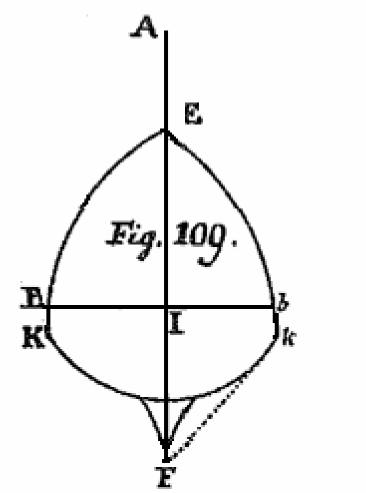
Chapter 17

A FULLER EXPLANATION OF THE MOTION OF TOPS ON A HORIZONTAL PLANE WITHOUT FRICTION

DEFINITION 14

850. The *axis* of a top is the line *AF* drawn from the point *F* through the centre of inertia *I*, which likewise is the singular principal axis, thus so that the moments of inertia about all the axis *IB* normal to that are equal to each other. (Fig. 109).



COROLLARY 1

851. Hence the most suitable figure of the top is a solid of revolution [Latin *tornata* : a figure turned on a lathe], which is generated if some shape is rotated about the axis *AF*, provided it comes to an end at the point *F*, which can be put in place above a horizontal plane.

COROLLARY 2

852. Moreover in the top it is required to know the following quantities, which are present in the calculation :

1. The mass or the weight of this, which is equal to *M ;*
2. The distance of the point from the centre of inertia, which is $IF = f;$
3. The moment of inertia about the axis *AF,* which is equal to *Maa,* and
4. The moment of inertia about all the axis normal to that, which are equal to *Mcc.*

COROLLARY 3

853. Hence since above in general we have put in place the principal moments of inertia *Maa, Mbb* and *Mcc,* and of this body we have set the latter two moments of inertia equal, so that $bb = cc.$

COROLLARY 4

854. Therefore while the top advances with the point *F* on the horizontal plane, the axis *AF* of this cannot be inclined to the horizontal beyond a certain angle, which is obtained be drawing the most distant line *Fk* from *F* to the body of the top; then indeed the angle *AFk* gives that limit.

SCHOLIUM

855. Above we have considered only tops of this kind, in which all the moments of inertia are equal to each other, which condition was too limiting. Therefore now we investigate the motion of tops in general, if indeed the condition that *AF* is the principal axis, and about which the two remaining moments of inertia are equal, since it is seen to be a necessary

property that they are taken together. But the principles from which the determined motion of this is desired now have been set out above in Chapter 14, where we have seen that the whole calculation depends on the force with which the top leans on the horizontal plane with its point F, while it is moving. Which pressing down force can be found, even if it is not brought through to the end in the solution, yet at once from the beginning it is present in the calculation. Hence let Π be that [normal reaction] force, the direction of this always points vertically upwards from the point F ; and we have shown above in §767 that, concerning this force, if the inclination of the axis AF to the horizontal is put equal to ϑ, which in the element of time dt increases by it own differential $d\vartheta$, with the element dt taken constant, then [then the vertical acceleration is given by :]

$$\frac{dd\vartheta\cos\vartheta - d\vartheta^2\sin\vartheta}{dt^2} = \frac{2g}{f}\left(\frac{\Pi}{M}-1\right),$$

or

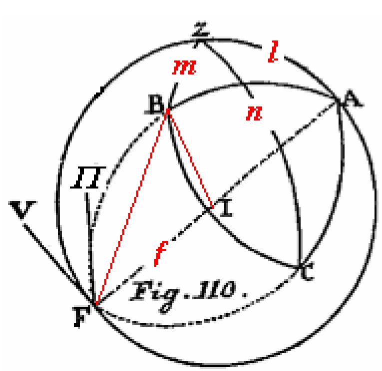
$$\frac{\Pi}{M} = 1 + \frac{f(dd\vartheta\cos\vartheta - d\vartheta^2\sin\vartheta)}{2gdt^2} = 1 + \frac{fd\cdot d\vartheta\cos\vartheta}{2gdt^2} = 1 + \frac{fdd\cdot\sin\vartheta}{2gdt^2}$$

Therefore since the top in addition to this force Π is only acted on by gravity put in place, the centre of inertia I is unable to take any other motion, except in the vertical direction either ascending or descending, while the distance of this from the horizontal plane is equal to $f\sin\vartheta$. But if initially a certain horizontal motion were impressed on that above, since it remains uniform constantly, and thus the whole question is reduced to the motion of rotation alone. Whereby since the weight confers nothing to that and all the disturbances arise from the force Π, it is necessary to define the moment of this force about the principal axis of the top.

PROBLEM 97

856. If the top holds some inclined position to the horizontal and likewise the force Π is given, by which its cusp [or point] is supported by the horizontal plane, to define the moment of the force of this about the principal axis of the top.

SOLUTION



With a sphere described about the centre of inertia of the top I (Fig. 110), on which Z is the vertical point, moreover the axis of the top cutting the sphere at the points A and F, then the two remaining principal axes now reach the sphere at the points B and C ; although in fact these two axes cannot be distinguished from each other, yet certainly it is agreed that the two lines are taken normal to each other as well as to the axis AF, from which in turn the position of the top can be defined at some time. The arcs of the great circles are put in place:

$$ZA = l,\ ZB = m \text{ et } ZC = n,$$

then $l = 90° - \vartheta$ with ϑ denoting the inclination of the axis *AF* to the horizontal. Now since the point *F*, the distance of which from the centre of inertia *I* is *FI* = *f*, is acted on in the vertical direction $F\Pi$ by the force equal to Π, in order that the angle $AF\Pi = l$, and that is resolved along the directions *FA* and *FV*, of which *FV* is normal to *AF* in the vertical plane *AZF*, then

$$\text{the force along } FA = \Pi \cos l$$

and

$$\text{the force along } FV = \Pi \sin l \,,$$

of which that component passing through the centre of inertia exerts no moments. Now neither does this force $FV = \Pi \sin l$ exert any moment about *AF*; but about the axis *IB* it gives a moment equal to $\Pi f \sin l \sin VFB$ in the sense *AC*, and likewise in a similar manner about the axis *1C*, the moment = $\Pi f \sin l \sin VFC$ in the sense *BA*. [The added red line *IB* comes out of the plane of the page or lies above the plane *ZAF*, at an inclination equal to the angle *ZAB* or to the equal spherical angle VFB at the other end of the segment *AZFBA* ; the force along *VF* can be resolved into components along *BI* and normal to *BI*, where *BI* lies in the plane *ABF* : the component of VF along BI is $\Pi \sin l \cos VFC$, while the component along the normal is $\Pi \sin l \sin VFC$, leading to the moment given, etc] and

$$\sin ZAB = -\cos ZAC = -\tfrac{\cos n}{\sin l}$$

then the angle *VFC* = *ZAC* and

$$\sin ZAC = \cos ZAB = \tfrac{\cos m}{\sin l}$$

On account of which we have

$$\text{the moment about the axis } IB = -\Pi f \cos n \text{, in the sense } AC,$$

$$\text{the moment about the axis } IC = \Pi f \cos m \text{, in the sense } BA$$

and since in general we have put above [§ 803] the moments of the forces *P*, *Q*, *R* about the axes *1A*, *1B*, *IC* in the sense *BC*, *CA*, *AB*, in our case then we have :

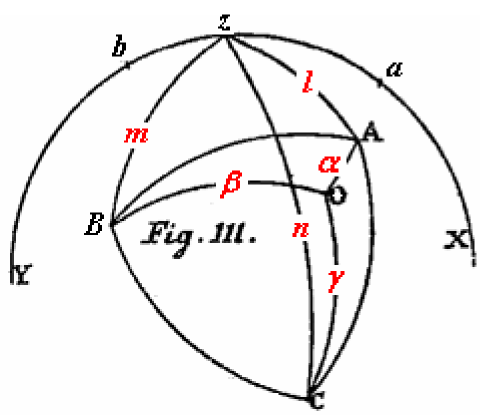
$$P = 0, \quad Q = \Pi f \cos n \quad \text{and} \quad R = -\Pi f \cos m.$$

PROBLEM 98

857. If a top inclined at some position is rotating about some axis passing through its centre of inertia, to define the momentary [*i. e.* instantaneous] variation produced both in the axis of rotation as well as in the speed of rotation.

SOLUTION

With a motionless sphere set up around the centre of inertia *I* (Fig. 111), on which *Z* is the point for the vertical and *ZX* is a fixed vertical circle ; now the top holds a position of this kind, so that the special axis of the top corresponds to the point of the sphere *A*, moreover with the points *B* and *C* for the two remaining principal axes, and for these there are put the declinations of the axes from the vertical, or the arcs



$$ZA = l,\ ZB = m,\ ZC = n\ ,$$

in order that

$$\cos^2 l + \cos^2 m + \cos^2 n = 1\ ;$$

then the angles

$$XZA = \lambda,\ XZB = \mu \quad \text{and} \quad XZC = \nu\ ,$$

[Thus, $l, \lambda;\ m, \mu;\ \&\ n, \nu$ correspond to the latitudes and longitudes of the three axes of the top w.r.t. the circle XZY, while $\cos l, \cos m,$ and $\cos n$ are the direction cosines of *IZ* w.r.t. the axis of the top.]
the relations of which to these arcs have to become known. But now the top rotates about the axes *IO* with an angular speed equal to γ' in the sense *ABC* and let the arcs for the pole of rotation *O* be

$$AO = \alpha,\ BO = \beta \quad \text{et} \quad CO = \gamma',$$

and on putting

$$\gamma' \cos\alpha = x,\quad \gamma' \cos\beta = y,\quad \gamma' \cos\gamma = z$$

on account of the moments of the forces :

$$P = 0,\quad Q = \Pi f \cos n \quad \text{and} \quad R = -\Pi f \cos m\ ,$$

and *bb* = *cc*, the variations produced in the element of time *dt* can be expressed from the following formulas [see Ch. XV, §808] :

$$\text{I. } dx = 0$$

$$\text{II. } dy + \frac{aa-cc}{cc} xzdt = \frac{2\Pi\, fgdt \cos n}{Mcc}$$

$$\text{111. } dz - \left(\frac{aa-cc}{cc}\right) xydt = -\frac{2\Pi\, fgdt \cos m}{Mcc}.$$

Now in addition it is necessary to add these equations for l, m, n, λ, μ, v [see Ch. X, Problem 68 for the earlier method, and Ch. XV, §810]:

$$dl \sin l = dt(y \cos n - z \cos m), \quad d\lambda \sin^2 l = -dt(y \cos m + z \cos n),$$
$$dm \sin m = dt(z \cos l - x \cos n), \quad d\mu \sin^2 m = -dt(z \cos n + x \cos l),$$
$$dn \sin n = dt(x \cos m - y \cos l), \quad dv \sin^2 n = -dt(x \cos l + y \cos m).$$

But since the inclination of the axis to the horizontal is equal to $90° - l$, which has been put equal to ϑ above, as $\sin\vartheta = \cos l$ then [from §854]

$$\frac{\Pi}{M} = 1 + \frac{fdd \cdot \cos l}{2gdt^2}.$$

So that these can be more compact, we put in place:

$$\cos l = p, \ \cos m = q, \ \cos n = r$$

and we find

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2},$$

and in addition these equations :

$$\text{I. } dx = 0$$
$$\text{II. } dy + \frac{(aa-cc)\,xzdt}{cc} = \frac{2\Pi\, fgrdt}{Mcc}$$
$$\text{III. } dz - \frac{(aa-cc)\,xydt}{cc} = \frac{-2\Pi\, fgqdt}{Mcc}.$$
$$\text{IV. } dp = dt(qz - ry), \quad \text{VI I. } d\lambda = \frac{-dt(qy+rz)}{1-pp}$$
$$\text{V. } dq = dt(rx - pz), \quad \text{VIII. } d\mu = \frac{-dt(rz+px)}{1-qq}$$
$$\text{VI. } dr = dt(py - qx), \quad \text{IX. } dv = \frac{-dt(px+qy)}{1-rr}$$

where it is to be noted that $pp + qq + rr = 1$.

COROLLARY 1

858. If the top is turning about the axis *IA*, so that

$$\alpha = 0 \text{ and } \beta = \gamma = 90°,$$

then

$$x = \gamma', \quad y = 0, \quad \text{and} \quad z = 0$$

and

$$dx = d\gamma', \quad dy = -\gamma' d\beta, \quad dz = -\gamma' d\gamma.$$

Hence there arises

$$d\gamma' = 0, \quad d\beta = \frac{-2\Pi fgrdt}{\gamma' Mcc}, \qquad d\gamma = \frac{2\Pi fgqdt}{\gamma' Mcc},$$

$$dp = 0, \quad dq = d\gamma' rdt, \quad dr = -d\gamma' qdt \quad \text{and} \quad d\lambda = 0,$$

and then in the first instant neither the angular speed γ' nor the position of the point *A* experiences a change.

COROLLARY 2

859. Since $dp = dt(qz - ry)$, then on differentiation

$$ddp = dt(qdz - rdy) + dt(zdq - ydr),$$

and with the given values substituted there is found :

$$\frac{ddq}{dt^2} = \frac{(aa-cc)x}{cc}(qy+rz) - \frac{2\Pi fg}{Mcc}(qq+rr) + x(qy+rz) - p(yy+zz),$$

from which there becomes

$$\frac{\Pi}{M} = 1 + \frac{f(aa-cc)x}{2gcc}(qy+rz) - \frac{\Pi ff}{Mcc}(qq+rr) + \frac{fx(qy+rz)}{2g} - \frac{fp(yy+zz)}{2g}$$

or

$$\frac{\Pi}{M}\left(1 + \frac{ff(qq+rr)}{cc}\right) = 1 + \frac{faax(qy+rz)}{2gcc} - \frac{fp(yy+zz)}{2g}$$

and hence

$$\frac{\Pi}{M} = \frac{2gcc + faax(qy+rz) - fccp(yy+zz)}{2gcc + 2gff(qq+rr)}.$$

[In the first edition the numerator is : $2gcc + faa(qy + rz) - fccp(yy + zz)$.
Corrected by the editor C. B.]

