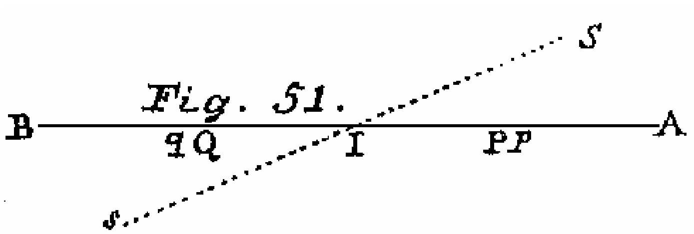
Chapter 6

THE INVESTIGATION OF THE MOMENT OF INERTIA FOR HOMOGENEOUS BODIES

PROBLEM 32

471. If the body should be the finest straight line filament *AIB* (Fig. 51), to find the principal axes of this, and the moments of inertia with respect to these.



SOLUTION

Let the whole length of the filament be $AB = 2a$, at the mid-point of this will be the centre of inertia *I*, such that there shall be $IA = IB = a$, moreover the mass of the filament, which geometrically is expressed by 2*a*, shall be equal to *M*. Now one principal axis clearly is the line *AB* itself, with respect to which the moment of inertia is zero and thus is the minimum ; the two remaining axes are normal to AB at *I* and about which the moments of inertia are equal, thus so that the position of these is not being determined. Therefore in the finding of the moment of inertia with respect to such an axes normal to *AB* at *I*, on choosing $IP = IQ = x,$ the moments of inertia of the elements $Pp = Qq = dx$ are *xxdx* and thus on taking both together, the integral $\frac{2}{3}x^3$ of this quantity, on putting $x = a$, gives the moment of inertia of the filament with respect to the axis normal to the filament at *I* equal to $\frac{2}{3}x^3 = \frac{1}{3}Maa$ on account of $M = 2a$.

COROLLARY 1

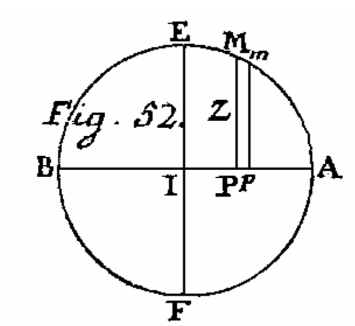
472. Hence the two remaining principal axes are not determined, and in addition to *AIB*, likewise it is the case that two right lines normal to each other as well as to the filament through *I* can be taken for these. And about these the moment of inertia $\frac{1}{3}Maa$ is a maximum, thus so that the mean agrees with the maximum.

COROLLARY 2

473. Since the moment of inertia about the axis *AB* is equal to zero, about any other axis *SIs* inclined at an angle $AIS = \vartheta$ to *AB* it will be equal to $\frac{1}{3}Maa\sin^2\vartheta$, which is evident from above, if of the two remaining principal axes one is taken in the plane *AIS*; for then the axis *Ss* is inclined to that at an angle $90^{\circ} - \vartheta$, and the other principal axes truly at right angles.

PROBLEM 33

474. If the body should be the finest filament curved in the periphery of a circle AEBF (Fig. 52), to find the principal axes of this, and the moments of inertia about these axes.



SOLUTION

Let the radius $IB = a$, and on putting the ratio of the diameter to the periphery equal to $1:\pi$ then the length of the filament is equal to $2\pi a$, which likewise refers to the mass of this, which shall be equal to *M*. Since the centre of inertia shall be at the centre of the circle *I*, in the first place a line perpendicular to the plane of the circle at *I* shall be one principal axis, and the moment of inertia about this axis shall be equal to Maa; the two remaining axes have been placed in the plane of the circle, for which any two diameters normal to each other are allowed to be taken for *AB* and *EF*. Now on choosing the abscissa $IP = x$ and the applied line $PM = y = XZ = \sqrt{(aa - xx)}$, on account of the element of the filament $Mm = \frac{adx}{y}$, the moment of this about the axis $AB = aydx$, and thus the total moment of inertia $a\int ydx$

$$= a \times \text{area of the circle} = \pi a^3,$$

since on account of $M = 2\pi a$ then this becomes $\frac{1}{2}Maa$.

Whereby the moment of inertia with respect to any diameter $= \frac{1}{2}Maa$.

COROLLARY 1

475. Therefore the moment of inertia about the axis normal to the plane of the circle is the maximum Maa and the mean agrees with the minimum and each is half of the maximum.

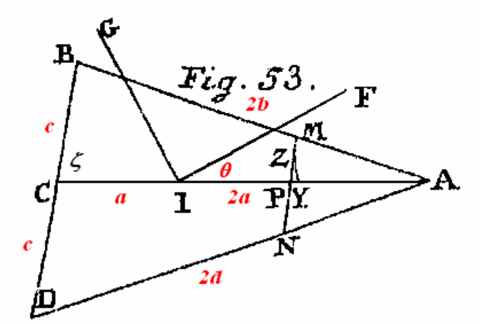
COROLLARY 2

476. If some other axis is taken inclined at an angle η to the plane of the circle through I, because this is inclined to the first axis at an angle $90^{\mathrm{o}} - \eta$, to the other of the remaining the axis is inclined at an η and to the third at a right angle, the moment of inertia about this axis will be equal to

$$Maa\,sin^2\,\eta + \tfrac{1}{2}Maa\,cos^2\,\eta = \tfrac{1}{2}Maa(1 + sin^2\,\eta).$$

PROBLEM 34

477. If the body should be the thinnest triangular sheet *ABD* (Fig. 53), to find the three principal axes of this shape, and the moments of inertia about these axes.



SOLUTION

In order that the centre of inertia *I* may be obtained, from the angle *A* the line *AC* is drawn bisecting the opposite side *BD* and on taking the third part *CI* of the whole *AC* the centre of inertia will be present at *I*. We put $CI = a, CB = CD = c$ and the angle $ACB = \zeta$, in order that

$$AI = 2a, AC = 3a \text{ and } BD = 2c.$$

Now it is evident that one principal axis it to be normal to the plane of the triangle through *I*, because, if on this line we should take the co–ordinate *x*, there becomes

$$\int xydM = 0 \text{ and } \int xzdM = 0 \text{ on account of } x = 0.$$

Whereby following problem 28 besides this axis the two remaining directrices are taken in the plane of the triangle, one of which shall be *IA*, and on taking some element *dM* at *Z* and thus on sending the perpendicular *ZY* to *IA*, let *IY* = *y* and *YZ* = *z* and the integrals may be called :

$$\int xxdM = A = 0,\quad \int yydM = B,\quad \int zzdM = C; \text{ then } \int yzdM = F;$$

thus, if *IF* and *IG* should be the two remaining principal axes, and on putting the angle *AIF* = ϑ, we have shown that [§ 442]:

$$\text{tang}\ 2\vartheta = \frac{2F}{B-C}$$

and the moment of inertia about the axis *IF* is equal to :

$$A + B\sin^2\vartheta + C\cos^2\vartheta - 2F\sin\vartheta\cos\vartheta,$$

where ϑ denotes the angle *AIF* as well as *AIG*. Then with respect to the first axis normal to the plane of the triangle, the moment of inertia is equal to *B* + *C*. In order that these values may be found, there is drawn through *Z* the line *MN* parallel to the side *BD*, and on putting in place *AP* = *t* and *PZ* = *u* then

$$PM = PN = \frac{ct}{3a}, YZ = u\sin\zeta \text{ and } PY = uc\cos\zeta$$

and the element at *Z* is equal to :

$$dtdu\sin\zeta = dM.$$

Hence therefore there is :

$$y = 2a - t + u\cos\zeta \text{ et } z = u\sin\zeta;$$

the other part is taken equal to the element $dtdu\sin\zeta$ for the former part, for which *u* shall be negative, and with these taken together it is possible to consider :

$$B = 2\int dt\sin\zeta\int du((2a-t)^2 + uu\cos^2\zeta), C = 2\int dt\sin\zeta\int uudu\sin^2\zeta$$

and

$$F = 2\int dt\sin\zeta\int(udu\sin\zeta(2a-t+u\cos\zeta) - udu\sin\zeta(2a-t-u\cos\zeta))$$

or

$$F = 2\int dt\sin\zeta\cdot\int uudu\sin\zeta\cos\zeta.$$

For the first integration to be performed, it is necessary to put $u = \frac{ct}{3a}$, thus there becomes :

$$B = 2\sin\zeta\int dt\left(\frac{ct}{3a}(2a-t)^2 + \frac{c^3t^3}{81a^3}\cos^2\zeta\right),$$
$$C = 2\sin\zeta\int dt\cdot\frac{c^3t^3\sin^2\zeta}{81a^3}$$

and

$$F = 2\sin^2\zeta\cos\zeta\int dt\cdot\frac{c^3t^3}{81a^3}$$

and thus

$$B = 2\sin\zeta\left(\frac{2actt}{3} - \frac{4ct^3}{9} + \frac{ct^4}{12a} + \frac{c^3t^4}{324a^3}\cos^2\zeta\right),$$
$$C = 2\sin\zeta\cdot\frac{c^3t^4\sin^2\zeta}{324a^3}$$

and

$$F = \frac{c^3t^4\sin^2\zeta\cos\zeta}{162a^3},$$

from which values extended through the whole triangle, on putting $t = 3a$, we have :

$$B = \tfrac{1}{2}ac\sin\zeta(3aa + cc\cos^2\zeta),$$
$$C = \tfrac{1}{2}ac^3\sin^3\zeta,$$
$$F = \tfrac{1}{2}ac^3\sin^3\zeta\cos\zeta.$$

And from these, there becomes for the position of the axes *IF* and *IG* :

$$\text{tang } 2\vartheta = \frac{2ac^3\sin^3\zeta\cos\zeta}{ac\sin\zeta(3aa+cc\cos 2\zeta)} = \frac{cc\sin 2\zeta}{3aa+cc\cos 2\zeta},$$

hence indeed the two values for ϑ are drawn forth. And then the moment of inertia about the principal axis normal to the plane of the triangle is equal to :

$$\tfrac{1}{2}ac\sin\zeta(3aa + cc\cos 2\zeta) = \tfrac{1}{6}M(3aa + cc)$$

on account of $M = 3ac\sin\zeta$, and about the axis *IF* or *IG*, since ϑ denotes either the angle *AIF* or *AIG*, the moment of inertia is then :

$$\tfrac{1}{6}M(3aa + cc\cos^2\zeta)\sin^2\vartheta + \tfrac{1}{6}Mcc\sin^2\zeta\cos^2\vartheta - \tfrac{1}{3}Mcc\sin\zeta\cos\zeta\sin\vartheta\cos\vartheta$$
$$= \tfrac{1}{2}Maa\sin^2\vartheta + \tfrac{1}{6}Mcc(\cos\zeta\sin\vartheta - \sin\zeta\cos\vartheta)^2$$
$$= \tfrac{1}{2}Maa\sin^2\vartheta + \tfrac{1}{6}Mcc\sin^2(\zeta - \vartheta).$$

COROLLARY 1

478. Since it is the case that :

$$AB^2 + AD^2 = 18aa + 2cc,$$

then

$$3aa = \frac{AB^2 + AD^2 - 2cc}{6}$$

and hence the moment of inertia about the axis normal to the plane of the triangle through I becomes equal to :

$$\frac{1}{36} M(AB^2 + AD^2 + BD^2),$$

thus as the thirty-sixth part of the mass of the triangle multiplied by the sum of the squares of the sides of the triangle.

COROLLARIUM 2

479. For the two remaining principal axes IF and IG placed in the plane of the triangle it is noted that ζ cannot be an angle greater than a right angle, thus $\sin 2\zeta$ is positive. Hence on putting the angle $AIF = \vartheta$ then

$$\text{tang } \vartheta = \frac{-3aa - cc\cos 2\zeta + \sqrt{(9a^4 + 6aacc\cos 2\zeta + c^4)}}{cc\sin 2\zeta} = \text{tang } AIF;$$

and

$$\text{tang } AIG = \frac{-3aa - cc\cos 2\zeta - \sqrt{(9a^4 + 6aacc\cos 2\zeta + c^4)}}{cc\sin 2\zeta}.$$

COROLLARY 3

480. The moment of inertia about these axes is equal to :

$$\frac{1}{6} M\left(\frac{3}{2}aa - \frac{3}{2}aa\cos 2\vartheta + \frac{1}{2}cc\cos 2(\zeta - \vartheta)\right);$$

since therefore

$$\text{tang } 2(\zeta - \vartheta) = \frac{3aa\sin 2\zeta}{3aa\cos 2\zeta + cc},$$

thus each moment can be expressed :

$$\tfrac{1}{12}M(3aa+cc\pm\sqrt{(9a^4+6aacc\cos 2\zeta+c^4)})$$
$$=\tfrac{1}{12}Mcc(1-\cos 2\zeta-\sin 2\zeta \operatorname{tang}\vartheta)=\tfrac{Mcc\sin\zeta\sin(\zeta-\vartheta)}{6\cos\vartheta};$$

since the angle *AIF* or *AIG* is assumed for ϑ, thus may be referred to each axis.

EXAMPLE

481. Let the triangle ABD be isosceles or the angle ζ right, and hence on account of $\tanh 2\vartheta=0$ either $\vartheta=0$ or $\vartheta=90^{\text{o}}$, thus the one axis lies on the line *AC* itself, the other now is normal to that. With respect to the one *AC* the moment of inertia is equal to $\tfrac{1}{6}Mcc$, and with respect to the other now it is equal to $\tfrac{1}{2}Maa$, while about the first axis, which is normal to the plane of the triangle, that is thus equal to $\tfrac{1}{2}Maa+\tfrac{1}{6}Mcc$, as this is equal to the sum of the two others. Now if in addition the triangle is equilateral, and the single side of this is equal to 2*c*, then
$3a=c\sqrt{3}$ or $aa=\tfrac{cc}{3}$, whereby all the axes in the plane of the triangle drawn through *I* present the same moment of inertia equal to $\tfrac{1}{6}Mcc$ and the moment about the axis normal to the plane of the triangle through *I* is twice as great, equal to $\tfrac{1}{3}Mcc$.

COROLLARY 4

482. This last property prevails so far in general ; for since the moment of inertia about the axis normal to the plane of the triangle is equal to $\tfrac{1}{6}M(3aa+cc)$, then about the axis *IF* it is equal to :

$$\tfrac{1}{12}M(3aa+cc-\sqrt{(9a^4+6aacc\cos 2\zeta+c^4)}),$$

and about the axis *IG*

$$=\tfrac{1}{12}M(3aa+cc+\sqrt{(9a^4+6aacc\cos 2\zeta+c^4)}),$$

it is clear that this sum must be equal to the first sum.

SCHOLIUM

483. Here it is worth noting, if the remaining sides of the triangle are put in place : $AB = 2b, AD = 2d$, as $BD = 2c$, then

$$9aa = 2bb + 2dd - cc \text{ and } \cos\zeta = \tfrac{dd-bb}{3ac},$$

thus the irrational formula $\sqrt{\left(9a^2 + 6aacc\cos 2\zeta + c^4\right)}$ becomes :

$$\tfrac{4}{3}\sqrt{\left(b^4 + c^4 + d^4 - bbcc - bbdd - ccdd\right)}.$$

Moreover this is not allowed to be defined in general, as either of the axis *IF* or *IG* may have a greater moment, as whenever this formula of the irrational part itself adopts a negative value, as is apparent from the case $\zeta = 90^o$, where the value of this [argument] $3aa - cc$ is negative, if $cc > 3aa$. Moreover in general these two moments of inertia cannot be equal to each other, because the irrational formula do not vanish, unless it is the case that $2\zeta = 180^o$ and $3aa = cc$. But the judgement in any case for the angles ϑ used in agreement with each axis is easily put in place from the formula

$$\tfrac{1}{2}Maa\sin^2\vartheta + \tfrac{1}{6}Mcc\sin^2(\zeta - \vartheta).$$

PROBLEM 35

484. If the body should be the thinnest sheet having the shape of the parallelogram *BDbd* (Fig. 54), to find the three principal axes of this, and the moments of inertia about these axes.

