Chapter 5

Concerning the Moment of Inertia.

DEFINITION 7

422. The moment of inertia of a body with respect to some axis is the sum of all the products which arise, if the individual elements of the body are multiplied by the square of their distances from the axis.

COROLLARY 1

423. Because both the elements of the body as well as the square of the distances are always positive, it is necessary that all these products are positive ; hence with an increase of the mass of the body clearly the moment of inertia of this is increased.

COROLLARY 2

424. Hence the moment of inertia can be considered as the product from the mass of the body by the square of a certain line; thus if the mass of the body is equal to *M*, then the moment of inertia with respect to some axis has a form of this kind *Mkk*.

COROLLARY 3

425. Therefore with the moment of inertia found with respect to an axis, around which that we assumed to be rotating about before, and that is equal to *Mkk*. Thus if the moment of the forces acting is *Vf* and the angular speed is equal to γ , then

$$d\gamma = \frac{2Vfgdt}{Mkk}.$$

EXPLANATION

426. The reasoning behind this derivation has been chosen from the similarity to progressive [i. e. linear] motion ; for just as in progressive motion, if a body is accelerated by a force along its own direction, the increment of the speed is as the force acting divided by the mass or the inertia, thus in gyratory motion, because in place of the force acting it is required to consider the moment of this, that expression $\int rrdM$, which is put in place of the inertia in the calculation, we can call the *moment of inertia*, as the increment of the angular speed in a similar manner becomes proportional to the moment of the force acting divided by the moment of inertia. Which is closer in its likeness to that, because it is required to multiply each by the element of the time *dt* and by twice the line 2*g*, in order that the increment in the speed may be expressed.

SCHOLIUM

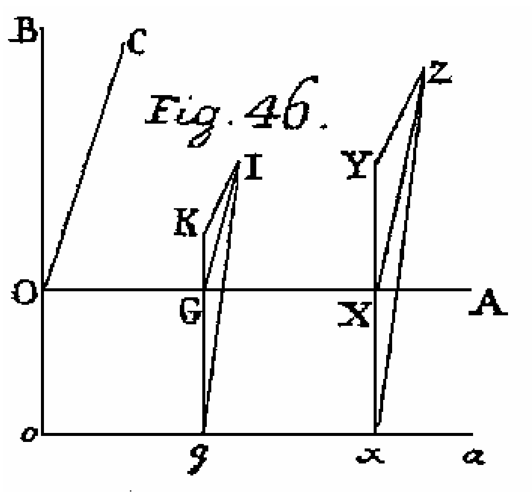
427. Since the same body can be referred to infinite axis, with respect to which it has certain moments of inertia, from which the moment of inertia cannot be defined absolutely, unless it is referred to a determined axis. Yet meanwhile it is not always a help, if the moment of inertia of the same body has to be found with respect to several axes, that the calculation is extricated afresh from the formula $\int rrdM$; but it happens often that, as we have come upon the moment of inertia with respect to one axis, from that we can easily gather together the moments of inertia of the same body with respect to an infinite number of other axes. But in the first place this [proposition] has an advantage, when the axes are parallel, thus in order that from a known moment of inertia for one axis from that it is possible to assign easily the moment of inertia for some other parallel axis, which we show in the following problem.

PROBLEM 25

428. For the given moment of inertia of a certain body with respect to the axis *OA*, to find the moment of inertia of the same body with respect to another axis *oa* parallel to that axis(Fig. 46).

SOLUTION

Let $Oo = c$ be the separation of these axis, in the plane of which there is taken the directrix *OB* normal to *OA* and in the third place *OC* perpendicular to each. Some element of the body *dM* of the whole mass *M* is considered at *Z*, thus with the perpendicular *ZY* sent to the plane *AOB* and with the normal *YX* drawn from *Y* to *OA*, which produced to the other axis *oa* crosses that at *x*; and there is put in place for the given axis *OA* the coordinates $OX = x$, $XY = y$ and $YZ = z$. Therefore since the moment of inertia is given with respect to this axis *OA*, let that be equal to *Mkk*, and then



$$\int (yy + zz)dM = Mkk .$$

Now for the new axis *oa*, on account of $ox = x$, $xY = c + y$ and $YZ = z$, then the moment of inertia is equal to

$$\int ((c+y)^2 + zz)dM = \int ccdM + 2\int cydM + \int (yy+zz)dM .$$

Therefore as the integral

$$\int (yy+zz)dM = Mkk \text{ et} \int ccdM = Mcc,$$

for the term $2\int cydM = 2c\int ydM$ the centre of inertia of the body is considered, that is at *I*, thus to the plane of the axes there is sent the perpendicular *IK* and from *K* normal to the axes *KGg*, and then

$$\int ydM = M \cdot GK .$$

Hence the moment of inertia with respect to the axis *oa* is equal to

$$Mkk + Mcc + 2Mc \cdot GK ,$$

but on account of $Gg = c$ and $cc + 2c.GK = gK^2 - GK^2$ thus is expressed so that it becomes :

$$Mkk + M \cdot gK^2 - M \cdot GK^2 ,$$

and thus from the known moment of inertia with respect to the axis *OA*, which is equal to *Mkk*, the moment of inertia with respect of other axis parallel to that *oa* is easily found.

COROLLARY 1

429. If the axis *oa* is more distant from the centre of inertia *I* than the axis *OA*, then the moment of inertia with respect to the axis *oa* is greater than with respect to the axis *OA*. Indeed the moment of inertia with respect to the axis

$$oa = Mkk + M \cdot gI^2 - M \cdot GI^2 .$$

COROLLARY 2

430. If therefore the infinitude of axes are considered parallel amongst themselves, the momentum if inertia will be a minimum with respect to that axis, which is itself drawn through the centre of inertia. Clearly, if the centre of inertia should be at *G* and the axis *OA* should pass through that, with respect to this the moment of inertia is equal to *Mkk*, and with respect to the axis *oa* the moment of inertia is equal to

$Mkk + M \cdot Gg^2$.

COROLLARIUM 3

431. Therefore if the moment of inertia *Mkk* is given with respect to a certain axis passing through the centre of inertia of the body, then the moment of inertia with respect to some other axis parallel to that, surpasses that by the product of the mass of the body by the square of the distance from the centre of inertia.

SCHOLIUM

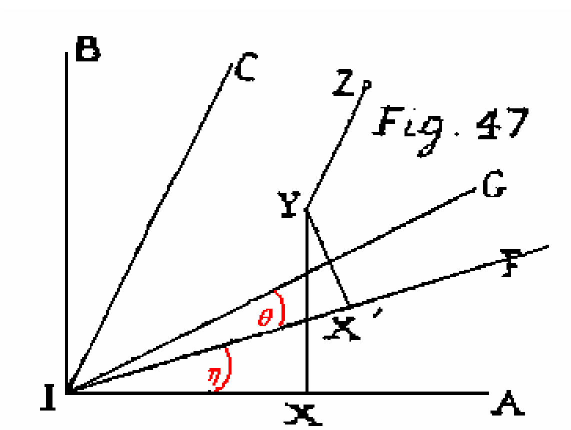
432. Hence the investigation of the moment of inertia of a certain body is restricted only to the axis of this drawn through the centre of inertia, with respect to which if moments of inertia were to be investigated, then for any other axis the moments of inertia could be gathered easily. And this property of the centre of inertia, because the moment of inertia with respect to the axis passing through that shall be the smallest among all with respect to the other parallel axis taken, altogether is remarkable, since also it indicates the outstanding nature of the rotational motion about this centre. Now it is possible to draw innumerable axes through the centre of inertia, of which there is strong disagreement with regard to the moments of inertia among themselves, nor is it apparent how the rest are able to be defined from the others given. Yet meanwhile, because none of these are able either to vanish or to grow indefinitely, among themselves it is necessary that both a maximum and a minimum are given, which enquiry as it is considered to be entirely worthwhile, as that may be undertaken more carefully. But in order that this may succeed more easily, it is convenient for the kinds of moment of inertia to be expressed by a calculation with respect to any axis drawn through the centre of inertia.

PROBLEM 26

433. If the nature of the body is expressed by an equation between three coordinates, to find the moment of inertia of this body with respect to any axis drawn through its centre of inertia.

SOLUTION

Let *I* be the centre of inertia of the body, in which likewise the coincidence of the three directrices *IA*, *IB*, *IC* is established, normal to each other (Fig. 47), from which for some element of the body *dM* placed at *Z* the parallel coordinates are : *IX* = *x*, *XY* = *y*, *YZ* = *z*; thus, if from which of the directrices should be taken for the axis, the moment of inertia can be assigned easily with respect to that. Now that [*i. e.* the new moment of inertia] has be defined with respect to some other axis *IG*, through which a plane drawn normal to *AIB* cutting this in the right line IF, and the angle is put in place $AIF = \eta$ and the angle $FIG = \theta$; hence the question arises, how the point *Z* may



be expressed by three other coordinates, of which one is taken with the axis *IG* itself. We may move the three directrices in the first place thus, so that one shall be *IF*, with *IC* remaining, while the third shall be normal to these, and on drawing *YX'* normal to *IF* the three coordinates will become *x', y', z'*,

$$IX' = x' = x\cos\eta + y\sin\eta,\ X'Y = y' = y\cos\eta - x\sin\eta \text{ et } YZ = z' = z\ ;$$

[Thus, in the first place, a rotation through the angle η takes place in the *AIB* or *xy* plane about the fixed *IC* or *z* axis.]
In a like manner hence the transition can be made to the three new coordinates *x'', y'', z''*, of which *x''* is taken on the axis *IG*, and there becomes :

$$x'' = x'\cos\theta + z'\sin\theta,\ z'' = z'\cos\theta - x'\sin\theta\ ,\ y'' = y'\ ,$$

[Thus, in the second place, a similar rotation through the angle θ takes place in the *FIG* or *x'z'* plane about the fixed *YX'* or *y'* axis.]
thus with the values substituted there is given :

$$x'' = x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta,$$

$$y'' = y\cos\eta - x\sin\eta\ ,$$

and

$$z'' = z\cos\theta - x\cos\eta\sin\theta - y\sin\eta\sin\theta.$$

And hence the square of the distance of the point *Z* from the axis *IG* produces :

$$\begin{aligned} &y''y'' + z''z'' = \\ &x^2\sin^2\eta + y^2\cos^2\eta + z^2\cos^2\theta - 2xy\sin\eta\cos\eta - 2xz\cos\eta\sin\theta\cos\theta \\ &-2yz\sin\eta\sin\theta\cos\theta + x^2\cos^2\eta\sin^2\theta + y^2\sin^2\eta\sin^2\theta + 2xy\sin\eta\cos\eta\sin^2\theta. \end{aligned}$$

Now we put in place the following integrals of the extended body :

$$\int xxdM = A,\ \int yydM = B,\ \int zzdM = C,$$
$$\int xydM = D,\ \int xzdM = E,\ \int yzdM = F,$$

and the moments of inertia sought with respect to the axis *IG* :

$$\begin{aligned} &A(\sin^2\eta + \cos^2\eta\sin^2\theta) + B(\cos^2\eta + \sin^2\eta\sin^2\theta) + C\cos^2\theta \\ &-2D\sin\eta\cos\eta\cos^2\theta - 2E\cos\eta\sin\theta\cos\theta - 2F\sin\eta\sin\theta\cos\theta. \end{aligned}$$

COROLLARY 1

434. Here the quantities *A, B, C* by necessity are positive, and the rest truly *D, E, F* on account of the body can be either positive or negative.

COROLLARY 2

435. The moment of inertia with respect to the axis *IA* is equal to $B+C$, with respect to the axis $IB = A+C$ and with respect to the axis $IC = A+B$. Therefore from these three moments the values *A, B*, and *C* are known.

COROLLARY 3

436. For however the angles η and θ are taken, the moment of inertia found is never able to vanish, but a positive value is always obtained.