COROLLARY 3

860. From equations IV, V, & VI it is deduced, as we have noted before, that

$$xdp + ydq + zdr = 0\ ,$$

which equation, since $pp + qq + rr = 1$, can be taken in place of equations V and VI. But it suffices to treat the equations VII, VIII, & IX one at a time, since the calculation will be examined in the following.

COROLLARY 4

861. But for the quantities found x, y and z, since

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

then the angular speed

$$\gamma' = \sqrt{(xx + yy + zz)}$$

and hence the arcs α, β, γ in turn can be inferred; clearly

$$cos\ \alpha = \frac{x}{\gamma'},\quad cos\beta = \frac{y}{\gamma'},\quad \text{and}\quad cos\gamma = \frac{z}{\gamma'}.$$

PROBLEM 99

862. The differential equations found before [§811], by which the motion of the top is expressed, are lead towards integration, whenever that is possible.

SOLUTION

It is at once apparent that x = const., and hence we put $x = A$, and the remaining equations to be integrated become :

$$\text{eq.1. } dy + \frac{A(aa-cc)zdt}{cc} = \frac{2\Pi\ fgrdt}{Mcc}$$

$$\text{eq.2. } dz - \frac{A(aa-cc)ydt}{cc} = -\frac{2\Pi\ fgqdt}{Mcc}$$

$$\text{eq.3. } dp = dt\ (qz - ry)$$

$$\text{eq.4. } ydq + zdr = -Adp$$

with the equation present :

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2}$$

and

$$pp + qq + rr = 1\ .$$

Now eq. 1. q + eq.2. r gives rise to this equation

$$qdy + rdz + \frac{A(aa-cc)}{cc}\, dt(qz - ry) = 0,$$

which on account of $dt(qz - ry) = dp$ becomes this equation :

$$qdy + rdz = \frac{-A(aa-cc)}{cc} dp\,,$$

to this eq.4 is added :

$$ydq + zdr = -Adp$$

then

$$qdy + ydq + rdz + zdr = \frac{-aa}{cc} Adp,$$

and the integral of this is

$$qy + rz = B - \frac{aa}{cc} Ap.$$

Again on taking eq.1. y + eq.2. z there is produced :

$$ydy + zdz = \frac{2\Pi\, fgdt}{Mcc}(ry - qz) = \frac{-2\Pi\, fgdp}{Mcc};$$

whereby since

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2},$$

it becomes

$$ydy + zdz = \frac{-2\, fgdp}{cc} - \frac{ffdpddp}{ccdt^2},$$

from which on integration we come upon :

$$yy + zz = C - \frac{4\, fgp}{cc} - \frac{ffdp^2}{ccdt^2}.$$

now since

$$\frac{dp}{dt} = qz - ry,$$

we obtain a new finite equation :

$$yy + zz = C - \frac{4\, fgp}{cc} - \frac{ff}{cc}(qz - ry)^2,$$

from which since :

$$(qz - ry)^2 = \frac{Ccc}{ff} - \frac{4gp}{f} - \frac{cc(yy+zz)}{ff},$$

moreover from being found before

$$(qy + rz)^2 = (B - \frac{Aaap}{cc})^2,$$

with these added there is produced :

$$(qq + rr)(yy + zz) = \frac{Ccc}{ff} - \frac{4gp}{f} - \frac{cc(yy+zz)}{ff} + \left(B - \frac{Aaap}{cc}\right)^2$$

from which as

$$qq + rr = 1 - pp$$

there is elicited

$$(1-pp+\frac{cc}{f})(yy+zz)=\frac{Ccc}{ff}-\frac{4gp}{f}+\left(B-\frac{Aaap}{cc}\right)^2$$

or

$$yy+zz=\frac{Ccc-4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp},$$

$$(qz-ry)^2=\frac{(Ccc-4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp}.$$

since now we have hence defined these $qy+rz$, $yy+zz$, $yy+zz$ and $qz-ry$ quantities by p alone, at once we find thus an expression for the [reaction] force Π by the same p

$$\frac{\Pi}{M}=\frac{2gcc+faaA\left(B-\frac{Aaap}{cc}\right)}{2g(cc+ff-ffpp)}-\frac{fccp\left(Ccc-4fgp+ff(B-\frac{Aaap}{cc})^2\right)}{g(cc+ff-ffpp)^2},$$

now in that case we obtain also the element of time dt

$$dt=\frac{dp\sqrt{(cc+ff-Ifpp)}}{\sqrt{\left((Ccc-4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2\right)}},$$

from which equally by p there is

$$d\lambda=\frac{-dt\left(B-\frac{Aaap}{cc}\right)}{1-pp}$$

and the angular speed γ' can thus be defined, in order that

$$\gamma'\gamma'=AA+\frac{Ccc-4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp}.$$

But from γ' again the arc $AO=\alpha$ is known, thus in order that, because the time t is given by p, the quantities γ',α, p and λ can be assigned at a given time. Finally although it matters little to know the quantities y and z separately, yet from eq.1 and eq. 2 there is made

$$zdy-ydz+\frac{A(aa-cc)(yy+zz)}{cc}\,dt=\frac{2\Pi fgdt}{Mcc}(rz+qy)$$

and therefore

$$\frac{ydz-zdy}{yy+zz}=\frac{A(aa-cc)dt}{cc}-\frac{2\Pi fgdt(B-\frac{Aaap}{cc})(cc+ff-ffpp)}{Mcc\left(Ccc-4fgp+ff(B-\frac{Aaap}{cc})^2\right)},$$

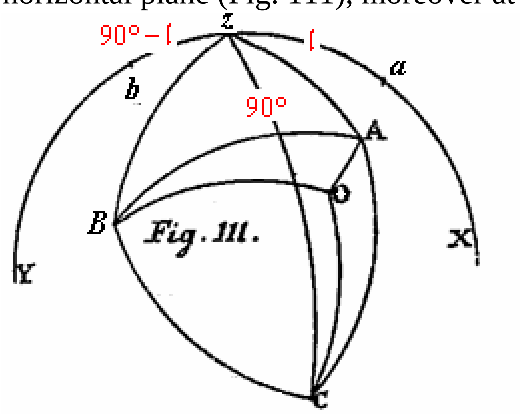
which since it is also integrable, gives A tang $\frac{z}{y}$ and thus the ratio between *y* and *z*, from which since *yy* + *zz* are taken together, both *y* and *z* are given separately; from which *q* and *r* are found also, being elicited separately from the values of the formulas *qy* + *rz* and *qz* – *ry*.

PROBLEM 100

863. If initially a rotational motion with a given angular speed were impressed about the special axis to the top, to define the position of this and the motion at some elapsed time.

SOLUTION

Initially we put, so that *t* = 0, the axis of the top to be at *a* with the distance or the arc present $Za = \mathfrak{l}$, and putting $\cos\mathfrak{l} = \mathfrak{p}$, so that $f\mathfrak{p}$ is the height of the centre of inertia above the horizontal plane (Fig. 111), moreover at the same time the arc *AB* is at *ab*, thus so that initially there is given



$$l = \mathfrak{l}, \quad m = 90° - \mathfrak{l}, \quad n = 90°, \text{ and } \quad \lambda = 0,$$

and thus

$$p = \mathfrak{p}, \quad q = \sqrt{(1 - \mathfrak{p}\mathfrak{p})} \quad \text{and} \quad r = 0 .$$

Then initially the top takes an angular motion about the axis *IA* itself in the sense *BC* with an angular speed equal to γ', thus so that there shall be

$$\alpha = 0, \quad \beta = 90°, \text{ and } \quad \gamma = 90°,$$

and thus $\gamma' = \varepsilon, x = \varepsilon, y = 0$ and $z = 0$. Hence therefore if the above constants entering on integration are defined, we obtain

$$1. \quad A = \varepsilon, \quad 2. \quad B = \frac{\varepsilon aa\mathfrak{p}}{cc} \quad \text{and} \quad 3. \quad C = \frac{4fg\mathfrak{p}}{cc}.$$

Moreover with these values first put in place between *t* and *p* this equation is found :

$$dt = \frac{cdp\sqrt{(cc + ff - ffpp)}}{\sqrt{(\mathfrak{p} - p)(4ccfg(l - pp) - \varepsilon\varepsilon a^4(\mathfrak{p} - p))}} .$$

Next the angle $XZA = \lambda$ is defined thus, in order that there shall be

$$d\lambda = \frac{-\varepsilon aadt(\mathfrak{p} - p)}{cc(1 - pp)} = \frac{-\varepsilon aadp\sqrt{(\mathfrak{p} - p)(cc + ff - ffpp)}}{c(1 - pp)\sqrt{(4ccfg(1 - pp) - eea^4(\mathfrak{p} - p))}}.$$

Again the angular speed γ' in the sense *ABC* is thus expressed

$$\gamma'\gamma' = \varepsilon\varepsilon + \frac{4c^4 fg(\mathfrak{p}-p)+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p)^2}{c^4(cc+ff-ffpp)}$$

and hence $\cos\alpha = \frac{\varepsilon}{\gamma'}$; but for $\cos\beta = \frac{y}{\gamma'}$ and $\cos\gamma = \frac{z}{\gamma'}$ there is first

$$yy+zz = \frac{4c^4 fg(\mathfrak{p}-p)+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p)^2}{c^4(cc+ff-ffpp)} = \gamma'\gamma'-\varepsilon\varepsilon.$$

In addition now we find :

$$qy+rz = \frac{\varepsilon aa}{cc}\left(\mathfrak{p}-p\right)$$

and

$$qz-ry = \frac{\sqrt{(\mathfrak{p}-p)\left(4ccfg(1-pp)-\varepsilon\varepsilon a^4(\mathfrak{p}-p)\right)}}{c\sqrt{(cc+ff-ffpp)}}$$

and the force, that the top with the point itself now exerts on the horizontal plane,

$$\frac{\Pi}{M} = \frac{2c^4 g+\varepsilon\varepsilon a^4 f(\mathfrak{p}-p)}{2ccg(cc+ff-ffpp)} - \frac{fp(\mathfrak{p}-p)\left(4c^4 fg+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p)\right)}{2ccg(cc-ff-ffpp)^2}.$$

And next for the definition of the quantities *y* and *z* separately, this equation is found:

$$\frac{ydz-zdy}{yy+zz} = \frac{\varepsilon(aa-cc)dt}{cc} - 2\frac{\Pi}{M}\cdot\frac{\varepsilon aafgdt(cc+ff-ffpp)}{4c^4 fg+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p)}$$

or

$$\frac{ydz-zdy}{yy+zz} = \frac{\varepsilon(aa-cc)\,dt}{cc} - \frac{\varepsilon aadt(2c^4 fg+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p))}{2cc(4c^4 fg+\varepsilon\varepsilon a^4 ff(\mathfrak{p}-p))} + \frac{\varepsilon aaffpdt(\mathfrak{p}-p)}{2cc(cc+ff-ffpp)}\ .$$

But with *y* and *z* found, also *q* and *r* can be determined through these.

COROLLARY 1

864. The arc *ZA* = *l* can be increased to a right angle or until the top collapses, as long as $\varepsilon\varepsilon a^4\mathfrak{p} < 4ccfg$. Hence it is necessary lest the top collapses, that the angular speed first impressed must be greater than

$$\frac{2c}{aa}\sqrt{\frac{fg}{\mathfrak{p}}},$$

[First edition: $-\frac{\Pi}{M}$ in place of $-2\frac{\Pi}{M}$. Corrected by C. B. in *O. O.* edition.]
where $\mathfrak{p} = \cos Za$. From which, if the top were vertical initially, then the condition must be :

$$\varepsilon > \frac{2c\sqrt{fg}}{aa};$$

for unless this condition is heeded, the slightest cause will be able to disturb the top.

COROLLARY 2

865. But if it should be the case that

$$\varepsilon\varepsilon a^4 \mathfrak{p} > 4ccfg ,$$

just as the quantity p cannot be greater than $\mathfrak{p}$, thus the limit is given, beyond which it can never be made smaller, which defined by the equation

$$4ccfgpp = 4ccfg - \varepsilon\varepsilon a^4 \mathfrak{p} + \varepsilon\varepsilon a^4 p$$

gives

$$p = \frac{\varepsilon\varepsilon a^4 - \sqrt{(e^4 a^8 - 16\varepsilon\varepsilon a^4 ccfg\mathfrak{p} + 64c^4\ ffgg)}}{8ccfg},$$

from which approximately there becomes

$$p = \mathfrak{p} - \frac{4ccfg(1-\mathfrak{pp})}{\varepsilon\varepsilon a^4 - 8ccfg}$$

for the smallest value of p itself, equal to *cos ZA* or with the maximum arc *ZA.*

COROLLARY 3

866. But if we look at the angle *IFk* in figure 109, the inclination of the axis to the horizontal cannot be made smaller than this, and the sine of this is equal to p, then the rotational motion of the top cannot continue, unless at this stage the smallest value of p is greater than the sine of the angle *IFk.* Whereby on putting $sinIFk = k$ it must be the case that

$$\varepsilon\varepsilon > \frac{4ccfg(1-kk)}{a^4(\mathfrak{p}-k)} .$$

SCHOLIUM 1

867. Hence it is agreed here that two cases are established, the one in which the rotational speed of the top impressed ε is less than

$$\frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}},$$

and the other in which this quantity is the greater. In the first case in which

$$\varepsilon < \frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}} ,$$

the top soon falls over, since it cannot arrive at the minimum inclination, without its body touching the horizontal plane, and thus the rotational motion is destroyed.
Now in the latter case, in which

$$\varepsilon > \frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}} ,$$

the rotational motion endures indefinitely, whenever we remove the motion from considerations of friction and all the other obstacles. Hence in order that the rotational motion shall be enduring perpetually, it is necessary that the top be given initially an angular speed greater than ε, as that formula shows. Moreover it is apparent that because a greater angular speed is impressed on the top, then the smaller the inclination to the horizontal is going to be, and if that speed should be infinite, the top perpetually maintains the same inclination. But when the rotational motion is everlasting, the top is inclined more to the horizontal from the start, while it arrives at the maximum inclination, then again it rights itself as far as to the initial position, so that when it arrives there it may be considered as if one period of its motion has been completed, then it will be progressing in a similar manner ; for the top at no time can be made more upright than it was initially, if it has not been influenced by friction. For if the top progresses while the point on the horizontal plane encounters friction, the effect of this wears away the ability of the top to erect itself, as long as it is not forced to collapse on account of the decreasing angular speed. Whereby it should not considered by anyone as miraculous, if the calculation agrees less with our experience, since the aberrations are to be attributed to friction.

