Chapter 19

CONCERNING THE MOTION OF CYLINDRICAL BODIES UPON A HORIZONTAL PLANE

THEOREM 11

926. While a cylindrical body is moving on a horizontal plane, the force with which it presses on the plane is vertical, and passes through the centre of a certain section of the cylinder made normal to the length.

DEMONSTRATION

The cylindrical body rests on the horizontal plane along a straight line parallel to the axis of the cylinder, in which the forces arise that are supporting the cylinder, and it happens that these forces are spread out along the whole length of that line. But since all these forces are normal to the horizontal plane and thus vertical and parallel to each other, one force gives a force equivalent to all these forces ; hence the direction of this equally is vertical and presses on the line at a certain point. If therefore the cylinder is cut normally to the length at this point, then the section is a circle and the force equivalent to all the pressing forces, because it is in this section to the point of contact, passes through the centre of this circle. Hence unless this section passes through the centre of inertia of the body, then the direction of the mean pressing force is not made in the section through the centre of inertia normal to the length, and the body rotates.

EXPLANATION

927. Therefore here we may consider cylinders, in which in the first place the axis of these are considered as if geometrical, to which all the sections made normally are equal circles, thus so that the body is a right cylinder, the motion of this, while we are to investigate the motion of this as it presses on the horizontal plane always. If the centre of inertia should be at the geometrical axis, then in any position the cylinder maintains a state of equilibrium (Fig. 117), but if the cut is considered a section of the cylinder made normal to the axis through the centre of inertia *I*, then let the centre of this cut be at *G*, and it is required for a state of equilibrium that the line *GI* is vertical, from which there is given a twofold state of equilibrium, the one in which the centre of inertia *I* lies below *G*, and the other in which it is situated above, of which that one is called the *stable* [position] and this other one the *slipping* position. But with

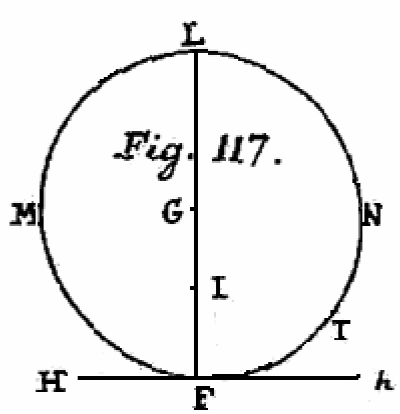


Fig. 117.

all the cylinders the geometric axis has been placed horizontal, hanging vertically over the line of contact. Next it is required to know the three mechanical axis of the cylinder, which if we grant the distribution of the material in some manner, that can be drawn in some manner different from the geometric axis or along the length, on the position of which the determination of the motion chiefly depends. Through the centre of inertia *I* there is considered drawn a right line parallel to the geometric axis, which in most cases is usually a principal axis and it always remains horizontal. Therefore with these noted, in whatever manner the cylinder presses on the plane, the point *T* is noted in the section *LMFN* made through the centre of inertia *I*, where this circle touches the horizontal plane, and then also that section parallel to this is observed properly, at which the mean direction of the forces is acting, which is parallel to the line *TG* ; and neither the magnitude of the pressing force nor the distance of the section from the section through the centre of inertia, about which the cylinder is turning, is known, and at last henceforth can be determined from the motion; if indeed it can be effected by this force, so that the longitudinal axis of the cylinder always remains horizontal and the cylinder rests on the horizontal plane.

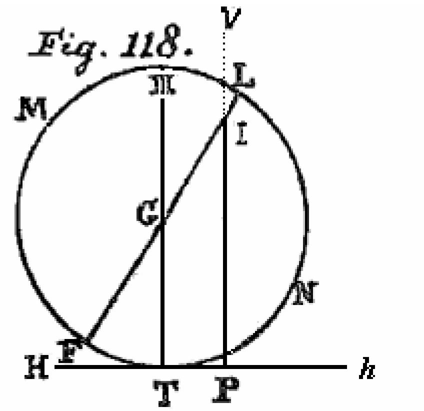
SCHOLIUM

928. In the two motions to be investigated there is no need that the whole body should be formed in the shape of a cylinder, but it suffices if at the places on which it rests on the horizontal plane, that it has such a shape. Therefore here the motion pertains to all these bodies, which can be designated in terms of cylinders, for which they rest equally raised on both sides on horizontal planes only, while the mass of the body within these is extended in some manner, as comes about in cones, which are put in motion in a to and fro manner on boundaries which are considered to be allowed as circles only ; then also here the motion of pendulums is to be referred to, which are oscillating not about a line axis as we have assumed previously, but being changed into a material cylinder on each side [of the pendulum, see Fig. 124], while the mass of the pendulum hangs between these two cylinders, each resting on horizontal planes. Therefore if in these cases the mean sections are not circles, yet these two sections, about which the other centre of inertia is rotating, now in the other the force of compression can be viewed as a circle, because the figure comes to be considered so far, as much as the body rests on a horizontal plane. Moreover now also, unless bodies of this kind execute whole revolutions, neither indeed is it worth the effort, that the whole boundaries of these are cylinders, but it suffices if a part of these, with which it makes contact, should have such a figure, for which it is agreed to note that the longitudinal axis extends properly through the whole length of the body. On which account this can be extended to more cases, concerning which motion, it is to be maintained that the centre of inertia cannot have any motion from the forces acting, apart from that taken on the same line to the vertical ; thus so that the horizontal motion is of no consequence, unless such motion should be accepted from the outside, but that is then to be pursued uniformly ; in which since no difficulty arises, we do not attend to that here.

PROBLEM 111

929. If a cylindrical body is moving on a horizontal plane and the force is given, by which it is presses on the plane, to define the progressive motion, with which the centre of inertia of the body proceeds.

SOLUTION



A normal section is made to the axis of the cylinder passing through the centre of inertia *I* (Fig. 118), which body is either a continuous cylinder or only has been provided with cylindrical boundaries, as the circle *LMFN* is considered with equal basis, the centre of this is at *G*, but the centre of inertia of the body is at *I*, with the interval $GI = f$ arising, so that in the position of rest the line *LIGF* maintains a vertical position, in which the centre of inertia *I* is shown above the centre of the circle *G* in the figure, because if it should be lower down, then the interval $GI = f$ has to be taken as negative. But now the point of contact corresponds to the point *T*, thus so that a line normal to the plane of the circle at *T* is the line of contact falling on the horizontal plane *Hh*. Therefore the line drawn through the centre of the circle ΠGT parallel to the contact *T* is in the direction of the pressing force, with which the body is repelled by the horizontal plane, which force, in whatever other section it is situated, is put equal to Π, as we consider that to be known. Again let the weight of the cylinder be equal to *M*, the radius of the circle $GF = GT = e$ and the angle of declination $\Pi GL = \rho$; the height of the centre of inertia *I* above the horizontal $IP = e + f\cos\rho$, which is put equal to *v*. Therefore because we are inquiring about progressive motion and the weight separately urges along *IP* with a force equal to *M*, the pressing force Π is considered to be applied upwards along *IV* to the centre of inertia *I*, thus so that the total force acting downwards is now equal to $M - \Pi$, and with the mass to be moved equal to *M;* and hence from the principles of motion there is found :

$$ddv = \frac{-2g(M-\Pi)}{M}\ dt^2$$

and hence

$$\frac{\Pi}{M} = 1 + \frac{ddv}{2gdt^2} \text{ or } \Pi = M\left(1+\frac{fdd\cdot\cos\rho}{2gdt^2}\right),$$

in which equation the angle $\Pi GL = \rho$ and therefore becomes known from the elevation of the centre of inertia $IP = e + f\cos\rho$; and unless a horizontal motion were impressed on the body, the point *I* is set in motion on the line *PIV*, on that either rising or falling, thus so that the point *P* remains fixed, from which the point of contact *T* is defined, because it is $PT = f\sin\rho$. But if the body takes a horizontal motion, it keeps constantly uniform in direction and the motion of the point *I* becomes composed from this uniform rectilinear motion and from that vertical motion.

SCHOLIUM

930. But in addition in this body rotational motion can be generated, thus yet in order that both the point *G* as well as the line normal to the circle *LMFN* at *G*, which is the special axis of the cylinder, always remains in the same horizontal plane. Towards the investigation of this rotational motion separately from the progressive motion, we consider the centre of inertia *I* as at rest, about which a sphere is described, in that the circle *LMFN* always remains vertical, to which if a normal line is drawn through *I*, that is the longitudinal axis of the cylinder. In which condition all the observed rotational motions, of which the cylinder is capable, can be easily represented in the figure. But here before everything it is required to attend to the principal axis properly, about which the moments from the pressing force arising are to be defined.

PROBLEM 112

931. If a cylindrical body is pressing on a horizontal plane in some position, and there is given both the pressing force Π as well as a transverse section of the cylinder, in which it is present, to find the moments about the principal axes.

SOLUTION

A section of the cylinder made normally through the centre of inertia *I* lies in the plane of the table (Fig. 119) [the circle shown is an absolute reference circle], to which the line *IZ* is normal, and the line *LIG* passes through the [geometric] centre of this section *G*, thus in order that the interval $IG = f$ and the angle $ZIL = \rho$. From *G* there is erected a normal *GH* to the plane of the table, as far as the section in which the pressing force Π is acting, and let the interval *GH* = *h*, and above we have seen to be that

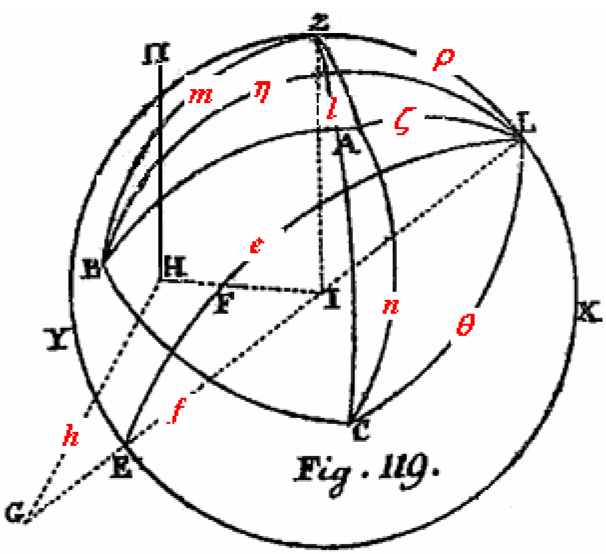
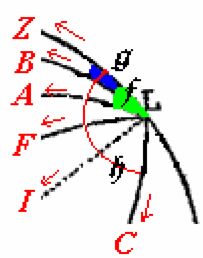


Fig. 119.

$$\Pi = M\left(1+\frac{fdd\cdot\cos\rho}{2gdt^2}\right),$$

the direction of the force is along the vertical $H\Pi$ [IZ lies in the plane of the table] and thus parallel to *IZ*. Now a sphere is considered to be described about the centre of inertia *I*, with the radius taken arbitrarily equal to 1, the principal axes of the body are directed towards the points *A*,*B*, *C* on the surface of this sphere, and the arcs for the determination of these points are called



$$LA = \zeta,\ LB = \eta,\ LC = \vartheta;$$

likewise $ZA = l, ZB = m,\ ZC = n$ with $ZL = \rho$ arise*;* then the angles $ZLA = \mathfrak{f}\ , ZLB = \mathfrak{g}, ZLC = \mathfrak{h},$ in order that [from the cosine rule for sides of sph. triangles :]

$$cos\, l = cos\,\zeta\, cos\,\rho + cos\,\mathfrak{f}\, sin\,\zeta\, sin\,\rho,$$
$$cos\, m = cos\,\eta\, cos\,\rho + cos\,\mathfrak{g}\, sin\,\eta\, sin\,\rho,$$
$$cos\, n = cos\,\vartheta\, cos\,\rho + cos\,\mathfrak{h}\, sin\,\vartheta\, sin\,\rho\ .$$

And indeed in the first place the force $H\Pi = \Pi$ is resolved along the directions parallel to the directions of the principal axes, and which resolution is likewise put in place, if this force should be applied at the centre *I* along the direction *IZ* ; from this moreover there arises :

the force along $IA = \Pi\ cos\, l$,
the force along $IB = \Pi\ cos\, m$,
the force along $IC = \Pi\ cos\, n$,

but which forces are understood now to be applied at the point *H*. The line *IH* is drawn which is equal to $\sqrt{(ff + hh)}$, [recall that *HGI* is the right angle in the triangle] cutting the sphere at *F*, then the $tang\ GIH = \frac{h}{f}$ and the arc *LF* with the arc *ZL* makes the right angle *ZLF* . Putting the arc $LF = \mathfrak{e}$, then $h = -f\ \ tang\,\mathfrak{e}$ and $IH = -\frac{f}{cos\,\mathfrak{e}}$, thus in order that in place of the interval *GH* = *h* we retain in the calculation the arc $LF = \mathfrak{e}$ in a more convenient manner. But now it is required to investigate, how the line *I F* is inclined to the principal axes, which inclination is defined through the arcs *FA*, *F B* and *F C*. Moreover it is found [from the sph. triangles FLA, etc, again from the cosine rule, noting complements of rt. angles, that]

$$cos\ AF = cos\,\zeta\, cos\,\mathfrak{e} + sin\,\mathfrak{f}\ \ sin\,\zeta\, sin\,\mathfrak{e}\ ,$$
$$cos\ BF = cos\,\eta\, cos\,\mathfrak{e} + sin\ \mathfrak{g}\ \ sin\,\eta\, sin\,\mathfrak{e}\ ,$$
$$cos\ CF = cos\,\vartheta\, cos\,\mathfrak{e} + sin\,\mathfrak{h}\, sin\,\vartheta\ \ sin\,\mathfrak{e}.$$

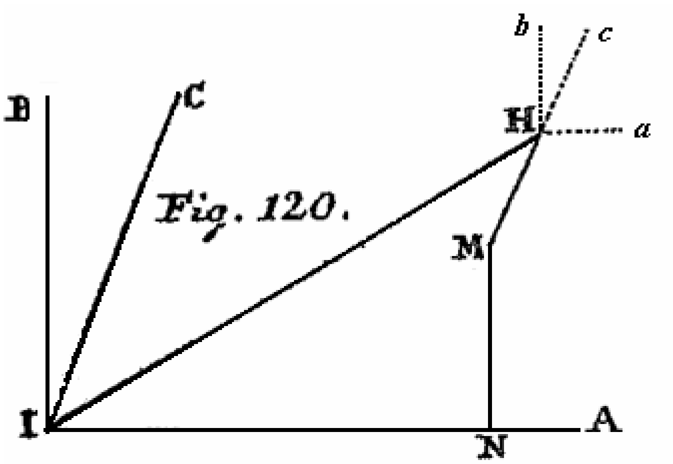


Fig. 120.

Now the lines *IA*, *I B*, *IC* between themselves represent the principal axes of the body (Fig.120), between which the line $IH = -\frac{f}{\cos\mathfrak{e}}$ is present, and the coordinates parallel to the axis for the point *H* are

$$IN = IH\cos AF,$$
$$NM = IH\cos BF,$$
$$MH = IH\cos CF$$

and the forces at *H* applied parallel to the axis are :

the force $Ha = \Pi\cos l$, the force $Hb = \Pi\cos m$, the force $Hc = \Pi\cos n$,
from which there arises the moments about the principal axes :

Mom. about the axis *IA* in the sense $BC = \Pi\cdot IH(\cos n\cos BF - \cos m\cos CF)$,
Mom. about the axis *IB* in the sense $CA = \Pi\cdot IH(\cos l\cos CF - \cos n\cos AF)$,
Mom. about the axis *IC* in the sense $AB = \Pi\cdot IH(\cos m\cos AF - \cos l\cos BF)$.

We have indicated these moments above by the letters *P*, *Q*, *R*, and if we substitute the values found above, we obtain :

$$P = \frac{-\Pi f}{\cos\mathfrak{e}}\left(\begin{array}{l}(\sin\mathfrak{e}\cos\rho(\sin\mathfrak{g}\sin\eta\cos\vartheta - \sin\mathfrak{h}\cos\eta\sin\vartheta) + \\ \quad + \cos\mathfrak{e}\sin\rho(\cos\mathfrak{h}\cos\eta\sin\vartheta - \cos\mathfrak{g}\sin\eta\cos\vartheta) + \\ \quad + \sin\eta\sin\vartheta\sin\mathfrak{e}\sin\rho(\sin\mathfrak{g}\cos\mathfrak{h} - \cos\mathfrak{g}\sin\mathfrak{h}))\end{array}\right).$$

But there is

$$\sin\mathfrak{g}\cos\mathfrak{h} - \cos\mathfrak{g}\sin\mathfrak{h} = \sin(\mathfrak{g}-\mathfrak{h}) = -\frac{\cos\zeta}{\sin\eta\sin\vartheta},$$

and then indeed

$$\sin\mathfrak{g}\sin\eta\cos\vartheta - \sin\mathfrak{h}\cos\eta\sin\vartheta = \cos\mathfrak{f}\sin\zeta,$$
$$\cos\mathfrak{h}\cos\eta\sin\vartheta - \cos\mathfrak{g}\sin\eta\cos\vartheta = \sin\mathfrak{f}\sin\zeta,$$

thus so that for both *P* as well as for *Q* and *R* we have from these correspondences :

$$P = -\frac{\Pi f}{\cos \mathfrak{e}}(\cos \mathfrak{f} \sin \zeta \sin \mathfrak{e} \cos \rho + \sin \mathfrak{f} \sin \zeta \cos \mathfrak{e} \sin \rho - \cos \zeta \sin \mathfrak{e} \sin \rho),$$
$$Q = -\frac{\Pi f}{\cos \mathfrak{e}}(\cos \mathfrak{g} \sin \eta \sin \mathfrak{e} \cos \rho + \sin \mathfrak{g} \sin \eta \cos \mathfrak{e} \sin \rho - \cos \eta \sin \mathfrak{e} \sin \rho),$$
$$R = -\frac{\Pi f}{\cos \mathfrak{e}}(\cos \mathfrak{h} \sin \vartheta \sin \mathfrak{e} \cos \rho + \sin \mathfrak{h} \sin \vartheta \cos \mathfrak{e} \sin \rho - \cos \vartheta \sin \mathfrak{e} \sin \rho).$$

COROLLARY 1

932. Since it is the case that $-f \operatorname{tang} \mathfrak{e} = h$ and h denotes the distance GH, by which the section, in which the pressing force lies, standing apart from the section in which the centre of inertia is present, then

$$P = -\Pi f \sin \mathfrak{f} \sin \zeta \cos \rho + \Pi h(\cos \mathfrak{f} \sin \zeta \cos \rho - \cos \zeta \sin \rho),$$
$$Q = -\Pi f \sin \mathfrak{g} \sin \eta \cos \rho + \Pi h(\cos \mathfrak{g} \sin \eta \cos \rho - \cos \eta \sin \rho),$$
$$R = -\Pi f \sin \mathfrak{h} \sin \vartheta \cos \rho + \Pi h(\cos \mathfrak{h} \sin \vartheta \cos \rho - \cos \vartheta \sin \rho).$$

COROLLARY 2

933. Hence while the motion of the body is found, it is required to define not only the magnitude of the pressing force Π but also the interval $GH = h$, in order that the place may be found, where the average position of the pressing force is applied.