SCHOLIUM

437. If we wish to determine not only the motion of the body about the axis *IG*, but also the forces sustained by the axis, we must know besides the moment of inertia with respect to this axis, also the values of the integral formulas
$\int x''y''\,dM$ and $\int x''z''\,dM$. Moreover the formulae in terms of the coordinates *x, y, z* become:

$$\int x''y''\,dM = \int dM\,(x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta)(y\cos\eta - x\sin\eta)$$

and

$$\int x''z''\,dM =$$
$$\int dM\,(x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta)(-x\cos\eta\,\sin\theta - y\sin\eta\sin\theta + z\cos\theta)$$

Whereby, if here the above values are substituted, we have :

$$\int x''y''\,dM = -A\sin\eta\cos\eta\cos\theta + B\sin\eta\cos\eta\cos\theta + D(\cos^2\eta - \sin^2\eta)\cos\theta$$
$$-E\sin\eta\sin\theta + F\cos\eta\sin\theta$$
$$\int x''z''\,dM = -A\cos^2\eta\sin\theta\cos\theta - B\sin^2\eta\sin\theta\cos\theta + C\sin\theta\cos\theta$$
$$-2D\sin\eta\cos\eta\sin\theta\cos\theta + E\cos\eta(\cos^2\theta - \sin^2\theta) + F\sin\eta(\cos^2\theta - \sin^2\theta),$$

which values from that are to be noted more, because they vanish in the cases, in which the moment of inertia shall be a maximum or minimum, as we shall soon see.

PROBLEM 27

438. Among all the axes drawn through the centre of inertia of a given body, to define that with respect to which the moment of inertia is either a maximum or a minimum.

SOLUTION

Everything remains as in the previous problem, and let *IG* be such an axis sought, so that it is thus required to determine the angles $AIF = \eta$ and $FIG = \theta$. Since the moment of inertia with respect to this axis shall be :

$$\int (y''y''+z''z'')dM = A\sin^2\eta + A\cos^2\eta\sin^2\theta + B\cos^2\eta + B\sin^2\eta\sin^2\theta + C\cos^2\theta$$
$$-2D\sin\eta\cos\eta\cos^2\theta - 2E\cos\eta\sin\theta\cos\theta - 2F\sin\eta\sin\theta\cos\theta,$$

it may be differentiated in two ways, on taking first η and then θ as the variable, and with each differential put equal to zero. Therefore from the first this equation is produced :

$$2A\sin\eta\cos\eta\cos^2\theta - 2B\sin\eta\cos\eta\cos^2\theta - 2D\cos^2\eta\cos^2\theta$$
$$+2D\sin^2\eta\cos^2\theta + 2E\sin\eta\sin\theta\cos\theta - 2F\cos\eta\sin\theta\cos\theta = 0,$$

which divided by $-2\cos\theta$ presents :

$$-(A-B)\sin\eta\cos\eta\cos\theta + D(\cos^2\eta - \sin^2\eta)\cos\theta$$
$$-E\sin\eta\sin\theta + F\cos\eta\sin\theta = 0$$

or $\int x''y''dM = 0;$ thus on collecting terms :

$$\frac{\sin\theta}{\cos\theta} = \text{tang}\,\theta = \frac{-(A-B)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta)}{E\sin\eta - F\cos\eta}.$$

But on taking θ for the variable we come upon this equation :

$$2A\cos^2\eta\sin\theta\cos\theta + 2B\sin^2\eta\sin\theta\cos\theta - 2C\sin\theta\cos\theta$$
$$+4D\sin\eta\cos\eta\sin\theta\cos\theta - 2E\cos\eta(\cos^2\theta - \sin^2\theta)$$
$$-2F\sin\eta(\cos^2\theta - \sin^2\theta) = 0,$$

which is the formula equal to $-2\int x''z''dM$. Now since

$$2\sin\theta\cos\theta = \sin 2\theta \text{ and } \cos^2\theta - \sin^2\theta = \cos 2\theta,$$

there then becomes:

$$A\cos^2\eta\sin 2\theta + B\sin^2\eta\sin 2\theta - C\sin 2\theta$$
$$+2D\sin\eta\cos\eta\sin 2\theta - 2E\cos\eta\cos 2\theta - 2F\sin\eta\cos 2\theta = 0,$$

thus it follows that :

$$\frac{\sin 2\theta}{\cos 2\theta} = \text{tang}2\theta = \frac{2E\cos\eta + 2F\sin\eta}{A\cos^2\eta + B\sin^2\eta - C + 2D\sin\eta\cos\eta},$$

from which values equated together there is

$$(E\cos\eta + F\sin\eta)(E\sin\eta - F\cos\eta)^2 = (E\cos\eta + F\sin\eta)((B-A)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta))^2$$
$$+(E\sin\eta - F\cos\eta)((B-A)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta))(A\cos^2\eta + B\sin^2\eta - C$$
$$+2D\sin\eta\cos\eta)$$
$$=((B-A)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta))(E(B\sin\eta - C\sin\eta + D\cos\eta) - F(A\cos\eta - C\cos\eta + D\sin\eta)).$$

Now since $\sin\eta$ and $\cos\eta$ everywhere occupy the same dimensions, if we put

$$\frac{\sin\eta}{\cos\eta} = \text{tang}\,\eta = t,$$

then we obtain this equation :

$$(E+Ft)(F-Et)^2 = (D+(B-A)t - Dt^2)(DE - AF + CF + (BE - CE - DF)t),$$

which on reduction in order gives

$$0 = EFF - DDE + (A-C)DF + t(F^3 - 2EEF + DDF + (A-2B+C)DE - (A-B)(A-C)F)$$
$$+t^2(E^3 - 2EFF + DDE + (A-2B+C)DF - (A-B)(A-C)E)$$
$$+t^3(EEF - DDF + (B-C)DE),$$

thus so that from this cubic equation the value of *t* must itself be elicited.

COROLLARY 1

439. Since the equation, from which the value of *t* must be found, is a cubic, certainly it always has one real root, which provides the tangent of the angle $AIF = \eta$, from which the other angle found $FIG = \theta$ thus is defined, so that it becomes

$$\text{tang}\,\theta = \frac{(B-A)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta)}{E\sin\eta - F\cos\eta} = \frac{\frac{1}{2}(B-A)\sin 2\eta + D\cos 2\eta}{E\sin\eta - F\cos\eta}.$$

COROLLARY 2

440. But it can happen that all the three roots are real, in which case three axis are given in the body, with respect to which axes the moments of inertia are either a maximum or a minimum.

SCHOLIUM

441. But from the nature of the problem it is understood that in any body with more than one such axis present, with respect to this, the moment of inertia shall be either a maximum or a minimum; for if one is axis is given, then with respect to this the moment shall be generally either a maximum or minimum, hence in some other case the axis by necessity is given, with respect of which the moment shall be either a minimum or a maximum. And hence it is possible to conclude, the cubic equation found not only has one but two real roots, from which thus all three roots are always real, which indeed can hardly be observed from the form of this equation. Now indeed with one such axis known, without difficulty the rest of the same kind can be found, which it will be worth the effort to show in the following problem.

PROBLEM 28

442. For one given axis passing through the centre of the body, with respect to which the moment of inertia is a maximum or a minimum, to find the remaining drawn through the centre of this, for which the same property is agreed.

SOLUTION

With the centre of inertia *I* of the body arising, let *IA* be the axis given for that, with respect of which the moment of inertia is either a maximum or a minimum, and from the preceding problem it is agreed that this property cannot be in place unless it shall be that $\int xydM = 0$ and $\int xzdM = 0$; whereby from the above formulas we have $D = 0$ and $E = 0$. But if now IG should be another axis of this kind, for which as before there is put the angle $AIF = \eta$ and $FIG = \theta$, as with respect to this the moment of inertia shall be

$$(A\sin^2\eta + \cos^2\eta\sin^2\theta) + B(\cos^2\eta + \sin^2\eta\sin^2\theta) + C\cos^2\theta - 2F\sin\eta\sin\theta\cos\theta,$$

the method of maxima and minima provides these two equations :

I. $(A-B)\sin\eta\cos\eta\cos^2\theta - F\cos\eta\sin\theta\cos\theta = 0,$

II. $(A\cos^2\eta + B\sin^2\eta)\sin\theta\cos\theta - C\sin\theta\cos\theta - F\sin\eta(\cos^2\theta - \sin^2\theta) = 0.$

Of which the former since divided by $\cos\eta\cos\theta$, is either $\cos\eta = 0$ or $\cos\theta = 0$; indeed the third root of this

$$\text{tang}\ \theta = \frac{(A-B)\sin\eta}{F}$$

defines nothing on substitution into the other equation, since the angle η clearly escaped from the calculation. Therefore let

$cos\,\eta = 0$ and thus $\eta = AIF$ a right-angle and $sin\,\eta = 1$, and the other equation presents:

$$B\,sin\,\theta\,cos\,\theta\; - C\,sin\,\theta\,cos\,\theta - F(cos^2\theta - sin^2\theta) = 0$$

or

$$\tfrac{1}{2}(B - C)\,sin\,2\theta = F\,cos\,2\theta$$

and

$$\text{tang } 2\theta = \tfrac{2F}{B-C};$$

thus for the angle *FIG* a twofold value is produced, either $FIG = \theta$ or $FIG = \theta + 90^{o}$. And thus from the one given axis *IA* always two of the new are gathered together rejoicing in the same property of maxima or minima, which three axis hence correspond to the three roots of the equation found before. But from the first equation the root $cos\,\theta = 0$ clearly makes nothing here; since indeed the angle *FIG* should be right, and the angle $AIF = \eta$ is varied in some manner, the right line *IG* keeps the same position *IC* always and nor from differentiation does it keep this place, now on account of $\eta = 90^{o}$ the momentum of inertia in respect of the axis *IG* is equal to

$$A + B\,sin^2\theta\; + C\,cos^2\theta - 2F\,sin\,\theta\,cos\,\theta,$$

but in respect of the given axis *IA* it is equal to $B + C$.

COROLLARY 1

443. Therefore since the angle $AIF = \eta$ becomes right, both the remaining axes are normal to *IA* and, since they make a right angle to that in turn, in every body three axes are given drawn through the centre of inertia *I* and normal to each other, in respect of which the moments of inertia are either a maximum or a minimum.

COROLLARY 2

444. But if therefore these lines themselves *IA*, *IB* and *IC* were these three axes, with respect of which the moments of inertia are either a maximum or a minimum, then

$$\int xydM = D = 0, \quad \int xzdM = E = 0, \quad \text{and} \int yzdM = F = 0.$$

SCHOLIUM

445. In these problems we have indeed selected the point *I* to be the centre of inertia of the body, because we have tied together firmly the calculation of the moment of inertia to axis of this kind only, which pass through the centre of inertia of the body ; now in the whole calculation of each problem there is nothing involved, because the nature of the centre of inertia is joined with the point *I*. Whereby these problems appear much wider, thus as on taking some point *I* among all the axes passing through it always three are able to be defined, with respect of which the moments of inertia shall be either a maximum or a minimum, and in order that these three axes are normal between themselves. Now here I consider only that property as convenient with the centre of inertia, and to know that for any body it [*i. e*. the centre of inertia] lies between these three axes, since from these the moments of inertia in respect of any axis can easily be found.

DEFINITION 8

446. The *principal axes* of any body are these three axes passing through the centre of inertia of this body, with respect to which the moments of inertia are either a maximum or a minimum.

COROLLARY 1

447. From the preceding, it is understood that for any body not only three such principle axes are to be given, but these also are to be normal amongst themselves ; thus these most conveniently may be taken for the three directrices to which the body is referred.

COROLLARY 2

448. But if therefore *IA, IB, IC* were the principal axis of any body (Fig. 47) and from these for the element of the body *dM* placed at *Z* there are put in place the coordinates $IX = x$, $XY = y$, $YZ = z$, then not only will

$$\int xdM = 0,\ \int ydM = 0,\ \int zdM = 0,$$

but also

$$\int xydM = 0,\ \int xzdM = 0,\ \text{et} \int yzdM = 0.$$

COROLLARY 3

449. Now moreover, if there is put

$$\int xxdM = A,\ \int yydM = B,\ \int zzdM = C.$$

then the moments of inertia of the body will be :

$$\text{with respect to the axis } IA = B + C,$$
$$\text{with respect to the axis } IB = A + C,$$
$$\text{with respect to the axis } IC = A + B,$$

which are either maxima or minima.

SCHOLIUM

450. Certainly it is true of the maximum moments, that in every body they are given by three such axes, and the demonstration of this has certainly been made clear from the preceding. For on taking some three directrices *IA*, *IB*, *IC*, which in turn intersect each other normally at the centre of inertia *I*, we have instructed how to defined one principle axis of this kind *IG* with the help of the resolution of a cubic equation, then with now knowing one the two left are assigned easily. Now truly it will scarcely happen for a body to be so irregular that not as much as a single axis will be unknown, thus so that henceforth the two remaining axes will be produced easily. Whereby previously we assume that these three axis are to be known by us; of which provided we know the moments of inertia with respect to which, for all other axes are able to be shown most promptly, as will be apparent from the following problem.