SCHOLIUM 2

868. Also from these the account of the construction of tops can be examined, in order that they can undertake rotational motion most easily, or in order that the smallest rotational speed suffices for this. Clearly since the angular speed initially impressed must be greater than $\frac{2c\sqrt{fg}}{aa}$, it is apparent that the shape of the top is required to be of this kind, in order that the moment of inertia about the axis *AF* is a maximum compared with the moment of inertia about an axis normal to this. Whereby the most appropriate figure is a disk transfixed by the most slender rod, in which case there arises $aa = 2cc$; and if the radius of this disk is equal to h, then $aa = \frac{1}{2}hh$ and $cc = \frac{1}{4}hh$, and hence $\varepsilon > \frac{2\sqrt{fg}}{h}$. Then since the point or the interval $IF = f$ should be shorter, from this the required greater angular speed ε for the continued rotations is made less ; while also now at the smallest inclination the top touches the horizontal plane with its own body. We may put $h = \frac{1}{2}$ in[ch] and $f = \frac{1}{4}$ in., since $g = 187\frac{1}{2}$ in. [if this is the case, then the acc. of gravity $2g$ is 31.25 ft./sec.^{2}], there must be taken $\varepsilon > \sqrt{750}$; whereby if ε is taken twice as large or $\varepsilon = 55$, the top completes an arc equal to 55 [radians] in one second, or $\frac{55}{2\pi}$ rev., that is nearly nine revolutions are completed. But for larger tops it is sufficient for the smaller angular speed to depend on the ratio of the square root of the sides.

PROBLEM 101

869. If initially, for the top holding a given inclination, a rotational motion was impressed with a designated angular speed, in order that the inclination of this top should undergo minimal mutations, and to define the rotational motion.

SOLUTION

With everything remaining, as have been set up in the preceding problem, here we assume $\varepsilon\varepsilon$ exceed by many turns the quantity $\frac{4ccfg}{a^4\mathfrak{p}}$. Hence we may put

$$\varepsilon\varepsilon=\frac{4nccfg}{a^4\mathfrak{p}}$$

so that here *n* denotes a number great enough, and first we have this equation for the relation between *t* and *p*, because from the initial amount *p* decreases, and

$$dt=\frac{-dp\sqrt{\mathfrak{p}(cc+ff-ffpp)}}{2\sqrt{fg(\mathfrak{p}-p)(\mathfrak{p}-\mathfrak{p}pp-n\mathfrak{p}+np)}}\,.$$

Therefore since *p* falls short by a small amount from $\mathfrak{p}$, we can put $p=\mathfrak{p}-u$, in order that *u* is an exceedingly small amount, and there is produced

$$dt=\frac{+du\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{fgu(\mathfrak{p}-\mathfrak{p}^3-nu)}}$$

and hence

$$t=\frac{+\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}}\int\frac{du}{\sqrt{\left(\frac{\mathfrak{p}-\mathfrak{p}^3}{n}u-uu\right)}}$$

$$=C+\frac{\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}}\,arc\,sin\,ver\,\frac{2nu}{\mathfrak{p}-\mathfrak{p}^3}\,,$$

where it must be that $C = 0$. [The function $arc\,sin\,ver\,\frac{2nu}{\mathfrak{p}-\mathfrak{p}^3}$ is taken to mean $arc\,sin\left(\frac{2nu}{\mathfrak{p}-\mathfrak{p}^3}-1\right)$, which arises from the integration, and also gives the correct value π on giving *u* its maximum value.]Whereby from the beginning when $u = 0$ or $p=\mathfrak{p}$ until the time when the inclination becomes a maximum

$$u=\frac{\mathfrak{p}(1-\mathfrak{p}\mathfrak{p})}{n}$$

or

$$p = \mathfrak{p} - \frac{\mathfrak{p}(1-\mathfrak{pp})}{n},$$

the time is equal to

$$\frac{\pi\sqrt{\mathfrak{p}(cc+ff-ff\,\mathfrak{pp})}}{2\sqrt{nfg}},$$

which hence is shorter here, when the number *n* is greater. Now there then becomes

$$d\lambda = \frac{-2\sqrt{nfg}}{c(1-\mathfrak{pp})\sqrt{\mathfrak{p}}}\cdot udt$$

or

$$\lambda = \frac{-\sqrt{(cc+ff-ff\,\mathfrak{pp})}}{c(1-\mathfrak{pp})}\int\frac{du\sqrt{u}}{\sqrt{\left(\frac{\mathfrak{p}-\mathfrak{p}^3}{n}-u\right)}},$$

from which there is elicited

$$\lambda = \frac{-\sqrt{(cc+ff-ff\,\mathfrak{pp})}}{c(1-\mathfrak{pp})}\left(\frac{\mathfrak{p}(1-\mathfrak{pp})}{2n}\,arcsin\,ver\,\frac{2nu}{\mathfrak{p}(\mathfrak{p}-\mathfrak{pp})} - \sqrt{\left(\frac{\mathfrak{p}(1-\mathfrak{pp})}{n}u-uu\right)}\right).$$

Clearly the arc *ZA* is progressing in the opposite sense, and in the elapsed time *t*

$$= \frac{\pi\sqrt{\mathfrak{p}(cc+ff-ff\,\mathfrak{pp})}}{2\sqrt{nfg}} = \frac{\pi c\sqrt{(cc+ff-ff\,\mathfrak{pp})}}{\varepsilon aa},$$

when the top is inclined maximally to the horizontal, there occurs

$$\lambda = -\frac{\pi\mathfrak{p}\sqrt{(cc+ff-ff\,\mathfrak{pp})}}{2nc}.$$

For the axis *A* is not carried around the vertex *Z* in a uniform motion, but with the inequality of the motion disregarded, the mean angular speed is

$$\frac{\sqrt{\mathfrak{p}fg}}{c\sqrt{n}} = \frac{2\,fg}{\varepsilon aa}$$

on account of

$$n = \frac{\varepsilon\varepsilon a^4\mathfrak{p}}{4ccfg},$$

thus so that this angular speed of the axis of the top itself around the vertex *Z* shall vary inversely as the angular speed of the top about its own axis. Then while the top has a maximum inclination, so that it is

$$p = \mathfrak{p} - \frac{4ccfg(1-\mathfrak{pp})}{\varepsilon\varepsilon a^4},$$

the angular speed γ' is defined thus, in order that

$$\gamma'\gamma' = \varepsilon\varepsilon + \frac{16\, ffgg(1-\mathfrak{pp})}{\varepsilon\varepsilon a^4}$$

or

$$\gamma' = \varepsilon + \frac{8\, ffgg(1-\mathfrak{pp})}{\varepsilon^3 a^4},$$

and then

$$\cos\alpha = \cos AO = 1 - \frac{8\, ffgg(1-\mathfrak{pp})}{\varepsilon^4 a^4} = 1 - \frac{1}{2}\alpha\alpha$$

or the minimum arc itself

$$\alpha = AO = \frac{4\, fg\sqrt{(1-\mathfrak{pp})}}{\varepsilon\varepsilon aa}.$$

But for the force of the point *F* on the horizontal plane, for the initial motion or when $p = \mathfrak{p}$ and the axis of the top is maximally upright :

$$\frac{\Pi}{M} = \frac{cc}{cc + ff - ff\,\mathfrak{pp}};$$

but when the top is inclined maximally :

$$\frac{\Pi}{M} = \frac{cc + 2\, ff\,(1-\mathfrak{pp})}{cc + ff - ff\,\mathfrak{pp}} - \frac{\gamma' ccf^3 g\mathfrak{p}(1-\mathfrak{pp})}{eea^4(cc + ff - ff\,\mathfrak{pp})}.$$

And these suffice for the motion to be known.

COROLLARY 1

870. If the axis of the top were initially at *a*, on putting $Za = \mathfrak{l}$ since there shall be $\mathfrak{p} = \cos\mathfrak{l}$, if *A* is at the maximum elongation of the axis from the vertex, on putting *ZA* = *l*, because there is now

$$p = \cos l = \cos\mathfrak{l} - \frac{4ccfg\sin^2\mathfrak{l}}{\varepsilon\varepsilon a^4}$$

then

$$l = \mathfrak{l} + \frac{4ccfg\sin\mathfrak{l}}{\varepsilon\varepsilon a^4}.$$

COROLLARY 2

871. Because at the maximum inclination of the top the arc *ZA* is a maximum, it is evident the pole of the rotation *O* then it must lie on the arc *ZA* itself, so that *ZO* < *ZA*, and then this interval *AO* is equal to

$$\frac{4\, fg(1-\mathfrak{pp})}{\varepsilon\varepsilon aa}.$$

SCHOLION

872. Up to now we have taken the initial motion to be impressed about the axis *AF*, which is the most common case. Yet now it is possible to happen, that the motion is itself impressed about another, which comes about, if the true axis *AF*, while the top is rotating about that, likewise is given an impulse, which either inclines the axis more to the horizontal or raises it more to the vertical. For this returns the same, as if the a rotational motion were impressed on the top about another axis, except perhaps then likewise a progressive motion arises, which we ignore as it presents nothing difficult. Indeed the case now treated can be referred to here, if a certain mean position, in which the top now is rotating about another axis in addition to *AF*, as we observe at the start, but since here the axis of the top is unable to erect itself to the vertical at any time, not all motions are encountered in that motion. Whereby it is agreed that the case is still to be examined, in which the axis of the top *AF* initially maintains a vertical position, but the rotational motion itself is impressed about another axis inclined to the horizontal, also we can resolve that case by the general formulas set out previously.

PROBLEM 102

873. If the axis of the top were vertical initially, and to that a rotational motion should be impressed about some inclined axis with a given angular speed, to determine the motion of the top.

SOLUTION

Hence in the beginning since the point *A* should be at *Z*, we may put the arc *AC* to be lying on the circle *ZX*, thus in order that the arc *AB* should be normal to *ZX*. Whereby on making $t = 0$ then

$$l = 0, \quad m = 90^\circ \quad \text{and} \quad n = 90^\circ$$

and thus [see fig. 112 below in the scholium]

$$p = 1, \quad q = 0 \quad \text{et} \quad r = 0;$$

and $\mu = 90^\circ, \quad \nu = 0$, with λ remaining undefined, while now initially a rotational motion is impressed on the top with an angular speed equal to ε in the sense *ABC* about a pole situated in the arc *AC*, thus so that on putting $t = 0$, then

$$\alpha = \mathfrak{a}, \quad \beta = 90^\circ \quad \text{and} \quad \gamma = 90^\circ - \mathfrak{a},$$

and thus

$$x = \varepsilon cos\mathfrak{a}, \quad y = 0 \quad \text{et} \quad z = \varepsilon\, sin\mathfrak{a}.$$

With these in place we can define the position of the top after a time equal to *t* from § 862, if we can determine conveniently the constants of integration introduced from these conditions. In the first place let $A = \varepsilon cos\mathfrak{a}$, then $B = \frac{aa}{cc}\varepsilon cos\mathfrak{a}$; and in the third place

$$\frac{Ccc}{ff} - \frac{4g}{f} - \frac{\varepsilon\varepsilon cc\, sin^2\mathfrak{a}}{ff} = 0,$$

or

$$C = \frac{4fg}{cc} + \varepsilon\varepsilon cc\, sin^2 \mathfrak{a}\,;$$

moreover with these values substituted we obtain :

$$qy + rz = \frac{\varepsilon aa\, cos\,\mathfrak{a}}{cc}\,(1-p)$$

$$qz - ry = \frac{\sqrt{\left((\varepsilon\varepsilon c^4(1-pp)\,sin^2\mathfrak{a} + 4ccfg(1-p)(1-pp) - \varepsilon\varepsilon a^4(1-p)^2\,cos^2\mathfrak{a}\right)}}{c\sqrt{(cc + ff - ffpp)}}$$

$$yy + zz = \frac{\varepsilon\varepsilon c^6\, sin^2\mathfrak{a} + 4c^4 fg(1-p) + \varepsilon\varepsilon a^4 ff(1-p)^2 cos^2\mathfrak{a}}{c^4(cc + ff - ffpp)}$$

and hence

$$dt = \frac{-\,cdp\sqrt{(cc + ff - ffpp)}}{\sqrt{(1-p)\,(4ccfg(1-pp) + \varepsilon\varepsilon c^4(1+p)\,sin^2\mathfrak{a} - \varepsilon\varepsilon a^4(1-p)cos^2\mathfrak{a})}}$$

since initially the quantity *p* is made smaller.
Again on account of

$$\gamma'\gamma' = xx + yy + zz$$

and $x = \varepsilon\, cos\,\mathfrak{a}$ then

$$\gamma'\gamma' = \varepsilon\varepsilon cos^2\mathfrak{a} + \frac{\varepsilon\varepsilon c^6\, sin^2\mathfrak{a} + 4c^4 fg(1-p) + \varepsilon\varepsilon a^4 ff(1-p)^2 cos^2\mathfrak{a}}{c^4(cc + ff - ffpp)}$$

and finally

$$d\lambda = \frac{-\varepsilon aadt\, cos\,\mathfrak{a}}{cc(1+p)}.$$

COROLLARY 1

874. From the formula found for *dt* it is allowed to judge, whether or not the top will fall over. For on putting *p* = 0, and whenever the factor of the denominator

$$4ccfg + \varepsilon\varepsilon c^4\, sin^2\mathfrak{a} - \varepsilon\varepsilon a^4\, cos^2\mathfrak{a},$$

is positive, this indicates that the top will fall down ; which therefore comes about if

$$4ccfg + \varepsilon\varepsilon c^4\, sin^2\mathfrak{a} > \varepsilon\varepsilon a^4\, cos^2\mathfrak{a}$$

COROLLARY 2

875. Hence lest the top should fall, in the first place it is necessary that

$$a^4\, cos^2\mathfrak{a} > c^4\, sin^2\mathfrak{a}$$

or

$$\text{tang}\ \mathfrak{a} < \frac{aa}{cc},$$

then it is now required that

$$\varepsilon\varepsilon > \frac{4ccfg}{a^4\cos^2\mathfrak{a} - c^4\sin^2\mathfrak{a}};$$

or the angular speed first impressed must be greater than the limit

$$\frac{2c\sqrt{fg}}{\sqrt{\left(a^4\cos^2\mathfrak{a} - c^4\sin^2\mathfrak{a}\right)}}$$

and certainly it is noteworthy, that the body of the top does not touch the horizontal while it is inclined.