EXPLANATION

934. The relation between the arcs ζ, η, ϑ and the angles $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ supplies certain significant properties, among which substitutions called upon can be put in place in the solution. For in the first place for the difference of these angles we find

$$\cos(\mathfrak{f}-\mathfrak{g}) = -\frac{\cos \zeta \cos \eta}{\sin \zeta \sin \eta}, \cos(\mathfrak{g}-\mathfrak{h}) = -\frac{\cos \eta \cos \vartheta}{\sin \eta \sin \vartheta}, \cos(\mathfrak{h}-\mathfrak{f}) = -\frac{\cos \zeta \cos \vartheta}{\sin \zeta \sin \vartheta},$$
$$\sin(\mathfrak{f}-\mathfrak{g}) = -\frac{\cos \vartheta}{\sin \zeta \sin \eta}, \quad \sin(\mathfrak{g}-\mathfrak{h}) = -\frac{\cos \zeta}{\sin \eta \sin \vartheta}, \quad \sin(\mathfrak{h}-\mathfrak{f}) = -\frac{\cos \eta}{\sin \zeta \sin \vartheta}.$$

Hence now the angles $\mathfrak{g}$ and $\mathfrak{h}$ can be reduced to the angle $\mathfrak{f}$ on account of $\mathfrak{g} = \mathfrak{f} - (\mathfrak{f}-\mathfrak{g})$ and
$\mathfrak{h} = \mathfrak{f} + (\mathfrak{h}-\mathfrak{f})$,
from which there is deduced

$$\sin \mathfrak{g} = \frac{-\sin \mathfrak{f} \cos \zeta \cos \eta + \cos \mathfrak{f} \cos \vartheta}{\sin \zeta \sin \eta}, \sin \mathfrak{h} = \frac{-\sin \mathfrak{f} \cos \zeta \cos \vartheta - \cos \mathfrak{f} \cos \eta}{\sin \zeta \sin \vartheta},$$
$$\cos \mathfrak{g} = \frac{-\cos \mathfrak{f} \cos \zeta \cos \eta - \sin \mathfrak{f} \cos \vartheta}{\sin \zeta \sin \eta}, \cos \mathfrak{h} = \frac{-\cos \mathfrak{f} \cos \zeta \cos \vartheta + \sin \mathfrak{f} \cos \eta}{\sin \zeta \sin \vartheta}.$$

But if with two taken together, either $\cos \mathfrak{f}$ or $\sin \mathfrak{f}$ is destroyed, and the following formulas are obtained :

I. $\sin\mathfrak{f}\sin\zeta\cos\zeta+\sin\mathfrak{g}\ \sin\eta\cos\eta+\sin\mathfrak{h}\sin\vartheta\ \cos\vartheta=\ 0$

11. $\cos\mathfrak{f}\sin\zeta\cos\zeta+\cos\mathfrak{g}\ \sin\eta\cos\eta+\cos\mathfrak{h}\sin\vartheta\cos\vartheta\ =\ 0$

111. $\sin\mathfrak{f}\sin\zeta=-\ \cos\mathfrak{g}\ \sin\eta\cos\vartheta+\cos\mathfrak{h}\cos\eta\sin\vartheta$

1V. $\sin\mathfrak{g}\sin\eta=-\ \cos\mathfrak{h}\sin\vartheta\cos\zeta+\cos\mathfrak{f}\cos\vartheta\sin\zeta$

V. $\sin\mathfrak{h}\sin\vartheta=-\ \cos\mathfrak{f}\sin\zeta\cos\eta+\cos\mathfrak{g}\cos\zeta\sin\eta$

VI. $\sin\mathfrak{f}\sin\zeta=\sin\mathfrak{g}\sin\eta\cos\vartheta-\sin\mathfrak{h}\cos\eta\sin\vartheta$

VII. $\cos\mathfrak{g}\sin\eta=\sin\mathfrak{h}\sin\vartheta\cos\zeta-\sin\mathfrak{f}\cos\vartheta\sin\zeta$

VIII. $\cos\mathfrak{h}\sin\vartheta=\sin\mathfrak{f}\sin\zeta\cos\eta-\sin\mathfrak{g}\cos\zeta\sin\eta$

IX. $\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta+\sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta+\sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta=0$

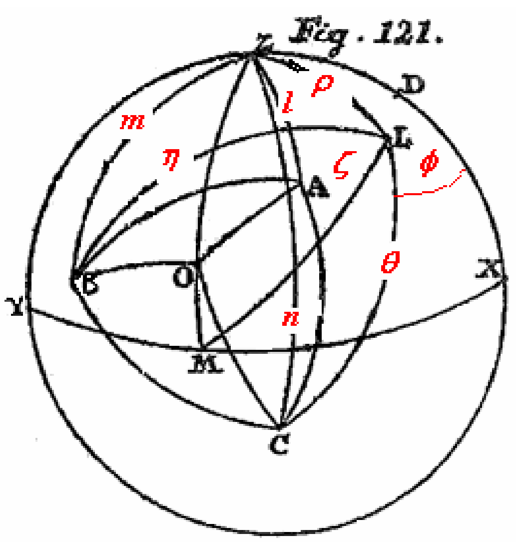
X. $\sin^2\mathfrak{f}\sin^2\zeta+\sin^2\mathfrak{g}\sin^2\eta+\sin^2\mathfrak{h}\sin^2\vartheta=1$

XI. $\cos^2\mathfrak{f}\sin^2\zeta+\cos^2\mathfrak{g}\sin^2\eta+\cos^2\mathfrak{h}\sin^2\vartheta=1;$

with the help of these, simpler equations can be returned leading to the determination of the motion.

PROBLEM 113

935. If in general a cylindrical body is moving in some manner on a horizontal plane, to show the equations, from which at some time the position of this and the rotational motion can be determined.



SOLUTION

With the denominations maintained made in the previous problem, the centre of inertia *I* is considered at rest, around which there is described a sphere, (Fig. 121), the vertical point of this is *Z* and *ZDX* is a fixed vertical circle, in which the central line *IL* initially has the position *ID* . But after the time *t* that arrives at *L*, and the arc $ZL=\rho$ is put in place and the angle $XZL=\varphi$, and hence the position of the principal axes, of which the poles are *A*, *B*, *C*, are defined thus, in order that the arcs are :

$LA=\zeta$, $LB=\eta$, $LC=\vartheta$ and the angles $ZLA=\mathfrak{f}$, $ZLB=\mathfrak{g}$, $ZLC=\mathfrak{h}$, which are constant quantities, from which with the variable arc $ZL=\rho$, thus the arcs *ZA* = *l* , *ZB* = *m*, *ZC* = *n*, are defined so that [from the cosine rule for sph. triangles]:

$$\begin{aligned}
\cos l &= \cos\zeta\cos\rho + \cos\mathfrak{f}\ \sin\zeta\sin\rho,\\
\cos m &= \cos\eta\cos\rho + \cos\mathfrak{g}\ \sin\eta\sin\rho,\\
\cos n &= \cos\vartheta\cos\rho + \cos\mathfrak{h}\ \sin\vartheta\sin\rho\ .
\end{aligned}$$

But if now the moments of inertia of the body about the principal axis *IA*, *IB*,*IC* are *Maa*, *Mbb*, *Mcc* with the mass of the body present *M*, moreover $\varPi$ is the pressing force, and the section, in which this arises, is separated from *I* by the interval equal to *s*, which because this is variable, must be written *s* in the above formulas in place of *h*. Now the body rotates about the pole *O* with an angular speed equal to γ' in the sense *ABC*, and on putting the arcs $OA = \alpha,\ OB = \beta,\ OC = \gamma$ there becomes

$$\gamma'\cos\alpha = x,\quad \gamma'\cos\beta = y,\quad \gamma'\cos\gamma = z\ ,$$

and in the first place we have :

$$\varPi = M\left(1\ + \frac{fdd\cdot\cos\rho}{2gdt^2}\right),$$

[Relating the unbalanced vertical force on the centre of inertia or mass to the vertical acceleration of this point.]

then these three equations :

$$\begin{aligned}
aadx + (cc-bb)\,yzdt &= \frac{-2\varPi\,fg}{M}\ dt\sin\mathfrak{f}\sin\zeta\sin\rho + \frac{2\varPi\,gs}{M}\ dt(\cos\mathfrak{f}\sin\zeta\cos\rho - \cos\zeta\sin\rho),\\
bbdy + (aa-cc)\,xzdt &= \frac{-2\varPi\,fg}{M}\ dt\sin\mathfrak{g}\sin\eta\sin\rho + \frac{2\varPi\,gs}{M}\ dt(\cos\mathfrak{g}\sin\eta\cos\rho - \cos\eta\sin\rho),,\\
ccdz + (bb-aa)\,xydt &= \frac{-2\varPi\,fg}{M}\ dt\sin\mathfrak{h}\sin\vartheta\sin\rho + \frac{2\varPi\,gs}{M}\ dt(\cos\mathfrak{h}\sin\vartheta\cos\rho - \cos\vartheta\sin\rho).
\end{aligned}$$

[Relating the rate of change of the angular momentum (a concept yet to be defined by Euler in a later work) to the unbalanced moments.]
In addition we have these three equations :

$$\begin{aligned}
dl\sin l &= dt(y\cos n - z\cos m) = d\rho(\cos\zeta\ \sin\rho - \cos\mathfrak{f}\sin\zeta\ \cos\rho\ ),\\
dm\sin m &= dt(z\cos l - x\cos n) = d\rho(\cos\eta\sin\rho - \cos\mathfrak{g}\sin\eta\cos\rho\ ),\\
dn\sin n &= dt(x\cos m - y\cos l) = d\rho(\cos\vartheta\sin\rho - \cos\mathfrak{h}\sin\vartheta\cos\rho\ ),
\end{aligned}$$

[For a general point on the unit sphere with the coordinates $\cos l, \cos m, \cos n$, expressing the kinematic relation between the components of the angular speeds about the principal axes and the corresponding linear tangential components of the speeds normal to the axes, on taking account of the senses of the velocities.]
but of which only two suffice to be taken, thus in order that the above six equations, from which it is required to determine just as many variables $x,\ y,\ z,\ \varPi, \gamma'$ and ρ at the given time *t*. Finally from the positions of the angles $XZA = \lambda,\quad XZB = \mu,\quad XZO = \nu$, there becomes

$$d\lambda\sin^2 l = \ -dt\ (y\cos m + z\cos n)$$

so that a single equation suffices to be resolved. But since $LZA = \lambda - \varphi$, there becomes

$$cos(\lambda - \varphi) = \frac{cos\,\zeta - cos\,l\,cos\,\rho}{sin\,l\,sin\,\rho} \quad \text{and} \quad sin(\lambda - \varphi) = \frac{sin\,\mathfrak{f}\,sin\,\zeta}{sin\,l},$$

from which

$$(d\lambda - d\varphi)\,cos(\lambda - \varphi) = \frac{-dl\,sin\,\mathfrak{f}\,sin\,\zeta\,cos\,l}{sin^2 l} = \frac{(d\lambda - d\varphi)(cos\,\zeta - cos\,l\,cos\,\rho)}{sin\,l\,sin\,\rho},$$

and thus

$$d\varphi = \frac{-dt(y\,cos\,m + z\,cos\,n)}{sin^2 l} + \frac{dl\,sin\,\mathfrak{f}\,sin\,\zeta\,cos\,l\,sin\,\rho}{sin\,l(cos\,\zeta - cos\,l\,cos\,\rho)},$$

and hence also at the given time the angle φ is defined; from which investigations the motion of the body is understood perfectly well.

COROLLARY I

936. Since it is the case that

$$cos\,\zeta - cos\,l\,cos\,\rho = sin\,\rho\left(cos\,\zeta\,sin\,\rho - cos\,\mathfrak{f}\,sin\,\zeta\,cos\,\rho\right) = \frac{dl\,sin\,l\,sin\,\rho}{d\rho},$$

then

$$d\varphi = \frac{-dt(y\,cos\,m + z\,cos\,n)}{sin^2 l} + \frac{d\rho\,sin\,\mathfrak{f}\,sin\,\zeta\,cos\,l}{sin^2 l},$$

and hence

$$d\varphi\,sin^2 l = -dt(y\,cos\,m + z\,cos\,n) + d\rho\,cos\,\rho\,sin\,\mathfrak{f}\,sin\,\zeta\,cos\,\zeta + d\rho\,sin\,\rho\,sin\,\mathfrak{f}\,cos\,\mathfrak{f}\,sin^2\zeta.$$

Moreover like expressions for $d\varphi\,sin^2 m$ and $d\varphi\,sin^2 n$ are found, which are collected into a single sum, in order that

$$sin^2 l + sin^2 m + sin^2 n = 2,$$

giving

$$d\varphi = -dt(x\,cos\,l + y\,cos\,m + z\,cos\,n)$$

by nos. I. and IX. § 934, where $x\cos l + y\cos m + z\cos n$ denotes the cosine of the arc *ZO* multiplied by γ'.

COROLLARY 2

937. From the equations found for $dl\,sin\,l,\ dm\,sin\,m,\ dn\,sin\,n$ we deduce

$$dl\,sin\,l\,cos\,\zeta + dm\,sin\,m\,cos\,\eta + dn\,sin\,n\,cos\,\vartheta = d\rho\,sin\,\rho$$

and with the values substituted for *dt* we arrive at

$$d\rho = -dt(x\,sin\,\mathfrak{f}\,sin\,\zeta + y\,sin\,\mathfrak{g}\,sin\,\eta + z\,sin\,\mathfrak{h}\,sin\,\vartheta)$$

with the help of the reduction treated above.

COROLLARY 3

938. Moreover from the three former equations we deduce on account of

$$xdl\,sin\,l + ydm\,sin\,m + zdn\,sin\,n = 0$$

this equation *aaxdx* + *bbydy* + *cczdz*

$$= -\frac{2\Pi\,fg}{M}\,dt\,sin\,\rho(x\,sin\,\mathfrak{f}\,sin\zeta + y\,sin\,\mathfrak{g}\,sin\,\eta + z\,sin\,\mathfrak{h}\,sin\,\vartheta)$$

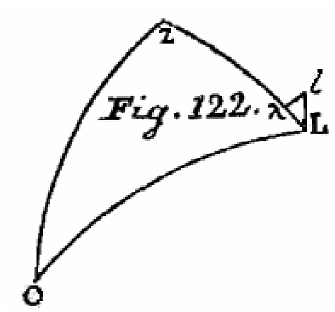
$$= \frac{2\Pi\,fg}{M}\,d\rho\,sin\,\rho = -2\,fgd\cdot cos\rho\left(1 + \frac{fdd\cdot cos\,\rho}{2gdt^2}\right),$$

hence the integral of this is :

$$aaxx + bbyy + cczz = C - 4\,fgcos\rho - \frac{ffd\rho^2\,sin^2\,\rho}{dt^2}.$$

SCHOLION

939. If in our sphere there is drawn the great horizontal circle *YMX*, then it is necessary that the longitudinal axis of the cylinder is always found on this. The anterior boundary of this can touch the circle at *M*, and because both *ML* and *MZ* are quadrants, the angles *MZL* and *MLZ* are right, and thus the angle $ZML = \rho$ and the arc XM equals the angle $XZM = 90° + \varphi$. Then, because the point *M* is able to move only on the circle XY, the pole of rotation *O* by necessity must be situated in the quadrant *ZM*. Hence if the arc *OM* is put equal to ω, on account of the angular speed equal to γ' pointing in the sense *ABC* in the element of time *dt* the point *M* proceeds backwards towards *X* along the element of arc equal to $\gamma' dt\,sin\,\omega$; now there is



$$sin\,\omega = cos\,OZ = cos\,\alpha\,cos\,l + cos\beta\,cos\,m + cos\,\gamma\,cos\,n,$$

and thus

$$\gamma'\,sin\,\omega = x\,cos\,l + y\,cos\,m + z\,cos\,n,$$

thus in order that

$$-d\varphi = dt(x\,cos\,l + y\,cos\,m + z\,cos\,n),$$

as we find in corollary 1. Then since in triangle *OZL* (Fig.122) let $ZO = 90° - \omega$, $ZL = \rho$ and $OZL = 90°$, then

$$cos\,OL = sin\,\omega\,cos\,\rho,\quad sin\,OLZ = \frac{cos\,\omega}{sin\,OL} \text{ and } cos\,OLZ = \frac{sin\omega\,sin\,\rho}{sin\,OL}$$

on account of $cot\,OLZ = \frac{sin\,\rho\,sin\,\omega}{cos\,\omega}$.

Whereby if in the element of time *dt* the point *L* is rotating about O at *l*, then $Ll = \gamma'\,dt\,sin\,OL$ and the angle *OLl* is right; hence with the circle $l\lambda$ drawn to the perpendicular *Zl* there arises

$$\lambda = Ll\,cos\,Z\;Ll = Ll\,sin\,OLZ = \gamma'\,dt\,cos\,\omega,$$

but then $L\lambda = -d\rho$ and thus $d\rho = -\gamma' dt\,cos\,\omega$. Which formula compared with that, which we found in § 937 gives

$$\gamma' \cos\omega = x \sin\mathfrak{f} \sin\zeta + y \sin\mathfrak{g} \sin\eta + z \sin\mathfrak{h} \sin\vartheta\,;$$

but then

$$xx + yy + zz = \gamma'\gamma' = (x\cos l + y\cos m + z\cos n)^2 + (x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta)^2 :$$

which equality is confirmed by the equation $xdl\sin l + ydm\sin m + zdn\sin n = 0$. Now lest we be overwhelmed by a multitude of symbols, we set out the case, in which the longitudinal axis of the cylinder is likewise a principal axis.

PROBLEMA 114

940. If the longitudinal axis of a cylindrical body drawn through the centre of inertia likewise should be a principal axis and that moving on a horizontal plane in some manner, to define the motion of this cylinder.

