Chapter 16

Concerning the Rotational or Turning Motions of Celestial Bodies

PROBLEM 91

815. If the individual elements of a rigid body is acted upon by forces towards some point *F*, which shall be as the masses and divided by the square of the distances of these from that point, to determine the moments of the forces about the principal axes of the body.

SOLUTION

Let *IA*, *IB*, *IC* be the principal axes of the body (Fig. 104), and the moments of inertia about these axes shall be *Maa*, *Mbb*, *Mcc*. Moreover the distance of the point *F* or of the centre of forces from the centre of inertia *I* of the body is put as *IF* = *s*, which thus shall be inclined to the three principal axes, in order that the angles shall be

$$AIF = \zeta, \quad BIF = \eta \quad \text{and} \quad CIF = \vartheta;$$

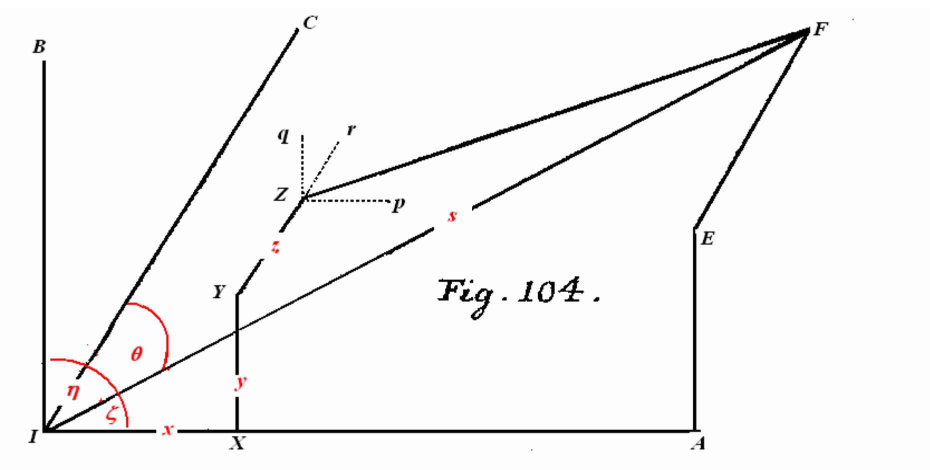


Fig. 104.

on sending a perpendicular *FE* from *F* to the plane *AIB* and the normal *EA* from *E* to the axis *IA*, then

$$IA = s\cos\zeta \ , \quad AE = s\cos\eta \quad \text{and} \quad EF = s\cos\vartheta.$$

Again the force pushing the individual bodies to the point *F* shall be so great, that at the distance equal to *e* it is equal to gravity, but at other distances it is diminishes following the square of their distances. Now some element of the body *dM* is considered at *Z*, for which the coordinates agreeing with the principal axes shall be

$$IX = x, \quad XY = y, \quad YZ = z$$

and the force, by which this element *dM* is urged towards *F* is then equal to $\frac{ee}{ZF^2}dM$.
Now this force is resolved along the directions of the axes *Zp*, *Zq*, *Zr*, and then

$$\text{force along } Zp = \frac{ee(s\cos\zeta - x)dM}{ZF^3},$$
$$\text{force along } Zq = \frac{ee(s\cos\eta - y)dM}{ZF^3},$$
$$\text{force along } Zr = \frac{ee(s\cos\vartheta - z)dM}{ZF^3}$$

and hence the moments of these forces about the principal axes become :

$$\text{moment about the axis } IA \text{ in the sense } BC = \frac{ees(y\cos\vartheta - z\cos\eta)dM}{ZF^3},$$
$$\text{moment about the axis } IB \text{ in the sense } CA = \frac{ees(z\cos\zeta - x\cos\vartheta)dM}{ZF^3},$$
$$\text{moment about the axis } IC \text{ iin the sense } AB = \frac{ees(x\cos\eta - y\cos\zeta)dM}{ZF^3}.$$

Therefore from these moments gathered together through the whole body we obtain the moments, which we indicate by the letters *P*, *Q*, *R*, thus in order that as the quantity *s* remains constant :

$$P = ees\int\frac{(y\cos\vartheta - z\cos\eta)dM}{ZF^3},$$
$$Q = ees\int\frac{(z\cos\zeta - x\cos\vartheta)dM}{ZF^3},$$
$$R = ees\int\frac{(x\cos\eta - y\cos\zeta)dM}{ZF^3}.$$

But since

$$ZF = \sqrt{((s\cos\zeta - x)^2 + (s\cos\eta - y)^2 + (s\cos\vartheta - z)^2)}$$

or, on account of $\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$,

$$ZF = \sqrt{(ss - 2sx\cos\zeta - 2sy\cos\eta - 2sz\cos\vartheta + xx + yy + zz)}.$$

But since in celestial bodies the distance *1F* = *s* is always much greater in comparison with the body itself or with the quantities *x*, *y*, *z*, it is completely satisfactory for us to set up

$$\frac{1}{ZF^3} = \frac{1}{s^3} + \frac{3x\cos\zeta + 3y\cos\eta + 3z\cos\vartheta}{s^4}.$$

Now because *I* is the centre of mass, and *IA*, *IB*, *IC* are the principal axes of this, we have

$$\int xdM = 0, \quad \int ydM = 0, \quad \int zdM = 0,$$

and

$$\int xydM = 0, \quad \int xzdM = 0, \quad \int yzdM = 0,$$

from these put in place there are produced :

$$P = ees\int \frac{(3yy\cos\eta\cos\vartheta - 3zz\cos\eta\cos\nu)\ dM}{s^4}$$

$$= \frac{3ee\cos\eta\cos\vartheta}{s^3}\int(yy - zz)dM$$

$$Q = \frac{3ee\cos\zeta\cos\vartheta}{s^3}\int(zz - xx)dM$$

$$R = \frac{3ee\cos\zeta\cos\eta}{s^3}\int(xx - yy)dM.$$

Now from the given moments of inertia there is

$$\int xxdM = \tfrac{1}{2}M(bb + cc - aa),$$

$$\int yydM = \tfrac{1}{2}M(aa + ce \sim bb)$$

and

$$\int zzdM = \tfrac{1}{2}M(aa + bb - cc),$$

on account of which,

$$P = \frac{3Mee(cc - bb)\ \cos\eta\cos\vartheta}{s^3}$$

$$Q = \frac{3Mee(aa - cc)\ \cos\zeta\cos\vartheta}{s^3}$$

$$R = \frac{3Mee(bb - aa)\ \cos\zeta\cos\eta}{s^3}.$$

COROLLARY 1

816. Therefore these moments of the forces have not been defined with geometrical rigor, but they only prevail, when the distance of the point of the attraction exceeds the magnitude of the body being attracted. And thus it readily comes about, that these can be expressed so neatly by the moments of inertia.

COROLLARY 2

817. If the attracting body has all the moments of inertia equal to each other, also these moments of the forces vanish ; hence so far only the rotational motion of the celestial bodies is affected by forces of this kind, in as much as those are not spherical, or perhaps they are provided with equal moments of inertia.

SCHOLlUM 1

818. If we want to define how much these forces are involved in the progressive motion, we must apply the elementary forces to the centre of inertia itself ; because if for some axis we gather these into one sum, then we have the whole body force acting towards progressive motion. But in order that we can rise to second order in the variables *x, y, z*, we must express the value *ZF* more accurately, so that it becomes :

$$\frac{1}{ZF^3} = \frac{1}{s^3} + \frac{3\left(x\cos\zeta + y\cos\eta + z\cos\vartheta\right)}{s^4}$$
$$+\frac{15\left(x\cos\zeta + y\cos\eta + z\cos\vartheta\right)^2}{2s^5} - \frac{3\left(xx + yy + zz\right)}{2s^5}.$$

This formula multiplied by $(s\cos\zeta - x)dM$ and integrated according to the above precepts, gives

$$\int\frac{(s\cos\zeta - x)dM}{ZF^3} = \frac{M\cos\zeta}{ss} + \frac{15\cos\zeta}{2s^4}\int dM(xx\cos^2\zeta + yy\cos^2\eta + zz\cos^2\vartheta)$$
$$-\frac{3\cos\zeta}{2s^4}\int dM(xx + yy + zz) - \frac{3\cos\zeta}{s^4}\int xxdM\ ,$$

which can be changed into this form :

$$\frac{M\cos\zeta}{ss} + \frac{3M\cos\zeta}{2s^4}\left(aa\left(3 - 5\cos^2\zeta\right) + bb\left(1 - 5\cos^2\eta\right) + cc\left(1 - 5\cos^2\vartheta\right)\right).$$

Whereby we obtain the three following forces :

I. Along $IA = \frac{Mee\cos\zeta}{ss} + \frac{3Mee\cos\zeta}{2s^4}\left(aa\left(3 - 5\cos^2\zeta\right) + bb\left(1 - 5\cos^2\eta\right) + cc\left(1 - 5\cos^2\vartheta\right)\right)$

1I. Along $IB = \frac{Mee\cos\eta}{ss} + \frac{3Mee\cos\eta}{2s^4}\left(bb\left(3 - 5\cos^2\eta\right) + cc\left(1 - 5\cos^2\vartheta\right) + aa\left(1 - 5\cos^2\zeta\right)\right)$

III. Along $IC = \frac{Mee\cos\vartheta}{ss} + \frac{3Mee\cos\vartheta}{2s^4}\left(cc\left(3 - 5\cos^2\vartheta\right) + aa\left(1 - 5\cos^2\zeta\right) + bb\left(1 - 5\cos^2\eta\right)\right).$

But these three forces can be replaced in the first place by a single force acting along the line *1F*, which is :

$$\frac{Mee}{ss} + \frac{3Mee}{2s^4}\left(aa\left(1 - 5\cos^2\zeta\right) + bb\left(1 - 5\cos^2\eta\right) + cc\left(1 - 5\cos^2\vartheta\right)\right),$$

to this above force these three are to be added

1. Along $IA = \frac{3Maaee\cos\zeta}{s^4}$
2. Along $IB = \frac{3Mbbee\cos\eta}{s^4}$
3. Along $IC = \frac{3Mccee\cos\vartheta}{s^4}$.

From which it is apparent, if the three principal moments of inertia should be equal to each other, then all the forces are reduced to the one force $\frac{Mee}{ss}$ acting along *1F*, which is considered in theoretical astronomy, now in the remaining cases that centripetal force will not be purely inversely proportional with the square of the distances, but they agree with that above with a small term inversely proportional to the fourth power of the distances, but which in addition depend on the position of the body with respect to the forces *F* ; which is of assistance in attending to the aberration in astronomical calculations, especially if bodies deviate notably from the spherical shape.

SCHOLlUM 2

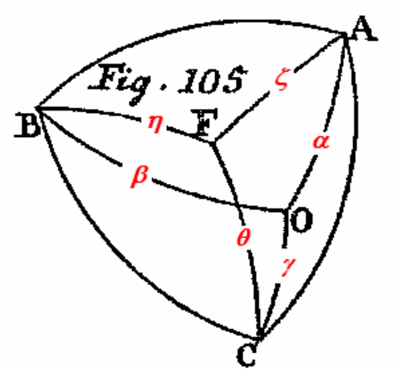
819. I have assumed here that the individual elements of the body are attracted towards a single point *F*, since in the hypothesis of attraction the individual elements of the body are still also being attracted. Now if the attracting body were a sphere, it is clear that between themselves the bodies are attracted in the same way, as if the whole mass were to be united at the centre of the sphere ; so that thus our problem also includes this case between the bodies. But if the body were not spherical, indeed in a short time there is a change in both the reciprocal square ratio, as well as the direction of the force, which no longer is directed towards a certain point ; now, it is to be agreed that these irregularities are to vanish completely at great distances, especially since celestial bodies may depart a little from spherical shapes. But here since I only consider rotational motion, and by disregarding progressive motion, I consider the centre of inertia to be at rest, and in the first place, if the body itself is at rest, I will investigate what rotational motion it shall undertake about some axis.

PROBLEM 92

820. If the body is at rest, and acted on by from the centre of forces *F* in the manner defined before, to define the axis about which in the first instant it begins to rotate, and the angular speed thus arising.

SOLUTION

Hence we consider the body at rest, or rather with the progressive motion of the body not borne in the mind; therefore with the centre of inertia of this in the centre of a sphere there are put in place the poles of the principal axes *A*, *B*, C (Fig. 105), and with respect to which, at this stage, the moments of inertia are *Maa*, *Mbb*, *Mcc*. Now a right line drawn from the centre of inertia to the centre of force cuts the surface at the point *F*, in order that the arcs are $AF = \zeta, BF = \eta, CF = \vartheta$; moreover the distance of the centre of forces is equal to *s* and the attracting force of this is of such a size that at a distance equal to *e* it is equal to gravity. Hence the moments of the forces *P*, *Q*, *R* about the principal axes *IA*, *IB*, *IC* are :



$$P = \frac{3Mee(cc-bb)\cos\eta\cos\vartheta}{s^3}$$

$$Q = \frac{3Mee(aa-cc)\cos\zeta\cos\vartheta}{s^3}$$

and

$$R = \frac{3Mee(bb-aa)\cos\zeta\cos\eta}{s^3}.$$

Whereby from § 806 the body begins to rotate about an axis of this kind *IO*, so that on putting the arcs

$$AO = \alpha, \quad BO = \beta, \quad CO = \gamma$$

there soon becomes

$$\cos\alpha = \frac{P}{aa} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}$$

$$\cos\beta = \frac{Q}{bb} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}$$

$$\cos\gamma = \frac{R}{cc} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)},$$

moreover in the increment of time *dt* there is acquired the angular speed arising :

$$d\gamma' = \frac{2gdt}{M}\sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)},$$

which is directed in the sense *ABC*. And from these the distance of the pole of rotation *O* from the point *F* thus can be found from the expression, in order that

$$\cos OF = \left(\frac{P\cos\zeta}{aa} + \frac{Q\cos\eta}{bb} + \frac{R\cos\vartheta}{cc}\right) : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}.$$

COROLLARY 1

821. This case is worth remembering, in which the centre of forces *F* falls between two principal poles; indeed if the point *F* falls on the arc *AB* and on account of $\cos\vartheta = 0$ and $\cos^2\zeta + \cos^2\eta = 1$, then

$$P = 0,\ Q = 0$$

and

$$R = \frac{3Mee(bb-aa)\sin\zeta\cos\zeta}{s^3}$$

from which also there arises $\cos\alpha = 0$, $\cos\beta = 0$ and $\cos\gamma = 1$, thus so that the pole of rotation *O* falls on the principal pole *C*.

COROLLARY 2

822. In the same case, when the centre of the forces is in the plane *AIB* and the body begins to rotate about the axis *IC*, in the first increment of time *dt* the angular speed arising is acquired

$$d\gamma' = \frac{6gee(bb\text{-}aa)\,dt\sin\zeta\cos\zeta}{ccs^3}$$

in the sense *AB*, or

$$d\gamma' = \frac{3gee(bb\text{-}aa)\,dt\sin 2\zeta}{ccs^3}.$$

COROLLARY 3

823. But hence in the same case if the body now has a rotary motion about the axis *IC* with a speed equal to γ' in the sense *AB*, this is accelerating towards the centre of forces *F* on account of the force acting, so that it becomes

$$d\gamma' = \frac{3gee(bb\text{-}aa)\,dt\sin 2\zeta}{ccs^3}.$$

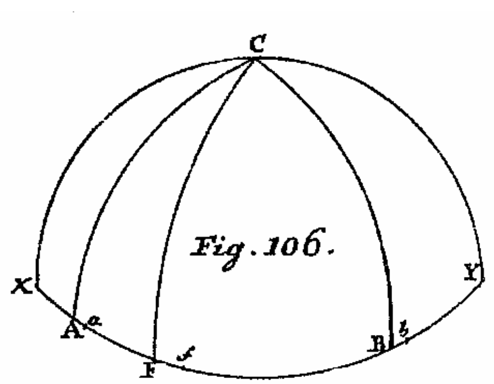
SCHOLIUM

824. Hence it is therefore evident, if the centre of forces *F* thus is carried around the body, so that it falls on great circle *AB* contained by the two principal axes *IA* and *IB* and the body begins to rotate about the remaining principal axis *IC*, so that it will always be rotating about the same axis *IC*, and only the angular speed γ' is going to be increased a little. This case is entirely worthy of note, which can be set out with great enthusiasm, because the motion of the moon's librations, where nearly always the same face is turned towards the earth, is seen to be included. Which investigation is rendered simpler and clearer, if in the first place we assume the centre of the forces to remain in uniform motion about the centre of inertia of the body, in the same plane and always to be carried around at the same distance.