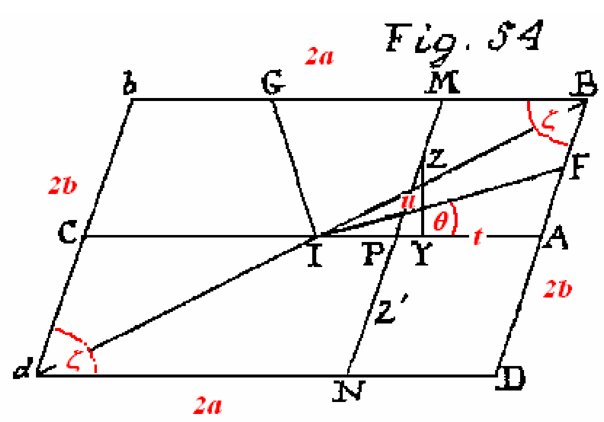
SOLUTION

With the two opposite sides *BD* and *bd* bisected at *A* and *C* and with the right line *AC* drawn, in the mid-point of this shall be the centre of inertia of the body *I*, and the mass of this is put equal to *M*. The sides are put :

$$Bb = Dd = AC = 2a,$$

$$BD = bd = 2b$$

and the acute angle $B = d = \zeta$, then the area = $4ab\sin\zeta = M$. Now one of the principal axis shall be normal to the plane of the lamina at *I* and the two remaining *IF* and *IG* placed in this plane itself ; in the finding of these some element *dM* is considered at *Z*, through which point in the first place there is drawn the right line *MN* parallel to the side *BD*, and let *AP* = *t* and *PZ* = *u*; then from *Z* with the perpendicular *ZY* sent to *AC* following problem 29 there is called $IY = y$ and $YZ = z$. On account of $APZ = \zeta$ then



$$ZY = u\sin\zeta \text{ et } PY = u\cos\zeta,$$

thus

$$y = a - t + u\cos\zeta \text{ and } z = u\sin\zeta;$$

while now $dM = dtdu\sin\zeta$; but in that calculation $x = 0$, in order that

$$\int xxdM = 0, \int xydM = 0, \int xzdM = 0.$$

Hence we have therefore :

$$\int yydM = B = \int dt \int du \sin\zeta \left(a - t + u\cos\zeta\right)^2,$$

$$\int zzdM = C = \int dt \int uudu \sin^3\zeta$$

and

$$\int yzdM = F = \int dt \int udu \sin^2\zeta \left(a - t + u\cos\zeta\right).$$

The similar element Z' placed in the other part is combined with these, for which u is negative, and there arises :

$$B = 2\sin\zeta \int dt \int du \left(\left(a-t\right)^2 + uu\cos^2\zeta\right),$$

$$C = 2\sin^3\zeta \int dt \int uudu$$

and

$$F = 2\sin^3\zeta\cos\zeta \int dt \int uudu.$$

With the first integration established, there is put in place $u = b$, and there is produced

$$B = 2\sin\zeta \int dt \int du \left(b\left(a-t\right)^2 + \tfrac{1}{3}b^3\cos^2\zeta\right),$$

$$C = \tfrac{2}{3}b^3\sin^3\zeta \int dt$$

and

$$F = \tfrac{2}{3}b^3\sin^3\zeta\cos\zeta \int dt.$$

Finally, with the second integration performed, $t = 2a$ is put in place, and there becomes

$$B = 4ab\sin\zeta\left(aa + bb\cos^2\zeta\right) = \tfrac{1}{3}M\left(aa + bb\cos^2\zeta\right),$$

$$C = \tfrac{4}{3}ab^3\sin^3\zeta = \tfrac{1}{3}Mbb\sin^3\zeta$$

and

$$F = \tfrac{4}{3}ab^3\sin^3\zeta\cos\zeta = \tfrac{1}{3}Mbb\sin\zeta\cos\zeta.$$

From these the moment of inertia is gathered about the first axis normal to the lamina at I :

$$B + C = \frac{1}{3}M\left(aa+bb\right);$$

which therefore does not depend on obliquity, but only on the lengths. But for the remaining axes IF and IG on putting the angle $AIF = \vartheta$ we find

$$\text{tang } 2\vartheta = \frac{2F}{B-C} = \frac{2bb\sin\zeta\cos\zeta}{aa+bb\cos 2\zeta}$$

or

$$\text{tang } 2\vartheta = \frac{bb\sin 2\zeta}{aa+bb\cos 2\zeta},$$

the two values of this [tangent function of] ϑ gives rise to each angle *AIF* and *AIG*. But the moment of inertia about these axes is

$$B\sin^2\vartheta + C\cos^2\vartheta - 2F\sin\vartheta\cos\vartheta$$
$$= \tfrac{1}{3}M\left(aa\sin^2\vartheta + bb\cos^2\zeta\sin^2\vartheta + bb\sin^2\zeta\cos^2\vartheta - 2bb\sin\zeta\cos\zeta\sin\vartheta\cos\vartheta\right)$$
$$= \tfrac{1}{6}M\left(aa - aa\cos 2\vartheta + bb - bb\cos 2\zeta\cos 2\vartheta - bb\sin 2\zeta\sin 2\vartheta\right)$$

and thus this moment of inertia therefore can be expressed:

$$\tfrac{1}{6}M\left(aa + bb - aa\cos 2\vartheta - bb\cos\left(2\zeta - 2\vartheta\right)\right).$$

Since therefore there becomes :

$$\sin 2\vartheta = \frac{bb\sin 2\zeta}{\sqrt{\left(a^4 + 2aabb\cos 2\zeta + b^4\right)}}$$

and

$$\cos 2\vartheta = \frac{aa+bb\cos 2\zeta}{\sqrt{\left(a^4 + 2aabb\cos 2\zeta + b^4\right)}},$$

that moment will be :

$$\tfrac{1}{6}M\left(aa+bb-\sqrt{\left(a^4+2aabb\cos 2\zeta+b^4\right)}\right),$$

where the ambiguity of the sign of the root gives rise to the moments of inertia about both the axes *IF* and *IG*. Hence it is apparent that the sum of these two is equal to the first moment of inertia.

COROLLARY 1

485. If $aa + bb\cos 2\zeta$ should have a positive value, with the positive root taken the angle 2ϑ is less than a right angle and thus the angle *AIF* is less than half a right angle; and the moment of inertia about the axis *IF* is the minimum equal to

$$\tfrac{1}{6}M\left(aa+bb-\sqrt{\left(a^4+2aabb\cos 2\zeta+b^4\right)}\right),$$

now about the axis *IG* it is the mean.

COROLLARY 2

486. If $aa + bb\cos 2\zeta$ should have a negative value and with the root for the axis *IF* taken positive, the angle 2ϑ is greater than a right angle and thus the angle *AIF* is greater than a half right angle, and the moment of inertia about the axis *IF* is a minimum.

COROLLARY 3

487. If the diagonal *Bd* is drawn through the acute angles *B* and *d*, on account of

$$\text{tang } AIB = \frac{b\sin\zeta}{a+b\cos\zeta}$$

there is found

$$\text{tang } 2\ BIF = \frac{2ab\sin\zeta(aa-bb)}{a^4+2a^3b\cos\zeta+2aabb\cos 2\zeta+2ab^3\cos\zeta+b^4},$$

thus it is apparent in the rhombus, where $a = b$, that both diagonals are to be principal axes, while in the rectangle the line *AC* is the principal axis.

EXAMPLE 1

488. If the parallelogram *BbdD* should be a rectangle, on account of $\zeta = 90^{\circ}$ then $\text{tang } 2\vartheta = 0$ and thus either $\vartheta = 0$ or $\vartheta = 90^{\circ}$; thus about the axis normal to the lamina at *I* the moment of inertia is equal to $\frac{1}{3}M(aa+bb)$; then truly the other principal axis is *AC*, about which the moment of inertia is equal to $\frac{1}{3}Mbb$; now the third principal axis is in the plane of the lamina normal to *AC*, about which the moment of inertia $=\frac{1}{3}Maa$ with the sides present $Bb = Dd = 2a$ and $BD = bd = 2b$.

EXAMPLE 2

489. If the parallelogram *BbdD* should be a rhombus, so that $b = a$ and the individual sides of this are equal to 2*a*, for the acute angles present ζ there is produced

$$\text{tang } 2\vartheta = \frac{\sin 2\zeta}{1+\cos 2\zeta} = \text{tang } \zeta$$

and hence either $\vartheta = \frac{1}{2}\zeta$ or $\vartheta = 90^{\circ} + \frac{1}{2}\zeta$. Whereby about the first principal axis normal to the plane of the rhombus through *I* the moment of inertia is equal to $\frac{2}{3}Maa$; both the remaining axes are the diagonals *Bd* and *Db*, of which that moment of inertia about *Bd* is equal to

$$\tfrac{1}{3}Maa\left(1-cos\,\zeta\right)=\tfrac{2}{3}Maa\,sin^2\frac{1}{2}\zeta,$$

now about the other diagonal *Db* it is equal to

$$\tfrac{1}{3}Maa\left(1+cos\,\zeta\right)=\tfrac{2}{3}Maa\,cos^2\frac{1}{2}\zeta.$$

COROLLARY 4

490. Therefore if the parallelogram becomes a square, the side of which is equal to 2*a*, all the lines drawn through the centre *I* in its plane are able to be taken as principal axes, and the moment of inertia about these is equal to $\tfrac{1}{3}Maa$, but about the axis normal to the square at *I* the moment of inertia is twice as large and equals $\tfrac{2}{3}Maa$.

PROBLEM 36

491. If the body should be the thinnest sheet formed in the shape of a circle (Fig. 52), to find the three principal axes of this shape, and the moments of inertia about these axes.

SOLUTION

Let the radius of the circle be equal to *a*, the area shall be equal to πaa, to which the mass *M* refers; and since one of the principal axes shall be normal to the plane of the circle at the centre *I*, there is put in place for some element *dM* at *Z* the coordinates $IP=y, PZ=z,$ on account of $dM=dydz$, then

$$\int yydM=\int dy\int yydz=\int dy\cdot yyz=\int yydy=\int yydy\sqrt{\left(aa-yy\right)}$$

on putting $z=\sqrt{\left(aa-yy\right)}$. But this integral is reduced to this form

$$\int yydM=\tfrac{1}{8}a^4\int\frac{dy}{\sqrt{\left(aa-yy\right)}}-\tfrac{1}{8}y\left(aa-2yy\right)\sqrt{\left(aa-yy\right)},$$

that is taken four times, and on putting $y = a$, this gives

$$B=\tfrac{\pi}{4}a^4=\tfrac{1}{4}Maa.$$

Now in a similar manner there arises

$$\int zzdM=C=\tfrac{1}{4}Maa.$$

Then $\int yzdM$, if an element is taken with a similar one from another part of the diameter, is reduced to zero, thus so that it becomes

$$\int yzdM=F=0.$$

Hence, since $B-C=0$, there arises tang $2\vartheta=\frac{0}{0}$ and thus the angle ϑ is indeterminate, from which we understand, which is clear by itself, that all the diameters can be taken for principal axes, about which the moment of inertia = $\frac{1}{4}Maa$. But about the first axis normal to the plane of the circle through I the moment of inertia $B+C=\frac{1}{2}Maa$.

SCHOLIUM

492. Since here the element of mass dM must equal $dydz$, it is to be noted that this always remains positive, even if y or z is taken negative, in which case also the differentials otherwise become negative. Therefore in this calculation rightly it must be advised, that when negative coordinates are taken, lest the expression of the element of the mass dM is brought into the calculation as negative. From which it may be agreed for the individual regions, where the coordinates are affected with contrary signs, that a separate calculation is put in place. Moreover the same value

$$B=\int yydM=\frac{1}{4}\pi a^4$$

is elicited, if there is put in place $IZ=r$ and the angle $AIZ=\varphi$, for then

$$dM=rdrd\varphi$$

and

$$y=r\cos\varphi,$$

hence

$$yydM=r^3drd\varphi\cos^2\varphi,$$

which following the variable r on integrating and putting $r=a$ gives $\frac{1}{4}a^4d\varphi\cos^2\varphi$, the integral of this

$$\text{on account of } \cos^2\varphi=\frac{1}{2}+\frac{1}{2}\cos 2\varphi$$

is produced :

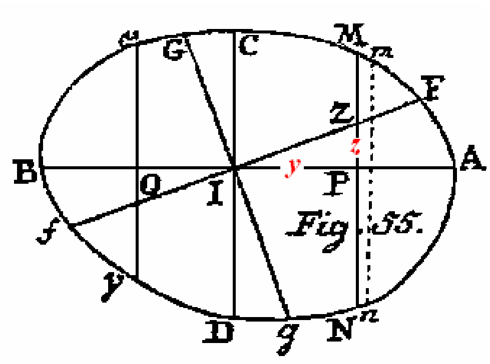
$$\frac{1}{4}a^4\left(\frac{1}{2}\varphi+\frac{1}{4}\sin 2\varphi\right).$$

Now there is put in place $\varphi=2\pi$, on account of $\sin 4\pi=0$ there is produced $\frac{1}{4}\pi a^2$ as before, from which it is apparent that the above caution is in agreement with the rule of continuity.

PROBLEM 37

493. If the body should be a lamina of the thinnest sheet having some shape *ABCD* (Fig. 55), to define the principal axis of this and the moments of inertia about these axes.

SOLUTION



Let *I* be the centre of inertia of the figure, and it is evident that the line normal to plane of the figure at *I* is one principal axis; then in the plane itself on taking the two directions *AB* and *CD* normal to each other for some element *dM* at *Z* the is put the coordinates *IP* = *y* and *PZ* = *z*, then *dM* = *dydz* and hence

$$\int yydM = \int dy \int yydz = \int dy \cdot yyz.$$

Hence on putting *z* = *PM*, there is produced $\int yydM = \int PM \cdot yyz,$

the value of this for the individual regions *AIC, AID, BIC* and *BID* must be elicited, and the sum of these shall be equal to *B*, so that

$$B = \int IP^2 \cdot MN \cdot d.IP + \int IQ^2 \cdot \mu\nu \cdot d.IQ.$$

Then

$$\int zzdM = \int dy \int zzdz = \tfrac{1}{3}\int PM^3 \cdot dy,$$

thus so that

$$C = \tfrac{1}{3}\int \left(PM^3 + PN^3\right) d.IP + \tfrac{1}{3}\int \left(Q\mu^3 + Q\nu^3\right) d.IQ.$$

Again it is the case that

$$\int yzdM = \int dy \int yzdz = \tfrac{1}{2}\int yzzdy = \tfrac{1}{2}\int PM^2 y \cdot dy,$$

the value of this in the regions *AID* and *BIC* is negative, in *BID* truly positive, thus there is obtained :

$$F = \tfrac{1}{2}\int IP\left(PM^2 - PN^2\right) d.IP - \tfrac{1}{2}\int IQ\left(Q\mu^2 - Q\nu^2\right) d.IQ.$$

But now the whole mass *M* is then

$$M = \int MN \cdot d.IP + \int \mu\nu \cdot d.IQ.$$

From these values found the moment of inertia about the axis at *I* normal to the plane equal to *B* + *C*, then the two remaining principal axes shall be *FIf* and *GIg*, and on putting the angle $AIF = \vartheta$ we find :

$$\text{tang } 2\vartheta = \frac{2F}{B-C}$$

and the moment of inertia about the axis *FIf*

$$= B\sin^2\vartheta + C\cos^2\vartheta - 2F\sin\vartheta\cos\vartheta$$
$$= \tfrac{1}{2}B + \tfrac{1}{2}C - \tfrac{1}{2}(B-C)\cos 2\vartheta - F\sin 2\vartheta.$$

Now on account of

$$\sin\ 2\vartheta = \frac{2F}{\sqrt{\left((B-C)^2 + 4FF\right)}}$$

and

$$\cos\ 2\vartheta = \frac{B-C}{\sqrt{\left((B-C)^2 + 4FF\right)}}$$

the moment of inertia about the axis *FIf* is obtained :

$$\text{about } FIf = \tfrac{1}{2}(B-C) - \tfrac{1}{2}\sqrt{\left((B-C)^2 + 4FF\right)}$$

and

$$\text{about } GIg = \tfrac{1}{2}(B+C) + \tfrac{1}{2}\sqrt{\left((B-C)^2 + 4FF\right)}.$$

COROLLARY 1

494. Therefore the moments of inertia about the axis *Ff* and *Gg* taken together are equal to the moment of inertia of the first principal axis, which is normal to the lamina at the point *I*.

COROLLARY 2

495. If the right line *AB* were a diameter of the figure, so that *PM* = *PN*, the value of the letter *F* vanishes, that which also comes about, if the right line *CD* were a diameter, so that on taking *IQ* = *IP* then $Q\mu = PM$. But as often as *F* = 0, so on account of $\text{tang } 2\vartheta = 0$, the lines *AB* and *CD* are themselves the principal axes.

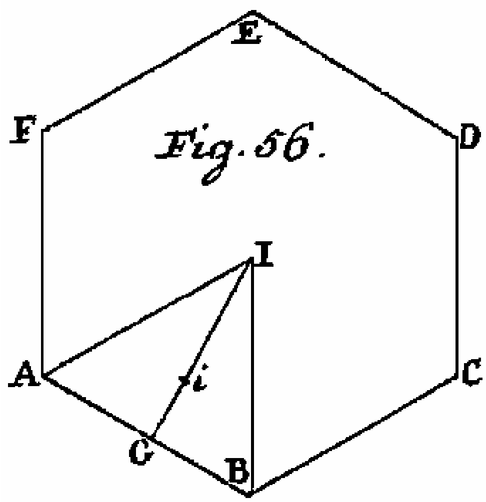
COROLLARY 3

496. In this case, in which *F* = 0 and both *AB* and *CD* are principal axes, the moment of inertia about the axis *Ff* = *C* and about the axis *CD* = *B*, which if the above were equal, on account of $\text{tang } 2\vartheta = \frac{0}{0}$ all the right lines drawn through *I* have equal moments of inertia equal to $B = C$.

COROLLARY 4

497. If in addition to the diameter *AB* there may be found other right lines drawn through *I*, about which the moment of inertia is equal to that, then all the right lines drawn through *I* in the plane enjoy the same property, and have equal moments of inertia.

PROBLEM 38



498. If the lamina should be a lamina of the thinnest sheet formed in the shape of a regular polygon (Fig. 56), to find the principal axes of this and the moments of inertia about these axes.