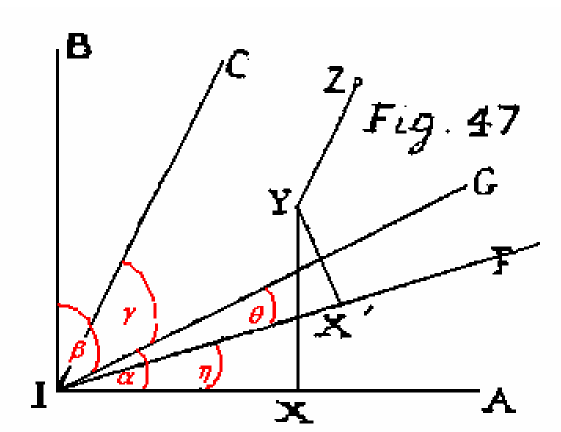
EXPLANATION

451. Just as the reasoning of the maxima and minima may be agreed upon for these three principal axes, thus it is not easily seen through. For since between these there is certainly one axis of which the moment of inertia shall be the greatest of all, and likewise one, with respect of which the moment of inertia shall be the least of all, but it is necessary that with respect to the third the moment of inertia shall be neither the greatest nor the least of all, unless perhaps it should agree with either of these, which sometimes can happen. Now the calculation of the maxima and minima on many occasions indicates quantities of the same kind, which absolutely are neither maxima nor minima, as from that calculation nothing more is declared, because, if you withdraw an infinitely small distance from the place found, neither an increase nor decrease is produced. Thus if *IA* shall be the axis of the maximum taken absolutely and *IC* the axis of the minimum taken absolutely (Fig. 47), with respect to the axis *IB* the moment of inertia will be neither the greatest nor the least of all, but yet it will possess a mean value of this kind, so that, if another axes an infinitely small distance from this is assumed in some direction, the moment of inertia of this neither increases nor decreases. And on this account between these three axis a large distinction intervenes, because it deserves especially to be observed, that of these one has the maximum moment, one the smallest, now the third a mean, that yet in the calculation is able to be seen as a maximum or a minimum, and the reason for this will be shown more in the following problem.

PROBLEM 29

452. Of the given moments of inertia of a certain body with respect to the three principal axes, to find the moment of inertia of this body with respect to any axis drawn through the centre of inertia.

SOLUTIO



Let *IA*, *IB*, *IC* be the three principal axes of the body (Fig. 47) mutually crossing each other normally at the centre of mass *I*, and with the mass of the body put equal to M let the moment of inertia of this with respect to the axis *IA* = *Maa*, with respect to the axis *IB* = *Mbb* and with respect to the axis *IC* = *Mcc* ; thus the moment of inertia with respect to some axis *IG* must be found, which is inclined to the plane *AIB* by the angle $GIF = \theta$, and let the angle $AIF = \eta$. Now the element of the body *dM* is considered at *Z*, and the coordinates of this point shall be *IX* = *x*, *XY* = *y* and *YZ* = *z*; and with the integrals put in place

$$\int xxdM = A, \ \int yydM = B, \ \int zzdM = C.$$

then

$$\int xydM = D = 0, \quad \int xzdM = E = 0, \quad \text{and} \int yzdM = F = 0.$$

Thus from § 433, the moment of inertia about the axis *IG* is equal to :

$$A(\sin^2\eta + \cos^2\eta\sin^2\theta) + B(\cos^2\eta + B\sin^2\eta\sin^2\theta) \ + C\cos^2\theta\,.$$

But since from the three given moments there shall be :

$$Maa = B + C, \quad Mbb = A + C, \quad Mcc = A + B,$$

hence in turn there is gathered :

$$A = \tfrac{1}{2}M(bb + cc - aa), \ B = \tfrac{1}{2}M(aa + cc - bb), \ C = \tfrac{1}{2}M(aa + bb - cc),$$

from which values substituted there will be the sought moment of inertia about the axis *IG* equal to :

$$M(aa\cos^2\eta\cos^2\theta + bb\sin^2\eta\cos^2\theta + cc\sin^2\theta).$$

Where it may be observed that

$$\cos\eta\cos\theta = \cos AIG,\quad \sin\eta\cos\theta = \cos BIG \text{ and } \sin\theta = \cos CIG.$$

Whereby, if the differences of the axis *IG* from the three given principal axes are put in place :

$$AIG = \alpha,\quad BIG = \beta,\quad CIG = \gamma,$$

then the moment of inertia about the axis *IG* is equal to

$$Maa\cos^2\alpha + Mbb\cos^2\beta + Mcc\cos^2\gamma,$$

but these angles α, β, γ are to be compared thus, so that always

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

COROLLARY 1

453. On putting in place the moment of inertia about the *IG* = *Mkk* this can be expressed in the following ways :

$$Mkk = Maa - M(aa-bb)\cos^2\beta - M(aa-cc)\cos^2\gamma,$$

$$Mkk = Mbb + M(aa-bb)\cos^2\alpha - M(bb-cc)\cos^2\gamma,$$

$$Mkk = Mcc - M(aa-cc)\cos^2\alpha + M(bb-cc)\cos^2\beta$$

and in any of these expressions two angles are able to be assumed freely.

COROLLARY 2

454. If it should be that *aa* > *bb* and *bb* > *cc*, then the momentum of inertia about the axis *IA* will be the greatest of all but about the axis *IC* it will be the smallest of all, moreover the mean holds the moment of inertia about the axis *IB*.

COROLLARIUM 3

455. If it should be that

$$(aa-bb)\cos^2\alpha > (bb-cc)\cos^2\gamma,$$

then the momentum of inertia about the *IG* is greater than the mean *Mbb*, with the opposite it is truly less. But if is should be that

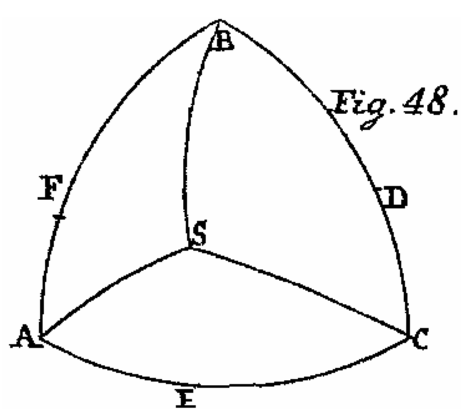
$$(aa-bb)\cos^2\alpha = (bb-cc)\cos^2\gamma,$$

that can happen in an infinite number of places, here all the moments of inertia are equal to each other.

COROLLARY 4

456. If it should be that $aa = bb = cc$, that is, if the principal moments of inertia are equal to each other, about all the axis drawn through the centre of inertia the principal moments of inertia are equal to each other, and thus any axis can be given for the principal axis.

SCHOLION



457. These are able to be represented elegantly in spherical trigonometry in the customary manner (Fig. 48). For with centre of inertia *I* at the centre of a sphere let the extremities of the principal axes be the points *A*, *B*, *C* that are defined on the surface of a sphere, thus so that the arcs *AB*, *AC*, and *BC* are quadrants, and to the end points at *A*, *B*, and *C* there correspond the moments of inertia *Maa*, *Mbb*, *Mcc*, of which the first shall be the maximum, the second the mean, and the third the minimum. But if now some other axis should be considered passing through the centre of inertia, which cuts the surface of the sphere at the point *S*, with respect to this the moment of inertia shall be :

$$Maa\cos^2 AS + Mbb\cos^2 BS + Mcc\cos^2 CS,$$

on account of which

$$\cos^2 AS + \cos^2 BS + \cos^2 CS = 1$$

from these rules it is possible to be expressed by :

$$Maa - M(aa - bb)\cos^2 BS - M(aa - cc)\cos^2 CS,$$

or $$Mbb + M(aa - bb)\cos^2 AS - M(bb - cc)\cos^2 CS,$$

or $$Mcc - M(aa - cc)\cos^2 AS + M(bb - cc)\cos^2 BS.$$

Hence, if S shall be considered in the quadrant BC at D, then the momentum of inertia with respect to the axis ID

$$= M(bb\cos^2 BD + cc\cos^2 CD),$$

$$= Mbb - M(bb - cc)\cos^2 CD,$$

$$= Mcc + M(bb - cc)\cos^2 BD$$

or the moment of inertia with respect to the axis ID will be

$$Mbb - M(bb - cc)\sin^2 BD = Mcc + M(bb - cc)\sin^2 CD.$$

In a similar manner the moment of inertia with respect to the axis IE is equal to

$$Maa - M(aa - cc)\sin^2 AE = Mcc + M(aa - cc)\sin^2 CE;$$

moreover the moment of inertia with respect to the axis IF becomes :

$$Maa - M(aa - bb)\sin^2 AF = Mbb + M(aa - bb)\sin^2 BF.$$

[All on account of the complementary angles on the arcs.]

PROBLEM 30

458. To find all the axes drawn through the centre of inertia, with respect to which all the moments of inertia are equal to each other.

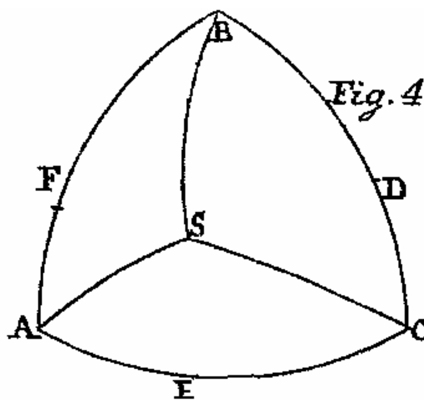


Fig. 48.

SOLUTION

Let the moments of inertia with respect to the principal axis *IA*, *IB*, *IC* (Fig. 48) be *Maa*, *Mbb*, *Mcc* respectively, and $aa > bb > cc$ and all the axes are sought drawn through the centre of inertia *I*, with respect to which the moments of inertia are equal to each other, and indeed equal to that which corresponds to the axis *IE* with *E* taken in the quadrant *AC*, since from *A* to *C* all the moments of this body from the maximum to the minimum occur. Let *IS* be such an axis, and we have this equation :

$$Maa - M(aa-cc)\sin^2 AE$$
$$= Maa - M(aa-bb)\cos^2 BS - M(aa-cc)\cos^2 CS$$

or

$$(aa-cc)\sin^2 AE = (aa-bb)\cos^2 BS + (aa-cc)\cos^2 CS;$$

now on account of

$$\cos^2 BS = \sin^2 AS - \cos^2 CS$$

then

$$(aa-cc)\sin^2 AE = (aa-bb)\sin^2 AS + (bb-cc)\cos^2 CS.$$

[Recall the cosine rule for spherical triangles, where *a*, *b*, and *c* are the arcs defining the triangle, and α, β, γ are the corresponding angles between the arcs :
$\cos a = \cos b \cos c + \sin a \sin b \cos \alpha$, etc.]