PROBLEM 93

825. If a body is rotating about its own principal axis *IC*, moreover the centre of forces *F* is carried around in the plane uniformly normal to that axis, with the distance of this from the centre of inertia remaining the same, to define the rotational motion of this body.

SOLUTION



Hence because the axis of rotation *IC* remains constant and as if fixed with respect to the sky (Fig. 106), let *XCY* be a celestial hemisphere and *XY* the great circle described with the pole C, in which the centre of forces *F* advances uniformly, and on this circle also there are present the two remaining poles of the principal axes *A* and *B* of the body. The angular speed of the centre of forces *F* is put equal to δ, since initially is was at *X*, then in the elapsed time equal to *t* the arc described by necessity is $XF = \delta t$ [note that here δ is an angular velocity]. But at the same instant of time *t* the other principal axes is found at *A*, and on putting the arc $XA = \lambda$, if the angular speed of the body about the axis *IO* is equal to γ' in the sense *AB*, then $d\lambda = \gamma' dt$. So now on account of $AF = \delta t - \lambda$, which above was ζ, this for us is $\delta t - \lambda$; but on retaining the remaining quantities *aa*, *bb*, *cc* and likewise *ee* and *s*, which are constants, as above, we have this equation :

$$d\gamma' = \frac{3gee(bb - aa)}{ccs^3} dt \sin 2\zeta .$$

But we introduce the angle $ACF = \zeta$, and because $\zeta = \delta t - \lambda$ we obtain

$$\lambda = \delta t - \lambda$$

and

$$d\lambda = \delta dt - d\zeta = \gamma' dt,$$

from which there becomes

$$\gamma' = \delta - \frac{d\zeta}{dt}.$$

On account of which on taking the element *dt* constant, this equation is produced to be resolved :

$$dd\zeta + \frac{3gee(bb - aa)}{ccs^3} dt^2 \sin 2\zeta = 0.$$

For brevity's sake we put

$$\frac{3gee(bb - aa)}{ccs^3} = N,$$

and by multiplying by $2d\zeta$ the equation becomes :

$$2d\zeta dd\zeta + 2Ndt^2 d\zeta \sin 2\zeta = 0,$$

the integral of this is :

$$d\zeta^2 - Ndt^2 \cos 2\zeta = Cdt^2,$$

from which there is deduced

$$dt = \frac{d\zeta}{\sqrt{(C + N\cos 2\zeta)}}$$

and

$$\gamma' = \delta - \sqrt{(C + N\cos 2\zeta)}.$$

It is necessary to define the arc $AF = \zeta$ from that equation at some time *t*, which if it should be constant, then the body remains for ever turning the same face towards the centre of forces *F*. Hence as far as *N* is not equal to 0 and the angle ζ in not liable to variation, the angular speed γ' is variable ; to which phenomena it is fitting that two cases are to be examined, provided either $bb > aa$ or $bb < aa$, each of which for the constant ratio *C* can be established in an infinite number of ways.

CASE I, in which $bb > aa$

826. Therefore let

$$\frac{3gee(bb - aa)}{ccs^3} = n$$

be a positive number, and while the centre of forces *F* is progressing along the circle *XFY* with the speed δ and at the time *t* the inclining arc *FA* in the preceding is called equal to ζ, then

$$dt = \frac{d\zeta}{\sqrt{(C + n\cos 2\zeta)}},$$

where I note the following in an account of the constant *C* :

1st. If $C = -n$ (for it cannot have a more negative value), then

$$dt = \frac{d\zeta}{\sqrt{n(-1 + \cos 2\zeta)}},$$

and thus by necessity the angle ζ is equal to zero ; clearly the point *A* always coincides with *F*, since it rotates uniformly with that about the axis *IA*.

2nd. If $C = 0$, then the angle ζ is less than a half a right–angle, either positive or negative and wanders between the limits +45° et – 45°. The point *A* hence never recedes beyond 45° from the point *F*, but only in the previous manner is found after that point, which motion must be deduced from the equation

$$dt = \frac{d\zeta}{\sqrt{n \ \cos 2\zeta}}.$$

3rd. If $C = n$, and the equation

$$dt = \frac{d\zeta}{\sqrt{n(1+ \ \cos 2\zeta)}}$$

is changed into this

$$dt = \frac{d\zeta}{\cos\zeta\sqrt{2n}},$$

which integrated gives

$$t = \frac{1}{\sqrt{2n}} \ l \ \text{tang}\left(45° \ + \tfrac{1}{2}\zeta\right),$$

if by taking $t = 0$ there should be $\zeta = 0$, from which it is apparent at last in an infinite lapse of time to become $\zeta = 90°$.

4th. If $C > n$, the point *A* moves away from *F* in a finite time to 90° and hence again it progresses to the point opposite to *F* and by circling around to the other part it returns again to *F*. For if $C = mmn$, with the number mm being a number greater than one, on account of

$$dt = \frac{d\zeta}{\sqrt{n(mm+ \ \cos 2\zeta)}}$$

then approximately,

$$dt = \frac{d\zeta}{\sqrt{n}}\left(\frac{1}{m} - \frac{\cos 2\zeta}{2m^3}\right)$$

and on integrating

$$t\sqrt{n} = \frac{\zeta}{m} - \frac{\sin 2\zeta}{4m^3},$$

from which it is apparent that the angle ζ moves successively through all the values.

5th. Up to this point we have put $\gamma' < \delta$, thus so that the motion of the point *F* is faster, than the rotation about the axis *IC;* if the opposite comes about, only the sign of the formula $\sqrt{(C + n\cos 2\zeta)}$ needs to be changed.

CASE 11, in which $bb < aa$.

827. Therefore let

$$\frac{3gee(aa-bb)}{ccs^3} = n,$$

then

$$dt = \frac{d\zeta}{\sqrt{(C - n\cos 2\zeta)}},$$

and

$$\gamma' = \delta - \sqrt{\left(C - n\cos 2\zeta\right)};$$

in which formulas, if there is put $\zeta = 90° + \varphi$, so that now φ denotes the distance of the pole *B* from the centre of forces *F* taken according to the foregoing, then the preceding formulas result, which therefore show the same phenomena.

COROLLARY 1

828. Therefore if we put the centre of forces *F* initially with the pole *A* agreeing with the condition arising, *bb* > *aa*, then the body always turns the same face to the point *F*, if the angular speed is equal to the angular speed of the centre of forces δ .

COROLLARY 2

829. But if initially, when *F* agrees with *A*, the angular speed of the body γ' is some amount greater or less than δ , in order that the difference does not exceed

$$\sqrt{2n} = \sqrt{\frac{6gee(bb-aa)}{ccs^3}},$$

the pole *A* moves away from *F* on both sides but not by more than a certain interval and appears as if it performs oscillations about the point *F* ; in which certainly the similarity with the librational motion of the moon is discerned.

COROLLARY 3

830. Hence in librational motion of this kind, the angular speed of the body is a maximum or a minimum while the point *A* is together with *F* itself, from that either as a consequence or in the foregoing it will be moving apart ; from which it follows that the smallest speed is greater than $\delta - \sqrt{2n}$. Therefore it can come about that such a motion arises, clearly while initially the body has no rotational motion.

SCHOLIUM 1

831. Hence no doubt remains, why the libratory motion of the moon should not arise from this reason, and so far it seems probable that the moon clearly has a place in that case in which initially no rotational motion of the moon has been impressed ; but then the principal axis of the moon *IA*, about which the moment of inertia *Maa* is a minimum, is to be directed towards the earth. Therefore since we know that the separation of the pole *A* from *F* to be a minimum, we are able to define the time of these oscillations ; for since the arc *AF* = ζ is certainly small, then $\cos 2\zeta = 1 - 2\zeta\zeta$ and hence

$$dt = \frac{d\zeta}{\sqrt{(C+n-2n\zeta\zeta)}},$$

from which on integration

$$t\sqrt{2n} = \text{A } \sin\frac{\zeta\sqrt{2n}}{\sqrt{(C+n)}}.$$

Whereby since the maximum digression becomes

$$\zeta = \sqrt{\frac{C+n}{2n}},$$

the time in which the point *A* moves apart maximally from *F* is equal to $\frac{\pi}{2\sqrt{2n}}$ seconds, the double of which $\frac{\pi}{\sqrt{2n}}$ gives the time, in which the pole *A* moving apart from *F* returns to the same again. But then the minimum angular speed, clearly when the pole *A* from *F* in the preceding digression, is then equal to $\delta - \sqrt{(C+n)}$ *;* which in order that is should vanish, the constant *C* must be equal to $\delta\delta - n$, from which it is necessary that the maximum digression in this case should be equal to $\frac{\delta}{\sqrt{2n}}$. Now we consider also the time of one revolution about the centre of forces *F,* which is equal to $\frac{2\pi}{\delta}$ seconds, half of this is equal to the time of one oscillation of the pole which is equal to $\frac{\pi}{\sqrt{2n}}$, there arises $\delta = \sqrt{2n}$ or $n = \frac{\delta\delta}{2}$ and thus $C = \frac{\delta\delta}{2} = n$; neither hence have we assumed that a greater digression should be the smallest.

SCHOLIUM 2

832. Hence therefore we will conclude that the motion of the moon's librations are not possible to be explained thus, clearly as we have set up [the problem] initially in which no rotational motion of the moon was to be impressed, but since rather it is exceedingly plausible that the moon, if that is carried uniformly around the earth in a circular orbit, which is the hypothesis of our problem, and clearly always has the same face directed towards us, and no nutation in that is going to be observed ; then we must establish in the same hypothesis : initially that such a motion of the moon be impressed, in order that the angular speed be precisely equal to δ, clearly equal to the speed of the earth around the moon, and likewise the axis of this *IA* to be pointed towards the earth. But this seems plausible enough ; since indeed the moment of inertia about the axis *1A* is a minimum, and thus of the moon, if the body of this is taken as an oblong spheroid, the maximum axis can be the cause, which initially this axis points towards the earth, and perhaps it is to be attributed to the same cause, because, while the moon takes the first motion, here the direction of the axis itself is conserved towards the earth; since it is the same, and if the angular speed at first should be equal to the angular speed of the earth δ. Therefore since the moon, if it describes a uniform motion around the earth, constantly turns the same face to us to be observed, the librations of this for the observed motion of the moon are irregular, for which the manner of the increasing or decreasing speed must be presented. Whereby also we have resolved the preceding problem according to this hypothesis, as we neither assume that the point *F* is carried around uniformly nor that it always holds the same distance from the centre of inertia of the body.

PROBLEM 94

833. If a body is rotating about its principal axis *1C*, moreover the centre of forces F is carried around in a plane normal to that neither uniformly nor at the same distance, if initially the axis *1A* is along the direction of the centre of forces F , then to define the motion of the librations of the body.

SOLUTION

The irregular motion of the point F of the body can be expressed thus, so that in the time t let the arc described be

$$XF = \delta t + \alpha \sin At ;$$

and let the variable distance be

$$\frac{1}{s^3} = \frac{1}{f^3}(1 + \beta \cos At) .$$

Whereby if now the angular speed is equal to γ' , on putting the arc $XA = \lambda$, then $d\lambda = \gamma' dt$ and on calling the arc $AF = \zeta$ we have

$$d\gamma' = \frac{3gee(bb - aa)dt \sin 2\zeta}{ccs^3} .$$

Therefore since

$$\lambda = \delta t + \alpha \sin At - \zeta ,$$

then

$$\gamma' = \delta + A\alpha \cos At - \frac{d\zeta}{dt}$$

and thus on putting

$$\frac{3gee(bb - aa)}{ccf^3} = n$$

then

$$-AA\alpha dt \sin At - \frac{dd\zeta}{dt} = ndt(1 + \beta \cos At) \sin 2\zeta .$$

But if now we assume that the arc ζ always remains very small, then we have this equation

$$\frac{dd\zeta}{dt^2} + AA\alpha \sin At + 2n\zeta(1 + \beta \cos At) = 0 ,$$

to this there is an approximate solution on putting $\zeta = m \sin At$, from which

$$-AAm \sin At + AA\alpha \sin At + 2mn \sin At = 0$$

on account of the term $\beta \cos At$ being very small besides 1. Hence we obtain therefore

$$m = \frac{AA\alpha}{AA-2n}$$

and thus

$$\zeta = \frac{AA}{AA-2n} sin\, \mathrm{A}t,$$

from which there arises

$$\gamma' = \delta + A\alpha\, cos\, At - \frac{A^3\, cos\, At}{AA-2n} = \delta - \frac{2A\alpha n}{AA-2n} cos\, At.$$

Here since there is

$$XF = \delta t + \alpha\, sin\, At \;,$$

the first part δt is called the mean location of the point *F* and the other part $\alpha\, sin\, At$ of this equation the prostapheresis, from which it is apparent that the digression *FA* is to be proportional to this prostapheresis, and greater than that on account of the positive number *n*. Thus with the vanishing of the prostaphaeresis, or to what extent the true position agrees with the mean position, so the body turns it face towards the centre of forces *F*, indeed with the smaller inequalities ignored, which ratio might produce the quantity β . Now this needs to be pursued further and determined more accurately if it is to agree with the greater known part of astronomy.

COROLLARY 1

834. Hence if the unequal motion of the point *F* is thus expressed, so that in the time *t* it completes the arc

$$XF = \delta t + \alpha\, sin\, At,$$

in the same time the arc of the libration is

$$FA = \zeta = \frac{AA\alpha}{AA-2n} sin\, At\, ,$$

with

$$n = \frac{3gee(bb-aa)}{ccf^3},$$

being present,
where f denotes the mean distance of the centre of forces *F*.

COROLLARY 2

835. If we wish to define this arc of the libration ζ more accurately, then the variable of the distance *FI* = *s* is also present in the calculation, thus in order that, if it were

$$\frac{1}{s^3} = \frac{1}{f^3}(1 + \beta\, cos\, At),$$

then it is found that

$$\zeta = \frac{AA\alpha}{AA-2n} sin\, At + \frac{n\alpha\beta}{4(AA-2n)} sin\, 2At.$$

[On assuming that $n << A^2$, as noted in the *O. O* edition.]

COROLLARY 3

836. In a like manner if more generally the arc completed in the time *t* were

$$XF = C + \delta t + \alpha \sin(At + \mathfrak{A}) + \alpha \sin(A't + 2\mathfrak{A}') + \text{ etc.}$$

and

$$\frac{1}{s^3} = \frac{1}{f^3}\left(1 + \beta \cos\left(At + \mathfrak{A}\right) + \beta' \cos\left(A't + \mathfrak{A}'\right) + \text{etc.}\right),$$

then the arc of the libration is found approximately :

$$. \; FA = \zeta = \frac{AA\alpha}{AA-2n} \sin\left(At + \mathfrak{A}\right) + \frac{A'A'\alpha'}{A'A'-2n} \sin\left(A't + \mathfrak{A}'\right) + \text{etc.}$$

