f*. Hinc, si corpus ita gyratur, ut recta *OI* accedat at verticalem *OC*, fuerit celeritas angularis = γ, quia ea a vi sollicitante augetur, per § 408 erit

$$d\gamma = \frac{2g\cdot Mf\ sin\varphi}{M\left(ff+kk\right)}dt$$

seu

$$d\gamma = \frac{2\ fgdt\ sin\varphi}{ff+kk};$$

sin autem recto OI recederet a verticali OI celeritate angulari = γ, foret

$$d\gamma = -\frac{2\ fgdt\ sin\varphi}{ff+kk}.$$

Cum autem illo casu sit $\gamma = \frac{-d\varphi}{dt}$, hoc vero $\gamma = \frac{d\varphi}{dt}$, sumto *dt* constante pro utroque erit

$$dd\varphi = -\frac{2\ fgdt^2\ sin\varphi}{ff+kk};$$

ubi signum – adest, quia momentum vis sollicitantis tendit ad angulum φ minuendum.

COROLLARIUM 1

523. Si corpus in situ *AEBF* nullum adhuc habeat motum, a gravitate ita rectam verticalem *OC* versus urgebitur, ut tempusculo *dt* eo sit accessurum per angulum =

$$\frac{2\ fgdt^2\ sin\varphi}{ff+kk},$$

qui est infinite parvus secundi ordinis.

COROLLARIUM 2

524. Si ergo corpus fuerit in quiete, in quiete persistere nequit, nisi sit $sin\varphi = 0$, hoc est nisi centrum inertiae *I* in recta verticali *OC* versetur. Quare, si corpus quodcunque hoc modo suspendatur, in quiete esse nequit, nisi recta *OI* sit verticalis, quod fit, si centrum inertiae locum vel imum vel summum obtineat.

COROLLARIUM 3

525. Quoties autem recta *OI* fuerit obliqua, corpus ob gravitatem ad motum sollicitabitur ac, si iam habuerit motum, eius motus perturbatur vel accelerando vel retardando, prout motus vel ad *OC* accedat vel ab eo recedat.

COROLLARIUM 4

526. Patet etiam, si axis per ipsum centrum inertiae *I* transeat, ut sit $OI = f = 0,$ momentum gravitatis evanescere motumque gyratorium propterea plane non turbari. Hoc ergo casu corpus vel quiescat vel uniformiter circa axem O gyrabitur.

SCHOLION

527. Hic statim notari convenit corpus non perinde moveri, ac si tota eiues massa in ipsius centro inertiae *I* esset collecta, quemadmodum in motu progressivo usu venire vidimus. Si enim hic tota corporis massa *M* revera in centro inertiae *I* esset collecta, eius momentum inertiae respectu axis per *I* ducti evanesceret foretque *kk* = 0; motus ergo ita perturbaretur, ut esset

$$dd\varphi = \frac{-2gdt^2 \sin\varphi}{f},$$

quae formula maior est quam casu proposito. Unde intelligitur motum corporis extensi, quale hic contemplamur, minus a gravitate perturbari, quam si tota corporis massa in centro inertiae esset collecta. Verum infra videbimus dari in recta *OI* aliud punctum magis ab axe *O* remotum, in quo si tota corporis massa esset collecta, motus eandem perturbationem esset passurus, quod punctum in motu gyratorio imprimis notari meretur, quoniam est id ipsum, quod vulgo *centrum oscillationis* appellari solet et de cuius inventione plurima passim occurrunt praecepta.

PROBLEMA 45

528. Si corpus rigidum *AEBF* fuerit mobile circa axem horizontalem (Fig. 67) eiusque detur situs et celeritas initio motus, ad tempus quodvis invenire eius situm et celeritatem.

SOLUTIO

Manentibus omnibus uti in praecendente problemate, scilicet massa corporis = *M*, distantia centri inertiae *I* ab axe gyrationis *O* scilicet *OI* = *f* et momento inertiae respectu axis ipsi axi gyrationis paralleli et per *I* transeuntis =

Mkk, teneat corpus elapso tempore = *t* situm in figura repraesentatum sitque angulus $COI = \varphi$ existente *CO* recta verticali; atque sumto elemento *dt* constante pervenimus ad hanc aequationem

$$dd\varphi = \frac{-2fgdt^2 \sin\varphi}{ff+kk},$$

quae per $2d\varphi$ multiplicata et integrata praebet

$$d\varphi^2 = \alpha dt^2 + \frac{4fgdt^2 \cos\varphi}{ff+kk},$$

unde cognoscitur quadratum celeritatis

$$\gamma\gamma = \alpha + \frac{4fg\cos\varphi}{ff+kk}.$$

Deinde posito brevitatis gratia

$$\frac{4fg}{ff+kk} = \lambda$$

ob

$$d\varphi^2 = dt^2(\alpha + \lambda\cos\varphi)$$

reperitur

$$dt = \frac{d\varphi}{\sqrt{(\alpha+\lambda\cos\varphi)}}$$

et

$$t = \int \frac{d\varphi}{\sqrt{(\alpha+\lambda\cos\varphi)}},$$

ubi constans α et altera in ultima integratione ingressa ex statu initiali dato debent definiri.

COROLLARIUM 1

529. Evanescante angulo $COI = \varphi$ sit celeritas angularis

$$\gamma\gamma = \sqrt{\left(\alpha + \frac{4fg}{ff+kk}\right)}$$

omnium maxima, in aequalibus autem elongationibus rectae *OI* a verticali *OC* celeritates sunt aequales; et nisi constans α sit minor quam $\frac{4fg}{ff+kk}$, corpus integras revolutiones circa axem absolvet, quoniam tum pro angulo $\varphi = 180^{\circ}$ celeritas angularis adhuc est realis.

COROLLARIUM 2

530. Si autem fuerit $\alpha < \frac{4fg}{ff+kk}$, angulus $COI = \varphi$ non ultra certum limitem crescere potest corpusque, cum eo pertinerit, rursus descendet motumque oscillatorium peraget; ac ducta *IK* horizontali, ob $OK = f\cos\varphi$ angelo elongationis *COI* respondebit celeritas angularis

$$\gamma = \sqrt{\left(\alpha + \frac{4g \cdot OK}{ff+kk}\right)}.$$

SCHOLION

531. Sive corpus integras revolutiones absolvat sive oscillando eat redeatque, determinatio motus eundem calculum postulat, atque motus penduli simplicis, quo corpusculum infinite parvum filo inertiae experti alligatum circa axem horizontalem gyratur. Quem motum cum iam fusius supra exposuerimus, superfluum foret eosdem calculos hic repetere; sufficiet igitur pro quovis casu pendulum simplex assignasse, quod pari motu angulari feratur. Atque hic quidem tantum longitudo huius penduli simplicis in computum venit, cum motus eius solum ab eius longitudine pendeat, siquidem initio utrique eundem motum angularem tribuimus.