SOLUTIO

Since the points *A* and *M* coincide in one point, the two remaining principal poles *B* and *C* are present in the vertical circle *ZL* (Fig. 123), and on this account:

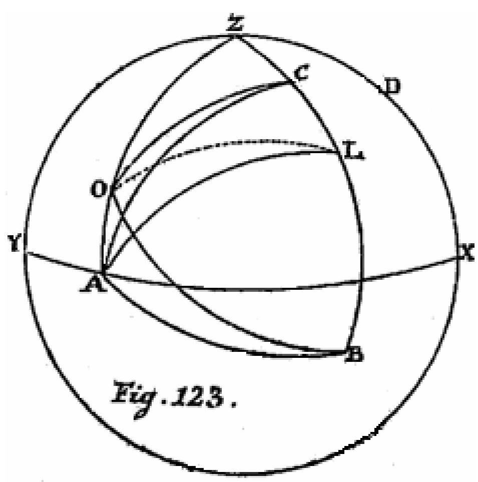


Fig. 123.

$$LA = \zeta = 90°,\quad LB = \eta,\qquad LC = \vartheta = 90° - \eta\ ;$$
$$ZLA = \mathfrak{f} = 90°,\quad ZLB = \mathfrak{g} = 180°,\quad ZLC = \mathfrak{h} = 0;$$

and hence

$$ZA = l = 90°,\quad ZB = m = \eta + \rho$$

and

$$ZC = n = \rho - \vartheta = \eta + \rho - 90°.$$

With which values put in place we have these equations :

$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos\rho}{2gdt^2}$$

$$aadx + (cc - bb)\,yzdt = -\frac{2\Pi\, fg}{M}\,dt\sin\rho,$$

$$bbdy + (aa - cc)\,xzdt = -\frac{2\Pi\, gs}{M}\,dt\sin(\eta + \rho),$$

$$ccdz + (bb - aa)\,xydt = \frac{2\Pi\, gs}{M}\,dt\cos(\eta + \rho),$$

$$y\sin(\eta + \rho) - z\cos(\eta + \rho) = 0\ ,$$

$$-\,xdt\sin(\eta + \rho) = d\rho\sin(\eta + \rho)$$

or $d\rho = -xdt$ et $d\varphi = -dt\big(y\cos(\eta + \rho) + z\sin(\eta + \rho)\big)$.

There is put
$y = u\cos(\eta+\rho)$ and $z = u\sin(\eta+\rho)$,
and for *dt* there is written $-\frac{d\rho}{x}$ or $x = -\frac{d\rho}{dt}$, with which changed made our equations become :

I. $\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos\rho}{2gdt^2}$,

11. $-aadd\rho + \frac{1}{2}(cc-bb)\,uudt^2\sin\,2(\eta+\rho) + \frac{2\Pi}{M}\,fgdt^2\sin\rho = 0$,

111. $bbdu\cos(\eta+\rho) - (aa+bb-cc)ud\rho\sin(\eta+\rho) = \frac{-2\Pi}{M}\,gsdt\sin(\eta+\rho)$,

1V. $ccdu\sin(\eta+\rho) + (aa-bb+cc)ud\rho\cos(\eta+\rho) = \frac{2\Pi}{M}\,gsdt\cos(\eta+\rho)$,

and

V. $d\varphi = -udt$.

From three and four on eliminating *s* we arrive at,

$$bbdu\cos^2(\eta+\rho) + ccdu\sin^2(\eta+\rho) - 2(bb-cc)\,ud\rho\sin(\eta+\rho)\cos(\eta+\rho) = 0,,$$

and the integral of this is

$$u = \frac{C}{bb+cc+(bb-cc)\cos(2\eta+2\rho)},$$

which value substituted in II. presents this equation :

$$-2aadd\rho + \frac{CC(cc-bb)\,dt^2\sin\,2(\eta+\rho)}{\left(bb+cc+(bb-cc)\cos2(\eta+\rho)\right)^2} + dt^2\sin\rho\left(4fg + \frac{2\,ffdd\cdot\cos\rho}{dt^2}\right) = 0 \;\_$$

which equation multiplied by $d\rho$ and integrated gives,

$$-aad\rho^2 + \frac{\frac{1}{2}CC\,dt^2}{bb+cc+(bb-cc)\cos2(\eta+\rho)} - 4fgdt^2\cos\rho - ffd\rho^2\sin^2\rho + Ddt^2 = 0$$

or

$$d\rho^2\left(aa + ff\sin^2\rho\right) = dt^2\left(D - 4fg\cos\rho - \frac{\frac{1}{2}CC\,dt^2}{bb+cc+(bb-cc)\cos2(\eta+\rho)}\right),$$

thus there is produced :

$$dt = \frac{d\rho\sqrt{\left(aa+ff\sin^2\rho\right)\left(bb+cc+(bb-cc)\cos2(\eta+\rho)\right)}}{\sqrt{\left((D-4fg\cos\rho)\left(bb+cc+(bb-cc)\cos2(\eta+\rho)\right)-\frac{1}{2}CC\right)}},$$

Now in the time *t* given equally by ρ and *u*, we deduce hence the pressing force Π and again the distance *s* from this equation:

$$\frac{2\Pi}{M}gsdt = \left(cc-bb\right)du\sin(\eta+\rho)\cos(\eta+\rho)+aaud\rho+(cc-bb)ud\rho\cos2(\eta+\rho).$$

Then we now obtain :

$$x=\frac{-d\rho}{dt},\quad y=u\cos\left(\eta+\rho\right)\ \text{et}\ \ z=u\sin\left(\eta+\rho\right)$$

and then also $\varphi=-\int udt$.

COROLLARY I

941. If initially the point *L* were at *D* in order that $ZD=\mathfrak{r}$ and with everything at rest, on putting t = 0 then $u=0,\ \ \frac{d\rho}{dt}=\ 0$, on account of $\gamma'\ =\ 0$; and thus it is necessary to define the constants, in order that C = 0 and $D=4\,fg\cos\mathfrak{r}$; thus there becomes

$$dt=d\rho\sqrt{\frac{aa+ff\sin^2\rho}{4\,fg(\cos\mathfrak{r}-\cos\rho)}},$$

and thus for $\rho>\mathfrak{r}$. Again u = 0, hence $\varphi=0$ and $\gamma'=0$; moreover the pressing force Π hence easily becomes known, and when ρ can be increased to 90°, as if the body falls forward. Hence here the motion neither depends on the position of the principal axes *IB* or *IC* nor on the radius of the base of the cylinder *e*.

COROLLARY 2

942. If initially the line *IL* were vertical or ρ = 0 and the body begins to rotate about that line with an angular speed ε in the sense *AB*, in order that *O* should be at *L* and thus $\alpha=\ 90°,\ \beta=\eta$ and $\gamma=\ 90°-\eta$ *:* then initially

$$x=-\frac{d\rho}{dt}=\ 0,\quad y=\varepsilon\cos\eta\quad \text{and}\ \ z=\varepsilon\sin\eta$$

Hence the constants become

$C=\varepsilon\left(bb\ +cc+\left(bb-cc\right)\cos2\eta\right)$ and $D=4\,fg+\ \frac{1}{2}\varepsilon\varepsilon(bb+cc+\left(bb-cc\right)\cos2\eta)$,

thus it is deduced that

$$u=\frac{\varepsilon(bb+cc+(bb-cc)\cos2\eta)}{bb+cc+(bb-cc)\cos2\left(\eta\ +\rho\right)}$$

and

$$\frac{d\rho^2(aa+ff\sin^2\rho)}{dt^2}=4fg\left(1-\cos\rho\right)+\frac{\frac{1}{2}\varepsilon\varepsilon(bb-cc)(bb+cc+(bb-cc)\cos2\eta)\left(\cos2\left(\eta+\rho\right)-\cos2\eta\right)}{bb+cc+(bb-cc)\cos2\left(\eta+\rho\right)}$$

COROLLARY 3

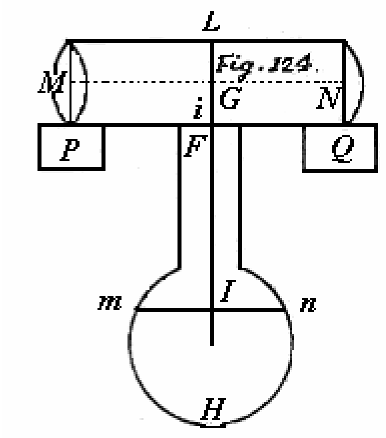
943. If is should be the case that bb = cc, there becomes :

$$dt=d\rho\sqrt{\frac{aa+ff\sin^2\rho}{4\,fg(1-\cos\rho)}}$$

and the line *IL* remains vertical always and the body proceeds to rotate uniformly about that line ; for since the denominator contains the term $\sqrt{\left(1-\cos\rho\right)}=\sin\frac{1}{2}\rho\sqrt{2}$, a finite arc ρ

cannot be completed in a finite time ; which likewise comes about, if either $\eta = 0$ or $\vartheta = 0$, that is if the line *IL* should be a principal axis.

SCHOLIUM



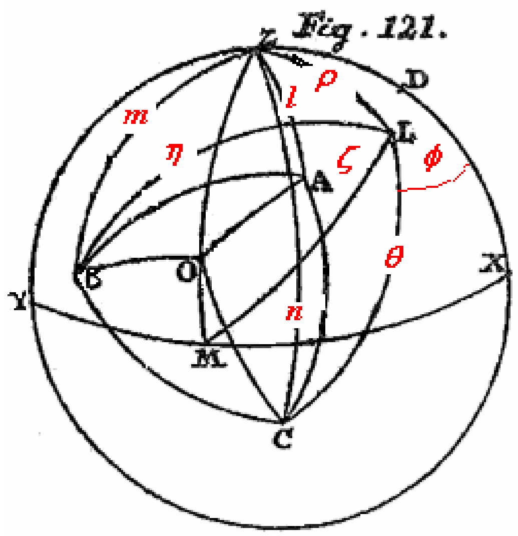
944. Unless the longitudinal axis is likewise a principal axis of the body, on account of the multitude of letters it is scarcely apparent, how the formulas elicited above can be set out generally, that yet we undertake below. Now if we consider only the motion of this kind of cylindrical bodies as if they were infinitely small, according to which it is required that the centre of inertia *I* lies below the centre of the circle *G* on the central line *LF* (fig. 118) and an infinitely small body is disturbed from the resting position, the smallest oscillations or vacillations are arising, the form of which it is possible to determine from our general formulas. Here there is no need that the whole body is a cylinder, but it suffices, if the ends of this around *M* and *N* are cylinders, firmly supported by horizontal planes *P* and *Q* (Fig. 124), that also suffices, if only about the point of each contact the shape should be cylindrical, if indeed we admit only infinitely small motions. Then between the supports *P* and *Q* it is possible to fix the body of some kind of pendulum *FmHn*, in order that the pendulum arises not about some fixed axis line, but about cylindrical ends resting freely on horizontal planes, and it is required to determine the oscillatory motion of this. Hence in such a pendulum in the first place the centre of inertia *I* is noted, through which the line *mn* is drawn parallel to the geometrical axis *MN* of the cylinder, which is the longitudinal axis that continually remains horizontal. Again there is drawn from *I* to *MN* the perpendicular line *IGL*, which if it were vertical, the body turns from rest, and if we put the interval *GI* equal to *f*, we must take the letter *f* negative in the above formulas. Then for the cylindrical shapes of the ends let the radius of the base be equal to *e*, but which, as we have seen, is completely missing from the computation, thus likewise the ends can be thicker or more slender. But if the line *IG* = *f* should be less than *GF* = *e* and the whole body is turning above the supports *P* and *Q*, a motion is produced in a like manner to these which are accustomed to arise from a disturbed cone. But whatever the centre of inertia shall be *I*, it remains in the same vertical line, from which the whole investigation is deduced about the rotational motion to be defined, in which we consider the centre of inertia as being at rest.

PROBLEM 115

945. If the body, which rests with the cylindrical basis on horizontal planes, if disturbed an infinitely small amount from the position of rest and to that perhaps likewise an infinitely small motion is impressed, to determine the vacillatory motion by which it is disturbed.

SOLUTION

In our general formulas the interval $GI = f$ at first is established as negative, then the arc $ZL = \rho$, by which the central line *LGI* declines from the vertical position (Fig. 121), must be considered as infinitely small, and likewise the angular speed γ', and thus from which the quantities $x = \gamma' \cos\alpha$, $y = \gamma' \cos\beta$, $z = \gamma' \cos\gamma$ as they must be treated as vanishing. Hence in whatever manner the principal axes *IA, IB, IC* should be arranged about the central axis *GI* and the longitudinal axis *mn*, the position of which is defined with the arcs $LA = \zeta$, $LB = \eta$, $LC = \vartheta$, and then with the angles



$$ZLA = \mathfrak{f}, \quad ZLB = \mathfrak{g}, \quad ZLC = \mathfrak{h},$$

in the first place we have

$$\sin\rho = \rho, \quad \cos\rho = 1,$$

whence the products *xy*, *xz* and *yz* can be omitted ; from which there becomes

$$\cos l = \cos\zeta, \quad \cos m = \cos\eta \text{ and } \cos n = \cos\vartheta;$$

and the equations containing the solution from problem 113 on account of

$$\frac{\Pi}{M} = 1 - \frac{fdd \cdot \cos\rho}{2gdt^2} = 1,$$

will be :

$$\text{I. } aadx = 2fg\rho dt \sin\mathfrak{f} \sin\zeta + 2fgsdt \cos\mathfrak{f} \sin\zeta,$$
$$\text{II. } bbdy = 2fg\rho dt \sin\mathfrak{g} \sin\eta + 2gsdt \cos\mathfrak{g} \sin\eta,$$
$$\text{III. } ccdz = 2fg\rho dt \sin\mathfrak{h} \sin\vartheta + 2gsdt \cos\mathfrak{h} \sin\vartheta.$$

from which from § 938 this integration has been derived

$$aaxx + bbyy + cczz = C - 2fg\rho\rho - \frac{ff\rho\rho d\rho^2}{dt^2}$$

on account of $\cos\rho = 1 - \frac{1}{2}\rho\rho$, since it is not possible to neglect here the infinitely small $\rho\rho$.

Then we have:

$$\text{IV.}\quad y\cos\vartheta - z\cos\eta = -\frac{d\rho}{dt}\cos\mathfrak{f}\sin\zeta,$$
$$\text{V.}\quad z\cos\zeta - x\cos\vartheta = -\frac{d\rho}{dt}\cos\mathfrak{g}\sin\eta,$$
$$\text{VI.}\ x\cos\eta - y\cos\zeta = -\frac{d\rho}{dt}\cos\mathfrak{h}\sin\vartheta$$

and from § 936 and §937

$$d\rho = -dt\left(x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta\right),$$
$$d\varphi = -dt\left(x\cos\zeta + y\cos\eta + z\cos\vartheta\right).$$

since now there arises

$$\text{IV.}\,x + \text{V.}\,y + \text{VI.}\,z = 0\ ,$$

then

$$x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta = 0.$$

Following from I., II., III. by calling upon the help of formulas I. and II., from § 934 we deduce

$$aax\cos\zeta + bby\cos\eta + ccz\cos\vartheta = A,$$

and for the interval s being determined :

$$aadx\cos\mathfrak{f}\sin\zeta + bbdy\cos\mathfrak{g}\sin\eta + ccdz\cos\mathfrak{h}\sin\vartheta = 2gsdt.$$

Now we put in place $d\rho = -udt$ and $d\varphi = -vdt$,
and on account of

$$x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\vartheta = 0$$

we follow with

$$x = u\sin\mathfrak{f}\sin\zeta + v\cos\zeta,\quad y = u\sin\mathfrak{g}\sin\eta + v\cos\eta,\quad z = u\sin\mathfrak{h}\sin\vartheta + v\cos\vartheta$$

and hence

$$A = u\left(aa\sin\mathfrak{f}\sin\zeta\cos\zeta + bb\sin\mathfrak{g}\sin\eta\cos\eta + cc\sin\mathfrak{h}\sin\vartheta\cos\vartheta\right)$$
$$+v\left(aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta\right).$$

We put as an abbreviation

$$bb\cos\mathfrak{h}\cos\eta\sin\vartheta - cc\cos\mathfrak{g}\sin\eta\cos\vartheta = \mathfrak{A}$$
$$cc\cos\mathfrak{f}\cos\vartheta\sin\zeta - aa\cos\mathfrak{h}\sin\vartheta\cos\zeta = \mathfrak{B}$$
$$aa\cos\mathfrak{g}\cos\zeta\sin\eta - bb\cos\mathfrak{f}\sin\zeta\cos\eta = \mathfrak{C}$$

and there arises

$$\mathfrak{A}\cos\mathfrak{f}\sin\zeta+\mathfrak{B}\cos\mathfrak{g}\sin\eta+\mathfrak{C}\cos\mathfrak{h}\sin\vartheta=0$$

$$aa\cos^2\zeta+bb\cos^2\eta+cc\cos^2\vartheta=\mathfrak{D}$$

$$aa\sin\mathfrak{f}\sin\zeta\cos\zeta+bb\sin\mathfrak{g}\sin\eta\cos\eta+cc\sin\mathfrak{h}\sin\vartheta\cos\vartheta=\mathfrak{F}$$

and we have :

$$v=\frac{A-\mathfrak{F}u}{\mathfrak{D}}\ ,\quad x=\frac{A\cos\zeta+\mathfrak{A}u}{\mathfrak{D}}\ ,\quad y=\frac{A\cos\eta+\mathfrak{B}u}{\mathfrak{D}},\quad z=\frac{A\cos\vartheta+\mathfrak{C}u}{\mathfrak{D}},$$

which values, to be substituted into the integrated equation embracing the *vim vivam* [*i. e*. double the kinetic energy], on account of $\frac{d\rho}{dt}=-u$ give:

$$\frac{AA\mathfrak{D}+2Au\left(\mathfrak{A}a^2\cos\zeta+\mathfrak{B}b^2\cos\eta+\mathfrak{C}c^2\cos\vartheta\right)+uu\left(\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2\right)}{\mathfrak{D}\mathfrak{D}}=C-2fg\rho\rho-ff\rho\rho uu$$

which equation on account of

$$\mathfrak{A}aa\cos\zeta+\mathfrak{B}bb\cos\eta+\mathfrak{C}cc\cos\vartheta=0$$

goes into this :

$$AA\mathfrak{D}+\left(\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2\right)uu=C\mathfrak{D}\mathfrak{D}-2\mathfrak{D}\mathfrak{D}fg\rho\rho-\mathfrak{D}\mathfrak{D}ff\rho\rho uu$$

where, if in place of $C\mathfrak{D}\mathfrak{D}-AA\mathfrak{D}$ there is put $B\mathfrak{D}\mathfrak{D}$, becomes

$$u=\frac{\mathfrak{D}\sqrt{(B-2fg\rho\rho)}}{\sqrt{\left(\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2+\mathfrak{D}\mathfrak{D}ff\rho\rho\right)}}$$

Again we may put in place

$$\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2=\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H},$$

and on rejecting the infinitely small term $\mathfrak{D}\mathfrak{D}ff\rho\rho$ we have :

$$u=\frac{\mathfrak{D}\sqrt{(B-2fg\rho\rho)}}{\mathfrak{H}}\quad\text{and}\quad dt=\frac{-\mathfrak{H}d\rho}{\sqrt{(B-2fg\rho\rho)}},$$

from which we deduce

$$t = \text{Const.} + \frac{\mathfrak{H}}{\sqrt{2fg}} A\cos\frac{\rho\sqrt{2fg}}{\sqrt{B}}$$

or

$$\rho = \frac{\sqrt{B}}{\sqrt{2fg}} \cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}} \quad \text{and} \quad u = \frac{\sqrt{B}}{\mathfrak{H}} \sin\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}$$

then indeed

$$v = \frac{A}{\mathfrak{D}} - \frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\mathfrak{H}} \sin\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}};$$

and hence

$$\varphi = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\sqrt{2fg}} \cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}} = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\rho}{\mathfrak{D}}.$$

Then we find :

$$s = \frac{-\sqrt{B\mathfrak{f}}}{\mathfrak{D}\mathfrak{H}\mathfrak{H}\sqrt{2g}}\left(aabb\sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta + aacc\sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta + bbcc\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta\right)\cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}.$$

And then indeed there is

$$\gamma'\gamma' = xx + yy + zz = \frac{AA - 2A\mathfrak{F}u + (\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C})uu}{\mathfrak{D}\mathfrak{D}}$$

and thus all have been defined at a given time. Furthermore here it is helpful to note that $\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} = \mathfrak{D}\mathfrak{D} + \mathfrak{F}\mathfrak{F}$, thus in order that $\gamma'\gamma' = vv + uu$.