SCHOLIUM 1

837. Now this likewise is the case, that the number

$$n = \frac{3gee(bb-aa)}{ccf^3}$$

can be either positive or negative, nor does the above required condition that *bb* > *aa* have a greater place, in order that the arc ζ vanishes. For in case II (§ 827) if there is put
C = *n,* then there becomes

$$dt\sqrt{2n} = \frac{d\zeta}{\sin\zeta}$$

and

$$t\sqrt{2n} = l \text{ tang} \tfrac{1}{2}\zeta - \text{Const.},$$

from which if initially *t* = 0, then there is ζ = 0, and the constant to be added must be infinite, [note of course that *l* means here the natural log function] and therefore *A* does not move away *F* unless an infinite time has elapsed. Whereby while the point *F* is carried uniformly in a circle, whichever principal axis initially is directed towards the point *F*, when it begins to rotate with an equal speed to *F*, each remains fixed to the other. And if hence the point *F* extends or decreases its motion, the pole *A* digresses from that according to the formulas found. Also it is apparent, if *n* = 0 or *bb* = *aa,* in which case the body is rotating uniformly around the pole *C*, that the digressions ζ of the difference between the mean position and the true value of *F* always remain equal in the future. Moreover if the number *n* is positive or *bb* > *aa,* the digressions are greater than this difference, but if *bb* < *aa* on the other hand they become less. Moreover the number *A* defining an inequality of the motion, from the time in which *sin At* reverts to the same values, it is possible to deduce, because if that comes about after the time equal to Θ min. sec., that

$$\sin A\Theta = \sin 2\pi$$

and thus

$$A = \frac{2\pi}{\Theta}.$$

SCHOLIUM 2

838. Hence it is apparent that the motion of the moons librations, in which the same face is not always turned towards the earth, mainly are due to defects in the uniformity of the motion, by which the earth is carried around the moon, or what is the same, the moon appears to be carried around the earth, and must not be attributed much to this inequality of the principal moments deduced for the moon, because that only affects the coefficients of the terms. Clearly a libration can be present, even if the moon were a spherical body, or its principal moments were equal. Now there is no apparent reason, why precisely so much initial rotational motion should be impressed on the moon, as great as our formulas show; but if the moon should be a spheroidal body either oblong or compressed, the reason can be understood in a certain way, and on account of that the motion begins with a certain principal axis being more notable than the others with respect to the earth. But whichever it shall be, an oblong or compressed spheroid, from the magnitude of the libration it is possible to judge, that if it exceeds the difference between the true position of the moon and the average, this indicates that $bb > aa$, or it is pleased to show the axis of the moon with the smallest moment turned towards the earth. Now here is not the place for anything to be defined, since the moon is also acted on by the sun and thus the librations may be disturbed; now in addition too, as the moon is not moving in the same plane around the earth, thus also in turn the motion of the centre of forces F may not be completed in the same plane around the moon, by which this enquiry becomes more complicated, as in the general treatment the position [of F] cannot be found. Moreover this always indicates a great mystery, because initially the moon accepts only a certain amount of rotational motion, as much as is here postulated for the cause of the librations ; for if it should take more or less, with the passage of time it must still be turned to present the face opposite us. Yet meanwhile the degree of the speed of this prescribed phenomenon is not so exactly set out, because although it should be a little larger or smaller, the librations must be touched on, on account of the previous problem; from which another deeper mystery is illustrated. But from such freedom nothing can be admitted , unless in the case in which $bb > aa$ or $n > 0$; for the more general differential equation

$$\frac{dd\zeta}{dt^2} + AA\alpha \sin At + 2n\zeta(1 + \beta \cos At) = 0$$

with the preceding constant thus can be integrated, in order that

$$\zeta = C \sin t\sqrt{2n} + \frac{AA\alpha}{AA-2n} \sin \mathrm{A}t,$$

from which the angular speed becomes

$$\gamma' = \delta - C\sqrt{2n} \cdot \cos t\sqrt{2n} - \frac{2A\alpha n}{AA-2n} \cos \mathrm{A}t,$$

where also for $t\sqrt{2n}$ there can be written $(t+\gamma)\sqrt{2n}$, thus so that C and γ can be assumed as you please. Whereby when initially $t = 0$ then $\zeta = C \sin \gamma\sqrt{2n}$, while the impressed angular speed is equal to

$$\delta - C\sqrt{2n}\cdot cos\,\gamma\sqrt{2n} - \frac{2A\alpha n}{AA-2n},$$

and C is a small enough fraction, the libratory motion follows, so that constantly a certain part of the moon remains hidden from us. But now also the fraction $\frac{AA\alpha}{AA-2n}$ must be very small, in order that the line 2ζ can be written for $sin\,2\zeta$.

SCHOLIUM 3

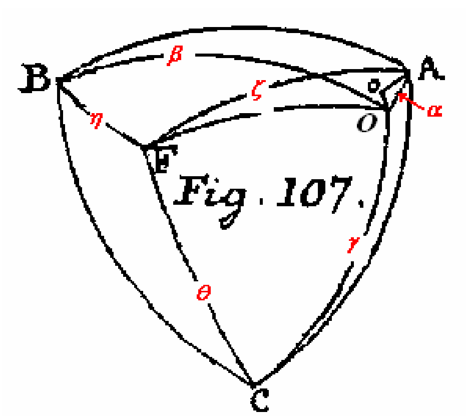
839. Therefore here an explanation of the libratory motion of the moon has been given, as we have stated the body of the moon is an oblong spheroid , the major axis of which, or that, with respect of which the moment of inertia is a minimum, initially is pointed towards the earth, moreover while the rotational motion of the moon is impressed about the axis normal to the plane of the orbit of the earth, just about equal to the angular speed of the mean motion of the moon, in which it is possible to have a certain latitude in its position. Then this is also sufficient, provided the axis of rotation is just about normal to the plane of the earth's orbit and the major axis of such a size is almost directed towards the earth ; in as much as also from these cases the reciprocal change in the disc of the moon must come about, even if that is not allowed to be determined. Whereby with this case left we may progress to the other perturbations of rotational motion arising from centripetal forces, from which it is possible to explain the nutation of the axis of the earth.
[Euler's explanation of the moon's librations does not form part of the modern explanation, although his equations could be the starting point for such an investigation, if such a thing can actually be found; for the moon is almost spherical, and the slow rotation rate on its axis rules out the case of an oblate spheroid. A good place to begin an investigation on the librations or swings is *Wikipedia*, as is often the case these days, where a number of sources are given. But one wonders why the fact that there are 13 lunar months in the year is not discussed in this article : each month on reaching full moon, the moon has turned approximately 30^{o} further on its axis relative to distant stars in order to keep the same face to the earth, and so in a year it turns round 12 + 1 times approximately, similar to the case of the earth with its solar and sidereal days. Thus there is perhaps little wonder that the moon has a longitudinal libration of a few degrees plus or minus as it tries to meet this deadline each month, while rotating at a steady rate on its axis, and orbiting around the earth in a slightly elliptical orbit with a varying speed. The latitudinal oscillation is due to the slight inclination of the moon's axis to the plane of its orbit around the earth, while in addition the diurnal rotation of the earth gives different views of the moon's disc as the observer rotates with the earth. Historically, it was just sheer bad luck that such a difficult problem was faced by Newton and those who followed him, as a test of their new theories of dynamics.]

PROBLEM 95

840. If a body is rotating about an axis that is close to some principal axis, and likewise it is subjected to the action of some central force, to determine the momentary change produced, both in the axis of rotation itself as well as in the angular speed.

SOLUTION

Let *A*, *B*, *C*, be the three principal poles of the body (Fig.107), and the moments of inertia of these about the respective axes are Maa, Mbb, Mcc ; moreover the body now rotates about the pole *O* close to *A* with an angular speed itself equal to γ' in the sense *ABC;* from which with the arcs put in place



$$OA = \alpha, \quad OB = \beta, \quad OC = \gamma$$

the arc α is very small, but β and γ disagree little with the quadrant, thus in order that $\cos\alpha = 1$ and $\cos\beta = \cos\gamma = 0$. Whereby on putting

$$x = \gamma'\cos\alpha, \; y = \gamma'\cos\beta \quad \text{and} \quad z = \gamma'\cos\gamma$$

these letters *y* and *z* can be taken as vanishing, yet not the differentials of these , which shall be

$$dy = -\gamma' d\beta \quad \text{and} \quad dz = -\gamma' d\gamma .$$

Now a right line is drawn crossing from the centre of forces to the centre of inertia at *F* and gives the arcs

$$AF = \zeta, \quad BF = \eta, \quad CF = \vartheta,$$

moreover the distance of the centre of forces is put equal to *s* and the force attracting is of such a size that on putting the distance equal to *e* it is equal to that of gravity. Hence from the action of this force, the magnitudes *x, y, z* thus are changed in the element of time *dt*, in order that

$$dx + \frac{cc-bb}{aa} yzdt = \frac{6gee(cc-bb)\,dt\cos\eta\cos\vartheta}{aas^3}$$

$$dy + \frac{aa-cc}{bb} xzdt = \frac{6gee(aa-cc)\,dt\cos\zeta\cos\vartheta}{bbs^3}$$

$$dz + \frac{bb-aa}{cc} xydt = \frac{6gee(bb-aa)\,dt\cos\zeta\cos\eta}{ccs^3}.$$

Since now on putting $dx = d\gamma'$, on account of *y* and *z* vanishing, then

$$d\gamma' = \frac{6gee(cc-bb)\,dt\cos\eta\cos\vartheta}{aas^3}$$

$$\gamma' d\beta = \frac{6gee(aa-cc)\,dt\cos\zeta\cos\vartheta}{bbs^3}$$

$$\gamma' d\gamma = \frac{6gee(bb-aa)\,dt\cos\zeta\cos\eta}{ccs^3}.$$

Which variation, so that we can examine it more carefully, we look for that arc *FO*, but on account of

$$\sin BAF = \frac{\cos\vartheta}{\sin\zeta}, \quad \cos BAF = \frac{\cos\eta}{\sin\zeta},$$

there becomes

$$\sin FAO = \frac{\cos\gamma\cos\eta - \cos\beta\cos\vartheta}{\sin\alpha\sin\zeta}$$

and

$$\cos FAO = \frac{\cos\beta\cos\eta + \cos\gamma\cos\vartheta}{\sin\alpha\sin\zeta}.$$

and hence

$$\cos FO = \cos\beta\cos\eta \ + \ \cos\gamma\cos\vartheta \ + \ \cos\alpha\cos\zeta;$$

with the differential of this giving

$$(Fo - FO)\sin FO = d\beta\cos\eta \ + d\gamma\cos\vartheta$$

on account of

$$\sin\beta = \sin\gamma = 1 \text{ et } \sin\alpha = 0.$$

Whereby since $FO = FA = \zeta$, there is found

$$(Fo - \ FO)\sin\zeta = \frac{6geedt\cos\zeta\cos\eta\cos\vartheta}{\gamma' s^3}\left(\frac{aa-cc}{bb} + \frac{bb-aa}{cc}\right)$$

or

$$Fo - FO = \frac{6gee(ce-bb)(bb+ce-aa)\,dt\cot\zeta\cos\eta\cos\vartheta}{\gamma' bbccs^3}$$

on account of

$$\text{tang } BAO = \frac{\cos\gamma}{\cos\beta}.$$

But for the position of the point *o* to be found, we have from differentiation :

$$-\frac{OAo}{\cos^2 BAO} = \frac{-d\gamma\cos\beta\sin\gamma + d\beta\cos\gamma\sin\beta}{\cos^2\beta}$$

and thus

$$OAo = \frac{d\gamma\cos\beta\sin\gamma - d\beta\cos\gamma\sin\beta}{\sin^2\alpha}$$

and

$$Oo = \sqrt{\left(d\beta^2 + d\gamma^2\right)}.$$

Then now, since

$$d\alpha = \frac{-d\beta \sin\beta \cos\beta - d\gamma \sin\gamma \cos\gamma}{\sin\alpha \cos\alpha},$$

there arises

$$\text{tang } OoA = \frac{d\beta \sin\beta \cos\gamma - d\gamma \sin\gamma \cos\beta}{d\beta \sin\beta \cos\beta + d\gamma \sin\gamma \cos\gamma}$$

an account of $\cos\alpha = 1$ or

$$\text{tang } OoA = \frac{d\beta \cos\gamma - d\gamma \cos\beta}{d\beta \cos\beta + d\gamma \cos\gamma}.$$

COROLLARY 1

841. If the moments of inertia about the axis *1B* and *IC* are equal or *bb* = *cc,* in the first place let $d\gamma' = 0$, or the angular speed suffers no change, then now we have

$$d\beta = \frac{-6gee(aa-cc)dt \cos\zeta \cos\vartheta}{\gamma' ccs^3}$$

and

$$d\gamma = \frac{6gee(aa-cc)dt \cos\zeta \cos\eta}{\gamma' ccs^3},$$

so that it becomes

$$d\beta \cos\eta + d\gamma \cos\vartheta = 0.$$

COROLLARIUM 2

842. Again in the case *bb* = *cc* if *Fo* – *FO* = 0 or the pole of rotation *o* thus is moved to O, in order that the interval *OO* is normal to the arc *FO* ; that is the interval

$$Oo = \frac{6gee(aa-cc)\, dt \cos\zeta \sin\zeta}{\gamma' ccs^3},$$

but now it is sought, whether is shall point from *O* towards *FA* or the contrary.

COROLLARY 3

843. But since then it is the case that

$$\sin FO : \sin FAO = \sin AO : \sin AFO,$$

then

$$\sin AFO = \frac{\cos\gamma \cos\eta - \cos\beta \cos\vartheta}{\sin\zeta \sin FO}.$$

Now because *FO* does not change, there becomes following the figure, where *O* is taken to approach *AF* :

$$-OFo \cdot \cos AFO = \frac{-d\gamma \cos\eta + d\beta \cos\vartheta}{\sin\zeta \sin FO} = -\frac{6gee(aa-cc)dt \sin\zeta \cos\zeta}{\gamma' ccs^3 \sin FO},$$

and thus

$$OFo = \frac{6gee(aa-cc)dt \sin\zeta \cos\zeta}{\gamma' ccs^3 \sin FO \cos AFO}.$$

Therefore since the angle *AFO* is infinitely small and *cos AFO* = 1 and *FO* = *FA* = ζ , then

$$OFo = \frac{6gee(aa-cc)\,dt\cos\zeta}{\gamma' ccs^3}.$$

Hence if *aa* > *cc,* the point *O* approaches the arc *AF*, or proceeds around *A* in the sense *CB.*

SCHOLIUM

843 [a]. This case, in which *bb* = *cc*, thus so that the body has two equal moments of inertia about the principal axes *I B* et *1C* and just as it rotates about the singular axis *IA* with an angular speed γ' in the sense *BC,* has an especial place in the rotational motion of the earth and thus merits to be considered further. So that this can be made easier, since *AO* = α , the angle *BAO* is put equal to ρ , then

$$90° - \beta = \alpha\cos\rho \quad \text{and} \quad 90° - \gamma = \alpha\sin\rho,$$

from which

$$\beta = 90° - \alpha\cos\rho \quad \text{and} \quad \gamma = 90° - \alpha\sin\rho.$$

But if for the sake of brevity we put hence

$$\frac{3gee(aa-cc)}{\gamma' ccs^3} = N,$$

so that

$$d\beta = -2Ndt\cos\zeta\cos\vartheta$$

and

$$d\gamma = 2Ndt\cos\zeta\cos\eta,$$

then

$$-d\alpha\cos\rho + \alpha d\rho\sin\rho = -2N\,dt\cos\zeta\cos\vartheta$$

and

$$-d\alpha\sin\rho - \alpha d\rho\cos\rho = 2Ndt\cos\zeta\cos\eta\ ;.$$

from which there is deduced

$$d\alpha = 2N\,dt\cos\zeta(\cos\rho\cos\vartheta - \sin\rho\cos\eta)$$

and

$$\alpha d\rho = -2Ndt\cos\zeta(\sin\rho\cos\vartheta + \cos\rho\cos\eta).$$

Now if we give the centre of forces *F* some kind of motion, also as long as we are able to use these formulas, for as long as the arc *AO* = α remains so small, so that the contractions put in place can be used.

PROBLEM 96

844. If a body has two equal principal moments and it is rotating almost about the third singular axis, moreover the centre of forces is carried uniformly in a circle about the centre of inertia, to determine the position and motion of the body at whatever time.