SOLUTION

The centre of inertia of such a regular polygon is in the middle of the circumscribed circle *I*, the radius of this is put as $IA = a$ and the number of sides is equal to *n*. Hence the angle $AIB = \frac{2\pi}{n}$, and with this bisected by the line IG the angle $AIG = \frac{2\pi}{n}$ and

$$AB = 2a\, sin\frac{\pi}{n} \text{ and } IG = 2a\, cos\frac{\pi}{n};$$

whereby the area of the triangle *AIB*

$$aa\, sin\frac{\pi}{n} cos\frac{\pi}{n} = \frac{1}{2} aa\, sin\frac{2\pi}{n},$$

and the whole area of the polygon

$$= \frac{n}{2} aa\, sin\frac{2\pi}{n},$$

bearing in turn the masses *M* . Now in the first place I note that all the lines in the plane of the lamina drawn through *I* (§497) shall have equal moments of inertia, of which two taken together make the moment about the axis normal to the plane through *I*. Now this moment can be gathered from the above. For the triangle *AIB* is considered, the mass of this is put equal to *m* and the centre of inertia at *i*, so that it becomes

$$Gi = \frac{1}{3} cos\frac{\pi}{n} \text{ and } Ii = \frac{2}{3} cos\frac{\pi}{n}$$

with $AG = a\, sin\frac{\pi}{n}$. Therefore since this triangle is isosceles, by § 481 the moment of inertia of this about the axis through i normal to the plane of the triangle

$$=\frac{1}{2}m\cdot Gi^2+\frac{1}{6}m\cdot AG^2=m\left(\frac{1}{18}aa\cos^2\frac{\pi}{n}+\frac{1}{6}aa\sin^2\frac{\pi}{n}\right)$$

and hence about the axis normal to the same plane at *I*

$$=m\left(\frac{1}{18}aa\cos^2\frac{\pi}{n}+\frac{1}{6}aa\sin^2\frac{\pi}{n}+\frac{4}{9}aa\cos^2\frac{\pi}{n}\right)=maa\left(\frac{1}{2}\cos^2\frac{\pi}{n}+\frac{1}{6}\sin^2\frac{\pi}{n}\right),$$

which multiplied by *n* on account of $mn = M$ gives the whole moment of the polygon about the axis of the normal at *I*

$$=Maa\left(\frac{1}{2}\cos^2\frac{\pi}{n}+\frac{1}{6}\sin^2\frac{\pi}{n}\right)=\frac{1}{3}Maa\left(1+\frac{1}{2}\cos\frac{2\pi}{n}\right).$$

Now about each axis in the plane of the lamina drawn through the point *I* the moment of inertia will be

$$=\frac{1}{6}Maa\left(1+\frac{1}{2}\cos\frac{2\pi}{n}\right)$$

clearly less than that by a factor of two.

COROLLARY 1

499. If besides on placing the side of the polygon $AB = c$, in order that $c=2a\sin\frac{\pi}{n}$, on account of $a=\frac{c}{2\sin\frac{\pi}{n}}$ the moment of inertia about the principal axis normal to the plane at *I*

$$=\frac{Mcc}{12\sin^2\frac{2\pi}{n}}\left(1+\frac{1}{2}\cos\frac{2\pi}{n}\right)=\frac{1}{12}Mcc\cdot\frac{2+\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}},$$

now of the remaining about the principal axes is half as great.

COROLLARY 2

500. If besides the radius of the circumscribed circle $IA = a$ the side of the polygon is introduced $AB = c$, on account of

$$\sin\frac{\pi}{n}=\frac{c}{2a}\text{ and }\cos\frac{2\pi}{n}=1-\frac{cc}{2aa},$$

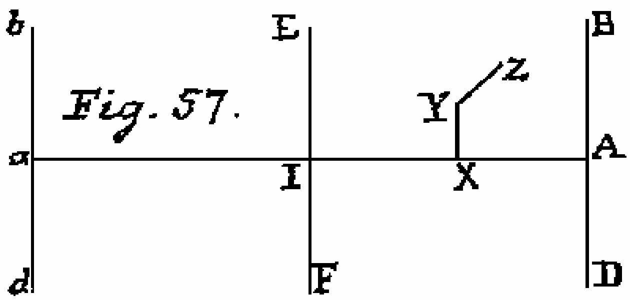
then the moment of inertia about the normal through *I*

$$=\frac{1}{3}Maa\left(1+\frac{1}{2}-\frac{cc}{4aa}\right)=\frac{1}{12}\left(6aa-cc\right),$$

now about the axes in the plane of the polygon itself drawn through *I* half as great.
minus.

PROBLEM 39

501. If the body were a right cylinder (Fig. 57), the axis of which $Aa = 2a$ and the radius of the base $AB = AD = c$, to find the principal axes of this, and to define the moments of inertia about these axes.



SOLUTION

Since the area of the base shall be equal to πcc, the volume of the cylinder or the mass shall be equal to $2\pi acc = M$. Moreover on the axis the middle point I will be the centre of inertia of this, in order that $AI = Ia = a$; but clearly here the axis Aa is one of the principal axes, through which on taking some plane $BDbd$, for some element of the mass dM placed at Z, there are the coordinates $IX = x$, $XY = y$, $YZ = z$, in order that $dM = dxdydz$. Hence the following values are gathered together:

1st
$$\int xxdM = \int xxdxdydz,$$
where in the first case on taking x and y constant and putting in place after the integration : $z = \sqrt{(cc - yy)}$, there is obtained :
$$z = \int xxdx \int dy\sqrt{(cc - yy)},$$
but $\int dy\sqrt{(cc - yy)}$ gives the area of the section made through X equal to πcc, so that there is obtained
$$\pi cc \int xxdx,$$
the integral of this extended to a as well as A produces $\frac{2}{3}\pi cca^3$, so that
$$\int xxdM = A = \frac{1}{3}Maa.$$

2nd
$$\int yydM = \int yydxdydz = \int dx \int yydy\sqrt{(cc - yy)},$$
but on putting $y = c$, there is obtained for the whole cylinder
$$\int yydy\sqrt{(cc - yy)} = \frac{1}{16}\pi c^4,$$
which must be taken four times, so that the integral becomes :
$$\int yydM = \frac{1}{4}\pi c^4 \int dx,$$

and hence there is obtained for the whole cylinder :

$$\int yydM = \tfrac{2}{4}\pi c^4 a = \tfrac{1}{4}Mcc = B.$$

3rd $$\int zzdM = \int zzdxdydz,$$

where, if initially x and z are taken as constants, on putting $y = \sqrt{(cc - zz)}$ there is had

$$\int dx \int zzdz\sqrt{(cc - zz)},$$

and the value of this is gathered as before :

$$\int zzdM = \tfrac{1}{4}Mcc = C = B.$$

4th $\int yzdM$, if likewise the element dM below the plane $BDbd$ is taken together with that, it results in zero, thus so that there is produced :

$$\int yzdM = F = 0.$$

With these in place with respect to the axis Aa, the moment of inertia will be equal to $B + C = \frac{1}{2}Mcc$; now for the two remaining axis normal to that there occurs :

$$\text{tang } 2\vartheta = \tfrac{2F}{B-C} = \tfrac{0}{0},$$

thus in order that all the diameters of the section normal to Aa at I can be considered as principal axes, around all of which the moment of inertia will be equal to

$$A + B = M\left(\tfrac{1}{3}aa + \tfrac{1}{4}cc\right).$$

COROLLARY 1

502. If some other axis is taken passing through I, which makes an angle equal to ζ with the axis Aa, the moment of inertia about this axis will be equal to

$$\begin{aligned}&(B+C)cos^2\zeta + (A+B)sin^2\zeta\\ &= M\left(\tfrac{1}{2}cc\cos^2\zeta + \tfrac{1}{3}aa\sin^2\zeta + \tfrac{1}{4}cc\sin^2\zeta\right)\\ &= M\left(\tfrac{1}{3}aa\sin^2\zeta + \tfrac{1}{2}cc - \tfrac{1}{4}cc\sin^2\zeta\right).\end{aligned}$$

COROLLARY 2

503. It can happen, that all the moments of inertia about the right lines drawn through *I* become equal to each other, which eventuates, if it should be that $\frac{1}{3}aa = \frac{1}{4}cc$ or $a = \frac{c\sqrt{3}}{2}$ and thus $\frac{c}{2a} = \frac{1}{\sqrt{3}}$ and the angle $AaB = 30^{o}$, or the triangle BaD is equilateral, in which case the individual moments are equal to $\frac{1}{2}Mcc = \frac{1}{8}M \cdot BD^2$.

PROBLEM 40

504. If the body were a right cone (Fig. 58), the vertex of this *A*, the altitude *AC* = *a* and the radius of the base *CB* = *CD* = *c*, to find the principal axes of this, and the moments of inertia about these axes.

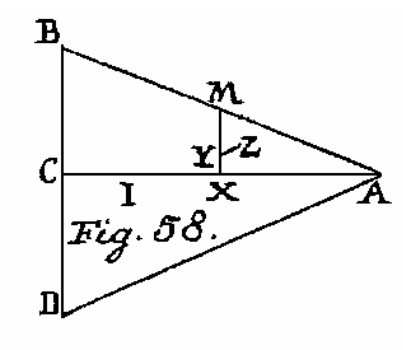


Fig. 58.

SOLUTION

Since the area of the basis shall be equal to πcc, the volume and the same mass of this will be equal to $M = \frac{1}{3}\pi acc$; then the centre of inertia *I* thus is placed on the axis, so that $CI = \frac{1}{4}a$ and $AI = \frac{3}{4}a$. Now some element *dM* is taken at *Z*, for which the coordinates are : *IX* = *x*, *XY* = *y* and *YZ* = *z*, then *dM* = *dxdydz*. Putting *AX* = *t*, then $XM = \frac{ct}{a}$ and $x = \frac{3}{4}a - t$, now with nothing less taken, it must be that *dM* = *dxdydz*.
Therefore the following formula are generated :

1st $$\int xxdM = A = \int \left(\tfrac{3}{4}a - t\right) xxdtdydz,$$

where initially *t* and *y* are taken as constants and putting in place :

$$z = \sqrt{\left(\frac{cctt}{aa} - yy\right)}$$

there is obtained :

$$\int\left(\tfrac{3}{4a} - t\right)^2 dt \int dy\sqrt{\left(\frac{cctt}{aa} - yy\right)};$$

where for the whole section X becomes :

$$\int dy\sqrt{\left(\frac{cctt}{aa} - yy\right)} = \frac{\pi cctt}{aa},$$

thus so that the integration remains :

$$\frac{\pi cc}{aa}\int ttdt\left(\tfrac{3}{4}a - t\right)^2 = \frac{\pi cc}{aa}\left(\frac{3}{16}aat^3 - \frac{3}{8}at^4 + \frac{1}{5}t^5\right).$$

Put $t = a$ and there becomes

$$A = \frac{1}{80}\pi cca^3 = \frac{3}{80}Ma^2.$$

2^{nd}

$$\int yydM = B = \int yydtdydz = \int dt\int yydy\sqrt{\left(\frac{cctt}{aa} - yy\right)}$$

by the first integration. But with t remaining constant up to this stage,

$$\int yydy\sqrt{\left(\frac{cctt}{aa} - yy\right)},$$

on putting $y = \frac{ct}{a}$ and taken four times is equal to $\frac{1}{4}\pi\frac{c^2t^2}{a^4}$,
as even now there must be integrated

$$\int \tfrac{1}{4}\pi \cdot \frac{c^4t^4}{a^4}dt,$$

hence on putting $t = a$ for the whole cone, this becomes

$$B = \frac{1}{20}\pi ac^4 = \frac{3}{20}Mcc.$$

3^{rd}

$$\int zzdM = C$$

in a like manner gives

$$C = \frac{3}{20}Mcc = B,$$

but

$$\int yzdM = F$$

clearly vanishes as before.

Therefore since AC is one principal axis, about this axis the moment of inertia is equal to

$$B + C = \frac{3}{10}Mcc.$$

The remaining principal axes are all the diameters of the section normal to the axis at I, about which the moment of inertia is equal to

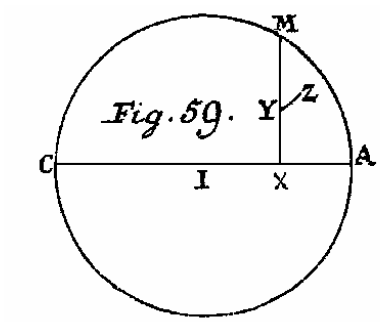
$$A+B=\tfrac{3}{80}M\left(aa+4cc\right).$$

COROLLARY

505. In the case, in which $aa+4cc=8cc$ or $a = 2c$, that is $AC = BD$, all the right lines drawn through I enjoy the property of being a principal axis, about which the moment of inertia is equal to $\tfrac{3}{10}Mcc$.

PROBLEM 41

506. If the body were a sphere made from some homogeneous material (Fig. 59), the centre of this I and the radius $IA = a$, to define the moment of inertia of this about some axis passing through the centre of this.



SOLUTION

On account of the radius $IA = a$ the area of the great circles is equal to πaa and the surface of the sphere is equal to $4\pi aa$, hence the volume of this or the mass $M=\tfrac{4}{3}\pi a^3$. Now on putting the coordinates for some element dM at Z : $IX = x$, $XY = y$ and $YZ = z$, the moment of inertia about the axis AC is equal to

$$\int dM\left(yy+zz\right).$$

There is put $XZ = r$ and $YXZ=\varphi$, then

$$y=r\cos\varphi,\quad z=r\sin\varphi$$

and

$$dM=rdrd\varphi dx,$$

thus

$$\int rrdM=\int r^3drd\varphi dx=2\pi\int r^3drdx$$

on account of $\int d\varphi=2\pi$; now on taking r for the variable and on putting

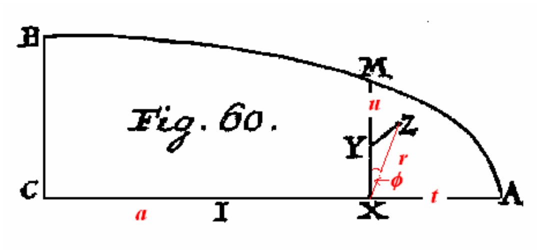
$$r=XM=\sqrt{\left(aa-xx\right)}.$$

we have

$$\tfrac{1}{2}\pi\int dx\left(aa-xx\right)^2=\tfrac{1}{2}\pi\left(a^4x-\tfrac{2}{3}aax^3+\tfrac{1}{5}x^5\right).$$

$x = a$ is put in place for either hemisphere, and twice this expression gives the moment of inertia sought: $\pi\cdot\tfrac{8}{15}a^5=\tfrac{2}{5}Maa$.

PROBLEM 42



507. If the body were some a conoid of some kind generated by revolving the line *AMB* about the axis AC (Fig. 60), to find the principal axes of this and the moments of inertia about these axes.

SOLUTION

Let *AC* = *a* and for the curve *AX* = *t* and *XM* = *u*, thus in order that the equation is given between *t* and *u*; let the volume or mass

$$M = \pi \int uudt$$

on putting *t* = *a* after integrating. Then the centre of inertia is now at *I*, in order that

$$AI = \frac{\int tuudt}{\int uudt}.$$

Hence for brevity put *AI* = *f*, in order that the equation becomes:

$$\int tuudt = f \int uudt,$$

now *AC* is one of the principal axes. Now for the element *dM* placed at *Z* the coordinates will be *IX* = *x* = *f* – *t*, *XY* = *y* and *YZ* = *z*, and putting *XZ* = *r*, the angle $YXZ = \varphi$, then

$$dM = rdrd\varphi dx,$$

$$y = r\cos\varphi, \quad z = r\sin\varphi$$

Now the following formulas are considered :

1st $$\int xxdM = \int (f-t)^2 rdrdtd\varphi = 2\pi \int (f-t)^2 rdrdt$$

as $\int d\varphi = 2\pi$. At this stage t is constant, and by putting $r = XM = u$ there arises

$$\int xxdM = \pi \int (f-t)^2 uudt = A$$

and thus

$$A = \pi ff \int uudt - 2\pi f \int tuudt + \pi \int ttuudt$$

$$= -\pi ff \int uudt + \pi \int ttuudt = M\left(-ff + \frac{\int ttuudt}{\int uudt}\right).$$

2^{nd} $$\int yydM = \int r^3 drdtd\varphi \cos^2\varphi = \pi\int r^3 drdt$$

on account of

$$\int d\varphi \cos^2\varphi = \int d\varphi\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\varphi\right) = \tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\varphi,$$

which on putting $\varphi = 2\pi$ becomes π. Again there is produced

$$\tfrac{\pi}{4}\int u^4 dt$$

on putting $r = u$, thus in order that

$$\int yydM = B = \tfrac{\pi}{4}\int u^4 dt = \frac{M\int u^4 dt}{4\int uudt},$$

to this also there is made equal

$$\int zzdM = C.$$

But

$$\int yzdM = F$$

vanishes.