The angle *CAS* is introduced, and because $\cos CS = \sin AS \cos CAS$, then

$$(aa-cc)\sin^2 AE = (aa-bb)\sin^2 AS + (bb-cc)\sin^2 AS \cos^2 CAS,$$

hence

$$\sin^2 AS = \frac{(aa-cc)\sin^2 AE}{aa-bb+(bb-cc)\cos^2 CAS}.$$

Moreover if we introduce the angle *ACS*, then we will discover

$$\sin^2 CS = \frac{(aa-cc)\sin^2 CE}{bb-cc+(aa-bb)\cos^2 ACS};$$

[from the first identity,] the angle *CAS* can be increased as far as to a right angle, then $(aa-cc)\sin^2 CE$ in not greater than *aa* – *bb*, that is, if it should be that

$$\sin AE < \sqrt{\frac{aa-bb}{aa-cc}};$$

but the angle *ACS* can be increased as far as a right angle, if it should be that

$$\sin CE < \sqrt{\frac{bb-cc}{aa-cc}};$$

or

$$\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}.$$

[*i. e*., since the arcs are complementary.]Whereby the point *S* shall be on a curve, which arising from *E* and crosses the quadrant *AB*, if it should be that

$$\sin AE < \sqrt{\frac{aa-bb}{aa-cc}};$$

but that curve crosses the quadrant *BC*, if it should be that

$$\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}.$$

But in the case, in which

$$\sin AE = \sqrt{\frac{aa-bb}{aa-cc}},$$

the curve passes through the point *B* itself and all the moments of inertia are equal to *Mbb*. Therefore in this case, then

$$\sin^2 AS = \frac{aa-bb}{aa-bb+(bb-cc)\cos^2 CAS}.$$

Hence on this account,

$$\cos AE = \sqrt{\frac{bb-cc}{aa-cc}}$$

and

$$\frac{aa-bb}{bb-cc} = \frac{\sin^2 AE}{\cos^2 AE}$$

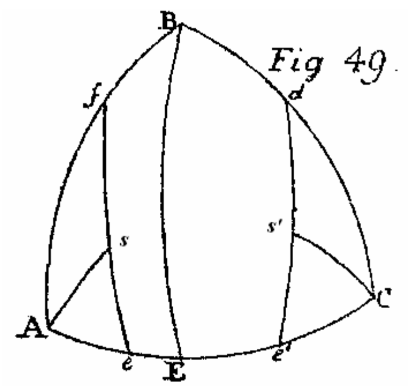
giving

$$\sin^2 AS = \frac{\sin^2 AE}{\sin^2 AE + \cos^2 AE \cos^2 CAS}$$

and thus

$$\operatorname{tang} AS = \frac{\operatorname{tang} AE}{\cos CAS},$$

thus it is understood [in this case] that the location of the points *S* are to be placed on a great circle drawn through the points *B* and *E* (Fig. 49).



In the case, in which $\sin AE < \sqrt{\frac{aa-bb}{aa-cc}}$ or the point *E* is taken nearer to *A*, let that be locoed at *e*, and in the quadrant *AB* there is given the point *f*, in which the moment is equally large. Hence the equation becomes :

$$\sin^2 Af = \frac{(aa-cc)\sin^2 Ae}{aa-bb},$$

thus, if there is put $Ae = e$, $Af = f$, $As = s$ and the angle $eAs = \varphi$, on account of $\frac{aa-cc}{aa-bb} = \frac{\sin^2 f}{\sin^2 e}$ and $\frac{bb-cc}{aa-bb} = \frac{\sin^2 f - \sin^2 e}{\sin^2 e}$ we will have this equation between *s* and φ:

$$\sin^2 s = \frac{\sin^2 e \sin^2 f}{\sin^2 e + (\sin^2 f - \sin^2 e)\cos^2 \varphi} = \frac{\sin^2 e \sin^2 f}{\sin^2 e \sin^2 \varphi + \sin^2 f \cos^2 \varphi},$$

by which equation the nature of the line *esf* is expressed,
and then,

$$\frac{\sin e}{\sin f} = \sin AE .$$

Then in the case, in which $\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}$, the point *E* falls at *e'* and the point *d* is given in the quadrant *BC*, where the moment is the same and at *e'*, so that it becomes

$$\sin^2 Cd = \frac{(aa-cc)\sin^2 Ce'}{bb-cc}.$$

Now there is put in place $Ce' = e, Cd = f, Cs' = s$ and the angle $e'Cs' = \varphi;$ on account of

$$\frac{aa-cc}{bb-cc}=\frac{\sin^2 f}{\sin^2 e} \text{ and } \frac{aa-bb}{bb-cc}=\frac{\sin^2 f-\sin^2 e}{\sin^2 e}$$

this equation is produced between s and φ :

$$\sin^2 s=\frac{\sin^2 e\sin^2 f}{\sin^2 e+(\sin^2 f-\sin^2 e)\cos^2\varphi}=\frac{\sin^2 e\sin^2 f}{\sin^2 e\sin^2\varphi+\sin^2 f\cos^2\varphi},$$

from which the nature of the line *e's'd* is expressed, and then

$$\frac{\sin e}{\sin f}=\sin CE .$$

COROLLARIUM 1

459. Hence a whole great circle is drawn through *B* and *E*, in order that

$$\sin AE=\sqrt{\frac{aa-bb}{aa-cc}},$$

the moment of inertia is equal to *Mbb*. And because the arc *AE* can be taken negatively as well as positively, two great circles as you please on the sphere give the same quality.

COROLLARY 2

460. In a similar manner both around the pole *A* and the opposite to itself, there are elliptic orbs on the surface of the sphere, of which the larger semi-axis is the arc *Af* and the semi-minor axis *Ae*, in which everywhere the same greater moment of inertia rules greater than *Mbb*. In the figure the line *fse* refers to the quadrant of these elliptic orbs.

COROLLARY 2

461. Moreover the lines, on which the moment of inertia is less than *Mbb*, are two elliptic orbs [i. e. on the spherical surface], of which the centres are at the pole *C* and at that opposite, and the major semi axis is the arc *Cd*, while the minor is the arc *Ce'*. In the figure the line *ds'e'* refers to the quadrant of these elliptical orbs.

SCHOLIUM 1

462. Although these lines *fse* and *ds'e'* drawn on the surface of the sphere are not in the same plane, because the tangential projections of these, that it pleases to distinguish by the name of elliptic orbs, have made ellipses in the plane of the sphere at the points *A* and *C* by lines normal to that plane, the centres of which are at the points *A* et *C*. For in the projection of the line *fse* in the plane made tangential at *A*, if there is put $\sin Af = m$, $\sin Ae = n$, in order that

$$\frac{mm}{nn} = \frac{aa-cc}{aa-bb},$$

and for the point *s* by projection, the abscissa at *m* taken equal to $x = sin\, s\, sin\, \varphi$ and with the applied line normal to this equal to $y = sin\, s\, cos\, \varphi$, this equation is had between *x* and *y* :

$$nnxx + mmyy = mmnn,$$

which is for an ellipse having centre at *A*, the semi-axis of which are *m* and *n*. And in a like manner the projection of the line *ds'e'* made in the tangent plane at *C* is found to be an ellipse. If it should be that *Mbb = Mcc*, in which case the point *E* lies at *C*, and letting *Ae = Af* and *m = n*, that ellipse becomes a circle, and the line *fse* becomes a small circle described around the pole *A*.

SCHOLIUM 2

463. Hence we have reduced the investigation of the moment of inertia from that, so that for any proposed body it is sufficient for three moments of inertia to be defined, which clearly shall be selected with respect to the three principal axis of this body. For with these known, the moment of inertia of the same body is easily able to be assigned with respect to any other axis passing through the centre of inertia, and hence again with respect to all these other parallel axes. And in this way, by finding the moments of inertia, which initially seemed as if infinite for any body, has been reduced wonderfully to a summary. Now in addition another distinct aid merits to be known in this development, and by its aid the momentum of inertia of any body can be gathered easily from the moments of inertia of its parts, that we explain in the following problem.

PROBLEM 31

464. From the given moments of inertia of two parts with respect to axis parallel between themselves and passing through each centre of inertia, to find the moment of inertia of the whole body with respect to that parallel axis passing through its centre of inertia.

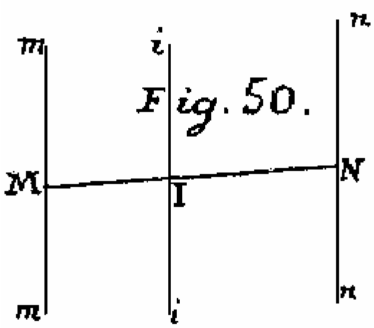


Fig. 50.

SOLUTION

Hence let a body be composed of two parts (Fig. 50), the mass of one shall be equal to *M* having its centre of inertia at *M*, and indeed the mass of the other shall be equal to *N* and its centre of inertia at *N*, and the interval *MN* is put equal to *c*. Now the given moments of inertia of the first part *M* with respect to the axis *mm*, which is equal to *Mmm*, and of the second part *N* with respect to the axis *nn*, which is equal to *Nnn*; and these axes *mm* and *nn* shall be parallel to each other, which pass through the centres of inertia of each; and thus the moment of inertia of the whole body with respect to the axis *ii* parallel to these and passing through its own centre of inertia *I* has to be determined. But the whole mass of the body $= M + N$ and its centre of inertia is found at the point *I* on the line *MN*, in order that it becomes :

$$IM = \frac{Nc}{M+N} \text{ and } IN = \frac{Mc}{M+N}.$$

Therefore with these three axes placed in the same plane, the inclination of these to the line *MN* or the angle $NIi = \delta$ is put in place and the distance between the axis *mm* and $ii = \frac{Mc\sin\delta}{M+N}$, thus the moment of inertia of the part *M* about the axis *ii* is equal to

$$Mmm + \frac{MNNcc\sin^2\delta}{(M+N)^2}.$$

Then also on account of the distance between the axes *nn* and $ii = \frac{Mc\sin\delta}{M+N}$, the moment of inertia of the second part *N* about the axis *ii* is equal to

$$Nnn + \frac{MMNcc\sin^2\delta}{(M+N)^2}.$$

Whereby the moment of inertia of the whole body about the axis *ii* is given

$$Mmm + Nnn + \frac{MNcc\sin^2\delta}{M+N}.$$

COROLLARY 1

465. Hence the moment of the whole body is greater than the moments of the parts likewise taken about their axis parallel to each other and drawn through the centre of inertia of each , and the excess $\frac{MNcc\sin^2\delta}{M+N}$ is proportional to the square of the separation of the axes.

COROLLARY 2

466. If the whole mass of the body is put equal to $I = M + N$ and its moment of inertia about the axis $ii = Iii$, then

$$Iii = Mmm + Nnn + \frac{MNcc\sin^2\delta}{I}.$$

Then on putting the distances $IM = a$ and $IN = b$ then

$$a = \frac{Nc}{I} \text{ and } b = \frac{Mc}{I},$$

thus there becomes

$$Iii = Mmm + Nnn + Iab\sin^2\delta.$$

COROLLARY 3

467. Hence with the moment of inertia of the whole body given *Iii* together with the moment of inertia of the one part *Mmm* the moment of inertia of the other part also is easily gathered

$$Nnn = Iii - Mmm - Iab\sin^2\delta$$

clearly with axis taken parallel to each other and passing through the centre of inertia of each.

COROLLARY 4

468. If the body should depend on more parts, of which the individual moments of inertia about their own axis are parallel to each other and passing through the centre of inertia of each are to be investigated, hence by joining together in twos finally the moment of inertia of the whole body around its axis and passing through its centre of inertia can be gathered.

SCHOLIUM 1

469. In this case of more parts, there is no need in the following problem for two parts to be joined, but the moment of the whole body can be gathered. For let *Mmm, Nnn, Ppp, Qqq* be the moments of the parts about their axes parallel to each other and passing through the centres of inertia of each, for the whole body moreover can be taken parallel to this axis and crossing its centre of inertia, from which the axes the parts *M, N, P, Q* are distant by the intervals *a, b, c, d* ; with which noted, the moment of inertia of the whole body is equal to :

$$M(mm+aa)+\ N(nn+bb)+\ P(pp+cc)+\ Q(qq+dd)\ .$$

Therefore in this manner often the moment of inertia of a very irregular bodies can be easily gathered, provided they are composed from parts of this kind, of which the moments of inertia are allowed to be assigned, with which agreed upon the calculation of the moments of inertia is not a little aided.

SCHOLIUM 2

470. Now it is not a sufficient method to treat all the moments of inertia of bodies to be come upon ; also it is necessary for each particular kind of body to be disclosed, so that as often as the use demands, thus they are able to be selected. But lest this effort should be indefinite, we have restricted this investigation to homogenous bodies, which are agreed to extend to similar material, thus in order that the calculation may be fitting only to geometrical bodies, where I am to consider only certain principal shapes. And in the first place, because it is allowed to consider the thinnest wires and plates as lines and planes, initially we lead from these, thus to various kinds of solids, and to be progressing in this way before others are in the habit of occurring. But for these individual bodies we must define the three principal axes and the moments of inertia around these, since from these the moments about all axes can be collected together in an easy manner. Hence also it becomes apparent likewise, how the calculation can be adapted most conveniently to all other kinds of bodies.

CAPUT V

DE MOMENTO INERTIAE

DEFINITIO 7

422. Momentum inertiae corporis respectu cuiuspiam axis est summa omnium productorum, quae oriuntur, si singula corporis elementa per quadrata distantiarum suarum ab axe multiplicentur.