SOLUTION

The centre of forces progresses along the great circle *XFY* (Fig. 108) with an angular speed equal to δ and in an elapsed time equal to *t* it arrives at *F* from *X*, so that *XF* = δt. Therefore on the sphere the fixed circle *XZY* is considered, on which *Z* is the pole of the circle *XFY*, so that the angle *XZF* = δt. Moreover now the singular axis of the body is turning about *A*, and there is placed the angle *XZA* = λ and the arc *ZA* = *p* [measuring longitude and latitude from the zenith *Z*]; then truly it is as if the first meridian of the body is *AB*, being distant from the arc *ZA* by the angle *ZAB* = *q*. Again the body now rotates about the axis *1O*, so that the smallest arc *AO* = α and the angle *BAO* = ρ, with the rotational speed equal to ε, since now we know that to be constant, and the point *A* in the element of time *dt* goes to *a*, so that

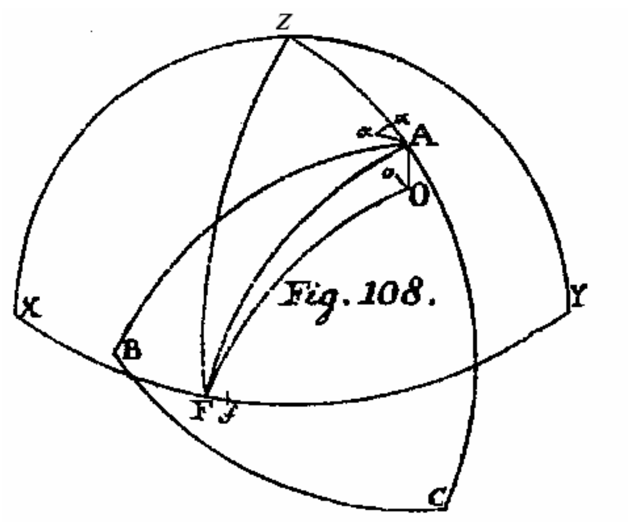


Fig. 108.

$$Aa = \varepsilon dt \sin\alpha = \alpha\varepsilon dt$$

[note as usual that the arc α is replaced by the linear section $\sin\alpha$, which here reverts to α as it is small.]
and the angle *aAO* is right; whereby on account of

$$ZAO = q + \rho$$

there becomes

$$ZAa = q + \rho - 90^\circ ,$$

and thus on sending the perpendicular $a\alpha$ to *ZA* there becomes

$$a\alpha = -\alpha\varepsilon dt \cos(q+\rho)$$

and

$$A\alpha = \alpha e dt \sin(q+\rho),$$

from which there is deduced

$$dp = -\alpha\varepsilon dt \sin(q+\rho)$$

and

$$d\lambda = \frac{\alpha\varepsilon dt \cos(q+\rho)}{\sin p};$$

indeed following, as if the body is rotating about the pole *A*, then $dq = \varepsilon dt$. And then in the triangle *AZF* on account of *ZA* = *p*, *ZF* = 90° and *AZF* = $\lambda - \delta t$ there is found

$$\cos FA = \cos\zeta = \sin p \cos(\lambda - \delta t)$$

and

$$cotZAF = - \cos p \cot(\lambda - \delta t).$$

For the sake of brevity putting the angle $ZAF = \varphi$, in order that

$$\tang\varphi = -\frac{\tang(\lambda - \delta t)}{\cos p},$$

then

$$BAF = \varphi - q$$

and hence

$$\cos BF = \cos(\varphi - q)\sin\zeta = \cos\eta$$

and

$$\cos CF = \sin(\varphi - q)\sin\zeta = \cos\vartheta .$$

Now there is

$$\sin\varphi\sin\zeta = \sin(\lambda - \delta t)$$

and

$$\cos\varphi\sin\zeta = - \cos p\cos(\lambda - \delta t)$$

and hence

$$\cos\eta = - \cos p\cos q\cos(\lambda - \lambda t) + \sin q\sin(\lambda - \delta t)$$

and

$$\cos\vartheta = \cos q\sin(\lambda - \delta t) + \cos p\sin q\cos(\lambda - \delta t).$$

From which there is put

$$\frac{3gee(aa-cc)}{\varepsilon ccs^3} = N,$$

and there is deduced to be

$$d\alpha = 2N\ dt\sin p\ \cos(\lambda - \delta t)(\cos p\sin(q+\rho)\cos(\lambda - \delta t) + \cos(q+\rho)\sin(\lambda - \delta t))$$

and

$$\alpha d\rho = -2Ndt\sin p\cos(\lambda - \delta t)(\sin(q+\rho)\sin(\lambda - \delta t) - \cos p\cos(q+\rho)\cos(\lambda - \delta t)),$$

from which if we join with

$$dq = \varepsilon dt$$

and

$$dp = -\alpha\varepsilon dt\sin(q+\rho),$$

from these four equations it is necessary for the four quantities p, q, α and ρ to be defined. But the first two are transformed into these simpler equations:

$$d\alpha\cos(q+\rho) - \alpha d\rho\sin(q+\rho) = 2Ndt\sin p\sin(A - \lambda t)\cos(A - \lambda t)$$

$$d\alpha\sin(q+\rho) + \alpha d\rho\cos(q+\rho) = 2Ndt\sin p\cos p\cos^2(A - \lambda t).$$

Since then $q = \varepsilon t + C$, we put $q + \rho = \omega$, in order that $\rho = \omega - q$, and on joining to the first four equations we thus have these equations:

$$dp = -\varepsilon\alpha dt \sin\omega$$
$$d\lambda = \frac{\varepsilon\alpha dt \cos\omega}{\sin p}$$
$$d\alpha\cos\omega - \alpha d\omega\sin\omega + \varepsilon\alpha dt\sin\omega = 2Ndt\sin p\sin(\lambda - \delta t)\cos(\lambda - \delta t)$$
$$d\alpha\sin\omega + \alpha d\omega\cos\omega - \varepsilon\alpha dt\cos\omega = 2Ndt\sin p\cos p\cos^2(\lambda - \delta t);$$

and if above we put $\lambda - \delta t = \varphi$, which letter is not to be confused with the previous φ, then these become :

$$dp = -\varepsilon\alpha dt\sin\omega$$
$$d\varphi = -\delta dt + \frac{\varepsilon\alpha dt\cos\omega}{\sin p}$$
$$d\alpha\cos\omega - \alpha d\omega\sin\omega + \varepsilon\alpha dt\sin\omega = 2N\,dt\sin p\sin\varphi\cos\varphi$$
$$d\alpha\sin\omega + \alpha d\omega\cos\omega - \varepsilon\alpha dt\cos\omega = 2N\,dt\sin p\cos p\cos^2\varphi.$$

Again we put $\alpha\cos\omega = x$ and $\alpha\sin\omega = y$, so that we have these equations :

1. $dp = -\varepsilon ydt$
2. $d\lambda = \frac{\varepsilon xdt}{\sin p}$
3. $d\varphi = -\delta dt + \frac{\varepsilon xdt}{\sin p}$
4. $dx + \varepsilon ydt = Ndt\sin p\sin 2\varphi$
5. $dy - \varepsilon xdt = Ndt\sin p\cos p + Ndt\sin p\cos p\cos 2\varphi$,

where since x and y are small quantities, we will be near enough to the truth, if in the two last equations we consider the arc p and the angle λ as constants. Hence we attribute to these as if the mean values, and then approximately $p = n$ and $\lambda = m$ and thus $d\varphi = -\delta dt$, so that we have the equations:

4. $dx - \frac{\varepsilon yd\varphi}{\delta} = -\frac{Nd\varphi}{\delta}\sin n\sin 2\varphi;$
5. $dy + \frac{\varepsilon xd\varphi}{\delta} = -Nd\varphi\sin n\cos n - \frac{Nd\varphi\sin n\cos n\cos 2\varphi}{\delta}$,

which it is evident they can be satisfied on putting

$x = E + F\cos 2\varphi$

$y = G\sin 2\varphi,$

and these coefficients are defined thus, so that there becomes

$$E = -\frac{N\sin n\cos n}{\varepsilon}$$
$$F = -\frac{N\sin n(2\delta + \varepsilon\cos n)}{\varepsilon\varepsilon - 4\delta\delta}$$
$$G = \frac{N\sin n(2\delta\cos n + \varepsilon)}{\varepsilon\varepsilon - 4\delta\delta}.$$

Then since this solution is only to be considered as a particular one, there is put

$$x = E + F\cos 2\varphi + u$$

and

$$y = G\sin 2\varphi + v,$$

and these equations are produced:

$$du - \frac{\varepsilon v d\varphi}{\delta} = 0$$

and

$$dv + \frac{\varepsilon v d\varphi}{\delta} = 0,$$

from which there is elicited

$$u = h\sin\tfrac{\varepsilon}{\delta}(\varphi + \zeta)$$

and

$$v = h\cos\tfrac{\varepsilon}{\delta}(\varphi + \zeta),$$

where h et ζ are arbitrary constants. Concerning which we have :

$$x = \alpha\cos\omega = -\frac{N\sin n\cos n}{\varepsilon} - \frac{N\sin n\,(\varepsilon\cos n + 2\delta)}{\varepsilon\varepsilon - 4\delta\delta}\cos 2\varphi + h\sin\tfrac{\varepsilon}{\delta}(\varphi + \zeta)$$

$$y = \alpha\sin\omega = \frac{N\sin n(\varepsilon - 2\delta\cos n)}{\varepsilon\varepsilon - 4\delta\delta}\sin 2\varphi + h\cos\tfrac{\varepsilon}{\delta}(\varphi + \zeta),,$$

where φ expresses the angle $FZA = \lambda - \delta t$. Then on account of

$$dp = -\varepsilon y dt = \frac{\varepsilon y d\varphi}{\delta}$$

we obtain on integration :

$$p = n - \frac{\varepsilon N\sin n(\varepsilon + 2\delta\cos n)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}\cos 2\varphi + h\sin\tfrac{\varepsilon}{\delta}(\varphi + \zeta) = ZA.$$

And then the equation $d\lambda = \frac{\varepsilon x dt}{\sin p} = -\frac{\varepsilon x d\varphi}{\delta\sin n}$ gives:

$$\lambda = m - Nt\cos n + \frac{\varepsilon N(\varepsilon\cos n + 2\delta)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}\sin 2\varphi + \frac{h}{\sin n}\cos\tfrac{\varepsilon}{\delta}(\varphi + \zeta) = XZA.$$

COROLLARY 1

845. Since by our putting in place $\alpha = \sqrt{(xx + yy)}$, it is apparent that in the succession of time the distance $AO = \alpha$ is unable to be increased beyond a certain limit, which if it is small enough, we can use our hypothesis without risk. Now likewise it is apparent that the interval α clearly at no time vanishes, except perhaps it happens when both $x = 0$ as well as $y = 0$.

COROLLARY 2

846. With the depending inequalities ignored in the angle

$$2\varphi = 2FZA$$

and

$$\frac{\varepsilon}{\delta}(\varphi+\zeta),$$

the pole *A* uniformly goes around the point *Z* in the preceding with an angular speed equal to *N* cos *n*, if indeed

$$N=\frac{3gee(aa-cc)}{eccs^3}$$

should be a positive number, and thus the whole revolution is completed in a time equal to

$$\frac{2\pi}{N\cos n}\text{ sec.},$$

while the centre of forces *F* completes a revolution in a time equal to $\frac{2\pi}{\delta}$; and the body itself in a time equal to $\frac{2\pi}{\varepsilon}$.

COROLLARY 3

847. Now in addition both the interval *ZA* as well as the angle *XZA* appear to be very small inequalities, partially from the angle $2\varphi=2FZA$, partially from the angle

$$\frac{\varepsilon}{\delta}(\varphi+\zeta)=C-\varepsilon t,$$

that is, partially from the motion of the centre of forces, partially from the rotational motion depending on the body itself. Whereby if we put the angle $ZAB=\psi$, then

$$ZA=n-\frac{\varepsilon N\sin n(\varepsilon+2\delta\cos n)}{2\delta(\varepsilon\varepsilon-4\delta\delta)}\cos 2\varphi-h\sin\frac{\varepsilon}{\delta}(\varphi+\zeta)$$

$$XZA=m-Nt\cos n+\frac{\varepsilon N(\varepsilon\cos n+2\delta)}{2\delta(\varepsilon\varepsilon-4\delta\delta)}\sin 2\varphi+\frac{h}{\sin n}\cos\frac{\varepsilon}{\delta}(\varphi+\zeta).$$

SCHOLIUM 1

848. Here we have accepted the body to be rotating in the same sense that the centre of forces *F* is carried around it, just as happens on the earth, which rotates from the west to the east, in which sense also the sun and moon by their own motion are discerned to be moving [on removing the diurnal motion]. As also we have considered the number

$$N=\frac{3gee(aa-cc)}{eccs^3}$$

as positive, or the body thus composed, in order that the moment of inertia of this with respect to the axis, about which it is turning nearby, is the maximum equal to *Maa*, while about the axis taken at the equator it is the minimum equal to *Mcc*, for which property observations in place indicate that the shape of the earth to be a compressed spheroid. Hence in this arrangement, the axis of the earth must be turning around the pole of the ecliptic *Z* in the preceding, just as that also from observation is constant. Now in addition neither is that motion of the axis uniform, nor is the distance of that axis from the pole of the ecliptic *Z* constant, but it is liable to two inequalities, of which one depends on twice the angle $FZA=\varphi$ and the other indeed depends on the rotational motion of the body itself, which latter may become greater or less, as initially the pole of the rotation *O* has been put in place both on account of the position of the pole *A* as well as of the centre of forces *F*. Clearly since ω can denote the angle *ZAO*, if initially or at some given time the quantities $AO = \alpha$, $ZAO = \omega$, $FZA = \varphi$ become known, and $ZAB = \psi$, on taking *AB* for the first meridian of the body, the two constants *h* and ζ are defined from the equations :

$$\alpha \cos\omega + \frac{N \sin n \cos n}{\varepsilon} + \frac{N \sin n(\varepsilon \cos n + 2\delta)}{\varepsilon\varepsilon - 4\delta\delta}\cos 2\varphi + h\sin(\varphi + \zeta) = 0$$

$$\alpha \sin\omega - \frac{N \sin n(\varepsilon + 2\delta \cos n)}{\varepsilon\varepsilon - 4\delta\delta}\sin 2\varphi - h\cos(\varphi + \zeta) = 0$$

Hence unless *h* = 0 is produced, the pole *A* also undergoes diurnal inequalities, thus in order that from the interval and the rotation of this alternately it approaches and recedes from the pole of the ecliptic, and likewise in turn it nutates in the preceding and subsequent motion. On this account the pole *A* describes an unequal circle in the individual revolutions ; since the centre of this is at rest, that it taken rather for the true pole of the earth, in order that these inequalities are not perceived. Then the remaining inequalities depending on the action of the centre of forces do not now appear at this pole, but they do affect the pole of the principal axis.

SCHOLIUM 2

849. But with these diurnal inequalities omitted, by which perhaps the axis of nutation is affected, if it should be the case that *aa* > *cc* and the body is rotating in the same sense as the centre of forces, thus these phenomena are themselves found :
In the first place the distance of the pole *A* from the point *Z*, which is the vertex or the pole of the orbit, that the centre of forces describes, is variable and a certain minima can be taken, if either the angle *FZA* becomes zero or 180°, but a maxima if the angle should be either 90° or 270°, difference between the maximum and the minimum distance present is equal to

$$\frac{\varepsilon N \sin n(\varepsilon + 2\delta \cos n)}{\delta(\varepsilon\varepsilon - 4\delta\delta)}.$$

Secondly the pole *A* in the preceding motion is returning non–uniformly about the point *Z*, which if, according to custom, to be represented by a correction of the prostapheresis of the mean motion, with the mean returning with an angular speed equal to *N* cos *n*, while now a correction or the maximum prostapheresis to be added equal to

$$\frac{\varepsilon N \sin n(\varepsilon + 2\delta \cos n)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}$$

[corrected from the original by C. H. in the *O. O*. edition], if the angle *FZA* is either 45° or 225°, and to be subtracted, if the angle itself becomes either 135° or 315°, where it is noted, this angle $FZA = \varphi$ is to be found if the longitude of the centre of forces *F* is subtracted from the longitude of the pole *A*. Moreover here we see that the speed of the rotational motion ε is much larger compared to the speed of the centre of forces δ; if indeed it should be given by $\varepsilon = 2\delta$, then the attempts found thus go to infinity; now in this case the integration of our equations are to be set up in a singular way, by putting

$$x = E + F\cos 2\varphi + A\varphi \sin 2\varphi;$$

and

$$y = G\,sin\,2\varphi + B\varphi\,cos\,2\varphi,,$$

and there being discovered

$$E = \frac{-N sinn\,cosn}{2\delta}$$

$$A = B = -\frac{N sinn(1 + \;cos\,n\,)}{2\delta}$$

$$F + G = \frac{N\,sin\,n(1 - \;cos\,n\,)}{4\delta}.$$

Since indeed here *x* and *y* continually increase, soon the hypothesis made is transgressed, and there is no longer a place for the whole calculation. Whereby unless $\varepsilon\varepsilon$ is notably different from $4\delta\delta$ our formulas are unable to be put to use.

[Here we may note that prior to discovering the 'Euler equations' used in this derivation, Euler had deduced similar equations from first principles in what is now E171, where he had gone as far as to predict the future yearly changes in the pole's coordinates due to perturbations of both the sun and moon :
Recherces sur la précherches sur la precession des equinoxes et sur la nutation de l'axe de la terr, in *Opera Omnia*, Series II, vol, 30, p. 92-123, published in *Mémoires de l'acad´emie dessciences de Berlin* **5** (1749), 1751, p. 289-325. A translation of this paper by Steven Jones, edited by Robert E. Bradley, is available at MathDL.]

CAPUT XVI

DE MOTU GYRATORIO SEU VERTIGINIS CORPORUM COELESTIUM

PROBLEMA 91

815. Si corporis rigidi singula elementa sollicitentur versus aliquod punctum *F* viribus, quae sint ut eorum massae per quadrata distantiarum ab eo puncto divisae, determinare harum virium momenta respectu axium principalium corporis.