From these derivations there is produced the moment of inertia about the axis *AC*

$$= B + C = \frac{M\int u^4 dt}{2\int uudt},$$

putting $t = a$ after the integration, then in the section normal to *AC* at *I*, all the diameters support the position of a principal axis and the moment of inertia about these is equal to

$$A + B = M\left(-ff + \frac{4\int ttuudt + \int u^4 dt}{4\int uudt}\right) = M\left(\frac{\int uudt(4tt+uu)}{4\int uudt} - ff\right).$$

EXAMPLE 1

508. Let the body be a hemisphere or *AMB* the quadrant of a circle of radius *CA* = *CB* = *a*; then $uu = 2at - tt$, hence

$$\int uudt = att - \tfrac{1}{3}t^3 = \tfrac{2}{3}a^3$$

putting $t = a$; again

$$\int tuudt = \tfrac{2}{3}t^3 - \tfrac{1}{4}t^4 = \tfrac{5}{12}a^4,$$

hence

$$f = AI = \tfrac{5}{8}a \text{ et CI} = \tfrac{3}{8}a.$$

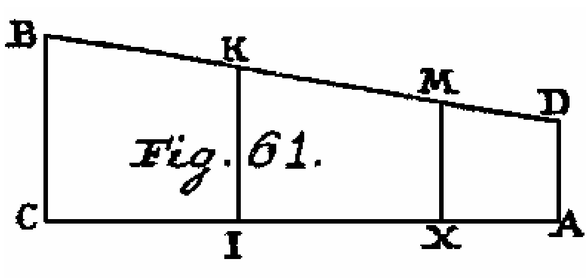
Whereby the moment of inertia about the axis *AC* is equal to

$$\frac{M\cdot 8a^5\cdot 3}{15\cdot 4a^3} = \tfrac{2}{5}Maa$$

and about any other axis normal to that at *I*, it is equal to

$$M\left(-\tfrac{25}{64}aa+\tfrac{13}{20}aa\right)=\tfrac{83}{320}Maa,$$

thus in order that moment shall be to this moment as 128 to 83.



EXAMPLE 2

509. Let the body be a truncated cone (Fig. 61), the axis of this $AC = a$, the radius of one be $BC = c$, of the other $AD = b$, and then $u=b+\frac{(c-b)t}{a}$ and $uu=bb+\frac{2(c-b)t}{a}+\frac{(c-b)^2tt}{aa}$, thus for the centre of inertia I to be found there is

$$\int uudt=bbt+\frac{(c-b)tt}{a}+\frac{(c-b)^2t^3}{3aaa}=\tfrac{1}{3}a\left(bb+bc+cc\right)$$

and thus the volume or mass is given by :

$$M=\tfrac{1}{3}\pi a\left(bb+bc+cc\right),$$

then

$$\int tuudt=\frac{1}{2}bbt+\frac{2(c-b)tt}{3a}+\frac{(c-b)^2t^4}{4aa}=\frac{1}{12}aa\left(bb+2bc+3cc\right),$$

thus the length arises

$$AI=f=\frac{a(bb+2bc+3cc)}{4(bb+bc+cc)}\text{ A }CI=\frac{a(cc+2bc+3bb)}{4(bb+bc+cc)}.$$

Again on account of

$$u^4=b^4+\frac{4b^3(c-b)t}{a}+\frac{6bb(c-b)^2tt}{aa}+\frac{4b(c-b)^3t^3}{a^3}+\frac{(c-b)^4t^4}{a^4}$$

there will be

$$\int u^4dt=b^4t+\frac{2b^3(c-b)tt}{a}+\frac{2bb(c-b)^2t^3}{aa}+\frac{b(c-b)^3t^4}{a^3}+\frac{(c-b)^4t^5}{5a^4}$$

and on making $t=a$

$$\int u^4dt=\tfrac{1}{5}a\left(b^4+b^3c+bbcc+bc^3+c^4\right),$$

and then

$$\int ttuudt=\tfrac{1}{3}bbt^3+\frac{b(c-b)t^4}{2a}+\frac{(c-b)^2t^5}{5aa}=\frac{1}{30}a^3\left(bb+3bc+6cc\right).$$

From these it is gathered that the moment of inertia about the axis AC is given by :

$$\frac{3}{10}M\,\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc}=\frac{3}{10}M\,\frac{b^5-c^5}{b^3-c^3};$$

but about the axis of the normal to AC at I , the moment of inertia becomes

$$\frac{3}{20}M\,\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc}+\frac{1}{80}Maa\left(\frac{8(bb+3bc+6cc)}{bb+bc+cc}-\frac{5(bb+2bc+3cc)^2}{(bb+bc+cc)^2}\right),$$

which is reduced to this form:

$$\frac{3}{20}M\cdot\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc}+\frac{3}{80}Maa\,\frac{(b+c)^4+4bbcc}{(bb+bc+cc)^2}.$$

COROLLARY 1

510. If $b = c$, the case of the cylinder is produced, in which there is made $AI = f = \frac{1}{2}a,$ the moment of inertia about the axis $AC = \frac{1}{2}Mcc$ and the moment of inertia about that axis of the normal at I is equal to $\frac{1}{4}Mcc+\frac{1}{12}Maa.$

COROLLARY 2

511. If $b = 0$, the case of the right cone is produced, in which there is made $AI = f = \frac{3}{4}a;$ the moment of inertia about the axis $AC = \frac{3}{10}Mcc$ and the moment of inertia about that axis of the normal at I is equal to $\frac{3}{20}Mcc+\frac{3}{80}Maa,$ as above.

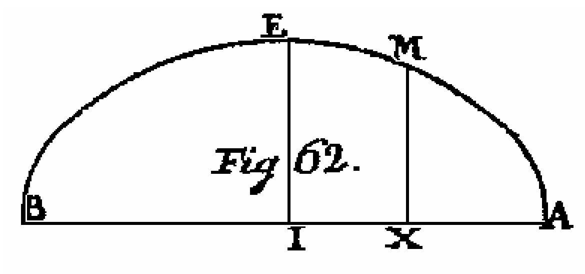
COROLLARY 3

512. In order that all the moments of inertia about the axes drawn through I may become equal, it must be that :

$$4\left(b^4+b^3c+bbcc+bc^3+c^4\right)=aa\cdot\frac{(b+c)^4+4bbcc}{bb+bc+cc}$$

and thus for the given bases of the truncated cone the height $AC = a$ thus has to be defined, in order that it becomes

$$aa=\frac{4(bb+bc+cc)\left(b^4+b^3c+bbcc+bc^3+c^4\right)}{(b+c)^4+4bbcc}.$$



EXAMPLE 3

508. Let the body be an elliptical spheroid arising from the rotation of the semi ellipse *AEB* about the axis *AB* (Fig. 62), therefore at the middle of this is the centre of inertia *I*. Putting the semi axis *AI* = *IB* = *a* and joining *IE* = *c*; then there arises $uu = \frac{cc}{aa}(2at - tt)$ and in the integration it is required to put $t = 2a$.

[Thus, $x = t - a = IX$ and $y = u = MX$ for the ellipse $\frac{(t-a)^2}{a^2} + \frac{u^2}{c^2} = 1$.] Hence we obtain:

$$\int uudt = \frac{cc}{aa}\left(att - \tfrac{1}{3}t^3\right) = \tfrac{4}{3}acc,$$

and thus the mass $M = \frac{4}{3}\pi acc$, then

$$\int tuudt = \frac{cc}{aa}\left(\tfrac{2}{3}at^3 - \tfrac{1}{4}t^4\right) = \tfrac{4}{3}aacc,$$

hence $AI = f = a$, again

$$\int ttuudt = \frac{cc}{aa}\left(\tfrac{1}{2}at^4 - \tfrac{1}{5}t^5\right) = \tfrac{8}{5}a^3cc$$

and as

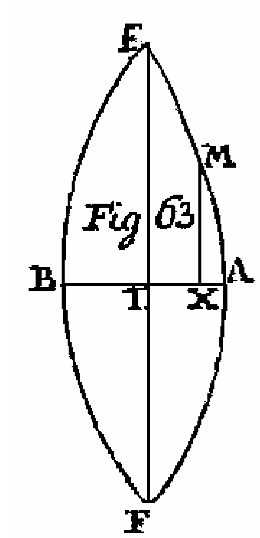
$$u^4 = \frac{c^4}{a^4}\left(4aatt - 4at^3 + t^4\right)$$

then

$$\int u^4dt = \frac{c^4}{a^4}\left(\tfrac{4}{3}aat^3 - at^4 + \tfrac{1}{5}t^5\right) = \tfrac{16}{15}ac^4.$$

From these it is gathered that the moment of inertia about the axis $AB = \frac{2}{5}Mcc$, but about the axis normal to *AB* at $I = \frac{1}{5}M(aa + cc)$.

EXAMPLE 4



514. If the body shall be a lens composed from two equals spherical segments (Fig. 63) or arising from the rotation of the figure *AEB*, formed from two equal semi segments *AIE* and *BIE* of a circle, about the axis *AB*, therefore the centre of inertia will be at the midpoint *I*. Putting the semi axis $AI = BI = a$ and $IE = IF = b$, then the diameter of the circle is equal to $\frac{aa+bb}{a}$, that for the time being we put equal to 2*c*, so that $c = \frac{aa+bb}{2a}$. Whereby there arises $uu = 2ct - tt$, and in the above integrations it is required to put $t = a$, with which done these must be doubled, unless since [*cf*. Fig. 60] $AI = f$, each $= a$, and thus

$$A = M\left(aa + \frac{-2a\int tuudt + \int ttuudt}{\int uudt}\right).$$

Hence we find

$$\int tuudt = \tfrac{2}{3}a^3c - \tfrac{1}{4}a^4,$$

$$\int uudt = aac - \tfrac{1}{3}a^3$$

and

$$M = 2\pi\left(aac - \tfrac{1}{3}a^3\right)$$

$$\int ttuudt = \tfrac{1}{2}a^4c - \tfrac{1}{5}a^5$$

and

$$\int u^4dt = \tfrac{4}{3}a^3cc - a^4c + \tfrac{1}{5}a^5.$$

From these the moment of inertia about the axis *AB* is gathered :

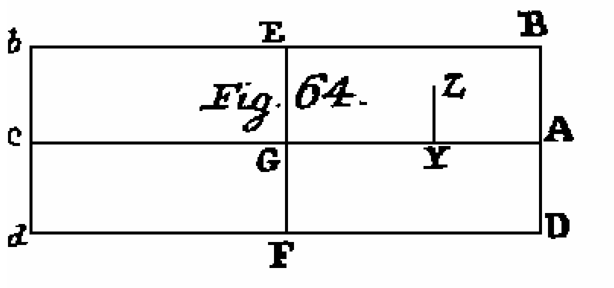
$$= \frac{1}{10}M \cdot \frac{20acc - 15aac + 3a^2}{3c - a} = \frac{1}{10}M \cdot \frac{a^4 + 5aabb + 10b^4}{aa + 3bb},$$

but about the axis *EF* normal to *AB* at *I* :

$$\frac{1}{20}M \cdot \frac{a^3 - 5aac + 20acc}{3c - a} = \frac{1}{20}M \cdot \frac{7a^4 + 15aabb + 10b^4}{aa + 3bb}.$$

PROBLEM 42a

515. If the body were a rectangular parallelepiped (Fig. 64), the find the principal axes of this and the moments of inertia about these.



SOLUTION

Let $BDbd$ be the rectangular base of the parallelepiped, the sides of this shall be $Bb = 2a$, $BD = 2b$, now with the height = $2c$, and it is clear that the mid-point of the parallelepiped is the centre of inertia of this and the principal axes are the three straight lines through this point parallel to the sides. Hence the moment of inertia is sought about the axis parallel to the height, which stands perpendicularly to the base at the middle point G. This rectangle $BDbd$ is considered, as a section parallel to the base being distant from the centre of inertia by an interval $= x$ and putting $GY = y$ and $YZ = z$, then $dxdydz$ is an element of the volume or of the mass dM, thus it becomes $M = 8abc$. Now we then have

$$\int xxdM = \int xxdxdydz$$

and on bang integrated twice by the variables y and z and on putting $y = a$ and $z = b$ the integrals are doubled, so that they extend through the whole section, then the integration becomes :

$$\int xxdM = 4ab\int xxdx = \tfrac{4}{3}abx^3;$$

now on putting $x = c$ and doubling then for the whole parallelepiped

$$\int xxdM = A = \tfrac{8}{3}abc^3 = \tfrac{1}{3}Mcc;$$

and in a like manner, there will be

$$\int yydM = B = \tfrac{1}{3}Maa;$$

and

$$\int zzdM = c = \tfrac{1}{3}Mbb$$

and

$$\int yzdM = F = 0.$$

From these it inferred that the moment of inertia about the principal axis parallel to the height or perpendicular to the base $BDbd$ is equal to

$$B + C = \tfrac{1}{3}M\left(aa + bb\right),$$

then the moment of inertia about the axis parallel to the side Bb

$$=\tfrac{1}{3}M\left(bb+cc\right)$$

and about the axis parallel to the side *BD*

$$=\tfrac{1}{3}M\left(aa+cc\right).$$

COROLLARIUM 1

516. Therefore if *ABCDabce* were such a rectangular parallelepiped (Fig. 65), the mass of this shall equal *M*, the principal axes of this are parallel to the sides *AB,AC,AD* and passing through the central point and the moment of inertia

about the axis parallel to the side *AB* will be equal to $\tfrac{1}{12}M\left(AC^2+AD^2\right)$;

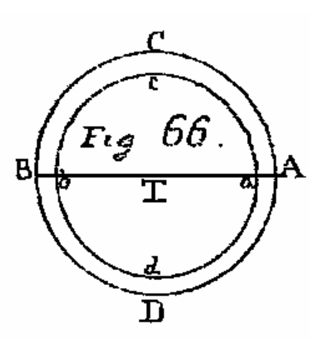
" " " " " " " " " " *AC* " " " " " $\tfrac{1}{12}M\left(AB^2+AD^2\right)$;

" " " " " " " " " " *AD* " " " " " $\tfrac{1}{12}M\left(AB^2+AC^2\right)$.

COROLLARY 2

517. If the body were a cube, the side of which = *a*, these three moments become equal to each other ; and thus the moments of inertia of all the axes drawn through the centre of the cube clearly are equal to each other and indeed = $\tfrac{1}{6}Maa$. Moreover such an equality must be in place for all the regular bodies.

PROBLEM 43



518. If the body were an empty sphere (Fig. 66), so that the cavity shall also be a sphere with the same given centre, to define the moment of inertia of this about all the axis passing through the centre.

SOLUTION

Let the centre be *I* and the radius of the sphere *IA* = *a*, now that of the cavity *Ia* = *b*, in order that the thickness of the spherical shell shall be equal to $a-b$ = *Aa*, and hence the mass of this spherical cavity $=\tfrac{4}{3}\pi\left(a^3-b^3\right)$, which is put equal to *M*; but it is evident that all the axes drawn through the centre *I* have moments of inertia equal to each other ; hence we may seek the moment of inertia about the axis *AB*. And if the sphere should be solid, on account of this, the mass is equal to $\tfrac{4}{3}\pi a^3$ then the moment of inertia of this becomes equal to

$$\pi a^3 \cdot \tfrac{2}{5} aa = \tfrac{8}{15} \pi a^5 ,$$

but with the middle taken away equal to $\tfrac{8}{15}\pi b^5$, where with that subtracted there must remain the moment of inertia of the hollow sphere, which therefore will be equal to

$$\tfrac{8}{15}\pi\left(a^5 - b^5\right) = \tfrac{2}{5} M \cdot \tfrac{a^5 - b^5}{a^3 - b^3} .$$

Hence the moment of inertia for a hollow sphere about all the axes drawn through the centre is obtained equal to

$$\tfrac{2}{5} M \cdot \tfrac{a^4 + a^3 b + a^2 b^2 + a b^3 + b^4}{aa + ab + bb} .$$

COROLLARY 1

519. If $b = 0$, the case of the solid sphere is produced, the radius of which is equal to a, from which the moment of inertia is $\tfrac{2}{5} Maa$ as above about all axes drawn through the centre.

COROLLARY 2

520. If this spherical shell should be the thinnest, so that $b = a$ approximately, the n the moment of inertia is equal to $\tfrac{2}{3} Maa$, which formula prevails for a spherical surface. But if we are unable to neglect the thickness Aa, which is equal to $c = a - b$, then the moment of inertia is equal to

$$= \tfrac{2}{5} M \cdot \tfrac{5a^4 c - 10 a^3 c^2}{3aac - 10acc} = \tfrac{2}{5} M \left(aa - ac\right).$$

SCHOLIUM

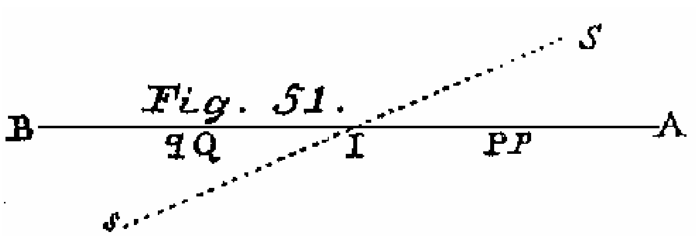
521. These cases are sufficient abundantly not only, as hence for several cases the moments of inertia have been brought forwards, but also other bodies occur, we may prevail in putting the calculation in place easily. On account of which, we can progress to defining the rotational motion of bodies acted on by gravity, since this is a particular case, to which this discourse becomes accustomed to be applied.

