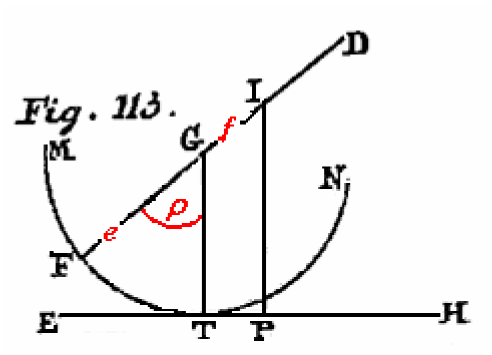
Chapter 18

CONCERNING THE MOTION OF BODIES WITH A SPHERICAL BASE ON A HORIZONTAL PLANE

PROBLEM 104

884. If a body with a given spherical base presses on a horizontal plane in some manner, to define the forces which are acting, and the effect of those on the motion of the body in the progressive disturbance.

SOLUTION



Let *EH* be a horizontal plane (Fig. 113) and *T* the point, where the body presses on that plane, but in the body first is noted the centre of the spherical base *MTN*, which is at *G*, then the centre of inertia of the body *I*, and the diagram represents a plane, on which these three points are situated. The radius *GT* is drawn, which since it is normal to the horizontal *EH*, has a vertical position, and likewise the plane *TGI* is itself vertical. Now, since for the progressive motion the whole mass of the body, which is equal to *M*, can be considered as gathered together at the centre of inertia *I*, with *IP* itself parallel to *GT*, then the body in the first place on account of gravity is urged in the direction *IP* by a force equal to *M* [at this time the distinction between mass and weight was apparent only in the equation relating force, mass, and acceleration, as in the equation below]; then where the body makes contact with the horizontal plane at *T*, it is urged upwards by that in a direction by a certain force in the direction *TG*, by a pressing [*i.e.* reactive or contact] force equal to Π . Whereby unless these two forces cancel each other out, then the body is unable to remain if a position of rest ; from which it is evident that the state of rest demands that the body presses on the horizontal plane through the line *GI* produced to *F* and thus the line *DIGF* becomes vertical. Hence the figure represents an inclined position of the body, and the inclination is indicated by the angle *FGT*, which is equal to ρ , with which vanishing the body becomes a position of equilibrium. Again we may put the radius of the spherical *GF* = *GT* = *e* and the distance between the points *G* and *I* clearly *GI* = *f*, in as much as the centre of inertia *I* is at a greater distance from the point *F* than the centre of the figure *G;* thus in order that, if the body falls on its own account, the quantity *f* cannot be taken as negative. Hence therefore the distance *IP* is equal to $e + f\cos\rho$ **,** which is the height of the centre of inertia *I* above the horizontal plane *EH* and which is only affected by the forces acting. But on translating the force $TG = \Pi$ to the centre of inertia *I*, the point *I* is acted on by a downwards force equal to $M - \Pi$; and because the downwards speed of this is equal to

$$\frac{f\,d\rho\sin\rho}{dt}$$

on putting the speed equal to *u* then

$$du = \frac{2g(M - \Pi)dt}{M}$$

with *dt* denoting the element of time, from which there is obtained :

$$f(dd\rho\, sin\,\rho + d\rho^2\, cos\,\rho) = 2g\left(1\ - \frac{\Pi}{M}\right)dt^2$$

on taking *dt* constant; and neither is it influenced by any other motion.

COROLLARY 1

885. Hence in turn, if an account of the progressive motion is given or perhaps it can be considered as given, then the pressing force Π can be defined, since the equation becomes

$$\frac{\Pi}{M} = 1 - \frac{f(dd\rho\, sin\,\rho + d\rho^2\, cos\,\rho)}{2gdt^2} \quad \text{or} \quad \frac{\Pi}{M} = 1 + \frac{fdd \cdot cos\,\rho}{2gdt^2}.$$

COROLLARY 2

886. If it is the case that $f = 0$ or the centre of inertia *I* falls on the spherical centre G, then $\Pi = M$ is produced, and the body is content to remain properly in all situations in a position of equilibrium.

COROLLARY 3

887. If it should be that $f > 0$ or $F1 > FG$, and suddenly the body is inclined a little, the inclination is increased by the force acting, but if it should be that $f < 0$ or $FI < FG$, then the inclination is diminished and the body is restored to the position of equilibrium, at which the point *F* stands on the plane: while in the one case it leans forward, in the other it seeks to return to the position of equilibrium.

SCHOLIUM 1

888. But whichever figure the body should have, in that there are always at least two positions of equilibrium given, of which the one is put in place in order that, if the body is moved away from that momentarily, then the body at once restores itself to its original position, but the other, so that without doubt it slips and falls forward; the first of which is to be called the *position of stable equilibrium*, and the second is usually called the *position of slipping*. For however the body rests on the horizontal plane, it is moving about equilibrium if the line drawn from the centre of inertia to the point of contact is vertical ; which can always come about at least in two ways. For if from the centre of inertia to all the points of the [hemispherical] surface straight lines are considered drawn, because none of these either vanish or are infinite, among these there is given both a maximum and a minimum [height of I] , but with each normal to the tangent plane ; whereby if the body rests on the horizontal plane from each of these points, from which the centre of inertia is either a maximum or a minimum distance, with the line drawn from the centre of inertia to the point of contact vertical and thus giving a position of equilibrium, and that is stable if that line is a minimum [length], and indeed if the opposite for slipping then a maximum ; from which it is understood that the centre of inertia always seeks the lowest point, where it may come to rest. But repeatedly more positions of equilibrium are given, some of stability and others of slipping, which follow each other alternately, since it is necessary that the body goes from a position of slipping to arrive at a position of stability.

SCHOLIUM 2

889. In the present case, in which we have put in place a spherical surface of the body, a line drawn through the centre of inertia I and the centre of the figure G gives these two points F and D, from which if the body rests on a horizontal plane, it can maintain a position of equilibrium ; and while it is a tangent to the plane at the point F, the position of equilibrium is stable, if $FI < FG$ or $f < 0$, but a position of slipping [or the top is inverted], if $FI > FG$ or $f > 0$; and besides these two positions of equilibrium no other is given, unless it happens that $f = 0$, in which case at once all the positions clearly take on the equilibrium property. But if I consider here the whole surface of the body as spherical, yet it is sufficient for our purpose, if at least that part should be spherical, during which it is in contact with the horizontal plane ; and hence it is apparent also from that treatment of these tops above, that the lower end of the axis is not a point as we have assumed before, but ends in a hemisphere, or instead it can be formed from a smaller spherical segment, thus so that hence the above shape is produced [as in Ch.17], if the radius of the sphere $GF = e$ vanishes, and thus this discussion includes the above within itself. Therefore the straight line $DIGF$ drawn through the centre of inertia I and the centre of the spherical base G shows the special axis of the top, which indeed, as tops are accustomed to be made, likewise is one principal axis of the body, and indeed the two remaining moments of inertia are had equal, thus in such a form as we set up above. Now in order that this discussion appears wider, and likewise it is able to be adapted to the staggering motion of bodies provided with some kind of spherical bases, I will consider the principal axes of the body somehow to be different from the special axis DF , and I will investigate the moments of the forces about these axes.

[Thus Euler had found a generalised form of the tippy top, still of interest to everyone for fun, as well as being a subject for more serious thought. Such tops have two orientations of equilibrium as above, and the top can flip from one to the other repeatedly.See, e. g. fysikbasin.dk ; Euler's top in general is asymmetric, leading to the stumbling motion that he refers to, but for which he cannot produce detailed mathematical solutions.]

PROBLEM 105

890. With the pressing force given Π, by which the body provided with the spherical base leans on the horizontal plane, to define the moments hence arising about the principal axes of the body, by whatever reasoning these have been disposed about the special axes of the body.

SOLUTION

With a sphere described about the centre of inertia of the body *I* (Fig.l14), let *Z* be the vertical point of this, and the special axis now holds the position *DIG*, in order that the declination of this from the position of the vertical is given by $DZ = \rho$. Hence since the direction of the pressing force Π is vertical and passing through the point *G* with *IG* = *f*, the vertical straight line $G\Pi$ refers to this force equal to Π, thus in order that $ZDG\Pi$ is a vertical plane, in which the force $G\Pi = \Pi$ is resolved along the directions *GI* and *GV*, which are normal to each other, and on account of the angle $DG\Pi = \rho$ the force along $GI = \Pi \cos\rho$ and the force along $GV = \Pi \sin\rho$, of which that passing through the centre of inertia *I* furnishes no moments. Now let

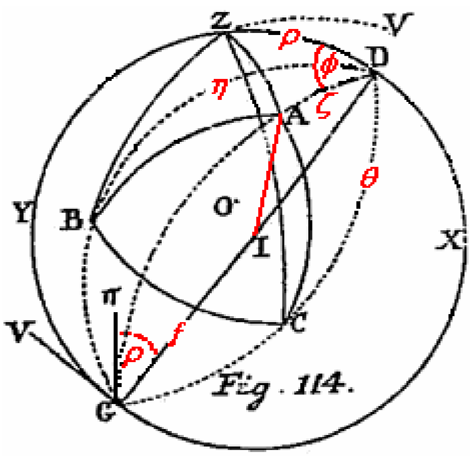


Fig . 114.

IA, *I B*, *IC* be the three principal axes of the body, holding a given position on account of the special axis *ID*, and from *D* the semicircles *DAG, DBG, DCG* are drawn through the points *A, B,* C. But if the axis *IA* should be normal to the plane *IGV*,[then the arc *GA* would be a quadrant, and] the moment of the force GV about this would be equal to $\Pi f \sin\rho$, but now [*i.e.* otherwise, $\Pi \sin\rho$ must be resolved into components parallel and perpendicular to *IA*, (according to the inclination of the segment *GIAZY* to the horizontal, the angle of which is VGA or ZGA) : *i. e.* $\Pi \sin\rho \cos VGA$ and $\Pi \sin\rho \sin VGA$, while the perpendicular distance of *G* from *IA*, or $f \sin AG$ must be introduced],
since it must be diminished in the ratio of the whole sine to the sine of the arc *GA* as well as by the sine of the angle *VGA*, thus in order that from the pressing force there arises :

the moment about the axis $IA = \Pi f \sin\rho \cdot \sin GA \cdot \sin VGA$ in the sense CB,
the moment about the axis $IB = \Pi f \sin\rho \cdot \sin GB \cdot \sin VGB$ in the sense AC,
the moment about the axis $1C = \Pi f \sin\rho \cdot \sin GC \cdot \sin VGC$ in the sense BA.

But we have indicated these three moments above by the letters *P, Q, R* [Problem 88, Ch. 15, § 803] since indeed they are set up to act in the opposite sense, whereby with everything transferred to the point *D* we have:

$$P = -\Pi f \sin DZ \cdot \sin DA \cdot \sin ZDA = -\Pi f \sin ZD \cdot \sin ZA \cdot \sin DZA,$$
$$Q = -\Pi f \sin DZ \cdot \sin DB \cdot \sin ZDB = -\Pi f \sin ZD \cdot \sin ZB \cdot \sin DZB,$$
$$R = -\Pi f \sin DZ \cdot \sin DC \cdot \sin ZDC = -\Pi f \sin ZD \cdot \sin ZC \cdot \sin DZC .$$

COROLLARY 1

891. Here we have assumed the centre of the base *G* to be closer to the lower end *F* than the centre of inertia *I;* but if it comes about otherwise, so that the interval *FI* is less than the interval *FG* = *e,* then the interval *GI* = *f* must be taken as negative. But if it should be that *GI* = 0, then the moments found vanish or the body is always held in a state of equilibrium.

COROLLARY 2

892. IF the arcs

$$AD = \zeta, \quad BD = \eta, \quad CD = \vartheta,$$

are put in place for the position of the special axis *ID* with respect to the principal axes of the body, then the angle $ZDA = \varphi$, with the arc present $ZD = \rho$, [see Ch.14, Problem 72 for some notes on solving spherical triangles, recall here that *AB*, *BC*, and *CA* are quadrants] on account of which

$$\cos ADB = \frac{\cos\zeta \cos\eta}{\sin\zeta \sin\eta} \quad \text{and} \quad \sin ADB = -\frac{\cos\vartheta}{\sin\zeta \sin\eta},$$

since

$$\sin ADB : \sin DAB \text{ or } \sin ADB : -\cos DAC = 1 : \sin BD = 1 : \sin\eta,$$

then

$$\sin ZDB = \frac{-\cos\zeta \cos\eta \sin\phi + \cos\vartheta \cos\varphi}{\sin\zeta \sin\eta},$$

but

$$\cos DAC = \frac{\cos CD}{\sin AD} = \frac{\cos\vartheta}{\sin\zeta},$$

and thus

$$P = -\Pi f \sin\rho \sin\zeta \sin\varphi,$$

and

$$Q = \frac{\Pi f \sin\rho(\cos\zeta \cos\eta \sin\phi - \cos\vartheta \cos\varphi)}{\sin\zeta},$$

and also

$$R = \frac{\Pi f \sin\rho(\cos\zeta \cos\vartheta \sin\phi - \cos\eta \cos\varphi)}{\sin\zeta}.$$

COROLLARY 3

893. If the special axis *ID* is congruent with the principal axis *IA*, then there arises $C = \vartheta$ and $\eta = \vartheta = 90°$, as it the case that $\cos\eta = \cos\vartheta = \sin\zeta$ and the angle φ remains indefinite. But from the first formulas the moments of the forces become:

$$P = 0, \quad Q = -\Pi f \sin\rho \sin ZAB \quad \text{and} \quad R = -\Pi f \sin\rho \sin ZAC$$

or

$$P = 0, \quad Q = \Pi f \sin ZC \quad \text{and} \quad R = -\Pi f \sin ZB.$$

COROLLARY 4

894. But if now as above we may put

$$ZA = l, \; ZB = m \quad \text{and} \quad ZC = n,$$

we can find the moments of the forces in general and
and

$$P = \Pi f(\cos\vartheta \cos m - \cos\eta \cos n),$$

$$Q = \Pi f(\cos\zeta \cos n - \cos\vartheta \cos l)$$

and also

$$R = \Pi f(\cos\eta \cos l - \cos\zeta \cos m),$$

from which it follows at once that, $\sin\zeta = 0$ and $\eta = \vartheta = 90°$..

EXPLANATION

895. An account of the derivation of these latter formulas may be presented thus: in the first place there is

$$\sin DZ \cdot \sin ZDA = \sin ZA \cdot \sin ZAD,$$

then

$$P = -\Pi f \sin DA \cdot \sin ZA \cdot \sin ZAD;$$

but

$$ZAD = BAD - BAZ,$$

and

$$\sin BAD = -\cos CAD = \frac{-\cos\vartheta}{\sin DA}, \quad \cos BAD = \frac{\cos\eta}{\sin DA},$$

$$\sin BAZ = -\cos CAZ = \frac{-\cos n}{\sin ZA}, \quad \cos BAZ = \frac{\cos m}{\sin ZA},$$

from which

$$\sin ZAD = \frac{-\cos m \cos\vartheta + \cos n \cos\eta}{\sin ZA \sin DA}$$

and

$$P = \Pi f(\cos\vartheta \cos m - \cos\eta \cos n).$$

The two remaining moments *Q* and *R* can be produced by analogy without further calculation. Then it is now the case that

$$\cos DZ = \cos\rho = \cos\zeta \cos l + \cos\eta \cos m + \cos\vartheta \cos n,$$

which expression, as it is nowhere greater than unity, thus cannot be equal to unity or $DZ = 0$, unless it is the case that $l = \zeta$, $m = \eta$ and $n = \vartheta$, clearly these three determinations likewise suppose that this equation is true:

$$\cos\zeta\cos l + \cos\eta\cos m + \cos\vartheta\cos n = 1.$$

Since now in addition, then

$$\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$$

and

$$\cos^2 l + \cos^2 m + \cos^2 n = 1\ ,$$

if from the sum of these twice that is taken away, there is produced

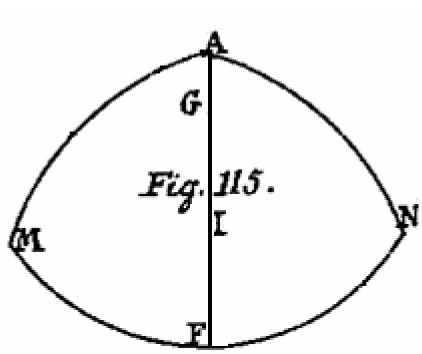
$$(\cos\zeta - \cos l)^2 + (\cos\eta - \cos m)^2 + (\cos\vartheta - \cos n)^2 = 0\ ,$$

but the sum of three squares cannot be equal to zero, unless each is zero.

SCHOLIUM

896. Since neither in the expressions for the moments of the forces *P, Q, R* found, nor in the expression for the force

$$\Pi = M\left(1 + \frac{fdd \cdot \cos\rho}{2gdt^2}\right)$$



is the radius of the base sphere *e* in place present, everything, which above that have been discussed concerning the motion of tops ending in points below, also prevail for tops of this kind, which end in a hemisphere or another part of a sphere, as long as the point *F*, that before denoted the point, here is put in place at the centre of the figure *G*. Likewise hence it is the case that the top rotates on the point or upon the hemisphere, provided *f* is the distance of the centre of inertia *I* from the centre of the spherical base, for whatever should be the size of the radius of this base *e*, it is not present in the computation, but with that vanishing the base of the top turns into a point. Therefore the whole preceding chapter is understood to be in place here, thus so that the theory of tops can be agreed upon without any labour, and not to be greatly enlarged on. But the case is excluded for making the base spherical occurring before, clearly as the centre of inertia *I* is nearer to the bottom, then the centre of the spherical part, and here the quantity *f* is negative [There appears to the translator to be a use of the wrong word here : *propius* means 'closer', which could be used to give a meaningful statement, while the term used here *proprius* means 'special', as referring to the special axis *GD*, which seems less relevant to the discussion]. Now either the body shall have a complete globe, or the base has the form of part of a sphere described *MFN* (Fig. 115), with the centre *G*, by which it rests on the horizontal plane, we investigate the motion of this, as far as the body makes contact with this base. But here we confine such bodies to be attributed with this quality, in order that the special axis *AGIF*, which if it should be vertical, remains in a state of rest, likewise it shall be a principal axis of the body, now with the two remaining principal axes having moments of inertia equal to each other. Clearly if the moment of inertia about the axis *IA* is equal to *Maa,* about the two remaining axes now *Mbb* and *Mcc,* we can put in place *bb* = *cc*. Hence we can determine how the motion of a body of this kind will continue when it receives a motion of some kind impressed on it.

PROBLEM 106

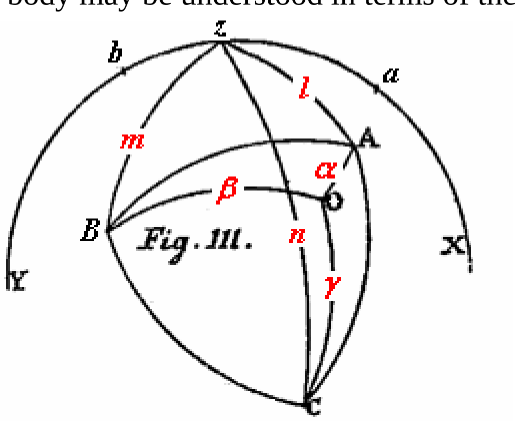
897. If the body constructed with the spherical base *MFN*, in which the axis of equilibrium *AGJF* is a principal axis, and about this the moment of inertia is equal to *Maa,* and moreover about the two remaining axes the moment of inertia is equal to *Mcc*, is given some kind of motion, to determine the continuation of the motion.

SOLUTION

Let the radius of the spherical base *FG* = *e*, but the centre of inertia *I* lies beyond the centre of the base *G* at a distance *GI* = *f.* For the progressive motion, if the centre of inertia *I* has a motion along the horizontal direction, that keeps pointing in the same direction, but to what extend the vertical motion is disturbed can thus be defined from the known force, which is equal to Π , since if the declination of the axis *AF* from the vertical is put equal to ρ , then

$$\frac{\Pi}{M} = 1 - \frac{fdd \cdot \cos\rho}{2gdt^2},$$

with *M* the mass or weight of the body present. Now this force Π , which the body exerts on the horizontal plane, is unable to be known except from the rotational motion. Hence our body may be understood in terms of the fixed sphere (Fig. lll), on which *Z* is the vertical point, now in the elapsed time *t* a situation of this kind arises, concerning the principal axes of this body at *A*, *B*, *C*, and the arcs



$$ZA = l\,,\ ZB=m,\ ZC=n,$$

and then the angles

$$XZA = \lambda, \quad XZB = \mu, \quad XZC = \nu\ ,$$

are put in place thus so that $l = \rho$. But now it rotates about the pole *O* in the sense *ABC* with an angular speed equal to γ', and on putting the arcs

$$OA = \alpha\ , \quad OB = \beta, \quad OC = \gamma$$

or for brevity as

$$\gamma' \cos\alpha = x, \quad \gamma' \cos\beta = y, \quad \gamma' \cos\gamma = z\ .$$

But the moments of the forces arising from the force Π are :

$$P = O, \quad Q = -\Pi\ f \cos n, \quad R = +\Pi\ f \cos m,$$

from which we deduce the following equations [See Ch. 16]:

$$dx = 0,$$

$$dy + \frac{aa-cc}{cc}\, xzdt = \frac{-2\Pi\, fgdt\cos n}{Mcc},$$

$$dz + \frac{cc-aa}{cc}\, xydt = \frac{2\Pi\, fgdt\cos m}{Mcc},$$

$dl\sin l = dt(y\cos n - z\cos m)$, $\quad d\lambda \sin^2 l = -dt(y\cos m + z\cos n)$,

$dm\sin m = dt(z\cos l - x\cos n)$, the remaining angles μ and ν hence are

$dn\sin n = dt(x\cos m - y\cos l)$, given at once.

If again we put

$$\cos l = p,\quad \cos m = q,\quad \cos n = r\ ,$$

since these equations agree with those, which we have integrated above in problem 99, unless because f it taken as negative, we have in the end these equations :

$x=A$ and

$$I.\ qy + rz = B - \frac{Aaap}{cc},$$

$$11.\ (qz - ry)^2 = \frac{(Ccc+4fgp)(1-pp) - cc\left(B - \frac{Aaap}{cc}\right)^2}{cc + ff(1-pp)},$$

$$111.\ yy + zz = \frac{Ccc + 4fgp + ff\left(B - \frac{Aaap}{cc}\right)^2}{cc + ff(1-pp)},$$

$$\text{IV.}\ \frac{\Pi}{M} = \frac{2gcc - Afaa\left(B - \frac{Aaap}{cc}\right)}{2g(cc + ff(1-pp))} + \frac{fccp\left(Ccc + 4fgp + ff\left(B - \frac{Aaap}{cc}\right)^2\right)}{2g(cc + ff(1-pp))^2},$$

$$\text{V.}\ dt = \frac{dp\sqrt{(cc + ff(1-pp))}}{\sqrt{\left((Ccc+4fgp)(1-pp) - cc\left(B - \frac{Aaap}{cc}\right)^2\right)}}\ ,$$

$$\text{VI.}\ d\lambda = \frac{-dt}{1-pp}\left(B - \frac{Aaap}{cc}\right),$$

$$\text{VII.}\ \gamma'\gamma' = AA + \frac{Ccc + 4fgp + ff\left(B - \frac{Aaap}{cc}\right)^2}{cc + ff(1-pp)}\ ,$$

$$\text{VIII.}\ \frac{ydz - zdy}{yy+zz} = \frac{A(aa-cc)\,dt}{cc} + \frac{2\Pi\, fgdt\left(B - \frac{Aaap}{cc}\right)(cc + ff - ffpp)}{Mcc\left(Ccc + 4fgp + ff\left(B - \frac{Aaap}{cc}\right)^2\right)}\ ,$$

where the constants A, B, C and the rest introduced from the integration must be defined from the initial state of the body.