SOLUTIO

Sint *IA*, *IB*, *IC* axes principales corporis (Fig. 104), eorumque respectu eius momenta inertiae *Maa*, *Mbb*, *Mcc*. Puncti autem *F* seu centri virium a centro inertiae corporis *I* distantia ponatur *IF* = *s*, quae ita ad ternos axes prineipales corporis sit inclinata, ut sint anguli

$$AIF = \zeta, \quad BIF = \eta \quad \text{et} \quad CIF = \vartheta;$$

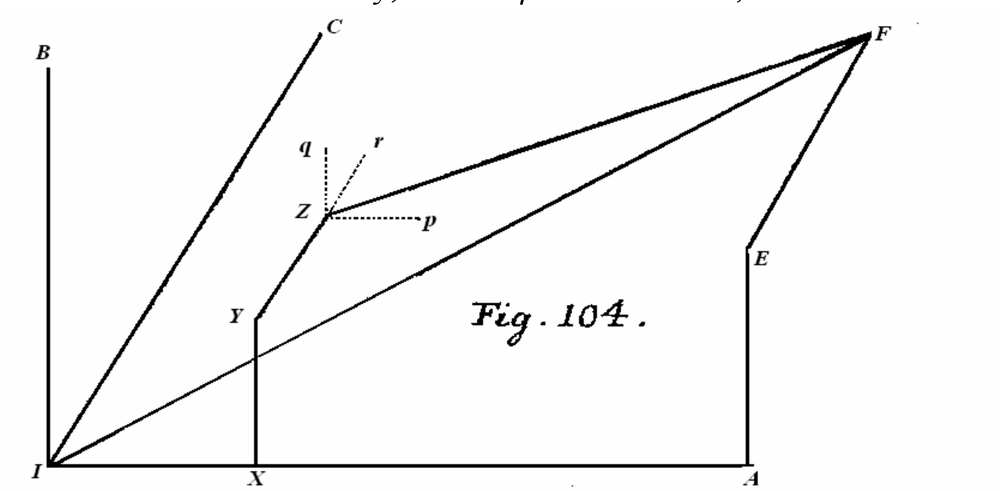


Fig. 104.

hinc demisso ex *F* ad planum *AIB* perpendiculo *FE* et ex *E* ad axem *IA* normali *EA*, erit

$$IA = s\cos\zeta \;, \quad AE = s\cos\eta \quad \text{et} \quad EF = s\cos\vartheta.$$

Vis porro singula corpora ad punctum *F* pellens tanta sit, ut in distantia = *e* aequetur gravitati, in aliis autem distantiis secundum quadrata earum diminuatur. Consideretur nunc corporis elementum quodcunque *dM* in *Z*, pro quo sint coordinatae axibus principalibus congruae

$$IX = x, \quad XY = y, \quad YZ = z$$

atque vis, qua hoc elementum dM ad punctum F urgetur, erit $=\frac{ee}{ZF^2}dM$.
Iam haec vis resolvatur secundum directiones axium Zp, Zq, Zr, eritque

$$\text{vis secundum } Zp = \frac{ee(s\cos\zeta - x)dM}{ZF^3},$$
$$\text{vis secundum } Zq = \frac{ee(s\cos\eta - y)dM}{ZF^3},$$
$$\text{vis secundum } Zr = \frac{ee(s\cos\vartheta - z)dM}{ZF^3}$$

atque hinc erunt momenta istarum virium respectu axium principalium:

$$\text{momentum axis } IA \text{ in sensum } BC = \frac{ees(y\cos\vartheta - z\cos\eta)dM}{ZF^3},$$
$$\text{momentum axis } IB \text{ in sensum } CA = \frac{ees(z\cos\zeta - x\cos\vartheta)dM}{ZF^3},$$
$$\text{momentum axis } IC \text{ in sensum } AB = \frac{ees(x\cos\eta - y\cos\zeta)dM}{ZF^3}.$$

His igitur momentis per totum corpus colligendis obtinebimus momenta, quae supra litteris P, Q, R indicavimus, ita ut sit ob s quantitatem constantem:

$$P = ees\int\frac{(y\cos\vartheta - z\cos\eta)dM}{ZF^3},$$
$$Q = ees\int\frac{(z\cos\zeta - x\cos\vartheta)dM}{ZF^3},$$
$$R = ees\int\frac{(x\cos\eta - y\cos\zeta)dM}{ZF^3}.$$

Est autem

$$ZF = \sqrt{((s\cos\zeta - x)^2 + (s\cos\eta - y)^2 + (s\cos\vartheta - z)^2)}$$

seu ob $\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$,

$$ZF = \sqrt{(ss - 2sx\cos\zeta - 2sy\cos\eta - 2sz\cos\vartheta + xx + yy + zz)}.$$

Cum autem in corporibus coelestibus distantia $IF = s$ sit semper vehementer magna prae ipso corpore seu quantitatibus x, y, z, erit satis exacte ad nostrum institutum

$$\frac{1}{ZF^3} = \frac{1}{s^3} + \frac{3x\cos\zeta + 3y\cos\eta + 3z\cos\vartheta}{s^4}.$$

Quoniam vero ob I centrum corporis inertiae et IA, IB, IC eius axes principales habemus

$$\int xdM = 0, \quad \int ydM = 0, \quad \int zdM = 0,$$

atque

$$\int xydM = 0, \quad \int xzdM = 0, \quad \int yzdM = 0,$$

prodibit his factis substitutionibus :

$$P = ees\int \frac{(3yy\cos\eta\cos\vartheta - 3zz\cos\eta\cos\nu)\ dM}{s^4}$$
$$= \frac{3ee\cos\eta\cos\vartheta}{s^3}\int(yy - zz)dM$$
$$Q = \frac{3ee\cos\zeta\cos\vartheta}{s^3}\int(zz - xx)dM$$
$$R = \frac{3ee\cos\zeta\cos\eta}{s^3}\int(xx - yy)dM.$$

Verum ob data momenta inertiae est

$$\int xxdM = \tfrac{1}{2}M(bb + cc - aa),$$
$$\int yydM = \tfrac{1}{2}M(aa + ce \sim bb)$$

et

$$\int zzdM = \tfrac{1}{2}M(aa + bb - cc),$$

quocirca erit

$$P = \frac{3Mee(cc - bb)\ \cos\eta\cos\vartheta}{s^3}$$
$$Q = \frac{3Mee(aa - cc)\ \cos\zeta\cos\vartheta}{s^3}$$
$$R = \frac{3Mee(bb - aa)\ \cos\zeta\cos\eta}{s^3}.$$

COROLLARIUM 1

816. Haec igitur momenta virium non rigore geometrico sunt definita, sed tantum valent, quando distantia puncti attrahentis magnitudinem corporis attracti longe superat. Atque sic commode evenit, ut ea per momenta inertiae tam concinne exprimi potuerint.

COROLLARIUM 2

817. Si corpus attractum omnia momenta inertiae habeat inter se aequalia, etiam haec virium momenta evanescunt; eatenus ergo tantum motus gyratorius corporum coelestium ab huiusmodi viribus afficitur, quatenus ea non sunt sphaerica, seu saltem momentis inertiae aequalibus praedita.

SCHOLlON

818. Si, quantum hae vires ad motum progressivum conferant, definire velimus, singulas vires elementares ipsi centro inertiae applicare debemus; quas si pro quolibet axe in unam summam colligamus, habebimus totam vim corpus ad motum progressivum sollicitantem. Ut autem ad binas dimensiones variabilium *x, y, z* ascendamus, accuratius valorem *ZF* exprimere debemus, ut sit:

$$\frac{1}{ZF^3}=\frac{1}{s^3}+\frac{3\left(x\cos\zeta+y\cos\eta+z\cos\vartheta\right)}{s^4}$$
$$\frac{15\left(x\cos\zeta+y\cos\eta+z\cos\vartheta\right)^2}{2s^5}-\frac{3\left(xx+yy+zz\right)}{2s^5}.$$

Haec formula per $(s\cos\zeta-x)dM$ multiplicata et secundum praecepta superiora integrata dabit

$$\int\frac{(s\cos\zeta-x)dM}{ZF^3}=\frac{M\cos\zeta}{ss}+\frac{15\cos\zeta}{2s^4}\int dM(xx\cos^2\zeta+yy\cos^2\eta+zz\cos^2\vartheta)$$
$$-\frac{3\cos\zeta}{2s^4}\int dM(xx+yy+zz)-\frac{3\cos\zeta}{s^4}\int xxdM\ ,$$

quae in hanc formam transmutatur:

$$\frac{M\cos\zeta}{ss}+\frac{3M\cos\zeta}{2s^4}\left(aa\left(3-5\cos^2\zeta\right)+bb\left(1-5\cos^2\eta\right)+cc\left(1-5\cos^2\vartheta\right)\right).$$

Quare nanciscemur sequentes tres vires:

I. $sec.IA=\frac{Mee\cos\zeta}{ss}+\frac{3Mee\cos\zeta}{2s^4}\left(aa\left(3-5\cos^2\zeta\right)+bb\left(1-5\cos^2\eta\right)+cc\left(1-5\cos^2\vartheta\right)\right)$

1I. $sec.IB=\frac{Mee\cos\eta}{ss}+\frac{3Mee\cos\eta}{2s^4}\left(bb\left(3-5\cos^2\eta\right)+cc\left(1-5\cos^2\vartheta\right)+aa\left(1-5\cos^2\zeta\right)\right)$

III. $sec.IC=\frac{Mee\cos\vartheta}{ss}+\frac{3Mee\cos\vartheta}{2s^4}\left(cc\left(3-5\cos^2\vartheta\right)+aa\left(1-5\cos^2\zeta\right)+bb\left(1-5\cos^2\eta\right)\right)$

Hac tres autem vires revocantur primo ad unicam in directione *1F* sollicitantem, quae est:

$$\frac{Mee}{ss}+\frac{3Mee}{2s^4}\left(aa\left(1-5\cos^2\zeta\right)+bb\left(1-5\cos^2\eta\right)+cc\left(1-5\cos^2\vartheta\right)\right),$$

cui insuper sunt adiungendae hae ternae

$$1°.\quad sec.\,IA=\frac{3Maaee\cos\zeta}{s^4}$$
$$2°.\quad sec.\,IB=\frac{3Mbbee\cos\eta}{s^4}$$
$$3°.\quad sec.\,IC=\frac{3Mccee\cos\vartheta}{s^4}.$$

Unde patet, si terna momenta inertiae principalia fuerint inter se aequalia, omnes vires ad unicam $\frac{Mee}{ss}$ secundum *1F* agentem reduci, quae in Theoria Astronomiae spectatur, reliquis vero casibus vis illa centripeta non erit pure quadrato distantiae reciproce proportionalis, sed eo accedunt insuper exiguae particulae biquadrato distantiae reciproce proportionales, quae autem praeterea a situ corporis respectu virium *F* pendent; ad quam aberrationem in calculo Astronomico attendisse iuvabit, praecipue si corpora notabiliter a figura sphaerica recedant.

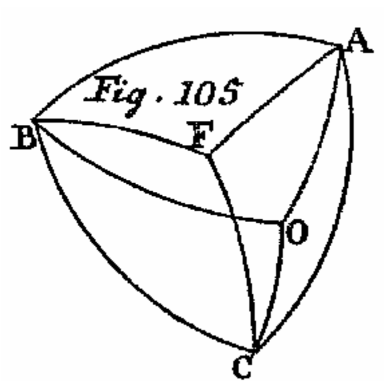
SCHOLlON 2

819. Assumsi hic singula corporis elementa versus unicum punctum *F* urgeri, cum tamen in hypothesi Attractionis etiam ad singula corporis attrahentis elementa sollicitentur. Verum si corpus attrahens fuerit sphaera, certum est id perinde ad se adtrahere, ac si tota eius massa in centro esset unita; ita ut nostrum problema etiam hos casus in se complectatur. At si corpus attrahens non fuerit sphaericum, mutabitur quidem paulisper tam ratio reciproca duplicata, quam directio vis, quae non amplius ad certum punctum erit directa; verum, haec irregularitas in ingenti distantia penitus evanescere est censenda, praecipue cum corpora coelestia parum a figura sphaerica discrepent. Hic autem quoniam tantum ad motum gyratorium respicio, a motu progressivo mentem abstrahendo, centrum inertiae corporis in quiete considero, ac primo, si ipsum corpus quiescat, circa quemnam axem motum gyratorium sit accepturum, investigabo.

PROBLEMA 92

820. Si corpus quiescat, idque a centro virium *F* modo ante definito sollicitetur, definire axem, circa quem primo instanti motum gyratorium accipiet, ac celeritatem angularem inde genitam.

SOLUTIO



Corpus ergo in quiete consideramus, seu potius ab eius motu progressivo mentem abstrahimus; eius igitur centro inertiae in centro sphaerae constituto sint *A, B,* C poli axium principalium (Fig. 105), eorumque respectu, ut hactenus, momenta inertiae *Maa, Mbb,Mcc.* Iam recta ex centro inertiae ad centrum virium ducta traiiciat superficiem sphaericam in puncto *F,* ut sint arcus $AF=\zeta, BF=\eta, CF=\vartheta;$ distantia autem centri virium sit $=s$ eiusque vis attrahens tanta, ut in distantia $=e$ aequetur gravitati. Hinc virium momenta *P, Q, R* respectu axium principalium *IA*, *IB*, *IC* sunt:

$$P=\frac{3Mee(cc-bb)\cos\eta\cos\vartheta}{s^3}$$

$$Q=\frac{3Mee(aa-cc)\cos\zeta\cos\vartheta}{s^3}$$

atque

$$R=\frac{3Mee(bb-aa)\cos\zeta\cos\eta}{s^3}.$$

Quare ex § 806 corpus gyrari incipiet circa eiusmodi axem *IO,* ut positis arcubus

$$AO = \alpha, \quad BO = \beta, \quad CO = \gamma$$

futurum sit

$$\cos\alpha = \frac{P}{aa} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}$$

$$\cos\beta = \frac{Q}{bb} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}$$

$$\cos\gamma = \frac{R}{cc} : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)},$$

tempusculo autem *dt* acquiret celeritatem angularem nascentem

$$d\gamma' = \frac{2gdt}{M}\sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)},$$

quae in sensum *ABC* erit directa. Atque ex his distantia poli gyrationis *O* a puncto *F* ita reperitur expressa, ut sit

$$\cos OF = \left(\frac{P\cos\zeta}{aa} + \frac{Q\cos\eta}{bb} + \frac{R\cos\vartheta}{cc}\right) : \sqrt{\left(\frac{PP}{a^4} + \frac{QQ}{b^4} + \frac{RR}{c^4}\right)}.$$

COROLLARIUM 1

821. Memoratu hic dignus est casus, quo centrum virium *F* cadit intra binos polos principales; cadat enim punctum *F* in arcum *AB* et ob $\cos\vartheta = 0$ et $\cos^2\zeta + \cos^2\eta = 1$, erit

$$P = 0,\ Q = 0$$

et

$$R = \frac{3Mee(bb-aa)\sin\zeta\cos\zeta}{s^3}$$

unde etiam fit $\cos\alpha = 0$ et $\cos\beta = 0$ et $\cos\gamma = 1$, ita ut polus gyrationis *O* cadat in polum principalem *C*.

COROLLARIUM 2

822. Eodem casu, quo centrum virium est in plano *AIB* et corpus circa axem *IC* gyrari incipit, primo tempusculo *dt* acquirit celeritatem angularem nascentem

$$d\gamma' = \frac{6gee(bb\text{-}aa)\,dt\sin\zeta\cos\zeta}{ccs^3}$$

in sensum *AB*, seu

$$d\gamma' = \frac{3gee(bb\text{-}aa)\,dt\sin 2\zeta}{ccs^3}.$$

COROLLARIUM 3

823. Quodsi ergo eodem casu corpus iam habuerit motum gyratorium circa istum axem *IC* celeritate $= \gamma'$ in sensum *AB*, is ob vim sollicitantem versus centrum virium *F* tendentem accelerabitur, ita ut fiat

$$d\gamma' = \frac{3gee(bb\text{-}aa)\, dt \sin 2\zeta}{ccs^3}.$$

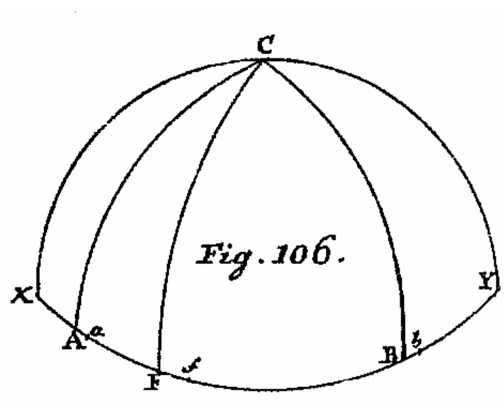
SCHOLION

824. Hinc ergo evidens est, si centrum virium *F* ita circa corpus circumferatur, ut per circulum maximum *AB* duos axes principales *IA* et *IB* continentem incedat corpusque circa reliquum axem principalem *IC* gyrari coeperit, tum perpetuo circa eundem axem *IC* esse gyraturum, solamque celeritatem angularem γ' modo auctum modo minutum iri. Casus hic omnino dignus est, qui omni studio evolvatur, quoniam motum libratorium lunae, quo semper fere eandem faciem terrae obvertit, complecti videtur. Quae investigatio quo facilior et clarior reddatur, primo centrum virium motu uniformi circa corporis centrum inertiae in eodem plano circumferri ac perpetuo eandem distantiam servare assumamus.