CAPUT VI

INVESTIGATIO MOMENTI INERTIAE IN CORPORIBUS HOMOGENEIS

PROBLEMA 32

471. Si corpus fuerit filum tenuissimum rectum AIB (Fig. 51), invenire eius axes principales eorumque respectu momenta inertiae.



SOLUTIO

Sit tota fili longitudo $AB = 2a$, in cuius medio puncto I erit eius centrum inertiae, ut sit $IA = IB = a$, massa autem fili, quae geometrice per $2a$ exprimitur, sit $= M$. Iam unus axium principalium certo est ipsa linea AB, cuius respectu momentum inertiae est nullum ideoque minimum; bini reliqui sunt ad AB in I normales eorumque respectu momenta inertiae aequalia, ita ut eorum situs non determinetur. Ad momentum ergo inertiae respectu talis axis ad AB in I normalis inveniendum, sumto $IP = IQ = x$, elementorum $Pp = Qq = dx$ momenta sunt $xxdx$ sicque amborum coniunctum, cuius integrale $\frac{2}{3}x^3$ posito $x = a$ dat momentum inertiae fili respectu axium ad filiu in I normalium $= \frac{2}{3}x^3 = \frac{1}{3}Maa$ ob $M = 2a$.

COROLLARIUM 1

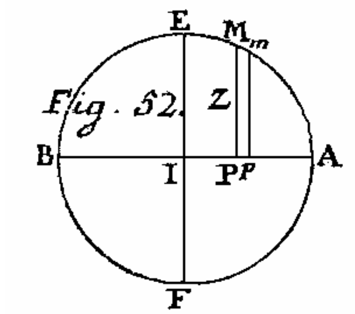
472. Bini ergo reliqui axes principales praeter AIB non determinantur perindeque est, quaenam duae rectae tam inter se quam ad filum in I normales pro iis accipiantur. Eorumque respectu mometum inertiae $\frac{1}{3}Maa$ est maximum, ita ut medium cum maximo congruat.

COROLLARIUM 2

473. Cum momentum inertiae respectu axis *AB* sit nihilo aequale, respectu alius cuiuscunque axis *SIs* ad *AB* angulo $AIS = \vartheta$ inclinato erit $\frac{1}{3}Maa\sin^2\vartheta$, quod ex superioribus evidens est, si binorum reliquorum axium principalium alter in plano *AIS* capiatur; tum enim axis *Ss* ad eum inclinatur angulo $90^{\circ} - \vartheta$, ad alterum vero angulo recto.

PROBLEMA 33

474. Si corpus fuerit filum tenuissimum in peripheriam circuli AEBF incurvatum (Fig. 52), invenire eius axes eorumque principales respectu momenta inertiae.



SOLUTIO

Sit radius $IB = a$, et posita ratione diametri ad peripheriam =1:π erit longitudo fili = $2\pi a$, quae simul eius massam refert, quae sit = *M*. Cum centrum inertiae sit in circuli centro *I*, primo recta ad planum circuli in *I* perpendicularis erit unus axis principalis, cuius respectu erit momentum inertiae = Maa; duo reliqui axes in plano circuli sunt siti, pro quibus binos diametros quoscunque inter se normales assumere licet *AB* et *EF*. Sumta iam abscissa $IP = x$ et applicata $PM = y = XZ = \sqrt{(aa - xx)}$, ob elementum fili $Mm = \frac{adx}{y}$ erit eius momentum respectu axis *AB* = *aydx* ideoque

momentum totum $= a\int ydx$

$$= a \times \text{aream circuli} = \pi a^3,$$

quod ob $M = 2\pi a$ erit $= \frac{1}{2}Maa$. Quare momentum respectu diametri cuiusvis est $= \frac{1}{2}Maa$.

COROLLARIUM 1

475. Momentum ergo inertiae respectu axis principalis ad planum circuli normalis *Maa* est maximum et momentum medium cum minimo congruit estque utrumque semissis maximi.

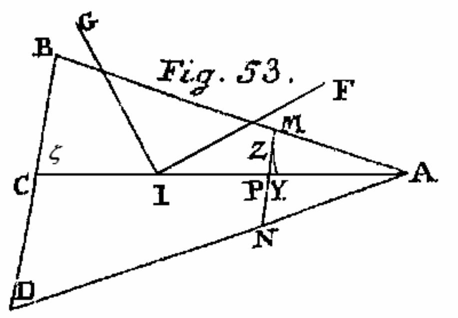
COROLLARIUM 2

476. Si alius axis quicunque concipiatur ad planum circuli in *I* inclinatus angulo $=\eta$, quia is ad axem primum inclinatur angulo $90^{\circ}-\eta$, ad reliquorum alterum angulo η et ad tertium angulo recto, erit eius respectu momentum inertiae =

$$Maa\,sin^{2}\eta+\tfrac{1}{2}Maa\,cos^{2}\eta=\tfrac{1}{2}Maa(1+sin^{2}\eta).$$

PROBLEMA 34

477. Si corpus fuerit lamina tenuissima plana triangularis *ABD* (Fig. 53), invenire eius tres axes principales eorumque respectu momenta inertiae.



SOLUTIO

Ut centrum inertiae *I* obtineatur, ex angulo *A* ducatur recta *AC* latus oppositum *BD* bisecans sumtaque *CI* parte tertia totius *AC* erit centrum inertiae in *I*. Ponamus $CI=a, CB=CD=c$ et angulum $ACB=\zeta$, ut sit

$$AI=2a, AC=3a \text{ et } BD=2c.$$

Iam perspicuum est unum axem principalem fore ad planum trianguli normalem in *I*, quoniam, si in hac recta coordinatam *x* sumeremus, foret

$$\int xydM=0 \text{ et } \int xzdM=0 \text{ ob } x=0.$$

Quare secundum problema 28 praeter istum axem sumantur in plano trianguli binae reliquae directrices, quarum altera sit *IA*, et sumto elemento quocumque *dM* in *Z* indeque ad *IA* demisso perpendiculo *ZY* sit *IY* = *y* et *YZ* = *z* vocenturque integralia

$$\int xxdM=A=0,\quad \int yydM=B,\quad \int zzdM=C;\ \text{tum}\ \int yzdM=F;$$

unde, si *IF* et *IG* sint bini reliqui axes principales ponaturque angulus *AIF* = ϑ, demonstravimus fore

$$\text{tang } 2\vartheta=\frac{2F}{B-C}$$

et respectu axis *IF* momenta inertiae =

$$A+B\,sin^{2}\vartheta+C\,cos^{2}\vartheta-2F\,sin\vartheta\,cos\vartheta,$$

ubi ϑ denotat tam angulum *AIF* quam *AIG*. Tum vero respectu primi axis ad planum trianguli normalis est moment inertiae = *B* + *C*. Ad hos valores inveniendos per *Z* ducatur lateri *BD* parallela *MN*, positisque $AP = t$ et $PZ = u$ erit

$$PM = PN = \frac{ct}{3a}, YZ = u\sin\zeta \text{ et } PY = uc\cos\zeta$$

atque elementum in $Z =$

$$dtdu\sin\zeta = dM.$$

Hinc igitur erit

$$y = 2a - t + u\cos\zeta \text{ et } z = u\sin\zeta;$$

concipiatur aliud aequale elementum $dtdu\sin\zeta$ ad alteram partem, pro quo sit *u* negativum, hisque iunctim consideratis fiet

$$B = 2\int dt\sin\zeta\int du((2a-t)^2 + uu\cos^2\zeta), C = 2\int dt\sin\zeta\int uudu\sin^2\zeta$$

et

$$F = 2\int dt\sin\zeta\int(udu\sin\zeta(2a-t+u\cos\zeta) - udu\sin\zeta(2a-t-u\cos\zeta))$$

seu

$$F = 2\int dt\sin\zeta\cdot\int uudu\sin\zeta\cos\zeta.$$

Prima integratione peracta poni debet $u = \frac{ct}{3a}$, unde fit

$$B = 2\sin\zeta\int dt(\frac{ct}{3a}(2a-t)^2 + \frac{c^3t^3}{81a^3}\cos^2\zeta),$$

$$C = 2\sin\zeta\int dt\cdot\frac{c^3t^3\sin^2\zeta}{81a^3}$$

et

$$F = 2\sin^2\zeta\cos\zeta\int dt\cdot\frac{c^3t^3}{81a^3}$$

ideoque

$$B = 2\sin\zeta(\frac{2actt}{3} - \frac{4ct^3}{9} + \frac{ct^4}{12a} + \frac{c^3t^4}{324a^3}\cos^2\zeta),$$

$$C = 2\sin\zeta\cdot\frac{c^3t^4\sin^2\zeta}{324a^3}$$

et

$$F = \frac{c^3t^4\sin^2\zeta\cos\zeta}{162a^3},$$

quibus valoribus per totum triangulum, ponendo $t = 3a$, extensis habebimus :

$$B = \tfrac{1}{2} ac \sin\zeta (3aa + cc \cos^2 \zeta),$$
$$C = \tfrac{1}{2} ac^3 \sin^3 \zeta,$$
$$F = \tfrac{1}{2} ac^3 \sin^3 \zeta \cos\zeta.$$

Ex his pro situ axium *IF* et *IG* fiet

$$\text{tang } 2\vartheta = \frac{2ac^3 \sin^3 \zeta \cos\zeta}{ac \sin\zeta (3aa + cc \cos 2\zeta)} = \frac{cc \sin 2\zeta}{3aa + cc \cos 2\zeta},$$

hinc enim duo valores pro ϑ eliciuntur. Denique momentum inertiae respectu axis principalis ad planum trianguli normalis est =

$$\tfrac{1}{2} ac \sin\zeta (3aa + cc \cos 2\zeta) = \tfrac{1}{6} M (3aa + cc)$$

ob $M = 3ac \sin\zeta$ et respectu axis *IF* vel *IG*, prout ϑ angulum *AIF* vel *AIG* denotat, est momentum inertiae

$$\tfrac{1}{6} M (3aa + cc \cos^2 \zeta) \sin^2 \vartheta + \tfrac{1}{6} Mcc \sin^2 \zeta \cos^2 \vartheta - \tfrac{1}{3} Mcc \sin\zeta \cos\zeta \sin\vartheta \cos\vartheta$$
$$= \tfrac{1}{2} Maa \sin^2 \vartheta + \tfrac{1}{6} Mcc (\cos\zeta \sin\vartheta - \sin\zeta \cos\vartheta)^2$$
$$= \tfrac{1}{2} Maa \sin^2 \vartheta + \tfrac{1}{6} Mcc \sin^2 (\zeta - \vartheta).$$

COROLLARIUM 1

478. Cum sit

$$AB^2 + AD^2 = 18aa + 2cc,$$

erit

$$3aa = \frac{AB^2 + AD^2 - 2cc}{6}$$

hincque momentum inertiae respectu axis ad planum trianguli in *I* normalis fiet =

$$\tfrac{1}{36} M (AB^2 + AD^2 + BD^2),$$

ita ut sit pars tricesima sexta massae per summam quadratorum laterum multiplicatae.

COROLLARIUM 2

479. Pro binis reliquis axibus principalibus in plano triangulari sitis *IF* et *IG* notetur ζ esse angulum recto non maiorem, unde sit $\sin 2\zeta$ erit positivus. Posito ergo angulo $AIF = \vartheta$ erit

$$\text{tang } \vartheta = \frac{-3aa - cc\cos 2\zeta + \sqrt{(9a^4 + 6aacc\cos 2\zeta + c^4)}}{cc\sin 2\zeta} = \text{tang } AIF;$$

ac

$$\text{tang } AIG = \frac{-3aa - cc\cos 2\zeta - \sqrt{(9a^4 + 6aacc\cos 2\zeta + c^4)}}{cc\sin 2\zeta}.$$

COROLLARIUM 3

480. Momentum inertiae respectu horum axium est =

$$\tfrac{1}{6}M\left(\tfrac{3}{2}aa - \tfrac{3}{2}aa\cos 2\vartheta + \tfrac{1}{2}cc\cos 2(\zeta - \vartheta)\right);$$

cum igitur sit

$$\text{tang } 2(\zeta - \vartheta) = \frac{3aa\sin 2\zeta}{3aa\cos 2\zeta + cc},$$

hoc utrumque momentum ita exprimetur :

$$\tfrac{1}{12}M\left(3aa + cc \pm \sqrt{(9a^4 + 6aacc\cos 2\zeta + c^4)}\right)$$
$$= \tfrac{1}{12}Mcc(1 - \cos 2\zeta - \sin 2\zeta \text{ tang } \vartheta) = \frac{Mcc\sin\zeta\sin(\zeta - \vartheta)}{6\cos\vartheta};$$

prout enim pro ϑ angulus *AIF* vel *AIG* assumitur, ita ad utrumque axem referetur.

EXEMPLUM

481. Sit triangulum ABD isosceles seu angulus ζ rectus, hincque ob $\tanh 2\vartheta = 0$ erit vel $\vartheta = 0$ vel $\vartheta = 90^{\circ}$, unde alter axis in ipsam rectam *AC* incidit, alter vero ad eum est normalis. Respectu prioris *AC* momentum inertiae erit = $\frac{1}{6}Mcc$, respectu posterioris vero $= \frac{1}{2}Maa$, dum respectu primi, qui ad planum trianguli est normalis, erat = $\frac{1}{2}Maa + \frac{1}{6}Mcc$ ita, ut hoc sit aequale summae binorum reliquorum. Si praeterea triangulum sit aequilaterum, cuius singula latera = 2*c*, erit $3a = c\sqrt{3}$ seu $aa = \frac{cc}{3}$, quare omnes axes in plano trianguli per *I* ducti aequalia praebent momenta inertiae = $\frac{1}{6}Mcc$ et momentum respectu axis ad triangulum in *I* normalis erit duplo maius = $\frac{1}{3}Mcc$.

COROLLARIUM 4

482. Haec postrema proprietas adeo in genere valet; cum enim sit momentum inertiae respectu axis ad planum triangulum normalis = $\frac{1}{6}M(3aa+cc)$, tum vero respectu axis IF =

$$= \frac{1}{12}M(3aa+cc-\sqrt{(9a^4+6aacc\cos 2\zeta+c^4)}),$$

respectu axis IG

$$= \frac{1}{12}M(3aa+cc+\sqrt{(9a^4+6aacc\cos 2\zeta+c^4)}),$$

evidens est horum summam priori esse aequalem.

SCHOLION

483. Notare hic meretur, si reliqua trianguli latera ponantur $AB=2b, AD=2d$, uti est $BD=2c$, fore

$$9aa=2bb+2dd-cc \text{ et } \cos\zeta=\frac{dd-bb}{3ac},$$

unde formula irrationalis $\sqrt{\left(9a^2+6aacc\cos 2\zeta+c^4\right)}$ abit in hanc

$$\frac{4}{3}\sqrt{\left(b^4+c^4+d^4-bbcc-bbdd-ccdd\right)}.$$

Ceterum hic in genere definire non licet, uter axium IF et IG maius praebeat momentum, cum haec ipsa formula irrationalis quandoque negativum valorem induere debeat, quemadmodum patet ex casu $\zeta=90^o$, ubi valor eius $3aa-cc$ fit negativus, si $cc > 3aa$. In genere autem haec duo momenta inter se aequalia fieri nequent, quia formula irrationalis evanescere non potest, nisi sit $2\zeta=180^o$ et $3aa=cc$. At iudicium hoc quovis casu adhibitis angulis ϑ utrique axi convenientibus facile instituetur ex formula

$$\frac{1}{2}Maa\sin^2\vartheta+\frac{1}{6}Mcc\sin^2(\zeta-\vartheta).$$

PROBLEMA 35

484. Si corpus fuerit lamina tenuissima plana figuram parallelogrammi *BDbd* habens (Fig. 54), invenire eius tres axes principales eorumque respectu momenta inertiae.