COROLLARY 1

898. If the body is at rest initially and the principal axis A of this is at a declination a with $Za = \mathfrak{l}$ and $cos\mathfrak{l} = \mathfrak{p}$ present, initially $x = 0$, $y = 0$ and $z = 0$, on account of $\gamma' = 0$; and $p = \mathfrak{p}$.
Hence there is put in place

$$A = O,\ B = O \quad \text{and} \quad Ccc = -4fg\mathfrak{p}.$$

Hence in the elapsed time t there becomes

$$x = 0,\ qy + rz = 0,$$

$$qz - ry = \frac{2\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{(cc+ff(1-pp))}},$$

$$yy + zz = \frac{4fg(p-\mathfrak{p})}{cc+ff(1-pp)} = \gamma'\gamma'$$

and

$$\frac{\Pi}{M} = \frac{cc}{cc+ff(1-pp)} + \frac{2ffccp(p-\mathfrak{p})}{(cc+ff(1-pp))^2}.$$

COROLLARIUM 2

899. Now in addition in the same case it arises that $d\lambda = 0$ and thus the axis of the body, which initially was at a, itself moves through the arc aZ, and then

$$dt = \frac{dp\sqrt{(cc+ff(1-pp))}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}},$$

thus, since $p > \mathfrak{p}$ or $l < \mathfrak{l}$, the axis progresses directly from a to Z. And then as

$$ydz - zdy = 0,$$

there becomes

$$z = \delta y \quad \text{and} \quad y = \frac{2\sqrt{fg(p-\mathfrak{p})}}{\sqrt{(1+\delta\delta)(cc+ff(1-pp))}}$$

and also

$$q(yy+zz) = \frac{2z\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{(cc+ff(1-pp))}}$$

or

$$q = \frac{\delta\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}} \quad \text{et} \quad r = \frac{-\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}},$$

thus there becomes

$$cos\,ZAB = \frac{q}{\sqrt{(1-pp)}} = \frac{\delta}{\sqrt{(1+\delta\delta)}},$$

hence which angle remains constant.

COROLLARY 3

900. Hence if the body is initially at rest and the principal axis *IA* of this holds the position of inclination *Ia*, then it erects itself directly on rising from *a* to *Z*, but it is rotating about the point *O*, so that as $x = \gamma' \cos\alpha = 0$ the arc *AO* is a quadrant, and because

$$\cos ZO = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n = \frac{qy+rz}{\gamma'} = 0,$$

then also *ZO* is a quadrant, and thus *O* is the pole of the circle *XZY*. And when it comes to the axis *Z*, then the angular speed

$$\gamma' = \frac{2\sqrt{fg(1-\mathfrak{p})}}{c}.$$

SCHOLIUM 1

901. If the body is not initially at rest, but has taken a motion of some kind, the continuation of the motion is determined from the same formulas, as long as the constants *A, B, C* to be put in place initially may be defined conveniently ; moreover where formulas of this kind turn up to be integrated, which are able to be obtained only from the conceded quadratures of the above orders. Since here also, even in the simplest case, in which the body initially is at rest in an inclined position, the integration depends on this formula

$$dt = \frac{dp\sqrt{(cc+ff(1-pp))}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}},$$

which is unable to be resolved either by logarithms or circular arcs. But if the initial inclination *Za* should be as it were infinitely small, the work leads on to circular arcs : for initially let $Za = \mathfrak{l}$, and in the elapsed time *t* the declination *Za* = *l*, on account of the smallest arcs $\mathfrak{l}$ and *l* , then $p = 1-\frac{1}{2}ll$, $dp = -l\,dl$ and $\mathfrak{p} = 1-\frac{1}{2}\mathfrak{l}\mathfrak{l}$, hence

$$dt = \frac{-cdl}{\sqrt{2fg(\mathfrak{l}\mathfrak{l}-ll)}} \quad \text{and} \quad t = \frac{c}{\sqrt{2fg}} A\cos\frac{l}{\mathfrak{l}} \quad \text{or} \quad l = \mathfrak{l}\cos\frac{t\sqrt{2fg}}{c}.$$

Whereby the axis *IA* is made vertical in the elapsed time equal to $\frac{\pi c}{2\sqrt{2fg}}$ and the body performs isochronous staggering motions, as a simple pendulum of length equal to $\frac{cc}{f}$.

[Thus, the general problem cannot be solved from these known integrals, and Euler has to be content with the s.h.m. arising from small oscillations.]

SCHOLIUM 2

902. If we attribute the same kind of nature to the body, so that the natural axis *FD* of this, which is vertical in a state of rest, likewise is a principal axis of this body and the two remaining axes have equal moments of inertia, then certain differential formulas containing the motion of this are assigned, but in no way should we be able to define the motion itself on account of defective analysis. Yet meanwhile, as in the case treated, where we have attributed an infinitely small declination to the body, as comes about from usage, so that the motion is made simple enough and by conforming to the motion of a pendulum, therefore in general there is a place for this idea, for whatever manner the principal axes have been arranged about the natural axis. Clearly in the situation of equilibrium, where the natural axis *DF* keeps a vertical position, I take the centre of inertia *I* to lie below the centre of the spherical base *G* by the interval $GI = f$; then we may put this body to be declined at an infinitely small amount from it resting position, so that the arc $ZD = \rho$ is infinitely small (Fig.114), and it is clear that the body soon restores itself by oscillations or staggering motions, yet then with the motion destroyed on account of resistance it remains in a state of equilibrium. Because the declination of the body here is always very small, no effort is required, as the whole shape of the body is spherical; moreover it suffices, if the lowest part of this is part of the surface of a sphere and that very small, the centre of which is at *G* which acts on the horizontal plane. We consider initially in this staggering motion to be investigated, how the above formulas in general are elicited for this case, and in which the natural axis of the body *DF* decline minimally from the vertical position, and thus the moments of the forces taken together *P*, *Q*, *R* thus are able to be defined conveniently, so that hence from these we are enabled to assign the motion.

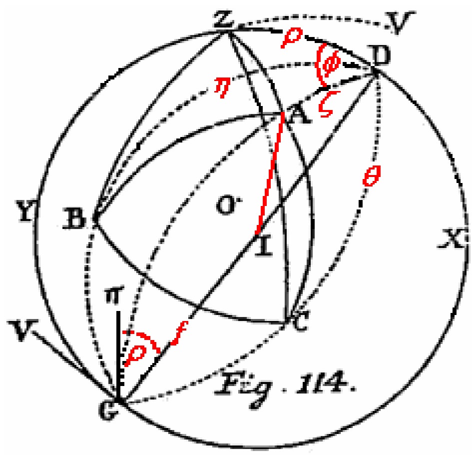


Fig. 114.

PROBLEM 107

902 [a]. If a body constructed with a spherical base is declined an infinitely small amount from equilibrium, to define the moments of the forces about the three principal axis of this body.

SOLUTION

With a sphere described around the centre of inertia *I* of the body (Fig.114), in which *Z* is the vertical point, the natural axis of the body *ID* keeps a position declining the least to the vertical, in order that the arc $ZD = \rho$ is the smallest ; moreover the principal axis of the body correspond to the points *A*, *B*, *C*, by reason of which the position of the point *D* thus itself is had, so that the arcs

$$DA = \zeta, \quad DB = \eta, \quad DC = \vartheta,$$

which are constant. But now with respect to the vertical point *Z* the arcs are

$$ZA = l, \quad ZB = m \quad \text{and} \quad ZC = n \ ,$$

which on account of the minimum $ZD = \rho$ hardly disagree with these ζ, η, ϑ; whereby if we put:

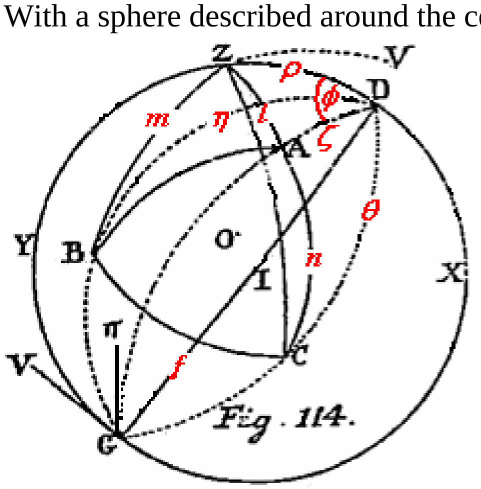


Fig. 114.

$$cos\, l = cos\, \zeta + p\ ,$$
$$cos\, m = cos\, \eta + q,$$
$$cos\, n = cos\, \vartheta + r\ ,$$

the quantities *p*, *q*, *r* are very small. Because now both

$$cos^2 \zeta + cos^2 \eta + cos^2 \vartheta = 1,$$

and

$$cos^2 l + cos^2 m + cos^2 n = 1,$$

making

$$2p\, cos\, \zeta + 2q\, cos\, \eta + 2r\, cos\, \vartheta + pp + qq + rr = 0\,.$$

But then since

$$cos\, \rho = cos\, \zeta\, cos\, l + cos\, \eta\, cos\, m + cos\, \vartheta\, cos\, n\,,$$

there is

$$cos\, \rho = 1 + p\, cos\, \zeta + q\, cos\, \eta + r\, cos\, \vartheta$$

and thus

$$cos\ p\, cos\, \zeta + q\, cos\, \eta + r\, cos\, \vartheta = -\tfrac{1}{2}\rho\rho$$

and

$$pp + qq + rr = \rho\rho\,.$$

Therefore now, on putting the force of the body on the horizontal plane equal to Π , from § 894 by attributing a negative value to f itself we obtain the moments of the forces about the principal axes :

$$P = IIf\,(r\, cos\, \eta - q\, cos\, \vartheta)\ ,$$
$$Q = IIf\,(p\, cos\, \vartheta - r\, cos\, \zeta)\ ,$$

and

$$R = IIf\,(q\, cos\, \zeta - p\, cos\, \eta),$$

besides we see now that

$$II = M\left(1 - \frac{fdd \cdot cos\, \rho}{2gdt^2}\right);$$

since now $cos\, \rho$ is approximately equal to 1 and undergoes the smallest variations, it satisfies exactly $\Pi = M$, thus so that is agreed that at first the body presses on the plane with its whole weight ; and thus we have

$$P = Mf\left(r\, cos\, \eta - q\, cos\, \vartheta\right)\ ,$$
$$Q = Mf\left(p\, cos\, \vartheta - r\, cos\, \zeta\right)\ ,$$
$$R = Mf\left(q\, cos\, \zeta - p\, cos\, \eta\right).$$

PROBLEM 108

903. If the body provided with the spherical base is declined briefly from its position of rest in which the axis *D1* is vertical, and sent off again, so that it reverts from rest to its position of equilibrium, to determine a motion of this kind.

SOLUTION

At the elapsed time *t* the body maintains the position represented in fig. 114 and keeps all the denominations fixed in the preceding problem, then the moments of inertia of the body about the principal axes *IA*, *IB*, *IC* are *Maa*, *Mbb*, *Mcc*. But now the body may rotate about the axis *IO* in the sense *ABC* with an angular speed equal to γ' and the arcs are

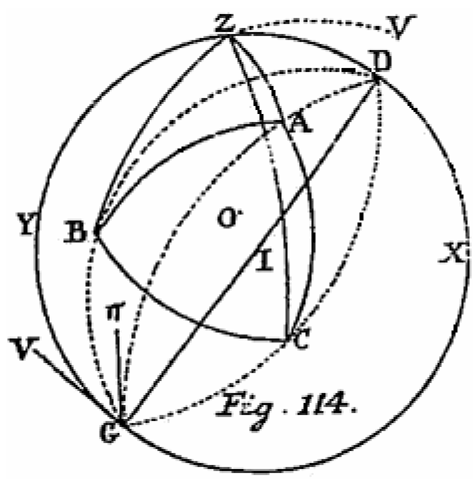


Fig. 114.

$$AO = \alpha, \quad BO = \beta, \quad CO = \gamma .$$

and there is put

$$\gamma' \cos\alpha = x, \quad \gamma' \cos\beta = y, \quad \gamma' \cos\gamma = z$$

Since therefore initially, when $t = 0$, the body is assumed to begin from rest, there is then $x = 0$, $y = 0$ and $z = 0$. Moreover now, since the motion of the body always remains the smallest, then the quantities *x*, *y*, *z* always must remain the smallest, thus so that the products of the two *xy*, *xz* and *yz* are able to be had as vanishing before the individual terms. Therefore since the moments of the forces acting *P*, *Q*, *R* are defined according to § 810, we arrive at the following equations :

$$dx = \frac{2\, fgdt}{aa}\left(r\cos\eta - q\cos\vartheta\right),$$
$$dy = \frac{2\, fgdt}{bb}\left(p\cos\vartheta - r\cos\zeta\right),$$
$$dz = \frac{2\, fgdt}{cc}\left(q\cos\zeta - p\cos\eta\right)$$

Then since

$$\cos l = \cos\zeta + p \ ,$$
$$\cos m = \cos\eta + q,$$

and

$$\cos n = \cos\vartheta + r \ ,$$

on account of the constants ζ, η, ϑ, then

$$dl \sin l = -dp,$$
$$dm \sin m = -dq$$

and

$$dn \sin n = -dr \ ,$$

from which above these three terms are agreed upon

$$-\,dp = dt\left(y\cos\vartheta - z\cos\eta\right),$$
$$-\,dq = dt\left(z\cos\zeta - x\cos\vartheta\right),$$
$$-\,dr = dt\left(x\cos\eta - y\cos\zeta\right),$$

where we have omitted the products *yr, zq, zp, xr, xq, yp* as small besides these terms shown here. And then if the arc *ZA* from a certain fixed circle now is agreed to be declined by the angle λ, on account of

$$\sin^2 l = \sin^2\zeta - 2p\cos\zeta$$

we have this equation :

$$d\lambda = \frac{-dt\left(y\cos\eta + z\cos\vartheta\right)}{\sin^2\zeta - 2p\cos\zeta}.$$

But because in the above equation the quantities *x, y, z* and *p, q, r* everywhere occupy a single dimension and *x, y, z* on putting $t = 0$ must vanish, it is evident on putting:

$$x = A\sin\delta t,\quad y = B\sin\delta t,\quad z = C\sin\delta t,$$
$$p = D\cos\delta t,\quad q = E\cos\delta t,\quad r = F\cos\delta t,$$

that both this condition as well as the these six equations can be satisfied, while indeed the first three equations divided by $\cos\delta t$ and the latter three equations divided by $\sin\delta t$ give

$$A\delta = \frac{2fg}{aa}\left(F\cos\eta - E\cos\vartheta\right),\quad D\delta = B\cos\vartheta - C\cos\eta,$$
$$B\delta = \frac{2fg}{bb}\left(D\cos\vartheta - F\cos\zeta\right),\quad E\delta = C\cos\zeta - A\cos\vartheta,$$
$$C\delta = \frac{2fg}{cc}\left(E\cos\zeta - D\cos\eta\right),\quad F\delta = A\cos\eta - B\cos\zeta.$$

From the latter the values of the coefficients *D, E, F* are substituted in the former and we obtain :

$$\frac{A\delta\delta aa}{2fg} = A\cos^2\eta - B\cos\zeta\cos\eta - C\cos\zeta\cos\vartheta + A\cos^2\vartheta,$$
$$\frac{B\delta\delta bb}{2fg} = B\cos^2\vartheta - C\cos\eta\cos\vartheta - A\cos\zeta\cos\eta + B\cos^2\zeta,$$
$$\frac{C\delta\delta cc}{2fg} = C\cos^2\zeta - A\cos\zeta\cos\vartheta - B\cos\eta\cos\vartheta + C\cos^2\eta.$$

But now if for brevity we put

$$A\cos\zeta + B\cos\eta + C\cos\vartheta = G,$$

on account of

$$\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$$

then

$$A\left(1-\frac{\delta\delta aa}{2fg}\right)=G\cos\zeta,\quad B\left(1-\frac{\delta\delta bb}{2fg}\right)=G\cos\eta \quad \text{et} \quad C\left(1-\frac{\delta\delta cc}{2fg}\right)=G\cos\vartheta.$$

For brevity we put $\frac{\delta\delta}{2fg}=u$, so that there arises

$$A=\frac{G\cos\zeta}{1-aau},\quad B=\frac{G\cos\eta}{1-bbu}\quad C=\frac{G\cos\vartheta}{1-ccu}.$$

But since

$$A\cos\zeta+B\cos\eta+C\cos\vartheta=G,$$

then

$$\frac{\cos^2\zeta}{1-aau}+\frac{\cos^2\eta}{1-bbu}+\frac{\cos^2\vartheta}{1-ccu}=1,$$

from which equation set out we follow on with, on dividing by u

$$\begin{aligned}aabbccuu-bbccu\sin^2\zeta+aa\cos^2\zeta\\ -aaccu\sin^2\eta+bb\cos^2\eta\\ -aabbu\sin^2\vartheta+cc\cos^2\vartheta=0.\end{aligned}$$

The known quantities are put in place

$$bbcc\sin^2\zeta+aacc\sin^2\eta+aabb\sin^2\vartheta=Kaabbcc,$$
$$aa\cos^2\zeta+bb\cos^2\eta+cc\cos^2\vartheta=Laabbcc,$$

in order that there becomes

$$uu-Ku+L=0,$$

and hence

$$u=\frac{\delta\delta}{2fg}=\frac{1}{2}K+\sqrt{\left(\frac{1}{4}KK-L\right)}$$

and the quantity G remains undefined and to be defined from the initial position, while on the other hand the quantities K and L have been given from the nature of the body, therefore since we have found the value of u, hence we have $\delta=\sqrt{2fgu}$ and

$$A=\frac{G\cos\zeta}{1-aau},\quad B=\frac{G\cos\eta}{1-bbu}\quad C=\frac{G\cos\vartheta}{1-ccu}$$

$$D=\frac{Gu(bb-cc)\cos\eta\cos\vartheta}{\delta(l-bbu)(1-ccu)},$$
$$E=\frac{Gu(cc-aa)\cos\zeta\cos\vartheta}{\delta(l-ccu)(1-aau)},$$

and

$$F=\frac{Gu(aa-bb)\cos\zeta\cos\eta}{\delta(l-aau)(1-bbu)}.$$

Now if initially the arc *ZD* = $\mathfrak{r}$, which now is ρ, since initially *p = D, q = E, r = F,* we have

$$DD + EE + FF = \mathfrak{rr},$$

thus the constant *G* is found in terms of $\mathfrak{r}$. And then for the angle λ to be found there is produced

$$d\lambda = \frac{-dt(B\cos\eta + C\cos\vartheta)\sin\delta t}{\sin^2\zeta}$$

and thus

$$\lambda = \frac{(B\cos\eta + C\cos\vartheta)(\cos\delta t - 1)}{\delta\sin^2\zeta},$$

if indeed the arc *ZA* should be fixed in the vertical initially, and then it is taken to be moving in the sense *XOY*, hence as much as this expression for λ is negative, it is agreed to be rotating in the contrary sense to the axis *IA* about *Z*. And then since

$$pp + qq + rr = \rho\rho$$

there arises

$$\rho = \mathfrak{r}\cos\delta t$$

on account of

$$\mathfrak{r} = \sqrt{(DD + EE + FF)},$$

thus it is apparent that the axis *ID* has erected itself into the vertical position in the elapsed time $\frac{\pi}{2\delta}$ and the times between falling over are isochronous with the oscillations of a pendulum, the length of which is equal to

$$= \frac{2g}{\delta\delta} = \frac{1}{fu} = \frac{K - \sqrt{(KK - 4L)}}{2Lf}.$$

COROLLARY 1

904. Since $DD + EE + FF = \mathfrak{rr}$, then it follows that

$$\delta\delta\mathfrak{rr} = AA\left(\cos^2\eta + \cos^2\vartheta\right) + BB\left(\cos^2\zeta + \cos^2\vartheta\right) + CC\left(\cos^2\zeta + \cos^2\eta\right)$$
$$-2BC\cos\eta\cos\vartheta - 2AC\cos\zeta\cos\vartheta - 2AB\cos\zeta\cos\eta$$

and because

$$G = A\cos\zeta + B\cos\eta + C\cos\vartheta,$$

the square of this added to that gives

$$\delta\delta\mathfrak{rr} + GG = AA + BB + CC,$$

where for brevity there is put

$$\frac{1}{1-aau}=\mathfrak{P},\quad \frac{1}{1-bbu}=\mathfrak{Q}\ ,\quad \frac{1}{1-ccu}=\mathfrak{R},$$

on account of

$$\mathfrak{P}\cos^2\zeta+\mathfrak{Q}\cos^2\eta+\mathfrak{R}\cos^2\vartheta=1,$$

and

$$A\ =G\mathfrak{P}\cos\zeta,\quad B=G\mathfrak{Q}\cos\eta\quad\text{and}\quad C=G\mathfrak{R}\cos\vartheta$$

there becomes

$$\delta\delta\mathfrak{rr}=GG\left(\mathfrak{PP}\cos^2\zeta+\mathfrak{QQ}\cos^2\eta+\mathfrak{RR}\cos^2\vartheta-1\right)$$

and thus on account of

$$\mathfrak{PP}-\mathfrak{P}=\frac{aau}{(1-aau)^2}$$

there is found

$$\delta\delta\mathfrak{rr}=GGu\left(\frac{aa\cos^2\zeta}{(1-aau)^2}+\frac{bb\cos^2\eta}{(1-bbu)^2}+\frac{cc\cos^2\vartheta}{(1-ccu)^2}\right).$$

COROLLARY 2

905. Since again there is the equation $\delta\delta=2\,fgu$, if in support the equation $uu-Ku+L=\ 0$ is called upon, there is found

$$GG=\frac{-2\,fg\mathfrak{rr}(1-aau)(1-bbu)(1-ccu)}{aa\cos^2\zeta+bb\cos^2\eta+cc\cos^2\vartheta-aabbccuu}\ .$$

EXPLANATION

906. This equation for *GG* can be elicited well enough in the following manner:
On putting for brevity

$$\frac{1}{aa}=\mathfrak{a},\quad \frac{1}{bb}=\mathfrak{b},\quad \frac{1}{cc}=\mathfrak{c},$$

we have:

$$\text{I. }\ K=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}-\mathfrak{a}\cos^2\zeta-\mathfrak{b}\cos^2\eta-\mathfrak{c}\cos^2\vartheta,$$

$$\text{II. }\ L=\mathfrak{bc}\cos^2\zeta+\mathfrak{ac}\cos^2\eta+\mathfrak{ab}\cos^2\vartheta,$$

$$\text{III. }\ 1=\cos^2\zeta+\cos^2\eta+\cos^2\vartheta.$$

Hence there is deduced on account of $uu - Ku + L = 0$,

$$\cos^2\zeta=\frac{\mathfrak{a}K-L-\mathfrak{aa}}{(\mathfrak{a}-\mathfrak{b})\,(\mathfrak{c}-\mathfrak{a})}$$

and

$$u\cos^2\zeta = \frac{(\mathfrak{a}-u)(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})(\mathfrak{c}-\mathfrak{a})}$$

and hence

$$\frac{\delta\delta\mathfrak{rr}}{GG} = \frac{\mathfrak{a}(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})(\mathfrak{c}-\mathfrak{a})(\mathfrak{a}-u)} + \frac{\mathfrak{b}(L-\mathfrak{b}u)}{(\mathfrak{b}-\mathfrak{c})(\mathfrak{a}-\mathfrak{b})(\mathfrak{b}-u)} + \frac{\mathfrak{c}(L-\mathfrak{c}u)}{(\mathfrak{c}-\mathfrak{a})(\mathfrak{b}-\mathfrak{c})(\mathfrak{c}-u)}$$

from which that expression is found from the reduced equation.

SCHOLIUM

907. Because these extend to the stumbling motions of all bodies, of which the base is part of a sphere, in whatever manner the principal axes of this are disposed in relation to the natural axis *DGIF*, and the moments of inertia of these about these are unequal, in order that we do not overextend ourselves, at first it is convenient to apply our formulas to simpler kinds of bodies, from which hence it is possible to progress to more complicated kinds of bodies. And indeed in the first case, in which all the moments of inertia are equal to each other or $aa = bb = cc$, everything is the most simple, since then the axis *DF* can be taken as the principal axis and the same stumbling motions appear, which we have now defined before. Following this, we put in place now at least two equal moments of inertia, thus it is permitted to put $bb = cc$.

CASE I : in which $aa = bb = cc$.