COROLLARY 3

876. But when there is both

$$\text{tang}\ \mathfrak{a} < \frac{aa}{cc},$$

and

$$\varepsilon\varepsilon > \frac{4ccfg}{a^4\cos^2\mathfrak{a} - c^4\sin^2\mathfrak{a}}$$

the axis of the top is not inclined as far as the horizontal, or the quantity p to be able to be diminished to zero ; but the smallest value of this appearing from the equation

$$4ccfgpp = \varepsilon\varepsilon\, p(a^4\cos^2\mathfrak{a} + c^4\sin^2\mathfrak{a}) - \varepsilon\varepsilon a^4\cos^2\mathfrak{a} + \varepsilon\varepsilon c^4\sin^2\mathfrak{a} + 4ccfg$$

and there is found

$$p = \frac{\varepsilon\varepsilon(a^4\cos^2\mathfrak{a} + c^4\sin^2\mathfrak{a}) - \sqrt{\varepsilon^4\left(a^4\cos^2\mathfrak{a} + c^4\sin^2\mathfrak{a}\right)^2 - 16\varepsilon\varepsilon ccfg\left(a^4\cos^2\mathfrak{a} - c^4\sin^2\mathfrak{a}\right)^2 + 64c^4 ffgg}}{8ccfg}.$$

COROLLARY 4

877. But if it should be the case that

$$\text{tang}\ \mathfrak{a} = \frac{aa}{cc}$$

or

$$a^4\cos^2\mathfrak{a} = c^4\sin^2\mathfrak{a},$$

then the equation between p and t becomes

$$dt = \frac{-dp\sqrt{(cc + ff - ffpp)}}{\sqrt{(1-p)\,(4fg(1-pp) + 2\varepsilon\varepsilon ccp\sin^2\mathfrak{a})}}$$

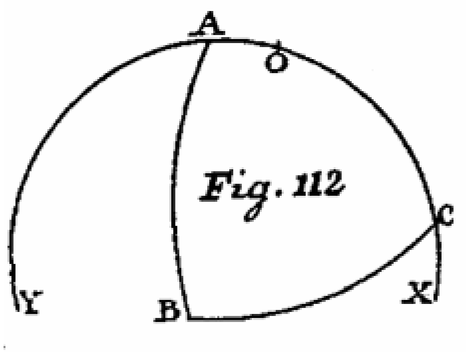
and not only can is p able to go to zero, but also to be diminished to a negative value, which becomes :

$$p=\frac{\varepsilon\varepsilon cc\sin^2\mathfrak{a}-\sqrt{\left(\varepsilon^4c^4\sin^2\mathfrak{a}+16ffgg\right)}}{4fg}$$

but so large that it excludes the inclination of the position of the question.

SCHOLIUM

878. The situation showing such a motion is represented in fig. 112, where the axis of the top *A* is turning at the vertex itself, now the remaining two axis at *B* and *C,* and thus we assume a certain fixed vertical circle *AX*; so that the quadrant AC is on that and the other *AB* is normal to that.



Hence in the beginning the motion is

$$l=0,\quad m=90°,\quad n=90°,$$

and thus

$$p=1,\quad q=0,\quad r=0,$$

then now $\mu=90°$ and $\nu=0$, with the value of λ being undefined. Then in the beginning the rotational motion of the top to be impressed I take now to be about the axis *IO,* with the arc $AO=\mathfrak{a}$ present, and that with a speed ε in the sense *ABC;* and thus on putting *t* = 0 then

$$\alpha=\mathfrak{a},\quad \beta=90°,\quad \gamma=90°-\mathfrak{a}\quad\text{and}\quad \gamma'=\varepsilon$$

and hence

$$x=\varepsilon\cos\mathfrak{a},\quad y=0\quad\text{and}\quad z=\varepsilon\sin\mathfrak{a}.$$

Therefore in the case that the top does not fall over, two conditions are required, the one being that

$$\text{tang }\mathfrak{a}\text{ or tang }AO<\frac{aa}{cc},$$

and the other that

$$\varepsilon>\frac{2c\sqrt{fg}}{\sqrt{\left(a^4\cos^2\mathfrak{a}-c^4\sin^2\mathfrak{a}\right)}}.$$

And if we wish, so that the axis may be inclined the least, and on making $p=1-\omega$ for an particular exceedingly small ω present, then there is found

$$\omega=\frac{2\varepsilon\varepsilon c^4\sin^2\mathfrak{a}}{\varepsilon\varepsilon a^4\cos^2\mathfrak{a}+\varepsilon\varepsilon c^4\sin^2\mathfrak{a}-8ccfg},$$

whereby the arc $AO=\mathfrak{a}$ is to be the smallest, then $\varepsilon\varepsilon a^4$ in this case must be much greater than *8ccfg* in order that

$$\varepsilon>\frac{2c\sqrt{2fg}}{aa}.$$

Since if that comes about, the motion is regular enough, which will be helpful in determining the motion more accurately.

PROBLEM 103

879. If, for a top established in an upright position, there is impressed a rotational motion fast enough about an axis at a minimum declination, in order that the top only leaves its position for a short time, to determine the motion of this.

SOLUTION

Hence we take the arc $AO = \mathfrak{a}$ in the beginning to be very small and the rotational speed ε impressed in the beginning to be so large that

$$\varepsilon\varepsilon a^4 \cos^2 \mathfrak{a} > 8ccfg .$$

Hence we may put

$$\varepsilon\varepsilon a^4 \cos^2 \mathfrak{a} = 8nccfg$$

so that $n > 1$, and we have

$$dt = \frac{-cdp\sqrt{(cc + ff - ffpp)}}{\sqrt{(1-p)(4ccfg(1-pp) - 8nccfg(1-p) + \varepsilon\varepsilon c^4(1+p)\sin^2 \mathfrak{a})}}$$

Hence as we know that p is made much less that one, we may put $p = 1-u$,
and with the smallest terms ignored,

$$dt = \frac{cdu}{\sqrt{u(8fgu - 8nfgu + 2\varepsilon\varepsilon cc\sin^2 \mathfrak{a} - \varepsilon\varepsilon ccu\sin^2 \mathfrak{a})}} ,$$

and the integral of this is

$$t = \frac{c}{\sqrt{(\varepsilon\varepsilon cc\sin^2 \mathfrak{a} + 8(n-1)fg)}} \, arcsin\,ver \frac{u(\varepsilon\varepsilon cc\sin^2 \mathfrak{a} + 8(n-1)fg)}{\varepsilon\varepsilon cc\sin^2 \mathfrak{a}}$$

as now the maximum value of u is equal to [i.e. to make the argument = 2]

$$\frac{2\varepsilon\varepsilon cc\sin^2 \mathfrak{a}}{\varepsilon\varepsilon cc\sin^2 \mathfrak{a} + 8(n-1)fg}$$

and the time to reach the maximum inclination of the top is equal to

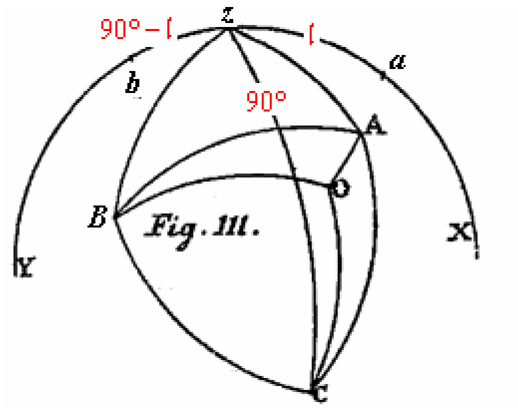
$$\frac{\pi c}{\sqrt{(\varepsilon\varepsilon cc\sin^2 \mathfrak{a} + 8(n-1)fg)}} = \frac{\pi cc}{\sqrt{(\varepsilon\varepsilon c^4\cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4\sin^2 \mathfrak{a} - 8ccfg)}}$$

and the top meanwhile falls from the vertical position by a small angle, of which the versed sine is equal to

$$\frac{2\varepsilon\varepsilon c^4 \sin^2 \mathfrak{a}}{\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg}$$

and the angle itself equal to

$$\frac{2\varepsilon cc \sin \mathfrak{a}}{\sqrt{\left(\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg\right)}}.$$



Then since

$$d\lambda = \frac{-\varepsilon aadt \cos \mathfrak{a}}{cc(1+p)}$$

and here it is able to consider p as a constant equal to 1, in the time, in which the top reaches its maximum inclination (Fig. 111), the axis of this turns in the vertical plane, since it declines from the circle ZX, by the angle

$$XZA = 90^{\mathrm{O}} - \frac{\pi\varepsilon aa \cos \mathfrak{a}}{2\sqrt{\left(\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg\right)}};$$

for in the beginning at first, when A is rotating about O, fig.112, the angle λ is right or equal to $\frac{\pi}{2}$.

COROLLARY 1

880. Since the initial arc $AO = \mathfrak{a}$ shall be as if infinitely small, and the angle $XZA = \lambda$ in the beginning is equal to 90°, in the elapsed time t, this angle becomes

$$\lambda = 90^{\mathrm{O}} - \frac{\varepsilon aat}{2cc}.$$

Hence the axis of the top is moving away from the point to the vertical in the preceding, and the whole circuit is completed in a time equal to $\frac{4\pi acc}{\varepsilon aa}$ sec.

COROLLARY 2

881. Since initially $u = 0$, in the elapsed time t there arises

$$1 - \frac{u(\varepsilon\varepsilon a^4 - 8ccfg)}{\varepsilon\varepsilon c^4 \sin^2 \mathfrak{a}} = \cos\frac{t\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}}{cc}.$$

But on putting the smallest arc ZA equal to l, on account of

$$p = \cos l = 1 - \tfrac{1}{2}ll\ ,$$

then $u = \frac{1}{2}ll$, and hence

$$l = \frac{2\varepsilon cc \sin \mathfrak{a}}{\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}} \sin\frac{t\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}}{2cc}.$$

thus so that at some time t we are able to assign the values λ and l.

COROLLARY 3

882. Since the axis of the top *Z* moves away to reach the maximum declination in the passing of time equal to

$$\frac{\pi cc}{\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}},$$

in which time this is carried around *Z* in the preceding through the angle

$$\frac{\pi\varepsilon aa}{2\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}},$$

which hence is greater than a right angle and returns to the vertical *Z* on passing through an absolute angle greater than two right angles

$$\frac{\pi\varepsilon aa}{\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}}.$$

SCHOLION

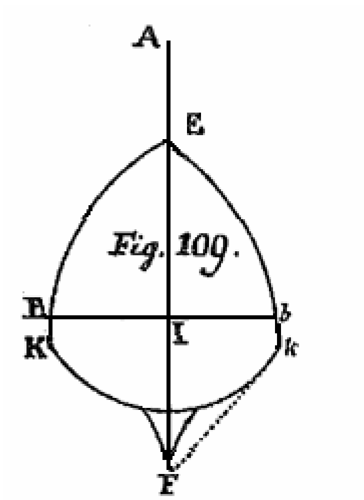
883. I will not linger to set out examples of further motions of this kind, since all the phenomena can be derived easily from the formulas found. But it is required to remember properly, that here no account of friction is been given, which however small that is put in place, strongly disturbs the phenomena defined here. Indeed from the friction, which the point *F* experiences in proceeding on the horizontal plane, there arises a horizontal force, which is impressed on the horizontal motion of the top, and because the direction of that force is changing continually, it is readily understood to be the reason why tops are observed to proceed along curved lines. Indeed the motions come to an end on account of the individual friction forces ; whereby we may proceed from separate impediments of this kind to certain other kinds of motion in which rotation occurs ; and because here I have considered bodies of this kind, which proceed with a point [or cusp] on a horizontal plane, thus in order that the cusp can be considered as if the base of these, here we are lead to consider other bodies in general, which proceed with some base on the plane. And indeed a plane or angular base is hardly worth any attention, since either no rotational motion is found to be present, and thus the determination of the motion presents no difficulties, or perhaps by a jump a rotation becomes confused itself, while the contact is transferred to another base, where a like conflict unfolds, and the explanation of this has to be referred to another branch of mechanics. Therefore here it is agreed to consider only bases of this kind, by which bodies proceed on a plane unmoved, which are provided with a continual curvature, lest another jump in the motion happens. But excessive circuits of the motion which lead to inextricable calculations are to be avoided , so we will pursue mainly only two kinds of bodies, clearly cylinders and spheres, without doubt the external figures of which, by which they are applied to the plane, shall be either cylindrical or spherical, in whatever manner the inner material should be distributed, the ratio of this is determined from the centre of inertia and from the principal axes. Hence they are referred to as a kind of cylinder, pendulums which are not suspended from the line of the axis, as we have assumed above, but from small axes lying on both sides on a horizontal plane. Then here also the motion extends to wandering motions similar to the to and fro motions of cradles, and bodies of this kind, as they lie in a plane, can be viewed as cylinders. Then also, how bodies of this kind descent on an inclined plane, is worth the effort of being looked. Again with spherical bodies, I refer not only to those in which the whole figure is globular, but also below which, where they reach a plane, they have formed hemispheres, just as they are tops, the axes of which do not end below in a point but as if in the hemisphere ; when indeed the centre of inertia is higher than the centre of the hemisphere, but when the centre of inertia has been placed deeper, another kind of motion can arise, in which the body by staggering performs oscillations, in which it is possible to have in place a wonderful perturbation of the rotational motion.

CAPUT XVII

PLENIOR EXPLICATIO MOTUS TURBINUM SUPER PLANO HORIZONTALI SEMOTA FRICTIONE

DEFINITIO 14

850. *Axis turbinis* est recta *AF* ex cuspide *F* per centrum inertiae *I* ducta, qui simul sit eius axis principalis singularis, ita ut respectu omnium axium ad eum normalium *I B* momenta inertiae sint inter se aequalia (Fig. 109).



COROLLARIUM 1

851. Aptissima ergo turbinis figura est tornata, quae generatur, si figura quaecunque circa axem *AF* revolvitur, dummodo ea in cuspidem *F* desinat, qua super plano horizontali incedere posset.