DEFINITIO 9

532. Pro motu gyratorio vel oscillatorio corporis cuiusvis gravis circa axem horizontalem *pendulum simplex isochronum* vocatur, quod cum semel in pari a recta verticali elongatione parem celeritatem angularem acceperit, deinceps continuo simili angulari feratur.

EXPLICATIO

533. Si corpus ponatur quodcunque *AEBF* (Fig. 67), quod a sola gravitate sollicitatum circa axem horizontalem *O* gyretur, primo eius centrum inertiae *I* spectandum est, quod si in recta verticali *OC* versatur, corporis situm naturalem, in quo acquiescat, indicat, angulus autem *COI elongatio a situ naturali* vocatur. Quodsi iam huic corpori in data elongatione datus motus angularis fuerit impressus, pendulum simplex isochronum ita debet esse comparatum, ut, si ei in pari elongatione aequalis motus angularis imprimatur, deinceps huius motus perpetuo sit responsurus motui corporis propositi. Vel quia totum negotium a longitudine huius penduli simplicis pendet, si id fuerit *OS* atque ex communi axe *O* suspensum concipiatur, motu suo perpetuo motum corporis *AEBF* comitabitur, dummodo semel aequalem motum gyratorium acceperit. Perinde quidem est, sive hoc pendulum simplex eidem axi applicatum concipiatur sive secus; sed quoniam utrinque elongationes a situ verticali *OC* perpetuo eaedem esse debent corporisque elongatio ex situ rectae *OI* est aestimanda, pendulum simplex commodissime in puncto *O* suspensum consideratur, ut eius situs *OS* perpetuo in rectam *OI* incidat totaque quaestio ad determinationem puncti S revocetur.

COROLLARIUM 1

534. Invento hoc puncto *S* in recta *OI* producta corpus perinde movebitur, ac si tota eius massa in ipso hoc puncto *S* esset collecta; tum enim ob extensionem evanescentem habetur pendulum simplex longitudinis *OS*.

COROLLARIUM 2

535. Hoc ergo punctum *S* quaeri debet in recta, quae per centrum inertiae corporis ad axem gyrationis normalite ducitur, etiamsi hic non sit necessarium ut pendulum simplex *OS* ex eodem axis puncto *O* suspensum statuatur.

SCHOLION

536. Cum istud pendulum simili motu latum ob massae evanescentiam *simplex* vocatur, ad hunc modum corpora quaevis extensa circa axem fixum mobilia vocari solent *penduli composita*, ita ut quaestio huc reducatur, ut proposito quocunque pendulo composo, quod scilicet sit corpus rigidum, assignetur pendulum simplex isochronum, quam quaestionem nunc quidem facillime resolvere poterimus. Ceterum monendum est filum, quo pendulum simplex axi alligatum intelligimus, non solum inertiae expers statui, sed etiam rigidum concipi oportere, ne ulla inflexio calculum turbare queat.

PROBLEMA 46

537. Proposito corpore quocunque rigido et gravi *AEBF* circa axem horizontalem fixum *O* mobili (Fig. 67), definire pendulum simplex isochronum *OS*.

SOLUTIO

Posita massa totius corporis = *M* eiusque centro inertiae in *I*, hinc ad axem ducatur recta normalis *IO = f*, quae iam a verticali *OC* distet angulo $COI = \varphi$; tum vero sit *Mkk* momentum inertiae corporis respectu axis per *I* ducti et axi gyrationis paralleli. His positis, quicunque motus corpori initio fuerit impressus, elapso tempore = *t* motus variatio hac formula exprimatur

$$dd\varphi = \frac{-2\,fgdt^2\,sin\,\varphi}{ff+kk}.$$

Ponatur nunc penduli simplicis isochroni longitudo *OS = l*, quod cum eodem angulo $COS = \varphi$ a situ verticali distet, eius motus hanc variationem patientur, ut sit

$$dd\varphi = \frac{-2gdt^2\,sin\,\varphi}{l},$$

quae quidem formula ex praecedente fluit ponento *k* = 0 et *f* = *l*. Quare, cum eadem variatio utrinque evenire debeat, obtinemus

$$l = \frac{ff+kk}{f} \text{ seu } l = f + \frac{kk}{f}.$$

COROLLARIUM 1

538. Longitudo ergo penduli simplicis isochroni *OS* superat distantiam centri inertiae *I* ab axe gyrationis *O* estque intervallum $IS = \frac{kk}{f}$. Cognita vero longitudine *OS* = *l* erit

$$kk = f\left(l-f\right) = OI \cdot IS,$$

ita ut pro eodem corpore rectangulum $OI \cdot IS$ sit constans.

COROLLARIUM 2

539. Si pro eodem corpore distantia *OI = f* varietur, patet tam casu *f* = 0 quam $f = \infty$ pendulum simplex isochronum *l* evadere inifinitum; brevissimum autem erit, si capiatur *f* = *k*, quo casu $l = 2k$; praeterea semper est $l > 2k$.

COROLLARIUM 3

540. Invento pendulo simplici isochrono l, quoniam oscillationes minimae corporis perinde atque istius penduli sunt isochronae, tempus cuiusque oscillationis erit = $\frac{\pi\sqrt{l}}{\sqrt{2g}}$ min. sec. (§ 215). Hinc si prodeat $l = \frac{2g}{\pi\pi}$, singulae oscillationes minimae corporis absolventur minutis secundis.

SCHOLION

541. Hinc collititur methodus facilis cuiusque corporis momentum inertiae practice definiendi. Suspenso enim corpore ex axe horizontali, circa quem liberrime gyrari queat, omni cura primo defineatur distantia centri inertiae I ab axe gyrationis O, nempe $OI = f$, quod etiam practise fieri potest; deinde corpus ad minimas oscillationes peragendas incitetur, pluribusque dato tempore numeratis inde colligatur tempus unius oscillationis, quod sit = τ min.sec., hinc habebitur $l = \frac{2g\tau\tau}{\pi\pi}$; quo invento erit $kk = f(l - f)$, et pondus corporis M per kk muliplicatum dabit momentum inertiae respectu axis per eius centrum inertiae transeuntis et axi gyrationis paralleli. Potest etiam hoc experimentum multiplicari, dum corpus successive ex variis axibus, qui tamen sint inter se paralleli, suspenditur, quo certiores de vero valore kk reddamur. Quin etiam hinc vicissim longitudo penduli simplicis singulia minutis secundis oscillationis explorare potest, quandoquidem neque pendulis simplicibus uti licet neque altitudo lapsus g uno minuto secundo absoluto satis accurate per experimenta lapsus determiniri potest. Hinc autem pro corpore suspenso quantitates f et kk accurate nosse oportet, unde colligitur $l = f + \frac{kk}{f}$; tum, si tempus unius oscillationis minimae τ sit observatum, habebitur $g = \frac{\pi\pi l}{2\tau\tau}$ hincque longitudo penduli simplicis singulis minutis secundis oscillantis $\frac{2g}{\pi\pi} = \frac{l}{\tau\tau}$.