908. Hence in this case we have :

$$A = \frac{G\cos\zeta}{1-aau}, \quad B = \frac{G\cos\eta}{1-bbu}, \quad C = \frac{G\cos\vartheta}{1-ccu}$$

and hence

$$G = \frac{G\cos^2\zeta + G\cos^2\eta + G\cos^2\vartheta}{1-aau} = \frac{G}{1-aau},$$

thus in order that $u = 0$. Now by putting $u = \frac{1}{aa}$ and $G = 0$ it is also possible to satisfy these same formulas, so that

$$A\cos\zeta + B\cos\eta + C\cos\vartheta = 0,$$

nor besides can this be determined in any way, and thus we have

$$\delta = \frac{\sqrt{2fg}}{a};$$

then now

$$D = \frac{B\cos\vartheta - C\cos\eta}{\delta}, \quad E = \frac{C\cos\zeta - A\cos\vartheta}{\delta}, \quad F = \frac{A\cos\eta - B\cos\zeta}{\delta}$$

and

$$\delta\delta\mathfrak{rr} = AA + BB + CC,$$

in order that

$$\rho = \mathfrak{r}\cos\delta t.$$

Now we can see, that the body is set to rotate about some pole *O*, and at first we have

$$\cos OD = \cos\alpha\cos\zeta + \cos\beta\cos\eta + \cos\gamma\cos\vartheta\,,$$

or

$$\gamma'\cos OD = x\cos\zeta + y\cos\eta + z\cos\vartheta = 0,$$

and thus *OD* is a quadrant arc. Then

$$\cos OZ = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n$$

or

$$\gamma'\cos OZ = x\cos l + y\cos m + z\cos n = 0 + px + qy + rz = 0$$

on account of

$$AD + BE + CF = 0\,,$$

and hence also *OZ* is a quadrant. From which the body is seen to be rotating about the point *O*, which is the pole of the vertical circle *ZDX*, and thus the axis is raised from *D* directly to the vertical *Z*, thus so that in the time elapsed *t* then

$$\rho = \mathfrak{r}\cos\frac{t\sqrt{2fg}}{a}.$$

Whereby these stumbling motions are isochronous with the oscillations of a pendulum, the length of which is equal to $\frac{aa}{f}$.

CASE 11 : in which only two principal moments of inertia are equal or *bb* = *cc*.

909. Hence in this case then

$$K = \frac{cc\sin^2\zeta + aa\sin^2\eta + aa\sin^2\vartheta}{aacc} = \frac{cc\sin^2\zeta + aa + aa\cos^2\zeta}{aacc}$$

and

$$L = \frac{aa\cos^2\zeta + cc\sin^2\zeta}{aac^4}\,;$$

or if with the equation, from which *u* must be define, there is

$$\frac{\cos^2\zeta}{1-aau} + \frac{\sin^2\zeta}{1-ccu} = 1\,,$$

then

$$aa\cos^2\zeta + cc\sin^2\zeta = aaccu$$

and thus

$$u = \frac{aa\cos^2\zeta + cc\sin^2\zeta}{aacc}\,,$$

which value is elicited also from the general equation, except because in this way the useless root $u = \frac{1}{cc}$ is excluded. On account of which we have

$$\delta = \frac{\sqrt{2fg}\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}{ac}$$

then indeed

$$A = \frac{Gcc}{(-aa+cc)\cos\zeta},$$

$$B = \frac{Gaa\cos\eta}{(aa-cc)\sin^2\zeta},$$

$$C = \frac{Gaa\cos\vartheta}{(aa-cc)\sin^2\zeta}.$$

From this in order that G can be found from $\mathfrak{r}$, put

$$\delta\delta\mathfrak{rr} + GG = \frac{GG\left(a^4\cos^2\zeta + c^4\sin^2\zeta\right)}{(aa-cc)^2\sin^2\zeta\cos^2\zeta}$$

or

$$\delta\delta\mathfrak{rr} = \frac{GG\left(a^4\cos^2\zeta + c^4\sin^2\zeta\right)^2}{(aa-cc)^2\sin^2\zeta\cos^2\zeta}$$

[Two small typographic errors have been corrected by the O. O. editor in these derivations]

and

$$G = \frac{(aa-cc)\delta\mathfrak{r}\sin\zeta\cos\zeta}{aa\cos^2\zeta + cc\sin^2\zeta}$$

or

$$G = \frac{(aa-cc)\mathfrak{r}\sin\zeta\cos\zeta\sqrt{2fg}}{ac\sqrt{\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}}$$

From which we now obtain

$$D = 0, \quad E = \frac{\mathfrak{r}\cos\vartheta}{\sin\zeta}, \quad F = \frac{-\mathfrak{r}\cos\eta}{\sin\zeta}$$

and

$$A=\frac{-c\mathfrak{r}\sin\zeta\sqrt{2fg}}{a\sqrt{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}},$$

$$B=\frac{a\mathfrak{r}\cos\zeta\cos\eta\sqrt{2fg}}{c\sin\zeta\sqrt{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}},$$

$$C=\frac{a\mathfrak{r}\cos\zeta\cos\vartheta\sqrt{2fg}}{c\sin\zeta\sqrt{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}},$$

from which we follow with

$$x=\gamma'\cos\alpha=A\sin\delta t\ ,$$
$$y=\gamma'\cos\beta=B\sin\delta t\ ,$$
$$z=\gamma'\cos\gamma=C\sin\delta t\ ,$$
$$p=\cos l-\cos\zeta=D\cos\delta t\ ,$$
$$q=\cos m-\cos\eta=E\cos\delta t\ ,$$
$$r=\cos n-\cos\vartheta=F\cos\delta t,$$

and

$$\lambda=\frac{-a\mathfrak{r}\cos\zeta\left(1-\cos\delta t\right)\sqrt{2fg}}{\delta c\sin\zeta\sqrt{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}}=\frac{-aa\mathfrak{r}\cos\zeta\left(1-\cos\delta t\right)}{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)\sin\zeta}$$

[The original has the factor $\sin\zeta$ on the numerator. Corrected by C. B. One has to admit to the possibility that Euler did all his calculations mentally, which may account for the nature of the errors he made occasionally.]

and λ is the angle *VZA*, with the circle *ZV* fixed to the vertical, from which we compute the declination of the pole *A*. From which now $\rho=\mathfrak{r}\cos\delta t$, and in order that we can obtain the angle *DZV*, we seek the angle *DZA* from the formula

$$\cos DZA=\frac{\cos\zeta-\cos l\cos\rho}{\rho\sin l}=\frac{\cos\zeta-\cos\zeta\cos\rho-p\cos\rho}{\rho\sin\zeta}=\frac{1}{2}\rho\frac{\cos\zeta}{\sin\zeta},$$

as *D* = 0 and thus *p* = 0, hence

$$\cos DZA=\frac{\mathfrak{r}\cos\zeta\cos\delta t}{2\sin\zeta},$$

which since it is infinitely small, it is apparent that the angle *DZA* is almost a right angle and equal to the angle *ZDA*. Whereby when the initial angle *ZDA* was not right, this solution does not extend to that, which clearly is a special case only. Now also moreover these stumbling motions are isochronous with the oscillations of a pendulum, the length of which is equal to

$$\frac{aacc}{f\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}.$$

From which since then

$$\gamma' = \sin\delta t\cdot\sqrt{\left(AA+BB+CC\right)},$$

there is produced

$$\gamma' = \frac{\mathfrak{r}\sqrt{2fg}\left(a^4\cos^2\zeta+c^4\sin^2\zeta\right)}{ac\sqrt{\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}}\cdot\sin\delta t\ .$$

But for the pole *O* of the rotation we find:

$$\gamma'\cos OD = (A\cos\zeta + B\cos\eta + C\cos\vartheta)\sin\delta t = G\sin\delta t$$

and

$$\gamma'\cos OZ = (A\cos l + B\cos m + C\cos n)\sin\delta t = G\sin\delta t,,$$

thus so that there becomes $OD = OZ$ on account of $Ap + Bq + Cr = 0$.

SCHOLION

910. It is no wonder that this solution is not general, since indeed from the given nature of the body, with the quantities *aa*, *bb*, *cc* and the angles ζ, η, ϑ dealt with, and from the initial position of the axis *DF*, or from the declination $\mathfrak{r}$ of this from a vertical position, all the coefficients *A*, *B*, *C*, *D*, *E*, *F* since the number δ can be determined, from these the angle *ADZ* , by which the arc *DA* deviates from the arc *DZ*, can be determined at once, nor can anything further be left from our choice, as the nature of the thing requires. Now since in general for the quantity *u* we could have made use of the twin value, of which neither you may say is to be discarded rather than the other, and if likewise we put each to use, we will obtain an enlarged solution, from which likewise it is possible to effect that the initial angle *ADZ* should be equal to the given angle. Since indeed in the differential equations the quantities *x*, *y*, *z* and *p*, *q*, *r* everywhere have a single dimension, if this can be satisfied in the two ways, for whatever the size, the sum of these two values can be put in place, and hence we obtain the general solution, as we set out here.

PROBLEM 109

911. If the body provided with the spherical base is declined an infinitely small amount from the state of equilibrium in some manner and suddenly released, to define the stumbling motion by which it is driven.

SOLUTION

With the denominations of the above problem retained, since on putting

$$\frac{sin^2\zeta}{aa}+\frac{sin^2\eta}{bb}+\frac{sin^2\vartheta}{cc}=K \quad \text{et} \quad \frac{cos^2\zeta}{bbcc}+\frac{cos^2\zeta}{aacc}+\frac{cos^2\vartheta}{aabb}=L$$

and for u we find the twin value, these become

$$u=\tfrac{1}{2}\mathrm{K}+\sqrt{\left(\tfrac{1}{4}KK-L\right)} \quad \text{and} \quad u'=\tfrac{1}{2}\mathrm{K}-\sqrt{\left(\tfrac{1}{4}KK-L\right)}$$

from which for δ also two values are arrived at, which are

$$\delta=V\sqrt{2fgu} \quad \text{and} \quad \delta'=\sqrt{2fgu'}\,,$$

and hence for the old quantities x, y, z and p, q, r we obtain the following values

$$x=\gamma' \cos\alpha=\frac{G cos\zeta\, sin\delta t}{1-aau}+\frac{H cos\zeta\, sin\delta' t}{1-aau'},$$
$$y=\gamma' \cos\beta=\frac{G cos\eta sin\delta t}{1-bbu}+\frac{H cos\eta sin\delta' t}{1-bbu'},$$
$$z=\gamma' \cos\gamma=\frac{G cos\vartheta sin\delta t}{1-ccu}+\frac{H cos\vartheta sin\delta' t}{1-ccu'},$$

then again

$$p=cos\,l-cos\zeta=\frac{Gu(bb-cc)cos\eta cos\vartheta\cos\delta t}{\delta(1-bbu)(1-ccu)}+\frac{Hu' cos\eta cos\vartheta \cos\delta' t}{\delta'(1-bbu')(1-ccu')},$$
$$q=cos\,m-cos\eta=\frac{Gu(cc-aa)cos\zeta cos\vartheta\cos\delta t}{\delta(1-ccu)(1-aau)}+\frac{Hu' cos\zeta cos\vartheta \cos\delta' t}{\delta'(1-ccu')(1-aau')},$$
$$r=cos\,n-cos\vartheta=\frac{Gu(aa-bb)cos\zeta cos\eta\cos\delta t}{\delta(1-aau)(1-bbu)}+\frac{Hu' cos\zeta cos\eta \cos\delta' t}{\delta'(1-aau')(1-bbu')}.$$

Now here we have two arbitrary constant quantities G and H, and thus these values are satisfied, as on making the substitution into the differential equations both the terms for G as for H are affected to cancel each other. Now if initially the arc ZD is equal to $\mathfrak{r}$, on putting $t = 0$, there must arise $pp+qq+rr=\mathfrak{rr}$. From which, if the initial angle $ZDA=\mathfrak{f}$, on account of

$$\cos\mathfrak{f} = \frac{\cos l - \cos\zeta\cos\mathfrak{r}}{\sin\zeta\sin\mathfrak{r}} = \frac{\cos\zeta(1-\cos\mathfrak{r})+p}{\sin\zeta\sin\mathfrak{r}} = \frac{\mathfrak{r}\cos\zeta}{2\sin\zeta} + \frac{p}{\mathfrak{r}\sin\mathfrak{r}}$$

[The penultimate denominator lacks the '2' in the first edition. Corrected by C.B.]
and because $\mathfrak{r}$ is infinitely small, then

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{f}.$$

If therefore here the above value for p is substituted on putting $t = 0$, another equation is had, from which joined with that the two constants G and H are determined. But from the position of the angle $VZA = \lambda$ then

$$d\lambda = \frac{-dt(y\cos\eta + z\cos\vartheta)}{\sin^2\zeta},$$

with the integral of this shown easily. In a similar manner on putting the angles $VZB = \mu$ and $VZ0 = \nu$, then

$$d\mu = \frac{-dt(z\cos\vartheta + x\cos\zeta)}{\sin^2\eta} \quad \text{and} \quad d\nu = \frac{-dt(x\cos\zeta + y\cos\eta)}{\sin^2\vartheta}.$$

But here it is appropriate to note, if $bb = cc$, that the two values are

$$u = \frac{1}{cc} \quad \text{and} \quad u' = \frac{\sin^2\zeta}{aa} + \frac{\cos^2\zeta}{cc}$$

and thus certain numerators and denominators of the former fractions likewise vanish. Hence in order to investigate the values of these there is put in place

$$\frac{1}{bb} = \frac{1}{cc} + \omega$$

with the quantity ω present vanishing, and there is found

$$u = \frac{1}{cc} + \frac{\omega\cos^2\vartheta}{\sin^2\zeta} \quad \text{and} \quad u' = \frac{\sin^2\zeta}{aa} + \frac{\cos^2\zeta}{cc},$$

and hence if $\frac{G}{\omega}$ is put equal to I, in order that $G = I\omega = 0$, there becomes

$$x = \gamma'\cos\alpha = \frac{H\cos\zeta\sin\delta' t}{1-aau'} = \frac{-Hcc\sin\delta' t}{(aa-cc)\cos\zeta},$$

$$y = \gamma'\cos\beta = \frac{I\sin^2\zeta\sin\delta t}{cc\cos\eta} + \frac{H\cos\eta\sin\delta' t}{1-ccu'} = \frac{I\sin^2\zeta\sin\delta t}{cc\cos\eta} + \frac{Haa\cos\eta\sin\delta' t}{(aa-cc)\sin^2\zeta},$$

$$z = \gamma'\cos\gamma = \frac{-I\sin^2\zeta\sin\delta t}{cc\cos\vartheta} + \frac{H\cos\vartheta\sin\delta' t}{1-ccu'} = \frac{-I\sin^2\zeta\sin\delta t}{cc\cos\vartheta} + \frac{Haa\cos\vartheta\sin\delta' t}{(aa-cc)\sin^2\zeta},$$

from which

$$p = \cos l - \cos\zeta = \frac{I \sin^4 \zeta \cos \delta t}{\delta cc \cos \eta \cos \vartheta},$$
$$q = \cos m - \cos\eta = \frac{-I \sin^2 \zeta \cos \zeta \cos \delta t}{\delta cc \cos \vartheta} - \frac{Hu'(aa-cc)\cos\zeta \cos\vartheta \cos \delta' t}{\delta'(1-aau')(1-ccu')},$$
$$r = \cos n - \cos\vartheta = \frac{-I \sin^2 \zeta \cos \zeta \cos \delta t}{\delta cc \cos \eta} + \frac{Hu'(aa-cc)\cos\zeta \cos\eta \cos \delta' t}{\delta'(1-aau')(1-ccu')},$$

where

$$\frac{aa-cc}{(1-aau')(1-ccu')} = \frac{-aacc}{(aa-cc)\sin^2 \zeta \cos^2 \zeta}.$$

Or if we put

$$I = \frac{\mathfrak{G} \delta cc \cos \eta \cos \vartheta}{\sin^2 \zeta} \quad \text{and} \quad H = \frac{\mathfrak{H} \delta'(aa-cc)\sin^2 \zeta \cos \zeta}{aacc}$$

on account of

$$u = \frac{1}{cc} \quad \text{and} \quad u' = \frac{aa \cos^2 \zeta + cc \sin^2 \zeta}{aacc},$$

and thus

$$\delta = \frac{\sqrt{2fg}}{c} \quad \text{and} \quad \frac{\sqrt{2fg}\left(aa \cos^2 \zeta + cc \sin^2 \zeta\right)}{ac},$$

then

$$x = \gamma' \cos\alpha = \frac{\mathfrak{H}\delta' \sin^2 \zeta \sin\delta' t}{aa},$$
$$y = \gamma' \cos\beta = \frac{\mathfrak{H}\delta' \cos \zeta \cos \eta \sin\delta' t}{cc} + \mathfrak{G}\delta \cos \vartheta \sin \delta t,$$
$$z = \gamma' \cos\gamma = \frac{\mathfrak{H}\delta' \cos \zeta \cos \vartheta \sin\delta' t}{cc} - \mathfrak{G}\delta \cos \eta \sin \delta t,$$
$$p = \cos l - \cos \zeta = \mathfrak{G} \sin^2 \zeta \cos \delta t,$$
$$q = \cos m - \cos \eta = -\mathfrak{G} \cos \zeta \cos \eta \cos \delta t + \mathfrak{H} u' \cos \vartheta \sin\delta' t,$$
$$r = \cos n - \cos \vartheta = -\mathfrak{G} \cos \zeta \cos \vartheta \cos \delta t - \mathfrak{H} u' \cos \eta \cos \delta' t,$$

which formulas now can be applied to all the cases without any difficulty.

COROLLARIUM 1

912. These integrals are able to be extended still further, since *x*, *y*, *z* and *p*, *q*, *r* may receive constant terms ; and with the form of the *G* and *H* letters changed, we have :

$$x = \cos\zeta\,(\mathfrak{E} + \mathfrak{G}(1-bbu)(1-ccu)\sin\delta t + \mathfrak{H}(1-bbu')(1-ccu')\sin\delta' t\,)$$
$$y = \cos\eta(\mathfrak{E} + \mathfrak{G}(1-aau)(1-ccu)\sin\delta t + \mathfrak{H}(1-aau')(1-ccu')\sin\delta' t\,)$$
$$z = \cos\vartheta\,(\mathfrak{E} + \mathfrak{G}(1-ccu)(1-ccu)\sin\delta t + \mathfrak{H}(1-ccu')(1-ccu')\sin\delta' t\,)$$

and

$$p = \mathfrak{F}cos\zeta + (bb - cc)cos\,\eta\,cos\,\vartheta\,\left(\frac{\mathfrak{G}u(1 - aau)cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1 - aau')cos\delta' t}{\delta'}\right),$$

$$q = \mathfrak{F}cos\eta + (cc - aa)cos\,\zeta\,cos\,\vartheta\,\left(\frac{\mathfrak{G}u(1 - bbu)cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1 - bbu')cos\delta' t}{\delta'}\right),$$

$$r = \mathfrak{F}cos\vartheta + (aa - bb)cos\,\zeta\,cos\,\eta\,\left(\frac{\mathfrak{G}u(1 - ccu)cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1 - ccu')cos\delta' t}{\delta'}\right).$$

COROLLARY 2

913. Also each of the angular quantities δt and $\delta' t$ is increased by a constant quantity, and if in place of these we write $\delta t + \mathfrak{g}$ and $\delta' t + \mathfrak{h}$, the integrals contain six arbitrary constants $\mathfrak{g}, \mathfrak{h}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ and thus they are the complete integrals of these six differential equations:

$$aadx = 2\,fgdt(rcos\eta - qcos\vartheta), dp = dt(zcos\eta - ycos\vartheta)$$

$$bbdy = 2\,fgdt(pcos\vartheta - rcos\zeta), dq = dt(xcos\vartheta - zcos\zeta)$$

$$ccdz = 2\,fgdt(qcos\zeta - pcos\eta), dr = dt(ycos\zeta - xcos\eta).$$

COROLLARY 3

914. If the body were at rest in the beginning, as we assume in the problem, thus in order that then it should be the case that $x = 0$, $y = 0$ and $z = 0$, and on putting $\mathfrak{E} = 0$, $\mathfrak{g} = 0$ and $\mathfrak{h} = 0$; but it is necessary that the remaining constants be defined from the initial position of the body.

COROLLARY 4

915. Clearly if for the beginning, when $t = 0$, the angles are put

$$ZDA = \mathfrak{l}, \quad ZDB = \mathfrak{m} \;\text{ et }\; ZDC = \mathfrak{n};$$

in order that

$$sin(\mathfrak{l} - \mathfrak{m}) = -\frac{cos\vartheta}{sin\,\zeta\,sin\,\eta}, \quad sin(\mathfrak{m} - \mathfrak{n}) = -\frac{cos\zeta}{sin\,\eta\,sin\,\vartheta}, \quad sin(\mathfrak{n} - \mathfrak{l}) = -\frac{cos\eta}{sin\,\zeta\,sin\,\vartheta},$$

$$cos(\mathfrak{l} - \mathfrak{m}) = -\frac{cos\zeta\,cos\,\eta}{sin\,\zeta\,sin\,\eta}, \quad cos(\mathfrak{m} - \mathfrak{n}) = -\frac{cos\eta\,cos\,\vartheta}{sin\,\eta\,sin\,\vartheta}, \quad cos(\mathfrak{n} - \mathfrak{l}) = -\frac{cos\zeta\,cos\,\vartheta}{sin\,\zeta\,sin\,\vartheta},$$

for the beginning $t = 0$, it is necessary thus to define the constants, so that if then $ZD = \mathfrak{r}$, there comes about

$$p = \mathfrak{r}\,sin\zeta cos\,\mathfrak{l}, \quad q = \mathfrak{r}\,sin\eta cos\,\mathfrak{m}, \quad r = \mathfrak{r}\,sin\vartheta cos\,\mathfrak{n}.$$

EXPLANATION

916. Towards defining the constants $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ in general from the initial position in the manner described for brevity we put

$$aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta = \mathfrak{A},$$

$$bbcc\cos^2\zeta + aacc\cos^2\eta + aabb\cos^2\vartheta = \mathfrak{B}$$

and then

$$\frac{\mathfrak{G}u\cos\mathfrak{g}}{\delta} = X$$

and

$$\frac{\mathfrak{H}u'\cos\mathfrak{h}}{\delta'} = Y,$$

by which the calculation is made easier. Moreover there is found from that absolutely

$$\mathfrak{F} = \mathfrak{r}\sin\zeta\cos\zeta\cos\mathfrak{l} + \mathfrak{r}\sin\eta\cos\eta\cos\mathfrak{m} + \mathfrak{r}\sin\vartheta\cos\vartheta\cos\mathfrak{n}$$

$$X + Y = \frac{\frac{+\mathfrak{r}\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\vartheta}(\mathfrak{B} - bbcc) + \frac{\mathfrak{r}\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\vartheta}(\mathfrak{B} - aacc) + \frac{\mathfrak{r}\sin\vartheta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{B} - aabb)}{(bb-cc)(cc-aa)(aa-bb)},$$

$$uX + u'Y = \frac{\frac{-\mathfrak{r}\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\vartheta}(\mathfrak{A} - aa) - \frac{\mathfrak{r}\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\vartheta}(\mathfrak{A} - bb) - \frac{\mathfrak{r}\sin\vartheta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{A} - cc)}{(bb-cc)(cc-aa)(aa-bb)}.$$

But from these values $\mathfrak{A}$ and $\mathfrak{B}$, from above there is found

$$L = \frac{\mathfrak{A}}{aabbcc} \quad \text{and} \quad K = \frac{aabb + aacc + bbcc - \mathfrak{B}}{aabbcc},$$

from which there becomes

$$u = \tfrac{1}{2}K + \sqrt{\left(\tfrac{1}{4}KK - L\right)} \quad \text{and} \quad u' = \tfrac{1}{2}K - \sqrt{\left(\tfrac{1}{4}KK - L\right)}$$

thus so that $u + u' = K$ and $u' - u = \sqrt{(KK - 4L)}$.

This analysis in general prevails, even if an initial motion were impressed on the body, because in place of the angles δt and $\delta' t$ here we have used $\delta t + \mathfrak{g}$ and $\delta' t + \mathfrak{h}$. In a similar manner, in order that here we have defined the constants $\mathfrak{F}, \mathfrak{G}$, and $\mathfrak{H}$ from the initial position, from the initial motion impressed the quantities *x, y, z* are given values, from which if the formulas related in corollary 1and for δt and $\delta' t$ on writing $\delta t + \mathfrak{g}$ and $\delta' t + \mathfrak{h}$ at length, on putting

$t = 0$ they are equal, and the remaining constants $\mathfrak{E}, \mathfrak{g}$ et $\mathfrak{h}$ are determined; which indeed, as noted before now vanish, if the motion starts from rest. .