PROBLEMA 93

825. Si corpus gyretur circa suum axem principalem *IC*, centrum virium autem *F* in plano ad eum normali uniformiter circumferatur, eius distantia a centro inertiae corporis eadem manente, definire motum gyratorium huius corporis.

SOLUTIO



Quoniam ergo axis gyrationis *IC* manet constans et coeli respectu quasi fixus (Fig. 106), sit *XCY* hemisphaerium coeleste et *XY* circulus maximus polo C descriptus, in quo centrum virium *F* uniformiter incedat, atque in hoc circulo quoque constanter inerunt bini reliqui poli principales corporis *A* et *B*. Ponatur celeritas angularis centri virium $F = \delta$, quod eum initio fuisset in *X*, tempore elapso = *t* arcum descripserit necesse est $XF = \delta t$. Eodem autem temporis momento alter axis principalis reperiatur in *A*, positoque arcu $XA = \lambda$, si celeritas angularis corporis circa axem *IO* sit = γ' in sensum *AB*, erit $d\lambda = \gamma' dt$. Tum vero ob $AF = \delta t - \lambda$, quod supra erat ζ, hic nobis est $\delta t - \lambda$; at retentis reliquis quantitatibus *aa*, *bb*,*cc* itemque *ee* et *s*, quae sunt constantes, ut supra, habebimus hanc aequationem

$$d\gamma' = \frac{3gee(bb - aa)}{ccs^3} dt \sin 2\zeta.$$

Introducamus autem angulum $ACF = \zeta$, et ob $\zeta = \delta t - \lambda$ nanciscimur

$$\lambda = \delta t - \lambda$$

et

$$d\lambda = \delta dt - d\zeta = \gamma' dt,$$

unde fit

$$\gamma' = \delta - \frac{d\zeta}{dt}.$$

Quocirca sumto elemento *dt* constante prodit haec aequatio resolvenda:

$$dd\zeta + \frac{3gee(bb - aa)}{ccs^3} dt^2 \sin 2\zeta = 0.$$

Statuamus brevitatis gratia

$$\frac{3gee(bb - aa)}{ccs^3} = N,$$

et multiplicando per $2d\zeta$ fit

$$2d\zeta dd\zeta + 2Ndt^2 d\zeta \sin 2\zeta = 0,$$

cuius integralis est:

$$d\zeta^2 - Ndt^2 \cos 2\zeta = Cdt^2,$$

unde colligitur

$$dt = \frac{d\zeta}{\sqrt{(C + N\cos 2\zeta)}}$$

atque

$$\gamma' = \delta - \sqrt{(C + N\cos 2\zeta)}.$$

Ex illa autem aequatione ad quodvis tempus *t* arcum $AF = \zeta$ definiri oportet, qui si esset constans, corpus perpetuo eandem faciem centro virium *F* obverteret. Quatenus ergo *N* non est = 0 et angulus ζ variationi obnoxius, celeritas angularis γ' est variabilis ; ad quae phaenomena exploranda binos casus evolvi decet, prout fuerit vel $bb > aa$ vel $bb < aa$, quorum uterque pro ratione constantis *C* infinitam varietatem complectitur.

CASUS I quo $bb > aa$

826. Sit igitur

$$\frac{3gee(bb - aa)}{ccs^3} = n$$

numero positivo, et dum centrum virium *F* celeritate δ per eirculum *XFY* progreditur et ad tempus *t* arcus *FA* in antecedentia vergens vocetur = ζ ,
erit

$$dt = \frac{d\zeta}{\sqrt{(C + n\cos 2\zeta)}},$$

ubi ratione constantis *C* sequentia annoto:

1°. Si $C = -n$ (nam valorem negativum maiorem habere nequit), erit

$$dt = \frac{d\zeta}{\sqrt{n(-1 + \cos 2\zeta)}},$$

ideoque angulus ζ ; necessario est = 0; scilicet punctum *A* cum *F* semper congruet, cum eoque uniformiter circa axem *IA* gyrabitur.

2°. Si $C = 0$, angulus ζ minor erit semirecto sive positivus sive negativus et intra limites +45° et – 45° vagabitur. Punctum *A* ergo nunquam ultra 45° a puncto *F* recedet, sed modo ante modo post id reperietur, qui motus ex aequatione

$$dt = \frac{d\zeta}{\sqrt{n \ \cos 2\zeta}}$$

eolligi debet.

3°. Si $C = n$, et aequatio

$$dt = \frac{d\zeta}{\sqrt{n(1+ \ \cos 2\zeta)}}$$

abit in hanc

$$dt = \frac{d\zeta}{\cos\zeta\sqrt{2n}},$$

quae integrata dat

$$t = \frac{1}{\sqrt{2n}} \ l \ \mathrm{tang}\left(45° \ + \tfrac{1}{2}\zeta\right),$$

si sumto $t = 0$ fuerit $\zeta = 0$, unde patet demum elapso tempore infinito fieri $\zeta = 90°$.

4°. Si $C > n$, punctum *A* ab *F* tempore finito ad 90° digredietur indeque porro in oppositum ipsi *F* punctum progredietur et ad alteram partem circumeundo iterum in *F* revertet. Sit enim $C = mmn$, existente mm numero unitate maiore, ob

$$dt = \frac{d\zeta}{\sqrt{n(mm+ \ \cos 2\zeta)}}$$

erit proxime

$$dt = \frac{d\zeta}{\sqrt{n}}\left(\frac{1}{m} - \frac{\cos 2\zeta}{2m^3}\right)$$

et integrando

$$t\sqrt{n} = \frac{\zeta}{m} - \frac{\sin 2\zeta}{4m^3},$$

unde patet angulum ζ successive per omnes valores migrare.

5°. Hactenus posuimus $\gamma' < \delta$, ita ut motus puncti *F* celerior sit, quam gyratorius circa axem *1C;* si contrarium eveniat, tantum signum formulae $\sqrt{(C + n\cos 2\zeta)}$ mutari debet.

CASUS 11 quo $bb < aa$

827. Sit igitur

$$\frac{3gee(aa-bb)}{ccs^3} \ = n,$$

erit

$$dt = \frac{d\zeta}{\sqrt{(C - n\cos 2\zeta)}},$$

et

$$\gamma' = \delta - \sqrt{(C - n\cos 2\zeta)};$$

in quibus formulis si ponatur $\zeta = 90° + \varphi$, ut iam φ denotet distantiam poli *B* a centro virium *F* antecedentia versus sumtam, resultabunt formulae praecedentes, quae propterea eadem phaenomena exhibebunt.

COROLLARIUM 1

828. Si ergo ponamus centrum virium *F* initio cum polo *A* convenisse existente $bb > aa$, corpus semper eandem faciem puncto *F* obvertet, si celeritas angularis ipsi celeritati centri virium δ fuerit aequalis.

COROLLARIUM 2

829. Sin autem initio, quo *F* cum *A* conveniebat, celeritas corporis angularis γ' sit aliquanto maior vel minor quam δ, ut differentia non superet

$$\sqrt{2n} = \sqrt{\frac{6gee(bb-aa)}{ccs^3}},$$

polus *A* utrinque ab *F* digredietur non ultra certum intervallum et circa punctum *F* quasi oscillationes peragere videbitur; In quo utique similitudo cum motu lunae libratorio cernitur.

COROLLARIUM 3

830. In huiusmodi ergo motu libratorio celeritas angularis eorporis est maxima vel minima, dum punctum *A* ipsi *F* coniungitur, ab eo vel in consequentia vel in antecedentia digressurum; unde celeritas minima maior est quam $\delta - \sqrt{2n}$. Fieri igitur potest, ut talis motus oriatur, dum initio corpus plane nullum habuit motum gyratorium.

SCHOLION 1

831. Dubium ergo relinquitur nullum, quin motus libratorius lunae hae ratione oriatur, atque adeo probabile videtur in luna eum casum locum habere, quo lunae initio nullus plane motus gyratorius fuerat impressus ; tum autem axem lunae principalem *IA*, cuius respectu momentum inertiae *Maa* est minimum, terram versus fuisse directum. Quoniam igitur novimus digressiones poli *A* ab *F* esse minimas, tempus harum oscillationum definire poterimus; cum enim arcus $AF = \zeta$ sit valde parvus, erit $cos\, 2\zeta = 1 - 2\zeta\zeta$ ideoque

$$dt = \frac{d\zeta}{\sqrt{(C+n-2n\zeta\zeta)}},$$

unde fit integrando

$$t\sqrt{2n} = \text{A } sin \frac{\zeta\sqrt{2n}}{\sqrt{(C+n)}}.$$

Quare cum in digressione maxima fiat

$$\zeta = \sqrt{\frac{C+n}{2n}},$$

erit tempus, quo punetum *A* ab *F* maxime digreditur, = $\frac{\pi}{2\sqrt{2n}}$ secundis, cuius duplam $\frac{\pi}{\sqrt{2n}}$ dabit tempus, quo polus *A* ab *F* digressus iterum eodem redit.

Tum autem celeritas angularis minima, quando scilicet polus *A* ab *F* in antecedentia digreditur, erit = $\delta-\sqrt{(C+n)}$ *;* quae ut evaneseat, constans *C* esse debet = $\delta\delta-n$, unde digressio maxima hoc casu fuerit = $\frac{\delta}{\sqrt{2n}}$ necesse est. Consideremus nunc etiam tempus unius revolutionis centri virium *F*, quod est = $\frac{2\pi}{\delta}$; min. sec., cuius dimidium si aequale sit uni oscillationi poli *A* = $\frac{\pi}{\sqrt{2n}}$, fiet $\delta=\sqrt{2n}$ seu $n=\frac{\delta\delta}{2}$ ideoque $C=\frac{\delta\delta}{2}=n$; neque ergo digressio amplius foret minima, uti assumseramus.

SCHOLION 2

832. Hinc igitur concludimus motum lunae libratorium non ita explicari posse, ut statuamus lunae initio nullum plane motum gyratorium fuisse impressum, sed potius cum vehementer verisimile sit, lunam, si ea circa terram in orbita circulari uniformiter circumferretur, quae est hypothesis nostri problematis, perpetuo eandem plane faciem nobis esse obversuram neque ullam nutationem in ea observatum iri; in eadem hypothesi statuere debemus; lunae initio talem motum gyratorium fuisse impressum, ut praecise fuerit celeritas angularis = δ, nempe celeritati terrae circa lunam, et simul axem eius *IA* terram versus fuisse directum. Hoc autem satis probabile videtur; cum enim respectu axis *IA* momentum inertiae sit minimum, ideoque lunae, si eius corpus sphaeroides oblongum statuatur, axis maximus, causa esse potuit, quae initio hunc axem ad terram direxerit, atque eidem causae fortasse tribuendum est, quod, dum luna primum motum accepit, hic ipse axis directionem suam versus terram conservaverit; quod idem est, ac si celeritas angularis prima ipsi celeritati terrae δ fuisset aequalis. Cum igitur luna, si circulum circa terram motu uniformi describeret, nobis constanter eandem faciem esset obversura, eius librationes observatae motui lunae irregulari, quo modo celerius modo tardius incedit, tribui debent. Quare etiam praecedens problema in hac hypothesi resolvamus, ut punctum *F* neque uniformiter circumferri neque perpetuo eandem distantiam a centro inertiae corporis tenere assumamus.

PROBLEMA 94

833. Si corpus gyretur circa suum axem principalem *1C*, centrum virium autem *F* in plano ad eum normali neque uniformiter neque in eadem distantia circumferatur, initio vero axis *1A* fuerit ad centrum virium *F* directus similemque motum acceperit, definire motum corporis libratorium.

SOLUTIO

Motus corporis irregularis puncti *F* ita exprimi poterit, ut tempore *t* descripserit arcum

$$XF=\delta t+\alpha\sin At;$$

ac pro distantia variabili sit

$$\frac{1}{s^3}=\frac{1}{f^3}(1+\beta\cos At).$$

Quare si iam celeritas angularis sit = γ', posito arcu $XA = \lambda$. erit $d\lambda = \gamma' dt$ et vocato arcu $AF = \zeta$ habebimus

$$d\gamma' = \frac{3gee(bb - aa)dt \sin 2\zeta}{ccs^3}.$$

Cum igitur sit

$$\lambda = \delta t + \alpha \sin At - \zeta ,$$

erit

$$\gamma' = \delta + A\alpha cosAt - \frac{d\zeta}{dt}$$

ideoque posito

$$\frac{3gee(bb - aa)}{ccf^3} = n$$

erit

$$-AA\alpha dt \sin At - \frac{dd\zeta}{dt} = ndt(1 + \beta cosAt) \sin 2\zeta .$$

Quodsi iam assumamus arcum ζ semper manere valde parvum, habebimus hanc aequationem

$$\frac{dd\zeta}{dt^2} + AA\alpha \sin At + 2n\zeta(1 + \beta cosAt) = 0,$$

cui proxime satisfieri potest ponendo $\zeta = m \sin At$, unde fit

$$-AAm \sin At + AA\alpha \sin At + 2mn \sin At = 0$$

ob terminum $\beta cosAt$ prae 1 valde parvum. Hinc ergo adipiscimur

$$m = \frac{AA\alpha}{AA-2n}$$

ideoque

$$\zeta = \frac{AA}{AA-2n} \sin \mathrm{A}t,$$

unde fit

$$\gamma' = \delta + A\alpha \cos At - \frac{A^3 \cos At}{AA-2n} = \delta - \frac{2A\alpha n}{AA-2n} \cos At.$$

Hic cum sit

$$XF = \delta t + \alpha \sin At ,$$

pars prior δt vocatur locus medius puncti *F* et pars altera $\alpha \sin At$ eius aequatio seu prostaphaeresis, unde patet digressionem *FA* huic prostaphaeresi esse proportionalem, eaque maiorem ob *n* numerum positivum. Ita evanescente prostaphaeresi, seu quoties locus verus cum medio congruit, toties corpus eandem faciem centro virium *F* obvertit, neglectis quidem minoribus inaequalitatibus, quas ratio quantitatis β inveheret. Verum haec fusius prosequi atque accuratius determinare sine maiori astronomiae cognitione haud convenit.

COROLLARIUM 1

834. Si ergo inaequalitas motus puncti *F* ita exprimitur, ut tempore *t* conficiat arcum

$$XF = \delta t + \alpha \sin At,$$

eodem tempore sit arcus librationis

$$FA = \zeta = \frac{AA\alpha}{AA-2n} \sin At,$$

existente

$$n = \frac{3gee(bb-aa)}{ccf^3},$$

ubi *f* distantianl mediam centri virium *F* denotat.