COROLLARIUM 2

852. In turbine autem sequentes quantitates cognitas esse oportet, quae in calculum ingrediuntur :

1°. eius massam vel pondus, quod sit = *M ;*
2°. distantiam cuspidis a centro inertiae, quae sit $IF = f;$
3°. momentum inertiae respectu axis *AF,* quod sit = *Maa,* et
4°. momentum inertiae respectu omnium axium ad illum normalium, quod sit = *Mcc.*

COROLLARIUM 3

853. Cum ergo supra in genere momenta inertiae principalia cuiusque corporis posuerimus *Maa*, *Mbb* et *Mcc*, hic bina posteriora aequalia statuemus, ut sit $bb = cc$.

COROLLARIUM 4

854. Dum igitur turbo cuspide *F* super plano horizontali incedit, eius axis *A.F* non ultra certum terminum ad horizontem inclinari potest, qui habebitur ducendo ab *F* ad corpus turbinis rectam extremam *Fk;* tum enim angulus *AFk* dabit illum terminum.

SCHOLION

855. Supra tantum eiusmodi turbines consideravimus, in quibus omnia momenta inertiae inter se essent aequalia, quae conditio nimium erat limitata. Nunc igitur motum turbinum in genere exploremus, siquidem conditio, quod *AF* sit axis principalis, et respectu binorum reliquorum axium momenta inertiae aequalia, cum indole turbinum necessario cuniuncta videtur. Principia autem, unde huius motus determinatio est petenda, supra in Capite 14 iam sunt exposita, ubi vidimus totum negotium a pressione, qua turbo, dum movetur, cuspide sua *F*

plano horizontali innititur, pendere. Quae pressio, etiamsi nonnisi solutione ad finem perducta cognosci queat, tamen statim ab initio in calculum ingreditur. Sit ergo Π ista pressio, cuius directio a cuspide *F* semper verticaliter sursum tendit; atque de hae pressione supra §767 ostendimus, si inclinatio axis *AF* ad horizontem ponatur = ϑ, quae tempusculo *dt* suo differentiali $d\vartheta$ crescat, sumto elemento *dt* constante, fore

$$\frac{dd\vartheta\cos\vartheta - d\vartheta^2\sin\vartheta}{dt^2} = \frac{2g}{f}\left(\frac{\Pi}{M} - 1\right),$$

sive

$$\frac{\Pi}{M} = 1 + \frac{f(dd\vartheta\cos\vartheta - d\vartheta^2\sin\vartheta)}{2gdt^2} = 1 + \frac{fd\cdot d\vartheta\cos\vartheta}{2gdt^2} = 1 + \frac{fdd\cdot\sin\vartheta}{2gdt^2}$$

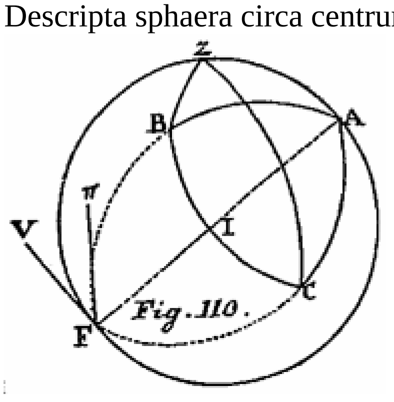
Cum igitur turbo praeter hanc vim Π a gravitate tantum urgeri statuatur, eius centrum inertiae *I* alium motum recipere nequit, nisi in directione verticali vel ascendendo vel descendendo, dum eius distantia a plano horizontali est = $f\sin\vartheta$. Sin autem initio ei insuper quidam motus horizontalis fuerit impressus, cum constanter acquabilem conservabit, sicque tota quaestio ad solum motum gyratorium reducitur. Quare cum gravitas ad eum nihil conferat eiusque perturbationes omnes a sola pressione Π oriantur, huius vis momenta respectu axium principalium turbinis definiri oportet.

PROBLEMA 97

856. Si turbo teneat situm quomodocunque inclinatum ad horizontem simulque detur pressio Π, qua eius cuspis horizontali plano innititur, definire huius vis momenta respectu axium principalium turbinis.

SOLUTIO

Descripta sphaera circa centrum inertiae turbinis *I* (Fig. 110), in qua sit *Z* punctum



verticale, axis turbinis autem sphaeram traiiciat in punctis *A* et *F*, bini reliqui vero axes principales pertingant ad sphaerae puncta *B* et *C;* etsi enim hi duo axes per se non determinantur, tamen certas duas lineas tam inter se quam ad axem *AF* normales accipi convenit, ex quibus deinceps situs turbinis ad quodvis tempus definiatur. Ponantur arcus circulorum maximorum

$$ZA = l,\ ZB = m \text{ et } ZC = n,$$

erit $l = 90° - \vartheta$ denotante ϑ inclinationem axis *AF* ad horizontem. Cum iam cuspis *F*, cuius distantia a centro inertiae *I* est *FI* = *f*, urgeatur in directione verticali $F\Pi$ vi = Π, ut sit angulus $AF\Pi = l$, resolvatur ea secundum directiones *FA* et *FV,* quarum haec *FV* sit in plano verticali *AZF*ad *AF* normalis, erit

$$\text{vis secundum } FA = \Pi\cos l$$

et

$$\text{vis secundum } FV = \Pi\sin l,$$

quarum illa per centrum inertiae transiens nulla praebet momenta. Haec vero vis $FV = \Pi \sin l$ respectu axis *AF* quoque nullum praebet momentum; at respectu axis *IB* dat momentum = $\Pi f \sin l \sin VFB$ in sensum *AC*, similique modo respectu axis *1C* momentum = $\Pi f \sin l \sin VFC$ in sensum *BA*. Verum est angulus *VFB* = *ZAR* et

$$\sin ZAB = -\cos ZAC = -\frac{\cos n}{\sin l}$$

tum vero angulus *VFC* = *ZAC* et

$$\sin ZAC = \cos ZAB = \frac{\cos m}{\sin l}$$

Quamobrem habebimus

momentum respectu axis $IB = -\Pi f \cos n$ in sensu *AC*,
momentum respectu axis $IC = \Pi f \cos m$, in sensu *BA*

et quia momenta virium respectu axium *1A*, *1B*, *IC* in sensum *BC*, *CA*, *AB* supra [§ 803] in genere posuimus *P*, *Q*, *R*, erit pro nostro casu:

$$P = 0, Q = \Pi f \cos n \quad \text{et} \quad R = -\Pi f \cos m.$$

PROBLEMA 98

857. Si turbo in situ quocunque inclinato gyretur circa axem quemcunque per eius centrum inertiae transeuntem, definire variationem momentaneam tam in axe gyrationis quam in celeritate angulari productam.

SOLUTIO

Circa centrum inertiae *I* constituta sphaera immobili (Fig. 111), in qua sit *Z* punctum verticale et *ZX* circulus verticalis fixus; teneat iam turbo eiusmodi situm, ut axis turbinis proprius respondeat sphaerae puncto *A*, bini reliqui autem axes principales punctis *B* et *C*, ponanturque horum axium declinationes a verticali seu arcus

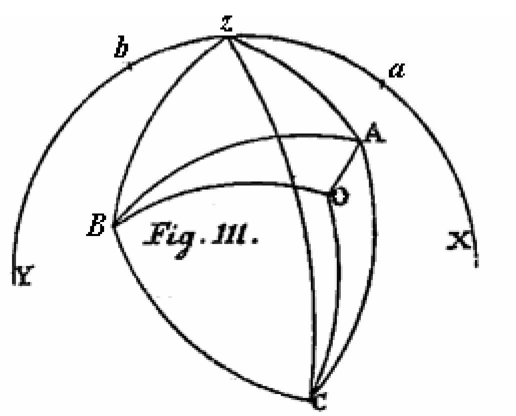


Fig. 111.

$$ZA = l,\ ZB = m,\ ZC = n\ ,$$

ut sit

$$\cos^2 l + \cos^2 m + \cos^2 n = 1\,;$$

tum vero anguli

$$XZA = \lambda,\ XZB = \mu \quad \text{et} \quad XZC = \nu\ ,$$

quorum relationes ad illos arcus sunt cognitae. Nunc autem turbo gyretur circa axem *IO* celeritate angulari = γ' in sensum *ABC* sintque pro polo gyrationis *O* arcus

$$AO = \alpha,\ BO = \beta \quad \text{et} \quad CO = \gamma',$$

atque ponendo

$$\gamma' \cos \alpha = x, \quad \gamma' \cos \beta = y, \quad \gamma' \cos \gamma = z$$

ob momenta virium

$$P = 0, \quad Q = \Pi f \cos n \quad \text{et} \quad R = -\Pi f \cos m \ ,$$

atque $bb = cc$, variationes tempusculo dt productae sequentibus formulis exprimentur :

$$\text{I. } dx = 0$$

$$\text{II. } dy + \frac{aa-cc}{cc} xzdt = \frac{2\Pi fgdt \cos n}{Mcc}$$

$$\text{111. } dz - \left(\frac{aa-cc}{cc}\right) xydt = -\frac{2\Pi fgdt \cos m}{Mcc}.$$

Praeterea vero has aequationes pro $l,\ m,\ n,\ \lambda,\ \mu,\ v$ adiungi oportet:

$$dl \sin l = dt(y \cos n - z \cos m), \quad d\lambda \sin^2 l = -dt(y \cos m + z \cos n),$$

$$dm \sin m = dt(z \cos l - x \cos n), \quad d\mu \sin^2 m = -dt(z \cos n + x \cos l),$$

$$dn \sin n = dt(x \cos m - y \cos l), \quad dv \sin^2 n = -dt(x \cos l + y \cos m).$$

Cum autem inclinatio axis ad horizontem sit = 90°- l, quae supra posita est ϑ, ob $\sin\vartheta = \cos l$ erit

$$\frac{\Pi}{M} = 1 + \frac{fdd \cdot \cos l}{2gdt^2}.$$

Ad haec magis contrahenda statuamus:

$$\cos l = p \ , \cos m = q \ , \cos n = r$$

et habebimus

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2},$$

ac praeterea has aequationes:

$$\text{I.} dx = 0$$

$$\text{II. } dy + \frac{(aa-cc)\, xzdt}{cc} = \frac{2\Pi fgrdt}{Mcc}$$

$$\text{III. } dz - \frac{(aa-cc)\, xydt}{cc} = \frac{-2\Pi fgqdt}{Mcc}.$$

$$\text{IV. } dp = dt(qz - ry), \qquad \text{VI I. } d\lambda = \frac{-dt(qy + rz)}{1-pp}$$

$$\text{V. } dq = dt(rx - pz), \qquad \text{VIII. } d\mu = \frac{-dt(rz + px)}{1-qq}$$

$$\text{VI. } dr = dt(py - qx), \qquad \text{IX. } dv = \frac{-dt(px + qy)}{1-rr}$$

ubi notandum est esse $pp + qq + rr = 1$.

COROLLARIUM 1

858. Si turbo circa ipsum axem *IA* gyretur, ut sit

$$\alpha = 0 \text{ et } \beta = y = 90°,$$

erit

$$x = \gamma', \quad y = 0, \quad \text{et} \quad z = 0$$

et

$$dx = d\gamma', \quad dy = -\gamma' d\beta, \quad dz = -\gamma' d\gamma .$$

Fiet ergo

$$d\gamma' = 0, \quad d\beta = \frac{-2\Pi\, fgrdt}{\gamma' Mcc}, \qquad d\gamma = \frac{2\Pi\, fgqdt}{\gamma' Mcc},$$
$$dp = 0, \quad dq = d\gamma' rdt, \quad dr = -d\gamma' qdt \quad \text{et} \quad d\lambda = 0,$$

tum ergo primo instanti neque celeritas angularis γ' neque situs puncti *A* mutationem patitur.

COROLLARIUM 2

859. Cum sit $dp = dt(qz-ry)$, erit differentiando

$$ddp = dt(qdz - rdy) + dt(zdq - ydr),$$

et substitutis valoribus datis reperietur:

$$\frac{ddq}{dt^2} = \frac{(aa-cc)x}{cc}(qy+rz) - \frac{2\Pi\, fg}{Mcc}(qq+rr) + x(qy+rz) - p(yy+zz),$$

unde fit

$$\frac{\Pi}{M} = 1 + \frac{f(aa-cc)x}{2gcc}(qy+rz) - \frac{\Pi\, ff}{Mcc}(qq+rr) + \frac{fx(qy+rz)}{2g} - \frac{fp(yy+zz)}{2g}$$

seu

$$\frac{\Pi}{M}\left(1 + \frac{ff(qq+rr)}{cc}\right) = 1 + \frac{faax(qy+rz)}{2gcc} - \frac{fp(yy+zz)}{2g}$$

hincque

$$\frac{\Pi}{M} = \frac{2gcc + faax(qy+rz) - fccp(yy+zz)}{2gcc + 2gff(qq+rr)}.$$

[In editione principe numerator est: $2gcc + faa(qy + rz) - fccp(yy + zz)$.
Correxit C. B.]

COROLLARIUM 3

860. Ex aequationibus IV. V. VI. colligitur, ut iam ante notavimus,

$$xdp + ydq + zdr = 0,$$

quae aequatio, cum sit

$$pp + qq + rr = 1\ ,$$

loco aequationum V. et VI. usurpari potest. At aequationum VII. VIII. IX unicam tractasse sufficiet, quod negotium postremo loco erit suscipiendum.

COROLLARIUM 4

861. Inventis autem quantitatibus *x, y* et *z*, ob

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

erit celeritas angularis

$$\gamma' = \sqrt{(xx + yy + zz)}$$

hincque vicissim arcus α, β, γ concluduntur; nempe

$$\cos\alpha = \frac{x}{\gamma'},\quad \cos\beta = \frac{y}{\gamma'},\quad \text{et}\quad \cos\gamma = \frac{z}{\gamma'}.$$

PROBLEMA 99

862. Aequationes differentiales ante inventas, [811] quibus motus turbinis exprimitur, ad integrationem perducere, quantum fieri licet.