SCHOLIUM

917. Hence for the case of bodies of this kind, for which $bb = cc$, then

$$u = \frac{1}{cc}, \quad u' = \frac{aa\cos^2\zeta + cc\sin^2\zeta}{aacc},$$

and

$$\delta = \sqrt{2fgu}, \quad \delta' = \sqrt{2fgu'},$$

the integrals in general thus may be had :

$$x = \mathfrak{E}\cos\zeta \qquad - \frac{\mathfrak{H}\delta'\sin^2\zeta\sin(\delta' t + \mathfrak{h})}{aa}$$

$$y = \mathfrak{E}\cos\eta + \mathfrak{G}\delta\cos\vartheta\sin(\delta' t + \mathfrak{g}) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\eta\sin(\delta' t + \mathfrak{h})}{cc},$$

$$z = \mathfrak{E}\cos\vartheta - \mathfrak{G}\delta\cos\eta\sin(\delta' t + \mathfrak{g}) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\vartheta\sin(\delta' t + \mathfrak{h})}{cc},$$

and

$$p = \mathfrak{F}\cos\zeta + \mathfrak{G}\sin^2\zeta\cos(\delta t + \mathfrak{g}),$$

$$q = \mathfrak{F}\cos\eta - \mathfrak{G}\cos\zeta\cos\eta\cos(\delta t + \mathfrak{g}) + \mathfrak{H}u'\cos\vartheta\cos(\delta' t + \mathfrak{h}),$$

$$r = \mathfrak{F}\cos\vartheta - \mathfrak{G}\cos\zeta\cos\vartheta\cos(\delta t + \mathfrak{g}) - \mathfrak{H}u'\cos\eta\cos(\delta' t + \mathfrak{h}).$$

Whereby if initially $t = 0$, then there becomes

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{l}, \quad q = \mathfrak{r}\sin\eta\cos\mathfrak{m}, \quad r = \mathfrak{r}\sin\vartheta\cos\mathfrak{n},$$

and here is found

$$\mathfrak{H} = \frac{\mathfrak{r}\sin\eta\cos\vartheta\cos\mathfrak{m} - \mathfrak{r}\cos\eta\sin\vartheta\cos\mathfrak{n}}{u'\sin^2\zeta\cos\mathfrak{h}},$$

$$\mathfrak{G} = \frac{\mathfrak{r}\sin^3\zeta\cos\mathfrak{l} - \mathfrak{r}\sin\eta\cos\zeta\cos\eta\cos\mathfrak{m} - \mathfrak{r}\sin\vartheta\cos\zeta\cos\vartheta\cos\mathfrak{n}}{\sin^2\zeta\cos\mathfrak{g}},$$

$$\mathfrak{F} = \mathfrak{r}\sin\zeta\cos\zeta\cos\mathfrak{l} + \mathfrak{r}\sin\eta\cos\eta\cos\mathfrak{m} + \mathfrak{r}\sin\vartheta\cos\vartheta\cos\mathfrak{n}.$$

But with the angles $\mathfrak{l}, \mathfrak{m}, \mathfrak{n}$ given, likewise there are given ζ, η, ϑ

$$cos^2\zeta = \frac{cos(\mathfrak{l}-\mathfrak{m})cos(\mathfrak{n}-\mathfrak{l})}{sin(\mathfrak{l}-\mathfrak{m})sin(\mathfrak{n}-\mathfrak{l})}, \quad cos^2\eta = \frac{cos(\mathfrak{m}-\mathfrak{n})cos(\mathfrak{l}-\mathfrak{m})}{sin(\mathfrak{m}-\mathfrak{n})sin(\mathfrak{l}-\mathfrak{m})}, \quad cos^2\vartheta = \frac{cos(\mathfrak{n}-\mathfrak{l})cos(\mathfrak{m}-\mathfrak{n})}{sin(\mathfrak{n}-\mathfrak{l})sin(\mathfrak{m}-\mathfrak{n})},$$

$$sin^2\zeta = \frac{-cos(\mathfrak{m}-\mathfrak{n})}{sin(\mathfrak{l}-\mathfrak{m})sin(\mathfrak{n}-\mathfrak{l})}, \quad cos^2\eta = \frac{-cos(\mathfrak{n}-\mathfrak{l})}{sin(\mathfrak{m}-\mathfrak{n})sin(\mathfrak{l}-\mathfrak{m})}, \quad cos^2\vartheta = \frac{-cos(\mathfrak{l}-\mathfrak{m})}{sin(\mathfrak{n}-\mathfrak{l})sin(\mathfrak{m}-\mathfrak{n})}.$$

From these formulas it can be deduced that

$$sin\zeta\, cos\zeta\, cos\,\mathfrak{l} + sin\eta\, cos\eta\, cos\,\mathfrak{m} + sin\vartheta\, cos\vartheta\, cos\,\mathfrak{n} = 0,$$

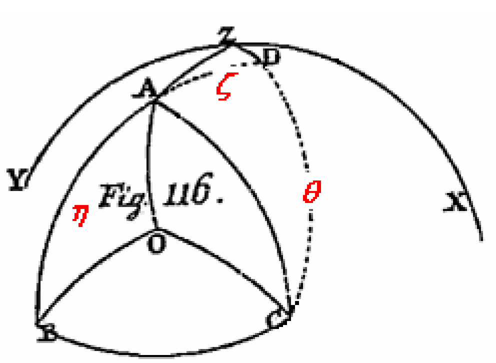
thus so that the constant defined above $\mathfrak{F}$ is always equal to 0. Now in a like manner,

$$\frac{sin\zeta cos\,\mathfrak{l}}{cos\eta cos\vartheta} + \frac{sin\eta cos\,\mathfrak{m}}{cos\zeta cos\vartheta} + \frac{sin\vartheta cos\,\mathfrak{n}}{cos\zeta cos\eta} = 0,$$

[In the first edition, the first sign is negative rather than positive. Corrected by the editor.] from which the values of the coefficients defined above are determined much more simply, that so that the letters $\mathfrak{A}$ and $\mathfrak{B}$ may be removed in a straight–forwards manner. These prevail in general, even if it is not the case that $bb = cc$.

PROBLEM 110

918. If the body provided with the spherical base has two equal principal axis and for that, when it should be declined an infinitely small amount from rest, some minimum motion should be impressed, the define the continued motion.



SOLUTION

Let *ID* be the axis of the body in equilibrium passing through the centre of inertia *I* and crossing the base *G* (Fig. 116), and let this be placed higher with that interval present *GI* = *f*. Again let *IA* be the special principal axis of the body about which the moment of inertia is equal to *Maa*, but about all the other axes and normal to that equal to *Mcc*, which all can be had equally for principal axes, one *IB* is taken on the arc *DA* produced, and the other on the arc *1C*, in order that the quadrant *AC* is normal to *AD* and likewise also the quadrant *DC* is normal to *AD*. Hence on putting $DA = \zeta$ then

$$DB = \eta = \zeta + 90^\circ$$

and $DC = \vartheta =$ 90°. Moreover initially, when *t* = 0, the arc $DZ = \mathfrak{r}$ and the angle $ZDA = \mathfrak{l}$, then $ZDB = \mathfrak{m} = \mathfrak{l}$ and $ZDC = \mathfrak{n} = \mathfrak{l} + 90^\circ$. From the preceding formulas we have hence

$$u=\frac{1}{cc},\quad u'=\frac{aa\cos^2\zeta+cc\sin^2\zeta}{aacc},\quad \delta=\sqrt{2fgu},\quad \delta'=\sqrt{2fgu'},$$

while now from the initial position in the first place there occurs $\mathfrak{F}=0$, and then

$$\mathfrak{r}\cos\zeta\cos\mathfrak{l}=\mathfrak{G}\sin^2\zeta\ \cos\mathfrak{g};$$

hence

$$\mathfrak{r}\cos\zeta\cos\mathfrak{l}=\mathfrak{G}\sin\zeta\cos\zeta\cos\mathfrak{g},\quad \mathfrak{G}=\frac{\mathfrak{r}\cos\mathfrak{l}}{\sin\zeta\cos\mathfrak{g}},$$

$$-\mathfrak{r}\sin\mathfrak{l}=\mathfrak{H}u'\sin\zeta\cos\mathfrak{h},\quad \mathfrak{H}=\frac{-\mathfrak{r}\sin\mathfrak{l}}{u'\sin\zeta\cos\mathfrak{h}}.$$

In the beginning a motion is then impressed on the body about the axis *IC*, with an angular speed equal to ε in the sense *ABC*, with $OA=\mathfrak{a}, OB=\mathfrak{b}$, and $OC=\mathfrak{c}$ present and there must be produced

$$\varepsilon\cos\mathfrak{a}=\mathfrak{G}\cos\zeta-\frac{\mathfrak{H}\delta'\sin^2\zeta\sin\mathfrak{h}}{aa},$$

$$\varepsilon\cos\mathfrak{b}=-\mathfrak{G}\sin\zeta-\frac{\mathfrak{H}\delta'\sin\zeta\cos\zeta\sin\mathfrak{h}}{cc},$$

$$\varepsilon\cos\mathfrak{c}=\mathfrak{G}\delta\sin\zeta\sin\mathfrak{g},$$

from which we conclude

$$\varepsilon\left(aa\cos\mathfrak{a}\cos\zeta-cc\cos\mathfrak{b}\sin\zeta\right)=\mathfrak{E}\left(aa\cos^2\zeta+cc\sin^2\zeta\right).$$

and

$$\varepsilon\left(\cos\mathfrak{a}\sin\zeta+\cos\mathfrak{b}\cos\zeta\right)=-\mathfrak{H}\delta'\sin\zeta\sin\mathfrak{h}\left(\frac{\sin^2\zeta}{cc}+\frac{\cos^2\zeta}{aa}\right).$$

Hence

$$\mathfrak{E}=\frac{\varepsilon\left(aa\cos\mathfrak{a}\cos\zeta-cc\cos\mathfrak{b}\sin\zeta\right)}{aa\cos^2\zeta+cc\sin^2\zeta},$$

$$\mathfrak{G}=\frac{\varepsilon\cos\mathfrak{c}}{\delta\sin\zeta\sin\mathfrak{g}},$$

$$\mathfrak{H}=\frac{-\varepsilon\left(\cos\mathfrak{a}\sin\zeta+\cos\mathfrak{b}\cos\zeta\right)}{\delta'u'\sin\zeta\sin\mathfrak{h}},$$

hence the arises

$$\tang\mathfrak{g}=\frac{\varepsilon\cos\mathfrak{c}}{\delta\mathfrak{r}\cos\mathfrak{l}}\quad\text{and}\quad \tang\mathfrak{h}=\frac{\varepsilon\left(\cos\mathfrak{a}\sin\zeta+\cos\mathfrak{b}\cos\zeta\right)}{\delta'\mathfrak{r}\sin\mathfrak{l}},$$

from which the angles $\mathfrak{g}$ and $\mathfrak{h}$ and hence the numbers $\mathfrak{G}$ and $\mathfrak{H}$ become known.

With these definitions after an elapsed time *t* the body holds the position shown in the figure and here becomes

$$ZD=\rho,\ ZA=l,\ ZB=m,\ ZC=n:$$

and there is put in place

$$cos l = cos\zeta + p, \quad cos m = cos\eta + q, \quad cos n = cos\vartheta + r$$

or

$$cos\, m = -\, sin\,\zeta + q \quad \text{et} \quad cos n = r.$$

Thereon the body now rotates about the axis *IO* with an angular speed equal to δ in the sense *ABC* with the arcs arising

$$OA = \alpha, \quad OB = \beta, \quad OC = \gamma\ ,$$

and on putting

$$\gamma'\, cos\alpha = x\ , \quad \gamma'\, cos\beta = y \quad \text{and} \quad \gamma'\, cos\gamma = z\ ,$$

we have from § 917

$$\gamma'\, cos\,\alpha = \frac{\varepsilon cos\zeta\left(aacos\,\mathfrak{a}\, cos\zeta\ -\ cc\ cos\,\mathfrak{b}\, sin\zeta\ \right)}{aacos^2\zeta + cc\ sin^2\zeta} + \frac{\delta'\,\mathfrak{r}\, sin\zeta\, sin\,\mathfrak{l}\, sin\left(\delta' t+\mathfrak{h}\right)}{aau'\, cos\,\mathfrak{h}},$$

$$\gamma'\, cos\,\beta = \frac{-\varepsilon\, sin\,\zeta\left(aacos\,\mathfrak{a}\, cos\zeta\ -\ cc\ cos\,\mathfrak{b}\, sin\zeta\ \right)}{aacos^2\zeta + cc\ sin^2\zeta} + \frac{\delta'\,\mathfrak{r}\, cos\,\zeta\, sin\,\mathfrak{l}\, sin\left(\delta' t+\mathfrak{h}\right)}{ccu'\, cos\,\mathfrak{h}},$$

$$\gamma'\, cos\,\delta = \frac{\delta\mathfrak{r}\, cos\,\mathfrak{l}\, sin\left(\delta t+\mathfrak{g}\right)}{cos\,\mathfrak{g}},$$

[In the first edition the denominator in the second equation is $aau'\, cos\,\mathfrak{h}$ rather than $ccu'\, cos\,\mathfrak{h}$. C. B.]
then now in addition :

$$p = \frac{\mathfrak{r}\, sin\,\zeta\, cos\,\mathfrak{l}\, cos\left(\delta t+\mathfrak{g}\right)}{cos\,\mathfrak{g}},$$

$$q = \frac{\mathfrak{r}\, cos\,\zeta\, cos\,\mathfrak{l}\, cos\left(\delta t+\mathfrak{g}\right)}{cos\,\mathfrak{g}},$$

$$r = \frac{-\mathfrak{r}\, sin\,\mathfrak{l}\, cos\left(\delta' t+\mathfrak{h}\right)}{cos\,\mathfrak{h}}.$$

From these if the arc is put in place $ZD = \rho$, then

$$\rho = \mathfrak{r}\sqrt{\left(\frac{cos^2\,\mathfrak{l}\, cos^2\left(\delta t+\mathfrak{g}\right)}{cos^2\,\mathfrak{g}} + \frac{sin^2\,\mathfrak{l}\, cos^2\left(\delta' t+\mathfrak{h}\right)}{cos^2\,\mathfrak{h}}\right)}.$$

Again from triangle *AZD* there is

$$cos\ ADZ = \frac{cos\, l - cos\zeta\, cos\rho}{sin\zeta\, sin\rho} = \frac{p+\frac{1}{2}\rho\rho cos\zeta}{\rho sin\zeta} = \frac{p}{\rho sin\zeta}$$

with the term $\frac{\rho cos\zeta}{2\rho sin\zeta}$ vanishing, and hence therefore

$$cos\ ADZ = \frac{cos\,\mathfrak{l}\, cos\left(\delta t+g\right)}{cos\,\mathfrak{g}} : \sqrt{\left(\frac{cos^2\,\mathfrak{l}\, cos^2\left(\delta t+\mathfrak{g}\right)}{cos^2\,\mathfrak{g}} + \frac{sin^2\,\mathfrak{l}\, cos^2\left(\delta' t+\mathfrak{h}\right)}{cos^2\,\mathfrak{h}}\right)}$$

and therefore

$$\text{tang } ADZ = \frac{\cos\mathfrak{g}\,\text{tang}\,\mathfrak{l}\cos(\delta' t+\mathfrak{h})}{\cos\mathfrak{h}\cos(\delta t+\mathfrak{g})}.$$

But besides the arc $DZ = \rho$ and the angle ADZ is necessary to know the angle XZD calculated from the fixed vertical circle ZX ; now $DZA = 180^{\circ} - ADZ$, or

$$\text{tang } ADZ = -\frac{\cos\mathfrak{g}\,\text{tang}\,\mathfrak{l}\cos(\delta' t+\mathfrak{h})}{\cos\mathfrak{h}\cos(\delta t+\mathfrak{g})}\text{tang}\,\mathfrak{l},$$

since initially there is present $DZA = 180^{\circ} - \mathfrak{l}$ and $\text{tang } DZA = -\text{tang}\,\mathfrak{l}$. From which now on putting the angle $XZA = \lambda,$ there is

$$d\lambda = -\frac{dt(y\cos\eta + z\cos\vartheta)}{\sin^2\zeta} = \frac{ydt}{\sin\zeta},$$

and hence

$$\lambda = \text{Const.} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\cos\zeta)}{aa\cos^2\zeta + cc\sin^2\zeta} - \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}\cos(\delta' t+\mathfrak{h})}{ccu'\sin\zeta\cos\mathfrak{h}}.$$

But if we put the initial angle XZD to be vanishing, initial there is $\lambda = 180^{\circ} - \mathfrak{l}$, and thus a constant is introduced here :

$$\text{Const.} = 180^{\circ} - \mathfrak{l} + \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}}{ccu'\sin\zeta},$$

from which we have :

$$\lambda = 180^{\circ} - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\cos\zeta)}{aa\cos^2\zeta + cc\sin^2\zeta} - \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}}{ccu'\sin\zeta}\left(1 - \frac{\cos(\delta' t+\mathfrak{h})}{\cos\mathfrak{h}}\right).$$

and hence $XZD = \lambda - DZA$, from which at the time t the position of the body is known perfectly, and in this determination the motion of the body is likewise present.

COROLLARY 1

919. If the initial motion impressed on the body vanishes, then there becomes $\mathfrak{g} = 0$ and $\mathfrak{h} = 0$; and hence

$$x = \gamma'\cos\alpha = \frac{\delta'\mathfrak{r}\sin\zeta\sin\mathfrak{l}\sin\delta' t}{aau'},$$

$$y = \gamma'\cos\beta = \frac{\delta'\mathfrak{r}\cos\zeta\sin\mathfrak{l}\sin\delta' t}{ccu'},$$

$$z = \gamma'\cos\delta = \delta\mathfrak{r}\cos\mathfrak{l}\sin\delta t,$$

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{l}\cos\delta t, q = \mathfrak{r}\cos\zeta\cos\mathfrak{l}\cos\delta t, r = -\mathfrak{r}\sin\mathfrak{l}\cos\delta' t$$

$$\text{tang } ADZ = \text{tang}\left(180^{\circ} - DZA\right) = \frac{\cos\delta' t}{\cos\delta t}\text{tang}\,\mathfrak{l}$$

and

$$\lambda = 180^{o} - \mathfrak{l} - \frac{\mathfrak{r}\cos\zeta \sin\mathfrak{l}}{ccu' \sin\zeta}\left(1 - \cos\delta' t\right).$$

COROLLARY 2

920. But if an initial motion were impressed on the body with an angular speed of ε, this must not be much greater than $\mathfrak{r}$. For if $\frac{\varepsilon}{\mathfrak{r}}$ should be a very large number, the angles $\mathfrak{g}$ and $\mathfrak{h}$ appearing are almost right and the cosines of these nearly equal to zero, and thus the numbers *p, q, r* become exceedingly large, while yet they can be considered as very small, as the nature of the solution examines. In as much as the arc $ZD = \rho$ must always be very small.

COROLLARY 3

921. Since it is the case that $\gamma' = \sqrt{(xx + yy + zz)}$, unless the three quantities *x, y, z* individually vanish, it cannot happen that the body can be returned to rest at any time. And even if the body begins to move from rest, it can happen still, that the body subsequently never returns to rest, and this therefore always comes about, unless it is the case that either $\sin\mathfrak{l} = 0$ or $\cos\mathfrak{l} = 0$; then also the natural axis of the body *DF* never comes to a vertical position.

COROLLARY 4

922. Since it is the case that the quantity $\mathfrak{r}$ is very small, if initially the body is given no motion, as then $\varepsilon = 0$, then it is exactly enough that $\lambda = 180^{o} - \mathfrak{l}$; clearly the angle *XZA* remains constant, and the motion of the axis *IA* thus put in place, as now at one time it approaches the vertical point *Z* along the arc *ZA* and now at another it recedes further from that, then moreover

$$AZ = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\delta t \quad \text{and} \quad \text{ang.} ZAD = \mathfrak{r}\frac{\sin\mathfrak{l}\cos\delta' t}{\sin\zeta}.$$

COROLLARY 5

923. But in general some kind of initial motion is impressed on the body, then

$$XZA = \lambda = 180^{o} - \mathfrak{l} - \frac{\varepsilon t\left(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\cos\zeta\right)}{aa\cos^{2}\zeta + cc\sin^{2}\zeta}$$

and thus the arc *ZA* is carried uniformly around the vertical point *Z* ; from which now, since

$$\sin AD : \sin DZA = ZD(\rho) : ZAD,$$

then

$$ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}\cos(\delta' t + \mathfrak{h})}{\sin\zeta\cos\mathfrak{h}} \text{ and the arc } ZA = \zeta - \frac{\mathfrak{r}\cos\mathfrak{l}\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}.$$

Or for $\mathfrak{g}$ and $\mathfrak{h}$ with the values put in place :

$$\text{the angle } ZAD = \frac{\mathfrak{r}\, sin\mathfrak{l} cos\delta' t}{sin\zeta} - \frac{\varepsilon\left(cos\,\mathfrak{a}\, sin\,\zeta + cos\,\mathfrak{b}\, cos\,\zeta\right) sin\,\delta' t}{\delta' sin\,\zeta}$$

and

$$\text{arcus } ZA = \zeta - \mathfrak{r}\, cos\,\mathfrak{l}\, cos\delta t + \frac{\varepsilon\, cos\,\mathfrak{c}\, sin\,\delta t}{\delta}.$$

SCHOLIUM 1

924. These three final formulas, showing the angles *XZA*, *ZAD* with the arc *Z A*, bring about the whole solution of the problem. For if indeed we are able to assign this at some time, we know perfectly the position of the body. Whereby if we substitute the above values found for δ and δ', then the general solution of this problem is contained in these formulas :

$$\text{the angle } XZA = 180^{\text{o}} - \mathfrak{l} - \frac{\varepsilon t(\,aacos\mathfrak{a}cos\zeta - cc\,cos\,\mathfrak{b}\, sin\zeta\,)}{aacos^2\zeta + ccsin^2\zeta}\ ,$$

$$\text{the arc } ZA = \zeta - cos\,\mathfrak{l}\, cos\frac{t\sqrt{2\,fg}}{c} + \frac{\varepsilon c\, cos\,\mathfrak{c}}{\sqrt{2\,fg}}\, sin\frac{t\sqrt{2\,fg}}{c}\,,$$

$$\text{and the angle } ZAD = \frac{\mathfrak{r}\, sin\mathfrak{l}}{sin\zeta} cos\frac{t\sqrt{2\,fg\left(aacos^2\zeta + ccsin^2\zeta\right)}}{ac} - \frac{\varepsilon ac\left(cos\,\mathfrak{a}\, sin\,\zeta + cos\,\mathfrak{b}\, cos\,\zeta\right)}{sin\,\zeta\sqrt{2\,fg\left(aacos^2\zeta + ccsin^2\zeta\right)}}\, sin\frac{t\sqrt{2\,fg\left(aacos^2\zeta + ccsin^2\zeta\right)}}{ac}.$$

Hence if all the moments of inertia were then equal, clearly *aa* = *cc*, then the angle

$$XZA = 180^{\text{o}} - \mathfrak{l} - \varepsilon t(\,cos\mathfrak{a}cos\zeta - cos\,\mathfrak{b}\, sin\zeta\,)\ ,$$

$$\text{the arc } ZA = \zeta - cos\,\mathfrak{l}\, cos\frac{t\sqrt{2\,fg}}{c} + \frac{\varepsilon c\, cos\,\mathfrak{c}}{\sqrt{2\,fg}}\, sin\frac{t\sqrt{2\,fg}}{c}$$

$$\text{and the angle } ZAD = \frac{\mathfrak{r}\, sin\mathfrak{l}}{sin\zeta} cos\frac{t\sqrt{2\,fg}}{c} - \frac{\varepsilon c\left(cos\,\mathfrak{a}\, sin\,\zeta + cos\,\mathfrak{b}\, cos\,\zeta\right)}{sin\,\zeta\sqrt{2\,fg}}\, sin\frac{t\sqrt{2\,fg}}{c}\,,$$

in which case the position of the point *A* is clearly arbitrary.

SCHOLIUM 2

925. The argument, which in this chapter we have undertaken chiefly to set out, clearly the rocking motion of bodies given with spherical bases, we have resolved completely, provided the rocking motions are small, which hypothesis is usually put in place in the theory of oscillations; for the formulas shown in § 912 and in the following provide a complete solution to this question, if indeed there the angles δt and $\delta' t$ are augmented by the constants $\mathfrak{g}$ and $\mathfrak{h}$. Moreover we have shown in § 916 how to define the constants from the initial position, in which the calculation is is strongly supported from the note added at the end of § 917; whereby we may move on to the explanation of the motion of cylindrical bodies.