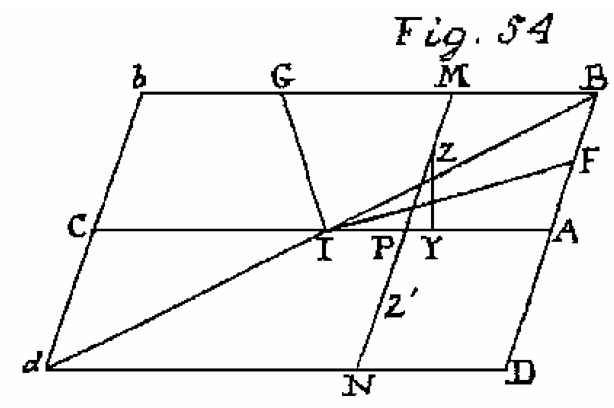
SOLUTIO

Bisectis lateribus binis oppositis *BD* et *bd* in *A* et *C* ductaque recta *AC*, in eius puncto medio *I* erit centum inertiae corporis, cuius massa ponatur = *M*. Ponatur latera

$$Bb = Dd = AC = 2a,$$

$$BD = bd = 2b$$

et angulus acutus $B = d = \zeta$, erit area = $4ab \sin\zeta = M$. Iam unus axium principalium erit ad planum laminae in *I* normalis binique reliqui *IF* et *IG* in ipso hoc plano siti; ad quos inveniendos concipiatur elementum quodcunque *dM* in *Z*, per quod punctum primo ducatur recta *MN* laterali *BD* parallela, sitque *AT* = *t* et *PZ* = *u*; tum ex *Z* ad *AC* demisso perpendiculo *ZY* vocetur secundum problema 29



$$IY = y \text{ et } YZ = z. \text{ Ob } APZ = \zeta \text{ erit}$$

$$ZY = u \sin\zeta \text{ et } PY = u \cos\zeta,$$

unde

$$y = a - t + u\cos\zeta \text{ et } z = u\sin\zeta;$$

tum vero $dM = dtdu \sin\zeta$; at in illo calculo fit $x = 0$, ut sit

$$\int xxdM = 0, \int xydM = 0, \int xzdM = 0.$$

Hinc ergo habemus :

$$\int yydM = B = \int dt \int du \sin\zeta\,(a - t + u\cos\zeta)^2,$$

$$\int zzdM = C = \int dt \int uudu \sin^3\zeta$$

et

$$\int yzdM = F = \int dt \int udu \sin^2\zeta\,(a - t + u\cos\zeta).$$

Combinetur cum his elementum simile *Z'* ad alteram partem situm, pro quo est *u* negativum, fietque :

$$B = 2\sin\zeta \int dt \int du\left((a-t)^2 + uu\cos^2\zeta\right),$$

$$C = 2\sin^3\zeta \int dt \int uudu$$

et

$$F = 2\sin^3\zeta\cos\zeta \int dt \int uudu.$$

Priori integratione instituata ponatur $u = b$, prodibitque

$$B = 2\sin\zeta \int dt \int du\left(b(a-t)^2 + \tfrac{1}{3}b^3\cos^2\zeta\right),$$

$$C = \tfrac{2}{3}b^3\sin^3\zeta \int dt$$

et

$$F = \tfrac{2}{3}b^3\sin^3\zeta\cos\zeta \int dt.$$

Denique posteriori integratione facta ponatur $t = 2a$ fietque

$$B = 4ab\sin\zeta\left(aa + bb\cos^2\zeta\right) = \tfrac{1}{3}M\left(aa + bb\cos^2\zeta\right),$$

$$C = \tfrac{4}{3}ab^3\sin^3\zeta = \tfrac{1}{3}Mbb\sin^3\zeta$$

et

$$F = \tfrac{4}{3}ab^3\sin^3\zeta\cos\zeta = \tfrac{1}{3}Mbb\sin\zeta\cos\zeta.$$

Ex his collititur momentum inertiae respectu primi axis ad laminam in *I* normalis

$$B + C = \frac{1}{3}M(aa+bb);$$

quod ergo non ab obliquitate, sed tantum a lateribus pendet. At pro reliquis axibus *IF* et *IG* posito angulo $AIF = \vartheta$ invenimus

$$\text{tang } 2\vartheta = \frac{2F}{B-C} = \frac{2bb \sin\zeta \cos\zeta}{aa+bb\cos 2\zeta}$$

seu

$$\text{tang } 2\vartheta = \frac{bb \sin 2\zeta}{aa+bb\cos 2\zeta} \ ,$$

cuius duplex valor ϑ praebet utrumque angulum *AIF* et *AIG*. Momentum autem inertiae respectu horum axium est

$$B \sin^2\vartheta + C\cos^2\vartheta - 2F \sin\vartheta\cos\vartheta$$

$$= \tfrac{1}{3}M\left(aa\sin^2\vartheta + bb\cos^2\zeta\sin^2\vartheta + bb\sin^2\zeta\cos^2\vartheta - 2bb\sin\zeta\cos\zeta\sin\vartheta\cos\vartheta\right)$$

$$= \tfrac{1}{6}M\left(aa - aa\cos 2\vartheta + bb - bb\cos 2\zeta\cos 2\vartheta - bb\sin 2\zeta\sin 2\vartheta\right)$$

sicque hoc momentum inertiae ita exprimi poterit

$$= \tfrac{1}{6}M\left(aa + bb - aa\cos 2\vartheta - bb\cos\left(2\zeta - 2\vartheta\right)\right).$$

Cum igitur sit

$$\sin 2\vartheta = \frac{bb\sin 2\zeta}{\sqrt{\left(a^4 + 2aabb\cos 2\zeta + b^4\right)}}$$

et

$$\cos 2\vartheta = \frac{aa+bb\cos 2\zeta}{\sqrt{\left(a^4 + 2aabb\cos 2\zeta + b^4\right)}},$$

istud momentum erit :

$$\tfrac{1}{6}M\left(aa+bb-\sqrt{\left(a^4+2aabb\cos 2\zeta+b^4\right)}\right),$$

ubi ambiguitas signi radicalis et ambos axes *IF* et *IG* et momenta inertiae eorum respectu praebet. Patet ergo summam horum binorum aequalem esse momento primo.

COROLLARIUM 1

485. Si $aa+bb\cos 2\zeta$ habeat valorem positivum, sumto radicali positivo angulus 2ϑ recto erit minor ideoque angulus *AIF* semirecto minor; ac respectu axis *IF* momentum inertiae erit minimum =

$$\tfrac{1}{6}M\left(aa+bb-\sqrt{\left(a^4+2aabb\cos 2\zeta+b^4\right)}\right)$$

respectu axis *IG* vero medium.

COROLLARIUM 2

486. Si $aa+bb\cos 2\zeta$ habeat valorem negativum et radicale pro axe *IF* capiatur positive, angulus 2ϑ recto maior ideoque angululs *AIF* semi-recto maior atque axis *IF* respectu momentum inertiae erit minimum.

COROLLARIUM 3

487. Si ducatur diagonalis *Bd* per angulos acutos *B* et *d*, ob

$$\text{tang } AIB = \frac{b \sin\zeta}{a + b\cos\zeta}$$

reperitur

$$\text{tang } 2\, BIF = \frac{2ab \sin\zeta(aa-bb)}{a^4 + 2a^3 b\cos\zeta + 2aabb\cos 2\zeta + 2ab^3 \cos\zeta + b^4},$$

unde patet in rhombo, ubi $a = b$, ambas diagonales fore axes principales, dum in rectangulo recta *AC* est axis principalis.

EXAMPLE 1

488. Si parallelogrammum *BbdD* sit rectangulum, ob $\zeta = 90^{\text{o}}$ fit $\text{tang } 2\vartheta = 0$ ideoque vel $\vartheta = 0$ vel $\vartheta = 90^{\text{o}}$; unde respectu axis at laminam in *I* normalis erit momentum inertiae $= \frac{1}{3} M\,(aa + bb)$; tum vero alter axis principalis est *AC*, cuius respectu momentum inertiae est $= \frac{1}{3} Mbb$; tertius vero axis principalis est in plano laminae ad *AC* normalis, cuius respectu $= \frac{1}{3} Maa$ existentibus lateribus $Bb = Dd = 2a$ et $BD = bd = 2b$.

EXAMPLUM 2

489. Si parallelogrammum *BbdD* sit rhombus, ut sit $b = a$ et singula eius latera = 2*a*, existentibus angulis acutis ζ fit

$$\text{tang } 2\vartheta = \frac{\sin 2\zeta}{1 + \cos 2\zeta} = \text{tang } \zeta$$

hincque vel $\vartheta = \frac{1}{2}\zeta$ vel $\vartheta = 90^{\text{o}} + \frac{1}{2}\zeta$. Quare respectu primi axis principalis ad planum rhombi in *I* normalis est momentum inertiae = $\frac{2}{3} Maa$; reliqui ambo axes sunt diagonales *Bd* et *Db*, quorum illius *Bd* respectu momentum inertiae est =

$$\frac{1}{3} Maa(1 - \cos\zeta) = \frac{2}{3} Maa \sin^2 \tfrac{1}{2}\zeta,$$

respectu vero alterius diagonalis *Db* est =

$$\frac{1}{3} Maa(1 + \cos\zeta) = \frac{2}{3} Maa \cos^2 \tfrac{1}{2}\zeta.$$

COROLLARIUM 4

490. Si igitur parallelogrammum abeat in quadratum, cuius latus = 2*a*, omnes rectae in eius plano per centrum inertiae *I* ductae pro axibus principalibus haberi possunt eritque eorum respectu momentum inertiae = $\frac{1}{3}Maa$, at respectu axis ad quadratum in I normalis duplo erit maius = $\frac{2}{3}Maa$.

PROBLEMA 36

491. Si corpus fuerit lamina tenuissima plana in figuram circuli efformata (Fig. 52), invenire eius tres axes principales eorumque respectu momenta inertiae.

SOLUTIO

Sit radius circuli = *a*, erit area = πaa, quae massam *M* refert; et cum unus axium principalium ad planum circuli in centro *I* sit normalis, ponatur pro elemento quocunque *dM* in *Z* sito coordinatae $IP = y, PZ = z,$ ob $dM = dydz$ erit

$$\int yydM = \int dy \int yydz = \int dy \cdot yyz = \int yydy = \int yydy\sqrt{(aa-yy)}$$

posito $z = \sqrt{(aa-yy)}$. At hoc integrale reducitur ad hanc formam

$$\int yydM = \tfrac{1}{8}a^4 \int \frac{dy}{\sqrt{(aa-yy)}} - \tfrac{1}{8}y(aa-2yy)\sqrt{(aa-yy)},$$

quod quater sumtum et posito *y* = *a* dat

$$B = \tfrac{\pi}{4}a^4 = \tfrac{1}{4}Maa.$$

Simili modo vero fit

$$\int zzdM = C = \tfrac{1}{4}Maa.$$

Deinde $\int yzdM$, si ex altera diametri parte simile elementum coniungatur, ad nihilum reducitur, ita ut sit

$$\int yzdM = F = 0.$$

Hinc, cum $B - C = 0,$ oritur tang $2\vartheta = \frac{0}{0}$ sicque angulus ϑ est indeterminatus, ex quo cognoscimus, quod per se est clarum, omnes diametros pro axibus principalibus haberi posse, quorum respectu sit momentum inertiae = $\frac{1}{4}Maa$. At respectu primi axis ad planum circuli in centro I normalis est momentum ineritae $B + C = \frac{1}{2}Maa.$

SCHOLION

492. Cum hic elementum massae dM esset = $dydz$, notandum est id semper manere positivum, etiamsi vel y vel z capiatur negative, quo casu etiam differentialia alioquin fierent negativa. In hoc ergo calculo probe cavendum est, ne, cum coordinatae negative accipiuntur, elementi massae dM expressio in calculum tanquam negativa inferetur. Ex quo conveniet pro singulis regionibus, ubi coordinatae signis contrariis afficiuntur, calculum seorsim institui. Ceterum idem valor

$$B = \int yydM = \frac{1}{4}\pi a^4$$

eruitur, si ponatur $IZ = r$ et angulus $AIZ = \varphi$, erit enim

$$dM = rdrd\varphi$$

et

$$y = r\cos\varphi,$$

unde

$$yydM = r^3 drd\varphi \cos^2\varphi,$$

quae secundum variabilem r integrata posita $r = a$ dat $\frac{1}{4}a^4 d\varphi \cos^2\varphi,$ cuius integrale

$$\text{ob } \cos^2\varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$$

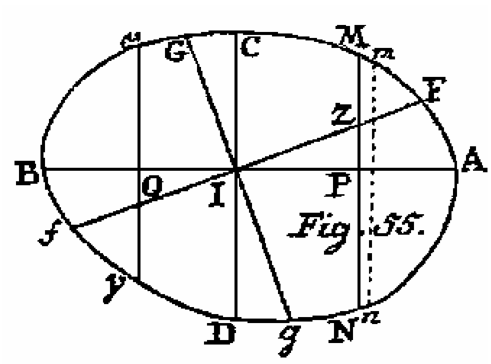
praebet

$$\frac{1}{4}a^4\left(\frac{1}{2}\varphi + \frac{1}{4}\sin 2\varphi\right).$$

Statuatur nunc $\varphi = 2\pi$, ob $\sin 4\pi = 0$ prodit $\frac{1}{4}\pi a^2$ ut ante, unde patet superiorem cautelam legi continuitatis non repugnare.

PROBLEMA 37

493. Si corpus sit lamina tenuissima plana in figuram habens quamcunque *ABCD* (Fig. 55), definire eius axes principales eorumque respectu momenta inertiae.



SOLUTIO

Sit *I* figurae centrum inertiae, manifestumque est rectam ad eius planum in *I* normalem fore unum axium principalium; tum in plano ipso sumtis binis directionibus *AB* et *CD* inter se normalibus pro elemento quovis *dM* in *Z* ponatur coordinatae $IP = y$ et $PZ = z$, erit $dM = dydz$ hincque

$$\int yydM = \int dy \int yydz = \int dy \cdot yyz.$$

Posito ergo $z = PM$, fit

$$\int yydM = \int PM \cdot yyz,$$

cuius valor pro singulis regionibus *AIC, AID, BIC* et *BID* erui debet, eorumque summa erit = *B*, ut sit

$$B = \int IP^2 \cdot MN \cdot d.IP + \int IQ^2 \cdot \mu\nu \cdot d.IQ.$$

Deinde est

$$\int zzdM = \int dy \int zzdz = \tfrac{1}{3} \int PM^3 \cdot dy,$$

ita ut sit

$$C = \tfrac{1}{3} \int \left(PM^3 + PN^3\right) d.IP + \tfrac{1}{3} \int \left(Q\mu^3 + Q\nu^3\right) d.IQ.$$

Porro est

$$\int yzdM = \int dy \int yzdz = \tfrac{1}{2} \int yzzdy = \tfrac{1}{2} \int PM^2 y \cdot dy,$$

cuius valor in regionibus *AID* et *BIC*est negativus, in *BID* vero positivus, unde habebitur

$$F = \tfrac{1}{2} \int IP \left(PM^2 - PN^2\right) d.IP - \tfrac{1}{2} \int IQ \left(Q\mu^2 - Q\nu^2\right) d.IQ.$$

At vero tota massa *M* erit

$$M = \int MN \cdot d.IP + \int \mu\nu \cdot d.IQ.$$

His valoribus inventis erit momentum inertiae respectu axis ad planum in *I* normalis = $B + C$, tum sint reliqui axes principales *FIf* et *GIg*, ac posito angulo $AIF = \vartheta$ reperimus

$$\text{tang } 2\vartheta = \frac{2F}{B-C}$$

et momentum inertiae respectu axis *FIf*

$$= B\sin^2\vartheta + C\cos^2\vartheta - 2F\sin\vartheta\cos\vartheta$$
$$= \tfrac{1}{2}B + \tfrac{1}{2}C - \tfrac{1}{2}(B-C)\cos 2\vartheta - F\sin 2\vartheta.$$

Verum ob

$$\sin\ 2\vartheta = \frac{2F}{\sqrt{\left((B-C)^2+4FF\right)}}$$

et

$$\cos\ 2\vartheta = \frac{B-C}{\sqrt{\left((B-C)^2+4FF\right)}}$$

obtinebitur momentum inertiae respectu

$$\text{axis } FIf = \tfrac{1}{2}(B-C) - \tfrac{1}{2}\sqrt{\left((B-C)^2+4FF\right)}$$

et

$$\text{axis } GIg = \tfrac{1}{2}(B+C) + \tfrac{1}{2}\sqrt{\left((B-C)^2+4FF\right)}.$$

COROLLARIUM 1

494. Momenta ergo inertiae respectu axium *Ff* et *Gg* simul sumta aequalia sunt momento inertiae respectu primi axis principalis, qui ad planum laminae in *I* est normalis.

COROLLARIUM 2

495. Si recta *AB* fuerit figurae diameter, ut sit *PM* = *PN*, valor litterae *F* evanescit, id quod etiam evenit, si recta *CD* fuerit diameter, ut sumto *IQ* = *IP* sit $Q\mu = PM$. At quoties sit *F* = 0, tam ob $\tang 2\vartheta = 0$ ipsae rectae *AB* et *CD* erunt axes principales.

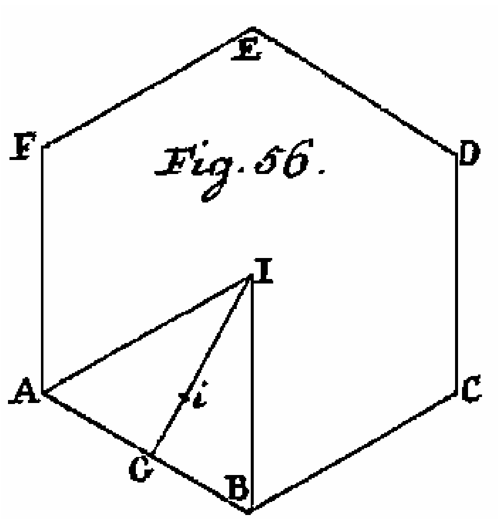
COROLLARIUM 3

496. Casu hoc, quo *F* = 0 et *AB* et *CD* sunt axes principales, erit momentum inertiae respectu axis *Ff* = *C* et respectu axis *CD* = *B*, quae si insuper fuerint aequalia, ob $\tang 2\vartheta = \frac{0}{0}$ omnes rectae per *I* ductae paria habent moment = *B* = *C*.

COROLLARIUM 4

497. Si praeter diametrum *AB* reperiatur alia recta per *I* ducta, cuius respectu momentum inertiae illi sit aequale, tum omnes plane rectae per I ductae eadem proprietate gaudebunt et momenta inertiae habebunt aequalia.

PROBLEMA 38



498. Si corpus sit lamina tenuissima plana in figuram polygoni regularis efformata (Fig. 56), eius axes principales eorumque respectu momenta inertiae definire.