COROLLARY 1

946. Since now

$$\rho = \frac{\sqrt{B}}{\sqrt{2fg}} \cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}},$$

it is apparent that the arc $ZL = \rho$ or the declination of the line *LI* varies from the vertical position according to the like manner of a pendulum and the rocking motion of this line *LI* to be isochronous with the oscillations of a pendulum, the length of which is equal to $\frac{\mathfrak{H}\mathfrak{H}}{\mathfrak{f}}$, which length is equal to :

$$\frac{\mathfrak{A}^2a^2 + \mathfrak{B}^2b^2 + \mathfrak{C}^2c^2}{\mathfrak{D}\mathfrak{D}\mathfrak{f}}.$$

COROLLARY 2

947. Again since

$$\varphi = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\rho}{\mathfrak{D}}$$

the point *L* if rotating in a mean motion around the vertical *Z* with an angular speed equal to $\frac{A}{\mathfrak{D}}$; now the mean position ought to be corrected by the small amount $\frac{\mathfrak{F}\rho}{\mathfrak{D}}$. But if the constant *A* = 0, then angle *DZL* is changed for a short while, unless $\mathfrak{F} = 0$.

COROLLARY 3

948. Hence if the rotations of the body about the axis vertical *IZ* are excluded, so that *A* = 0, and initially $\varphi = 0$, $\rho = \mathfrak{r}$ and the angular speed $\gamma' = \varepsilon$, the constants may be defined thus, so that

$$D = \frac{\mathfrak{F}\mathfrak{r}}{\mathfrak{D}}, \quad \mathfrak{r} = \frac{\sqrt{B}}{\sqrt{2fg}} cos \frac{\delta\sqrt{2fg}}{\mathfrak{H}} \quad \text{and} \quad \varepsilon\varepsilon = \frac{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})B}{\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}} \left(sin \frac{\delta\sqrt{2fg}}{\mathfrak{H}} \right)^2 .$$

Hence

$$\sqrt{B} = \frac{\mathfrak{r}\sqrt{2fg}}{cos\frac{\delta\sqrt{2fg}}{\mathfrak{H}}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{sin\frac{\delta\sqrt{2fg}}{\mathfrak{H}}\sqrt{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}}$$

and thus

$$\text{tang}\frac{\delta\sqrt{2fg}}{\mathfrak{H}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{\mathfrak{r}\sqrt{2fg(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}},$$

and from which the constant *B* can be found. But if $\varepsilon = 0$, there is produced

$$\sqrt{B} = \mathfrak{r}\sqrt{2fg} \quad \text{and} \quad \delta = 0.$$

EXAMPLE

949. We can put the line *IM*, which is drawn through the centre of inertia *I* parallel to the geometric axis of the cylinder (*MN* fig. 124), likewise to be a principal axis of the body, and we have as § 940 $\mathfrak{f} = 90°$, $\mathfrak{g} = 180°$, $\mathfrak{h} = 0$ and $\zeta = 90°$ and also $\vartheta = 90° - \eta$. Moreover we may deduce hence:

$\mathfrak{A} = bb\,cos^2\eta + cc\,sin^2\eta$, $\mathfrak{B} = 0$, $\mathfrak{C} = 0$, $\mathfrak{D} = \mathfrak{A}$ et $\mathfrak{F} = 0$

hence $aa = \mathfrak{H}\mathfrak{H}$, from which the length of the isochronous simple pendulum to become equal to $\frac{aa}{\mathfrak{f}}$. Then the horizontal axis *IA* remains motionless. And if initially, when $\rho = \mathfrak{r}$, the body begins to move from rest, then $\delta = 0$ and $\sqrt{B} = \mathfrak{r}\sqrt{2fg}$, from which the remaining variable quantities can be deduced

$$\rho = \mathfrak{r}\, cos \frac{t\sqrt{2fg}}{a}, \quad u = \frac{\mathfrak{r}\sqrt{2fg}}{a} sin \frac{t\sqrt{2fg}}{a}, \quad v = 0,$$

on account of

$$A = 0 \quad \text{and} \quad x = u = \frac{\mathfrak{r}\sqrt{2fg}}{a} sin \frac{t\sqrt{2fg}}{a}, \quad y = 0 \quad \text{and} \quad z = 0 \quad \text{and also} \quad \gamma' = x.$$

But actually with a progressive motion adjoined, the centre of inertia *I* in turn rises and falls along a straight line, with the above cylinder *MN* following this motion, while it is able to advance freely on the planes *P* and *Q*, assuming it is not impeded by friction.

SCHOLIUM

950. Because the size of the cylinder *MN* is not present in the computation, the same solution prevails if the thickness of this vanishes and the adjoining body is suspended from a line axis. From which it appears that this motion ought to agree with the oscillatory motion defined above, since yet in length otherwise it usually arises that for the oscillatory motion the length of the isochronous simple pendulum emerging is equal to $f + \frac{aa}{f} = \frac{aa+ff}{f}$, since here it is only equal to $\frac{aa}{f}$. The reason for this distinction lies in this, since above in the theory of the oscillations we assumed the axis *MN* fixed, while here it has been put in place to move freely. Hence it is apparent, on account of the free axis, and if it lies on a horizontal plane, that the oscillations are much more rapidly than if the axis were retained firmly in the same place. And this also is in general agreement with the theory ; if indeed (fig. 118) the circle ΠMTN must touch the plane at the same point *T*, besides the pressing force Π a certain horizontal force should be introduced into the calculation, which if this is put equal to Θ, acting along *TH*, in order that T remains constant, as $TP = f\ sin\,\rho$ it is required that [the first edition in error, has the angle squared]

$$\frac{fdd\ sin\,\rho}{dt^2} = -\frac{2\Theta g}{M}.$$

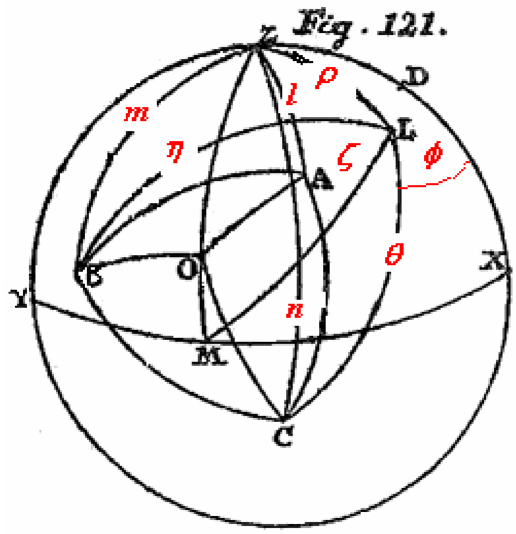
Moreover from this force the moment about the principal axis also arises, by which therefore the rotational motion is affected, in order that such an amount is produced as we have determined above in the investigation of the oscillatory motion. Moreover here must be noted rightly, if small axes lie on a highly polished horizontal plane, then the motion of the oscillations are able to disagree greatly from that which may arise if they are held in place firmly, and indeed are destined to be made much quicker. But the least friction lifts this distinction and the law prevails to reduce the oscillatory motion. But the solution of this problem leads us to the general solution of problem no. 113.

PROBLEM 116

951. If whatever cylinder is moving in some manner on a horizontal plane, to resolve and lead to integration the equations found above for which the motion of this cylinder is defined.

SOLUTION

Here everything remains as have been set up above in problem 113, and we take a point *F* on the central line *LIGF* more distant from the centre of inertia *I* of the cylinder than the centre of the section of the cylinder *G*, on putting the interval *GI* = *f* (Fig. 121 [& Fig. 119]). Therefore from the differential equations shown there we now extract a single differential equation, which is :



$$aaxx+bbyy+cczz=\ C-4\,fg\cos\rho-\frac{ffd\rho^2\sin^2\rho}{dt^2}.$$

Now in addition the three first equations with the aid of the three last equations in the final terms of the application go over into these forms :

$$aadx+(cc-bb)\,yzdt=-\frac{2\Pi\,fg}{M}\cdot dt\sin\mathfrak{f}\sin\zeta\sin\rho-\frac{2\Pi\,gs}{M}\cdot\frac{dtdl\sin l}{d\rho},$$

$$bbdy+(aa-cc)\,xzdt=-\frac{2\Pi\,fg}{M}\cdot dt\sin\mathfrak{g}\sin\eta\sin\rho-\frac{2\Pi\,gs}{M}\cdot\frac{dtdm\sin m}{d\rho},$$

$$ccdz+(bb-aa)\,xydt=-\frac{2\Pi\,fg}{M}\cdot dt\sin\mathfrak{h}\sin\vartheta\sin\rho-\frac{2\Pi\,gs}{M}\cdot\frac{dtdn\sin n}{d\rho}.$$

Hence there is now collected together the form : $\mathrm{I}.\cos l+\mathrm{II}.\cos m+\mathrm{III}.\cos n$, and because

$$dl\sin l\cos l+dm\sin m\cos m+dn\sin n\cos n=0,$$

the final terms involving *s* cancel each other out; then also by the relations treated in § 934 there is found

$$\sin\mathfrak{f}\sin\zeta\cos l+\sin\mathfrak{g}\sin\eta\cos m+\sin\mathfrak{h}\sin\vartheta\cos n=0,$$

and thus also the penultimate are removed. On account of which we arrive at this equation :

$$aadx\cos l+bbdy\cos m+ccdz\cos n+aaxzdt\cos m+bbxydt\cos n$$
$$+ccyzdt\cos l-aaxydt\cos n-bbyzdt\cos l-ccxzdt\cos m=0,$$

but from the three final equations :

$$z\cos m - y\cos n = -\frac{dl\sin l}{dm\sin m}, \quad x\cos n - z\cos l = -\frac{dm\sin m}{dt},$$
$$y\cos l - x\cos m = -\frac{dn\sin n}{dt},$$

with which values put in place we obtain :

$$aadx\cos l + bbdy\cos m + ccdz\cos n$$
$$-\ aaxdl\sin l - bbydm\sin m - cczdn\sin n = 0,$$

and the integral of this is :

$$aax\cos l + bby\cos m + ccz\cos n = D.$$

Next in place of x, y and z we introduced the new variables defined hence :

$$x\cos\zeta + y\cos\eta + z\cos\vartheta = p\ ,$$
$$x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\vartheta = q\ ,$$
$$x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta = r\ ,$$

and in the first $d\rho = -rdt$; again on account of

$$x\cos l + y\cos m + z\cos n = p\cos\rho + q\sin\rho\,,$$

then there arises $d\varphi = -dt(p\cos\rho + q\sin\rho)$. Besides on account of

$$xdl\sin l + ydm\sin m + zdn\sin n = 0$$

there becomes $p\sin\rho - q\cos\rho = 0$. On account of which we put

$$p = u\cos\rho \quad \text{and} \quad q = u\sin\rho$$

and then

$$d\varphi = -udt \quad \text{and} \quad d\rho = -rdt\,,$$

and from these assumed equations, we elicit :

$$x = r\sin\mathfrak{f}\sin\zeta + u\cos l\ ,$$
$$y = r\sin\mathfrak{g}\sin\eta + u\cos m,$$
$$z = r\sin\mathfrak{h}\sin\vartheta\ + u\cos n$$

and hence

$$xx + yy + zz = rr + uu = \gamma'\gamma'.$$

Now the integrated equation just found before presents itself :

$$D = r(aa\sin\mathfrak{f}\sin\zeta\cos l + bb\sin\mathfrak{g}\sin\eta\cos m + cc\sin\mathfrak{h}\sin\vartheta\cos n)$$
$$+\ u(aa\cos^2 l + bb\cos^2 m + cc\cos^2 n),$$

in which u is determined by r and ρ ; and thus also x, y, z. Finally the equation of the first integral found

$$aaxx + bbyy + cczz = C - 4fg\cos\rho - ffrr\sin^2\rho,$$

because it contains only r and ρ, determines r through ρ, and thus the equation $dt = -\frac{d\rho}{r}$ for the given time t shows the magnitudes containing all the motion. But if as an abbreviation we put the constants :

$$\begin{aligned}
&aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta = \mathfrak{A},\\
&aa\cos\mathfrak{f}\sin\zeta\cos\zeta + bb\cos\mathfrak{g}\sin\eta\cos\eta + cc\cos\mathfrak{h}\sin\vartheta\cos\vartheta = \mathfrak{B},\\
&aa\cos^2\mathfrak{f}\sin^2\zeta + bb\cos^2\mathfrak{g}\sin^2\eta + cc\cos^2\mathfrak{h}\sin^2\vartheta = \mathfrak{C},\\
&aa\sin\mathfrak{f}\sin\zeta\cos\zeta + bb\sin\mathfrak{g}\sin\eta\cos\eta + cc\sin\mathfrak{h}\sin\vartheta\cos\vartheta = \mathfrak{D},\\
&aa\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta + bb\sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta + cc\sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta = \mathfrak{E},\\
&aa\sin^2\mathfrak{f}\sin^2\zeta + bb\sin^2\mathfrak{g}\sin^2\eta + cc\sin^2\mathfrak{h}\sin^2\vartheta = \mathfrak{F},
\end{aligned}$$

our integrated equations become

$$D = r(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho) + u(\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho - t + \mathfrak{C}\sin^2\rho),$$

$$\begin{aligned}
C - 4fg\cos\rho - ffrr\sin^2\rho &= \mathfrak{F}rr + 2ru(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)\\
&\quad + uu(\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho),
\end{aligned}$$

from which it is concluded

$$rr = \frac{DD - (\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho)(C - 4fg\cos\rho)}{(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)^2 - (\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho)(\mathfrak{F} + \mathfrak{ff}\sin^2\rho)}.$$

Hence we arrive at the time $t = \int \frac{-d\rho}{r}$,

and since

$$u = \frac{D - r(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)}{\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho},$$

then the angle

$$\varphi = -\int udt = \int \frac{ud\rho}{r}.$$

Moreover since we have determined both the arc ρ as well as the angle φ at some time t [i. e, the latitude and longitude of the central axis on the sphere], then the whole motion is know perfectly.

COROLLARY 1

952. Hence the quantities $\mathfrak{A}, \mathfrak{C}$ and $\mathfrak{F}$ are positive by necessity and $\mathfrak{B}$ to $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{A}$ and $\mathfrak{C}$ thus is referred to , in order that

$$\mathfrak{AC} - \mathfrak{BB} = aabb\, sin^2 \mathfrak{h}\, sin^2 \vartheta + aacc\, sin^2 \mathfrak{g}\, sin^2 \eta + bbcc\, sin^2 \mathfrak{f}\, sin^2 \zeta ,$$

from which it is apparent that the form

$$\mathfrak{A}\, cos^2 \rho + 2\mathfrak{B}\, sin\, \rho\, cos\, \rho + \mathfrak{C}\, sin^2 \rho$$

cannot be resolved into two simple factors.

COROLLARY 2

953. From this general solution the case in the preceding problem set out is easily deduced on taking f negative and the arc ρ infinitely small, from which there becomes

$$rr = \frac{DD - \mathfrak{A}C - 4\mathfrak{A}fg\, cos\, \rho}{\mathfrak{DD} - \mathfrak{AF}} = \frac{Const. + 4\mathfrak{A}fg\, cos\, \rho}{\mathfrak{AF} - \mathfrak{DD}}.$$

Moreover there is found from the values established:

$$\mathfrak{AF} - \mathfrak{BB} = aabb\, cos^2 \mathfrak{h}\, sin^2 \vartheta + aacc\, cos^2 \mathfrak{g}\, sin^2 \eta + bbcc\, cos^2 \mathfrak{f}\, sin^2 \zeta ,$$

from which the length of the simple isochronous pendulum is shown more simply than above, so that it is equal to

$$\frac{aabb\, cos^2 \mathfrak{h}\, sin^2 \vartheta + aacc\, cos^2 \mathfrak{g}\, sin^2 \eta + bbcc\, cos^2 \mathfrak{f}\, sin^2 \zeta}{f\left(aa\, cos^2 \zeta + bb\, cos^2 \eta + cc\, cos^2 \vartheta\right)}.$$

SCHOLIUM

954. From these problems concerned with the motion of cylindrical bodies on a horizontal plane set out I will add a little concerning the motion on inclined planes ; now if the motion should be simple, then there is no difficulty in the exercise, but if is should be complicated, in we may happen upon inconvenient calculations. Whereby in practice it is not permitted to separate friction from these motions, anyhow thus we will deal with the more simple motions on an inclined plane, in order that likewise we have an account of friction, from which it is convenient to add a separate section concerning the motion of bodies disturbed by friction.