COROLLARIUM 1

423. Quoniam tam elementa corporis quam quadrata distantiarum semper sunt positiva, omnia haec producta positiva sint necesse est; hinc aucta corporis massa certe eius momentum inertiae augetur.

COROLLARIUM 2

424. Momentum ergo inertiae spectari potest tanquam productum ex massa corporis in quadratum cuiuspiam lineae; ita si massa corporis fuerit = *M*, eius momentum respectu cuiusvis axis habebit huiusmodi formam *Mkk*.

COROLLARIUM 3

425. Invento ergo momento inertiae corporis respectu axis, circa quem id ante gyrari assumsimus, idque fuerit = *Mkk*. Ita si momentum virium sollicitantium sit *Vf* et celeritas angularis = γ, erit $d\gamma = \frac{2Vfgdt}{Mkk}$.

EXPLICATIO

426. Ratio huius denominationis ex similitudine motus progressivi est desumta; quemadmodum enim in motu progressivo, si a vi secundum suam directionem sollicitante acceleretur, est incrementum celeritatis ut vis sollicitans divisa per massam seu inertiam, ita in motu gyratorio, quoniam loco ipsius vis sollicitantis eius momentum considerari oportet, eam expressionem $\int rrdM$, quae loco inertiae in calculum ingreditur, *momentum inertiae* appellemus, ut incrementum celeritas angularis simili modo proportionale fiat momento vis sollicitantis diviso per momentum inertiae. Quae similitudo eo est perfectior, quod utrinque per elementum temproris dt et duplam lineam $2g$ multiplicari oporteat, ut ipsum celeritatis incrementum exprimatur.

SCHOLION

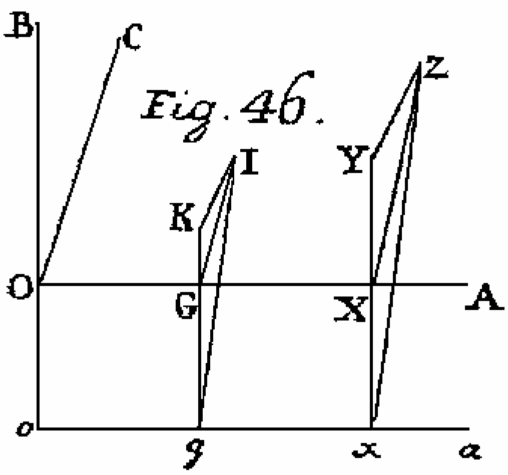
427. Cum idem corpus ad infinitos axes referri possit, respectu cuiuslibet peculiare habebit momentum inertiae, ex quo momentum inertiae absolute definiri nequit, nisi ad determinatum axem referatur. Interim tamen non semper opus est, si eiusdem corporis momentum inertiae successive respectu plurium axium investigari debeat, ut calculus de novo ex formula $\int rrdM$ evolvatur; sed saepe evenit, ut, cum momentum inertiae respectu unius axis invenerimus, ex eo facile momenta inertiae eiusdem corporis respectu infinitorum aliorum axium colligere queamus. Haec autem commoditas imprimis locum habet, quando axes fuerint paralleli, ita ut cognito momento inertiae pro uno axe ex eo facile momentum inertiae pro quovis alio axe illi parallelo assignari possit, id quod sequente problemate ostendamus.

PROBLEMA 25

428. Dato corporis cuiusdam momento inertiae respectu axis *OA*, invenire eiusdem corporis momentum inertiae respectu alius axis *oa* illi paralleli (Fig. 46).

SOLUTIO

Sit $Oo = c$ distantia horum axium, in quorum plano accipiatur directrix *OB* et *OA* normalis et tertia *OC* ad utrumque perpendicularis. Consideretur corporis, cuius tota massa = *M*, elementum quodvis *dM* in *Z*, unde ad planum *AOB* demisso perpendiculo *ZY* et ex *Y* ducta ad *OA* normalite *YX*, quae producta alteri axi *oa* occurat in x; ponanturque pro axe dato *OA* coordinatae $OX = x$, $XY = y$ et $YZ = z$. Quoniam igitur respectu huius axis *OA* momentu inertiae datur, sit id = Mkk, eritque



$$\int (yy+zz)dM = Mkk .$$

Iam pro novo axe *oa*, ob $ox = x,\ xY = c+y$ et $YZ = z$, erit momentum inertiae =

$$\int ((c+y)^2 + zz)dM = \int ccdM + 2\int cydM + \int (yy+zz)dM .$$

Cum igitur sit

$$\int (yy + zz)dM = Mkk \text{ et} \int ccdM = Mcc,$$

pro membro $2\int cydM = 2c\int ydM$ consideretur centrum inertiae corporis, quod sit in I, unde ad planum axium demittatur perpendiculum IK et ex K ad axes normalis KGg, eritque

$$\int ydM = M \cdot GK \,.$$

Hinc erit momentum inertiae respectu axis *oa* =

$$Mkk + Mcc + 2Mc \cdot GK \,,$$

quod ob $Gg = c$ et $cc + 2c.GK = gK^2 - GK^2$ ita exprimetur, ut sit

$$Mkk + M \cdot gK^2 - M \cdot GK^2 \,,$$

sicque cognito momento inertiae respectu axis *OA*, quod est = *Mkk*, facile invenitur momentum inertiae respectu alius cuiusque axis *oa* illi paralleli.

COROLLARIUM 1

429. Si axis *oa* longius distat a centro inertiae *I* quam axis *OA*, momentum inertiae respectu axis *oa* maius est quam respectu axis *OA*. Est enim momentum inertiae respectu axis $oa = Mkk + M \cdot gI^2 - M \cdot GI^2$.

COROLLARIUM 2

430. Si igitur infiniti axes inter se paralleli concipiantur, momentum inertiae erit minimum respectu eius axis, qui per ipsum centrum inertiae ducitur. Scilicet, si centrum inertiae esset in *G* axisque *OA* per id transiret, cuius respectu momentum inertiae fuerit = *Mkk*, erit respectu axis *oa* momentum inertiae = $Mkk + M \cdot Gg^2$.

COROLLARIUM 3

431. Si igitur detur momentum inertiae *Mkk* respectu cuiuspiam axis per centrum inertiae corporis transeuntis, momentum inertiae respectu alius cuiusvis axis illi paralleli superat illud producto ex massa in quadratum distantiae huius axis a centro inertiae.

SCHOLION

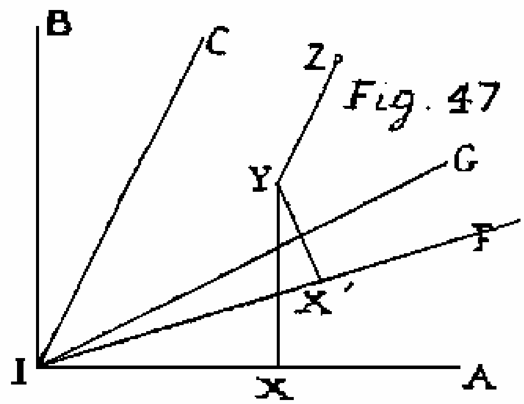
432. Hinc investigatio momentorum inertiae pro quovis corpore restringitur tantum ad axes per eius centrum inertiae ductos, quorum respectu si explorata fuerint momenta inertiae, inde pro alias quibuscunque axibus momenta inertiae facile colliguntur. Atque haec proprietas centri inertiae, quod momenta inertiae respectu axium per id transeuntium sint minima inter omnia respectu aliorum axium parallelerum sumta, omnino est memorabilis, cum etiam pro motu gyratorio insignem huius centri praestantiam declaret. Verum per centrum inertiae innumerabiles axes ducere licet, quorum respectu momenta inertiae vehementer inter se discrepare possunt, neque patet, quomodo ex datis aliquibus reliqua definiri queant. Interim tamen, quoniam eorum nullum vel evanescere vel in infinitu excrescere potest, inte ea tam maximum detur quam minimum necesse est, quae investigatio omnino digna videtur, ut diligentius suscipiatur. Sed quo ea facilius succedat, conveniet in gerere momentum inertiae respectu axis cuiuscunque per centrum inertiae ducti calculo exprimi.

PROBLEMA 26

433. Si natura corporis exprimatur aequatione inter ternas coordinatas, invenire eius momentum inertiae respectu axis cuiusque per centrum inertiae ducti.

SOLUTIO

Sit *I* centrum inertiae corporis, in quo simul concursus ternarum directricium *IA*, *IB*, *IC* inter se normalium constituatur (Fig. 47), quibus pro elemento corporis quocunque *dM* in *Z* sito coordinatae parallelae sint $IX = x$, $XY = y$, $YZ = z$; unde, si qua directricium pro axe sumeretur, eius respectu momentum inertiae facile assignaretur. Verum id definiendum sit respectu axis cuiuscunque *IG*, per quem planum ad *AIB* normale ductum hoc secet in recta IF, ac ponatur angulus $AIF = \eta$ et angulus $FIG = \theta$; quaestio ergo redit, ut punctum *Z* per alias ternas coordinatas exprimatur, quarum una sit in ipso axe *IG* sumta. Mutemus ternas directrices primo ita, ut una sit *IF*, manente *IC*, dum tertia ad has sit normalis, et ducta *YX'* ad *IF* normali erunt ternae coordinatae, quae sint *x'*, *y'*, *z'*,



$$IX' = x' = x\cos\eta + y\sin\eta,\ X'Y = y' = y\cos\eta - x\sin\eta \text{ et } YZ = z' = z\ ;$$

simili modo hinc transitus fiat ad novas ternas coordinatas *x''*, *y''*, *z''*, quarum *x''* in axe *IG* capiatur, eritque

$$x'' = x'\cos\theta + z'\sin\theta,\ z'' = z'\cos\theta - x'\sin\theta\ ,\ y'' = y'\ ,$$

unde valoribus substitutis habebitur

$$x'' = x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta,$$

$$y'' = y\cos\eta - x\sin\eta\ ,$$

et

$$z'' = z\cos\theta - x\cos\eta\ \sin\theta - y\sin\eta\sin\theta.$$

Atque hinc puncti *Z* ab axe *IG* distantiae quadratum prodibit

$$y''y''+z''z''=$$
$$x^2 sin^2\eta + y^2 cos^2\eta + z^2 cos^2\theta - 2xy\sin\eta\cos\eta - 2xz\cos\eta\sin\theta\cos\theta$$
$$-2yz\sin\eta\sin\theta\cos\theta + x^2 cos^2\eta\, sin^2\theta + y^2 sin^2\eta\, sin^2\theta + 2xy\sin\eta\cos\eta\, sin^2\theta.$$

Ponamus iam sequentia integralia per totum corpus extensa :

$$\int xxdM = A,\ \int yydM = B,\ \int zzdM = C,$$
$$\int xydM = D,\ \int xzdM = E,\ \int yzdM = F,$$

eritque momentum inertiae respectu axis *IG* quaesitum

$$A(sin^2\eta + cos^2\eta\, sin^2\theta) + B(cos^2\eta + sin^2\eta\, sin^2\theta)\ + C\, cos^2\theta$$
$$-2D\sin\eta\cos\eta\, cos^2\theta - 2E\cos\eta\sin\theta\cos\theta - 2F\sin\eta\sin\theta\cos\theta.$$

COROLLARIUM 1

434. Hic quantitates *A, B, C* necessario sunt quantitates positivae, relique vero *D, E, F* pro ratione corporis vel positivae vel negativae esse possunt.

COROLLARIUM 2

435. Momentum inertiae respectu axis *IA* est = $B+C$, respectu axis $IB=A+C$ et respectu axis $IC=A+B$. Cognitis ergo his tribus momentis innotescunt valores *A, B,* et *C*.

COROLLARIUM 3

436. Quomodocunque autem accipiantur anguli η et θ, momentum inertiae inventum nunquam evanescere potest, sed semper valorem positivum obtinet.