DEFINITIO 9

542. Centrum oscillationis in pendulo composito est punctum, in quo si tota corporis massa esset collecta, idem motus oscillatorius esset prodituros. Sumitur autem in recta, quae per centrum inertiae corporis transiens ad axem gyrationis est normalis.

COROLLARIUM 1

543. Distantia ergo centri oscillationis ab axe gyrationis aequalis est longitudini penduli simplicis isochroni; ac semper ab axe gyrationis *O* magis distat, quam centrum inertiae, intervallo $IS = \frac{kk}{f}$.

COROLLARIUM 2

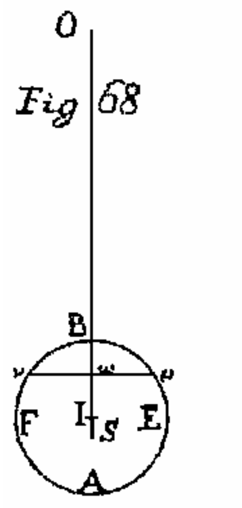
544. Ad centrum igitur oscillationis *S* inveniendum nosse oportet momentum inertiae corporis respectu axis per eius centrum inertiae *I* transeuntis et axi gyrationis paralleli, quod si fuerit = *Mkk*, dividi debet per *Mf*, hoc est per productum ex massa corporis *M* in distantiam axis gyrationis a centro inertiae $OI = f$, et quotus $\frac{Mkk}{Mf}$ ostendet distantiam centri oscillationis a centro inertiae.

SCHOLION

545. Hoc modo investigatio motus pendulorum compositorum ad centri oscillationis investigationem perduci solet, etsi ad hoc sufficit longitudinem penduli simplicis isochroni nosse, neque ulla ratio urget, ut hoc pendulum eidem axi suspensionis et quidem secundum rectam per centrum inertiae ad axem suspensionis normaliter ductam applicatum concipiatur. Verum hic modus rem concipiendi est commodissimus et, si corpus in situ quietis pendeat, ut recta per centrum inertiae ad axem normalite ducta simul sit verticalis, centrum oscillationis in eadem recta profundius quam centrum inertiae erit situm; neque enim hic opus est, ut corpus tanquam in motu spectetur. Ita recta *OI* in verticalem *OC* incidens consideratur, in qua erit centrum oscillationis *S* profundius situm centro inertia *I*, quod hic revera nomen centri gravitatis obtinet, ita ut sit intervallum $IS = \frac{Mkk}{Mf} = \frac{kk}{f}$. Quare calculus centri oscillationis facillime expeditur calculo, quem supra pro momento inertiae inveniendo tradidimus.

EXEMPLUM

546. Experimenta ante memorata globo ex materia homogenoa confecto institui solent (Fig. 68), qui ope fili *OB* suspensus ad minimas oscillationes incitatur,



ubi quidem filum tam tenue est sumendum, ut eius massa prae globo pro nihilo haberi liceat. Sit igitur radius globi $BI = b$ et distantia puncti suspensionis *O* a centro globi *I*, quod simul eius est centrum inertiae vel gravitatis, nempe $OI = f$, erit, ut supra invenimus, $kk = \frac{2}{5}bb.$ Quare centrum oscillationis erit in *S*, ut sit $IS = \frac{2bb}{5f}$, seu oscillationes convenient cum oscillationibus penduli simplicis, cuius longitudo est $= f + \frac{2bb}{5f}$. Ut ergo hoc pendulum singulis minutis secundis oscilletur, necesse est sit

$$f + \frac{2bb}{5f} = \frac{2g}{\pi\pi}$$

seu

$$ff = \frac{2gf}{\pi\pi} - \frac{2}{5}bb,$$

unde

$$f = \frac{g}{\pi\pi} \pm \sqrt{\left(\frac{g}{\pi^4} - \frac{2}{5}bb\right)},$$

ita ut pro *f* duplex habeatur valor, qui simul sumti dent $\frac{2g}{\pi\pi}$. Hi ambo valores fient aequales, si globus tantus accipiatur, ut sit

$$bb = \frac{5gg}{2\pi^4} \text{ et } b = \frac{g}{\pi\pi}\sqrt{\frac{5}{2}};$$

hoc est, in pedibus Rhenanis debet esse radius globi = 2,50317, ac tum distantia $OI = f$ fit = 1,583144 ped., ita ut punctum suspensionis seu axis gyrationis intra globum capi debeat. Cum autem sit

$$f = \frac{g}{\pi\pi} = b\sqrt{\frac{5}{2}}$$

seu $f = k$, evidens est hoc casu globum celerrime oscillari. Scilicet si sit $I\omega = b\sqrt{\frac{5}{2}}$, ducta horizontali $\mu\nu$, quae axem gyrationis referet, erit $\cos B\mu = \sqrt{\frac{5}{2}}$ ideoque arcus $B\mu = 50^{\circ}\ 46'$. Sin autem globus fuerit valde parvus, ut fieri solet, ad minuta secunda producenda sumi debet

$$OI = \frac{2g}{\pi\pi} - \frac{\pi\pi bb}{5g};$$

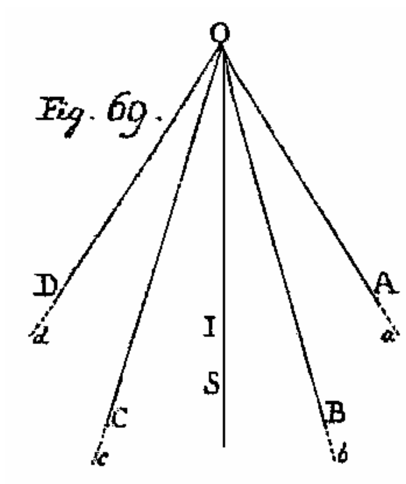
quare, ut globus ex ipso puncto B suspensus hoc praestet, eius radius debet esse

$$b = \frac{\left(\sqrt{65}-5\right)g}{2\pi\pi} = 0{,}155136g \text{ proxime.}$$

PROBLEMA 47

547. Si corpus rigidum circa axem horizontalem mobile pluribus constet partibus, quarum singularum centra inertiae et momenta inertiae sint cognita, definire totius corporis centrum oscillationis.