CAPUT XVIII

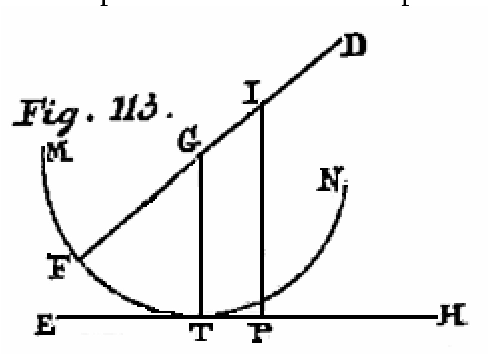
DE MOTU CORPORUM BASI SPHAERICA PRAEDITORUM SUPER PLANO HORIZONTALI

PROBLEMA 104

884. Si corpus basi sphaerica praeditum plano horizontali quomodocunque incumbat, definire vires, quibus sollicitatur, earumque effectum in motu corporis progressivo turbando.

SOLUTIO

Sit *EH* planum horizontale (Fig. 113) et *T* punctum, ubi corpus ei insistit, in corpore autem notetur prima centrum baseos sphaericae *MTN*,quod sit in *G*, deinde centrum inertiae corporis *I*, ac tabula repraesentet planum, in quo haec tria puncta sunt sita. Ducatur radius *GT*, qui cum sit ad horizontalem *EH* normalis, situm habebit verticalem, ideoque ipsum planum *TGI* erit verticale. Iam quia pro motu progressivo totam corporis massam, quae sit = *M*, tanquam in centro inertiae *I* collectam concipere licet, ducta *IP* ipsi *GT* parallela corpus primo ob gravitatem urgetur in directione *IP* vi = *M;* deinde vero ubi planum horizontale in *T* tangit, ab eo certa quadam vi urgebitur sursum in directione *TG*, et pressioni aequali, quae vis sit = Π . Quare nisi hae duae vires se destruant, corpus in quiete persistere nequit; ex quo perspicuum est statum quietis exigere, ut producta recta *GI* in *F* corpus puncto *F* plano horizontali insistat sicque recta *DIGF* fiat verticalis. Figura ergo repraesentat statum corporis inclinatum, et inclinatio indicatur angulo *FGT*, qui sit = ρ , quo evanescente corpus in statu aequilibrii versatur. Ponamus porro radium basis sphaericae *GF* = *GT* = *e* et intervallum punctorum *G* et *I* nempe *GI* = *f*, quatenus centrum inertiae *I* longius distat a puncto *F* quam centrum figurae *G;* ita ut, si proprius caderet, quantitas *f* negative esset accipienda. Hinc ergo erit $IP = e + f\cos\rho$, quae est altitudo centri inertiae *I* supra planum horizontale *EH* et quae a viribus sollicitantibus sola afficitur. Translata autem vi *TG* = Π in centrum inertiae *I*, punctum *I* deorsum sollicitatur vi = $M - \Pi$; et quia eius celeritas deorsum directa est



$$= \frac{fd\rho\sin\rho}{dt}$$

posita ea = *u* erit

$$du = \frac{2g(M - \Pi)dt}{M}$$

denotante *dt* elementum temporis, ex quo habetur

$$f\,(dd\rho\sin\rho + d\rho^2\cos\rho) = 2g\left(1\ -\frac{\Pi}{M}\right)dt^2$$

sumto *dt* constante; neque aliter motus progressivus afficietur.

COROLLARIUM 1

885. Vicissim ergo, si ratio motus progressivi detur vel saltem ut data consideretur, inde pressio Π definietur, cum sit

$$\frac{\Pi}{M} = 1 - \frac{f\,(dd\rho\sin\rho + d\rho^2\cos\rho)}{2gdt^2} \quad \text{seu} \quad \frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos\rho}{2gdt^2}.$$

COROLLARIUM 2

886. Si fuerit $f = 0$ seu centrum inertiae *I* in ipsum centrum sphaerae G incidat, prodit $\Pi = M$, et corpus in omni situ aequilibrii proprietate gaudet.

COROLLARIUM 3

887. Si fuerit $f > 0$ seu $F1 > FG$, statim ac corpus tantillum inclinatur, a vi sollicitante inclinatio augebitur, sin autem sit $f < 0$ seu $FI < FG$, inclinatio minuetur corpusque in situm aequilibrii, quo punctum *F* plano insistit, restituetur: dum priori casu procumbit, alium quaerens aequilibrii situm.

SCHOLION 1

888. Quamcunque autem corpus habuerit figuram, in eo semper ad minimum duo dantur aequilibrii situs, quorum alter ita est comparatus, ut, si corpus ex eo parumper declinetur, sponte sua se restituat, alter vero, ut penitus prolabatur; quorum prior *status aequilibrii stabilis*, posterior vero *labilis* vocari solet. Quodcunque enim corpus plano horizontali incumbit, in aequilibrio versatur, si recta a centro inertiae ad punctum contactus ducta fuerit verticalis; id quod semper duplici saltem modo evenire potest. Namque si ex centro inertiae ad omnia superficiei puncta rectae concipiantur ductae, quoniam nulla earum vel evanescit vel sit infinita, inter eas necesse est dari et maximam et minimam; utraque autem ad planum tangens erit normalis; quare si corpus alterutro eorum punctorum, a quibus centrum inertiae vel maxime vel minime distat, plano horizontali incumbat, recta ex centro inertiae ad punctum contactus ducta erit verticalis ideoque situm aequilibrii dabit, eumque stabilem, si recta ista fuerit minima, contra vero labilem, si maxima; unde intelligitur centrum inertiae semper infimum locum quaerere, ubi acquiescat. Saepenumero autem plures dantur aequilibrii situs, alii stabiles alii labiles, qui se alternatim excipere debent, quoniam corpus ex situ labili digressum in stabilem perveniat necesse est.

SCHOLION 2

889. In praesente casu, quo corporis superficiem sphaericam statuimus, recta per centrum inertiae *I* et centrum figurae *G* ducta dabit duo illa puncta *F* et *D*, quibus si corpus plano horizontali incumbat, situm aequilibrii teneat ; ac dum puncto *F* planum horizontale tangit, situs aequilibrii erit stabilis, si $FI < FG$ seu $f < 0$, labilis autem, si $FI > FG$ seu $f > 0$; neque praeter hos duos situs aequilibrii alius hic dabitur, nisi fuerit $f = 0$, quo casu subito omnes plane situs aequilibrii indolem recipiunt. Etsi autem hic totam corporis superficiem ut sphaericam considero, tamen ad institutum nostrum sufficit, si ea saltem portio, qua durante motu planum horizontale contingit, fuerit sphaeriea; atque hinc ista tractatio etiam ad eos turbines patet, quorum axes inferius non in cuspidem, ut ante assumsimus, sed in haemisphaerium vel etiam minus sphaerae segmentum efformantur, ita ut forma supra considerata hinc prodeat, si radius sphaerae $GF = e$ evanescat, sicque haec tractatio superiorem in se complectatur. Recta igitur *DIGF* per centrum inertiae *I* et centrum basis sphaerieae *G* dueta proprium turbinis axem exhibet, quae quidem, uti turbines construi solent, simul unus est axium principalium corporis, bini vero reliqui momenta inertiae habent paria, qualem formam iam supra statuimus. Verum quo haec tractatio latius pateat simulque ad titubationes corporum quorumcumque basi sphaerica praeditorum accommodari queat, axes corporis principales utcumque ab axe proprio *DF* diversos considerabo eorumque respectu momenta virium explorabo.

PROBLEMA 105

890. Data pressione Π, quo corpus basi sphaerica praeditum plano horizontali incumbit, definire momenta inde orta respectu axium principalium corporis, quomodocunque hi ratione axis proprii corporis fuerint dispositi.

SOLUTIO

Circa corporis centrum inertiae *I* descripta sphaera (Fig.l14), sit *Z* eius punctum

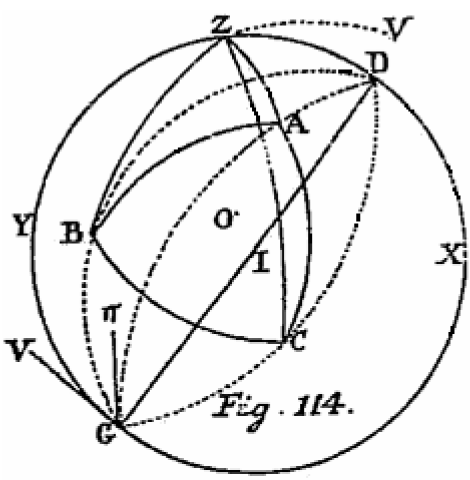


Fig. 114.

verticale, axisque proprius teneat iam situm *DIG*, ut eius declinatio a situ verticali sit $DZ = \rho$. Cum ergo directio pressionis Π sit verticalis et per punctum *G* transeat existente $IG = f$, referat recta verticalis $G\Pi$ hanc pressionem = Π, ita ut *ZDGII* sit planum verticale, in quo resolvatur vis $G\Pi = \Pi$ secundum directiones *GI* et *GV*, quarum haec ad illam sit normalis, et ob angulum $DG\Pi = \rho$ prodit vis secundunl $GI = \Pi \cos\rho$ et vis secundum $GV = \Pi \sin\rho$, quarum illa per centrum inertiae *I* transiens nulla suggerit momenta. Sint nunc *IA, I B, IC* corporis tres axes principales, datum situm ratione axis proprii *ID* tenentes, ac per puncta *A, B,* C ex *D* ducantur semicirculi *DAG, DBG, DCG.* Quodsi axis *IA* esset normalis ad planum *IGV*, eius respectu foret momentum vis $GV = \Pi f \sin\rho$, quod autem nunc in ratione sinus totius tam ad sinum arcus *GA* quam ad sinum anguli *VGA* minui debet, ita ut ex vi pressionis resultet

mom. respectu axis $IA = \Pi f \sin\rho \cdot \sin GA \cdot \sin VGA$ in sensum CB,
mom. respectu axis $IB = \Pi f \sin\rho \cdot \sin GB \cdot \sin VGB$ in sensum AC,
mom. respectu axis $IC = \Pi f \sin\rho \cdot \sin GC \cdot \sin VGC$ in sensum BA.

Haec autem terna momenta supra litteris *P, Q, R* indicavimus [803] quatenus quidem in sensum contrarium agere statuuntur, quare omnibus ad punctum *D* translatis habebimus:

$$P = -\Pi f \sin DZ \cdot \sin DA \cdot \sin ZDA = -\Pi f \sin ZD \cdot \sin ZA \cdot \sin DZA,$$
$$Q = -\Pi f \sin DZ \cdot \sin DB \cdot \sin ZDB = -\Pi f \sin ZD \cdot \sin ZB \cdot \sin DZB,$$
$$R = -\Pi f \sin DZ \cdot \sin DC \cdot \sin ZDC = -\Pi f \sin ZD \cdot \sin ZC \cdot \sin DZC .$$

COROLLARIUM 1

891. Assumsimus hic centrum basis *G* propius esse termino imo *F* quam centrum inertiae *I;* sin autem secus eveniat, ut intervalIum *FI* minus sit intervallo *FG = e*, intervalIum *GI = f* negative capi debet. At si fuerit *GI* = 0, momenta inventa evanescunt seu corpus in omni situ aequilibrium tenebit.

COROLLARIUM 2

892. Si pro situ axis proprii *ID* respeetu axium eorporis principalium ponatur arcus

$$AD = \zeta, \quad BD = \eta, \quad CD = \vartheta,$$

tum vero angulus $ZDA = \varphi$, existente arcu $ZD = \rho$, ob

$$\cos ADB = \frac{\cos\zeta \cos\eta}{\sin\zeta \sin\eta} \quad \text{et} \quad \sin ADB = -\frac{\cos\vartheta}{\sin\zeta \sin\eta},$$

quia

$\sin ADB : \sin DAB$ seu $\sin ADB : -\cos DAC = 1 : \sin BD = 1 : \sin\eta$,

erit

$$\sin ZDB = \frac{-\cos\zeta \cos\eta \sin\phi + \cos\vartheta \cos\varphi}{\sin\zeta \sin\eta},$$

at

$$\cos DAC = \frac{\cos CD}{\sin AD} = \frac{\cos\vartheta}{\sin\zeta},$$

ideoque

$$P = -\Pi f \sin\rho \sin\zeta \sin\varphi,$$

atque

$$Q = \frac{\Pi f \sin\rho(\cos\zeta \cos\eta \sin\phi - \cos\vartheta \cos\varphi)}{\sin\zeta},$$

et

$$R = \frac{\Pi f \sin\rho(\cos\zeta \cos\vartheta \sin\phi - \cos\eta \cos\varphi)}{\sin\zeta}.$$

COROLLARIUM 3

893. Si axis proprius *ID* eongrueret cum axe principali *IA*, foret $C = \vartheta$ atque $\eta = \vartheta = 90°$, ut esset $\cos\eta = \cos\vartheta = \sin\zeta$ et angulus φ maneret indefinitus. At ex prioribus formulis fient momenta virium:

$$P = 0, \quad Q = -\Pi f \sin\rho \sin ZAB \quad \text{et} \quad R = -\Pi f \sin\rho \sin ZAC$$

seu

$$P = 0, \quad Q = \Pi f \sin ZC \quad \text{et} \quad R = -\Pi f \sin ZB.$$

COROLLARIUM 4

894. Quodsi vero ut supra ponamus

$$ZA = l,\ ZB = m \quad \text{et} \quad ZC = n\ ,$$

reperiemus momenta virium in genere
atque

$$P = \Pi f(\cos\vartheta\cos m - \cos\eta\cos n),$$

$$Q = \Pi f(\cos\zeta\cos n - \cos\vartheta\cos l)$$

et

$$R = \Pi f(\cos\eta\cos l - \cos\zeta\cos m),$$

unde illa, $\sin\zeta = 0$ et $\eta = \vartheta = 90°$, sponte sequuntur.

EXPLICATIO

895. Ratio investigationis harum posteriorum formularum ita se habet:
primo cum sit

$$\sin DZ \cdot \sin ZDA = \sin ZA \cdot \sin ZAD\ ,$$

erit

$$P = -\Pi f \sin DA \cdot \sin ZA \cdot \sin ZAD\ ;$$

at est

$$ZAD = BAD - BAZ\ ,$$

et

$$\sin BAD = -\cos CAD = \frac{-\cos\vartheta}{\sin DA}, \quad \cos BAD = \frac{\cos\eta}{\sin DA},$$

$$\sin BAZ = -\cos CAZ = \frac{-\cos n}{\sin ZA}, \quad \cos BAZ = \frac{\cos m}{\sin ZA},$$

unde

$$\sin ZAD = \frac{-\cos m\cos\vartheta + \cos n\cos\eta}{\sin ZA \sin DA}$$

et

$$P = \Pi f(\cos\vartheta\cos m - \cos\eta\cos n).$$

Reliqua duo momenta *Q* et *R* praebet analogia sine ulteriori calculo. Deinde vero est

$$\cos DZ = \cos\rho = \cos\zeta\cos l + \cos\eta\cos m + \cos\vartheta\cos n,$$

quae expressio, uti unitatem nunquam superare potest, ita unitati aequalis esse nequit seu

$DZ = 0$, nisi sit $l = \zeta$, $m = \eta$ et $n = \vartheta$, scilicet has ternas determinationes simul suppeditat haec aequatio:

$$\cos\zeta \cos l + \cos\eta \cos m + \cos\vartheta \cos n = 1.$$

Cum enim praeterea sit

$$\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$$

et

$$\cos^2 l + \cos^2 m + \cos^2 n = 1,$$

si a summa harum illius duplum subtrahatur, prodit

$$(\cos\zeta - \cos l)^2 + (\cos\eta - \cos m)^2 + (\cos\vartheta - \cos n)^2 = 0,$$

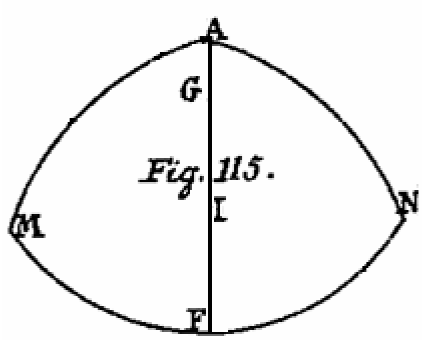
trium autem quadratorum summa nihilo aequari nequit, nisi singula sint nulla.

SCHOLION

896. Cum neque in his expressionibus pro momentis virium, *P, Q, R* inventis, neque in pressione

$$\Pi = M\left(1 + \frac{fdd \cdot \cos\rho}{2gdt^2}\right)$$

radius sphaerae *e* basin constituentis insit, omnia, quae supra de motu turbinis infra in cuspidem desinentis sunt tradita, etiam valent de eiusmodi turbinibus, qui desinunt in haemisphaerium seu aliam sphaerae partem, dummodo punctum *F*, quod ante cuspidem denotabat, hic in centro figurae sphaericae *G* constituatur. Perinde ergo est, sive turben gyretur super cuspide, sive super hemisphaerio, dummodo *f* sit distantia centri inertiae *I* a centro basis sphaericae, quantuscunque enim fuerit radius huius basis *e*, is in computum non ingreditur, eo autem evanescente basis turbinis abit in cuspidem. Totum igitur caput praecedens hic inseri intelligatur, ita ut Theoria turbinum sine ullo labore haud mediocriter amplificata sit censenda. Basin autem sphaericam faciendo casus occurrit ante exclusus, scilicet quo centrum inertiae *I* fundo proprius est, quam centrum sphaericitatis, hicque sit quantitas *f* negativa. Sive iam tale corpus sit globus completus, sive basin habeat *MFN* portionem sphaerae (Fig. 115), centro *G* descriptae, qua plano horizontali incumbat, eius motum, quatenus contactus in hanc basin cadit, investigemus. Hic autem cogimur corpori talem indolem tribuere, ut axis proprius *AGIF*, qui si fuerit verticalis, statum quietis exhibet, simul sit corporis axis principalis, reliqui vero bini axes principales habeant momenta inter se aequalia. Scilicet si respectu axis *IA* momentum inertiae sit = *Maa*, respectu binorum reliquorum vero *Mbb* et *Mcc*, statuemus *bb* = *cc*. Huiusmodi ergo corpus quemcunque receperit motum impressum, quomodo motum sit continuaturum, determinemus.



PROBLEMA 106

897. Si corpus basi sphaerica *MFN* instructum, in quo axis aequilibrii *AGJF* sit axis principalis, eiusque respeetu momentum inertiae = *M aa*,respectu binorum reliquorum autem aequalia = *Mcc,* motum acceperit quemcunque, determinare motus continuationem.

SOLUTIO

Sit radius basis sphaericae *FG* = *e,* centrum autem inertiae *I* cadat infra centrum basis *G* ad intervalIum *GI* = *f.* Pro motu progressivo, si centrum inertiae *I* habuerit motum secundum directionem horizontalem, eum constantem in directum conservabit, quatenus autem motu verticali cietur, is cognita pressione, quae sit = Π , inde definietur, quod si declinatio axis *AF* a situ verticali ponatur = ρ , sit

$$\frac{\Pi}{M} = 1 - \frac{fdd \cdot cos\,\rho}{2gdt^2},$$

existente *M* corporis massa seu pondere. Verum ipsa haec pressio Π , quam corpus in planum horizontale exerit, non nisi ex motu gyratorio cognosci potest. Teneat ergo corpus nostrum respectu sphaerae fixae (Fig.lll), in qua *Z* est punctum verticale, nunc elapso tempore *t* eiusmodi situm, ut eius axes principales in *A*, *B*, *C* pertingant, ponanturque arcus

$$ZA = l\,,\ ZB = m,\ ZC = n,$$

tum vero anguli

$$XZA = \lambda, \quad XZB = \mu, \quad XZC = \nu\ ,$$

ita ut sit $l = \rho$. Nunc autem gyretur circa polum *O* in sensum *ABC* celeritate angulari = γ' , ac positis arcubus

$$OA = \alpha\ , \quad OB = \beta, \quad OC = \gamma$$

sit brevitatis gratia

$$\gamma' cos\,\alpha = x, \quad \gamma' cos\,\beta = y, \quad \gamma' cos\,\gamma = z\ .$$

Momenta autem virium ex pressione Π orta sunt

$$P = O, \quad Q = -\Pi\ fcosn, \quad R = +\Pi\ fcosm,$$

unde colligimus sequentes aequationes:

$$dx = 0,$$

$$dy + \frac{aa-cc}{cc}\ xzdt = \frac{-2\Pi\ fgdt\cos n}{Mcc},$$

$$dz + \frac{cc-aa}{cc}\ xydt = \frac{2\Pi\ fgdt\cos m}{Mcc},$$

$$dl\,sin\,l = dt(ycosn - zcosm), \qquad d\lambda sin^2 l = -dt(\,ycosm + zcosn\,),$$

$$dm\,sin\,m = dt(z\cos l - xcosn\,),$$ reliqui anguli μ et ν hinc sponte

$$dn\,sin\,n = dt(xcosm - y\cos l),$$ dantur.

Si porro ponamus

$$\cos l = p, \quad \cos m = q, \quad \cos n = r \;,$$

quoniam hae aequationes congruunt cum iis, quas supra problema 99 integravimus, nisi quod *f* capiatur negative, habebimus in finitis has aequationes:
$x=A$ et

$$I.\ qy+rz=B-\frac{Aaap}{cc},$$

$$11.\ (qz-ry)^2=\frac{(Ccc+4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff(1-pp)},$$

$$111.\ yy+zz=\frac{Ccc+4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff(1-pp)},$$

$$\text{IV.}\ \frac{\Pi}{M}=\frac{2gcc-Afaa\left(B-\frac{Aaap}{cc}\right)}{2g(cc+ff(1-pp))}+\frac{fccp\left(Ccc+4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2\right)}{2g(cc+ff(1-pp))^2},$$

$$\text{V.}\ dt=\frac{dp\sqrt{(cc+ff(1-pp))}}{\left((Ccc+4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2\right)},$$

$$\text{VI.}\ d\lambda=\frac{-dt}{1-pp}\left(B-\frac{Aaap}{cc}\right),$$

$$\text{VII.}\ \gamma'\gamma'=AA+\frac{Ccc+4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff(1-pp)},$$

$$\text{VIII.}\ \frac{ydz-zdy}{yy+zz}=\frac{A(aa-cc)dt}{cc}+\frac{2\Pi fgdt\left(B-\frac{Aaap}{cc}\right)(cc+ff-ffpp)}{Mcc\left(Ccc+4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2\right)},$$

ubi constantes *A, B, C* et reliquae per integrationem ingressurae ex statu corporis initiali debent definiri.

COROLLARIUM 1

898. Si corpus initio quieverit axisque principalis *A* fuerit in *a* declinatione eius existente $Za=\mathfrak{l}$ et $\cos\mathfrak{l}=\mathfrak{p}$, initio erat $x=0$, $y=0$ et $z=0$, ob $\gamma'=0$; atque $p=\mathfrak{p}$. Fit ergo

$$A=O,\ B=O \quad \text{et} \quad Ccc=-4fg\mathfrak{p}.$$

Hinc elapso tempore *t* erit

$$x=0,\ qy+rz=0,$$

$$qz-ry=\frac{2\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{(cc+ff(1-pp))}},$$

$$yy+zz=\frac{4fg(p-\mathfrak{p})}{cc+ff(1-pp)}=\gamma'\gamma'$$

et

$$\frac{\Pi}{M}=\frac{cc}{cc+ff(1-pp)}+\frac{2ffccp(p-\mathfrak{p})}{\left(cc+ff(1-pp)\right)^2}$$

COROLLARIUM 2

899. Praeterea vero in eodem casu est $d\lambda=0$ ideoque axis, qui initio in *a* erat, per ipsum arcum *aZ* movebitur, eritque

$$dt=\frac{dp\sqrt{\left(cc+ff(1-pp)\right)}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}},$$

unde, quia $p>\mathfrak{p}$ seu $l<\mathfrak{l}$, axis ab *a* recta ad *Z* progreditur. Denique ob

$$ydz-zdy=0,$$

fit

$$z=\delta y \quad \text{et} \quad y=\frac{2\sqrt{fg(p-\mathfrak{p})}}{\sqrt{(1+\delta\delta)\left(cc+ff(1-pp)\right)}}$$

atque

$$q\left(yy+zz\right)=\frac{2z\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{\left(cc+ff(1-pp)\right)}}$$

seu

$$q=\frac{\delta\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}} \quad \text{et} \quad r=\frac{-\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}},$$

unde fit

$$\cos ZAB=\frac{q}{\sqrt{(1-pp)}}=\frac{\delta}{\sqrt{(1+\delta\delta)}},$$

qui ergo angulus manet constans.