CAPUT XIX

DE MOTU CORPORUM CYLINDRICORUM SUPER PLANO HORIZONTALI

THEOREMA 11

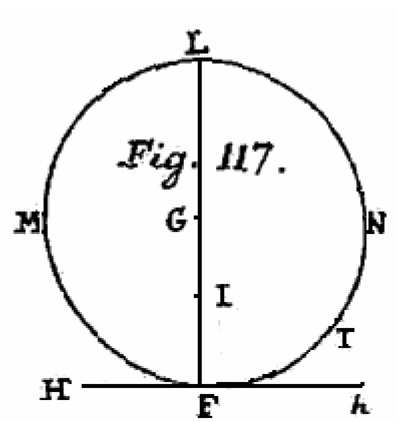
926. Dum corpus cylindricum plano horizontali movetur, pressio, qua plano innititur, est verticalis, et per centrum cuiusdam sectionis cylindri ad longitudinem normaliter factae transit.

DEMONSTRATIO

Corpus cylindricum plano horizontali incumbit secundum lineam rectam axi cylindri parallelam, in qua vires existunt cylindrum sustentantes, fierique potest, ut hae vires per totam illam rectam sint dispersae. Cum autem istae vires omnes sint ad planum horizontale normales ideoque verticales ac parallelae inter se, una dabitur vis iis omnibus aequivalens; cuius ergo directio pariter erit verticalis certoque rectae contactus puncto insistet. Quodsi igitur in hoc puncto cylindrus ad longitudinem normaliter secetur, sectio erit circulus et vis omnibus pressionibus aequivalens, quia est in hac sectione ad punctum contactus verticalis per eius centrum transibit. Nisi ergo haec sectio transeat per centrum inertiae corporis, directio media pressionis non in sectione per centrum inertiae ad longitudinem normaliter facta versabitur.

EXPLICATIO

927. Corpora igitur hic consideramus cylindrica, in quibus primo notetur eorum axis quasi geometricus, ad quem omnes sectiones normaliter factae sint circuli aequales, ita ut corpus sit cylindrus rectus, cuius motus, dum plano horizontali perpetuo incumbit; sumus exploraturi. Si centrum inertiae in ipso axe geometrico esset, in omni situ cylindrus aequilibrium teneret (Fig. 117), sin autem secus, consideretur sectio cylindri ad axem normalis per centrum inertiae I facta, cuius centrum sit in G, atque ad statum aequilibrii requiritur, ut recta GI sit verticalis, ex quo duplex datur aequilibrii situs, alter quo centrum inertiae I infra G, alter quo supra G versatur, quorum ille *stabilis*, hic *labilis* vocetur. In omni autem cylindri situ axis geometricus est horizontalis, rectae contactus verticaliter imminens. Deinde ternos axes



mechanicos cylindri nosse oportet, qui si qualemcunque materiae distributionem admittimus, utcunque ab axe geometrico seu secundum longitudinem ducto differre possunt, a quorum positione motus determinatio potissimum pendet. Per centrum inertiae I etiam ducta concipiatur recta axi geometrico parallela, quae plerumque axis principalis esse solet et semper manet horizontalis. His igitur notatis, quomodocunque cylindrus plano horizontali incumbat, in sectione *LMFN* per centrum inertiae *I* facta notetur punctum *T*, ubi hic circulus planum horizontale tangit, deinde etiam illa sectio huic parallela probe notetur, in qua media directio pressionum versatur, quae rectae *TG* erit parallela; ac tam quantitas pressionis, quam distantia sectionis, in qua versatur, a sectione per centrum inertiae facta, erit incognita, demum ex motu deinceps determinanda; siquidem hac vi effici debet, ut axis cylindri longitudinalis perpetuo maneat horizontalis et cylindrus plano horizontali incumbat.

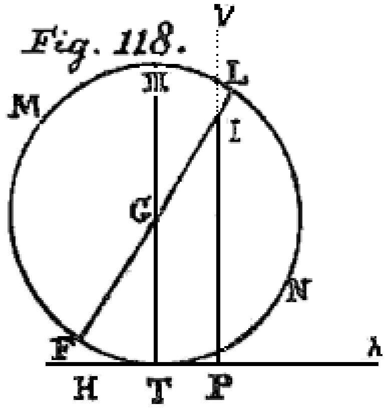
SCHOLION

928. In bis motibus investigandis non opus est, ut totum corpus in figuram cylindri sit efformatum, sed sufficit, si in locis, quibus plano horizontali incumbit, talem habuerit formam. Huc ergo pertinent motus omnium eorum corporum, quae in terminos cylindricos desinunt, quibus tantum planis horizontalibus aeque elevatis utrinque incumbant, dum intra eos moles corporis utcunque fuerit extensa, quemadmodum evenit in cunis, quae motu vacillatorio super terminis, quos tanquam circulos spectare licet, agitantur; deinde etiam huc referendus est motus pendulorum, quae non circa axem linearem, uti supra assumsimus, sed materialem utrinque in cylindrum abeuntem oscillantur, dum his binis cylindris planis horizontalibus incumbunt, intra quae massa penduli dependet. Etsi igitur his casibus sectiones mediae non sunt circuli, tamen binas illas sectiones, in quarum altera centrum inertiae versatur, in altera vero vis pressionis tanquam circulos spectare licet, quoniam figura eatenus tantum in censum venit, quatenus corpus plano horizontali incumbit. Tum vero etiam, nisi huiusmodi corpora integras circumvolutiones peragant, ne opus quidem est, ut toti termini sint cylindrici, sed sufficit, si eorum portio, qua sit contactus, talem habeat figuram, cuius axem longitudinalem per totum corpus extensum probe notasse convenit. Quocirca haec tractatio ad plurimos casus extenditur, de quo motu primum tenendum, centrum inertiae a viribus sollicitantibus alium motum nisi in eadem recta verticali recipere non posse; ita ut nullum consequatur motum horizontalem, nisi extrinsecus talem acceperit, quem autem deinceps uniformiter esset prosecuturum; in quo cum nulla insit difficultas, ad eum hic non attendimus.

PROBLEMA 111

929. Si corpus cylindricum super plano horizontali moveatur deturque pressio, qua plano inititur, definire motum progressivum, quo centrum inertiae corporis incedet.

Fig. 118.



SOLUTIO

Ad axem cylindri fiat sectio normalis per centrum inertiae *I* (Fig. 118), quae sive corpus sit cylindrus continuus sive tantum terminis cylindricis sit praeditum, spectetur tanquam circulus *LMFN* basibus aequalis, cuius centrum sit in *G*, centrum autem inertiae corporis in *I*, existente intervallo $GI = f$, ita ut in statu quietis recta *LIGF* teneat situm verticalem, in quo centrum inertiae *I* supra centrum circuli *G* in figura repraesentatur, quod si fuerit profundius, intervallum $GI = f$ negative est capiendum. Nunc autem contactus respondeat puncto *T*, ita ut recta ad planum circuli in *T* normalis sit linea contactus in planum horizontale *Hh* cadens. Ducta igitur per centrum circuli recta ΠGT ad contactum *T* parallela erit directioni pressionis, qua corpus a plano horizontali repellitur, quae vis, in quacunque alia sectione versetur, ponatur = *Il*, quam tanquam cognitam spectamus. Sit porro pondus cylindri = *M*, radius circuli *GF* = *GT* = *e* et angulus declinationis $\Pi GL = \rho$ *;* erit elevatio centri inertiae *I* supra horizontem $IP = e + f\cos\rho$, quae ponatur =*v*. Quoniam igitur in motum progressivum inquirimus et gravitas deorsum urget secundum *IP* vi = *M*, pressionis vis Π ipsi centro inertiae *I* sursum secundum *IV* applicata concipiatur, ita ut iam tota vis deorsum sollicitans sit $= \mathrm{M} - \Pi$,et massa movenda = *M;* unde ex principiis motus habetur hincque

$$ddv = \frac{-2g(M-\Pi)}{M}\ dt^2$$

hincque

$$\frac{\Pi}{M} = 1 + \frac{ddv}{2gdt^2} \text{ seu } \Pi = M(1+\frac{fdd\cdot\cos\rho}{2gdt^2}),$$

qua aequatione angulus $\Pi GL = \rho$ ex eoque elevatio centri inertiae $IP = e + f\cos\rho$ innotescit; ac nisi corpori motus horizontalis fuerit impressus, punctum *I* in recta *PIV* agitaretur, in ea vel ascendens vel descendens, ita ut punctum *P* maneret fixum, ex quo punctum contactus *T* definitur, quia est $PT = f\sin\rho$. Sin autem corpus acceperit motum horizontalem, eum constanter servaret uniformem in directum motusque puncti *I* ex hoc aequabili rectilineo horizontali et illo verticali foret compositus.

SCHOLION

930. Praeterea autem in hoc corpore motus gyratorius generari potest, ita tamen, ut tam punctum *G* quam recta ad circulum *LMFN* in *G* normalis, quae est axis proprius cylindri, perpetuo maneat in eodem plano horizontali. Ad hunc motum gyratorium investigandum seposito motu progressivo centrum inertiae *I* tanquam in quiete spectabimus, circa quod sphaera descripta, in ea circulus *LMFN* perpetuo erit verticalis, ad quem si recta normalis per *I* ducatur, erit ea axis cylindri longitudinalis. Qua eonditione observata omnes motus gyratorii, quorum cylindrus est capax, facile in figura repraesentari possunt. Hic autem ante omnia ad situm axium principalium probe attendi oportet, quorum respectu momenta ex vi pressionis nata sunt definienda.

PROBLEMA 112

931. Si corpus cylindricum plano horizontali incumbens habeat situm quemcunque deturque tam pressio Π quam sectio cylindri transversa, in qua versatur, invenire eius momenta respectu axium principalium.

SOLUTIO

Sectio cylindri per centrum inertiae *I* ad longitudinem normaliter facta cadat in planum tabulae (Fig. 119), in qua recta *IZ* sit normalis, et recta *LIG* per centrum huius sectionis *G* transeat, ita ut sit intervallum $IG = f$ et angulus $ZIL = \rho$. Ex *G* erigatur ad planum tabulae normalis *GH* usque ad sectionem, in qua vis pressionis Π versatur, sitque intervallum *GH* = *h*, ac supra vidimus esse

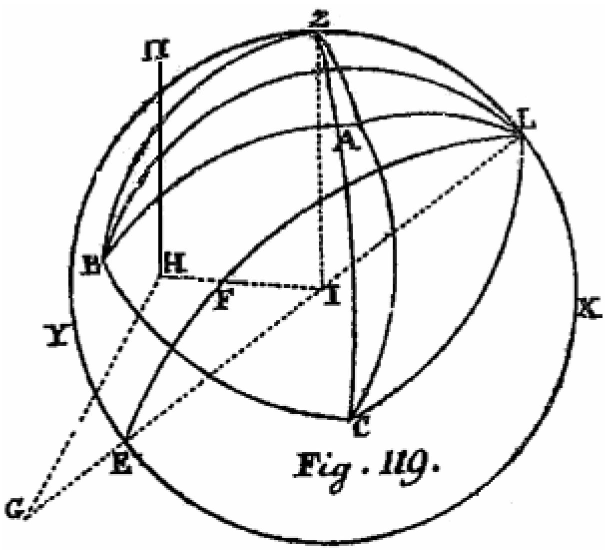


Fig. 119.

$$\Pi = M(1+\frac{fdd\cdot cos\,\rho}{2gdt^2}),$$

cuius vis directio erit $H\Pi$ verticalis ideoque parallela ipsi *IZ* . Iam radio arbitrario = 1 circa centrum inertiae *I* sphaera concipiatur descripta, ad cuius superficiei puncta *A*,*B*, *C* axes corporis principales dirigantur, vocenturque arcus pro horum punctorum determinatione

$$LA=\zeta,\ LB=\eta,\ LC=\vartheta;$$

item $ZA=l, ZB=m, ZC=n$ existente $ZL=\rho$; tum vero anguli $ZLA=\mathfrak{f}$,$ZLB=\mathfrak{g}$, $ZLC=\mathfrak{h}$, ut sit

$$cos\,l = cos\,\zeta\ cos\,\rho + cos\,\mathfrak{f}\ sin\,\zeta\ sin\,\rho,$$
$$cos\,m = cos\,\eta\ cos\,\rho + cos\,\mathfrak{g}\ sin\,\eta\ sin\,\rho,$$
$$cos\,n = cos\,\vartheta\ cos\,\rho + cos\,\mathfrak{h}\ sin\,\vartheta\ sin\,\rho\ .$$

Ac primo quidem vis $H\Pi = \Pi$ resolvatur secundum directiones axibus principalibus parallelas, quae resolutio perinde instituitur, ac si vis haec in centro *I* secundum directionem *IZ* esset applicata; inde autem nascitur

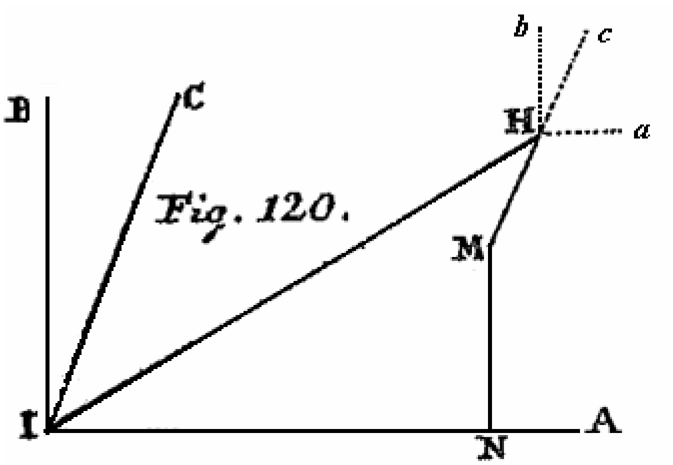
vis secundum $IA = \Pi\ cos\,l$,

vis secundum $IB = \Pi\cos m$,

vis secundum $IC = \Pi\cos n$,

quae autem vires iam in puncto *H* applicatae sunt intelligendae. Ducatur recta *IH* quae erit $=\sqrt{(ff+hh)}$, secans sphaeram in *F*, erit $tang\ GIH = \frac{h}{f}$ et arcus *LF* cum arcu *ZL* faciet angulum *Z LF* rectum. Ponatur arcus , $LF = \mathfrak{e}$, erit $h = -f\ tang\mathfrak{e}$ et $IH = -\frac{f}{cos\,\mathfrak{e}}$, ita ut loco intervalli $GH = h$ commode mode arcum $LF = \mathfrak{e}$ in calculo retineamus. Nunc autem investigari oportet, quomodo recta IF ad axes principales inclinetur, quae inclinatio per arcus FA, FB et FC definitur. Reperitur autem

$$\cos AF = \cos\zeta \cos\mathfrak{e} + \sin\mathfrak{f}\ \sin\zeta \sin\mathfrak{e}\ ,$$
$$\cos BF = \cos\eta \cos\mathfrak{e} + \sin\mathfrak{g}\ \sin\eta \sin\mathfrak{e}\ ,$$
$$\cos CF = \cos\vartheta \cos\mathfrak{e} + \sin\mathfrak{h} \sin\vartheta\ \sin\mathfrak{e}.$$



Repraesentent iam rectae inter se normales IA, IB, IC axes principales corporis (Fig.120), inter quos existat recta $IH = -\frac{f}{\cos\mathfrak{e}}$ eruntque coordinatae pro puneto H axibus parallelae

$$IN = IH \cos AF,$$
$$NM = IH \cos BF,$$
$$MH = IH \cos CF$$

et vires in H applicatae axibusque parallelae erunt

vis $Ha = \Pi \cos l$, vis $Hb = \Pi \cos m$, vis $Hc = \Pi \cos n$,
unde respectu axium principalium nascuntur momenta:

Mom. respectu axis IA in sensum $BC = \Pi \cdot IH(\cos n\ \cos BF - \cos m \cos CF)$
Mom. respectu axis IB in sensum $CA = \Pi \cdot IH(\cos l \cos CF - \cos n \cos AF)$
Mom. respectu axis IC in sensum $AB = \Pi \cdot IH(\cos m \cos AF - \cos l \cos BF)$.

Quae momenta cum supra litteris P, Q, R indicaverimus, si valores supra exhibitos substituamus, obtinebimus:

$$P = \frac{-\Pi f}{\cos\mathfrak{e}} \begin{pmatrix} (\sin\mathfrak{e} \cos\rho(\sin\mathfrak{g} \sin\eta \cos\vartheta - \sin\mathfrak{h} \cos\eta \sin\vartheta) + \\ + \cos\mathfrak{e} \sin\rho(\cos\mathfrak{h} \cos\eta \sin\vartheta - \cos\mathfrak{g} \sin\eta \cos\vartheta) + \\ + \sin\eta \sin\vartheta \sin\mathfrak{e} \sin\rho(\sin\mathfrak{g} \cos\mathfrak{h} - \cos\mathfrak{g} \sin\mathfrak{h})) \end{pmatrix}.$$

At est

$$\sin\mathfrak{g} \cos\mathfrak{h} - \cos\mathfrak{g} \sin\mathfrak{h} = \sin(\mathfrak{g} - \mathfrak{h}) = -\frac{\cos\zeta}{\sin\eta \sin\vartheta},$$

tum vero

$$\sin\mathfrak{g} \sin\eta \cos\vartheta - \sin\mathfrak{h} \cos\eta \sin\vartheta = \cos\mathfrak{f} \sin\zeta,$$
$$\cos\mathfrak{h} \cos\eta \sin\vartheta - \cos\mathfrak{g} \sin\eta \cos\vartheta = \sin\mathfrak{f} \sin\zeta,$$

ita ut tam pro P quam pro Q et R ex analogia habeamus

$$P = -\frac{\Pi f}{\cos\mathfrak{e}}(\cos\mathfrak{f}\sin\zeta\sin\mathfrak{e}\cos\rho + \sin\mathfrak{f}\sin\zeta\cos\mathfrak{e}\sin\rho - \cos\zeta\sin\mathfrak{e}\sin\rho),$$
$$Q = -\frac{\Pi f}{\cos\mathfrak{e}}(\cos\mathfrak{g}\sin\eta\sin\mathfrak{e}\cos\rho + \sin\mathfrak{g}\sin\eta\cos\mathfrak{e}\sin\rho - \cos\eta\sin\mathfrak{e}\sin\rho),$$
$$R = -\frac{\Pi f}{\cos\mathfrak{e}}(\cos\mathfrak{h}\sin\vartheta\sin\mathfrak{e}\cos\rho + \sin\mathfrak{h}\sin\vartheta\cos\mathfrak{e}\sin\rho - \cos\vartheta\sin\mathfrak{e}\sin\rho).$$

COROLLARIUM 1

932. Cum sit $-f\,\mathrm{tang}\,\mathfrak{e} = h$ et h denotet intervalIum *GH*, quo sectio, in quam cadit pressio, antrorsum distat a sectione, in qua est centrum inertiae, erit

$$P = -\Pi f\sin\mathfrak{f}\sin\zeta\cos\rho + \Pi h(\cos\mathfrak{f}\sin\zeta\cos\rho - \cos\zeta\sin\rho),$$
$$Q = -\Pi f\sin\mathfrak{g}\sin\eta\cos\rho + \Pi h(\cos\mathfrak{g}\sin\eta\cos\rho - \cos\eta\sin\rho),$$
$$R = -\Pi f\sin\mathfrak{h}\sin\vartheta\cos\rho + \Pi h(\cos\mathfrak{h}\sin\vartheta\cos\rho - \cos\vartheta\sin\rho).$$

COROLLARIUM 2

933. Dum ergo motus corporis determinatur, non solum quantitatem pressionis Π sed etiam intervalIum $GH = h$,definiri oportet, ut habeatur locus, ubi media directio pressionum est applicata.