SOLUTIO



Axis gyrationis horizontalis ad planum figurae in puncto *O* normalis concipiatur (Fig. 69) sintque *A*, *B*, *C*, *D* centra inertiae partium, ex quibus corpus est compositum, quarum partium massae sint *A*, *B*, *C*, *D* et momenta inertiae respectu axium ipsi axi gyrationis parallelorum et per cuiusque centrum inertiae transeuntium $Aa^2, Bb^2, Cc^2, Dd^2;$ centra autem inertiae distent ab axe gyrationis intervallis *AO*, *BO*, *CO*, *DO*; perinde enim est, sive haec intervalla ad idem axis punctum *O* tendant sive ad diversa, quoniam tam momenta gravitatis quam momenta inertiae tantum a distantiis ab axe pendent, neque diversitas punctorum *O* quicquam eo confert. Primum ergo centrum inertiae I totius corporis, cuius massa sit = $M = A + B + C + D$, definiatur, quod in tali recta *OI* erit situm, ut sit

$$A \cdot OA \cdot \sin AOI + B \cdot OB \cdot \sin BOI = C \cdot OC \cdot \sin COI + D \cdot OD \cdot \sin DOI;$$

tum vero erit :

$$M \cdot OI = A \cdot AO \cdot \cos AOI + B \cdot BO \cdot \sin BOI + C \cdot CO \cdot \cos COI + D \cdot DO \cdot \cos DOI;$$

quae quantitas in superiori formula $IS = \frac{Mkk}{Mf}$ loco *Mf* scribi debet. At momentum inertiae totius corporis respectu axis gyrationis $M\left(ff + kk\right)$ ex partibus ita componitur, ut sit :

$$A\left(AO^2 + aa\right) + B\left(BO^2 + bb\right) + C\left(CO^2 + cc\right) + D\left(DO^2 + dd\right).$$

Quare, cum sit

$$OS = \frac{M(ff+kk)}{Mf},$$

erit

$$OS = \frac{A(AO^2+aa)+B(BO^2+bb)+C(CO^2+cc)+D(DO^2+dd)}{A\cdot AO\cdot \cos AOI+B\cdot BO\cdot \cos BOI+C\cdot CO\cdot \cos COI+D\cdot DO\cdot \cos DOI}.$$

COROLLARIUM 1

548. Si singulae partes seorsim considerentur earumque centra oscillationis statuantur in punctis *a*, *b*, *c*, *d*, ob

$$Oa = \frac{A(AO^2+aa)}{A\cdot AO}$$

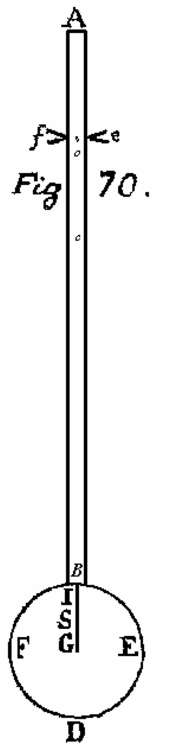
erit

$$OS = \frac{A\cdot OA\cdot Oa+B\cdot OB\cdot Ob+C\cdot OC\cdot Oc+D\cdot OD\cdot Od}{A\cdot AO\cdot \cos AOI+B\cdot BO\cdot \cos BOI+C\cdot CO\cdot \cos COI+D\cdot DO\cdot \cos DOI}.$$

COROLLARIUM 2

549. Invento autem centro inertiae seu gravitatis totius corporis *I* loco denominatoris poni potest $M\cdot OI$; per praecepta autem statica centrum gravitatis totius corporis ex datis centris gravitatis partium facile colligitur.

EXEMPLUM



550. Sit pendulum compositum ex virga cylindrica recta *ACB* et globo illi annexo *BEDF*, quod circa axem horizontalem *eOf* sit mobile (Fig. 70), cuius centrum oscillationis *S* quaeratur. Virga autem et globus constent ex materia uniforma ponaturque virgae longitudo $AB = a$, pondus = *A* et extremitatis *B* ab aex gyrationis *O* distantia $BO = b$, basis autem huius cylindri radius = c ; erit eius centrum inertiae in *C*, ut sit $AC = BC = \frac{1}{2}a$ et $OC = b - \frac{1}{2}a$, momentum vero inertiae respectu axis per C ducti et axi gyrationis paralleli = $A\left(\frac{1}{12}aa + \frac{1}{4}cc\right)$. Porro globi annexi sit massa = *E*, radius $BG = e$, erit eius centrum inertiae in *G* et momentum inertiae $= \frac{2}{5}Eee$. Sit iam totius corporis centrum inertiae in I, erit

$$(A+E)\cdot OI = A\left(b-\tfrac{1}{2}a\right)+E\left(e+b\right)=Mf;$$

deinde momentum inertiae respectu axis gyrationis =

$$A\left(\frac{1}{12}aa+\frac{1}{4}cc+\left(b-\frac{1}{2}a\right)^2\right)+E\left(\frac{2}{5}ee+(b+e)^2\right),$$

quod loco $M\left(ff+kk\right)$ substituti debet. Sicque centrum oscillationis erit in S, ut sit

$$OS=\frac{A\left(\frac{1}{3}aa-ab+bb+\frac{1}{4}cc\right)+E\left(bb+2be+\frac{7}{5}ee\right)}{A\left(b-\frac{1}{2}a\right)+E\left(b+e\right)},$$

ergo ob $OG = b + e$ fiet

$$GS=\frac{A\left(be+\frac{1}{2}ab-\frac{1}{2}ae-\frac{1}{3}aa-\frac{1}{4}cc\right)-E\cdot\frac{2}{5}ee}{A\left(b-\frac{1}{2}a\right)+E\left(b+e\right)}.$$

COROLLARIUM 1

551. Si axis gyrationis O capiatur in summitate virgae A, ut sit $b = a$, erit

$$OS = \frac{A\left(\frac{1}{3}aa+\frac{1}{4}cc\right)+E\left(aa+2ae+\frac{7}{5}ee\right)}{A\cdot\frac{1}{2}a+E\left(a+e\right)}$$

et

$$GS = \frac{A\left(\frac{1}{2}ae+\frac{1}{6}aa-\frac{1}{4}cc\right)-E\cdot\frac{2}{5}ee}{A\cdot\frac{1}{2}a+E\left(a+e\right)},$$

siquidem sumamus punctum S supra G cadere.

COROLLARIUM 2

552. Si sit exempli gratia $E=30A, a=b=3$ ped., $e=\frac{1}{12}$ ped., et $c=\frac{1}{500}$ ped., ita ut cc tuto negligi possit, erit

$$OG = 3\frac{1}{12} = 3{,}0833$$

et

$$OS = \frac{3+285\frac{7}{24}}{1\frac{1}{2}+92\frac{1}{2}} = \frac{288\frac{7}{24}}{94} = 3{,}0669,$$

hocque casu punctum S supra G cadit; sin autem massa virgae evanesceret, foret OS = 3,0842 sicque S infra G caderet.