COROLLARIUM 3

900. Si ergo corpus initio quiescat eiusque axis principalis *IA* tenuerit situm inclinatum *Ia*, inde recta se eriget ex *a* ad *Z* ascendens, gyrabitur autem circa punctum *O*, ut ob $x=\gamma'\cos\alpha=0$ arcus *AO* sit quadrans, et quia

$$\cos ZO=\cos\alpha\cos l+\cos\beta\cos m+\cos\gamma\cos n=\frac{qy+rz}{\gamma'}=0,$$

erit etiam *ZO* quadrans, sicque *O* erit polus circuli *XZY*. Et cum axis in *Z* pervenerit, erit

$$\text{celeritas angularis } \gamma'=\frac{2\sqrt{fg(1-\mathfrak{p})}}{c}.$$

SCHOLION 1

901. Si corpus initio non quieverit, sed motum quemcunque acceperit, continuatio motus ex iisdem formulis determinatur, dummodo constantes *A*, *B*, *C* statui initiali convenienter definiantur; ubi autem ad eiusmodi formulas integrandas devenitur, quae nonnisi concessis quadraturis superioris ordinis expediri possunt. Quin etiam casus hic simplicissimus, quo corpus initio in situ inclinato quievit, ab integratione formulae huius

$$dt = \frac{dp\sqrt{(cc + ff(1-pp))}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}},$$

pendet, quae neque per logarithmos neque arcus circulares absolvi potest. At si declinatio initialis *Za* fuerit quasi infinite parva, negotium ad arcus circulares perducitur: fit enim initio $Za = \mathfrak{l}$, et elapso tempore *t* declinatio *Za* = *l*, ob $\mathfrak{l}$ 1et *l* arcus minimos, erit

$p = 1 - \frac{1}{2}ll, \quad dp = -l\ dl$ et $\mathfrak{p} = 1 - \frac{1}{2}\mathfrak{l}\mathfrak{l}$, unde

$$dt = \frac{-cdl}{\sqrt{2fg(\mathfrak{l}\mathfrak{l} - ll)}} \quad \text{et} \quad t = \frac{c}{\sqrt{2fg}} A\cos\frac{l}{\mathfrak{l}} \quad \text{seu} \quad l = \mathfrak{l}\cos\frac{t\sqrt{2fg}}{c}.$$

Quare axis *IA* fiet verticalis elapso tempore = $\frac{\pi c}{2\sqrt{2fg}}$ et corpus titubationes isochronas conficiet, uti pendulum simplex longitudinis = $\frac{cc}{f}$.

SCHOLION 2

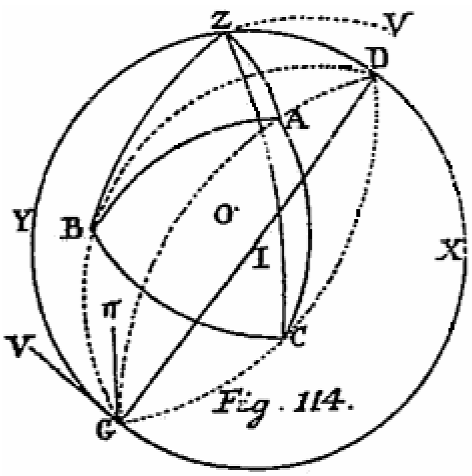
902. Nisi corpori eiusmodi indolem tribuissemus, ut eius axis naturalis *FD*, qui in statu quietis sit verticalis, simuI esset eius axis principalis, binique reliqui haberent momenta inertiae aequalia, formulas quidem differentiales motum eius continentes assignare, nullo autem modo ob analyseos defectum ipsum motum definire potuissemus. Interim tamen, quemadmodum in casu tractato, ubi corpori infinite parvam declinationem tribuimus, usu venit, ut motus fieret satis simplex motuique penduli conformis, id adeo in genere locum habet, quomodocunque axes principales respectu axis naturalis fuerint dispositi. In situ scilicet aequilibrii, ubi axis naturalis *DF* situm tenet verticalem, assumo centrum inertiae *I* infra centrum basis sphaericae *G* ad intervallum *GI* = *f* cadere; tum vero hoc corpus infinite parum de situ suo quietis declinari ponamus, ut arcus $ZD = \rho$ sit infinite parvus (Fig.114), atque evidens est corpus se restituendo oscillationes seu titubationes esse peracturum, donec tandem motu ob resistentiam extincto in statu aequilibrii requiescat. Quoniam declinatio corporis hic perpetuo est minima, non opus est, ut tota corporis figura sit sphaerica, sed sufficit, si infima eius portio eaque minima, qua plano horizontali applicatur, sit pars superficiei sphaerieae, cuius centrum est in *G*. Hunc igitur motum titubatorium investigaturi primo dispiciamus, quomodo formulae supra in genere erutae pro hoc casu, quo axis corporis naturalis *DF* quam minime a situ verticali declinat, contrahi indeque momenta virium *P*, *Q*, *R* ita commode definiri queant, ut deinceps ex iis motum assignare valeamus.

PROBLEMA 107

902 [a]. Si corpus basi sphaerica instructum infinite parum a situ aequilibrii declinet, definire momenta virium respeetu ternorum eius axium principalium.

SOLUTIO

Circa corporis centrum inertiae *I* descripta sphaera (Fig.114), in qua *Z* sit punctum verticale, teneat axis corporis naturalis *ID* situm a verticali minime declinans, ut sit arcus



$ZD = \rho$ minimus; axes autem corporis principales respondeant punctis *A*, *B*, *C*, quorum situs ratione puncti *D* ita se habeat, ut sint arcus

$$DA = \zeta, \quad DB = \eta, \quad DC = \vartheta,$$

qui sunt constantes. Nunc autem respectu puncti verticalis *Z* sint arcus

$$ZA = l, \quad ZB = m \quad \text{et} \quad ZC = n \ ,$$

qui ob arcum $ZD = \rho$ minimum vix discrepabunt ab illis ζ, η, ϑ; quare si ponamus:

$$cos\, l = cos\, \zeta + p \ ,$$

$$cos\, m = cos\, \eta + q,$$

$$cos\, n = cos\, \vartheta + r \ ,$$

quantitates *p, q, r* erunt minimae. Quia vero est tam

$$cos^2 \zeta + cos^2 \eta + cos^2 \vartheta = 1,$$

quam

$$cos^2 l + cos^2 m + cos^2 n = 1,$$

fiet

$$2p\, cos\, \zeta + 2q\, cos\, \eta + 2r\, cos\, \vartheta + pp + qq + rr = 0 .$$

Deinde autem cum sit

$$cos\, \rho = cos\, \zeta\, cos\, l + cos\, \eta\, cos\, m + cos\, \vartheta\, cos\, n \ ,$$

erit

$$cos\, \rho = 1 + p\, cos\, \zeta + q\, cos\, \eta + r\, cos\, \vartheta$$

ideoque

$$cos\ p\, cos\, \zeta + q\, cos\, \eta + r\, cos\, \vartheta = -\tfrac{1}{2}\rho\rho$$

et

$$pp + qq + rr = \rho\rho .$$

Nunc igitur, posita pressione corporis in planum horizontale $= \Pi$, ex § 894 tribuendo ipsi *f* valorem negativum obtinebimus momenta virium respectu axium principalium:

$$P = IIf\,(r\, cos\, \eta - q\, cos\, \vartheta) \ ,$$

$$Q = IIf\,(p\, cos\, \vartheta - r\, cos\, \zeta) \ ,$$

et

$$R = \Pi f\,(q\cos\zeta - p\cos\eta),$$

tum vero vidimus esse

$$\Pi = M\left(1 - \frac{fdd\cdot\cos\rho}{2gdt^2}\right);$$

quia autem $\cos\rho$ proxime est = 1 et minimas variationes subit, erit satis exacte $\Pi = M$, ita ut corpus toto suo pondere planum horizontale premere sit censendum; sicque habebimus

$$P = Mf\left(r\cos\eta - q\cos\vartheta\right),$$

$$Q = Mf\left(p\cos\vartheta - r\cos\zeta\right),$$

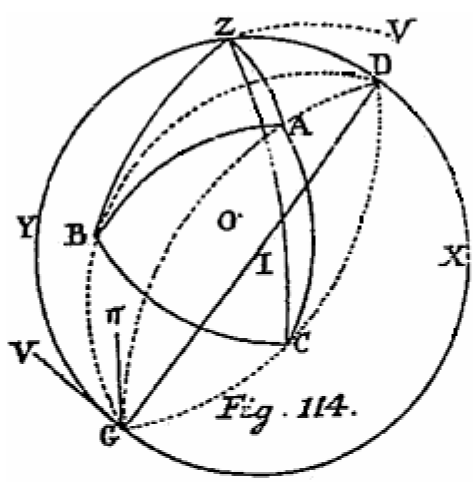
$$R = Mf\left(q\cos\zeta - p\cos\eta\right).$$

PROBLEMA 108

903. Si corpus basi sphaerica praeditum de situ quietis, in quo axis *D1* est verticalis, parumper declinetur iterumque demittatur, ut ex quiete versus statum aequilibrii revertatur, determinare eius motum.

SOLUTIO

Elapso tempore *t* teneat corpus situm in fig. 114 repraesentatum maneantque omnes denominationes in problemate praecedente stabilitae, tum vero sint corporis momenta inertiae *Maa*, *Mbb*, *Mcc* respectu axium principalium *IA*, *IB*, *IC*. Nunc autem corpus gyretur circa axem *IO* in sensum *ABC* celeritate angulari = γ' sintque arcus



$$AO = \alpha, \quad BO = \beta, \quad CO = \gamma.$$

ac ponatur

$$\gamma'\cos\alpha = x, \quad \gamma'\cos\beta = y, \quad \gamma'\cos\gamma = z$$

Quoniam igitur initio, ubi *t* = 0, corpus ex quiete motum incipere assumitur, erat tum *x* = 0, *y* = 0 et *z* = 0. Tum vero, quia motus corporis perpetuo manet tardissimus, quantitates *x, y, z* semper manebunt minimae, ita ut binarum producta *xy*, *xz* et *yz* prae singulis pro evanescentibus haberi queant. Cum ergo momenta virium sollicitantium *P, Q, R* modo sint definita ex § 810, sequentes adipiscimur aequationes

$$dx = \frac{2\,fgdt}{aa}\left(r\cos\eta - q\cos\vartheta\right),$$

$$dy = \frac{2\,fgdt}{bb}\left(p\cos\vartheta - r\cos\zeta\right),$$

$$dz = \frac{2\,fgdt}{cc}\left(q\cos\zeta - p\cos\eta\right)$$

Deinde quia est

$$\cos l = \cos\zeta + p,$$

$$\cos m = \cos\eta + q,$$

et

$$\cos n = \cos \vartheta + r \ ,$$

ob ζ, η, ϑ constantes erit

$$dl \sin l = -dp,$$
$$dm \sin m = -dq$$

et

$$dn \sin n = -dr \ ,$$

unde insuper hae ternae aequationes accedunt

$$-\ dp = dt\left(y \cos \vartheta - z \cos \eta\right),$$
$$-\ dq = dt\left(z \cos \zeta - x \cos \vartheta\right),$$
$$-\ dr = dt\left(x \cos \eta - y \cos \zeta\right),$$

ubi producta *yr, zq, zp, xr, xq, yp* ut minima prae terminis hic exhibitis omittimus. Denique si arcus *ZA* a circulo quodam verticali fixo nunc declinare statuatur angulo λ, ob

$$\sin^2 l = \sin^2 \zeta - 2p \cos \zeta$$

habebimus hanc aequationem:

$$d\lambda = \frac{-dt\left(y \cos \eta + z \cos \vartheta\right)}{\sin^2 \zeta - 2p \cos \zeta}.$$

Quia autem in superioribus aequationibus quantitates *x, y, z* et *p, q, r* ubique unam dimensionem occupant atque *x, y, z* posito $t = 0$ evanescere debent, manifestum est tam huic conditioni quam sex illis aequationibus satisfieri posse ponendo:

$$x = A \sin \delta t, \quad y = B \sin \delta t, \quad z = C \sin \delta t,$$
$$p = D \cos \delta t, \quad q = E \cos \delta t, \quad r = F \cos \delta t,$$

tum enim ternae priores aequationes per $\cos \delta t$ divisae et ternae posteriores per $\sin \delta t$ divisae dabunt

$$A\delta = \frac{2fg}{aa}\left(F \cos \eta - E \cos \vartheta\right), \quad D\delta = B \cos \vartheta - C \cos \eta,$$
$$B\delta = \frac{2fg}{bb}\left(D \cos \vartheta - F \cos \zeta\right), \quad E\delta = C \cos \zeta - A \cos \vartheta,$$
$$C\delta = \frac{2fg}{cc}\left(E \cos \zeta - D \cos \eta\right), \quad F\delta = A \cos \eta - B \cos \zeta.$$

Ex posterioribus substituantur valores coefficientium *D, E, F* in prioribus et obtinebimus:

$$\frac{A\delta\delta aa}{2fg} = A \cos^2 \eta - B \cos \zeta \cos \eta - C \cos \zeta \cos \vartheta + A \cos^2 \vartheta,$$
$$\frac{B\delta\delta bb}{2fg} = B \cos^2 \vartheta - C \cos \eta \cos \vartheta - A \cos \zeta \cos \eta + B \cos^2 \zeta,$$
$$\frac{C\delta\delta cc}{2fg} = C \cos^2 \zeta - A \cos \zeta \cos \vartheta - B \cos \eta \cos \vartheta + C \cos^2 \eta.$$

Quodsi iam brevitatis gratia ponamus

$$A\cos\zeta + B\cos\eta + C\cos\vartheta = G\ ,$$

ob

$$\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$$

erit

$$A\left(1-\frac{\delta\delta aa}{2fg}\right) = G\cos\zeta,\quad B\left(1-\frac{\delta\delta bb}{2fg}\right) = G\cos\eta \quad \text{et} \quad C\left(1-\frac{\delta\delta cc}{2fg}\right) = G\cos\vartheta\,.$$

Ponamus brevitatis causa $\frac{\delta\delta}{2fg} = u$, ut fiat

$$A = \frac{G\cos\zeta}{1-aau},\quad B = \frac{G\cos\eta}{1-bbu} \quad C = \frac{G\cos\vartheta}{1-ccu}$$

Cum autem sit

$$A\cos\zeta + B\cos\eta + C\cos\vartheta = G\ ,$$

erit

$$\frac{\cos^2\zeta}{1-aau} + \frac{\cos^2\eta}{1-bbu} + \frac{\cos^2\vartheta}{1-ccu} = 1,$$

qua aequatione evoluta consequimur per *u* dividendo

$$aabbccuu - bbccu\sin^2\zeta + aa\cos^2\zeta - aaccu\sin^2\eta$$
$$+bb\cos^2\eta - aabbu\sin^2\vartheta + cc\ \cos^2\vartheta = 0.$$

Statuantur quantitates cognitae

$$bbcc\sin^2\zeta + aacc\sin^2\eta + aabb\sin^2\vartheta = Kaabbcc\ ,$$
$$aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta = Laabbcc,$$

ut sit

$$uu - Ku + L = 0,$$

hincque

$$u = \frac{\delta\delta}{2fg} = \frac{1}{2}K + \sqrt{\left(\frac{1}{4}KK - L\right)}$$

et quantitas *G* manet indefinita ex statu initiali definienda, dum contra quantitates *K* et *L* sunt ex natura corporis datae, cum igitur hinc inventus sit valor ipsius *u*, inde habemus $\delta = \sqrt{2fgu}$ et

$$A = \frac{G\cos\zeta}{1-aau},\quad B = \frac{G\cos\eta}{1-bbu} \quad C = \frac{G\cos\vartheta}{1-ccu}$$

$$D = \frac{Gu(bb-cc)\cos\eta\cos\vartheta}{\delta(l-bbu)(1-ccu)},$$

$$E = \frac{Gu(cc-aa)\cos\zeta\cos\vartheta}{\delta(l-ccu)(1-aau)},$$

et

$$F = \frac{Gu(aa-bb)\cos\zeta\cos\eta}{\delta(l-aau)(1-bbu)}.$$

Si iam initio fuerit arcus $ZD = \mathfrak{r}$, qui nunc est ρ, cum sit initio $p = D$, $q = E$, $r = F$, habebimus

$$DD + EE + FF = \mathfrak{rr},$$

unde per $\mathfrak{r}$ invenitur constans G. Denique pro angulo λ inveniendo prodit

$$d\lambda = \frac{-dt(B\cos\eta + C\cos\vartheta)\sin\delta t}{\sin^2\zeta}$$

ideoque

$$\lambda = \frac{(B\cos\eta + C\cos\vartheta)(\cos\delta t - 1)}{\delta\sin^2\zeta},$$

si quidem arcus ZA initio fuerit in verticali fixo, indeque in sensum XOY moveri sumatur, quatenus ergo haec expressio pro λ est negativa, in sensum contrarium axis IA circa Z gyrari est censendus. Denique cum sit

$$pp + qq + rr = \rho\rho$$

erit

$$\rho = \mathfrak{r}\cos\delta t$$

ob

$$\mathfrak{r} = \sqrt{(DD + EE + FF)},$$

unde patet axem ID in situm verticalem erigi elapso tempore $= \frac{\pi}{2\delta}$ et titubationes isochronas fore oscillationibus penduli, cuius longitudo est

$$= \frac{2g}{\delta\delta} = \frac{1}{fu} = \frac{K - \sqrt{(KK - 4L)}}{2Lf}.$$

COROLLARIUM 1

904. Cum sit $DD + EE + FF = \mathfrak{rr}$, erit

$$\delta\delta\mathfrak{rr} = AA\left(\cos^2\eta + \cos^2\vartheta\right) + BB\left(\cos^2\zeta + \cos^2\vartheta\right) + CC\left(\cos^2\zeta + \cos^2\eta\right)$$
$$-2BC\cos\eta\cos\vartheta - 2AC\cos\zeta\cos\vartheta - 2AB\cos\zeta\cos\eta$$

et quia

$$G = A\cos\zeta + B\cos\eta + C\cos\vartheta,$$

huius quadratum eo additum dabit

$$\delta\delta\mathfrak{rr} + GG = AA + BB + CC,$$

ubi si brevitatis gratia ponatur

$$\frac{1}{1-aau}=\mathfrak{P},\quad \frac{1}{1-bbu}=\mathfrak{Q}\;,\quad \frac{1}{1-ccu}=\mathfrak{R},$$

ob

$$\mathfrak{P}\cos^2\zeta+\mathfrak{Q}\cos^2\eta+\mathfrak{R}\cos^2\vartheta=1,$$

et

$$A=G\mathfrak{P}\cos\zeta,\quad B=G\mathfrak{Q}\cos\eta\quad \text{et}\quad C=G\mathfrak{R}\cos\vartheta$$

fiet

$$\delta\delta\mathfrak{rr}=GG\left(\mathfrak{PP}\cos^2\zeta+\mathfrak{QQ}\cos^2\eta+\mathfrak{RR}\cos^2\vartheta-1\right)$$

ideoque ob

$$\mathfrak{PP}-\mathfrak{P}=\frac{aau}{(1-aau)^2}$$

habebitur

$$\delta\delta\mathfrak{rr}=GGu\left(\frac{aa\cos^2\zeta}{(1-aau)^2}+\frac{bb\cos^2\eta}{(1-bbu)^2}+\frac{cc\cos^2\vartheta}{(1-ccu)^2}\right).$$

COROLLARIUM 2

905. Quia porro est $\delta\delta=2fgu$, si in subsidium vocetur aequatio $uu-Ku+L=0$, reperietur

$$GG=\frac{-2fg\mathfrak{rr}(1-aau)(1-bbu)(1-ccu)}{aa\cos^2\zeta+bb\cos^2\eta+cc\cos^2\vartheta-aabbccuu}\,.$$

EXPLICATIO

906. Haec expressio pro *GG* satis concinna sequenti modo eruitur: Posito brevitatis gratia

$$\frac{1}{aa}=\mathfrak{a},\quad \frac{1}{bb}=\mathfrak{b},\quad \frac{1}{cc}=\mathfrak{c},$$

habemus:

$$\text{I. } K=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}-\mathfrak{a}\cos^2\zeta-\mathfrak{b}\cos^2\eta-\mathfrak{c}\cos^2\vartheta,$$

$$\text{II. } L=\mathfrak{bc}\cos^2\zeta+\mathfrak{ac}\cos^2\eta+\mathfrak{ab}\cos^2\vartheta,$$

$$\text{III. } 1=\cos^2\zeta+\cos^2\eta+\cos^2\vartheta.$$

Hinc deducitur ob *uu* - *Ku* + *L* = 0 ,

$$\cos^2\zeta=\frac{\mathfrak{a}K-L-\mathfrak{aa}}{(\mathfrak{a}-\mathfrak{b})\,(\mathfrak{c}-\mathfrak{a})}$$

et

$$u\cos^2\zeta=\frac{(\mathfrak{a}-u)(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})\,(\mathfrak{c}-\mathfrak{a})}$$

ideoque

$$\frac{\delta\delta\mathfrak{rr}}{GG}=\frac{\mathfrak{a}(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})\,(\mathfrak{c}-\mathfrak{a})(\mathfrak{a}-u)}+\frac{\mathfrak{b}(L-\mathfrak{b}u)}{(\mathfrak{b}-\mathfrak{c})\,(\mathfrak{a}-\mathfrak{b})(\mathfrak{b}-u)}+\frac{\mathfrak{c}(L-\mathfrak{c}u)}{(\mathfrak{c}-\mathfrak{a})(\mathfrak{b}-\mathfrak{c})\,(\mathfrak{c}-u)}$$

ex qua aequatione reducta illa expressio obtinetur.

SCHOLION

907. Quoniam haec ad titubationes omnium corporum, quorum basis est portio sphaerica, patent, quomodocunque eius axes principales ratione axis naturalis *DGIF* fuerint dispositi eorumque respectu momenta inertiae inaequalia, ne in tanta amplitudine confundamur, conveniet prima formulas nostras ad species corporum simpliciores accommodari, quo inde facilius ad species magis complicatas progredi liceat. Ac prima quidem casus, quo omnia momenta inertiae sunt inter se aequalia seu $aa = bb = cc$, omnium est simplicissimus, quia tum axis *DF* pro principali haberi potest et titubationes eaedem prodire debent, quas iam ante definivimus. Tum vero duo saltern momenta inertiae aequalia statuamus, sicilicet $bb = cc$.

CASUS I
quo $aa = bb = cc$.

908. Hoc ergo casu habemus:

$$A=\frac{G\cos\zeta}{1-aau},\quad B=\frac{G\cos\eta}{1-bbu},\quad C=\frac{G\cos\vartheta}{1-ccu}$$

hincque

$$G=\frac{G\cos^2\zeta+G\cos^2\eta+G\cos^2\vartheta}{1-aau}=\frac{G}{1-aau},$$

ita ut sit $u = 0$. Verum iisdem quoque formulis satisfit ponendo $u=\frac{1}{aa}$ et $G = 0$, ut sit

$$A\cos\zeta+B\cos\eta+C\cos\vartheta=0,$$

neque quicquam praeterea determinetur, sicque habebimus

$$\delta=\frac{\sqrt{2fg}}{a};$$

tum vero

$$D=\frac{B\cos\vartheta-C\cos\eta}{\delta},\quad E=\frac{C\cos\zeta-A\cos\vartheta}{\delta},\quad F=\frac{A\cos\eta-B\cos\zeta}{\delta}$$

atque

$$\delta\delta\mathfrak{rr}=AA+BB+CC,$$

ut sit

$$\rho=\mathfrak{r}\cos\delta t.$$

Videamus iam, circa quemnam polum *O* corpus sit gyraturum, ac primo habemus

$$\cos OD=\cos\alpha\cos\zeta+\cos\beta\cos\eta+\cos\gamma\cos\vartheta,$$

seu

$$\gamma'\cos OD=x\cos\zeta+y\cos\eta+z\cos\vartheta=0,$$

sicque arcus *OD* quadrans. Deinde est

$$\cos OZ = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n$$

seu

$$\gamma'\cos OZ = x\cos l + y\cos m + z\cos n = 0 + px + qy + rz = 0$$

ob

$$AD + BE + CF = 0\ ,$$

eritque ergo etiam *OZ* quadrans. Ex quo perspicitur corpus circa punctum *O*, quod est polus circuli verticalis *ZDX*, gyrari sicque axem ex *D* recta in situm verticalem *Z* erigi, ita ut elapso tempore *t* sit

$$\rho = \mathfrak{r}\cos\frac{t\sqrt{2fg}}{a}.$$

Quare hae titubationes isochronae erunt oscillationibus penduli, cuius longitudo est $= \frac{aa}{f}$.