SOLUTIO

Centrum inertiae talis polygoni regularis erit in centro circuli circumscripti I, cuius radius ponatur $IA = a$ numerusque laterum $= n$. Hinc fit angulus $AIB = \frac{2\pi}{n}$, eoque per rectam IG bisecto angulus $AIG = \frac{2\pi}{n}$ atque

$$AB = 2a\sin\frac{\pi}{n} \text{ et } IG = 2a\cos\frac{\pi}{n};$$

quare area trianguli AIB

$$aa\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}aa\sin\frac{2\pi}{n}$$

et area polygoni totius

$$= \frac{n}{2}aa\sin\frac{2\pi}{n}$$

vicem massae M gerens. Iam primo observo (§497) omnes rectas in plano laminae per I ductas aequalia esse habituras momenta, quorum bina simul sumta efficiant mometum respectu axis ad planum laminae in I normalis. Hoc vero momentum ex superioribus colligi potest. Consideretur enim triangulum AIB, cuius massa ponatur = m et centrum inertiae in i, ut sit

$$Gi = \frac{1}{3}\cos\frac{\pi}{n} \text{ et } Ii = \frac{2}{3}\cos\frac{\pi}{n}$$

existente $AG = a\sin\frac{\pi}{n}$. Quia igitur hoc triangulum est isosceles, per § 481 erit eius momentum inertiae respectu axis ad planum trianguli in i normalis

$$= \frac{1}{2}m\cdot Gi^2 + \frac{1}{6}m\cdot AG^2 = m\left(\frac{1}{18}aa\cos^2\frac{\pi}{n} + \frac{1}{6}aa\sin^2\frac{\pi}{n}\right)$$

hincque respectu axis ad idem planum in I normalis

$$= m\left(\frac{1}{18}aa\cos^2\frac{\pi}{n} + \frac{1}{6}aa\sin^2\frac{\pi}{n} + \frac{4}{9}aa\cos^2\frac{\pi}{n}\right) = maa\left(\frac{1}{2}\cos^2\frac{\pi}{n} + \frac{1}{6}\sin^2\frac{\pi}{n}\right),$$

quod per n multiplicatum ob $mn = M$ dabit momentum totius polygoni respectu axis ad id in I normalis

$$= Maa\left(\frac{1}{2}cos^2\frac{\pi}{n} + \frac{1}{6}sin^2\frac{\pi}{n}\right) = \frac{1}{3}Maa\left(1 + \frac{1}{2}cos\frac{2\pi}{n}\right).$$

Respectu vero cuiusque axis in plano laminae per punctum I ducti erit momentum inertiae

$$= \frac{1}{6}Maa\left(1 + \frac{1}{2}cos\frac{2\pi}{n}\right)$$

illo scilicet duplo minus.

COROLLARIUM 1

499. Si praeterea latus polygoni ponatur $AB = c$, ut sit $c = 2a\,sin\frac{\pi}{n}$, ob $a = \frac{c}{2\,sin\frac{\pi}{n}}$ erit momentum inertiae respectu axis principalis ad planum in I normalis

$$= \frac{Mcc}{12\,sin^2\frac{2\pi}{n}}\left(1 + \frac{1}{2}cos\frac{2\pi}{n}\right) = \frac{1}{12}Mcc\cdot\frac{2+cos\frac{2\pi}{n}}{1-cos\frac{2\pi}{n}},$$

respectu reliquorum vero axium principalium est duplo minus.

COROLLARIUM 2

500. Si praeter radium circuli circumscripti $IA = a$ latus polygoni $AB = c$ introducatur, ob

$$sin\frac{\pi}{n} = \frac{c}{2a} \text{ et } cos\frac{2\pi}{n} = 1 - \frac{cc}{2aa}$$

erit momentum respectu axis in I normalis

$$= \frac{1}{3}Maa\left(1 + \frac{1}{2} - \frac{cc}{4aa}\right) = \frac{1}{12}\left(6aa - cc\right),$$

respectu axium vero in ipso plano polygoni per I ductorum est duplo minus.

PROBLEMA 39

501. Si corpus fuerit cylindrus rectus (Fig. 57), eius axes $Aa = 2a$ et radius basis $AB = AD = c$, invenire eius axes principales eorumque respectu momenta inertiae definire.

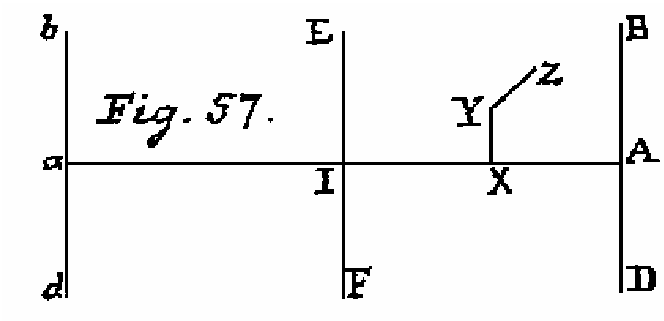


Fig. 57.

SOLUTIO

Cum area basis sit $= \pi cc$, erit cylindri solidas seu massa $= 2\pi acc = M$. In axis autem puncto medio I erit eius centrum inertiae, ut sit $AI = Ia = a$; at ipse hic axis Aa unus manifesto est axium principalium, per quem sumto plano quocunque $BDbd$ pro elemento quovis dM in Z sito habebuntur coordinatae $IX = x, XY = y, YZ = z$, ut sit $dM = dxdydz$. Hinc colligantur valores sequentes :

1°
$$\int xxdM = \int xxdxdydz,$$

ubi sumtis primo x et y constantibus et posito post integrationem $z = \sqrt{(cc-yy)}$, habetur

$$z = \int xxdx \int dy\sqrt{(cc-yy)},$$

at $\int dy\sqrt{(cc-yy)}$ data aream sectionis per X factae $= \pi cc$, ut habeatur

$$\pi cc \int xxdx,$$

cuius integrale tam ad A quam a extensum praebet $\frac{2}{3}\pi cca^3$, ut sit

$$\int xxdM = A = \frac{1}{3}Maa.$$

2°
$$\int yydM = \int yydxdydz = \int dx \int yydy\sqrt{(cc-yy)},$$

at posito $y = c$ est

$$\int yydy\sqrt{(cc-yy)} = \frac{1}{16}\pi c^4,$$

quod quater sumi debet, ut sit

$$\int yydM = \frac{1}{4}\pi c^4 \int dx,$$

hincque habebitur per totum cylindrum

$$\int yydM = \tfrac{2}{4}\pi c^4 a = \tfrac{1}{4}Mcc = B.$$

3° $$\int zzdM = \int zzdxdydz,$$

ubi, si primo x et z pro constantibus sumantur, posito $y = \sqrt{(cc - zz)}$ habetur

$$\int dx \int zzdz\sqrt{(cc - zz)},$$

cuius valor ut ante colligitur

$$\int zzdM = \tfrac{1}{4}Mcc = C = B.$$

4° $\int yzdM$, si simile elementum dM infra planum $BDbd$ cum eo coniungatur, in nihilum abit, ita ut prodeat

$$\int yzdM = F = 0.$$

His positis respectu axis Aa erit momentum inertiae $= B + C = \frac{1}{2}Mcc$; pro reliquis vero binis axibus ad illum normalibus fit

$$\text{tang } 2\vartheta = \tfrac{2F}{B-C} = \tfrac{0}{0},$$

ita ut omnes diametri sectionis in I ad Aa normalis tanquam axes principales spectari possunt, quorum omnium respectu erit momentum inertiae

$$= A + B = M\left(\frac{1}{3}aa + \frac{1}{4}cc\right).$$

COROLLARIUM 1

502. Si alius axis quicunque per I transiens accipiatur, qui faciat cum axe Aa angulum $= \zeta$, eius respectu momentum inertiae erit

$$= (B + C)\cos^2\zeta + (A + B)\sin^2\zeta$$
$$= M\left(\tfrac{1}{2}cc\cos^2\zeta + \tfrac{1}{3}aa\sin^2\zeta + \tfrac{1}{4}cc\sin^2\zeta\right)$$
$$= M\left(\tfrac{1}{3}aa\sin^2\zeta + \tfrac{1}{2}cc - \tfrac{1}{4}cc\sin^2\zeta\right).$$

COROLLARIUM 2

503. Fieri potest, ut omnia momenta respectu rectarum per I ductarum fiant inter se aequalia, quod evenit, si

fuerit $\frac{1}{3}aa = \frac{1}{4}cc$ seu $a = \frac{c\sqrt{3}}{2}$ ideoque $\frac{c}{2a} = \frac{1}{\sqrt{3}}$ et angulus $AaB = 30^\circ$ sive triangulum BaD aequilaterum, quo casu singulu momenta sunt $= \frac{1}{2}Mcc = \frac{1}{8}M \cdot BD^2$.

PROBLEMA 40

504. Si corpus fuerit conus rectus (Fig. 58), cuius vertex *A*, altitudo $AC = a$ et radius basis $CB = CD = c$, invenire eius axes principales eorumque respectu momenta inertiae.

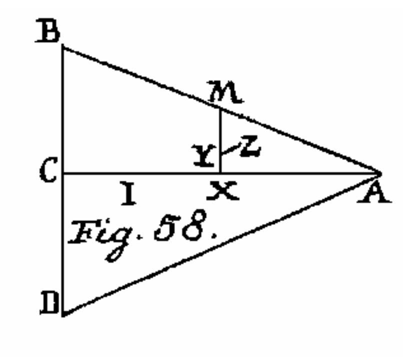


Fig. 58.

SOLUTIO

Cum area basis sit $=\pi cc$, erit solidas eademque massa $M = \frac{1}{3}\pi acc$; tum vero centrum inertiae *I* ita in axe est situm, ut sit $CI = \frac{1}{4}a$ et $AI = \frac{3}{4}a$. Sumatur iam elementum quodcunque *dM* in *Z*, pro quo sint coordinatae $IX = x$, $XY = y$ et $YZ = z$, erit $dM = dxdydz$. Ponatur $AX = t$, erit $XM = \frac{ct}{a}$ et $x = \frac{3}{4}a - t$, nihilo vero minus capi debet $dM = dxdydz$.

Evoluntur ergo sequentes formuale :

1°
$$\int xxdM = A = \int \left(\tfrac{3}{4}a - t\right) xxdtdydz,$$

ubi sumtis primo *t* et *y* constantibus positoque

$$z = \sqrt{\left(\frac{cctt}{aa} - yy\right)}$$

habebitur :

$$\int \left(\frac{3}{4a} - t\right)^2 dt \int dy \sqrt{\left(\frac{cctt}{aa} - yy\right)};$$

ubi pro tota sectione in *X* est

$$\int dy \sqrt{\left(\frac{cctt}{aa} - yy\right)} = \frac{\pi cctt}{aa},$$

ita ut integrandum supersit

$$\frac{\pi cc}{aa} \int ttdt \left(\frac{3}{4}a - t\right)^2 = \frac{\pi cc}{aa}\left(\frac{3}{16}aat^3 - \frac{3}{8}at^4 + \frac{1}{5}t^5\right).$$

Ponatur $t = a$ fietque

$$A = \frac{1}{80}\pi cca^3 = \frac{3}{80}Ma^2.$$

2°
$$\int yydM = B = \int yydtdydz = \int dt \int yydy \sqrt{\left(\frac{cctt}{aa} - yy\right)}$$

per primam integrationem. At manente adhuc *t* constante est

$$\int yydy \sqrt{\left(\frac{cctt}{aa} - yy\right)},$$

posito $y = \frac{ct}{a}$ et quater sumtum $= \frac{1}{4}\pi\frac{c^2t^2}{a^4}$,
ut etiamnum integrari debeat

$$\int \frac{1}{4}\pi \cdot \frac{c^4t^4}{a^4}dt,$$

unde posito pro toto cono $t = a$ fit

$$B = \frac{1}{20}\pi ac^4 = \frac{3}{20}Mcc.$$

3° $$\int zzdM = C$$

pari modo dat

$$C = \frac{3}{20}Mcc = B,$$

at

$$\int yzdM = F$$

manifesto evanescit ut ante.

Cum ergo AC sit unus axium principalium, eius respectu momentum inertiae est =

$$B + C = \frac{3}{10}Mcc.$$

Reliqui axes principales sunt diametri omnes sectionis in I ad axem normalis, quorum respectu momentum inertiae est

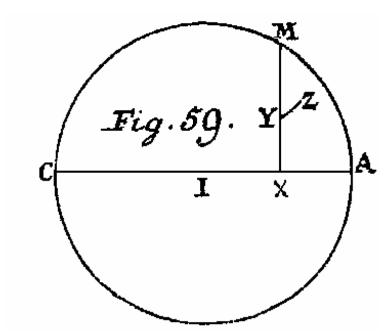
$$A + B = \frac{3}{80}M\left(aa + 4cc\right).$$

COROLLARIUM

505. Casu, quo $aa + 4cc = 8cc$ seu $a = 2c$, hoc est $AC = BD$, omnes rectae per I ductae axium principalium proprietate gaudent eorumque respectu erit momentum inertiae = $\frac{3}{10}Mcc$.

PROBLEMA 41

506. Si corpus fuerit globus ex materia homogenea confectus (Fig. 59), cuius centrum I et radius $IA = a$, definire eius momentum inertiae respectu axis cuisvis per eius centrum transeuntis.



SOLUTIO

Ob radium $IA = a$ erit area circuli maximi = πaa et superficies globi = $4\pi aa$, hinc eius solidas seu massa $M = \frac{4}{3}\pi a^3$. Iam positis pro elemento quocunque dM in Z posito coordinatis $IX = x$, $XY = y$ et $YZ = z$ erit respectu axis AC momentum inertiae

$$=\int dM\left(yy+zz\right).$$

Ponatur $XZ = r$ et $YXZ=\varphi$, erit

$$y=r\cos\varphi,\quad z=r\sin\varphi$$

et

$$dM=rdrd\varphi dx,$$

unde

$$\int rrdM=\int r^3drd\varphi dx=2\pi\int r^3drdx$$

ob $\int d\varphi=2\pi$; nunc sumto r variabli positoque

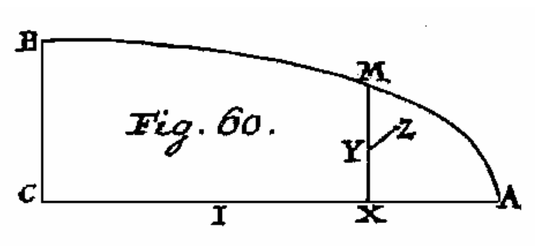
$$r=XM=\sqrt{\left(aa-xx\right)}.$$

habebimus

$$\tfrac{1}{2}\pi\int dx\left(aa-xx\right)^2=\tfrac{1}{2}\pi\left(a^4x-\tfrac{2}{3}aax^3+\tfrac{1}{5}x^5\right).$$

Statuatur $x = a$ pro altero hemisphaerio, et duplum huius expressionis dabit momentum inertiae quaesitum

$$=\pi\cdot\tfrac{8}{15}a^5=\tfrac{2}{5}Maa.$$



PROBLEMA 42

507. Si corpus fuerit conoides quodcunque revolutione linea *AMB* circa axem AC gentitum (Fig. 60), eius axes principales eorumque respectu momentum inertiae invenire.

SOLUTIO

Sit $AC = a$ et pro curva $AX = t$ et $XM = u$, ita ut detur aequatio inter t et u; erit soliditas seu massa

$$M=\pi\int uudt$$

posito post integrationem $t = a$. Tum vero centrum inertiae erit in I, ut sit

$$AI=\frac{\int tuudt}{\int uudt}.$$

Ponatur brevitatis ergo $AI = f$, ut sit

$$\int tuudt=f\int uudt,$$

est vero AC unus axium principalium. Iam pro elemento dM in Z posito sint coordinatae $IX = x = f - t$, $XY = y$ et $YZ = z$, ac ponatur $XZ = r$, angulus $YXZ=\varphi$, erit

$$dM = rdrd\varphi dx,$$
$$y = r\cos\varphi, \quad z = r\sin\varphi$$

Nunc considerentur formulae sequentes :

1° $$\int xxdM = \int (f-t)^2 rdrdtd\varphi = 2\pi \int (f-t)^2 rdrdt$$

ob $\int d\varphi = 2\pi$. Sit adhuc t constans, et posito $r = XM = u$ fiet

$$\int xxdM = \pi \int (f-t)^2 uudt = A$$

ideoque

$$A = \pi ff \int uudt - 2\pi f \int tuudt + \pi \int ttuudt$$
$$= -\pi ff \int uudt + \pi \int ttuudt = M\left(-ff + \frac{\int ttuudt}{\int uudt}\right).$$

2° $$\int yydM = \int r^3 drdtd\varphi \cos^2\varphi = \pi \int r^3 drdt$$

ob

$$\int d\varphi \cos^2\varphi = \int d\varphi\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\varphi\right) = \tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\varphi,$$

quae posito $\varphi = 2\pi$ abit in π. Porro prodit

$$\tfrac{\pi}{4}\int u^4 dt$$

posito $r = u$, ita ut sit

$$\int yydM = B = \tfrac{\pi}{4}\int u^4 dt = \frac{M\int u^4 dt}{4\int uudt},$$

cui etiam aequale fit

$$\int zzdM = C.$$

At

$$\int yzdM = F$$

evanescit.