SCHOLION

553. Hic postremus casus ideo est notatu dignus, quod vulgo filum, si fuerit valde tenue ac leve respectu globi, vix quicquam ad centrum oscillationis conferre videatur hic enim certe, etsi globus tricies ponderosier est filo, huius ratio sine insigni errore negligi non posset. Ponamus enim hoc pendulum oscillationes absolvisse minutis secundis hincque longitudinem penduli simplicis isochroni determinare oportere. Haec igitur neglecta fili massa prodiret = 3,0842 ped., cum tamen revera tantum fit 3,0669 ped., ita ut error 0,0173 ped. = $2\frac{1}{2}$ lin. committeretur minime certe tolerandus. Sin autem manentibus reliquis dimensionibus filum adhuc levius atque E = 60A esset, foret $OS = \frac{3+570\frac{7}{12}}{1\frac{1}{2}+185} = 3{,}0755,$ cuius loco si sumeretur 3,0842, error committeretur = 0,0087 ped = $\frac{5}{4}$ lin.

PROBLEMA 48

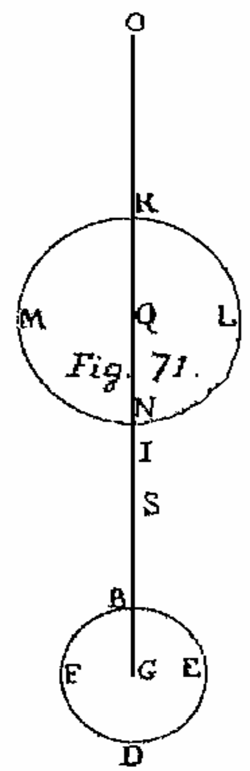


Fig. 71.

554. Si pendulum constet ex virga tenuissima *OB* inertiae experte, rigida tamen, et globo *BDEF* (Fig. 71), invenire locum, ubi alius globus datus eidem virgae affigi debeat, ut oscillationes fiant promtissimae.

SOLUTIO

Cum in *O* sit axis gyrationis, sit distantia *OG* = *b* et radius globi infra affixi *BG* = *c* massaque huius globi = *B*; tum alterius globi affigendi sit massa = *L* et radius *QK* = *e*, pro loco autem eius quaesito distantia *OQ* = *q*. His positis sit *I* centrum inertiae commune, erit $(B+L)OI = Bb + Lq = Mf$, tum vero momentum inertiae totius penduli respectu axis gyrationis =

$$B\left(\tfrac{2}{5}cc+bb\right)+L\left(\tfrac{2}{5}ee+qq\right)=M\left(ff+kk\right).$$

Quare, si centrum oscillationis statuatur in *S*, erit

$$OS=\frac{B\left(\tfrac{2}{5}cc+bb\right)+L\left(\tfrac{2}{5}ee+qq\right)}{(Bb+Lq)},$$

quae longitudo minima esse debet, ut oscillationes fiant promtissimae. Hinc prodit ista aequatio :

$$2BLbq-BL\left(\tfrac{2}{5}cc+bb\right)-\tfrac{2}{5}LLee+LLqq=0$$

seu

$$Lq=-Bb+\sqrt{\left(BBbb+BLbb+\tfrac{2}{5}BLcc+\tfrac{2}{5}LLee\right)},$$

unde innotescit distantia *OQ* = *q*, ex qua porro colligitur longitudo penduli simplis isochroni

$$OS=\tfrac{2}{L}\sqrt{\left(BBbb+BLbb+\tfrac{2}{5}BLcc+\tfrac{2}{5}LLee\right)}-\tfrac{2Bb}{L}=2q.$$

Hinc, si ambo globi ex eadem materia fuerint confecti, ob $B{:}L=c^3{:}e^3$ erit

$$OS = \frac{2\sqrt{\left(c^6bb + c^3e^3 + \frac{2}{5}c^5e^3 + \frac{2}{5}e^8\right)} - 2c^3b}{e^3}$$

et

$$OQ = q = \frac{\sqrt{\left(c^6bb + c^3e^3bb + \frac{2}{5}c^5e^3 + \frac{2}{5}e^8\right)} - c^3b}{e^3}.$$

COROLLARIUM 1

555. Si diametri globorum fuerint minimi, ut *cc* et *ee* prae *bb* negligi queant, distantia $OQ = q$ ita capi debet, ut sit

$$OQ = \frac{\sqrt{B(B+L)} - B}{L} b,$$

et longitudo penduli simplicis isochroni erit

$$2 \cdot OQ = 2b \cdot \frac{\sqrt{B(B+L)} - B}{L}.$$

COROLLARIUM 2

556. Si globus alter *KLMN* plane omitteretur, foret

$$OS = b + \frac{2cc}{5b},$$

quae maior est quam adiuncto isto globo, si fuerit

$$b + \frac{2cc}{5b} > 2e\sqrt{\frac{2}{5}}.$$

Unde, nisi sit

$$e > \frac{\sqrt{5}}{2\sqrt{2}}\left(b + \frac{2cc}{5b}\right),$$

hoc altero globo adiungendo oscillationes promtiores reddi possunt.

COROLLARIUM 3

557. Si autem fuerit

$$e = \frac{\sqrt{5}}{2\sqrt{2}}\left(b + \frac{2cc}{5b}\right),$$

quantacunque etiam fuerit huius globi massa *L*, pro oscillationibus celerrimis obtinendis sumi debet

$$OQ = q = \frac{1}{2}b + \frac{cc}{5b},$$

et tum longitudo pendui simplicis isochroni erit $=\frac{1}{2}b+\frac{cc}{5b}$, omnia ac si globus KLMN removeretur.

COROLLARIUM 4

558. Si ambo globi fuerint aequales, ut sit $L = B$ et $e = c$, oscillationes promtissimae evadent capiendo

$$OQ=q=\sqrt{\left(2bb+\tfrac{4}{5}cc\right)}-b\,;$$

ac si *cc* prae *bb* negligere liceat,

$$OQ=OG\left(\sqrt{2}-1\right);$$

hincque longitudo penduli simplicis isochroni =

$$2\cdot OG\left(\sqrt{2}-1\right)=0{,}828427\;OG.$$

COROLLARIUM 5

559. Si ambo globi ex eadem materia constent, definiri potest globi *KLMN* radius *e*, ut eo rite adiungendo oscillationes fiant promtissimae; scilicet *e* quaeri debet ex hac aequatione

$$16e^{10}-48c^5e^5-600bbc^6ee+9c^6\left(5bb+2cc\right)^2-120bbc^3e^5=0.$$

SCHOLION

560. Ceterum patet, quo minor sit radius e globi *KLMN* manente eius massa *L*, eo minorem prodire distantiam $OQ = q$, ideoque eo promtiores fore oscillationes. At vero manente radio *e* oscillationes fient celerrimae, si massa *L* globi affigendi fuerit quam maxima; nam si esset $L = 0$, foret

$$OS=b+\frac{2cc}{5b},$$

qui est valor maximus, siquidem affigendo altero globo oscillationes crebriores reddi possunt [Opera Omnia nota: Euler omittit hypothesin $e<\frac{5bb+2cc}{2b\sqrt{10}}$.] At vero, si fuerit

$$5bb+2cc=2be\sqrt{10}\ \text{ seu }\ e=\frac{5bb+2cc}{2b\sqrt{10}},$$

quantacunque fuerit huius globi massa *L*, eo rite annexo oscillationes manent eiusdem durationis, et si hic globus adhuc fuerit maior, oscillationes adeo tardiores evadent. Quodsi ambo globi ex materia aeque gravi fuerint confecti, magnitudo affigendi, ut motus oscillationes fiat rapidissimus, ex aequatione decimi gradus definiri debet; verum axis per centrum *G* globi *BCDF* transeat, ut sit $b = 0$, inde prodit $e = c\sqrt[5]{\frac{3}{2}}$ pro radio globi affigendi et pro eius loco