SCHOLION

437. Si non solum motum corporis circa axem *IG*, sed etiam vires ab axe sustentatas determinare velimus, praeter momentum inertiae respectu huius axis quoque valores formularum integralium $\int x''y''dM$ et $\int x''z''dM$ nosse debemus. Fiunt autem formulae per coordinatas *x, y, z* :

$$\int x''y''dM = \int dM(x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta)(y\cos\eta - x\sin\eta)$$

et

$$\int x''z''dM =$$
$$\int dM(x\cos\eta\cos\theta + y\sin\eta\cos\theta + z\sin\theta)(-x\cos\eta\ \sin\theta - y\sin\eta\sin\theta + z\cos\theta)$$

Quare, si hic valores supra assumti substituantur, habebimus

$$\int x''\, y''\, dM = -A\sin\eta\cos\eta\cos\theta + B\sin\eta\cos\eta\cos\theta + D(\cos^2\eta - \sin^2\eta)\cos\theta$$
$$-E\sin\eta\sin\theta + F\cos\eta\sin\theta$$
$$\int x''\, z''\, dM = -A\cos^2\eta\sin\theta\cos\theta - B\sin^2\eta\sin\theta\cos\theta + C\sin\theta\cos\theta$$
$$-2D\sin\eta\cos\eta\sin\theta\cos\theta + E\cos\eta(\cos^2\theta - \sin^2\theta) + F\sin\eta(\cos^2\theta - \sin^2\theta),$$

qui valores sunt eo magis notandi, quod casibus, quibus momentum inertiae fit maximum vel minimum, evanescunt, uti mox videbimus.

PROBLEMA 27

438. Inter omnes axes per centrum inertiae dati corporis ductos definire eum, cuius respectu momentum inertiae est vel maximum vel minimum.

SOLUTIO

Maneant omnia, uti in problema praecedente, sitque *IG* axis talis quaesitus, ita ut determinari oporteat angulos $AIF = \eta$ et $FIG = \theta$. Momentum ergo inertiae respectu huius axis cum sit

$$\int (y''\, y'' + z''\, z'')dM = A\sin^2\eta + A\cos^2\eta\sin^2\theta + B\cos^2\eta + B\sin^2\eta\sin^2\theta + C\cos^2\theta$$
$$-2D\sin\eta\cos\eta\cos^2\theta - 2E\cos\eta\sin\theta\cos\theta - 2F\sin\eta\sin\theta\cos\theta,$$

differentietur duplici modo, sumendo primum η deinde θ variabile, et utrumque differentiale nihilo aequale ponatur. Ex priore igitur prodibit haec aequatio :

$$2A\sin\eta\cos\eta\cos^2\theta - 2B\sin\eta\cos\eta\cos^2\theta - 2D\cos^2\eta\cos^2\theta$$
$$+2D\sin^2\eta\cos^2\theta + 2E\sin\eta\sin\theta\cos\theta - 2F\cos\eta\sin\theta\cos\theta = 0,$$

quae per $-2\cos\theta$ divisa praebet

$$-(A-B)\sin\eta\cos\eta\cos\theta + D(\cos^2\eta - \sin^2\eta)\cos\theta$$
$$-E\sin\eta\sin\theta + F\cos\eta\sin\theta = 0$$

sive $\int x''\, y''\, dM = 0;$ unde colligitur

$$\frac{\sin\theta}{\cos\theta} = \text{tang}\,\theta = \frac{-(A-B)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta)}{E\sin\eta - F\cos\eta}.$$

Sumendo autem θ variable pervenimus ad hanc aequationem :

$$2A\cos^2\eta\sin\theta\cos\theta+2B\sin^2\eta\sin\theta\cos\theta-2C\sin\theta\cos\theta$$
$$+4D\sin\eta\cos\eta\sin\theta\cos\theta-2E\cos\eta(\cos^2\theta-\sin^2\theta)$$
$$-2F\sin\eta(\cos^2\theta-\sin^2\theta)=0,$$

quae formula est $=-2\int x''z''dM$. Cum nunc sit

$$2\sin\theta\cos\theta=\sin 2\theta \text{ et } \cos^2\theta-\sin^2\theta=\cos 2\theta,$$

erit

$$A\cos^2\eta\sin 2\theta+B\sin^2\eta\sin 2\theta-C\sin 2\theta$$
$$+2D\sin\eta\cos\eta\sin 2\theta-2E\cos\eta\cos 2\theta-2F\sin\eta\cos 2\theta=0,$$

unde sequitur

$$\frac{\sin 2\theta}{\cos 2\theta}=\text{tang}2\theta=\frac{2E\cos\eta+2F\sin\eta}{A\cos^2\eta+B\sin^2\eta-C+2D\sin\eta\cos\eta},$$

quibus valoribus coaequatis erit

$$(E\cos\eta+F\sin\eta)(E\sin\eta-F\cos\eta)^2=(E\cos\eta+F\sin\eta)((B-A)\sin\eta\cos\eta+D(\cos^2\eta-\sin^2\eta))^2$$
$$+(E\sin\eta-F\cos\eta)((B-A)\sin\eta\cos\eta+D(\cos^2\eta-\sin^2\eta))(A\cos^2\eta+B\sin^2\eta-C$$
$$+2D\sin\eta\cos\eta)$$
$$=((B-A)\sin\eta\cos\eta+D(\cos^2\eta-\sin^2\eta))(E(B\sin\eta-C\sin\eta+D\cos\eta)-F(A\cos\eta-C\cos\eta+D\sin\eta)).$$

Cum iam $\sin\eta$ et $\cos\eta$ ubique totidem compleant dimensiones, si ponamus

$$\frac{\sin\eta}{\cos\eta}=\text{tang}\,\eta=t,$$

obtinebimus hanc aequationem

$$(E+Ft)(F-Et)^2=(D+(B-A)t-Dt^2)(DE-AF+CF+(BE-CE-DF)t),$$

quae in ordinem redacta dat

$$0=EFF-DDE+(A-C)DF+t(F^3-2EEF+DDF+(A-2B+C)DE-(A-B)(A-C)F)$$
$$+t^2(E^3-2EFF+DDE+(A-2B+C)DF-(A-B)(A-C)E)$$
$$+t^3(EEF-DDF+(B-C)DE),$$

ita ut ex hac aequatione cubica valor ipsius t erui debeat.

COROLLARIUM 1

439. Cum aequatio, ex qua valor ipsius t inveniri debet, sit cubica, semper unam certe habet radicem realem, quae praebet tangentem anguli $AIF=\eta$, quo angulo invento alter $FIG=\theta$ ita definitur, ut sit

$$\operatorname{tang}\theta = \frac{(B-A)\sin\eta\cos\eta + D(\cos^2\eta - \sin^2\eta)}{E\sin\eta - F\cos\eta} = \frac{\frac{1}{2}(B-A)\sin 2\eta + D\cos 2\eta}{E\sin\eta - F\cos\eta}.$$

COROLLARIUM 2

440. Fieri autem potest, ut omnes tres radices sint reales, quo casu tres in corpore dabuntur axes, quorum respectu momenta inertiae sunt vel maxima vel minima.

SCHOLION

441. Ex rei autem natura intelligitur in quovis corpore plus uno tali axe inesse, cuius respectu momentum inertiae sit vel maximum vel minimum; si enim unicus daretur, eius respectu momentum esset omnium vel maximum vel minimum, utrovis ergo casu alius daretur axis necesse est, cuius respectu momentum inertiae foret vel minimum vel maximum. Atque hinc concludere licet aequationem cubicam inventam non solum unam, sed duas habere radices reales, ex quo adeo omnes tres radices semper erunt reales, quod quidem difficulter ex eius forma perspici potest.Verum cognitio iam uno tali axi haud difficulter reliqui eiusdem indolis reperiuntur, id quod sequente problemate ostendisse operae erit primum.

PROBLEMA 28

442. Dato uno corporis axe per centrum inertiae transeunte, cuius respectu momentum inertiae est maximum vel minimum, invenire reliquos eius axes per centrum inertiae ductos, quibus eadem proprietas conveniat.

SOLUTIO

Existente *I* centro inertiae corporis sit *IA* axis ille datus, cuius respectu momentum inertiae est maximum vel minimum, atque ex praecedente problemate constat hanc proprietatem locum habere non posse, nisi sit $\int xydM = 0$ et $\int xzdM = 0$; quare pro formulis superioribus erit $D = 0$ et $E = 0$. Quodsi iam *IG* alius fuerit eiusmodi axis, pro quo ponatur ut ante angulus $AIF = \eta$ et $FIG = \theta$, ut sit eius respectu momentum inertiae

$$(A\sin^2\eta + \cos^2\eta\sin^2\theta) + B(\cos^2\eta + \sin^2\eta\sin^2\theta) + C\cos^2\theta - 2F\sin\eta\sin\theta\cos\theta,$$

methodus maximoram et minimoram has duas suppeditat aequationes :

$$\text{I. } (A-B)\sin\eta\cos\eta\cos^2\theta - F\cos\eta\sin\theta\cos\theta = 0,$$

$$\text{II. } (A\cos^2\eta + B\sin^2\eta)\sin\theta\cos\theta - C\sin\theta\cos\theta - F\sin\eta(\cos^2\theta - \sin^2\theta) = 0.$$

Quarum prior cum sit divisibilis per $\cos\eta\cos\theta$, erit vel $\cos\eta = 0$ vel $\cos\theta = 0$; tertia enim eius radix

$$\operatorname{tang}\theta = \frac{(A-B)\sin\eta}{F}$$

in altera aequatione substituta nihil definit, quoniam angulus η prorsus ex calculo egreditur. Sit ergo $\cos\eta = 0$ ideoque $\eta = AIF$ rectus et $\sin\eta = 1$, atque altera aequatio praebet :

$$B\sin\theta\cos\theta - C\sin\theta\cos\theta - F(\cos^2\theta - \sin^2\theta) = 0$$

seu

$$\tfrac{1}{2}(B-C)\sin 2\theta = F\cos 2\theta$$

et

$$\text{tang } 2\theta = \tfrac{2F}{B-C};$$

unde pro angulo *FIG* duplex prodit valor, alter $FIG = \theta$ alter $FIG = \theta + 90^{\circ}$. Sicque ex uno axe *IA* dato duo semper novi colliguntur eadem maximi minimive proprietate gaudentes, qui ergo tres axes respondent tribus radicibus aequationis cubicae ante inventae. Prioris autem aequationis radix $\cos\theta = 0$ nihil plane huc facit; cum enim angulus *FIG* esset rectus, utcunque angulus $AIF = \eta$ variatur, recta *IG* eundem situm *IC* perpetuo servat neque differentiatio hic locum habet, erit vero ob $\eta = 90^{\circ}$ momentum inertiae respectu axis *IG* =

$$A + B\sin^2\theta + C\cos^2\theta - 2F\sin\theta\cos\theta,$$

at respectu axis dati $IA = B + C$.

COROLLARIUM 1

443. Cum igitur sit angulus $AIF = \eta$ rectus, ambo reliqui axes sunt ad *IA* normales et, quia illi etiam invicem angulum rectum constituunt, in omni corpore tres dantur axes per centrum inertiae *I* ducti et inter se normales, quorum respectu momenta inertiae sunt vel maxima vel minima.

COROLLARIUM 2

444. Quodsi ergo ipsae rectae *IA*, *IB* et *IC* fuerint his tres axes, quorum respectu momenta inertiae sunt vel maxima vel minima, erit

$$\int xydM = D = 0, \quad \int xzdM = E = 0, \quad \text{et} \int yzdM = F = 0.$$

SCHOLION

445. In his quidem problematibus sumsimus punctum *I* esse corporis centrum inertiae, quoniam calculum momenti inertiae tantum ad eiusmodi axis, qui per corporis centrum inertiae transeunt, adstrinximus; verum in toto calculo utriusque problematis nihil inest, quod naturam centri inertiae cum puncto *I* coniungat. Quare haec problemata multo latius patent, ita ut sumto quocunque puncto *I* inter omnes axes per id transeuntes semper tres definiri queant, quorum respectu momenta inertiae sint vel maxima vel minima, atque ut hi tres axes sint inter se normales. Verum hic tantum istam proprietatem tanquam centro inertiae convenientem considero, ac pro quolibet corpore plurimum intererit hos ternos axes nosse, quoniam ex iis momenta inertiae respectu omnium axium facillime inveniri poterunt.

DEFINITIO 8

446. *Axes principales* cuiusque corporis sunt tres illi axes per eius centrum inertiae transeuntes, quorum respectu momenta inertiae sunt vel maxima vel minima.

COROLLARIUM 1

447. Ex praecedentibus intelligitur pro quolibet corpore non solum dari tales ternos axes principales, sed eos etiam inter se esse normales; unde ii commodissime pro ternis directribus, ad quas corpus referatur, accipientur.