CASUS 11

quo duo tantum momenta principalia sunt aequalia seu *bb* = *cc*.

909. Hoc ergo casu est

$$K = \frac{cc\sin^2\zeta + aa\sin^2\eta + aa\sin^2\vartheta}{aacc} = \frac{cc\sin^2\zeta + aa + aa\cos^2\zeta}{aacc}$$

et

$$L = \frac{aa\cos^2\zeta + cc\sin^2\zeta}{aac^4};$$

sive cum aequatio, unde *u* definiri debet, sit

$$\frac{\cos^2\zeta}{1-aau} + \frac{\sin^2\zeta}{1-ccu} = 1,$$

erit

$$aa\cos^2\zeta + cc\sin^2\zeta = aaccu$$

ideoque

$$u = \frac{aa\cos^2\zeta + cc\sin^2\zeta}{aacc},$$

qui valor etiam ex generali forma elicitur, nisi quod hoc modo radix inutilis $u = \frac{1}{cc}$ excluditur.
Quamobrem habebimus

$$\delta = \frac{\sqrt{2fg}\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}{ac}$$

tum vero

$$A = \frac{Gcc}{(-aa+cc)\cos\zeta},$$

$$B = \frac{Gaa\cos\eta}{(aa-cc)\sin^2\zeta},$$

$$C = \frac{Gaa\cos\vartheta}{(aa-cc)\sin^2\zeta}.$$

Deinde pro *G* ex $\mathfrak{r}$ inveniendo fit

$$\delta\delta\mathfrak{rr} + GG = \frac{GG\left(a^4 cos^2 \zeta + c^4 sin^2 \zeta\right)}{\left(aa - cc\right)^2 sin^2 \zeta cos^2 \zeta}$$

sive

$$\delta\delta\mathfrak{rr} = \frac{GG\left(a^4 cos^2 \zeta + c^4 sin^2 \zeta\right)^2}{\left(aa - cc\right)^2 sin^2 \zeta cos^2 \zeta}$$

1) In editione principe primum membrum numeratoris est: $cc\, sin\, \zeta$.
2) Editio princeps: *(ac* + cc) loco *(-aa + cc)*
Correxit C. B.
et

$$G = \frac{\left(aa - cc\right)\delta\mathfrak{r}\, sin\,\zeta\, cos\,\zeta}{aa\, cos^2 \zeta + cc\, sin^2 \zeta}$$

vel

$$G = \frac{\left(aa - cc\right)\mathfrak{r}\, sin\,\zeta\, cos\,\zeta \sqrt{2\, fg}}{ac\sqrt{\left(aa\, cos^2 \zeta + cc\, sin^2 \zeta\right)}}$$

Deinde vero obtinemus

$$D = 0, \quad E = \frac{\mathfrak{r} cos\vartheta}{sin\,\zeta}, \quad F = \frac{-\mathfrak{r} cos\eta}{sin\,\zeta}$$

atque

$$A = \frac{-c\mathfrak{r}\, sin\,\zeta \sqrt{2\, fg}}{a\sqrt{\left(aa\, cos^2 \zeta + cc\, sin^2 \zeta\right)}},$$

$$B = \frac{a\mathfrak{r}\, cos\,\zeta\, cos\,\eta \sqrt{2\, fg}}{c\, sin\,\zeta \sqrt{\left(aa\, cos^2 \zeta + cc\, sin^2 \zeta\right)}},$$

$$C = \frac{a\mathfrak{r}\, cos\,\zeta\, cos\,\vartheta \sqrt{2\, fg}}{c\, sin\,\zeta \sqrt{\left(aa\, cos^2 \zeta + cc\, sin^2 \zeta\right)}},$$

ex quibus consequimur

$$x = \gamma' cos\,\alpha = A\, sin\,\delta t \ ,$$
$$y = \gamma' cos\,\beta = B\, sin\,\delta t \ ,$$
$$z = \gamma' cos\,\gamma = C\, sin\,\delta t \ ,$$
$$p = cos\, l - cos\,\zeta = D\, cos\,\delta t \ ,$$
$$q = cos\, m - cos\,\eta = E\, cos\,\delta t \ ,$$
$$r = cos\, n - cos\,\vartheta = F\, cos\,\delta t,$$

atque

$$\lambda = \frac{-a\mathfrak{r}\cos\zeta(1-\cos\delta t)\sqrt{2fg}}{\delta c\sin\zeta\sqrt{(aa\cos^2\zeta+cc\sin^2\zeta)}} = \frac{-aa\mathfrak{r}\cos\zeta(1-\cos\delta t)}{(aa\cos^2\zeta+cc\sin^2\zeta)\sin\zeta}$$

estque λ angulus *VZA*, existente *ZV* circulo verticali fixo, a quo declinationem poli *A.* computamus. Deinde vero est $\rho = \mathfrak{r}\cos\delta t$, et ut obtineamus angulum *DZV,* quaeramus angulum *DZA* ex formula

$$\cos DZA = \frac{\cos\zeta - \cos l\cos\rho}{\rho\sin l} = \frac{\cos\zeta - \cos\zeta\cos\rho - p\cos\rho}{\rho\sin\zeta} = \frac{1}{2}\rho\frac{\cos\zeta}{\sin\zeta},$$

ob *D* = 0 ideoque *p* = 0, ergo

$$\cos DZA = \frac{\mathfrak{r}\cos\zeta\cos\delta t}{2\sin\zeta},$$

I) Editio princeps:

$$\lambda = \frac{-a\mathfrak{r}\sin\zeta\cos\zeta(1-\cos\delta t)\sqrt{2fg}}{\delta c\sqrt{(aa\cos^2\zeta+cc\sin^2\zeta)}} = \frac{-aa\mathfrak{r}\sin\zeta\cos\zeta(1-\cos\delta t)}{aa\cos^2\zeta+cc\sin^2\zeta}.$$

Correxit C. B.

qui cum sit infinite parvus, patet angulum *DZA* esse rectum proxime et angulo *ZDA* aequalem. Quare cum initio angulus *ZDA* fuerit non rectus, haec solutio, quippe quae manifesto tantum est particularis, eo non extenditur. Ceterum vero et hae titubationes erunt isochronae oscillationibus penduli, cuius longitudo est

$$= \frac{aacc}{f(aa\cos^2\zeta+cc\sin^2\zeta)}.$$

Denique cum sit

$$\gamma' = \sin\delta t\cdot\sqrt{(AA+BB+CC)},$$

prodibit

$$\gamma' = \frac{\mathfrak{r}\sqrt{2fg}(a^4c\,os^2\zeta+c^4s\,in^2\zeta)}{ac\sqrt{(aa\cos^2\zeta+cc\sin^2\zeta)}}\cdot\sin\delta t$$

Pro polo autem gyrationis *O* invenimus:

$$\gamma'\cos OD = (A\cos\zeta + B\cos\eta + C\cos\vartheta)\sin\delta t = G\sin\delta t$$

et

$$\gamma'\cos OZ = (A\cos l + B\cos m + C\cos n)\sin\delta t = G\sin\delta t,,$$

ita ut sit *OD* = *OZ* ob $Ap + Bq + Cr = 0$.

SCHOLION

910. Mirum non est hanc solutionem non esse generalem, cum enim ex data indole corporis, quantitatibus *aa, bb, cc* et angulis ζ, η, ϑ comprehensa, et ex situ axis *DF* initiali eiusve declinatione $\mathfrak{r}$ a situ verticali omnes coeffieientes *A, B, C, D, E, F* cum numero δ determinentur, ex iis angulus *ADZ* , quo arcus *DA* initio ab arcu *DZ* deviabat, sponte determinatur, neque amplius arbitrio nostro, uti natura rei postulat, relinquitur. Verum cum in genere pro quantitate *u* geminum valorem elicuerimus, quorum neutrum prae altero reiicere fas est, si utrumque simul adhibeamus, solutionem ampliorem obtinebimus, unde simul effici potest, ut angulus *ADZ* initio fuerit dato angulo aequalis. Cum enim in aequationibus differentialibus quantitates *x,y, z* et *p, q, r* ubique unicam habeant dimensionem, si iis duplici modo satisfieri queat, pro qualibet quantitate summa binorum eius valorum statui poterit, hincque solutionem generalem impetrabimus, quam hic exponamus.

PROBLEMA 109

911. Si corpus basi sphaerica praeditum de situ aequilibrii quomodocunque infinite parum declinetur subitoque demittatur, definire motum titubatorium, quo agitabitur.

SOLUTIO

Retentis denominationibus superioris problematis, quoniam posito

$$\frac{sin^2\zeta}{aa}+\frac{sin^2\eta}{bb}+\frac{sin^2\vartheta}{cc}=K \quad \text{et} \quad \frac{cos^2\zeta}{bbcc}+\frac{cos^2\zeta}{aacc}+\frac{cos^2\vartheta}{aabb}=L$$

et pro *u* geminum invenimus valorem, sint ii

$$u=\frac{1}{2}K+\sqrt{\left(\frac{1}{4}KK-L\right)} \quad \text{et} \quad u'=\frac{1}{2}K-\sqrt{\left(\frac{1}{4}KK-L\right)}$$

unde pro δ etiam binos adipiscimur valores, qui sint

$$\delta=V\sqrt{2fgu} \quad \text{et} \quad \delta'=\sqrt{2fgu'}\,,$$

atque hinc pro senis quantitatibus *x, y, z* et *p, q, r* sequentes impetrabimus valores

$$x=\gamma' cos\,\alpha=\frac{Gcos\zeta\, sin\delta t}{1-aau}+\frac{Hcos\zeta\, sin\delta' t}{1-aau'}\,,$$
$$y=\gamma' cos\beta=\frac{Gcos\eta sin\delta t}{1-bbu}+\frac{Hcos\eta sin\delta' t}{1-bbu'}\,,$$
$$z=\gamma' cos\gamma=\frac{Gcos\vartheta sin\delta t}{1-ccu}+\frac{Hcos\vartheta sin\delta' t}{1-ccu'}\,,$$

tum vero porro

$$p = \cos l - \cos\zeta = \frac{Gu(bb-cc)\cos\eta\cos\vartheta\cos\delta t}{\delta(1-bbu)(1-ccu)} + \frac{Hu'\cos\eta\cos\vartheta\cos\delta' t}{\delta'(1-bbu')(1-ccu')},$$

$$q = \cos m - \cos\eta = \frac{Gu(cc-aa)\cos\zeta\cos\vartheta\cos\delta t}{\delta(1-ccu)(1-aau)} + \frac{Hu'\cos\zeta\cos\vartheta\cos\delta' t}{\delta'(1-ccu')(1-aau')},$$

$$r = \cos n - \cos\vartheta = \frac{Gu(aa-bb)\cos\zeta\cos\eta\cos\delta t}{\delta(1-aau)(1-bbu)} + \frac{Hu'\cos\zeta\cos\eta\cos\delta' t}{\delta'(1-aau')(1-bbu')}.$$

Hic iam habemus binas quantitates constantes arbitrarias *G* et *H*, atque hi valores ita satisfaciunt, ut facta substitutione in aequationibus differentialibus termini tam per *G* quam per *H* affecti seorsim se destruant. Verum si initio arcus *ZD* fuerit $= \mathfrak{r}$, posito $t = 0$, fieri debet $pp + qq + rr = \mathfrak{rr}$. Deinde vero, si initio fuerit angulus $ZDA = \mathfrak{f}$, ob

$$\cos\mathfrak{f} = \frac{\cos l - \cos\zeta\cos\mathfrak{r}}{\sin\zeta\sin\mathfrak{r}} = \frac{\cos\zeta(1-\cos\mathfrak{r})+p}{\sin\zeta\sin\mathfrak{r}} = \frac{\mathfrak{r}\cos\zeta}{2\sin\zeta} + \frac{p}{\mathfrak{r}\sin\mathfrak{r}}$$

et ob $\mathfrak{r}$ infinite parvum erit

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{f}.$$

Si hic ergo pro *p* eius valor superior posito $t = 0$ substituatur, habebitur alia aequatio, ex qua cum illa coniuncta binae constantes *G* et *H* determinabuntur. At posito angulo $VZA = \lambda$ erit

$$d\lambda = \frac{-dt(y\cos\eta + z\cos\vartheta)}{\sin^2\zeta},$$

cuius integrale facile exhibetur. Simili autem modo positis angulis $VZB = \mu$ et $VZ0 = \nu$, erit

$$d\mu = \frac{-dt(z\cos\vartheta + x\cos\zeta)}{\sin^2\eta} \quad \text{et} \quad d\nu = \frac{-dt(x\cos\zeta + y\cos\eta)}{\sin^2\vartheta}.$$

Hic autem notari convenit, si sit $bb = cc$, fore binos valores

$$u = \frac{1}{cc} \quad \text{et} \quad u' = \frac{\sin^2\zeta}{aa} + \frac{\cos^2\zeta}{cc}$$

ideoque pro priore fractionum superiorum quasdam numeratores ac denominatores simul evanescere. Ad earum ergo valores investigandos ponatur

$$\frac{1}{bb} = \frac{1}{cc} + \omega$$

existente ω quantitate evanescente, reperieturque

$$u = \frac{1}{cc} + \frac{\omega\cos^2\vartheta}{\sin^2\zeta} \quad \text{et} \quad u' = \frac{\sin^2\zeta}{aa} + \frac{\cos^2\zeta}{cc},$$

hincque si $\frac{G}{\omega}$ ponatur = *I*, ut sit $G = I\omega = 0$, fiet

1) Editio princeps : $\frac{\mathfrak{r}\cos\zeta}{\sin\zeta}$.

Correxit C. B.

$$x=\gamma' \cos\alpha=\frac{H\cos\zeta \sin\delta' t}{1-aau'}=\frac{-Hcc\sin\delta' t}{(aa-cc)\cos\zeta},$$
$$y=\gamma' \cos\beta=\frac{I\sin^2\zeta\sin\delta t}{cc\cos\eta}+\frac{H\cos\eta\sin\delta' t}{1-ccu'}=\frac{I\sin^2\zeta\sin\delta t}{cc\cos\eta}+\frac{Haa\cos\eta\sin\delta' t}{(aa-cc)\sin^2\zeta},$$
$$z=\gamma' \cos\gamma=\frac{-I\sin^2\zeta\sin\delta t}{cc\cos\vartheta}+\frac{H\cos\vartheta\sin\delta' t}{1-ccu'}=\frac{-I\sin^2\zeta\sin\delta t}{cc\cos\vartheta}+\frac{Haa\cos\vartheta\sin\delta' t}{(aa-cc)\sin^2\zeta},$$

deinde vero

$$p=\cos l-\cos\zeta=\frac{I\sin^4\zeta\cos\delta t}{\delta cc\cos\eta\cos\vartheta},$$
$$q=\cos m-\cos\eta=\frac{-I\sin^2\zeta\cos\zeta\cos\delta t}{\delta cc\cos\vartheta}-\frac{Hu'(aa-cc)\cos\zeta\cos\vartheta\cos\delta' t}{\delta'(1-aau')(1-ccu')},$$
$$r=\cos n-\cos\vartheta=\frac{-I\sin^2\zeta\cos\zeta\cos\delta t}{\delta cc\cos\eta}+\frac{Hu'(aa-cc)\cos\zeta\cos\eta\cos\delta t}{\delta'(1-aau')(1-ccu')},$$

ubi est

$$\frac{aa-cc}{(1-aau')(1-ccu')}=\frac{-aacc}{(aa-cc)\sin^2\zeta\cos^2\zeta}.$$

Vel si ponamus

$$I=\frac{\mathfrak{G}\delta cc\cos\eta\cos\vartheta}{\sin^2\zeta} \quad \text{et} \quad H=\frac{\mathfrak{H}\delta'(aa-cc)\sin^2\zeta\cos\zeta}{aacc}$$

ob

$$u=\frac{1}{cc} \quad \text{et} \quad u'=\frac{aa\cos^2\zeta+cc\sin^2\zeta}{aacc},$$

ideoque

$$\delta=\frac{\sqrt{2fg}}{c} \quad \text{et} \quad \frac{\sqrt{2fg}\left(aa\cos^2\zeta+cc\sin^2\zeta\right)}{ac},$$

erit

$$x=\gamma' \cos\alpha=\frac{\mathfrak{H}\delta'\sin^2\zeta\sin\delta' t}{aa},$$
$$y=\gamma' \cos\beta=\frac{\mathfrak{H}\delta'\cos\zeta\cos\eta\sin\delta' t}{cc}+\mathfrak{G}\delta\cos\vartheta\sin\delta t,$$
$$z=\gamma' \cos\gamma=\frac{\mathfrak{H}\delta'\cos\zeta\cos\vartheta\sin\delta' t}{cc}-\mathfrak{G}\delta\cos\eta\sin\delta t,$$
$$p=\cos l-\cos\zeta=\mathfrak{G}\sin^2\zeta\cos\delta t,$$
$$q=\cos m-\cos\eta=-\mathfrak{G}\cos\zeta\cos\eta\cos\delta t+\mathfrak{H}u'\cos\vartheta\sin\delta' t,$$
$$r=\cos n-\cos\vartheta=-\mathfrak{G}\cos\zeta\cos\vartheta\cos\delta t-\mathfrak{H}u'\cos\eta\cos\delta' t,$$

quae formulae iam sine ulla difficultate ad omnes casus accommodari possunt.

COROLLARIUM 1

912. Haec integralia adhuc latius extendi possunt, cum *x, y, z* et *p,q, r* partes constantes recipiant; ac forma litterarum *G* et *H* mutata habebimus:

$$x = \cos\zeta\,(\mathfrak{E} + \mathfrak{G}(1-bbu)(1-ccu)\sin\delta t + \mathfrak{H}(1-bbu')(1-ccu')\sin\delta' t\,)$$
$$y = \cos\eta(\mathfrak{E} + \mathfrak{G}(1-aau)(1-ccu)\sin\delta t + \mathfrak{H}(1-aau')(1-ccu')\sin\delta' t\,)$$
$$z = \cos\vartheta\,(\mathfrak{E} + \mathfrak{G}(1-ccu)(1-ccu)\sin\delta t + \mathfrak{H}(1-ccu')(1-ccu')\sin\delta' t\,)$$

atque

$$p = \mathfrak{F}\cos\zeta + (bb-cc)\cos\eta\cos\vartheta\,\left(\frac{\mathfrak{G}u(1-aau)\cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1-aau')\cos\delta' t}{\delta'}\right),$$
$$q = \mathfrak{F}\cos\eta + (cc-aa)\cos\zeta\cos\vartheta\,\left(\frac{\mathfrak{G}u(1-bbu)\cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1-bbu')\cos\delta' t}{\delta'}\right),$$
$$r = \mathfrak{F}\cos\vartheta + (aa-bb)\cos\zeta\cos\eta\,\left(\frac{\mathfrak{G}u(1-ccu)\cos\delta t}{\delta} + \frac{\mathfrak{H}u'(1-ccu')\cos\delta' t}{\delta'}\right).$$

COROLLARIUM 2

913. Angulorum etiam δt et $\delta' t$ uterque quantitate constante augeri potest, ac si eorum loco scribamus $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$, integralia continebunt sex constantes arbitrarias $\mathfrak{g}, \mathfrak{h}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ ideoque erunt integralia completa harum sex aequationum differentialium:

$$aadx = 2\,fgdt(r\cos\eta - q\cos\vartheta), dp = dt(z\cos\eta - y\cos\vartheta)$$
$$bbdy = 2\,fgdt(p\cos\vartheta - r\cos\zeta), dq = dt(x\cos\vartheta - z\cos\zeta)$$
$$ccdz = 2\,fgdt(q\cos\zeta - p\cos\eta)\;, dr = dt(y\cos\zeta - x\cos\eta).$$

COROLLARIUM 3

914. Si corpus initio quieverit, ut in problemate assumsimus, ita ut tum fuerit $x = 0$, $y = 0$ et $z = 0$, poni debet $\mathfrak{E} = 0$, $\mathfrak{g} = 0$ et $\mathfrak{h} = 0$; reliquas autem constantes ex situ corporis initiali definiri oportet.

COROLLARIUM 4

915. Nempe si pro initio, quo $t = 0$, ponantur anguli

$$ZDA = \mathfrak{l}, \quad ZDB = \mathfrak{m} \text{ et } ZDC = \mathfrak{n};$$

ut sit

$$\sin(\mathfrak{l}-\mathfrak{m}) = -\frac{\cos\vartheta}{\sin\zeta\sin\eta}, \quad \sin(\mathfrak{m}-\mathfrak{n}) = -\frac{\cos\zeta}{\sin\eta\sin\vartheta}, \quad \sin(\mathfrak{n}-\mathfrak{l}) = -\frac{\cos\eta}{\sin\zeta\sin\vartheta},$$
$$\cos(\mathfrak{l}-\mathfrak{m}) = -\frac{\cos\zeta\cos\eta}{\sin\zeta\sin\eta}, \quad \cos(\mathfrak{m}-\mathfrak{n}) = -\frac{\cos\eta\cos\vartheta}{\sin\eta\sin\vartheta}, \quad \cos(\mathfrak{n}-\mathfrak{l}) = -\frac{\cos\zeta\cos\vartheta}{\sin\zeta\sin\vartheta},$$

pro initio $t = 0$, constantes ita definiri oportet, ut si tum fuerit $ZD = \mathfrak{r}$, fiat

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{l}, \quad q = \mathfrak{r}\sin\eta\cos\mathfrak{m}, \quad r = \mathfrak{r}\sin\vartheta\cos\mathfrak{n}.$$

EXPLICATIO

916. Ad constantes $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ in genere ex statu initiali modo descripto definiendas ponamus brevitatis gratia

$$aa\cos^2\zeta + bb\cos^2\eta + cc\cos^2\vartheta = \mathfrak{A},$$
$$bbcc\cos^2\zeta + aacc\cos^2\eta + aabb\cos^2\vartheta = \mathfrak{B}$$

sitque

$$\frac{\mathfrak{G}u\cos\mathfrak{g}}{\delta} = X$$

et

$$\frac{\mathfrak{H}u'\cos\mathfrak{h}}{\delta'} = Y,$$

quo calculus facilius expediatur. Eo autem absoluto reperietur

$$\mathfrak{F} = \mathfrak{r}\sin\zeta\cos\zeta\cos\mathfrak{l} + \mathfrak{r}\sin\eta\cos\eta\cos\mathfrak{m} + \mathfrak{r}\sin\vartheta\cos\vartheta\cos\mathfrak{n}$$

$$X + Y = \frac{\frac{+\mathfrak{r}\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\vartheta}(\mathfrak{B}-bbcc) + \frac{\mathfrak{r}\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\vartheta}(\mathfrak{B}-aacc) + \frac{\mathfrak{r}\sin\vartheta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{B}-aabb)}{(bb-cc)(cc-aa)(aa-bb)},$$

$$uX + u'Y = \frac{\frac{-\mathfrak{r}\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\vartheta}(\mathfrak{A}-aa) - \frac{\mathfrak{r}\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\vartheta}(\mathfrak{A}-bb) - \frac{\mathfrak{r}\sin\vartheta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{A}-cc)}{(bb-cc)(cc-aa)(aa-bb)}.$$

Ex his autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ est pro superioribus

$$L = \frac{\mathfrak{A}}{aabbcc} \quad \text{et} \quad K = \frac{aabb+aacc+bbcc-\mathfrak{B}}{aabbcc},$$

ex quibus fit

$$u = \tfrac{1}{2}K + \sqrt{\left(\tfrac{1}{4}KK - L\right)} \quad \text{et} \quad u' = \tfrac{1}{2}K - \sqrt{\left(\tfrac{1}{4}KK - L\right)}$$

ita ut sit $u + u' = K$ et $u' - u = \sqrt{(KK - 4L)}$.