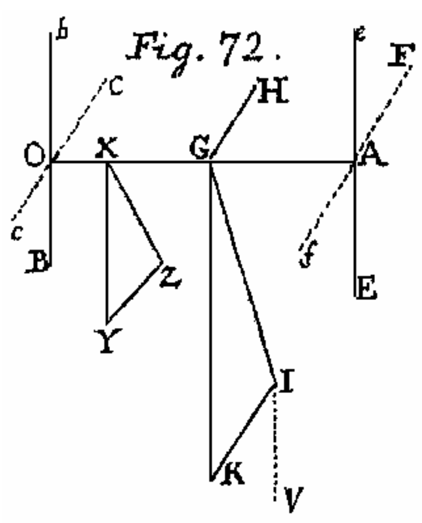
$$OQ = q = \sqrt{\left(\frac{2}{5}\frac{c^5}{e^3} + \frac{2}{5}e^2\right)} = c\sqrt[10]{\frac{8}{27}}$$

et longitudo penduli simplicis isochroni $= c\sqrt[10]{\frac{8}{27}}$. Axis ergo gyrationis per centrum prioris globi transiens alterum ita traiicere debet, ut ab eius centro distet intervallo
$OQ = c\sqrt[10]{\frac{8}{27}}$, quod minus est eius radio $e = c\sqrt[5]{\frac{3}{2}}$. Huiusmodi autem quaestiones circa motum oscillatorium plures proponi possent, quae autem ex stabilitis hic principiis non difficulter solventur. Plurimum autem intererit investigare, quamtas vires ipse axis gyrationis inter motum sustineat.

PROBLEMA 49

561. Dum corpus rigidum grave circa axem horizontalem fixum *OA* gyratur, ad quodvis tempus definire vires, quas axis in datis duobus punctis *O* et *A* sustinet.

SOLUTIO



Repraesentet tabula planum verticale per axem gyrationis *OA* transiens (Fig. 72), verseturque iam centrum inertiae corporis extra hoc planum in *I*, unde tam ad planum verticale quam ad axem ducantur perpendiculares *IK* et *IG*, erit angulus $IGK = \varphi$ elongatio corporis a situ naturali, ac posita distantia $IG = f$ erit $KI = f\ sin\varphi$ et $GK = f\ cos\varphi$. Tum sit massa corporis = *M*, quae cum simul eius pondus exprimat, vis sollicitans erit = *M* in directione verticali *IV* urgens, cuius momentum $= Mf\ sin\varphi$ tendit ad angulum *IGK* minuendum. Deinde consideretur elementum corporis quodcunque *dM* in *Z* , unde ad planum verticale et axem ductis perpendicularis *ZY, ZX* vocentur coordinatae

$$OX = x, \quad XY = y \quad \text{et} \quad YZ = z,$$

eritque

$$OG = \frac{\int xdM}{M}, \quad GK = \frac{\int ydM}{M}, \quad \text{et} \quad KI = \frac{\int zdM}{M};$$

posita autem distantia

$$XZ = \sqrt{(yy + zz)} = r$$

exprimit $\int rrdM$ momentum inertiae corporis respectu axis *OA*, quod sit = *Mkk*; denique ponatur distantia punctorum axis *OA* = *a* et per ambo ducantur rectae *BOb*, *COc* et *EAe*, *FAf* ipsis *KG* et *KI* parallelae. His praeparatis secundum ductum problema 23 primum observo nullum adesse vim, cuius directio cum axe sit in eodem plano; cum autem hic momentum vis $Mf \ sin\varphi$ in sensum contrarium vergat, atque ibi sumsimus, erit $Vf = -Mf \ sin\varphi$.

Nunc igitur ob vim *IV* = *M*, quae axi in *G* secundum directonem *GK* applicata est concipienda, axis in punctis *O* et *A* has sustinebit vires

$$\text{sec. } OB \text{ vim} = \frac{M \cdot AG}{a}, \quad \text{sec. } AE \text{ vim} = \frac{M \cdot OG}{a}.$$

Quibuscum coniungendae sunt illae, quae ex viribus elementaribus contrarie applicatis nascuntur, quae sunt

pro termino O $\begin{cases} \text{sec. } Ob \text{ vis} = \dfrac{f \ sin\varphi \cdot \int (a-x) zdM}{akk} \\ \text{sec. } Oc \text{ vis} = \dfrac{f \ sin\varphi \cdot \int (a-x) ydM}{akk} \end{cases}$

pro termino A $\begin{cases} \text{sec. } Ae \text{ vis} = \dfrac{f \ sin\varphi \cdot \int xzdM}{akk} \\ \text{sec. } Ae \text{ vis} = \dfrac{f \ sin\varphi \cdot \int xydM}{akk} \end{cases}$

hasque vires axis ob actionem gravitatis corporis sustinet, verum ob motum, quo iam gyratur, si celeritas gyratoria vocetur = γ, axis in punctis O et A has vires sustinet :

pro termino O $\begin{cases} \text{sec. } OB \text{ vis} = \dfrac{\gamma\gamma\int(a-x)zdM}{2ag} \\ \text{sec. } OC \text{ vis} = \dfrac{\gamma\gamma\int(a-x)ydM}{2ag} \end{cases}$

pro termino A $\begin{cases} \text{sec. } AE \text{ vis} = \dfrac{\gamma\gamma\int xzdM}{2ag} \\ \text{sec. } AE \text{ vis} = \dfrac{\gamma\gamma\int xydM}{2ag} \end{cases}$

COROLLARIUM 1

562. Si distantia terminorum *O* et *A* a puncto *G* vocentur *OG* = *b* et *AG* = *c*, ut sit $a = b + c$, tum vero ponatur *GX* = *u*, erit $x = b - u$ et $a - x = c + u$; ideo

$$\int(a-x)zdM = \int(c+u)zdM = Mc\cdot KI + \int uzdM\,,$$
$$\int(a-x)ydM = \int(c+u)ydM = Mc\cdot GK + \int uydM\,,$$
$$\int xzdM = \int(b-u)zdM = Mb\cdot KI - \int uzdM\,,$$
$$\int xydM = \int(b-u)ydM = Mb\cdot GK - \int uydM\,.$$