EXPLICATIO

934. Relatio inter arcus ζ, η, ϑ et anguIos $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ insignes suppeditat proprietates, inter quas substitutiones in solutione adhibitae continentur. Primo enim pro illorum angulorum differentiis invenimus

$$\cos(\mathfrak{f}-\mathfrak{g}) = -\frac{\cos\zeta\cos\eta}{\sin\zeta\sin\eta}, \ \cos(\mathfrak{g}-\mathfrak{h}) = -\frac{\cos\eta\cos\vartheta}{\sin\eta\sin\vartheta}, \ \cos(\mathfrak{h}-\mathfrak{f}) = -\frac{\cos\zeta\cos\vartheta}{\sin\zeta\sin\vartheta},$$
$$\sin(\mathfrak{f}-\mathfrak{g}) = -\frac{\cos\vartheta}{\sin\zeta\sin\eta}, \ \sin(\mathfrak{g}-\mathfrak{h}) = -\frac{\cos\zeta}{\sin\eta\sin\vartheta}, \ \sin(\mathfrak{h}-\mathfrak{f}) = -\frac{\cos\eta}{\sin\zeta\sin\vartheta}.$$

Hinc iam anguli $\mathfrak{g}$ et $\mathfrak{h}$ ad angulum $\mathfrak{f}$ reduci possunt ob $\mathfrak{g} = \mathfrak{f} - (\mathfrak{f}-\mathfrak{g})$ et $\mathfrak{h} = \mathfrak{f} + (\mathfrak{h}-\mathfrak{f})$, unde colligitur

$$\sin\mathfrak{g} = \frac{-\sin\mathfrak{f}\cos\zeta\cos\eta + \cos\mathfrak{f}\cos\vartheta}{\sin\zeta\sin\eta}, \ \sin\mathfrak{h} = \frac{-\sin\mathfrak{f}\cos\zeta\cos\vartheta - \cos\mathfrak{f}\cos\eta}{\sin\zeta\sin\vartheta},$$
$$\cos\mathfrak{g} = \frac{-\cos\mathfrak{f}\cos\zeta\cos\eta - \sin\mathfrak{f}\cos\vartheta}{\sin\zeta\sin\eta}, \ \cos\mathfrak{h} = \frac{-\cos\mathfrak{f}\cos\zeta\cos\vartheta + \sin\mathfrak{f}\cos\eta}{\sin\zeta\sin\vartheta}.$$

Quodsi binis coniungendis vel $\cos\mathfrak{f}$ vel $\sin\mathfrak{f}$ elidatur, obtinentur sequentes formulae:

I. $\sin\mathfrak{f}\sin\zeta\cos\zeta + \sin\mathfrak{g}\sin\eta\cos\eta + \sin\mathfrak{h}\sin\vartheta\cos\vartheta = 0$

11. $\cos\mathfrak{f}\sin\zeta\cos\zeta + \cos\mathfrak{g}\sin\eta\cos\eta + \cos\mathfrak{h}\sin\vartheta\cos\vartheta = 0$

111. $\sin\mathfrak{f}\sin\zeta = -\cos\mathfrak{g}\sin\eta\cos\vartheta + \cos\mathfrak{h}\cos\eta\sin\vartheta$

1V. $\sin\mathfrak{g}\sin\eta = -\cos\mathfrak{h}\sin\vartheta\cos\zeta + \cos\mathfrak{f}\cos\vartheta\sin\zeta$

V. $\sin\mathfrak{h}\sin\vartheta = -\cos\mathfrak{f}\sin\zeta\cos\eta + \cos\mathfrak{g}\cos\zeta\sin\eta$

VI. $\sin\mathfrak{f}\sin\zeta = \sin\mathfrak{g}\sin\eta\cos\vartheta - \sin\mathfrak{h}\cos\eta\sin\vartheta$

VII. $\cos\mathfrak{g}\sin\eta = \sin\mathfrak{h}\sin\vartheta\cos\zeta - \sin\mathfrak{f}\cos\vartheta\sin\zeta$

VIII. $\cos\mathfrak{h}\sin\vartheta = \sin\mathfrak{f}\sin\zeta\cos\eta - \sin\mathfrak{g}\cos\zeta\sin\eta$

IX. $\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta + \sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta + \sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta = 0$

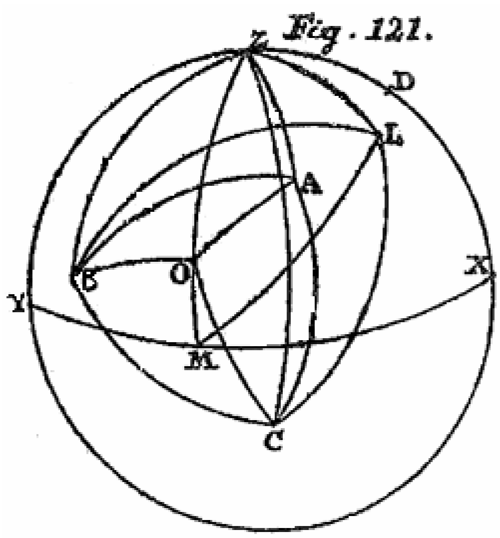
X. $\sin^2\mathfrak{f}\sin^2\zeta + \sin^2\mathfrak{g}\sin^2\eta + \sin^2\mathfrak{h}\sin^2\vartheta = 1$

XI. $\cos^2\mathfrak{f}\sin^2\zeta + \cos^2\mathfrak{g}\sin^2\eta + \cos^2\mathfrak{h}\sin^2\vartheta = 1;$

quarum ope aequationes, ad quas motus determinatio perducitur, simpliciores reddi possunt.

PROBLEMA 113

935. Si corpus cylindricum quodcunque super plano horizontali moveatur utcunque, aequationes exhibere, quibus ad quodvis tempus eius situs et motus gyratorius determinetur.



SOLUTIO

Manentibus denominationibus in praecedente problemate factis, consideretur centrum inertiae *I* ut quiescens, circa quod descripta sit sphaera (Fig. 121), cuius punctum verticale *Z* et circulus verticalis fixus *ZDX*, in quo recta centralis *IL* initio situm *ID* tenuerit. Elapso autem tempore *t* ea pervenerit in *L*, ac ponatur arcus $ZL = \vartheta$ et angulus $XZL = \varphi$, atque hinc situs axium principalium, quorum poli sint *A, B, C*, ita definitur, ut sint arcus

$LA = \zeta$, $LB = \eta$, $LC = \vartheta$ et anguli $ZLA = \mathfrak{f}$, $ZLB = \mathfrak{g}$, $ZLC = \mathfrak{h}$, qui sunt quantitates constantes, ex quibus cum arcu variabili $ZL = \rho$ ita definiuntur arcus *ZA = 1* , *ZB = m*, *ZC = n*, ut sit:

$$\cos l = \cos\zeta\cos\rho + \cos\mathfrak{f}\sin\zeta\sin\rho,$$
$$\cos m = \cos\eta\cos\rho + \cos\mathfrak{g}\sin\eta\sin\rho,$$
$$\cos n = \cos\vartheta\cos\rho + \cos\mathfrak{h}\sin\vartheta\sin\rho .$$

Quodsi iam momenta inertiae corporis respectu axium principalium *IA*, *IB*,*IC* sint *Maa*, *Mbb*, *Mcc* existente *M* massa corporis, Π autem sit pressio, et sectio, in qua ea versatur, ab *I* antrorsum distet intervallo = *s*, quod quia est variabile, in superioribus formulis loco *h* scribi debet *s*. Gyretur nunc corpus circa polum *O* celeritate angulari = γ' in sensum *ABC*, positisque arcubus $OA = \alpha$, $OB = \beta$, $OC = \gamma$ fit

$$\gamma' \cos\alpha = x, \quad \gamma' \cos\beta = y, \quad \gamma' \cos\gamma = z\ ,$$

ac prima habemus

$$\Pi = M\left(1 + \frac{fdd \cdot \cos\rho}{2gdt^2}\right),$$

tum vero has tres aequationes

$$aadx + (cc - bb)\, yzdt = \frac{-2\Pi\, fg}{M}\ dt \sin\mathfrak{f} \sin\zeta \sin\rho + \frac{2\Pi\, gs}{M}\ dt(\cos\mathfrak{f} \sin\zeta \cos\rho - \cos\zeta \sin\rho),$$
$$bbdy + (aa - cc)\, xzdt = \frac{-2\Pi\, fg}{M}\ dt \sin\mathfrak{g} \sin\eta \sin\rho + \frac{2\Pi\, gs}{M}\ dt(\cos\mathfrak{g} \sin\eta \cos\rho - \cos\eta \sin\rho),,$$
$$ccdz + (bb - aa)\, xydt = \frac{-2\Pi\, fg}{M}\ dt \sin\mathfrak{h} \sin\vartheta \sin\rho + \frac{2\Pi\, gs}{M}\ dt(\cos\mathfrak{h} \sin\vartheta \cos\rho - \cos\vartheta \sin\rho).$$

Praeterea habemus has tres aequationes

$$dl \sin l = dt(y\cos n - z\cos m) = d\rho(\cos\zeta \sin\rho - \cos\mathfrak{f} \sin\zeta \cos\rho),$$
$$dm \sin m = dt(z\cos l - x\cos n) = d\rho(\cos\eta \sin\rho - \cos\mathfrak{g} \sin\eta \cos\rho),,$$
$$dn \sin n = dt(x\cos m - y\cos l) = d\rho(\cos\vartheta \sin\rho - \cos\mathfrak{h} \sin\vartheta \cos\rho),$$

quarum autem binas tantum sumsisse sufficit, ita ut supersint sex aequationes, ex quibus variabiles totidem $x,\ y,\ z,\ \Pi, \gamma'$ et ρ ad datum tempus *t* determinari oporteat. Denique vero positis angulis $XZA = \lambda$, $XZB = \mu$, $XZO = \nu$, fit

$$d\lambda \sin^2 l = -dt\ (y\cos m + z\cos n)$$

quam unam resolvisse sufficit. At cum sit $LZA = \lambda - \varphi$, erit

$$\cos(\lambda - \varphi) = \frac{\cos\zeta - \cos l \cos\rho}{\sin l \sin\rho} \quad \text{et} \quad \sin(\lambda - \varphi) = \frac{\sin\mathfrak{f} \sin\zeta}{\sin l},$$

unde

$$(d\lambda - d\varphi)\cos(\lambda - \varphi) = \frac{-dl \sin\mathfrak{f} \sin\zeta \cos l}{\sin^2 l} = \frac{(d\lambda - d\varphi)(\cos\zeta - \cos l \cos\rho)}{\sin l \sin\rho},$$

ideoque

$$d\varphi = \frac{-dt(y\cos m + z\cos n)}{\sin^2 l} + \frac{dl \sin\mathfrak{f} \sin\zeta \cos l \sin\rho}{\sin l(\cos\zeta - \cos l \cos\rho)},$$

hincque etiam ad datum tempus angulus φ definitur; ex quibus rebus motus corporis perfecte cognoscitur.

COROLLARIUM I

936. Cum sit

$$\cos\zeta - \cos l\cos\rho = \sin\rho\left(\cos\zeta\sin\rho - \cos\mathfrak{f}\sin\zeta\cos\rho\right) = \frac{dl\,\sin l\sin\rho}{d\rho},$$

erit

$$d\varphi = \frac{-dt(y\cos m + z\cos n)}{\sin^2 l} + \frac{d\rho\sin\mathfrak{f}\,\sin\zeta\,\cos l}{\sin^2 l},$$

hincque

$$d\varphi\sin^2 l = -dt(\,y\cos m + z\cos n\,) + d\rho\cos\rho\sin\mathfrak{f}\sin\zeta\cos\zeta + d\rho\sin\rho\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta\,.$$

Similes autem expressiones pro $d\varphi\sin^2 m$ et $d\varphi\sin^2 n$ reperiuntur, quae in unam summam collectae, ob

$$\sin^2 l + \sin^2 m + \sin^2 n = 2\,,$$

dabunt

$$d\varphi = -dt(x\cos l + y\cos m + z\cos n)$$

per n° I. et IX. § 934, ubi $x\cos l + y\cos m + z\cos n$ denotat cosinum arcus *ZO* per γ' multiplicatum.

COROLLARIUM 2

937. Ex aequationibus pro $dl\sin l,\ dm\sin m,\ dn\sin n$ inventis colligimus

$$dl\sin l\cos\zeta + dm\sin m\cos\eta + dn\sin n\cos\vartheta = d\rho\sin\rho$$

ac valoribus per *dt* substitutis impetrabimus

$$d\rho = -dt(x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta)$$

ope reductionum supra traditarum.

COROLLARIUM 3

938. Ex tribus autem prioribus aequationibus deducimus ob

$$xdl\sin l + ydm\sin m + zdn\sin n = 0$$

hanc aequationem *aaxdx* + *bbydy* + *cczdz*

$$= -\frac{2\Pi fg}{M}\,dt\sin\rho(x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta)$$

$$= \frac{2\Pi fg}{M}d\rho\sin\rho = -2fgd\cdot\cos\rho\left(1 + \frac{fdd\cdot\cos\rho}{2gdt^2}\right)$$

cuius ergo integrale est

$$aaxx + bbyy + cczz = C - 4fg\cos\rho - \frac{\mathfrak{ff}d\rho^2\sin^2\rho}{dt^2}.$$

SCHOLION

939. Si in sphaera nostra ducatur circulus maximus horizontalis *YMX*,in eo perpetuo axis cylindri longitudinalis reperiatur necesse est. Pertingat eius terminus anterior in *M*, et quia tam *ML* quam *MZ* sunt quadrantes, erunt anguli *MZL* et *MLZ* recti, ideoque angulus $ZML = \rho$ et arcus $XM =$ angulo $XZM = 90° + \varphi$. Tum vero, quia punctum *M* aliter nisi in circulo *XY* moveri nequit, polus gyrationis *O* necessario in quadrante *ZM* situs sit necesse est. Hinc si arcus *OM* ponatur $= \omega$, ob celeritatem angularem $= \gamma'$ in sensum *ABC* tendentem punctum *M* tempusculo *dt* regreditur versus *X* per arculum $= \gamma' dt \sin\omega$; est vero

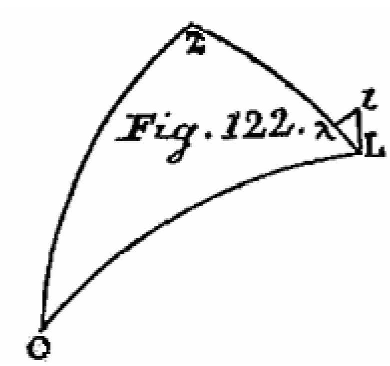
$$\sin\omega = \cos OZ = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n,$$

ideoque

$$\gamma'\sin\omega = x\cos l + y\cos m + z\cos n,$$

ita ut sit

$$-d\varphi = dt(x\cos l + y\cos m + z\cos n),$$



uti in corollario 1 invenimus. Deinde cum in triangulo *OZL* (Fig.122) sit $ZO = 90° - \omega$, $ZL = \rho$ et $OZL = 90°$, erit

$$\cos OL = \sin\omega\cos\rho, \quad \sin OLZ = \frac{\cos\omega}{\sin OL} \text{ et } \cos OLZ = \frac{\sin\ldots}{s\ldots}$$

ob $\cot OLZ = \frac{\sin\rho\sin\omega}{\cos\omega}$.

Quare si tempusculo *dt* punctum *L* circa O gyretur in *l*, erit $Ll = \gamma' dt \sin OL$ et angulus *OLl* rectus; hinc ducto circulo $l\lambda$ ad *Zl* perpendiculari fiet

$$\lambda = Ll\cos Z \; Ll = Ll\sin OLZ = \gamma' dt\cos\omega,$$

at est $L\lambda = -d\rho$ ideoque $d\rho = -\gamma' dt\cos\omega.$ Quae formula comparata cum ea, quam § 937 invenimus, dat

$$\gamma'\cos\omega = x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta;$$

at est

$$xx + yy + zz = \gamma'\gamma' = (x\cos l + y\cos m + z\cos n)^2 + (x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta)^2:$$

quae aequalitas per aequationem $xdl\sin l + ydm\sin m + zdn\sin n = 0$ confirmatur. Verum ne multitudine litterarum obruamur, evolvamus casum, quo axis cylindri longitudinalis simul est axis principalis.

PROBLEMA 114

940. Si corporis cylindrici axis longitudinalis per eius centrum inertiae ductus simul fuerit axis principalis idque super plano horizontali utcunque moveatur, definire eius motum.