SOLUTIO

Primo statim patet esse *x* = const., ponamus ergo *x* = *A*, et reliquae aequationes integrandae erunt

$$1^{\circ}.\ dy + \frac{A(aa-cc)zdt}{cc} = \frac{2\Pi fgrdt}{Mcc}$$

$$2^{\circ}.\ dz - \frac{A(aa-cc)ydt}{cc} = -\frac{2\Pi fgqdt}{Mcc}$$

$$3^{\circ}.\ dp = dt\ (qz - ry)$$

$$4^{\circ}.\ ydq + zdr = -Adp$$

existente

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2}$$

et

$$pp + qq + rr = 1\ .$$

Nunc 1°. *q* +2°. *r* suppeditat hanc aequationem

$$qdy + rdz + \frac{A(aa-cc)}{cc}\ dt(qz - ry) = 0,$$

quae ob *dt(qz – ry)* = *dp* abit in hanc:

$$qdy + rdz = \frac{-A\ (aa-cc)}{cc}dp\ ,$$

huc addatur 4°

$$ydq + zdr = -Adp$$

erit

$$qdy + ydq + rdz + zdr = \frac{-aa}{cc}Adp,$$

cuius integrale est

$$qy + rz = B - \frac{aa}{cc}Ap.$$

Porro colligendo 1° . y + 2°. z prodit:

$$ydy + zdz = \frac{2\Pi\, fgdt}{Mcc}(ry - qz) = \frac{-2\Pi\, fgdp}{Mcc};$$

quare cum sit

$$\frac{\Pi}{M} = 1 + \frac{fddp}{2gdt^2},$$

erit

$$ydy + zdz = \frac{-2\, fgdp}{cc} - \frac{ffdpddp}{ccdt^2},$$

unde integrando nanciscimur:

$$yy + zz = C - \frac{4\, fgp}{cc} - \frac{ffdp^2}{ccdt^2}.$$

cum iam sit

$$\frac{dp}{dt} = qz - ry,$$

obtinemus novam aequationem finitam:

$$yy + zz = C - \frac{4\, fgp}{cc} - \frac{ff}{cc}(qz - ry)^2,$$

ex qua cum sit

$$(qz - ry)^2 = \frac{Ccc}{ff} - \frac{4gp}{f} - \frac{cc(yy+zz)}{ff},$$

ex ante inventa autem

$$(qy + rz)^2 = (B - \frac{Aaap}{cc})^2,$$

prodibit his addendis

$$(qq + rr)(yy + zz) = \frac{Ccc}{ff} - \frac{4gp}{f} - \frac{cc(yy+zz)}{ff} + \left(B - \frac{Aaap}{cc}\right)^2$$

unde ob

$$qq + rr = 1 - pp$$

elicitur

$$(1 - pp + \frac{cc}{f})(yy + zz) = \frac{Ccc}{ff} - \frac{4gp}{f} + \left(B - \frac{Aaap}{cc}\right)^2$$

seu

$$yy+zz=\frac{Ccc-4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp},$$

$$(qz-ry)^2=\frac{(Ccc-4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp}.$$

cum ergo iam has quantitates $qy + rz$, $yy + zz$, $yy + zz$ et $qz - ry$ per solam p definiverimus, statim pressionem Π per eandem solam p ita reperimus expressam

$$\frac{\Pi}{M}=\frac{2gcc+faaA\left(B-\frac{Aaap}{cc}\right)}{2g(cc+ff-ffpp)}-\frac{fccp\left(Ccc-4fgp+ff(B-\frac{Aaap}{cc})^2\right)}{g(cc+ff-ffpp)^2},$$

deinde vero etiam elementum temporis dt obtinebimus

$$dt=\frac{dp\sqrt{(cc+ff-Ifpp)}}{\sqrt{\left((Ccc-4fgp)(1-pp)-cc\left(B-\frac{Aaap}{cc}\right)^2\right)}},$$

ex quo pariter per p erit

$$d\lambda=\frac{-dt\left(B-\frac{Aaap}{cc}\right)}{1-pp}$$

atque celeritas angularis γ' ita definietur, ut sit

$$\gamma'\gamma'=AA+\frac{Ccc-4fgp+ff\left(B-\frac{Aaap}{cc}\right)^2}{cc+ff-ffpp}.$$

Ex γ' autem porro cognoscitur arcus $AO=\alpha$, ita ut, quoniam tempus t per p datur, quantitates γ', α, p et λ ad datum tempus assignari queant. Denique etsi parum refert nosse quantitates y et z seorsim, tamen ex 1° et 2° fit

$$zdy-ydz+\frac{A(aa-cc)(yy+zz)}{cc}\,dt=\frac{2\Pi fg}{Mcc}dt(rz+qy)$$

ideoque

$$\frac{ydz-zdy}{yy+zz}=\frac{A(aa-cc)dt}{cc}-\frac{2\Pi fgdt(B-\frac{Aaap}{cc})(cc+ff-ffpp)}{Mcc\left(Ccc-4fgp+ff(B-\frac{Aaap}{cc})^2\right)},$$

quae cum etiam sit integrabilis, dabit $\text{A tang}\frac{z}{y}$ ideoque rationem inter y et z, ex qua cum $yy + zz$ coniunctim, utraque y et z seorsim datur; quibus inventis etiam q et r seorsim ex valoribus formularum $qy + rz$ et $qz - ry$ eliciuntur.

PROBLEMA 100

863. Si turbini initio in data inclinatione impressus fuerit motus gyratorius circa proprium axem data celeritate angulari, definire eius situm et motum ad quodvis tempus inde elapsum.

SOLUTIO

Ponamus initio, quo $t = 0$, axem turbinis fuisse in *a* distantia seu arcu existente $Za = \mathfrak{l}$, ac ponatur $cos\,\mathfrak{l} = \mathfrak{p}$, ut fuerit $f\mathfrak{p}$ altitudo centri inertiae supra planum horizontale (Fig. 111), eodem autem tempore arcus *AB* fuerit in *ab*, ita ut pro initio habeatur

$$l = \mathfrak{l}, \quad m = 90° - \mathfrak{l}, \quad n = 90° \text{ et } \quad \lambda = 0,$$

ideoque

$$p = \mathfrak{p}, \quad q = \sqrt{(1 - \mathfrak{p}\mathfrak{p})} \quad \text{et} \quad r = 0\,.$$

Deinde initio turbo circa ipsum axem *IA* acceperit in sensum *BC* motum gyratorium celeritate angulari = γ', ita ut fuerit

$$\alpha = 0, \quad \beta = 90° \text{ et } \quad \gamma = 90°,$$

ideoque $\gamma' = \varepsilon, x = \varepsilon, y = 0$ et $z = 0$. Hinc ergo si constantes supra per integrationem ingressae definiantur, obtinemus

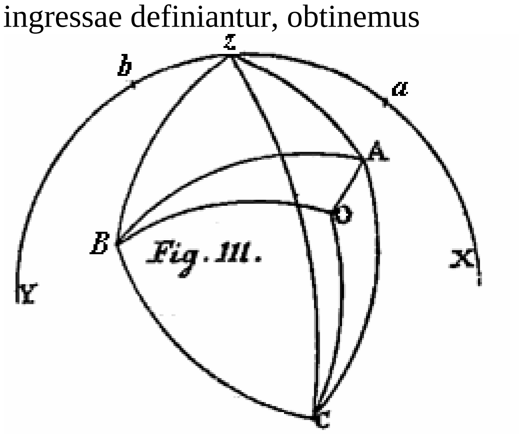


Fig. 111.

$$1^{\text{o}}. \quad A = \varepsilon, \quad 2^{\text{o}}. \quad B = \frac{\varepsilon aa\mathfrak{p}}{cc} \quad \text{et} \quad 3^{\text{o}}. \quad C = \frac{4fg\mathfrak{p}}{cc}.$$

His autem valoribus substitutis prima inter *t* et *p* haec reperitur aequatio

$$dt = \frac{cdp\sqrt{(cc + ff - ffpp)}}{\sqrt{(\mathfrak{p} - p)\left(4ccfg(l - pp) - \varepsilon\varepsilon a^4\,(\mathfrak{p} - p)\right)}}\,.$$

Deinde angulus $XZA = \lambda$ ita definitur, ut sit

$$d\lambda = \frac{-\varepsilon aadt(\mathfrak{p} - p)}{cc(1 - pp)} = \frac{-\varepsilon aadp\sqrt{(\mathfrak{p} - p)(cc + ff - ffpp)}}{c(1 - pp)\sqrt{\left(4ccfg(1 - pp) - eea^4(\mathfrak{p} - p)\right)}}.$$

Porro celeritas angularis γ' in sensum *ABC* ita exprimitur

$$\gamma'\gamma' = \varepsilon\varepsilon + \frac{4c^4 fg(\mathfrak{p} - p) + \varepsilon\varepsilon a^4 ff(\mathfrak{p} - p)^2}{c^4(cc + ff - ffpp)}$$

hincque $cos\,\alpha = \frac{\varepsilon}{\gamma'}$; at pro $cos\,\beta = \frac{y}{\gamma'}$ et $cos\,\gamma = \frac{z}{\gamma'}$ est primo

$$yy+zz=\frac{4c^4fg(\mathfrak{p}-p)+\varepsilon\varepsilon a^4ff(\mathfrak{p}-p)^2}{c^4(cc+ff-ffpp)}=\gamma'\gamma'-\varepsilon\varepsilon.$$

Praeterea vero invenimus:

$$qy+rz=\frac{\varepsilon aa}{cc}\left(\mathfrak{p}-p\right)$$

et

$$qz-ry=\frac{\sqrt{(\mathfrak{p}-p)\left(4ccfg(1-pp)-\varepsilon\varepsilon a^4(\mathfrak{p}-p)\right)}}{c\sqrt{(cc+ff-ffpp)}}$$

atque pressionem, quam nunc turbo cuspide sua exerit in planum horizontale,

$$\frac{\Pi}{M}=\frac{2c^4g+\varepsilon\varepsilon a^4f(\mathfrak{p}-p)}{2ccg(cc+ff-ffpp)}-\frac{fp(\mathfrak{p}-p)\left(4c^4fg+\varepsilon\varepsilon a^4ff(\mathfrak{p}-p)\right)}{2ccg(cc-ff-ffpp)^2}.$$

Denique ad quantitates *y* et *z* seorsim definiendas habetur haec aequatio

$$\frac{ydz-zdy}{yy+zz}=\frac{\varepsilon(aa-cc)dt}{cc}-2\frac{\Pi}{M}\cdot\frac{\varepsilon aafgdt(cc+ff-ffpp)}{4c^4fg+\varepsilon\varepsilon a^4ff(\mathfrak{p}-p)}$$

seu

$$\frac{ydz-zdy}{yy+zz}=\frac{\varepsilon(aa-cc)\,dt}{cc}-\frac{\varepsilon aadt(2c^4fg+\varepsilon\varepsilon a^4ff(\mathfrak{p}-p))}{2cc(4c^4fg+\varepsilon\varepsilon a^4ff(\mathfrak{p}-p))}+\frac{\varepsilon aaffpdt(\mathfrak{p}-p)}{2cc(cc+ff-ffpp)}.$$

Inventis autem *y* et *z*, etiam *q* et *r* per eas determinantur.

COROLLARIUM 1

864. Arcus $ZA = l$ usque ad angulum rectum augeri seu turbo procidere potest, quamdiu $\varepsilon\varepsilon a^4\mathfrak{p}<4ccfg$. Ne ergo turbo prolabatur, necesse est, ut eius celeritas angularis prima impressa maior sit quam

$$\frac{2c}{aa}\sqrt{\frac{fg}{\mathfrak{p}}},$$

[Editio princeps: $-\frac{\Pi}{M}$ loco $-2\frac{\Pi}{M}$. Correxit C. B.]

ubi est $\mathfrak{p}=\cos Za$. Unde, si turbo initio fuerit verticalis, debet esse

$$\varepsilon>\frac{2c\sqrt{fg}}{aa};$$

nisi enim haec conditio observetur, levissima causa turbinem deturbare valebit.

COROLLARIUM 2

865. Sin autem fuerit

$$\varepsilon\varepsilon a^4\mathfrak{p} > 4ccfg ,$$

quemadmodum quantitas p nunquam superare potest $\mathfrak{p}$, ita dabitur limes, infra quem nunquam diminuetur, qui definitus ex aequatione

$$4ccfgpp = 4ccfg - \varepsilon\varepsilon a^4\mathfrak{p} + \varepsilon\varepsilon a^4 p$$

prodit

$$p = \frac{\varepsilon\varepsilon a^4 - \sqrt{(e^4a^8 - 16\varepsilon\varepsilon a^4 ccfg\mathfrak{p} + 64c^4\ ffgg)}}{8ccfg},$$

unde fit proxime

$$p = \mathfrak{p} - \frac{4ccfg(1 - \mathfrak{pp})}{\varepsilon\varepsilon a^4 - 8ccfg}$$

pro minimo valore ipsius $p = \cos ZA$ seu pro maximo arcu ZA.

COROLLARIUM 3

866. Sin autem in figura 109 spectemus ad angulum IFk, quo inclinatio axis ad horizontem, cuius sinus est = p, minor fieri non potest, motus turbinis gyratorius perennis esse nequit, nisi valor minimus ipsius p adhuc fuerit maior quam sinus anguli IFk. Quare posito $\sin IFk = k$ debet esse

$$\varepsilon\varepsilon > \frac{4ccfg(1-kk)}{\mathrm{a}^4(\mathfrak{p}-k)} .$$

SCHOLION 1

867. Hic ergo duos casus constitui convenit, alterum quo celeritas angularis turbini primum impressa ε minor est quam

$$\frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}},$$

alterum quo hac quantitate est maior. Priori casu quo

$$\varepsilon < \frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}} ,$$

turbo mox procidet, quoniam ad minimam inclinationem pervenire nequit, quin corpore suo planum horizontale attingat, sicque motus gyratorius destruatur.
Posteriori vero casu, quo

$$\varepsilon > \frac{2c\sqrt{fg(1-kk)}}{aa\sqrt{(\mathfrak{p}-k)}} ,$$

motus gyratorius perpetuo durabit, quandoquidem a frictione omnibusque motus obstaculis mentem abstrahimus. Ut ergo motus gyratorius prodeat perennis, necesse est turbini primum maiorem celeritatem angularem ε imprimi, quam ista formula exhibet. Patet autem, quo maior turbini celeritas angularis imprimatur, eo minus eum ad horizontem inclinatum iri, ac si celeritas illa foret infinita, turbo eandem inclinationem perpetuo conservaret. Quando autem motus gyratorius est perennis, turbo ab initio magis ad horizontem inclinabitur, donec maximam inclinationem attigerit, tum iterum se eriget usque ad statum initialem, quo ubi pervenerit, quasi unam motus sui periodum absolvisse est censendus, deinceps simili modo progressurus; nunquam enim turbo magis fiet erectus, quam fuerat initio, si quidem nulla affuerit frictio. Namque si turbo, dum cuspide super plano horizontali incedit, frictionem offendat, eius effectus in erigendo turbine consumetur, quatenus is ob minutam eeleritatem angularem non prolabi cogitur. Quare nemini mirum videri debet, si experientia nostro calculo minus conveniat, cum aberrationes frictioni sint tribuendae.