COROLLARIUM 2

448. Quodsi ergo *IA*, *IB*, *IC* fuerint cuiuspiam corporis axes principales (Fig. 47) iisque pro elemento corporis *dM* in *Z* sito parallelae constituantur coordinatae $IX = x$, $XY = y$, $YZ = z$, non solum erit

$$\int xdM = 0,\ \int ydM = 0,\ \int zdM = 0,$$

sed etiam

$$\int xydM = 0,\ \int xzdM = 0, \text{ et } \int yzdM = 0.$$

COROLLARIUM 3

449. Tum vero, si ponatur

$$\int xxdM = A,\ \int yydM = B,\ \int zzdM = C.$$

erit corporis momentum inertiae

respectu axis $IA = B + C$,

respectu axis $IB = A + C$,

et respectu axis $IC = A + B$,

quae sunt maxima vel minima.

SCHOLION

450. Veritas utique est maximi momenti, quod in omni corpore tales tres axes principales dentur, cuius demonstratio ex praecedentibus utique est manifesta. Sumtis enim ternis directricibus *IA*, *IB*, *IC* utcunque, quae in centro inertiae I se invicem normaliter intersecent, unum eiusmodi axem principalem *IG* definire docuimus ope resolutionis aequationis cubicae, tum vero cognito uno facili calculo duo reliqui assignantur. Iam vero vix occurret corpus tam irregulare, cuius non saltem unus axis principalis innotescat, ita ut deinceps bini reliqui facillime se prodant. Quare in postremum assumam in quovis corpore hos ternos axes principales nobis esse cognitos; quorum respectu dummodo momenta inertiae novimus, pro omnibus aliis axibus promtissime exhiberi possunt , uti ex sequente problemate patebit.

EXPLICATIO

451. Quomodo ratio maximi ac minimi his tribus axibus principalibus conveniat, haud ita facile perspicitur. Cum enim inter se eos certe sit unus, cuius respectu momentum inertiae sit omnium maximum, itemque unus, cuius respectu momentum inertiae sit omnium minimum, necesse est, ut respectu tertii momentum inertiae sit neque omnium maximum neque omnium minimum, nisi forte cum alterutro illorum conveniat, quod aliquando fieri potest. Verum calculus maximorum et minimorum saepenumero eiusmodi quantitates indicit, quae absolute neque sint maxima neque minima, quoniam eo calculo plus non declaratur, quam, si infinite parum ab loco invento recesseris, neque augementum neque decrementum prodire. Ita si *IA* sit axis maximi absolute sumti et *IC* axis minimi absolute sumti (Fig. 47), respectu axis *IB* momentum inertia neque omnium erit maxium neque minimum, verumtamen eiusmodi medium tenebit, ut, si alius axis ab eo infinitie parum distans in quamcunque plagam assumtur, eius momentum inertiae neque crescat neque decrescat. Atque hanc ob rem inter hos tres axes principales ingens discremin interdit, quod imprimis observari meretur, ut eorum unus habeat maximum momentum, unus minimum, tertius vero medium, quod tamen in calculo tanquam maximum vel minimum spectari possit, cuius rei ratio in sequenti problemate magis illustrabitur.

PROBLEMA 29

452. Datis cuiusdam corporis momentis inertiae respectu trium axium principalium, invenire eius momentum inertiae respectu cuiusvis axis per eius centrum inertiae ducti.

SOLUTIO

Sint *IA*, *IB*, *IC* tres corporis axes principales (Fig. 47) sibi mutuo in centro inertiae *I* normaliter occurrentes, et posita corporis massa = *M* sit eius momentum inertiae respectu axis *IA* = *Maa*, respectu axis *IB* = *Mbb* et respectu axis *IC* = *Mcc* ; unde quaeri debeat momentum inertiae respectu axis cuiuscunque *IG*, qui ad planum *AIB* inclinetur angulo $GIF = \theta$, sitque angulus $AIF = \eta$. Consideretur nunc elementum corporis *dM* in *Z*, cuius puncti coordinatae sint *IX* = *x*, *XY* = *y* et *YZ* = *z*; ac positis integralibus

$$\int xxdM = A, \ \int yydM = B, \int zzdM = C.$$

erit

$$\int xydM = D = 0, \quad \int xzdM = E = 0, \quad \text{et} \int yzdM = F = 0.$$

Unde ex § 433 erit momentum inertiae respectu axis *IG* =

$$A(\sin^2\eta+\cos^2\eta\sin^2\theta)+B(\cos^2\eta+B\sin^2\eta\sin^2\theta)\ +C\cos^2\theta\,.$$

Cum autem ex datis ternis momentis sit

$$Maa=B+C,\quad Mbb=A+C,\quad Mcc=A+B,$$

hinc vicissim colligitur

$$A=\tfrac{1}{2}M(bb+cc-aa),\ B=\tfrac{1}{2}M(aa+cc-bb),\ C=\tfrac{1}{2}M(aa+bb-cc),$$

quibus valoribus substitutis erit quaesitum momentum inertiae respectu axis *IG* =

$$M(aa\cos^2\eta\cos^2\theta+bb\sin^2\eta\cos^2\theta+cc\sin^2\theta).$$

Ubi notetur esse

$$\cos\eta\cos\theta=\cos AIG,\quad \sin\eta\cos\theta=\cos BIG\ \text{ et }\ \sin\theta=\cos CIG\,.$$

Quare, si distantiae axis IG a ternis axibus principalibus ponantur :

$$AIG=\alpha,\quad BIG=\beta,\quad CIG=\gamma,$$

erit momentum inertiae respectu axis *IG* =

$$Maa\cos^2\alpha+Mbb\cos^2\beta+Mcc\cos^2\gamma,$$

illi autem anguli α,β,γ ita sunt comparati, ut sit semper

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

COROLLARIUM 1

453. Positio momento inertiae respectu axis *IG* = *Mkk* id sequentibus modis exprimi potest :

$$Mkk=Maa-M(aa-bb)\cos^2\beta-M(aa-cc)\cos^2\gamma,$$

$$Mkk=Mbb+M(aa-bb)\cos^2\alpha-M(bb-cc)\cos^2\gamma,$$

$$Mkk=Mcc-M(aa-cc)\cos^2\alpha+M(bb-cc)\cos^2\beta$$

et in qualibet harum expressionum binos angulos pro lubitu assumere licet.

COROLLARIUM 2

454. Si fuerit *aa* > *bb* et *bb* > *cc*, momentum inertiae respectu axis *IA* omnium erit maximum at respectu axis *IC* omnium erit minimum, medium autem tenebit momentum inertiae respectu axis *IB*.

COROLLARIUM 3

455. Si fuerit

$$(aa-bb)\cos^2\alpha > (bb-cc)\cos^2\gamma,$$

momentum inertiae respectu axis *IG* maius est quam medium *Mbb*, contra vero est minus. Sin autem sit

$$(aa-bb)\cos^2\alpha = (bb-cc)\cos^2\gamma,$$

quod infinitis locis fieri potest, ibi omnia momenta inertiae sunt inter se aequalia.

COROLLARIUM 4

456. Si fuerit $aa = bb = cc$, hoc est, si momenta inertiae principalia fuerint inter se aequalia, respectu omnium axium per centrum inertiae ductorum momenta inertiae sunt inter se aequalia ideoque quilibet axis pro principali haberi potest.

SCHOLION

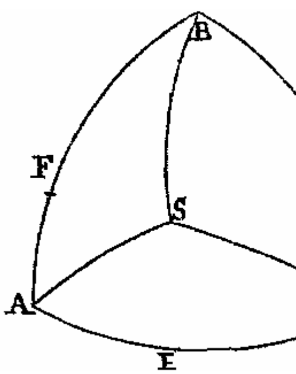


Fig. 48.

457. Eleganter haec more in trigonometria sphaerica recepto repraesentari possunt (Fig. 48). Sint enim constituto centro inertiae *I* in centro sphaerea puncta *A, B, C* extremitates axium principalium in superficiem sphaefica terminatae, ita ut arcus *AB, AC*, et *BC* sint quadrantes, quibusque in *A, B, C* terminatis respondeant momenta inertiae *Maa, Mbb, Mcc*, quorum primum sit maximum, secundum medium et tertium minimum. Quodsi iam alius axis quicunque per centrum inertiae transiens, qui superficiem sphaericam in puncto *S* traiiciat, consideretur, eius respectu momentum inertiae erit :

$$Maa\cos^2 AS + Mbb\cos^2 BS + Mcc\cos^2 CS,$$

quod ob

$$\cos^2 AS + \cos^2 BS + \cos^2 CS = 1$$

his modis exprimi potest :

$$Maa - M(aa-bb)\cos^2 BS - M(aa-cc)\cos^2 CS,$$

$$\text{vel}\quad Mbb + M(aa-bb)\cos^2 AS - M(bb-cc)\cos^2 CS,$$

$$\text{vel}\quad Mcc - M(aa-cc)\cos^2 AS + M(bb-cc)\cos^2 BS.$$

Hinc, si *S* sit in quadrante *BC* puta in *D*, erit momentum inertiae respectu axis *ID*

$$= M(bb\cos^2 BD + cc\cos^2 CD),$$
$$= Mbb - M(bb-cc)\cos^2 CD,$$
$$= Mcc + M(bb-cc)\cos^2 BD$$

seu momentum inertiae respectu axis *ID* erit =

$$Mbb - M(bb-cc)\sin^2 BD = Mcc + M(bb-cc)\sin^2 CD.$$

Simili modo momentum inertiae respectu axis *IE* est

$$Maa - M(aa-cc)\sin^2 AE = Mcc + M(aa-cc)\sin^2 CE;$$

momentum autem inertiae respectu axis *IF* fit

$$Maa - M(aa-bb)\sin^2 AF = Mbb + M(aa-bb)\sin^2 BF.$$

PROBLEMA 30

458. Invenire omnes axes per centrum inertiae ductos, quarum respectu momenta inertiae sint inter se aequalia.

SOLUTION

Sint momenta inertiae respectu axium principalium *IA*, *IB*, *IC* (Fig. 48) respective *Maa*, *Mbb*, *Mcc* et $aa > bb > cc$ et quaerantur omnes axes per centrum inertiae *I* ducendi, quarum respectu momenta inertiae sint inter se aequalia et quidem aequalia ei, quod respondet axi *IE* sumto *E* in quadrante *AC*, quoniam ab *A* ad *C* omnia momenta huius corporis a maximo ad minimum occurrunt. Sit *IS* talis axis, et habebimus hanc aequationem :

$$Maa - M(aa-cc)\sin^2 AE$$
$$= Maa - M(aa-bb)\cos^2 BS - M(aa-cc)\cos^2 CS$$

seu

$$(aa-cc)\sin^2 AE = (aa-bb)\cos^2 BS + (aa-cc)\cos^2 CS;$$

ergo ob

$$\cos^2 BS = \sin^2 AS - \cos^2 CS$$

ergo

$$(aa-cc)\sin^2 AE = (aa-bb)\sin^2 AS + (bb-cc)\cos^2 CS.$$

Introducatur angulus *CAS*, et cum sit $\cos CS = \sin AS \cos CAS$, erit

$$(aa-cc)\sin^2 AE = (aa-bb)\sin^2 AS + (aa-cc)\sin^2 AS \cos^2 CAS,$$

ergo

$$\sin^2 AS = \frac{(aa-cc)\sin^2 AE}{aa-bb+(bb-cc)\cos^2 CAS}.$$

Sin autem angulum ACS introducamus, reperiemus

$$\sin^2 CS = \frac{(aa-cc)\sin^2 CE}{bb-cc+(aa-bb)\cos^2 ACS};$$

angulus *CAS* usque ad rectum augeri potest, dum $(aa-cc)\sin^2 CE$ non excedat $aa-bb$, hoc est, si fuerit

$$\sin AE < \sqrt{\frac{aa-bb}{aa-cc}};$$

at angulus *ACS* usque ad rectum crescere potest, si sit

$$\sin CE < \sqrt{\frac{bb-cc}{aa-cc}};$$

seu

$$\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}.$$

Quare punctum *S* erit in curva, quae ex *E* assurgens per quadrantem *AB* transibit, si fuerit

$$\sin AE < \sqrt{\frac{aa-bb}{aa-cc}};$$

curva autem illa per quadrantem *BC* transibit, si fuerit

$$\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}.$$

Casu autem, quo

$$\sin AE = \sqrt{\frac{aa-bb}{aa-cc}},$$

curva per ipsum punctum *B* transibit omniaque momenta inertiae erunt = *Mbb*. Hoc igitur casu erit

$$\sin^2 AS = \frac{aa-bb}{aa-bb+(bb-cc)\cos^2 CAS}.$$

Hinc ob

$$\cos AE = \sqrt{\frac{bb-cc}{aa-cc}}$$

et

$$\frac{aa-bb}{bb-cc} = \frac{\sin^2 AE}{\cos^2 AE}$$

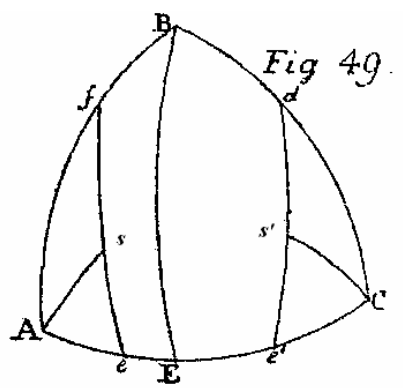
fiet

$$\sin^2 AS = \frac{\sin^2 AE}{\sin^2 AE + \cos^2 AE \cos^2 CAS}$$

ideoque

$$\text{tang } AS = \frac{\text{tang } AE}{\cos CAS},$$

unde intelligitur loca punctorum *S* sita esse in circulo maximo per puncta *B* et *E* traducto (Fig. 49).