SOLUTIO

Cum puncta *A* et *M* in unum incidant, bini reliqui poli principales *B* et *C* in circulo verticali *ZL* existent (Fig. 123), eritque propterea:

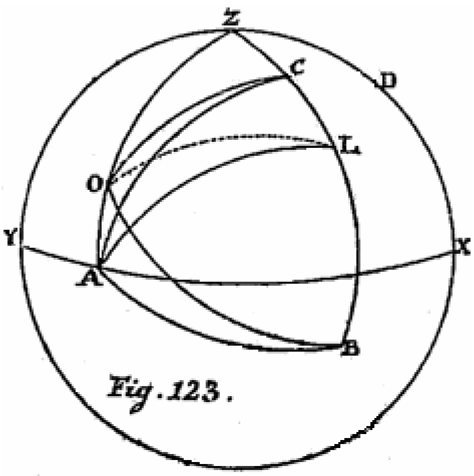


Fig. 123.

$$LA=\zeta=90°,\quad LB=\eta,\qquad LC=\vartheta=90°-\eta\;;$$
$$ZLA=\mathfrak{f}=90°,\quad ZLB=\mathfrak{g}=180°,\quad ZLC=\mathfrak{h}=0;$$

hincque

$$ZA=l=90°,\quad ZB=m=\eta+\rho$$

et

$$ZC=n=\rho-\vartheta=\eta+\rho-90°.$$

Quibus valoribus substitutis habebimus istas aequationes

$$\frac{\Pi}{M}=1+\frac{fdd\cdot cos\rho}{2gdt^2}$$
$$aadx+(cc-bb)yzdt=-\frac{2\Pi fg}{M}dt\,sin\,\rho,$$
$$bbdy+(aa-cc)xzdt=-\frac{2\Pi gs}{M}dt\,sin(\eta+\rho),$$
$$ccdz+(bb-aa)\,xydt=\frac{2\Pi gs}{M}dt\,cos(\eta+\rho),$$
$$ysin\,(\eta+\rho)-z\,cos\,(\eta+\rho)=0\;,$$
$$-\,xdtsin\,(\eta+\rho)=d\rho sin\,(\eta+\rho)$$
$$\text{seu}\;\;d\rho=-xdt\quad\text{et}\quad d\varphi=-dt\big(y\,cos\,(\eta+\rho)+z\,sin\,(\eta+\rho)\big).$$

Ponatur

$y=u\,cos\,(\eta+\rho)$ et $z=u\,sin\,(\eta+\rho)$,

ac pro *dt* scribatur $-\frac{d\rho}{x}$ seu $x=-\frac{d\rho}{dt}$, quo facto nostrae aequationes erunt

I. $\frac{\Pi}{M} = 1 + \frac{fdd \cdot cos\rho}{2gdt^2}$,

11. $-aadd\rho + \frac{1}{2}(cc-bb)\, uudt^2 \sin\, 2(\eta+\rho) + \frac{2\Pi}{M}\, fgdt^2 \sin\rho = 0$,

111. $bbdu\cos(\eta+\rho) - (aa+bb-cc)ud\rho\sin(\eta+\rho) = \frac{-2\Pi}{M} gsdt\sin(\eta+\rho)$,

1V. $ccdu\sin(\eta+\rho) + (aa-bb+cc)ud\rho\cos(\eta+\rho) = \frac{2\Pi}{M} gsdt\cos(\eta+\rho)$,

et

V. $d\varphi = -udt$.

Ex tertia et quarta eliminando s nanciscimur,

$$bbdu\cos^2(\eta+\rho) + ccdu\sin^2(\eta+\rho) - 2(bb-cc)\, ud\rho\sin(\eta+\rho)\cos(\eta+\rho) = 0,,$$

cuius integrale est

$$u = \frac{C}{bb+cc+(bb-cc)\cos(2\eta+2\rho)},$$

qui valor in 11. substitutus praebet

$$-2aadd\rho + \frac{CC(cc-bb)\, dt^2 \sin\, 2(\eta+\rho)}{\left(bb+cc+(bb-cc)\cos 2(\eta+\rho)\right)^2} + dt^2 \sin\rho\left(4fg + \frac{2ffdd \cdot \cos\rho}{dt^2}\right) = 0 \;\_$$

quae aequatio per $d\rho$ multiplicata et integrata dat,

$$-aad\rho^2 + \frac{\frac{1}{2}CC\, dt^2}{bb+cc+(bb-cc)\cos 2(\eta+\rho)} - 4fgdt^2\cos\rho - ffd\rho^2\sin^2\rho + Ddt^2 = 0$$

seu

$$d\rho^2\left(aa + ff\sin^2\rho\right) = dt^2\left(D - 4fg\cos\rho - \frac{\frac{1}{2}CC\, dt^2}{bb+cc+(bb-cc)\cos 2(\eta+\rho)}\right),$$

unde fit

$$dt = \frac{d\rho\sqrt{\left(aa+ff\sin^2\rho\right)\left(bb+cc+(bb-cc)\cos 2(\eta+\rho)\right)}}{\sqrt{\left((D-4fg\cos\rho)\left(bb+cc+(bb-cc)\cos 2(\eta+\rho)\right)-\frac{1}{2}CC\right)}},$$

Nunc dato tempore t per ρ pariter ac u, inde colligimus pressionem Π porroque intervalIum s ex hac aequatione

$$\frac{2\Pi}{M} gsdt = (cc-bb)du\sin(\eta+\rho)\cos(\eta+\rho) + aaud\rho + (cc-bb)ud\rho\cos 2(\eta+\rho).$$

Tum vero obtinemus

$$x = \frac{-d\rho}{dt},\quad y = u\cos(\eta+\rho) \text{ et } z = u\sin(\eta+\rho)$$

ac denique $\varphi = -\int udt$.

COROLLARIUM I

941. Si initio punctum *L* fuerit in *D* ut sit $ZD = \mathfrak{r}$ ibique quieverit, posito $t = 0$ erat $u = 0$, $\frac{d\rho}{dt} = 0$, ob $\gamma' = 0$; ideoque constantes ita definiri oportet, ut sit $C = 0$ et $D = 4fg\cos\mathfrak{r}$; unde fit

$$dt = d\rho\sqrt{\frac{aa + ff\sin^2\rho}{4fg(\cos\mathfrak{r} - \cos\rho)}},$$

sicque $\rho > \mathfrak{r}$. Porro est $u = 0$, hinc $\varphi = 0$ et $\gamma' = 0$; pressio autem Π hinc facile innotescit, et cum ρ ad 90° augeri possit, corpus quasi procumbet. Hic ergo motus neque a positione axium principalium *IB* et *IC* neque a radio basium cylindri *e* pendet.

COROLLARY 2

942. Si initio recta *IL* fuerit verticalis seu $\rho = 0$ et corpus circa eam gyrari coeperit celeritate angulari ε in sensum *AB*, ut fuerit *O* in *L* ideoque $\alpha = 90°$, $\beta = \eta$ et $\gamma = 90° - \eta$: initio erat

$$x = -\frac{d\rho}{dt} = 0, \quad y = \varepsilon\cos\eta \quad \text{et} \quad z = \varepsilon\sin\eta$$

Hinc fiunt constantes

$$C = \varepsilon\big(bb + cc + (bb - cc)\cos 2\eta\big) \quad \text{et} \quad D = 4fg + \tfrac{1}{2}\varepsilon\varepsilon(bb + cc + (bb - cc)\cos 2\eta),$$

unde colligitur

$$u = \frac{\varepsilon(bb + cc + (bb - cc)\cos 2\eta)}{bb + cc + (bb - cc)\cos 2(\eta + \rho)}$$

atque

$$\frac{d\rho^2(aa + ff\sin^2\rho)}{dt^2} = 4fg\big(1 - \cos\rho\big) + \frac{\frac{1}{2}\varepsilon\varepsilon(bb - cc)(bb + cc + (bb - cc)\cos 2\eta)\big(\cos 2(\eta + \rho) - \cos 2\eta\big)}{bb + cc + (bb - cc)\cos 2(\eta + \rho)}$$

COROLLARIUM 3

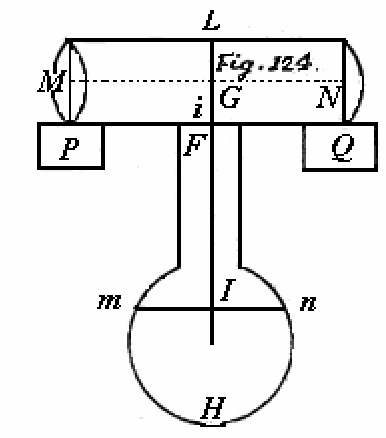
943. Si esset $bb = cc$, fieret

$$dt = d\rho\sqrt{\frac{aa + ff\sin^2\rho}{4fg(1 - \cos\rho)}}$$

et recta *I L* perpetuo maneret verticalis corpusque circa eam uniformiter gyrari pergeret; cum enim denominator contineat $\sqrt{(1 - \cos\rho)} = \sin\frac{1}{2}\rho\sqrt{2}$, nonnisi tempore elapso infinito arcus ρ finitus evaderet; quod idem evenit, si fuerit vel $\eta = 0$ vel $\vartheta = 0$, hoc est si recta *IL* fuerit axis principalis.

SCHOLION

944. Nisi axis longitudinalis simul sit axis principalis corporis, ob multitudinem literarum vix patet, quomodo formulae supra erutae generaliter evolvi queant, quod tamen inferius suscipiemus. Verum si huiusmodi corporum cylindricorum tantum motus quasi infinite parvos consideremus, ad quod necesse est, ut in recta centrali *LF* (fig. 118) centrum inertiae 1 infra centrum circuli *G* cadat corpusque infinite parum de statu quietis deturbetur, oscillationes vel vacillationes minimae orientur, quarum indolem ex formulis nostris generalibus determinare licebit. Hic non opus est, ut totum corpus sit cylindricum, sed sufficit, si eius termini circa *M* et *N* sint cylindrici, quibus super planis horizontalibus firmis *P* et *Q* sustentetur (Fig. 124), quin etiam sufficit, sitantum circa contactum utriusque termini figura fuerit cylindrica, siquidem motus tantum admittimus infinite parvos. Deinde inter sustentacula *P* et *Q* annexum esse potest corpus pendulum quodcunque *FmHn,* ut oriatur pendulum non circa axem fixum linearem, sed circa terminos cylindricos planis horizontalibus incumbentes mobile, euius motum oscillatorium definiri oporteat. In tali ergo pendulo primo notetur eius centrum inertiae *I*, per quod ducatur recta *mn* axi geometrico cylindri *MN* parallela, quae est axis longitudinalis iugiter manens horizontalis. Ducatur porro ex *I* ad *MN* recta perpendicularis *IGL*, quae si fuerit verticalis, corpus in quiete versabitur, ac si intervalIum *GI* ponamus = *f*, in superioribus formulis litteram *f* negative sumere debemus. Tum pro figura cylindrica terminorum sit radius basis = *e*, qui autem, ut vidimus, prorsus non in computum ingreditur, ita ut perinde sit sive termini sint crassiores sive graciliores. Quodsi recta *IG = f* minor fuerit quam *GF=e* totumque corpus supra sustentacula *P* et *Q* versetur, motus prodit similis ei, quo cunae agitari solent. Quicquid autem sit centrum inertiae *I*, perpetuo in eadem recta verticali manebit, unde tota investigatio ad motum gyratorium definiendum perducitur, in quo centrum inertiae *I* ut quiescens consideramus.



PROBLEMA 115

945. Si corpus, quod basibus cylindricis super planis horizontalibus incumbit, infinite parum de situ quietis deturbetur eique forte simul motus infinite parvus imprimatur, determinare motum vacillatorium, quo agitabitur.

SOLUTIO

In formulis nostris generalibus primo intervallum *GI* = *f* negativum statuatur, deinde arcus $ZL = \rho$, quo recta centralis *LGI* a situ verticali declinat (Fig. 121), ut infinite parvus spectari debet, perinde atque celeritas angularis γ' , unde quantitates
$x = \gamma' \cos\alpha, \; y = \gamma' \cos\beta, \; z = \gamma' \cos\gamma$ ut evanescentes tractari debent. Quomodocunque ergo

axes principales *IA*, *IB*, *IC* respectu rectae centralis *GI* et axis longitudinalis *mn* fuerint dispositi, quorum situs cum arcubus $LA=\zeta$, $LB=\eta$, $LC=\vartheta$, tum angulis $ZLA=\mathfrak{f}$, $ZLB=\mathfrak{g}$, $ZLC=\mathfrak{h}$, definitur, primo habebimus

$$\sin\rho=\rho,\quad \cos\rho=1,$$

deinde producta *xy*, *xz* et *yz* omitti poterunt; unde fiet

$$\cos l=\cos\zeta,\quad \cos m=\cos\eta \text{ et } \cos n=\cos\vartheta;$$

et aequationes solutionem continentes ex problemate 113 ob

$$\frac{\Pi}{M}=1-\frac{fdd\cdot\cos\rho}{2gdt^2}=1,$$

erunt :

$$\text{I. } aadx=2fg\rho dt\sin\mathfrak{f}\sin\zeta+2fgsdt\cos\mathfrak{f}\sin\zeta,$$
$$\text{II. } bbdy=2fg\rho dt\sin\mathfrak{g}\sin\eta+2gsdt\cos\mathfrak{g}\sin\eta,$$
$$\text{III. } ccdz=2fg\rho dt\sin\mathfrak{h}\sin\vartheta+2gsdt\cos\mathfrak{h}\sin\vartheta.$$

unde ex § 938 haec integralis est derivata

$$aaxx+bbyy+cczz=C-2fg\rho\rho-\frac{ff\rho\rho d\rho^2}{dt^2}$$

ob $\cos\rho=1-\frac{1}{2}\rho\rho$, quia hic infinite parvum $\rho\rho$ negligere non licet. Deinde habemus:

$$\text{IV. } y\cos\vartheta-z\cos\eta=-\frac{d\rho}{dt}\cos\mathfrak{f}\sin\zeta,$$
$$\text{V. } z\cos\zeta-x\cos\vartheta=-\frac{d\rho}{dt}\cos\mathfrak{g}\sin\eta,$$
$$\text{VI. } x\cos\eta-y\cos\zeta=-\frac{d\rho}{dt}\cos\mathfrak{h}\sin\vartheta$$

atque ex § 936 et §937

$$d\rho=-dt\left(x\sin\mathfrak{f}\sin\zeta+y\sin\mathfrak{g}\sin\eta+z\sin\mathfrak{h}\sin\vartheta\right),$$
$$d\varphi=-dt\left(x\cos\zeta+y\cos\eta+z\cos\vartheta\right).$$

cum nunc sit

$$\text{IV. } x+\text{V. } y+\text{VI. } z=0,$$

erit

$$x\sin\mathfrak{f}\sin\zeta+y\sin\mathfrak{g}\sin\eta+z\sin\mathfrak{h}\sin\vartheta=0.$$

Deinde ex I., II., III. in subsidium vocando formulas I. et II. ex § 934 colligimus

$$aax\cos\zeta+bby\cos\eta+ccz\cos\vartheta=A,$$

et pro intervallo *s* determinando

$$aadx\cos\mathfrak{f}\sin\zeta+bbdy\cos\mathfrak{g}\sin\eta+ccdz\cos\mathfrak{h}\sin\vartheta=2gsdt.$$

Statuamus $d\rho = -udt$ et $d\varphi = -vdt$,
atque ob

$$x \cos \mathfrak{f} \sin \zeta + y \cos \mathfrak{g} \sin \eta + z \cos \mathfrak{h} \sin \vartheta = 0$$

consequimur

$$x = u \sin \mathfrak{f} \sin \zeta + v \cos \zeta, \quad y = u \sin \mathfrak{g} \sin \eta + v \cos \eta, \quad z = u \sin \mathfrak{h} \sin \vartheta + v \cos \vartheta$$

hincque

$$A = u\left(aa \sin \mathfrak{f} \sin \zeta \cos \zeta + bb \sin \mathfrak{g} \sin \eta \cos \eta + cc \sin \mathfrak{h} \sin \vartheta \cos \vartheta\right)$$
$$+ v\left(aa \cos^2 \zeta + bb \cos^2 \eta + cc \cos^2 \vartheta\right).$$

Ponamus ad abbreviandum

$$bb \cos \mathfrak{h} \cos \eta \sin \vartheta - cc \cos \mathfrak{g} \sin \eta \cos \vartheta = \mathfrak{A}$$
$$cc \cos \mathfrak{f} \cos \vartheta \sin \zeta - aa \cos \mathfrak{h} \sin \vartheta \cos \zeta = \mathfrak{B}$$
$$aa \cos \mathfrak{g} \cos \zeta \sin \eta - bb \cos \mathfrak{f} \sin \zeta \cos \eta = \mathfrak{C}$$

eritque

$$\mathfrak{A} \cos \mathfrak{f} \sin \zeta + \mathfrak{B} \cos \mathfrak{g} \sin \eta + \mathfrak{C} \cos \mathfrak{h} \sin \vartheta = 0$$

$$aa \cos^2 \zeta + bb \cos^2 \eta + cc \cos^2 \vartheta = \mathfrak{D}$$

$$aa \sin \mathfrak{f} \sin \zeta \cos \zeta + bb \sin \mathfrak{g} \sin \eta \cos \eta + cc \sin \mathfrak{h} \sin \vartheta \cos \vartheta = \mathfrak{F}$$

et habebimus

$$v = \frac{A - \mathfrak{F}u}{\mathfrak{D}}, \quad x = \frac{A \cos \zeta + \mathfrak{A}u}{\mathfrak{D}}, \quad y = \frac{A \cos \eta + \mathfrak{B}u}{\mathfrak{D}}, \quad z = \frac{A \cos \vartheta + \mathfrak{C}u}{\mathfrak{D}},$$

qui valores, in aequatione integrali vim vivam eomplectente substituti, ob
$\frac{d\rho}{dt} = -u$ dabunt:

$$\frac{AA\mathfrak{D} + 2Au\left(\mathfrak{A}a^2 \cos \zeta + \mathfrak{B}b^2 \cos \eta + \mathfrak{C}c^2 \cos \vartheta\right) + uu\left(\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2\right)}{\mathfrak{D}\mathfrak{D}} = C - 2fg\rho\rho - ff\rho\rho uu$$

quae aequatio ob

$$\mathfrak{A}aa \cos \zeta + \mathfrak{B}bb \cos \eta + \mathfrak{C}cc \cos \vartheta = 0$$

abit in hanc

$$AA\mathfrak{D} + \left(\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2\right)uu = C\mathfrak{D}\mathfrak{D} - 2\mathfrak{D}\mathfrak{D}fg\rho\rho - \mathfrak{D}\mathfrak{D}ff\rho\rho uu$$

ubi, si loco $C\mathfrak{D}\mathfrak{D} - AA\mathfrak{D}$ ponatur $B\mathfrak{D}\mathfrak{D}$, fiet

$$u = \frac{\mathfrak{D}\sqrt{(B-2fg\rho\rho)}}{\sqrt{(\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2+\mathfrak{D}\mathfrak{D}ff\rho\rho)}}$$

Statuamus porro

$$\mathfrak{A}^2a^2+\mathfrak{B}^2b^2+\mathfrak{C}^2c^2=\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H},$$

et reiecto termino infinite parvo $\mathfrak{D}\mathfrak{D}ff\rho\rho$ habebimus

$$u=\frac{\mathfrak{D}\sqrt{(B-2fg\rho\rho)}}{\mathfrak{H}} \quad \text{et} \quad dt=\frac{-\mathfrak{H}d\rho}{\sqrt{(B-2fg\rho\rho)}},$$

unde colligimus

$$t=\text{Const.}+\frac{\mathfrak{H}}{\sqrt{2fg}}A\cos\frac{\rho\sqrt{2fg}}{\sqrt{B}}$$

seu

$$\rho=\frac{\sqrt{B}}{\sqrt{2fg}}\cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}} \quad \text{et} \quad u=\frac{\sqrt{B}}{\mathfrak{H}}\sin\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}$$

tum vero

$$v=\frac{A}{\mathfrak{D}}-\frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\mathfrak{H}}\sin\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}};$$

hincque

$$\varphi=D-\frac{At}{\mathfrak{D}}-\frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\sqrt{2fg}}\cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}=D-\frac{At}{\mathfrak{D}}-\frac{\mathfrak{F}\rho}{\mathfrak{D}}.$$

Deinde reperiemus:

$$s=\frac{-\sqrt{B}\mathfrak{f}}{\mathfrak{D}\mathfrak{H}\mathfrak{H}\sqrt{2g}}\left(aabb\sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta+aacc\sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta+bbcc\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta\right)\cos\frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}.$$

Denique vero erit

$$\gamma'\gamma'=xx+yy+zz=\frac{AA-2A\mathfrak{F}u+(\mathfrak{A}\mathfrak{A}+\mathfrak{B}\mathfrak{B}+\mathfrak{C}\mathfrak{C})uu}{\mathfrak{D}\mathfrak{D}}$$

sicque omnia ad datum tempus sunt definita. Ceterum hie notasse iuvat esse $\mathfrak{A}\mathfrak{A}+\mathfrak{B}\mathfrak{B}+\mathfrak{C}\mathfrak{C}=\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F}$, ita ut sit $\gamma'\gamma'=vv+uu$.