SCHOLION 2

868. Ex his etiam ratio constructionis turbinum perspicitur, ut facillime motum gyratorium recipiant, seu ut minima celeritas angularis ad hoc sufficere possit. Scilicet cum celeritas angularis initio impressa maior esse debeat quam $\frac{2c\sqrt{fg}}{aa}$, patet turbinis figuram eiusmodi esse oportere, ut eius momentum inertiae respeetu axis *AF* sit maximum prae momento respectu axium ad hunc normalium. Quare figura aptissima erit discus planus hasta tenuissima transfixus, quo casu fit $aa = 2cc$; ae si radius eius disci fuerit $= h$, erit $aa = \frac{1}{2}hh$ et $cc = \frac{1}{4}hh$, hincque $e > \frac{2\sqrt{fg}}{h}$. Deinde quo brevior fuerit cuspis seu intervallum $IF = f$, eo magis celeritas angularis ε ad durationem gyrationis requisita minuitur; verum tum etiam in minore inclinatione turbo planum horizontale corpore suo attinget. Ponamus $h = \frac{1}{2}$ dig. et $f = \frac{1}{4}$ dig., quoniam $g = 187\frac{1}{2}$ dig., sumi debet $\varepsilon > \sqrt{750}$; quare si capiatur ε dupla maior vel $\varepsilon = 55$, turbo uno minuto secundo conficiet arcum = 55, seu $\frac{55}{2\pi}$, hoc est fere novem revolutiones absolvet. Pro turbinibus autem maioris moduli celeritas angularis minor secundum rationem subduplicatam laterum sufficiet.

PROBLEMA 101

869. Si turbini initio datam inclinationem tenenti impressus fuerit motus gyratorius satis insigni celeritate angulari, ut inclinatio eius minimas subeat mutationes, definire eius motum gyratorium.

SOLUTIO

Manentibus omnibus, uti in problemate praecedente sunt constituta, assumimus hic $\varepsilon\varepsilon$ multis vicibus excedere quantitatem $\frac{4ccfg}{a^4\mathfrak{p}}$. Ponamus ergo

$$\varepsilon\varepsilon = \frac{4nccfg}{a^4\mathfrak{p}}$$

ut *n* hic denotet numerum satis magnum, ac primo pro relatione inter *t* et *p* hanc habebimus aequationem, quia ab initio quantitas *p* decrescit,

$$dt = \frac{-dp\sqrt{\mathfrak{p}(cc+ff-ffpp)}}{2\sqrt{fg(\mathfrak{p}-p)(\mathfrak{p}-\mathfrak{p}pp-n\mathfrak{p}+np)}}$$

cum igitur p quam minimum a $\mathfrak{p}$ deficiat, ponamus $p = \mathfrak{p} - u$, ut u sit particula vehementer exigua, fietque

$$dt = \frac{+du\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{fgu(\mathfrak{p}-\mathfrak{p}^3-nu)}}$$

hincque

$$t = \frac{+\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}}\int\frac{du}{\sqrt{\left(\frac{\mathfrak{p}-\mathfrak{p}^3}{n}u-uu\right)}}$$
$$= C + \frac{+\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}} A\cdot sin\ vers. \frac{2nu}{\mathfrak{p}-\mathfrak{p}^3},$$

ubi debet esse $C = 0$. Quare ab initio ubi $u = 0$ seu $p = \mathfrak{p}$ usque ad tempus, quo inclinatio fit maxima

$$u = \frac{\mathfrak{p}(1-\mathfrak{p}\mathfrak{p})}{n}$$

seu

$$p = \mathfrak{p} - \frac{\mathfrak{p}(1-\mathfrak{p}\mathfrak{p})}{n},$$

erit tempus

$$= \frac{\pi\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}},$$

quod ergo eo est brevius, quo maior fuerit numerus n. Deinde vero fit

$$d\lambda = \frac{-2\sqrt{nfg}}{c(1-\mathfrak{p}\mathfrak{p})\sqrt{\mathfrak{p}}}\cdot udt$$

sive

$$\lambda = \frac{-\sqrt{(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{c(1-\mathfrak{p}\mathfrak{p})}\int\frac{du\sqrt{u}}{\sqrt{\left(\frac{\mathfrak{p}-\mathfrak{p}^3}{n}-u\right)}},$$

unde elicitur

$$\lambda = \frac{-\sqrt{(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{c(1-\mathfrak{p}\mathfrak{p})}\left(\frac{\mathfrak{p}(1-\mathfrak{p}\mathfrak{p})}{2n} A\cdot sin\ vers. \frac{2nu}{\mathfrak{p}(\mathfrak{p}-\mathfrak{p}\mathfrak{p})} - \sqrt{\left(\frac{\mathfrak{p}(1-\mathfrak{p}\mathfrak{p})}{n}u-uu\right)}\right).$$

Arcus scilicet ZA in sensum oppositum progreditur, et elapso tempore t

$$= \frac{\pi\sqrt{\mathfrak{p}(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{2\sqrt{nfg}} = \frac{\pi c\sqrt{(cc+ff-ff\mathfrak{p}\mathfrak{p})}}{\varepsilon aa},$$

quo turbo maxime ad horizontem inclinatur, fit

$$\lambda = -\frac{\pi\mathfrak{p}\sqrt{(cc+ff-ff\mathfrak{pp})}}{2nc}.$$

Non quidem axis *A* motu aequabili circa verticem *Z* circumferetur, sed neglecta motus inaequalitate erit celeritas angularis media

$$\frac{\sqrt{\mathfrak{p}fg}}{c\sqrt{n}} = \frac{2\,fg}{\varepsilon aa}$$

ob

$$n = \frac{\varepsilon\varepsilon a^4\mathfrak{p}}{4ccfg},$$

ita ut haec celeritas angularis ipsius axis turbinis circa verticem *Z* sit reciproce, ut celeritas angularis turbinis circa proprium axem. Deinde dum turbo maximam habet inclinationem, ut sit

$$p = \mathfrak{p} - \frac{4ccfg(1-\mathfrak{pp})}{\varepsilon\varepsilon a^4},$$

celeritas angularis γ' ita definitur, ut sit

$$\gamma'\gamma' = \varepsilon\varepsilon + \frac{16\,ffgg(1-\mathfrak{pp})}{\varepsilon\varepsilon a^4}$$

seu

$$\gamma' = \varepsilon + \frac{8\,ffgg(1-\mathfrak{pp})}{\varepsilon^3 a^4},$$

eritque

$$\cos\alpha = \cos AO = 1 - \frac{8\,ffgg(1-\mathfrak{pp})}{\varepsilon^4 a^4} = 1 - \frac{1}{2}\alpha\alpha$$

seu ipse arcus minimus

$$\alpha = AO = \frac{4\,fg\sqrt{(1-\mathfrak{pp})}}{\varepsilon\varepsilon aa}.$$

Pro pressione autem cuspidis *F* in planum horizontale habetur pro motus initio, seu ubi $p = \mathfrak{p}$ et axis turbinis maxime erectus

$$\frac{\Pi}{M} = \frac{cc}{cc+ff-ff\mathfrak{pp}};$$

at quando turbo maxime inclinatur:

$$\frac{\Pi}{M} = \frac{cc+2\,ff(1-\mathfrak{pp})}{cc+ff-ff\mathfrak{pp}} - \frac{\gamma' ccf^3 g\mathfrak{p}(1-\mathfrak{pp})}{eea^4(cc+ff-ff\mathfrak{pp})}.$$

Haeque ad motus cognitionem sufficiunt.

COROLLARIUM 1

870. Si axis turbinis initio fuerit in *a*, posito $Za = \mathfrak{l}$ cum sit $\mathfrak{p} = \cos\mathfrak{l}$, si *A* sit maxima elongatio axis a vertice, posito *ZA* = *l*, quia est

$$p = cos\, l = cos\, \mathfrak{l} - \frac{4ccfg\sin^2\mathfrak{l}}{\varepsilon\varepsilon a^4}$$

erit

$$l = \mathfrak{l} + \frac{4ccfg\sin\mathfrak{l}}{\varepsilon\varepsilon a^4}.$$

COROLLARIUM 2

871. Quia in maxima turbinis inclinatione arcus *ZA* est maximus, evidens est polum gyrationis *O* tum in ipsum arcum *ZA* cadere debere, ut sit *ZO* < *ZA*, et tum intervalIum hoc *AO* erit

$$= \frac{4\,fg(1-\mathfrak{pp})}{\varepsilon\varepsilon aa}.$$

SCHOLION

872. Hactenus sumsimus turbini initio motum gyratorium imprimi circa ipsum axem *AF*, qui est casus maxime communis. Verum tamen fieri potest, ut ipsi circa alium motus imprimatur, quod evenit, si axis verus *AF*, dum turbo circa eum gyratur, simul impulsionem accipiat, qua ad horizontem vel magis inclinetur, vel inde magis erigatur. Hoc enim eodem redit, ac si turbini circa alium axem motus gyratorius imprimeretur, nisi quatenus inde simul motus progressivus oritur, qui cum nihil habeat difficultatis, ad eum non respiciamus. Casus quidem iam ante tractatus huc referri potest, si statum quendam medium, quo turbo iam circa alium axem praeter *AF* gyratur, tanquam initialem spectemus, sed quoniam ibi axis turbinis se nunquam ad situm verticalem erigere potest, in eo non omnes motus continentur. Quare conveniet adhuc eum casum pertractari, quo turbinis axis *AF* primo quidem tenet situm verticalem, ipsi autem motus gyratorius circa alium axem ad horizontem inclinatum imprimitur, quem casum etiam per formulas generales ante evolutas resolvere poterimus.

PROBLEMA 102

873. Si turbinis axis initio fuerit verticalis, eique circa axem quendam inclinatum impressus sit motus gyratorius data celeritate angulari, determinare motum turbinis.

SOLUTIO

Cum ergo initio punctum *A* fuerit in *Z*, ponamus arcum *AC* in circulum *ZX* incidisse, ita ut arcus *AB* fuerit ad *ZX* normalis. Quare facto $t = 0$ erat

$$l = 0, \quad m = 90° \quad \text{et} \quad n = 90°$$

ideoque

$$p = 1, \quad q = 0 \quad \text{et} \quad r = 0;$$

ac $\mu = 90°, \quad \nu = 0$, manente λ indefinito, tum vero initio turbini impressus fuerit motus gyratorius celeritate angulari = ε in sensum *ABC* circa polum in arcu *AC* situm, ita ut posito $t = 0$, fuerit

$$\alpha = \mathfrak{a}, \quad \beta = 90° \quad \text{et} \quad \gamma = 90° - \mathfrak{a},$$

ideoque

$$x = \varepsilon cos\mathfrak{a}, \quad y = 0 \quad \text{et} \quad z = \varepsilon \, sin\mathfrak{a} \; .$$

His positis statum turbinis post tempus = t ex § 862 definiemus, si constantes per integrationem ingressas his conditionibus convenienter determinemus.

Primo ergo fiet $A = \varepsilon cos\mathfrak{a}$, deinde $B = \frac{aa}{cc}\varepsilon cos\mathfrak{a}$; tertio

$$\frac{Ccc}{ff} - \frac{4g}{f} - \frac{\varepsilon\varepsilon cc\, sin^2 \mathfrak{a}}{ff} = 0,$$

sive

$$C = \frac{4fg}{cc} + \varepsilon\varepsilon cc\, sin^2 \mathfrak{a}$$

his autem valoribus substitutis obtinebimus

$$qy + rz = \frac{\varepsilon aa\, cos\,\mathfrak{a}}{cc}\,(1-p)$$

$$qz - ry = \frac{\sqrt{\left((\varepsilon\varepsilon c^4 (1-pp)\, sin^2 \mathfrak{a} + 4ccfg(1-p)(1-pp) - \varepsilon\varepsilon a^4 (1-p)^2 \, cos^2 \mathfrak{a}\right)}}{c\sqrt{(cc + ff - ffpp)}}$$

$$yy + zz = \frac{\varepsilon\varepsilon c^6 \, sin^2 \mathfrak{a} + 4c^4 fg(1-p) + \varepsilon\varepsilon a^4 ff(1-p)^2 cos^2\mathfrak{a}}{c^4(cc + ff - ffpp)}$$

hincque

$$dt = \frac{-cdp\sqrt{(cc + ff - ffpp)}}{\sqrt{(1-p)\,(4ccfg(1-pp) + \varepsilon\varepsilon c^4 (1+p)\, sin^2 \mathfrak{a} - \varepsilon\varepsilon a^4 (1-p) cos^2\mathfrak{a})}}$$

quoniam initio quantitas p minuitur.
Porro ob

$$\gamma'\gamma' = xx + yy + zz$$

et $x = \varepsilon \, cos\,\mathfrak{a}$ erit

$$\gamma'\gamma' = \varepsilon\varepsilon cos^2\mathfrak{a} + \frac{\varepsilon\varepsilon c^6 \, sin^2 \mathfrak{a} + 4c^4 fg(1-p) + \varepsilon\varepsilon a^4 ff(1-p)^2 cos^2\mathfrak{a}}{c^4(cc + ff - ffpp)}$$

ac tandem

$$d\lambda = \frac{-\varepsilon aadt\, cos\,\mathfrak{a}}{cc(1+p)}.$$

COROLLARIUM 1

874. Ex formula pro dt inventa iudicare licet, utrum turbo sit prolapsurus, necne. Ponatur enim $p = 0$, et denominatoris factor

$$4ccfg + \varepsilon\varepsilon c^4 \, sin^2 \mathfrak{a} - \varepsilon\varepsilon a^4 \, cos^2 \mathfrak{a},$$

quoties est positivus, turbinem ad lapsum proclivem indicat; quod ergo evenit, si

$$4ccfg + \varepsilon\varepsilon c^4 \, sin^2 \mathfrak{a} > \varepsilon\varepsilon a^4 \, cos^2 \mathfrak{a}$$

COROLLARIUM 2

875. Ne ergo turbo prolabatur, primo necesse est, ut sit

$$a^4 \cos^2 \mathfrak{a} > c^4 \sin^2 \mathfrak{a}$$

seu

$$\operatorname{tang} \mathfrak{a} < \frac{aa}{cc},$$

deinde vero esse oportet

$$\varepsilon\varepsilon > \frac{4ccfg}{a^4 \cos^2 \mathfrak{a} - c^4 \sin^2 \mathfrak{a}};$$

seu celeritas angularis primum impressa superare debet limitem

$$\frac{2c\sqrt{fg}}{\sqrt{\left(a^4 \cos^2 \mathfrak{a} - c^4 \sin^2 \mathfrak{a}\right)}}$$

et quidam notabiliter, ne turbo, dum inclinatur, corpore suo horizontem attingat.