COROLLARIUM 2

563. His valoribus introductis axis in puncto *O* has vires sustinet : primo secundum directionem *OB* vim

$$\frac{Mc}{a} - \frac{Mcf\sin\varphi\cdot KI}{akk} - \frac{f\sin\varphi\cdot\int uzdM}{akk} + \frac{\gamma\gamma\cdot Mc\cdot GK}{2ag} + \frac{\gamma\gamma\cdot\int uydM}{2ag}\,,$$

deinde secundum directionem *OC* vim

$$\frac{Mcf\sin\varphi\cdot GK}{akk} + \frac{f\sin\varphi\cdot\int uydM}{akk} + \frac{\gamma\gamma\cdot Mc\cdot KI}{2ag} + \frac{\gamma\gamma\cdot\int uzdM}{2ag}\,.$$

At vero in puncto *A* istas :

primo secundum directionem *AE* vim

$$\frac{Mb}{a}-\frac{Mbf\ sin\varphi\cdot KI}{akk}+\frac{f\ sin\varphi\cdot\int uzdM}{akk}+\frac{\gamma\gamma\cdot Mb\cdot GK}{2ag}+\frac{\gamma\gamma\cdot\int uydM}{2ag},$$

deinde secundum directionem AF vim

$$\frac{Mbf\ sin\varphi\cdot GK}{akk}-\frac{f\ sin\varphi\cdot\int uydM}{akk}+\frac{\gamma\gamma\cdot Mb\cdot KI}{2ag}-\frac{\gamma\gamma\cdot\int uzdM}{2ag}.$$

COROLLARIUM 3

564. Si corpus ita fuerit comparatum, ut in plano IGK in duas partes similes et aequales dividatur, sitque $GO=GA=\frac{1}{2}a$, ob

$$\int uzdM=0 \quad \text{et} \quad \int uydM=0$$

axis in puncto O sustinebit has vires

$$\text{sec.}OB \text{ vim}=\frac{1}{2}M-\frac{Mf\ sin\varphi\cdot KI}{2kk}+\frac{\gamma\gamma\cdot M\cdot GK}{4g},$$

$$\text{sec.}OC \text{ vim}=\frac{Mf\ sin\varphi\cdot GK}{2kk}+\frac{\gamma\gamma\cdot M\cdot KI}{4g},$$

in puncto autem A sustinebit has vires

$$\text{sec.}AE \text{ vim}=\frac{1}{2}M-\frac{Mf\ sin\varphi\cdot KI}{2kk}+\frac{\gamma\gamma\cdot M\cdot GK}{4g},$$

$$\text{sec.}AF \text{ vim}=\frac{Mf\ sin\varphi\cdot GK}{2kk}+\frac{\gamma\gamma\cdot M\cdot KI}{4g};$$

hoc ergo casu vires non a magintudine distantiae $OA = a$ pendent.

COROLLARIUM 4

565. Hoc ergo casu, quo $\int uydM=0$ et $\int uzdM=0$, nihil impedit, quominus distantia $OA = a$ evanescens accipiatur atque axis in unico puncto G retineri poterit ; hic quippe sustinet binas vires

$$\text{alteram secundum } GK =M-\frac{Mff\ sin^2\varphi}{kk}+\frac{M\gamma\gamma f\ cos\varphi}{2g},$$

$$\text{alteram secundum } GH =\frac{Mff\ sin\varphi\, cos\varphi}{kk}+\frac{M\gamma\gamma f\ sin\varphi}{2g}$$

existente GH ipsi KI parallela.

SCHOLION

566. Corpora, quae vulgo ad motum oscillatorium adhiberi solent, ita sunt comparata, ut plano, quod per eorum centrum inertiae ad axem gyrationis normaliter ducitur, in duas portiones aequales et similes secentur; de iis igitur locum habet, quod axis in unico puncto retineri queat. Scilicet, si figura 67 repraesentet planum verticale per talis corporis centrum inertiae I ductum et ad axem gyrationis normale, qui figurae in O normaliter insistere concipiatur, existente OC recta verticali et OH in hoc plano horizontali, axis in puncto O vires modo indicatas substinebit. Nempe se angulus COI ponatur = φ, distantia $OI = f$, massa corporis = M, eius momentum inertiae respectu axis gyrationis = Mkk et celeritas angularis in hoc statu sit = γ, sive ad angulum COI augendum tendat sive minuendum, axis O sustinet duas vires,

$$\text{alteram secundum } OC = M - \frac{Mff\,sin^2\varphi}{kk} + \frac{M\gamma\gamma f\,cos\varphi}{2g},$$

$$\text{alteram secundum } OH = \frac{Mff\,sin\varphi\,cos\varphi}{kk} + \frac{M\gamma\gamma f\,sin\varphi}{2g}.$$

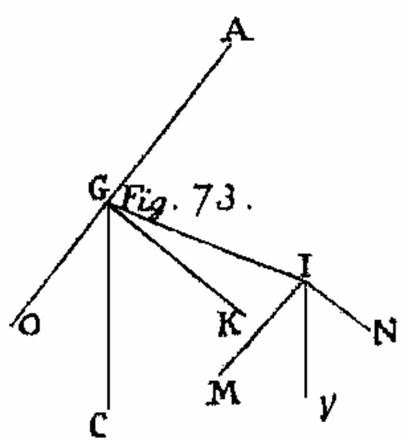
Priori ergo vi deorsum sollicitatur eamque sustentaculum sustinet; ob alteram vero vim axis in eam plagam, in qua centrum inertiae versatur, horizontaliter super sustentaculo procedere conatur, quem effectum obice arceri convenit. Quando centrum inertiae in contrariam plagam divagatur, haec vis horizontalis in contrarium dirigitur. Ceterum ambae vires ex duabus constant partibus, quarum altera actioni gravitatis, altera motui gyratorio ipsi debetur, ac ducta OL ad OI normali hae partes ad paucioreres redigentur, ut axis in puncto O ab his viribus sollicitetur :

$$\text{sec.}OC \text{ vi} = M, \quad \text{sec.}OL \text{ vi} = \frac{Mff\,sin\varphi}{kk}, \quad \text{sec.}OI \text{ vi} = \frac{Mf\gamma\gamma}{2g}.$$

Si non fuerit $\int uydM = 0$ et $\int uzdM = 0$, tum praeter istas vires axis insuper in punctis O et A figurae 72 eas virium § 563 partes sustinet, quae has formulas integrales involvunt, quoniam reliquas partes immunes ad unicum punctum reducere licuit.

PROBLEMA 50

567. Si axis *OA*, circa quem corpus rigidum grave est mobile, non fuerit horizontalis, definire motum gyratorium ut et vires, quas axis inde sustinet.