COROLLARIUM 1

946. Cum sit

$$\rho = \frac{\sqrt{B}}{\sqrt{2fg}} \cos \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}},$$

patet arcum $ZL = \rho$ seu declinationem rectae *LI* a situ verticali ad similitudinem penduli variari huiusque lineae *LI* vacillationes isochronas fore oscillationibus penduli, cuius longitudo est = $\frac{\mathfrak{H}\mathfrak{H}}{\mathfrak{f}}$, quae longitudo est

$$= \frac{\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2}{\mathfrak{D}\mathfrak{D}\mathfrak{f}}.$$

COROLLARIUM 2

947. Deinde cum sit

$$\varphi = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\rho}{\mathfrak{D}}$$

punctum *L* motu medio revolvitur circa verticem *Z* celeritate angulari = $\frac{A}{\mathfrak{D}}$;

verum locus medius corrigi debet particula $\frac{\mathfrak{F}\rho}{\mathfrak{D}}$. Sin autem sit constans $A = 0$, angulus *DZL* parumper mutatur, nisi sit $\mathfrak{F} = 0$.

COROLLARIUM 3

948. Si ergo revolutiones corporis circa axem verticalem *IZ* excludantur, ut sit $A = 0$, atque initio fuerit $\varphi = 0$, $\rho = \mathfrak{r}$ et celeritas angularis $\gamma' = \varepsilon$, constantes ita definientur, ut sit

$$D = \frac{\mathfrak{F}\mathfrak{r}}{\mathfrak{D}}, \quad \mathfrak{r} = \frac{\sqrt{B}}{\sqrt{2fg}} \cos \frac{\delta\sqrt{2fg}}{\mathfrak{H}} \quad \text{et} \quad \varepsilon\varepsilon = \frac{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})B}{\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}} \left(\sin \frac{\delta\sqrt{2fg}}{\mathfrak{H}} \right)^2.$$

Ergo

$$\sqrt{B} = \frac{\mathfrak{r}\sqrt{2fg}}{\cos \frac{\delta\sqrt{2fg}}{\mathfrak{H}}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{\sin \frac{\delta\sqrt{2fg}}{\mathfrak{H}} \sqrt{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}}$$

ideoque

$$\text{tang} \frac{\delta\sqrt{2fg}}{\mathfrak{H}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{\mathfrak{r}\sqrt{2fg(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}}$$

unde et constans *B* innotescit. Sin autem fuerit $\varepsilon = 0$, prodit

$$\sqrt{B} = \mathfrak{r}\sqrt{2fg} \quad \text{et} \quad \delta = 0.$$

EXEMPLUM

949. Ponamus rectam *IM*, quae per centrum inertiae *I* axi geometrico cylindri (*MN* fig. 124) parallela ducitur, simul esse corporis axem principalem, et habebimus uti § 940
$\mathfrak{f} = 90°$, $\mathfrak{g} = 180°$, $\mathfrak{h} = 0$ et $\zeta = 90°$ atque $\vartheta = 90° - \eta$. Hinc autem colligimus:

$\mathfrak{A} = bb\cos^2\eta + cc\sin^2\eta$, $\mathfrak{B} = 0$, $\mathfrak{C} = 0$, $\mathfrak{D} = \mathfrak{A}$ et $\mathfrak{F} = 0$

ergo $aa = \mathfrak{H}\mathfrak{H}$, unde longitudo penduli simplicis isochroni fit $= \frac{aa}{\mathfrak{f}}$. Tum vero axis *IA* horizontalis manebit immotus. Ac si initio, ubi $\rho = \mathfrak{r}$, corpus motum a quiete inceperit, erit $\delta = 0$ et $\sqrt{B} = \mathfrak{r}\sqrt{2fg}$ ex quibus reliquae quantitates variabiles colliguntur

$$\rho = \mathfrak{r}\cos\frac{t\sqrt{2fg}}{a}, \quad u = \frac{\mathfrak{r}\sqrt{2fg}}{a}\sin\frac{t\sqrt{2fg}}{a}, \quad v = 0,$$

ob

$$A = 0 \quad \text{et} \quad x = u = \frac{\mathfrak{r}\sqrt{2fg}}{a}\sin\frac{t\sqrt{2fg}}{a}, \quad y = 0 \quad \text{et} \quad z = 0 \quad \text{atque} \quad \gamma' = x.$$

Revera autem adiuncto motu progressivo centrum inertiae *I* in recta verticali alternatim ascendet ae descendet, cylindro superiore *MN* hunc motum sequente, dum super planis *P* et *Q* liberrime incedere potest, neque a frictione impediri assumitur.

SCHOLION

950. Quia magnitudo cylindri *MN* in computum non ingreditur, eadem solutio valebit, si eius crassities evaneseat corpusque annexum ab axe lineari esset suspensum. Ex quo hic motus convenire debere videtur cum motu oscillatorio supra definito, quod tamen longe aliter usu venit, quoniam pro motu oscillatorio vero longitudo penduli simplicis isochroni prodiit $= f + \frac{aa}{f} = \frac{aa+ff}{f}$, cum hic tantum sit $= \frac{aa}{f}$. Causa huius discriminis in eo est sita, quod supra in doctrina oscillationum axem *MN* fixum assumsimus, dum hic liberrime mobilis statuitur. Hinc patet, ob libertatem axis, etsi plano horizontali incumbat, oscillationes multo promptiores fieri, quam si axis in eodem loco firmiter detineretur. Atque hoc etiam Theoriae omnino est conforme; si enim (fig. 118) circulus ΠMTN planum semper in eodem puncto *T* tangere debeat, praeter pressionem Π vis quaedam horizontalis in calculum introduci debet, quae si ponatur = Θ secundum *TH* urgens, ut punctum *T* maneat constans, ob $TP = f\sin\rho$ esse oportet

$$\frac{fdd\sin\rho}{dt^2} = -\frac{2\Theta g}{M}.$$

Ex hac autem vi quoque nascitur momentum respectu axium principalium, qua propterea motus gyratorius afficitur, ut talis prodeat, qualem supra in motus oscillatorii investigatione determinavimus. Ceterum hic probe notasse iuvabit, si axiculi penduli planis horizontalibus politissimis incumbant, motum oscillatorium plurimum discrepare posse ab eo, qui oriretur, si firmiter detinerentur, et multo quidem promptiorem esse futurum. Minima autem frictio hoc discrimen tollere motumque ad oscillationum legem reducere valebit. Huius autem problematis solutio nos ad solutionem problematis generalis no. 113 manuducet.

PROBLEMA 116

951. Si corpus cylindricum quodcunque super plano horizontali moveatur utcunque, aequationes supra inventas, quibus eius motus definitur, resolvere atque ad integrationem perducere.

SOLUTIO

Maneant hic omnia, uti supra in problemate 113 sunt constituta, atque in recta centrali *LIGF* sumamus ut ibi centrum inertiae *I* a puncto *F* magis remotum, quam centrum sectionis cylindri *G*, ponendo intervalIum $GI = f$ (Fig. 121). Ex aequationibus igitur differentialibus ibi exhibitis iam unam aequationem integralem eruimus, quae est:

$$aaxx + bbyy + cczz = C - 4fg\cos\rho - \frac{ffd\rho^2 \sin^2\rho}{dt^2}.$$

Praeterea vero ternae priores aequationes ope ternarum posteriorum in postremis terminis applicatarum abeunt in has formas

$$aadx + (cc - bb)\, yzdt = -\frac{2\Pi fg}{M} \cdot dt\sin\mathfrak{f}\sin\zeta\sin\rho - \frac{2\Pi gs}{M} \cdot \frac{dtdl\sin l}{d\rho},$$
$$bbdy + (aa - cc)\, xzdt = -\frac{2\Pi fg}{M} \cdot dt\sin\mathfrak{g}\sin\eta\sin\rho - \frac{2\Pi gs}{M} \cdot \frac{dtdm\sin m}{d\rho},$$
$$ccdz + (bb - aa)\, xydt = -\frac{2\Pi fg}{M} \cdot dt\sin\mathfrak{h}\sin\vartheta\sin\rho - \frac{2\Pi gs}{M} \cdot \frac{dtdn\sin n}{d\rho}.$$

Hinc iam colligatur forma $\text{I.}\cos l + \text{II.}\cos m + \text{III.}\cos n$, et quia

$$dl\sin l\cos l + dm\sin m\cos m + dn\sin n\cos n = 0,$$

termini ultimi intervalIum *s* involventes se destruent; tum vero etiam per relationes § 934 traditas reperitur

$$\sin\mathfrak{f}\sin\zeta\cos l + \sin\mathfrak{g}\sin\eta\cos m + \sin\mathfrak{h}\sin\vartheta\cos n = 0,$$

ita ut quoque penultimi tollantur. Quocirca perveniemus ad hanc aequationem

$$aadx\cos l + bbdy\cos m + ccdz\cos n + aaxzdt\cos m + bbxydt\cos n$$
$$+ ccyzdt\cos l - aaxydt\cos n - bbyzdt\cos l - ccxzdt\cos m = 0,$$

at ex ternis posterioribus est

$$z\cos m - y\cos n = -\frac{dl\sin l}{dm\sin m}, \quad x\cos n - z\cos l = -\frac{dm\sin m}{dt},$$
$$y\cos l - x\cos m = -\frac{dn\sin n}{dt},$$

quibus valoribus substitutis obtinemus

$$aadx\cos l + bbdy\cos m + ccdz\cos n$$
$$-\ aaxdl\sin l - bbydm\sin m - cczdn\sin n = 0,$$

cuius integralis est

$$aax\cos l + bby\cos m + ccz\cos n = D.$$

Deinde loco *x*, *y* et *z* introducamus novas variabiles hinc definiendas

$$x\cos\zeta + y\cos\eta + z\cos\vartheta = p\ ,$$
$$x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\vartheta = q\ ,$$
$$x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\vartheta = r\ ,$$

eritque primo $d\rho = -rdt$; porro ob

$$x\cos l + y\cos m + z\cos n = p\cos\rho + q\sin\rho\ ,$$

erit $d\varphi = -dt(p\cos\rho + q\sin\rho)$. Praeterea ob

$$xdl\sin l + ydm\sin m + zdn\sin n = 0$$

fit $p\sin\rho - q\cos\rho = 0$. Quamobrem ponamus $p = u\cos\rho$ et $q = u\sin\rho$ eritque

$$d\varphi = -udt \quad \text{et} \quad d\rho = -rdt\ ,$$

at ex illis aequationibus assumtis elicimus

$$x = r\sin\mathfrak{f}\sin\zeta + u\cos l\ ,$$
$$y = r\sin\mathfrak{g}\sin\eta + u\cos m,$$
$$z = r\sin\mathfrak{h}\sin\vartheta + u\cos n$$

hincque

$$xx + yy + zz = rr + uu = \gamma'\gamma'.$$

Nunc aequatio integralis modo ante inventa praebet

$$D = r(aa\sin\mathfrak{f}\sin\zeta\cos l + bb\sin\mathfrak{g}\sin\eta\cos m + cc\sin\mathfrak{h}\sin\vartheta\cos n)$$
$$+\ u(aa\cos^2 l + bb\cos^2 m + cc\cos^2 n),$$

qua *u* determinatur per *r* et ρ ; ideoque et *x*, *y*, *z*. Denique aequatio integralis primum inventa

$$aaxx + bbyy + cczz = C - 4fg\cos\rho - ffrr\sin^2\rho,$$

quia tantum *r* et ρ continet, determinabit *r* per ρ , indeque aequatio

$dt = -\frac{d\rho}{r}$ pro dato tempore *t* omnes quantitates motum continentes manifestabit. Quodsi ad abbreviandum ponantur constantes:

$$\begin{aligned}
&aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta = \mathfrak{A},\\
&aa\cos\mathfrak{f}\sin\zeta\cos\zeta + bb\cos\mathfrak{g}\sin\eta\cos\eta + cc\cos\mathfrak{h}\sin\vartheta\cos\vartheta = \mathfrak{B},\\
&aa\cos^2\mathfrak{f}\sin^2\zeta + bb\cos^2\mathfrak{g}\sin^2\eta + cc\cos^2\mathfrak{h}\sin^2\vartheta = \mathfrak{C},\\
&aa\sin\mathfrak{f}\sin\zeta\cos\zeta + bb\sin\mathfrak{g}\sin\eta\cos\eta + cc\sin\mathfrak{h}\sin\vartheta\cos\vartheta = \mathfrak{D},\\
&aa\sin\mathfrak{f}\cos\mathfrak{f}\sin^2\zeta + bb\sin\mathfrak{g}\cos\mathfrak{g}\sin^2\eta + cc\sin\mathfrak{h}\cos\mathfrak{h}\sin^2\vartheta = \mathfrak{E},\\
&aa\sin^2\mathfrak{f}\sin^2\zeta + bb\sin^2\mathfrak{g}\sin^2\eta + cc\sin^2\mathfrak{h}\sin^2\vartheta = \mathfrak{F},
\end{aligned}$$

nostrae aequationes integrales erunt

$$\begin{aligned}
&D = r(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho) + u(\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho - t + \mathfrak{C}\sin^2\rho),\\
&C - 4fg\cos\rho - ffrr\sin^2\rho = \mathfrak{F}rr + 2ru(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)\\
&\qquad + uu(\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho),
\end{aligned}$$

ex quibus concluditur

$$rr = \frac{DD - (\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho)(C - 4fg\cos\rho)}{(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)^2 - (\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho)(\mathfrak{F} + \mathfrak{ff}\sin^2\rho)}.$$

Hinc pro tempore adipiscimur $t = \int \frac{-d\rho}{r}$,
et cum sit

$$u = \frac{D - r(\mathfrak{D}\cos\rho + \mathfrak{E}\sin\rho)}{\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho},$$

erit angulus

$$\varphi = -\int u\,dt = \int \frac{u\,d\rho}{r}.$$

Cum autem ad quodvis tempus *t* tam arcum ρ quam angulum φ determinaverimus, totus motus erit perfecte cognitus.

COROLLARIUM 1

952. Quantitates ergo $\mathfrak{A}, \mathfrak{C}$ et $\mathfrak{F}$ necessario sunt positivae et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{C}$ ita refertur, ut sit

$$\mathfrak{AC} - \mathfrak{BB} = aabb\sin^2\mathfrak{h}\sin^2\vartheta + aacc\sin^2\mathfrak{g}\sin^2\eta + bbcc\sin^2\mathfrak{f}\sin^2\zeta,$$

unde patet formam

$$\mathfrak{A}\cos^2\rho + 2\mathfrak{B}\sin\rho\cos\rho + \mathfrak{C}\sin^2\rho$$

in duos factores simplices resolvi non posse.

COROLLARIUM 2

953. Ex hac solutione generali casus in praecedente problemate evolutus facile deducitur sumendo f negative et arcum ρ infinite parvum, unde fit

$$rr = \frac{DD - \mathfrak{A}C - 4\mathfrak{A}fg\cos\rho}{\mathfrak{D}\mathfrak{D} - \mathfrak{A}\mathfrak{F}} = \frac{Const. + 4\mathfrak{A}fg\cos\rho}{\mathfrak{A}\mathfrak{F} - \mathfrak{D}\mathfrak{D}}.$$

Reperitur autem valoribus evolutis

$$\mathfrak{A}\mathfrak{F} - \mathfrak{B}\mathfrak{B} = aabb\cos^2\mathfrak{h}\sin^2\vartheta + aacc\cos^2\mathfrak{g}\sin^2\eta + bbcc\cos^2\mathfrak{f}\sin^2\zeta,$$

unde longitudo penduli simplicis isochroni simplicius quam supra ita exhibetur,
ut sit =

$$\frac{aabb\cos^2\mathfrak{h}\sin^2\vartheta + aacc\cos^2\mathfrak{g}\sin^2\eta + bbcc\cos^2\mathfrak{f}\sin^2\zeta}{f\left(aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta\right)}.$$

SCHOLION

954. His de motu corporum cylindricorum super plano horizontali expeditis instituеram pauca de motu super plano inclinato adiungere; verum si motus fuerit simplex, res nullam habet difficultatem, sin autem sit complicatus, in calculos incommodos incideremus. Quare cum in praxi frictionem ab his motibus separare haud liceat, motus saltem simpliciores super plano inclinato ita pertractabimus, ut simul frictionis rationem habeamus, ex quo peculiarem tractatum de motu corporum rigidorum a frictione perturbato adiungi conveniet.