His evolutis prodit momentum inertiae respectu axis AC

$$= B + C = \frac{M\int u^4 dt}{2\int uudt},$$

positio post integrationem $t = a$, tum vero in sectione ad AC in I normali omnes diametri locum axium principalium sustinent eorumque respectu reperitur momentum inertiae =

$$A + B = M\left(-ff + \frac{4\int ttuudt + \int u^4 dt}{4\int uudt}\right) = M\left(\frac{\int uudt(4tt+uu)}{4\int uudt} - ff\right).$$

EXEMPLUM 1

508. Sit corpus hemisphaerium seu AMB quadrans circuli radii $CA = CB = a$; erit $uu = 2at - tt$, hinc

$$\int uudt = att - \tfrac{1}{3}t^3 = \tfrac{2}{3}a^3$$

posito $t = a$; porro

$$\int tuudt = \tfrac{2}{3}t^3 - \tfrac{1}{4}t^4 = \tfrac{5}{12}a^4,$$

ergo

$$f = AI = \tfrac{5}{8}a \text{ et CI} = \tfrac{3}{8}a.$$

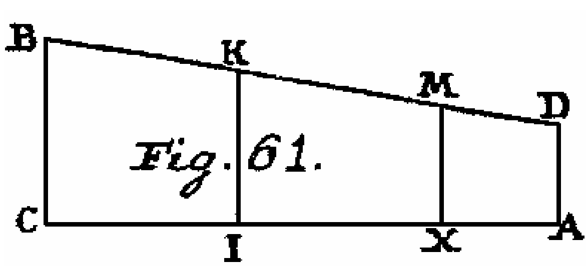
Quare respectu axis AC est momentum inertiae

$$= \frac{M\cdot 8a^5\cdot 3}{15\cdot 4a^3} = \frac{2}{5}Maa$$

et respectu axis cuiusvis alius ad istum in I normalis

$$= M\left(-\tfrac{25}{64}aa + \tfrac{13}{20}aa\right) = \tfrac{83}{320}Maa,$$

ita ut illud momentum sit ad hoc ut 128 ad 83.



EXEMPLUM 2

509. Sit corpus conus truncatus (Fig. 61), cuius axis $AC = a$, radius alterius basis $BC = c$, alterius $AD = b$, eritque $u = b + \frac{(c-b)t}{a}$ et

$uu = bb + \frac{2(c-b)t}{a} + \frac{(c-b)^2 tt}{aa}$, unde pro centro inertiae I inveniendo erit

$$\int uudt = bbt + \frac{(c-b)tt}{a} + \frac{(c-b)^2 t^3}{3aaa} = \frac{1}{3}a\left(bb + bc + cc\right)$$

ideoque soliditas seu massa

$$M = \tfrac{1}{3}\pi a\left(bb + bc + cc\right),$$

deinde

$$\int tuudt = \tfrac{1}{2}bbt + \frac{2(c-b)tt}{3a} + \frac{(c-b)^2 t^4}{4aa} = \tfrac{1}{12}aa\left(bb + 2bc + 3cc\right),$$

unde oritur intervallum

$$AI = f = \frac{a(bb+2bc+3cc)}{4(bb+bc+cc)} \text{ et } CI = \frac{a(cc+2bc+3bb)}{4(bb+bc+cc)}.$$

Porro ob

$$u^4 = b^4 + \frac{4b^3(c-b)t}{a} + \frac{6bb(c-b)^2 tt}{aa} + \frac{4b(c-b)^3 t^3}{a^3} + \frac{(c-b)^4 t^4}{a^4}$$

erit

$$\int u^4 dt = b^4 t + \frac{2b^3(c-b)tt}{a} + \frac{2bb(c-b)^2 t^3}{aa} + \frac{b(c-b)^3 t^4}{a^3} + \frac{(c-b)^4 t^5}{5a^4}$$

et facto $t = a$

$$\int u^4 dt = \tfrac{1}{5}a\left(b^4 + b^3c + bbcc + bc^3 + c^4\right),$$

denique

$$\int ttuudt = \tfrac{1}{3}bbt^3 + \frac{b(c-b)t^4}{2a} + \frac{(c-b)^2 t^5}{5aa} = \tfrac{1}{30}a^3\left(bb + 3bc + 6cc\right).$$

Ex his colligitur momentum inertiae respectu axis AC

$$\tfrac{3}{10}M\,\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc} = \tfrac{3}{10}M\,\frac{b^5-c^5}{b^3-c^3};$$

at respectu axium ad AC in I normalium fit momentum

$$\tfrac{3}{20}M\,\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc} + \tfrac{1}{80}Maa\left(\frac{8(bb+3bc+6cc)}{bb+bc+cc} - \frac{5(bb+2bc+3cc)^2}{(bb+bc+cc)^2}\right),$$

quae reducitur ad hanc formam:

$$\tfrac{3}{20}M\cdot\frac{b^4+b^3c+bbcc+bc^3+c^4}{bb+bc+cc} + \tfrac{3}{80}Maa\,\frac{(b+c)^4+4bbcc}{(bb+bc+cc)^2}.$$

COROLLARIUM 1

510. Si $b = c$, prodit casus cylindri, quo fit $AI = f = \tfrac{1}{2}a$, momentum inertiae respectu $AC = \tfrac{1}{2}Mcc$ et momentum inertiae respectu axium ad illum in I normalium $= \tfrac{1}{4}Mcc + \tfrac{1}{12}Maa$.

COROLLARIUM 2

511. Si $b = 0$, prodit casus coni recti, quo fit $AI = f = \frac{3}{4}a$; momentum inertiae respectu $AC = \frac{3}{10}Mcc$ et momentum inertiae respectu axium ad illum in I normalium $= \frac{3}{20}Mcc + \frac{3}{80}Maa$, ut supra.

COROLLARIUM 3

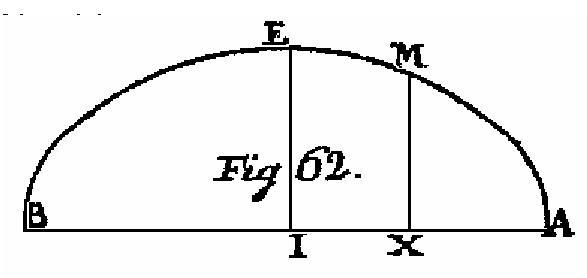
512. Ut omnia momenta respectu axium per I ductorum fiant aequalia, debet esse

$$4\left(b^4 + b^3c + bbcc + bc^3 + c^4\right) = aa \cdot \frac{(b+c)^4 + 4bbcc}{bb+bc+cc}$$

ideoque datis basibus coni truncati altitudo $AC = a$ ita debet definiri, ut sit

$$aa = \frac{4(bb+bc+cc)\left(b^4+b^3c+bbcc+bc^3+c^4\right)}{(b+c)^4+4bbcc}.$$

EXEMPLUM 3



508. Sit corpus sphaeroides ellipticum conversione semiellisis AEB circa axem AB natum (Fig. 62), in cuius ergo medio I est centrum inertiae. Ponatur semiaxis $AI = IB = a$ et coniungatus $IE = c$; erit $uu = \frac{cc}{aa}(2at - tt)$ et in integralibus poni oportet $t = 2a$.

Hinc habebimus

$$\int uudt = \frac{cc}{aa}\left(att - \tfrac{1}{3}t^3\right) = \tfrac{4}{3}acc,$$

ideoque massam $M = \frac{4}{3}\pi acc$, deinde

$$\int tuudt = \frac{cc}{aa}\left(\tfrac{2}{3}at^3 - \tfrac{1}{4}t^4\right) = \tfrac{4}{3}aacc,$$

ergo $AI = f = a$, porro

$$\int ttuudt = \frac{cc}{aa}\left(\tfrac{1}{2}at^4 - \tfrac{1}{5}t^5\right) = \tfrac{8}{5}a^3cc$$

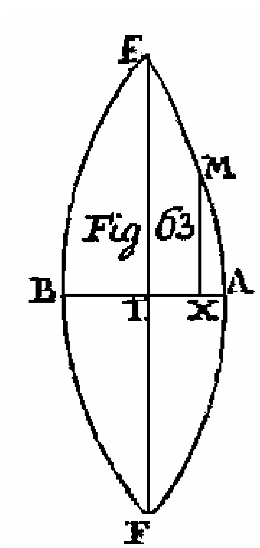
et ob

$$u^4 = \frac{c^4}{a^4}\left(4aatt - 4at^3 + t^4\right)$$

erit

$$\int u^4dt = \frac{c^4}{a^4}\left(\tfrac{4}{3}aat^3 - at^4 + \tfrac{1}{5}t^5\right) = \tfrac{16}{15}ac^4.$$

Ex his colligitur momentum inertiae respectu axis $AB = \frac{2}{5}Mcc$, at respectu axium ad AB in normalium $\frac{1}{5}M\left(aa + cc\right)$.



EXEMPLUM 4

514. Si corpus sit lens ex duabus segmentis sphaerae aeqalibus compositas (Fig. 63) seu ortum ex conversione figurae AEB, ex duobus semisegmentis circuli aequalibus *AIE* et *BIE* formatae, circa axem *AB*, in cuius ergo medio I erit centrum inertiae. Ponatur semiaxis $AI = BI = a$ et $IE = IF = b$, erit diameter circuli $= \frac{aa+bb}{a}$, quem tantisper ponamus = $2c$, ut sit $c = \frac{aa+bb}{2a}$. Quare fiet $uu = 2ct - tt$, et in integralibus superioribus poni debet $t = a$, quo facto ea debebunt duplicari, nisi quod $AI = f$ per se sit $= a$, ideoque

$$A = M\left(aa + \frac{-2a\int tuudt + \int ttuudt}{\int uudt}\right).$$

Hinc nanciscimur

$$\int tuudt = \tfrac{2}{3}a^3c - \tfrac{1}{4}a^4,$$
$$\int uudt = aac - \tfrac{1}{3}a^3$$

et

$$M = 2\pi\left(aac - \tfrac{1}{3}a^3\right)$$
$$\int ttuudt = \tfrac{1}{2}a^4c - \tfrac{1}{5}a^5$$

et

$$\int u^4dt = \tfrac{4}{3}a^3cc - a^4c + \tfrac{1}{5}a^5.$$

Ex his colligitur momentum inertiae respectu axis *AB*

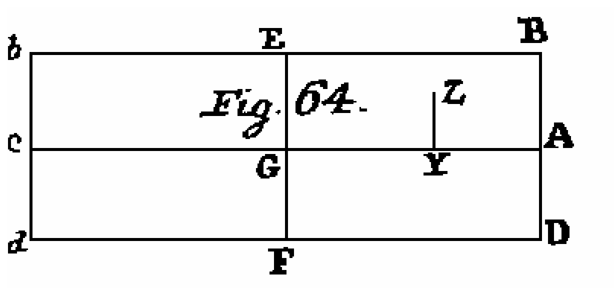
$$= \frac{1}{10}M \cdot \frac{20acc - 15aac + 3a^2}{3c - a} = \frac{1}{10}M \cdot \frac{a^4 + 5aabb + 10b^4}{aa + 3bb},$$

at respectu axium *EF* ad *AB* in *I* normalium :

$$\frac{1}{20}M \cdot \frac{a^3 - 5aac + 20acc}{3c - a} = \frac{1}{20}M \cdot \frac{7a^4 + 15aabb + 10b^4}{aa + 3bb}.$$

PROBLEMA 42a

515. Si corpus fuerit parallelepipedum rectangulum (Fig. 64), invenire eius axes principales eorumque respectu momentum inertiae.



SOLUTIO

Sit rectangulum *BDbd* basis parallelepipedi, cuius latera sint *Bb = 2a, BD = 2b*, altitudo vero = 2*c*,atque manifestum est in puncto medio parallelepipedi fore eius centrum inertiae et axes principales fore tres rectas per id punctum lateribus parallelas. Quaeratur ergo momentum inertiae respectu axis altitudini paralleli, qui basi in puncto medio *G* perpendiculariter insistet. Consideretur hoc rectangulum *BDbd* tanquam sectio basi parallela a centro inertiae distans intervallo = x ac ponatur $GY = y$ et $YZ = z$, erit $dxdydz$ elementum soliditatis seu massae dM, unde fit $M = 8abc$. Tum vero habebimus

$$\int xxdM = \int xxdxdydz$$

et bis integrando per y et z variabiles ponendoque $y = a$ et $z = b$ duplicentur integralia, ut per totam sectionem extendantur, erit

$$\int xxdM = 4ab\int xxdx = \tfrac{4}{3}abx^3;$$

iam posito x = c ac duplicando erit per totum parallelepipedum

$$\int xxdM = A = \tfrac{8}{3}abc^3 = \tfrac{1}{3}Mcc;$$

simili modo erit

$$\int yydM = B = \tfrac{1}{3}Maa;$$

et

$$\int zzdM = c = \tfrac{1}{3}Mbb$$

atque

$$\int yzdM = F = 0.$$

Ex his concluditur momentum inertiae respectu axis principalis altitudini paralleli seu ad basin $BDbd$ perpendicularis

$$= B + C = \tfrac{1}{3}M\left(aa + bb\right),$$

deinde momentum inertiae respectu axis lateri Bb paralleli

$$= \tfrac{1}{3}M\left(bb + cc\right)$$

et respectu axis lateri BD paralleli

$$= \tfrac{1}{3}M\left(aa + cc\right).$$

COROLLARIUM 1

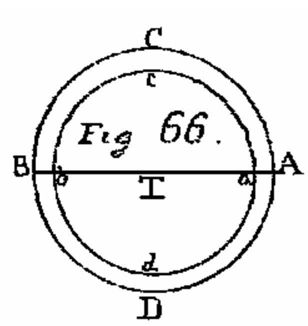
516. Si ergo $ABCDabce$ fuerit tale parallelepipedum rectangulum (Fig. 65), cuius massa sit = M, erunt eius axes principales lateribus AB, AC, AD paralleli per punctum medium transeuntes eritque momentum inertiae

respectu axis lateri $\begin{matrix} AB \\ AC \\ AD \end{matrix}$ parallel = $\begin{matrix} \tfrac{1}{12}M\left(AC^2 + AD^2\right) \\ \tfrac{1}{12}M\left(AB^2 + AD^2\right) \\ \tfrac{1}{12}M\left(AB^2 + AC^2\right). \end{matrix}$

COROLLARIUM 2

517. Si corpus fuerit cubus, cuius latus = a, haec tria momenta fiunt inter se aequalia; ideoque momenta inertiae respectu omnium plane axium per centrum cubi ductorum erunt inter se aequalia et quidem = $\frac{1}{6}Maa$. Talis autem aequalitas in omnibus corporibus regularibus locum habere debet.

PROBLEMA 43



518. Si corpus fuerit globus excavatus (Fig. 66), ut cavitas sit etiam sphaera eodem centro praedita, definire eius momentum inertiae respectu omnium axium per eius centrum ductorum.

SOLUTIO

Sit I centrum et radius globi $IA = a$, cavitatis vero $Ia = b$, ut crassities crustae sphaericae sit = $a - b$ = Aa, erit ergo massa huius globi cavi $= \frac{4}{3}\pi\left(a^3 - b^3\right)$, quae ponatur = M; omnes autem axes per centrum I ductos paria habere momenta inertiae per se est manifestum; quaeramus ergo momentum inertiae respectu axis AB. Ac si globus esset solidus, ob eius massam = $\frac{4}{3}\pi a^3$ foret eius momentum inertiae

$$= \pi a^3 \cdot \frac{2}{5}aa = \frac{8}{15}\pi a^5,$$

globi autem e medio sublati $= \frac{8}{15}\pi b^5$, quo ab illo subtracto remanere debet momentum inertiae globi cavi, quod ergo erit

$$= \frac{8}{15}\pi\left(a^5 - b^5\right) = \frac{2}{5}M \cdot \frac{a^5 - b^5}{a^3 - b^3}.$$

Habebitur ergo momentum inertiae pro globo excavato respectu omnium axium per centrum ductorum

$$= \frac{2}{5}M \cdot \frac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{aa + ab + bb}.$$

COROLLARIUM 1

519. Si $b = 0$, prodit casus globi solidi, cuius radius = a, pro quo momentum inertiae est ut supra $\frac{2}{5}Maa$ respectu omnium axium per centrum ductorum.

COROLLARIUM 2

520. Si crusta haec sphaerica fuerit tenuissima, ut sit proxime $b = a$, erit momentum inertiae = $\frac{2}{3}Maa$, quae formula valet pro superficie sphaerica. Sin autem crassitiem Aa, quae sit = $c = a - b$, omnino negligere nolimus, erit momentum

$$= \frac{2}{5}M \cdot \frac{5a^4c - 10a^3c^2}{3aac - 10acc} = \frac{2}{5}M\left(aa - ac\right).$$

SCHOLION

521. Hi casus abunde sufficiunt non solum, ut hinc pro pluribus corporibus momenta inertiae depromere, sed etiam, si alia corpora occurrant, calculum eo facilius instituere valeamus. Quamobrem progrediamur ad motus gyratios corporum a gravitate sollicatorum definiendos, quandoquidem hic est praecipuus casus, ad quem haec tractatio accommodare solet.