SOLUTIO

Per axem *OA* ductum concipiatur planum verticale, in quo sit *GC* recta verticalis (Fig. 73), ponaturque angulus $OGC = \zeta$, cuius complementum $90^{o} - \zeta$ dat axis *OA* inclinationem ad horizontem. Reperiatur nunc corporis centrum inertiae *I* extra hoc planum verticale, unde ad axem ducta normali *IG* = *f* et ex *G* in plano verticali ad axem pariter normaliter *GK* erit ipsum planum *IGK* ad planum verticale normale, ponaturque angulus $IGK = \varphi$, elongationem corporis a situ suo naturali metiens; recta enim *GI* in plano *IGK* movebitur. Statuatue massa corporis eademque eius pondus = *M* eiusque momentum inertiae respectu axis *OA* = *Mkk*, quod perinde colligitur, ac si axis esset horizontalis; inclinatio enim tantum ad vim sollicitantem spectat. Effectus autem gravitatis eo redit, ut corpus in puncto *I* sollicitetur in directione verticali *IV* a vi = *M*; ad quam resolvendam ducantur *IM* et *IN* parallelae ipsis *GO* et *GK*, eruntque rectae *IM*, *IV* et *IN* in plano verticali angulusque $MIV = \zeta$. Hinc ex vi *IV* = *M* nascuntur duae vires, altera secundum $IM = M \cos\zeta$ et altera secundum $IN = M \sin\zeta$. Prior cum sit axi parallela, nihil plane ad motum confert, sed tota in axem impenditur, quemadmodum supra docuimus. Pro motu ergo restat sola vis $IN = M \sin\zeta$, cuius directio cum sit ipsi *GK* parallela, orietur momentum = $Mf \sin\zeta$, cuius directio cum sit ipsi GK parallela, orietur momentum = $Mf \sin\zeta \sin\varphi$ tendens ad angulum *IGK* minuendum,atque pro motu definiendo formuale superiores pro axe horizontali inventae valebunt, nisi, quod loco momenti vis sollicitantis, quod ante erat $= Mf \sin\zeta$, hic scribi debeat $Mf \sin\zeta \sin\varphi$, vel quatenus *M* pondus corporis denotat, eius loco scribi debet $M \sin\zeta$, quatenus autem in momentum inertiae ingreditur, immutatum relinqui debet. Quare motus similis erit motui penduli simplicis circa axem horizontalem, cuius longitudo $= \frac{Mkk}{Mf \sin\zeta} = \frac{kk}{f \sin\zeta}$, quo ipso motus perfecte determinatur. Quod autem ad vires attinet, quas axis interea sustinet in datis si placet punctis *O* et *A*, primo ob vim $IM = M \cos\zeta$ axis secundum suam directionem *AO* a tanta vi urgetur preaterea vero in utroque *O* et *A* a vi $\frac{GI}{OA}\cdot M\cos\zeta$, in puncto *A* scilicet secundum directionem ipsi *GI* parallelam, in *O* vero secundum oppositam. Tum vero praeter has vires in punctis *O* et *A* ab iisdem viribus sollicitabitur, quas in problemate praecedente determinavimus, hoc tantum observato, quod pro *M* scribi debeat $M\sin\zeta$ et $f\sin\zeta\sin\varphi$ loco $f\sin\varphi$. Nempe, si in figura 72 *OA* sit noster axis inclinatus et reliqua maneant ut in problemate praecedente, tum axis praeter vires a vi $IM = M\cos\zeta$ natas sustinet insuper has vires. Primo in puncto *O* secundum directionem *OB* vim

$$\frac{Mc\sin\zeta}{a}-\frac{Mcff\sin\zeta\sin^2\varphi}{akk}-\frac{f\sin\zeta\sin\varphi\cdot\int uzdM}{akk}+\frac{Mcf\gamma\gamma\cos\varphi}{2ag}+\frac{\gamma\gamma\int uydM}{2ag}$$

et secundum directionem *OC* vim

$$\frac{Mcff\sin\zeta\sin\varphi\cos\varphi}{akk}+\frac{f\sin\zeta\sin\varphi\cdot\int uydM}{akk}+\frac{Mcf\gamma\gamma\sin\varphi}{2ag}+\frac{\gamma\gamma\int uzdM}{2ag}.$$

Deinde in puncto *A* secundum directionem *AE* vim

$$\frac{Mb\sin\zeta}{a}-\frac{Mbff\sin\zeta\sin^2\varphi}{akk}+\frac{f\sin\zeta\sin\varphi\cdot\int uzdM}{akk}+\frac{Mbf\gamma\gamma\cos\varphi}{2ag}-\frac{\gamma\gamma\int uydM}{2ag}$$

et secundum directionem *AF* vim

$$\frac{Mbff\sin\zeta\sin\varphi\cos\varphi}{akk}-\frac{f\sin\zeta\sin\varphi\cdot\int uydM}{akk}+\frac{Mbf\gamma\gamma\sin\varphi}{2ag}-\frac{\gamma\gamma\int uzdM}{2ag},$$

ubi est $OA = a$, $OG = b$, $AG = c$ et celeritas angularis = γ, integralibus sumtis, ut ibi definivimus.

COROLLARIUM 1

568. Cum longitudo pendulo simplicis isochroni sit = $\frac{kk}{f\sin\zeta}$, corpus circa axem inclinatum tardius oscillationes suas absolvit, quam si axis esset horizontalis, ac si oscillationes fuerint minimae, tempus unius erit = $\pi\sqrt{\frac{kk}{2fg\sin\zeta}}$ min.sec.

COROLLARIUM 2

569. Si axis est inclinatus, etiam vim sustinet secundum suam directionem *AO* (Fig. 73), quae est $M\cos\zeta$, reliquae vires omnes ad axem sunt normales et ad duo data puncto *O* et *A* revocari possunt.

COROLLARIUM 3

570. Si corpus a plano *IGK* in duas partes similes et aequales bisecetur, valores integralium $\int uydM$ et $\int uzdM$ evanescunt et omnes vires praeter eas, quae ex vi *IM* nascuntur, ad unicum punctum *G* reduci possunt, ut supra.

SCHOLION

571. Haec sunt, quae de motu gyratorio corporum rigidorum circa axem fixum proponenda videbantur, ubi quidem ipsius motus determinatio eo est reducta, ut plus difficultatis non habeat quam motus corpusculi circa axem fixum, si modo momentum inertiae fuerit exploratum. Vires autem, quas axis gyrationis inter motum sustinet, molestiorem calculum plerumque exigunt, cum ex corporis figura valores binorum integralium $\int xydM$ et $\int xzdM$ erui debeant. Verum haec investigatio maximi est momenti, si ad ad motum corporum rigidorum circa axes non fixos progredi velimus; ubi primo quidem eos casus diligentius evolvi convenit, quibus axis sponte manet immobilis, etiamsi extrinsecus non retineatur. Proposito ergo corpore quocunque rigido inquirendum est, utrum in eo dentur eiusmodi axes, circa quos si corpus motum gyratorium receperit, ipsi inde nullas sustineant vires; deinde etiam videndum est, a quibusnam viribus corpus circa talem axem motum sollicitari debeat, ut etiam hinc nullae vires ad axem dimovendum nascuntur.