COROLLARIUM 2

835. Si hunc arcum librationis ζ accuratius definire velimus, variabilitas distantiae $FI = s$ etiam in computum ingreditur, ita ut, si fuerit

$$\frac{1}{s^3} = \frac{1}{f^3}(1 + \beta \cos At),$$

reperiatur

$$\zeta = \frac{AA\alpha}{AA-2n} \sin At + \frac{n\alpha\beta}{4(AA-2n)} \sin 2At.$$

COROLLARIUM 3

836. Simili modo si generalius fuerit arcus tempore *t* confectus

$$XF = C + \delta t + \alpha \sin(At + \mathfrak{A}) + \alpha \sin(A't + 2\mathfrak{A}') + \text{ etc.}$$

et

$$\frac{1}{s^3} = \frac{1}{f^3}\left(1 + \beta \cos\left(At + \mathfrak{A}\right) + \beta' \cos\left(A't + \mathfrak{A}'\right) + \text{ etc.}\right),$$

invenitur proxime arcus librationis

$$. \; FA = \zeta = \frac{AA\alpha}{AA-2n} \sin\left(At + \mathfrak{A}\right) + \frac{A'A'\alpha'}{A'A'-2n} \sin\left(A't + \mathfrak{A}'\right) + \text{etc.}$$

SCHOLION 1

837. Hic iam perinde est, sive numerus

$$n = \frac{3gee(bb-aa)}{ccf^3}$$

sit positivus sive negativus, neque conditio superius requisita, ut pro arcu ζ evanescente esse debeat $bb > aa$, amplius locum habet. Casu enim II (§ 827) si ponatur $C = n$, fit

$$dt\sqrt{2n} = \frac{d\zeta}{\sin\zeta}$$

et

$$t\sqrt{2n} = l \text{ tang } \tfrac{1}{2}\zeta \ - \ \text{Const.},$$

unde si initio $t = 0$, fuerit $\zeta = 0$, constans addenda fit infinita, ideoque nonnisi elapso tempore infinito punctum *A* ab *F* digredietur. Quare dum punctum *F* uniformiter in circulo circumfertur, quicunque axis principalis initio ad punctum *F* fuerit directus, cum eoque pari celeritate gyrari coeperit, is ei constanter manebit annexus. Ac si deinceps punctum *F* motum suum vel intendat vel remittat, polus *A* ab eo digredietur secundum formulas inventas. Quin etiam patet, si fuerit $n = 0$ seu $bb = aa$, quo casu corpus uniformiter circa polum *C* gyraretur, digressiones ζ perpetuo differentiae inter locum medium et verum puncti *F* futuras esse aequales. At si numerus *n* sit positivus seu $bb > aa$, digressiones ista differentia essent maiores, contra autem sin $bb < aa$ minores. Ceterum numerus *A* inaequalitatem motus definiens, ex tempore, quo inaequalitas *sin At* ad eosdem valores revertitur, colligi potest, quod si eveniat post tempus = Θ min. sec., erit

$$\sin A\Theta = \sin 2\pi$$

ideoque

$$A = \frac{2\pi}{\Theta}.$$

SCHOLION 2

838. Hinc patet motum libratorium lunae, quo non semper eandem faciem terrae obvertit, potissimum defectui uniformitatis motus, quo terra circa lunam, seu quod idem est, luna circa terram circumferri videtur, tribui debere neque huc inaequalitatem momentorum principalium in luna multum conferre, quoniam ea tantum coefficientes terminorum afficiuntur. Libratio scilicet adesse posset, etiamsi luna esset corpus sphaericum, seu eius momenta principalia aequalia. Verum tum nulla ratio patet, cur lunae initio praecise tantus motus gyratorius fuisset impressus, quantum formulae nostrae exhibent; sin autem luna sit corpus sphaeroidicum sive oblongum sive compressum, rationem quodammodo intelligere licet, ob quam initio quidam axis principalis reliquis notabilior terram respicere inceperit. Utrum autem sit sphaeroides oblongum an compressum, ex quantitate librationis diiudicare licet, quae si excedat differentiam inter locum lunae verum ac medium, indicet esse $bb > aa$, seu axem lunae terrae obversum momento minimo gaudere. Verum hic non est locus quicquam definiendi, cum etiam luna ad solem urgeatur indeque libratio turbetur; praeterea vero quoque, uti luna non in eodem plano circa terram movetur, ita etiam vicissim motus centri virium *F* non in eodem plano circa lunam absolvetur, ex quo haec investigatio vehementer intricata reddetur, ut in tractatu generali locum invenire nequeat. Ceterum hoc semper insigne foret mysterium, quod luna initio praecise tantum motum gyratorium, quantum hic librationis casus postulat, acceperit; si enim vel maiorem vel minorem accepisset, labente tempore tandem facies opposita nobis obverti debuisset. Interim tamen hoc phaenomenon praescriptum celeritatis gradum non tam exacte postulat, quoniam etsi fuerit tantillo vel maior vel minor, librationes tamen ob problema praecedens contingere deberent; unde illud mysterium haud leviter illustratur. Talis autem latitudo admitti nequit, nisi casu quo $bb > aa$ seu $n > 0$; aequatio enim differentialis

$$\frac{dd\zeta}{dt^2} + AA\alpha \sin At + 2n\zeta(1 + \beta \cos At) = 0$$

generalius accedente constante arbitraria ita integrari potest, ut sit

$$\zeta = C \sin t\sqrt{2n} + \frac{AA\alpha}{AA-2n} \sin \mathrm{A}t,$$

unde fit celeritas angularis

$$\gamma' = \delta - C\sqrt{2n} \cdot \cos t\sqrt{2n} - \frac{2A\alpha n}{AA-2n} \cos \mathrm{A}t,$$

ubi etiam pro $t\sqrt{2n}$ scribi potest $(t+\gamma)\sqrt{2n}$, ita ut C et γ pro lubitu assumi queant. Quare cum initio t = 0 fuerit $\zeta = C \sin \gamma\sqrt{2n}$, dum celeritas angularis impressa sit

$$= \delta - C\sqrt{2n} \cdot \cos \gamma\sqrt{2n} - \frac{2A\alpha n}{AA-2n},$$

atque C sit fractio satis parva, motus libratorius sequetur, ut constanter pars quaedam lunae nobis maneat abscondita. At vero etiam fractio $\frac{AA\alpha}{AA-2n}$ esse debet valde parva, ut pro $\sin 2\zeta$ recte scribere liceat 2ζ .

SCHOLION 3

839. Explicatio ergo motus libratorii lunae huc redit, ut statuamus lunae corpus esse sphaeroides oblongum, cuius maior axis, vel is, cuius respectu momentum inertiae est minimum, initio terram versus directus, lunae autem tum circa axem ad planum orbitae terrestris normalem impressus fuerit motus gyratorius, cuius celeritas angularis propemodum motui lunae medio fuerat aequalis, in quo quidem insignis latitudo locum habere potest. Quin etiam sufficit, dummodo axis gyrationis propemodum fuerit ad planum orbitae terrestris normalis et axis maior propemodum tantum terram versus directus; namque etiam his casibus nutatio disci lunae reciproca evenire debet, etiamsi eam haud facile determinare liceat. Quare hoc casu relicto ad alias motus gyratorii perturbationes a viribus centripetis ortas progrediamur, unde nutatio axis terrae explicari possit.

PROBLEMA 95

840. Si corpus gyretur circa axem, qui alicui axi principali fuerit proximus, ac simul actioni centri virium subiiciatur, determinare mutationem momentaneam, tam in ipso axe gyrationis quam in celeritate angulari productam.

SOLUTIO

Sint *A*, *B*, *C*, terni poli principales corporis (Fig.107), eorumque respectu momenta inertiae

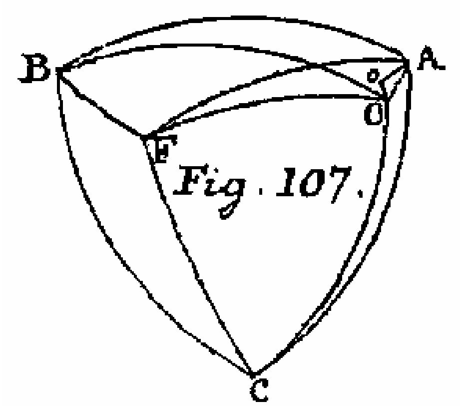


Fig. 107.

Maa, *Mbb*, *Mcc* ; corpus autem nunc gyretur circa polum *O* ipsi *A* proximum celeritate angulari γ' in sensum *ABC;* unde positis ternis arcubus

$$OA=\alpha,\quad OB=\beta,\quad OC=\gamma$$

erit arcus α valde parvus, at β et γ minime a quadrante discrepabunt, ita ut sit $\cos\alpha = 1$ et $\cos\beta=\cos\gamma = 0$. Quare posito

$$x = \gamma'\cos\alpha,\ y = \gamma'\cos\beta \quad \text{et} \quad z = \gamma'\cos\gamma$$

hae litterae *y* et *z* pro evanescentibus haberi poterunt, neque tamen earum differentialia, quae erunt $dy = -\gamma' d\beta$ et $dz = -\gamma' d\gamma$. Transeat iam recta ad centrum virium ducta per punctum *F* sintque arcus

$$AF=\zeta,\quad BF=\eta,\quad CF=\vartheta,$$

distantia autem centri virium ponatur = *s* eiusque vis attrahens tanta, ut in distantia = *e* aequetur gravitati. Ab actione ergo huius vis quantitates *x*, *y*, *z* tempusculo *dt* ita immutabuntur, ut sit

$$dx+\frac{cc-bb}{aa}yzdt=\frac{6gee(cc-bb)\,dt\cos\eta\cos\vartheta}{aas^3}$$

$$dy+\frac{aa-cc}{bb}xzdt=\frac{6gee(aa-cc)\,dt\cos\zeta\cos\vartheta}{bbs^3}$$

$$dz+\frac{bb-aa}{cc}xydt=\frac{6gee(bb-aa)\,dt\cos\zeta\cos\eta}{ccs^3}.$$

Cum nunc sit $dx = d\gamma'$, ob *y* et *z* evanescentes erit

$$d\gamma'=\frac{6gee(cc-bb)\,dt\cos\eta\cos\vartheta}{aas^3}$$

$$\gamma' d\beta=\frac{6gee(aa-cc)\,dt\cos\zeta\cos\vartheta}{bbs^3}$$

$$\gamma' d\gamma=\frac{6gee(bb-aa)\,dt\cos\zeta\cos\eta}{ccs^3}.$$

Quam variationem quo diligentius exploremus, quaeramus arcum *FO*, at ob

$$\sin BAF=\frac{\cos\vartheta}{\sin\zeta},\quad \cos BAF=\frac{\cos\eta}{\sin\zeta},$$

fit

$$\sin FAO = \frac{\cos\gamma\cos\eta - \cos\beta\cos\vartheta}{\sin\alpha\sin\zeta}$$

et

$$\cos FAO = \frac{\cos\beta\cos\eta + \cos\gamma\cos\vartheta}{\sin\alpha\sin\zeta}.$$

hincque

$$\cos FO = \cos\beta\cos\eta \ + \ \cos\gamma\cos\vartheta \ + \ \cos\alpha\cos\zeta;$$

cuius differentiale dat

$$(Fo - FO)\sin FO = d\beta\cos\eta \ + d\gamma\cos\vartheta$$

ob

$$\sin\beta = \sin\gamma = 1 \text{ et } \sin\alpha = 0.$$

Quare cum sit $FO = FA = \zeta$, habebitur

$$(Fo - \ FO)\sin\zeta = \frac{6geedt\cos\zeta\cos\eta\cos\vartheta}{\gamma' s^3}\left(\frac{aa-cc}{bb} + \frac{bb-aa}{cc}\right)$$

seu

$$Fo - FO = \frac{6gee(ce-bb)(bb+ce-aa)\,dt\cot\zeta\cos\eta\cos\vartheta}{\gamma' bbccs^3}$$

ob

$$\text{tang } BAO = \frac{\cos\gamma}{\cos\beta}.$$

Pro situ autem puncti *o* inveniendo habemus differentiando:

$$-\frac{OAo}{\cos^2 BAO} = \frac{-d\gamma\cos\beta\sin\gamma + d\beta\cos\gamma\sin\beta}{\cos^2\beta}$$

ideoque

$$OAo = \frac{d\gamma\cos\beta\sin\gamma - d\beta\cos\gamma\sin\beta}{\sin^2\alpha}$$

et

$$Oo = \sqrt{\left(d\beta^2 + d\gamma^2\right)}.$$

Tum vero, cum sit

$$d\alpha = \frac{-d\beta\sin\beta\cos\beta - d\gamma\sin\gamma\cos\gamma}{\sin\alpha\cos\alpha},$$

oritur

$$\text{tang } OoA = \frac{d\beta\sin\beta\cos\gamma - d\gamma\sin\gamma\cos\beta}{d\beta\sin\beta\cos\beta + d\gamma\sin\gamma\cos\gamma}$$

ob $\cos\alpha = 1$ seu

$$\text{tang } OoA = \frac{d\beta \cos\gamma - d\gamma \cos\beta}{d\beta \cos\beta + d\gamma \cos\gamma}.$$

COROLLARIUM 1

841. Si momenta inertiae respectu axium*1B* et *IC* sint aequalia seu *bb* = *cc*, primo fit $d\gamma' = 0$, seu celeritas angularis nullam patitur mutationem, tum vero erit

$$d\beta = \frac{-6gee(aa-cc)dt \cos\zeta \cos\vartheta}{\gamma' ccs^3}$$

et

$$d\gamma = \frac{6gee(aa-cc)dt \cos\zeta \cos\eta}{\gamma' ccs^3},$$

ita ut sit

$$d\beta \cos\eta + d\gamma \cos\vartheta = 0.$$

COROLLARIUM 2

842. Hoc porro casu *bb* = *cc* fit *Fo – FO* = 0 seu polus gyrationis *o* ita transfertur in O, ut spatiolum *OO* sit normale ad arcum *FO* ; est hoc
spatiolum

$$Oo = \frac{6gee(aa-cc)\, dt \cos\zeta \sin\zeta}{\gamma' ccs^3},$$

sed iam quaeritur, utrum ab *O* versus *FA* an contra sit directum.

COROLLARIUM 3

843. Cum autem sit

$$\sin FO : \sin FAO = \sin AO : \sin AFO,$$

erit

$$\sin AFO = \frac{\cos\gamma \cos\eta - \cos\beta \cos\vartheta}{\sin\zeta \sin FO}.$$

Iam quia *FO* non variatur, fiet secundum figuram, ubi *O* ad *AF* accedere sumitur:

$$-OFo \cdot \cos AFO = \frac{-d\gamma \cos\eta + d\beta \cos\vartheta}{\sin\zeta \sin FO} = -\frac{6gee(aa-cc)dt \sin\zeta \cos\zeta}{\gamma' ccs^3 \sin FO},$$

ideoque

$$OFo = \frac{6gee(aa-cc)dt \sin\zeta \cos\zeta}{\gamma' ccs^3 \sin FO \cos AFO}.$$

Cum igitur sit angulus *AFO* infinite parvus et *cosAFO* = 1 et *FO* = *FA* = ζ ,
erit

$$OFo = \frac{6gee(aa-cc)\, dt \cos\zeta}{\gamma' ccs^3}.$$

Ergo si *aa* > *cc*, punctum *O* ad arcum *AF* accedit, vel circa *A* in sensum *CB* procedit.

SCHOLION

843 [a]. Casus hic, quo $bb = cc$, ita ut corpus duo habeat momenta principalia respectu axium IB et IC aequalia et propemodum circa axem singularem IA gyretur celeritate angulari γ' in sensum BC, praecipue locum habet in motu vertiginis terrae ideoque meretur plenius evolvi. Quod quo facilius fieri possit, cum sit $AO = \alpha$, ponatur angulus $BAO = \rho$, erit

$$90° - \beta = \alpha \cos\rho \quad \text{et} \quad 90° - \gamma = \alpha \sin\rho,$$

unde

$$\beta = 90° - \alpha \cos\rho \quad \text{et} \quad \gamma = 90° - \alpha \sin\rho.$$

Quodsi ergo brevitatis gratia ponamus

$$\frac{3gee(aa-cc)}{\gamma' ccs^3} = N,$$

ut sit

$$d\beta = -2Ndt \cos\zeta \cos\vartheta$$

et

$$d\gamma = 2Ndt \cos\zeta \cos\eta,$$

erit

$$-d\alpha \cos\rho + \alpha d\rho \sin\rho = -2N\, dt \cos\zeta \cos\vartheta$$

et

$$-d\alpha \sin\rho - \alpha d\rho \cos\rho = 2Ndt \cos\zeta \cos\eta \;;.$$

unde colligitur

$$d\alpha = 2N\, dt \cos\zeta (\cos\rho \cos\vartheta - \sin\rho \cos\eta)$$

et

$$\alpha d\rho = -2Ndt \cos\zeta (\sin\rho \cos\vartheta + \cos\rho \cos\eta).$$

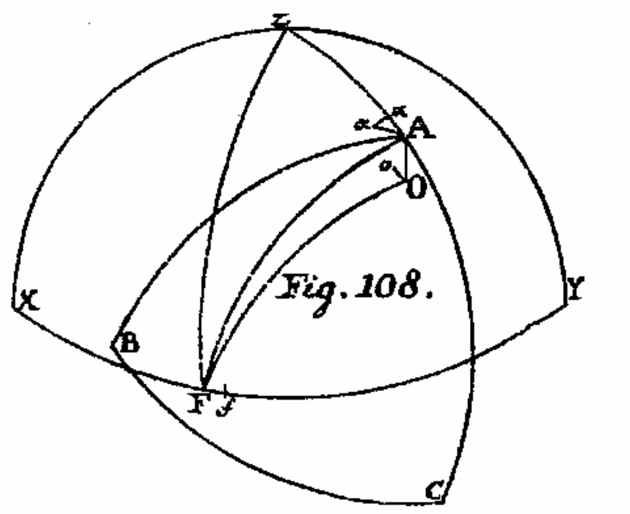
Si iam centro virium F motum quemcunque tribuamus, etiam tamdiu his formulis uti poterimus, quamdiu arcus $AO = \alpha$ manet tam parvus, ut contractiones adhibitae locum habere possint.

PROBLEMA 96

844. Si corpus habeat duo momenta principalia aequalia ac circa tertium axem singularem propemodum gyretur, centrum autem virium uniformiter in circulo circa centrum inertiae corporis circumferatur, ad quodvis tempus situm et motum corporis determinare.