COROLLARIUM 3

876. Quando autem est tam

$$\operatorname{tang} \mathfrak{a} < \frac{aa}{cc},$$

quam

$$\varepsilon\varepsilon > \frac{4ccfg}{a^4 \cos^2 \mathfrak{a} - c^4 \sin^2 \mathfrak{a}}$$

axis turbinis non ad horizontem usque inclinari, seu quantitas p ad nihilum usque diminui potest; sed minimus eius valor prodiens ex aequatione

$$4ccfgpp = \varepsilon\varepsilon p(a^4 \cos^2 \mathfrak{a} + c^4 \sin^2 \mathfrak{a}) - \varepsilon\varepsilon a^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} + 4ccfg$$

reperitur

$$p = \frac{\varepsilon\varepsilon(a^4 \cos^2 \mathfrak{a} + c^4 \sin^2 \mathfrak{a}) - \sqrt{\varepsilon^4\left(a^4 \cos^2 \mathfrak{a} + c^4 \sin^2 \mathfrak{a}\right)^2 - 16\varepsilon\varepsilon ccfg\left(a^4 \cos^2 \mathfrak{a} - c^4 \sin^2 \mathfrak{a}\right)^2 + 64c^4 ffgg}}{8ccfg}.$$

COROLLARIUM 4

877. Sin autem fuerit

$$\operatorname{tang} \mathfrak{a} = \frac{aa}{cc}$$

seu

$$a^4 \cos^2 \mathfrak{a} = c^4 \sin^2 \mathfrak{a},$$

aequatio inter p et t erit

$$dt = \frac{-\,dp\sqrt{(cc+ff-ffpp)}}{\sqrt{(1-p)\,(4\,fg(1-pp)+2\varepsilon\varepsilon ccp\,sin^2\,\mathfrak{a})}}$$

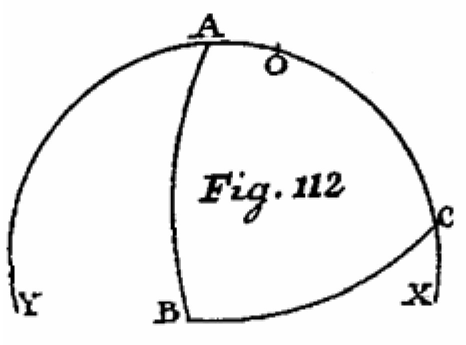
atque *p* non solum ad nihilum usque, sed etiam ad valorem negativum minui poterit, qui foret:

$$p = \frac{\varepsilon\varepsilon cc\,sin^2\,\mathfrak{a}\ -\sqrt{\left(\varepsilon^4c^4\,sin^2\,\mathfrak{a}+16\,ffgg\right)}}{4\,fg}$$

sed tantam inclinationem status quaestionis excludit.

SCHOLION

878. Status initialis talem motum exhibens in fig. 112 repraesentatur, ubi axis turbinis



A in ipso vertice versatur, bini reliqui vero in *B* et *C*, et circulum quidem verticalem fixum *AX* ita assumsimus; ut in eo esset quadrans *AC* et alter *AB* ad istum normalis. Initio motus ergo erat

$$l = 0,\quad m = 90°\ ,\quad n = 90°,$$

ideoque

$$p = 1,\quad q = 0\ ,\quad r = 0\ ,$$

tum vero $\mu = 90°$ et $\nu = 0$, existente λ indefinito. Deinde vero turbini initio motum gyratorium impressum esse sumo circa axem *IO*, existente arcu $AO = \mathfrak{a}$, eumque celeritate ε in sensum *ABC;* sicque posito *t* = 0 erat

$$\alpha = \mathfrak{a},\quad \beta = 90°,\quad \gamma = 90°-\mathfrak{a}\quad \text{et}\quad \gamma' = \varepsilon$$

hinque

$$x = \varepsilon cos\,\mathfrak{a},\quad y = 0\quad \text{et}\quad z = \varepsilon\,sin\,\mathfrak{a}\ .$$

Ne igitur hoc casu turbo prolabatur, binae conditiones requiruntur, altera ut sit

$$\text{tang}\mathfrak{a}\ \text{seu tang}\ AO < \frac{aa}{cc},$$

altera ut sit

$$\varepsilon > \frac{2c\sqrt{fg}}{\sqrt{\left(a^4\,cos^2\,\mathfrak{a}-c^4\,sin^2\,\mathfrak{a}\right)}}.$$

Ac si velimus, ut axis quam minime inclinetur, fiatque $p = 1-\omega$ existente ω particula valde parva, reperitur

$$\omega = \frac{2\varepsilon\varepsilon c^4\,sin^2\,\mathfrak{a}}{\varepsilon\varepsilon a^4\,cos^2\,\mathfrak{a}+\varepsilon\varepsilon c^4\,sin^2\,\mathfrak{a}-8ccfg},$$

quare arcus $AO = \mathfrak{a}$ quam minimus esse, deinde vero $\varepsilon\varepsilon a^4$ multum excedere debet $8ccfg$ ut sit

$$\varepsilon > \frac{2c\sqrt{2fg}}{aa}.$$

Quod si eveniat, motus satis erit regularis, quem accuratius determinasse iuvabit.

PROBLEMA 103

879. Si turbini in situ erecto constituto circa axem quam minime declinantem impressus fuerit motus gyratorius satis celer, ut turbo parumper tantum a statu erecto recedat, eius motum determinare.

SOLUTIO

Sumimus ergo arcum $AO = \mathfrak{a}$ initio fuisse valde parvum et celeritatem angularem initio impressam ε tantam, ut fuerit

$\varepsilon\varepsilon a^4 \cos^2 \mathfrak{a} > 8ccfg$.

Ponamus ergo

$$\varepsilon\varepsilon a^4 \cos^2 \mathfrak{a} = 8nccfg$$

ut sit $n > 1$, et habebimus

$$dt = \frac{-cdp\sqrt{(cc+ff-ffpp)}}{\sqrt{(1-p)(4ccfg(1-pp)-8nccfg(1-p)+\varepsilon\varepsilon c^4(1+p)\sin^2\mathfrak{a})}}$$

Quia ergo novimus p parum infra unitatem diminui, statuamus $p = 1-u$,
fietque neglectis terminis minimis

$$dt = \frac{cdu}{\sqrt{u(8fgu-8nfgu+2\varepsilon\varepsilon cc\sin^2\mathfrak{a}-\varepsilon\varepsilon ccu\sin^2\mathfrak{a})}},$$

cuius integrale est

$$t = \frac{c}{\sqrt{(\varepsilon\varepsilon cc\sin^2\mathfrak{a}+8(n-1)fg)}}\text{ A sin vers.}\frac{u(\varepsilon\varepsilon cc\sin^2\mathfrak{a}+8(n-1)fg)}{\varepsilon\varepsilon cc\sin^2\mathfrak{a}}$$

cum nunc maximus valor ipsius u sit

$$= \frac{2\varepsilon\varepsilon cc\sin^2\mathfrak{a}}{\varepsilon\varepsilon cc\sin^2\mathfrak{a}+8(n-1)fg}$$

tempus usque ad maximam turbinis inclinationem est

$$\frac{\pi c}{\sqrt{(\varepsilon\varepsilon cc\sin^2\mathfrak{a}+8(n-1)fg)}} = \frac{\pi cc}{\sqrt{(\varepsilon\varepsilon c^4\cos^2\mathfrak{a}+\varepsilon\varepsilon c^4\sin^2\mathfrak{a}-8ccfg)}}$$

atque turbo tum declinabit a situ erecto angulo exiguo, cuius sinus versus est

$$= \frac{2\varepsilon\varepsilon c^4 \sin^2 \mathfrak{a}}{\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg}$$

ipse angulus

$$= \frac{2\varepsilon cc \sin \mathfrak{a}}{\sqrt{\left(\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg\right)}}.$$

Deinde cum sit

$$d\lambda = \frac{-\varepsilon aadt \cos \mathfrak{a}}{cc(1+p)}$$

hicque p ut constans = 1 considerari possit, tempore, quo turbo ad maximam inclinationem pertingit (Fig. 111), eius axis versabitur in plano verticali, quod a circulo ZX declinat angulo

$$XZA = 90^{\circ} - \frac{\pi\varepsilon aa \cos \mathfrak{a}}{2\sqrt{\left(\varepsilon\varepsilon c^4 \cos^2 \mathfrak{a} + \varepsilon\varepsilon c^4 \sin^2 \mathfrak{a} - 8ccfg\right)}};$$

prima enim initio, quo A circa O gyratur fig. 112, angulus λ est rectus seu = $\frac{\pi}{2}$.

COROLLARIUM 1

880. Cum arcus initialis $AO = \mathfrak{a}$ sit quasi infinite parvus, et angulus $XZA = \lambda$ initio fuerit = 90°, elapso tempore t, fiet hic angulus

$$\lambda = 90^{\circ} - \frac{\varepsilon aat}{2cc}.$$

Axis ergo turbinis ex puncto verticali egressus in antecedentia movetur, et integrum circuitum absolvit tempore = $\frac{4\pi acc}{\varepsilon aa}$ min. sec.

COROLLARIUM 2

881. Cum initio esset u = 0, elapso tempore t fiet

$$1 - \frac{u(\varepsilon\varepsilon a^4 - 8ccfg)}{\varepsilon\varepsilon c^4 \sin^2 \mathfrak{a}} = \cos \frac{t\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}}{cc}.$$

Posito autem arcu ZA minimo = l, ob

$$p = \cos l = 1 - \tfrac{1}{2} ll ,$$

erit u = $\frac{1}{2} ll$, hincque

$$l = \frac{2\varepsilon cc \sin \mathfrak{a}}{\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}} \sin \frac{t\sqrt{(\varepsilon\varepsilon a^4 - 8ccfg)}}{2cc}.$$

ita ut ad quodvis tempus t assignare valeamus λ et l.

COROLLARIUM 3

882. Cum axis turbinis ex Z digressus ad maximam declinationem pertingit, praeterlabitur tempus

$$=\frac{\pi cc}{\sqrt{(\varepsilon\varepsilon a^4-8ccfg)}},$$

quo tempore is circa Z in antecedentia circumfertur per angulum

$$\frac{\pi\varepsilon aa}{2\sqrt{(\varepsilon\varepsilon a^4-8ccfg)}},$$

qui ergo recto est maior atque in verticem Z revertetur absoluto angulo

$$\frac{\pi\varepsilon aa}{\sqrt{(\varepsilon\varepsilon a^4-8ccfg)}}$$

maiore duobus rectis.

SCHOLION

883. Huiusmodi motibus evolvendis fusius non immoror, cum omnia phaenomena facile ex formulis inventis derivari queant. Probe autem meminisse oportet, hic nullam frictionis rationem esse habitam, quae quamvis parva statuatur, phaenomena hic definita vehementer perturbat. Ex frictione enim, quam cuspis F super plano horizontali incedens patitur, nascitur vis horizontalis, qua turbini motus progressivus imprimitur, et quia directio illius vis continuo mutatur, facile causa perspicitur, cur turbines motu curvilineo incedere observentur. Verum motus ob frictionem perturbati singularem exigunt tractionem; quare sepositis huiusmodi impedimentis ad alia quaedam motus genera, in quibus gyratio occurrit, progrediamur; et quoniam hic eiusmodi corpora sum contemplatus, quae cuspide super plano horizontali incedunt, ita ut cuspis sit quasi basis eorum censenda, hic ad alia corporum genera ducimur, quae basi quacunque super plano incedant. Ac de basi quidem plana vel angulosa vix quicquam proferri potest attentione dignum, cum vel nullus motus gyratorius locum inveniat, ideoque motus determinatio nihil habeat difficultatis, vel saltem per saltus gyratio se immisceat, dum contactus ad aliam hedram transfertur, ubi simul conflictus se exerit, cuius explicatio ad aliam Mechanicae partem est referenda. Hic igitur eiusmodi tantum bases, quibus corpora super plano immobili incedant, contemplari convenit, quae curvatura continua sint praeditae, ne ulius saltus in motu occurrat. Nimias autem ambages, quae in calculos inextricabiles perducerent, evitaturi, duo tantum corporum genera, cylindrica scilicet ac sphaerica, potissimum evolvamus, quorum nimirum figura externa, qua plano applicantur, sit vel cylindrica vel sphaerica, quomodocunque materia intrinsecus fuerit distributa, cuius ratio ex centro inertiae et axibus prineipalibus determinatur. Hinc ad genus cylindricum referuntur ea pendula, quae non ab axe lineari, uti supra assumsimus, sunt suspensa, sed axiculis cylindricis utrinque plano horizontali incumbant. Deinde etiam huc pertinet motus vacillatorius motui cunarum reciproco similis, cuiusmodi corpora, quatenus super plano incumbunt, tanquam cylindrica spectari possunt. Deinde etiam, quomodo huiusmodi corpora super plano inclinato descendant, operae pretium erit scrutari. Ad corpora porro sphaerica refero non solum ea, quorum tota figura est globosa, sed etiam quae inferius, ubi planum attingunt, in hemisphaerium sunt efformata, veluti sunt turbines, quorum axes infra non in cuspidem sed quasi in hemisphaerium desinunt; ubi quidem centrum inertiae magis est elevatum, quam centrum hemisphaerii, quando autem profundius est situm, aliud motus genus oriri potest, quo corpus quasi titubando oscillationes peragit, in quo motu mira motus gyratorii perturbatio locum habere potest.