Casu, quo $\sin AE < \sqrt{\frac{aa-bb}{aa-cc}}$ seu punctum *E* proprius ad *A* sumitur, sit id in *e*, et in quadrante A*B* dabitur punctum *f*, in quo momentum sit aeque magnum. Erit ergo:

$$\sin^2 Af = \frac{(aa-cc)\sin^2 Ae}{aa-bb},$$

unde, si ponatur *Ae* = *e*, *Af* = *f*, *As* = *s* et angulus $eAs = \varphi$, ob

$$\frac{aa-cc}{aa-bb} = \frac{\sin^2 f}{\sin^2 e} \text{ et } \frac{bb-cc}{aa-bb} = \frac{\sin^2 f - \sin^2 e}{\sin^2 e}$$

havebimus inter *s* et φ hanc aequationem :

$$\sin^2 s = \frac{\sin^2 e \sin^2 f}{\sin^2 e + (\sin^2 f - \sin^2 e)\cos^2 \varphi} = \frac{\sin^2 e \sin^2 f}{\sin^2 e \sin^2 \varphi + \sin^2 f \cos^2 \varphi},$$

qua aequatione natura lineae *esf* exprimitur, estque

$$\frac{\sin e}{\sin f} = \sin AE .$$

Casu denique, quo $\sin AE > \sqrt{\frac{aa-bb}{aa-cc}}$, cadat punctum *E* in *e'* dabiturque in quadrante *BC* punctum *d*, ubi momentum est idem atque in *e'*, ut sit

$$\sin^2 Cd = \frac{(aa-cc)\sin^2 Ce'}{bb-cc}.$$

Ponatur iam $Ce' = e, Cd = f, Cs' = s$ et angulus $e'Cs' = \varphi;$ ob

$$\frac{aa-cc}{bb-cc} = \frac{\sin^2 f}{\sin^2 e} \text{ et } \frac{aa-bb}{bb-cc} = \frac{\sin^2 f - \sin^2 e}{\sin^2 e}$$

inter s et φ haec prodit aequatio :

$$sin^2 s = \frac{sin^2 e\, sin^2 f}{sin^2 e + (sin^2 f - sin^2 e) cos^2 \varphi} = \frac{sin^2 e\, sin^2 f}{sin^2 e\, sin^2 \varphi + sin^2 f\, cos^2 \varphi},$$

qua natura lineae *e's'd* exprimitur, estque

$$\frac{sin\, e}{sin\, f} = sin\, CE .$$

COROLLARIUM 1

459. Per totum ergo circulum maximum ex B per E ductum, ut sit

$$sin\, AE = \sqrt{\frac{aa-bb}{aa-cc}},$$

momentum inertiae est = Mbb. Et quia arcus AE tam negative quam positive accipi potest, duo in sphaera dantur circuli maximi eadem proprietate gaudentes.

COROLLARIUM 2

460. Simili modo tam circa polum A quam ipsi oppositum erunt in superficie sphaerae orbes elliptici, quorum semiaxis maior est arcus Af et semiaxis minor arcus Ae, in quibus ubique idem regnabit momentum inertiae maius quam Mbb. In figura linea *fse* refert quadrantem horum orbium ellipticorum.

COROLLARIUM 2

461. Lineae autem, in quibus momentum inertiae minus est quam Mbb, erunt bini orbes elliptici, quarum centra sunt in polo C eique opposito et semiaxis maior arcus Cd, minor vero arcus Ce'. In figura linea *ds'e'* refert quadrantem horum orbium ellipticorum.

SCHOLION 1

462. Etsi hae lineae *fse* et *ds'e'* in superficie sphaerae ductae non sunt in eodem plano, tamen eas orbium ellipticorum nomine insignire lubet, quoniam earum proiectiones in plana sphaeram in punctis A et C tangentia per rectas eo normales factae sunt ellipses, quarum centra sunt in punctis A et C. In proiectione enim lineae *fse* in planum ad A tangens facta si ponatur $sin\, Af = m$, $sin\, Ae = n$, ut sit

$$\frac{mm}{nn} = \frac{aa-cc}{aa-bb},$$

et pro puncti s proiectione abscissa in m sumta = $x = sin\, s\, sin\, \varphi$ et applicata eo normalis = $y = sin\, s\, cos\, \varphi$, habebitur inter x et y haec aequatio

$$nnxx + mmyy = mmnn,$$

quae est pro ellipsi centrum in A habente, cuius semiaxis sunt m et n. Parique modo proiectio lineae *ds'e'* in planum ad C tangens facta reperietur esse ellipsis. Si fuerit

$Mbb = Mcc$, quo casu punctum E in C cadit, sitque $Ae = Af$ et $m = n$, ellipsis illa abit in circulum , eritque linea *fse* circulus minor circa polum A descriptus.

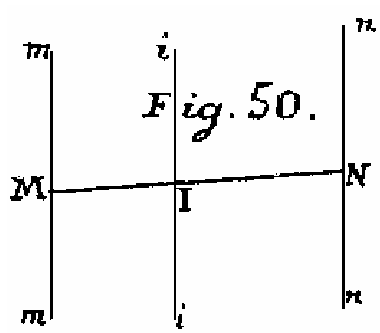
SCHOLION 2

463. Investigationem ergo momenti inertiae eo reduximus, ut pro quolibet corpore proposito sufficiat terna momenta inertiae definivisse, quae scilicet sumta sint respectu ternorum eius axium principalium. His enim cognitis facile momentum inertiae eiusdem corporis respectu aliuscuiscunque axis per eius centrum inertiae transeuntis atque hinc porro respectu aliorum omnium illi parallelorum assignari potest. Hocque modo inventio momentorum inertiae, quae initio pro quovis corpore quasi infinita videbatur, mirifice in compendium est redacta. Praeterea vero notari meretur in hoc negotio alius insigne subsidium, cuius ope momentum inertiae alicuius corporis facile colligi potest ex momentis eius partium, id quod sequente problemate explicemus.

PROBLEM 31

464. Datis momentis inertiae duarum partium respectu axium inter se parallelorum et per cuiusque centrum inertiae transeuntium, invenire momentum inertiae totius corporis respectu axis illis paralleli et per huius centrum inertiae transeuntis.

SOLUTIO



Sit ergo corpus compositum ex duabus partibus (Fig. 50), quarum alterius massa sit = M habens suum centrum inertiae in M, alterius vero massa sit = N eiusque centrum inertiae in N, ponaturque intervallum $MN = c$. Data iam sint momenta inertiae priores partis M respectu axis mm, quodsit = Mmm, et posterioris partis N respectu axis *nn*, quod sit = Nnn; sintque hi axes *mm* et *nn*, qui per utriusque partis centum inertiae transeant, inter se paralleli; unde totius corporis momentum inertiae respectu axis ii illis paralleli et per suum centrum inertiae I transeuntis determinari debet. Totius autem corporis massa est = $M + N$ eiusque centrum inertiae in rectae MN puncto I reperitur, ut sit

$$IM = \frac{Nc}{M+N} \quad \text{et } IN = \frac{Mc}{M+N}.$$

Cum igitur hi tres axes in eodem plano sint siti, ponatur eorum inclinatio ad rectam MN seu angulus $NIi = \delta$ eritque distantia axium *mm* et *ii* = $\frac{Mc\sin\delta}{M+N}$, unde partis M momentum inertiae respectu axis *ii* erit

$$= Mmm + \frac{MNNcc\sin^2\delta}{(M+N)^2}.$$

Tum vero ob distantiam axium *nn* et *ii* = $\frac{Mc\sin\delta}{M+N}$ prodit partis *N* momentum inertiae respectu axis *ii* =

$$Nnn+\frac{MMNcc\sin^2\delta}{(M+N)^2}.$$

Quare totius corporis momentum inertiae respectu axis *ii* habebitur

$$Mmm+Nnn+\frac{MNcc\sin^2\delta}{M+N}.$$

COROLLARIUM 1

465. Momentum ergo totius corporis maius est quam momenta partium simul sumta respectu axium inter se parallelorum et per cuiusque centrum inertiae traductorum: atque excessus $\frac{MNcc\sin^2\delta}{M+N}$ proportionalis est quadrato distantiae axium.

COROLLARIUM 2

466. Si massa totius corporis ponatur = *I* = *M* + *N* eiusque momentum inertiae respectu axis *ii* = *Iii*, erit

$$Iii=Mmm+Nnn+\frac{MNcc\sin^2\delta}{M+N}.$$

Tum vero positis distantiis *IM* =*a* et *IN* = *b* erit

$$a=\frac{Nc}{I} \text{ et } b=\frac{Mc}{I},$$

unde fit

$$Iii=Mmm+Nnn+Iab\sin^2\delta.$$

COROLLARIUM 3

467. Hinc dato momento totius corporis *Iii* una cum momento alterius partis *Mmm* facile quoque collititur momentum alterius partis

$$Nnn=Iii-Mmm-Iab\sin^2\delta$$

sumtis scilicet axibus inter se parallelis et per ciusque centrum inertiae transeuntibus.

COROLLARY 4

468. Si corpus constet pluribus partibus, quarum singularum momenta inertiae respectu axium inter se parallelorum et per cuiusque centrum inertiae transeuntium sint explorata, hinc binis coniungendis tandem momentum inertiae totius corporis respectu axis illis paralleli et per suum centum inertia transeuntis colligetur.

SCHOLION 1

469. Hoc casu plurium partium non opus est secundum problema bina coniungere, sed statim momentum totius corporis colligi potest. Sint enim Mmm, Nnn, Ppp, Qqq momenta partium respectu axium inter se parallelorum et per cuiusque centra inertiae transeuntium, pro toto autem corpore concipiatur axis illis parallelus per eius centrum inertiae transiens, a quo axes partium M, N, P, Q distent intervallis a, b, c, d ; quibus cognitis erit momentum inertiae totius corporis =

$$M(mm+aa) + N(nn+bb) + P(pp+cc) + Q(qq+dd) .$$

Hoc igitur modo saepe corporum admodum irregularium momenta inertiae facile colligi poterunt, dummodo ex eiusmodi partibus fuerint composita, quarum momenta inertiae assignare liceat, quo pacto calculus momentorum inertiae non mediocriter adiuvatur.

SCHOLION 2

470. Verum non sufficit methodum tradisse omnium corporum momenta inertiae inveniendi; necesse est etiam ea pro praecipuis corporum generibus evolvere, ut quoties usus postulat, inde desumi queant. Ne autem opus sit infinitum, hanc investigationem ad corpora homogenea, quae per totam extensionem similari constent materia, restringamus, ita ut calculus quasi ad corpora geometrica tantum sit accommodandus, ubi quidem figuras solum principales sum consideraturus. Ac primo, quoniam fila tensuissima et laminas tenuissimas tanquam lineas et superficies considerare licet, ab iis initium ducamus, inde ad varias species solidorum, cuiusmodi prae ceteris occurrere solent, progressuri. In singulis autem his corporis ternos axes principales eorumque respectu momenta inertiae definiamus, quandoquidem ex his momenta respectu omnium axium facili negotio colligi possunt. Hinc etiam simul patebit, quomodo calculum ad omnia alia corporum genera quam commodissime accommodari conveniat.