Haec analysis in genere valet, etiamsi corpori initio motus fuerit impressus, quoniam loco angulorum δt et $\delta' t$ hic adhibuimus $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$. Simili modo, quo hic ex situ initiali constantes $\mathfrak{F}, \mathfrak{G}$, et $\mathfrak{H}$ definivimus, ex motu initio impresso quantitates *x, y, z* datos obtinebunt valores, quibus si formulae corollario 1. traditae et pro δt et $\delta' t$ scribendo $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$. extensae, posito $t = 0$ aequentur, determinabuntur reliquae constantes $\mathfrak{E}, \mathfrak{g}$ et $\mathfrak{h}$ quae quidem, uti iam ante notavimus, evanescunt, si motus a quiete incipiat.

SCHOLION

917. Pro casu ergo eiusmodi corporum, pro quibus est *bb* = *cc*, erit

$$u=\frac{1}{cc},\quad u'=\frac{aa\cos^2\zeta+cc\sin^2\zeta}{aacc},$$

atque

$$\delta=\sqrt{2fgu},\quad \delta'=\sqrt{2fgu'},$$

integralia in genere ita se habebunt:

$$x=\mathfrak{E}\cos\zeta \qquad -\frac{\mathfrak{H}\delta'\sin^2\zeta\sin(\delta't+\mathfrak{h})}{aa}$$
$$y=\mathfrak{E}\cos\eta+\mathfrak{G}\delta\cos\vartheta\sin(\delta't+\mathfrak{g})+\frac{\mathfrak{H}\delta'\cos\zeta\cos\eta\sin(\delta't+\mathfrak{h})}{cc},$$
$$z=\mathfrak{E}\cos\vartheta-\mathfrak{G}\delta\cos\eta\sin(\delta't+\mathfrak{g})+\frac{\mathfrak{H}\delta'\cos\zeta\cos\vartheta\sin(\delta't+\mathfrak{h})}{cc},$$

atque

$$p=\mathfrak{F}\cos\zeta+\mathfrak{G}\sin^2\zeta\cos(\delta t+\mathfrak{g}),$$
$$q=\mathfrak{F}\cos\eta-\mathfrak{G}\cos\zeta\cos\eta\cos(\delta t+\mathfrak{g})+\mathfrak{H}u'\cos\vartheta\cos(\delta't+\mathfrak{h}),$$
$$r=\mathfrak{F}\cos\vartheta-\mathfrak{G}\cos\zeta\cos\vartheta\cos(\delta t+\mathfrak{g})-\mathfrak{H}u'\cos\eta\cos(\delta't+\mathfrak{h}).$$

Quare si initio *t* = 0 fuerit

$$p=\mathfrak{r}\sin\zeta\cos\mathfrak{l},\quad q=\mathfrak{r}\sin\eta\cos\mathfrak{m},\quad r=\mathfrak{r}\sin\vartheta\cos\mathfrak{n},,$$

reperitur

$$\mathfrak{H}=\frac{\mathfrak{r}\sin\eta\cos\vartheta\cos\mathfrak{m}-\mathfrak{r}\cos\eta\sin\vartheta\cos\mathfrak{n}}{u'\sin^2\zeta\cos\mathfrak{h}},$$

$$\mathfrak{G}=\frac{\mathfrak{r}\sin^3\zeta\cos\mathfrak{l}-\mathfrak{r}\sin\eta\cos\zeta\cos\eta\cos\mathfrak{m}-\mathfrak{r}\sin\vartheta\cos\zeta\cos\vartheta\cos\mathfrak{n}}{\sin^2\zeta\cos\mathfrak{g}},$$

$$\mathfrak{F}=\mathfrak{r}\sin\zeta\cos\zeta\cos\mathfrak{l}+\mathfrak{r}\sin\eta\cos\eta\cos\mathfrak{m}+\mathfrak{r}\sin\vartheta\cos\vartheta\cos\mathfrak{n}.$$

At datis angulis $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$ simul dantur ζ, η, ϑ

$$\cos^2\zeta=\frac{\cos(\mathfrak{l}-\mathfrak{m})\cos(\mathfrak{n}-\mathfrak{l})}{\sin(\mathfrak{l}-\mathfrak{m})\sin(\mathfrak{n}-\mathfrak{l})},\quad \cos^2\eta=\frac{\cos(\mathfrak{m}-\mathfrak{n})\cos(\mathfrak{l}-\mathfrak{m})}{\sin(\mathfrak{m}-\mathfrak{n})\sin(\mathfrak{l}-\mathfrak{m})},\quad \cos^2\vartheta=\frac{\cos(\mathfrak{n}-\mathfrak{l})\cos(\mathfrak{m}-\mathfrak{n})}{\sin(\mathfrak{n}-\mathfrak{l})\sin(\mathfrak{m}-\mathfrak{n})},$$

$$\sin^2\zeta=\frac{-\cos(\mathfrak{m}-\mathfrak{n})}{\sin(\mathfrak{l}-\mathfrak{m})\sin(\mathfrak{n}-\mathfrak{l})},\quad \cos^2\eta=\frac{-\cos(\mathfrak{n}-\mathfrak{l})}{\sin(\mathfrak{m}-\mathfrak{n})\sin(\mathfrak{l}-\mathfrak{m})},\quad \cos^2\vartheta=\frac{-\cos(\mathfrak{l}-\mathfrak{m})}{\sin(\mathfrak{n}-\mathfrak{l})\sin(\mathfrak{m}-\mathfrak{n})}.$$

Ex his autem formulis colligitur esse

$$\sin\zeta\cos\zeta\cos\mathfrak{l}+\sin\eta\cos\eta\cos\mathfrak{m}+\sin\vartheta\cos\vartheta\cos\mathfrak{n}=0,$$

ita ut constans supra definita $\mathfrak{F}$ semper sit = 0. Simili vero modo est

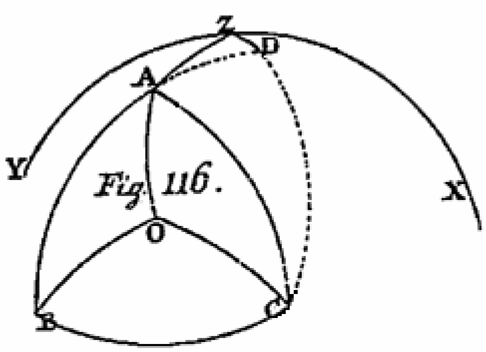
$$\frac{\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\vartheta}+\frac{\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\vartheta}+\frac{\sin\vartheta\cos\mathfrak{n}}{\cos\zeta\cos\eta}=0,$$

unde valores coefficientium supra definiti multo simplicius determinantur, ita ut litterae $\mathfrak{A}$ et $\mathfrak{B}$ ex illis prorsus elidantur. Valent haec in genere, etiamsi non sit *bb = cc*.

PROBLEMA 110

918. Si corpus basi sphaerica praeditum habeat duos axes principales pares eique, cum de situ quietis infinite parum fuerit declinatum, motus minimus quicunque fuerit impressus, definire motus continuationem.

SOLUTIO



Sit *ID* axis corporis aequilibrii per centrum inertiae *I* et centrum basis *G* transiens (Fig. 116), sitque hoc illo altius situm existente intervallo *GI = f*. Sit porro *IA* axis corporis singularis principalis eiusque respectu momentum inertiae = *Maa*, respectu axium omnium autem ad hunc normalium = *Mcc*, quos omnes cum aeque pro principalibus habere liceat, sumatur alter *IB* in arcu *DA* producto, eritque alter *1C*, ut quadrans *AC* sit ad *AD* normalis ideoque *DC* etiam quadrans ad *AD* normalis. Posito ergo $DA=\zeta$ erit $DB=\eta=\zeta+90°$

1) Editio princeps – loco ,+ . Correxit C. B.

et $DO=\vartheta=90°$. Initio autem, quo *t* = 0, fuerit arcus $DZ=\mathfrak{r}$ et angulus $ZDA=\mathfrak{l}$, erit $ZDB=\mathfrak{m}=\mathfrak{l}$ et $ZDC=\mathfrak{n}=\mathfrak{l}+90°$. Ex formulis ergo praecedentibus habebimus

$$u=\frac{1}{cc},\quad u'=\frac{aa\cos^2\zeta+cc\sin^2\zeta}{aacc},\quad \delta=\sqrt{2fgu},\quad \delta'=\sqrt{2fgu'},$$

tum vero ex hoc situ initiali fiet primo $\mathfrak{F}=0$, tum vero

$$\mathfrak{r}\cos\zeta\cos\mathfrak{l}=\mathfrak{G}\sin^2\zeta\cos\mathfrak{g};$$

ergo

$$\mathfrak{r}\cos\zeta\cos\mathfrak{l}=\mathfrak{G}\sin\zeta\cos\zeta\cos\mathfrak{g},\quad \mathfrak{G}=\frac{\mathfrak{r}\cos\mathfrak{l}}{\sin\zeta\cos\mathfrak{g}},$$

$$-\mathfrak{r}\sin\mathfrak{l}=\mathfrak{H}u'\sin\zeta\cos\mathfrak{h},\quad \mathfrak{H}=\frac{-\mathfrak{r}\sin\mathfrak{l}}{u'\sin\zeta\cos\mathfrak{h}}.$$

Deinde initio corpori motus sit impressus circa axem *IC* celeritate angulari $=\varepsilon$ in sensum *ABC*, existente $OA=\mathfrak{a}, OB=\mathfrak{b}, OC=\mathfrak{c}$, fierique debet

$$\varepsilon cos\,\mathfrak{a}=\mathfrak{G}cos\zeta-\frac{\mathfrak{H}\delta' sin^2\zeta\, sin\,\mathfrak{h}}{aa},$$

$$\varepsilon cos\,\mathfrak{b}=-\mathfrak{G}\,sin\,\zeta-\frac{\mathfrak{H}\delta' sin\,\zeta\, cos\,\zeta\, sin\,\mathfrak{h}}{cc},$$

$$\varepsilon cos\,\mathfrak{c}=\mathfrak{G}\delta\, sin\,\zeta\; sin\,\mathfrak{g},$$

unde concludimus

$$\varepsilon\left(aacos\,\mathfrak{a}cos\zeta-cccos\,\mathfrak{b}sin\zeta\right)=\mathfrak{E}\left(aacos^2\zeta+ccsin^2\zeta\right).$$

et

$$\varepsilon\left(cos\,\mathfrak{a}\,sin\,\zeta+cos\,\mathfrak{b}\,cos\,\zeta\right)=-\mathfrak{H}\delta'\,sin\,\zeta\;sin\,\mathfrak{h}\left(\frac{sin^2\zeta}{cc}+\frac{cos^2\zeta}{aa}\right).$$

Ergo

$$\mathfrak{E}=\frac{\varepsilon\left(aacos\,\mathfrak{a}cos\zeta-cccos\,\mathfrak{b}sin\zeta\right)}{aacos^2\zeta+ccsin^2\zeta},$$

$$\mathfrak{G}=\frac{\varepsilon cos\,\mathfrak{c}}{\delta\,sin\,\zeta\;sin\,\mathfrak{g}},$$

$$\mathfrak{H}=\frac{-\varepsilon\left(cos\,\mathfrak{a}\,sin\,\zeta+cos\,\mathfrak{b}\,cos\,\zeta\right)}{\delta' u'\,sin\,\zeta\;sin\,\mathfrak{h}},$$

hinc erit

$$tang\,\mathfrak{g}=\frac{\varepsilon cos\,\mathfrak{c}}{\delta\mathfrak{r}\,cos\,\mathfrak{l}}\quad \text{et}\quad tang\,\mathfrak{h}=\frac{\varepsilon\left(cos\,\mathfrak{a}\,sin\,\zeta+cos\,\mathfrak{b}\,cos\,\zeta\right)}{\delta'\,\mathfrak{r}\,sin\,\mathfrak{l}},$$

unde anguli $\mathfrak{g}$ et $\mathfrak{h}$ hincque numeri $\mathfrak{G}$ et $\mathfrak{H}$ innotescunt.
His definitis teneat corpus elapso tempore *t* situm in figura repraesentatum sitque

$$ZD=\rho,\ ZA=l,\ ZB=m,\ ZC\ =n:$$

ac ponatur

$$cosl=cos\zeta+p,\quad cosm=cos\eta+q,\quad cosn=cos\vartheta+r$$

seu

$$cos\,m=-\,sin\,\zeta+q\quad \text{et}\quad cosn=r.$$

Deinde gyretur nunc circa axem *IO* celeritate angulari $=\ \delta$ in sensum *ABC* existentibus arcubus

$$OA=\alpha,\quad OB=\beta,\quad OC=\gamma\ ,$$

ac ponendo

$$\gamma' cos\alpha=x\ ,\quad \gamma' cos\beta=y\quad \text{et}\quad \gamma' cos\gamma=z\ ,$$

habebimus ex § 917

$$\gamma' \cos\alpha = \frac{\varepsilon \cos\zeta\left(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta\right)}{aa\cos^2\zeta + cc\sin^2\zeta} + \frac{\delta'\mathfrak{r}\sin\zeta\sin\mathfrak{l}\sin(\delta't+\mathfrak{h})}{aau'\cos\mathfrak{h}},$$

$$\gamma' \cos\beta = \frac{-\varepsilon \sin\zeta\left(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta\right)}{aa\cos^2\zeta + cc\sin^2\zeta} + \frac{\delta'\mathfrak{r}\cos\zeta\sin\mathfrak{l}\sin(\delta't+\mathfrak{h})}{ccu'\cos\mathfrak{h}},$$

$$\gamma' \cos\delta = \frac{\delta\mathfrak{r}\cos\mathfrak{l}\sin(\delta t+\mathfrak{g})}{\cos\mathfrak{g}},$$

[In editione principe denominator est $aau'\cos\mathfrak{h}$. Correxit C. B.]
tum vero praeterea :

$$p = \frac{\mathfrak{r}\sin\zeta\cos\mathfrak{l}\cos(\delta t+\mathfrak{g})}{\cos\mathfrak{g}},$$

$$q = \frac{\mathfrak{r}\cos\zeta\cos\mathfrak{l}\cos(\delta t+\mathfrak{g})}{\cos\mathfrak{g}},$$

$$r = \frac{-\mathfrak{r}\sin\mathfrak{l}\cos(\delta't+\mathfrak{h})}{\cos\mathfrak{h}}.$$

Ex his si ponatur arcus $ZD = \rho$, erit

$$\rho = \mathfrak{r}\sqrt{\left(\frac{\cos^2\mathfrak{l}\cos^2(\delta t+\mathfrak{g})}{\cos^2\mathfrak{g}} + \frac{\sin^2\mathfrak{l}\cos^2(\delta't+\mathfrak{h})}{\cos^2\mathfrak{h}}\right)}.$$

Porro ex triangulo *AZD* est

$$\cos ADZ = \frac{\cos l - \cos\zeta\cos\rho}{\sin\zeta\sin\rho} = \frac{p+\frac{1}{2}\rho\rho\cos\zeta}{\rho\sin\zeta} = \frac{p}{\rho\sin\zeta}$$

evanescente termino $\frac{\rho\cos\zeta}{2\rho\sin\zeta}$, hinc ergo erit

$$\cos ADZ = \frac{\cos\mathfrak{l}\cos(\delta t+g)}{\cos\mathfrak{g}} : \sqrt{\left(\frac{\cos^2\mathfrak{l}\cos^2(\delta t+\mathfrak{g})}{\cos^2\mathfrak{g}} + \frac{\sin^2\mathfrak{l}\cos^2(\delta't+\mathfrak{h})}{\cos^2\mathfrak{h}}\right)}$$

ideoque

$$\text{tang}\, ADZ = \frac{\cos\mathfrak{g}\,\text{tang}\,\mathfrak{l}\cos(\delta't+\mathfrak{h})}{\cos\mathfrak{h}\cos(\delta t+\mathfrak{g})}.$$

Praeter arcum autem $DZ = \rho$ et angulum *ADZ* nosse oportet angulum *XZD* a circulo verticali fixo *ZX* computatum; est vero $DZA = 180^{\circ} - ADZ$, seu

$$\text{tang}\, ADZ = -\frac{\cos\mathfrak{g}\,\text{tang}\,\mathfrak{l}\cos(\delta't+\mathfrak{h})}{\cos\mathfrak{h}\cos(\delta t+\mathfrak{g})}\,\text{tang}\,\mathfrak{l},$$

cum initio esset $DZA = 180^{\circ} - \mathfrak{l}$ et $\text{tang}\, DZA = -\text{tang}\,\mathfrak{l}$. Deinde vero posito angulo $XZA = \lambda$, est

$$d\lambda = -\frac{dt(y\cos\eta + z\cos\vartheta)}{\sin^2\zeta} = \frac{ydt}{\sin\zeta},$$

hincque

$$\lambda = \text{Const.} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\cos\zeta)}{aa\cos^2\zeta + cc\sin^2\zeta} - \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}\cos(\delta' t + \mathfrak{h})}{ccu'\sin\zeta\cos\mathfrak{h}}.$$

Quodsi ponamus initio angulum *XZD* evanuisse, initio fuerat $\lambda = 180^{\text{o}} - \mathfrak{l}$,
sicque constans hic ingressa est:

$$\text{Const.} = 180^{\text{o}} - \mathfrak{l} + \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}}{ccu'\sin\zeta},$$

unde habebimus:

$$\lambda = 180^{\text{o}} - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\cos\zeta)}{aa\cos^2\zeta + cc\sin^2\zeta} - \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}}{ccu'\sin\zeta}\left(1 - \frac{\cos(\delta' t + \mathfrak{h})}{\cos\mathfrak{h}}\right).$$

hincque $XZD = \lambda - DZA$, ex quibus ad tempus *t* situs corporis perfecte cognoscitur, in hacque determinatione simul motus continetur.

COROLLARIUM 1

919. Si motus corpori initio impressus evanescat, est $\mathfrak{g} = 0$ et $\mathfrak{h} = 0$; hincque

$$x = \gamma'\cos\alpha = \frac{\delta'\mathfrak{r}\sin\zeta\sin\mathfrak{l}\sin\delta' t}{aau'},$$
$$y = \gamma'\cos\beta = \frac{\delta'\mathfrak{r}\cos\zeta\sin\mathfrak{l}\sin\delta' t}{ccu'},$$
$$z = \gamma'\cos\delta = \delta\mathfrak{r}\cos\mathfrak{l}\sin\delta t,$$

$$p = \mathfrak{r}\sin\zeta\cos\mathfrak{l}\cos\delta t, q = \mathfrak{r}\cos\zeta\cos\mathfrak{l}\cos\delta t, r = -\mathfrak{r}\sin\mathfrak{l}\cos\delta' t$$

$$\text{tang } ADZ = \text{tang}\left(180^{\text{o}} - DZA\right) = \frac{\cos\delta' t}{\cos\delta t}\text{tang}\,\mathfrak{l}$$

et

$$\lambda = 180^{\text{o}} - \mathfrak{l} - \frac{\mathfrak{r}\cos\zeta\sin\mathfrak{l}}{ccu'\sin\zeta}(1 - \cos\delta' t).$$

COROLLARIUM 2

920. Sin autem corpori initio motus fuerit impressus celeritate angulari ε, haec non multo maior esse debet quam $\mathfrak{r}$. Si enim $\frac{\varepsilon}{\mathfrak{r}}$ esset numerus praemagnus, anguli $\mathfrak{g}$ et $\mathfrak{h}$ prodirent fere recti eorumque cosinus fere evanescentes, sicque numeri *p*, *q*, *r* nimis fierent magni, quam ut tanquam valde parvae, uti natura solutionis exigit, considerari possent. Namque arcus $ZD = \rho$ semper minimus esse debet.

COROLLARIUM 3

921. Cum sit $\gamma' = \sqrt{(xx+yy+zz)}$, nisi ternae quantitates *x*, *y*, *z* seorsim evanescant, fieri nequit, ut corpus unquam ad quietem redigatur. Atque etiamsi corpus a quiete moveri inceperit, tamen fieri potest, ut corpus deinceps nunquam ad quietem revertatur, hocque adeo semper eveniet, nisi fuerit vel $sin\,\mathfrak{l} = 0$ vel $cos\,\mathfrak{l} = 0$; quin etiam tum axis corporis naturalis *DF* nunquam in situm verticalem perveniet.

COROLLARIUM 4

922. Cum sit $\mathfrak{r}$ quantitas valde exigua, si corpus initio nullum motum acceperit, ut sit $\varepsilon = 0$, erit satis exacte $\lambda = 180^{\circ} - \mathfrak{l}$; scilicet angulus *XZA* manebit constans, motusque axis *IA* ita comparatus, ut in arcu *ZA*modo ad punctum verticale *Z* propius accedat modo ab eo longius recedat, erit autem

$$AZ = \zeta - \mathfrak{r}\, cos\,\mathfrak{l}\, cos\delta t \quad \text{et} \quad \text{ang.} ZAD = \mathfrak{r}\frac{sin\,\mathfrak{l} cos\delta' t}{sin\,\zeta}.$$

COROLLARIUM 5

923. In genere autem quicunque motus corpori initio fuerit impressus, erit

$$XZA = \lambda = 180^{\circ} - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a} cos\zeta - cc\cos\mathfrak{b} cos\zeta)}{aa\cos^2\zeta + cc sin^2\zeta}$$

sicque arcus *ZA* uniformiter circa punctum verticale *Z* circumferetur; deinde vero, cum sit

$$sin\ AD : sin\ DZA = ZD(\rho) : ZAD,$$

erit

$$ZAD = \frac{\mathfrak{r}\, sin\,\mathfrak{l} cos(\delta' t + \mathfrak{h})}{sin\zeta cos\,\mathfrak{h}} \text{ atque arcus } ZA = \zeta - \frac{\mathfrak{r}\, cos\,\mathfrak{l} cos(\delta t + \mathfrak{g})}{cos\,\mathfrak{g}}.$$

Seu pro $\mathfrak{g}$ et $\mathfrak{h}$ substituendis valoribus:

$$\text{angulus } ZAD = \frac{\mathfrak{r}\, sin\,\mathfrak{l} cos\delta' t}{sin\,\zeta} - \frac{\varepsilon(cos\,\mathfrak{a}\, sin\,\zeta + cos\,\mathfrak{b}\, cos\,\zeta) sin\,\delta' t}{\delta' sin\,\zeta}$$

et

$$\text{arcus } ZA = \zeta - \mathfrak{r}\, cos\,\mathfrak{l}\, cos\delta t + \frac{\varepsilon\, cos\,\mathfrak{c}\, sin\,\delta t}{\delta}.$$

SCHOLION 1

924. Hae tres postremae formulae, angulos *XZA, ZAD* cum arcu *Z A* exhibentes, totam problematis solutionem complectuntur. Quodsi enim has res ad quodvis tempus assignare possimus, situm corporis perfecte cognoscimus. Quare si pro δ et δ' valores supra inventos substituamus, universa problematis solutio his formulis continebitur:

$$\text{angulus } XZA = 180^{\circ} - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos^2\zeta + cc\sin^2\zeta},$$

$$\text{arcus } ZA = \zeta - \cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c}$$

$$\text{angulus } ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}}{\sin\zeta}\cos\frac{t\sqrt{2fg\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}}{ac}$$

$$-\frac{\varepsilon ac\left(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta\right)}{\sin\zeta\sqrt{2fg\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}}\sin\frac{t\sqrt{2fg\left(aa\cos^2\zeta + cc\sin^2\zeta\right)}}{ac}.$$

Quodsi ergo omnia momenta inertiae fuerint aequalia, scilicet $aa = cc$, erit

$$\text{angulus } XZA = 180^{\circ} - \mathfrak{l} - \varepsilon t(\cos\mathfrak{a}\cos\zeta - \cos\mathfrak{b}\sin\zeta),$$

$$\text{arcus } ZA = \zeta - \cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c}$$

$$\text{angulus } ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}}{\sin\zeta}\cos\frac{t\sqrt{2fg}}{c} - \frac{\varepsilon c\left(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta\right)}{\sin\zeta\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c},$$

quo casu positio punCti *A* est plane arbitraria.

SCHOLION 2

925. Argumentum, quod in hoc capite potissimum evolvendum suscepimus, motum scilicet titubatorium corporum basi sphaerica praeditorum, perfecte pertractavimus, dummodo titubationes fuerint quam minimae, quae hypothesis etiam in doctrina oscillationum statui solet; formulae enim § 912 et seqq. exhibitae perfectam continent huius quaestionis solutionem, si quidem ibi anguli δt et $\delta' t$ constantibus $\mathfrak{g}$ et $\mathfrak{h}$ augeantur. Constantes autem ex statu initiali definire docuimus in § 916, quae operatio vehementer sublevatur annotatione sub finem § 917 adiuncta; quare ad motum corporum cylindricorum explicandum progrediamur.