SOLUTIO

Progrediatur centrum virium per circulum maximum *XFY* (Fig. 108) celeritate angulari = δ ac tempore elapso = *t* ex *X* pervenerit in *F*, ut sit $XF = \delta t$. In sphaera igitur consideretur circulus fixus *XZY*, in quo sit *Z* polus circuli *XFY*, ut sit angulus $XZF = \delta t$. Nunc autem versetur axis corporis singularis in *A*, ponaturque angulus $XZA = \lambda$ et arcus *ZA* = *p* ; tum vero corporis quasi primus meridianus sit *AB*, distans ab arcu *ZA* angulo *ZAB* = *q*. Porro gyretur nunc corpus circa axem *1O*, ut sit arcus minimus $AO = \alpha$ et angulus $BAO = \rho$, celeritate angulari = ε, quoniam iam novimus eam fore constantem, et punctum *A* abibit tempusculo *dt* in *a*, ut sit



$$Aa = \varepsilon dt \sin\alpha = \alpha\varepsilon dt$$

et angulus *aAO* rectus; quare ob

$$ZAO = q + \rho$$

erit

$$ZAa = q + \rho - 90° ,$$

ideoque demisso $a\alpha$ perpendiculo ad *ZA* fiet

$$a\alpha = -\alpha\varepsilon dt\cos(q+\rho)$$

et

$$A\alpha = \alpha edt\sin(q+\rho),$$

unde colligimus

$$dp = -\alpha\varepsilon dt\sin(q+\rho)$$

et

$$d\lambda = \frac{\alpha\varepsilon dt\cos(q+\rho)}{\sin p};$$

deinde vero, quia corpus quasi circa polum *A* gyratur~ erit $dq = \varepsilon dt$. Denique in triangulo *AZF* ob *ZA* = *p*, *ZF* = 90° et $AZF = \lambda - \delta t$ reperitur

$$\cos FA = \cos\zeta = \sin p\cos(\lambda - \delta t)$$

et

$$\cot ZAF = -\cos p\cot(\lambda - \delta t).$$

Ponamus brevitatis gratia angulum $ZAF = \varphi$, ut sit

$$\tan\varphi = -\frac{\tan(\lambda - \delta t)}{\cos p},$$

erit

$$BAF = \varphi - q$$

hincque

$$\cos BF = \cos(\varphi - q)\sin\zeta = \cos\eta$$

et

$$\cos CF = \sin(\varphi - q)\sin\zeta = \cos\vartheta\ .$$

Est vero

$$\sin\varphi \sin\zeta = \sin(\lambda - \delta t)$$

et

$$\cos\varphi \sin\ \zeta = -\ \cos p \cos(\lambda - \delta t)$$

ideoque

$$\cos\eta\ =\ -\ \cos p \cos q \cos(\lambda - \lambda t) + \sin q \sin\left(\lambda - \delta t\right)$$

et

$$\cos\vartheta\ =\ \cos q \sin(\lambda - \delta t) + \cos p \sin q \cos(\lambda - \delta t).$$

Unde si ponatur

$$\frac{3gee(aa-cc)}{\varepsilon ccs^3} = N,$$

colligitur fore

$$d\alpha = 2N\ dt \sin p\ \cos\left(\lambda - \delta t\right)(\cos p \sin(q + \rho)\cos\left(\lambda - \delta t\right) + \cos(q + \rho)\sin\left(\lambda - \delta t\right))$$

et

$$\alpha d\rho = -2Ndt \sin p \cos\left(\lambda - \delta t\right)(\sin(q + \rho)\sin\left(\lambda - \delta t\right) - \cos p \cos(q + \rho)\cos\left(\lambda - \delta t\right)),$$

quibus si adiungamus

$$dq = \varepsilon dt$$

et

$$dp = -\alpha\varepsilon dt \sin\ (q + \rho)\ ,$$

ex his quatuor aequationibus quatuor quantitates $p,\ q,\ \alpha$ et ρ definiri oportet.
Binae autem priores transformantur in has simpliciores:

$$d\alpha \cos(q + \rho) - \alpha d\rho \sin(q + \rho) = 2Ndt \sin p \sin(A - \lambda t)\cos(A - \lambda t)$$

$$d\alpha \sin(q + \rho) + \alpha d\rho \cos(q + \rho) = 2Ndt \sin p \cos p \cos^2\left(A - \lambda t\right).$$

Cum sit $q = \varepsilon t + C$, ponamus $q + \rho = \omega$, ut sit $\rho = \omega - q$, et adiungendo aequationes priores quaternas adhuc habebimus aequationes:

$$dp = -\varepsilon\alpha dt \sin\omega$$

$$d\lambda = \frac{\varepsilon\alpha dt \cos\omega}{\sin p}$$

$$d\alpha \cos\omega\ - \alpha d\omega \sin\omega\ + \varepsilon\alpha dt \sin\omega\ = 2Ndt \sin p \sin(\lambda - \delta t)\cos(\lambda - \delta t)$$

$$d\alpha \sin\omega\ + \alpha d\omega \cos\omega\ - \varepsilon\alpha dt \cos\omega\ =\ 2Ndt \sin p \cos p \cos^2(\lambda - \delta t);$$

ac si insuper ponamus $\lambda - \delta t = \varphi$, quae littera cum praecedente φ non est confundenda, erunt:

$$dp = -\varepsilon\alpha dt \sin\omega$$

$$d\varphi = -\delta dt + \frac{\varepsilon\alpha dt \cos\omega}{\sin p}$$

$$d\alpha \cos\omega - \alpha d\omega \sin\omega + \varepsilon\alpha dt \sin\omega = 2N\, dt \sin p \sin\varphi \cos\varphi$$

$$d\alpha \sin\omega + \alpha d\omega \cos\omega - \varepsilon\alpha dt \cos\omega = 2N\, dt \sin p \cos p \cos^2\varphi.$$

Ponamus porro $\alpha \cos\omega = x$ et $\alpha \sin\omega = y$, ut habeamus has aequationes

1°. $dp = -\varepsilon y dt$

2°. $d\lambda = \frac{\varepsilon x dt}{\sin p}$

3°. $d\varphi = -\delta dt + \frac{\varepsilon x dt}{\sin p}$

4°. $dx + \varepsilon y dt = N dt \sin p \sin 2\varphi$

5°. $dy - \varepsilon x dt = N dt \sin p \cos p + N dt \sin p \cos p \cos 2\varphi$,

ubi cum *x* et *y* sint quantitates minimae, ad veritatem satis appropinquabimus, si in binis postremis aequationibus arcum *p* et angulum λ ut constantes spectemus. Tribuamus ergo illis valores quasi medios sitque proxime $p = n$ et $\lambda = m$ ideoque $d\varphi = -\delta dt$, ut habeamus aequationes:

4°. $dx - \frac{\varepsilon y d\varphi}{\delta} = -\frac{N d\varphi}{\delta} \sin n \sin 2\varphi;$

5°. $dy + \frac{\varepsilon x d\varphi}{\delta} = -N d\varphi \sin n \cos n - \frac{N d\varphi \sin n \cos n \cos 2\varphi}{\delta}$,

quibus evidens est satisfieri posse ponendo

$x = E + F \cos 2\varphi$

$y = G \sin 2\varphi,$

ac hi coefficientes ita definiuntur, ut sit

$$E = -\frac{N \sin n \cos n}{\varepsilon}$$

$$F = -\frac{N \sin n (2\delta + \varepsilon \cos n)}{\varepsilon\varepsilon - 4\delta\delta}$$

$$G = \frac{N \sin n (2\delta \cos n + \varepsilon)}{\varepsilon\varepsilon - 4\delta\delta}.$$

Tum vero quia haec solutio tantum esset particularis, ponatur

$$x = E + F \cos 2\varphi + u$$

et

$$y = G \sin 2\varphi + v$$

orienturque hae aequationes

$$du - \frac{\varepsilon v d\varphi}{\delta} = 0$$

et

$$dv + \frac{\varepsilon v d\varphi}{\delta} = 0,$$

ex quibus elicitur

$$u = h \sin \tfrac{\varepsilon}{\delta}(\varphi + \zeta)$$

et

$$v = h \cos \tfrac{\varepsilon}{\delta}(\varphi + \zeta),$$

ubi h et ζ sunt constantes arbitrariae. Quocirca habebimus

$$x = \alpha \cos \omega = -\frac{N \sin n \cos n}{\varepsilon} - \frac{N \sin n\,(\varepsilon \cos n + 2\delta)}{\varepsilon\varepsilon - 4\delta\delta} \cos 2\varphi + h \sin \tfrac{\varepsilon}{\delta}(\varphi + \zeta)$$

$$y = \alpha \sin \omega = \frac{N \sin n(\varepsilon - 2\delta \cos n)}{\varepsilon\varepsilon - 4\delta\delta} \sin 2\varphi + h \cos \tfrac{\varepsilon}{\delta}(\varphi + \zeta),,$$

ubi φ exprimit angulum $FZA = \lambda - \delta t$. Deinde ob

$$dp = -\varepsilon y dt = \frac{\varepsilon y d\varphi}{\delta}$$

nanciscimur integrando:

$$p = n - \frac{\varepsilon N \sin n(\varepsilon + 2\delta \cos n)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)} \cos 2\varphi + h \sin \tfrac{\varepsilon}{\delta}(\varphi + \zeta) = ZA.$$

Denique aequatio $d\lambda = \frac{\varepsilon x dt}{\sin p} = -\frac{\varepsilon x d\varphi}{\delta \sin n}$ praebebit:

$$\lambda = m - Nt \cos n + \frac{\varepsilon N(\varepsilon \cos n + 2\delta)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)} \sin 2\varphi + \frac{h}{\sin n} \cos \tfrac{\varepsilon}{\delta}(\varphi + \zeta) = XZA.$$

COROLLARIUM 1

845. Cum sit ex nostris positionibus $\alpha = \sqrt{(xx + yy)}$,patet successu temporis distantiam $AO = \alpha$ non ultra certum limitem augeri posse, qui si fuerit satis exiguus, hypothesi nostra tuto utimur. Simul vero patet hanc distantiam α nunquam plane evanescere, nisi forte fiat tam $x = 0$ quam $y = 0$.

COROLLARIUM 2

846. Neglectis inaequalitatibus ab angulis

$$2\varphi = 2FZA$$

et

$$\tfrac{\varepsilon}{\delta}(\varphi + \zeta)$$

pendentibus, polus A uniformiter circa punctum Z in antecedentia regreditur celeritate angulari = $N \cos n$, si quidem

$$N = \frac{3gee(aa - cc)}{eccs^3}$$

fuerit numerus positivus, sicque integram revolutionem absolvet tempore

$$= \frac{2\pi}{N\cos n}\text{ min.sec. },$$

dum centrum virium *F* revolutionem absolvit tempore $= \frac{2\pi}{\delta}$; et ipsum corpus tempore $= \frac{2\pi}{\varepsilon}$.

COROLLARIUM 3

847. Praeterea vero tam distantia *ZA* quam angulus *XZA* exiguas inaequalitates patientur, partim ab angulo $2\varphi = 2FZA$, partim ab angulo

$$\frac{\varepsilon}{\delta}(\varphi+\zeta) = C - \varepsilon t,$$

hoc est, partim a motu centri virium, partim a motu vertiginis ipsius corporis pendentes. Quare si ponamus angulum $ZAB = \psi$, erit

$$ZA = n - \frac{\varepsilon N \sin n(\varepsilon + 2\delta\cos n)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}\cos 2\varphi - h\sin\frac{\varepsilon}{\delta}(\varphi+\zeta)$$

$$XZA = m - Nt\cos n + \frac{\varepsilon N(\varepsilon\cos n + 2\delta)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}\sin 2\varphi + \frac{h}{\sin n}\cos\frac{\varepsilon}{\delta}(\varphi+\zeta).$$

SCHOLION 1

848. Sumsimus hic corpus in eundem sensum gyrari, in quem centrum virium *F* circa id circumfertur, quemadmodum fit in terra, quae ab occidente in orientem gyratur, in quem sensum etiam sol et luna motu proprio promoveri cernuntur. Deinde etiam spectavimus numerum

$$N = \frac{3gee(aa-cc)}{eccs^3}$$

ut positivum, seu corpus ita comparatum, ut eius momentum inertiae respectu axis, circa quem proxime gyratur, sit maximum = *Maa*, dum respectu axium in aequatore sumtorum est minimum = *Mcc*, qua proprietate terram esse praeditam observationes circa figuram terrae sphaeroidicam compressam institutae declarant. In hac ergo constitutione axis terrae circa polum eclipticae *Z* in antecedentia regredi debet, quemadmodum etiam per observationes constat. Praeterea vero neque iste axis motus est aequabilis, neque eius distantia a polo eclipticae *Z* constans, sed duplici inaequalitati est obnoxia, quarum altera ab angulo $FZA = \varphi$ duplicato pendet, altera vero ab ipso motu vertiginis corporis, quae posterior maior minorve esse potest, prout initio polus gyrationis *O* tam ratione poli *A* quam ratione situs centri virium *F* fuerit constitutus. Scilicet cum ω denotet angulum *ZAO*, si initio vel dato saltem tempore innotuerint quantitates $AO = \alpha$, $ZAO = \omega$, $FZA = \varphi$ et $ZAB = \psi$, sumto *AB* pro corporis prima meridiano, ex aequationibus

$$\alpha\cos\omega + \frac{N\sin n\cos n}{\varepsilon} + \frac{N\sin n(\varepsilon\cos n + 2\delta)}{\varepsilon\varepsilon - 4\delta\delta}\cos 2\varphi + h\sin(\varphi+\zeta) = 0$$

$$\alpha\sin\omega - \frac{N\sin n(\varepsilon + 2\delta\cos n)}{\varepsilon\varepsilon - 4\delta\delta}\sin 2\varphi - h\cos(\varphi+\zeta) = 0$$

binae constantes *h* et ζ definiuntur. Nisi ergo prodeat *h* = 0, polus *A* inaequalitates etiam diurnas patietur, ita ut intervallo cuiusque revolutionis ad polum eclipticae alternatim accedat ab eoque recedat, simulque alternatim in antecedentia et consequentia nutet. Ob hanc scilicet inaequalitatem polus *A* singulis revolutionibus circulum describeret; cuius centrum cum quiescat, id potius pro vero polo terrae habebitur, ita ut hae inaequalitates non percipiantur. Tum vero reliquae inaequalitates ab actione centri virium pendentes non hunc polum apparentem, sed ipsum polum axis principalis afficient.

SCHOLION 2

849. Praetermissis autem his inaequalitatibus diurnis, quibus forte nutatio axis afficitur, si fuerit $aa > cc$ corpusque in eundem sensum gyratur ac centrum virium, phaenomena ita se habebunt :

Primo distantia poli *A* a puncto *Z*, quod est vertex seu polus orbitae, quam centrum virium describit, erit variabilis ac minima quidem deprehedetur, si angulus *FZA* vel evanescit, vel sit 180°, maxima autem, si iste angulus fuerit vel 90° vel 270°, differentia inter maximam minimamque distantiam existente =

$$\frac{\varepsilon N\, sin\, n(\varepsilon + 2\delta \cos n)}{\delta(\varepsilon\varepsilon - 4\delta\delta)}.$$

Secundo polus *A* circa punctum *Z* in antecedentia motu non uniformi regredietur, qui si, ut moris est, per motum medium prostaphaeresi corrigendu repraesentetur, motu medio regredietur celeritate angulari = $N \cos n$, tum vero correctio seu prostaphaeresis maxima erit =

$$\frac{\varepsilon N\, sin\, n(\varepsilon + 2\delta \cos n)}{2\delta(\varepsilon\varepsilon - 4\delta\delta)}.$$

addenda, si angulus *FZA* sit ve1 45° ve1 225°, subtrahenda vero, si iste angulus fiat vel 135° vel 315°, ubi notandum est, hunc angulum $FZA = \varphi$; reperiri si longitudo centri virium *F* a longitudine poli *A* subtrahatur. Ceterum hic celeritatem motus vertiginis ε prae celeritate centri virium δ ut multo maiore spectamus; si enim esset $\varepsilon = 2\delta$, conationes inventae adeo in infinitum abirent; verum hoc casu integratio nostrarum aequationum singulari modo esset instituenda, ponendo

$$x = E + F\, cos\, 2\varphi + A\varphi\, sin\, 2\varphi ;$$

et

$$y = G\, sin\, 2\varphi + B\varphi\, cos\, 2\varphi,,$$

reperireturque

$$E = \frac{-Nsinn\, cosn}{2\delta}$$

$$A = B = -\frac{Nsinn(1 + cos\, n)}{2\delta}$$

$$F + G = \frac{N\, sin\, n(1 - cos\, n)}{4\delta}.$$

Verum quia hic *x* et *y* continuo crescerent, mox hypothesin factam trangrederentur, totusque calculus non amplius locum haberet. Quare nisi $\varepsilon\varepsilon$ notabiliter discrepet a $4\delta\delta$ formulae nostrae adhiberi nequeunt.