Chapter 12

CONCERNING THE FREE MOTION OF THE SAID BODIES WITH TWO EQUAL PRINCIPAL AXES, AND ACTED ON BY NO FORCES.

DEFINITION 12

706. Two rigid bodies are said to have *equal principal axes*, when the moments of inertia about the two principal axes are equal to each other. said

COROLLARY 1

707. Hence bodies of this kind have innumerable principal axes; and indeed at once the two axis have equal moments of inertia, all the lines that can be drawn in the same plane passing through the centre of inertia can be taken equally for principal axes, and they are provided with the same moment of inertia.

COROLLARY 2

708. Therefore here that principal axis, the moment of inertia of which is unequal to the others, is singular, and all the lines drawn normally to it have equal moments of inertia, and can be considered as principal axes.

COROLLARY 3

709. And thus with the singular axis known, the position of the two remaining axes is not determined, but the location of these can be taken by any two lines as it pleases as long as they are taken normal to each other and to that axis, while they pass through the centre of inertia.

SCHOLIUM

710. Therefore above in general for the principal axes *IA, IB, IC* we were able to take the moments of inertia *Maa, Mbb, Mcc,* in this chapter we establish two of these equal. Therefore let the first axis *IA* be singular, and the moments of inertia of the remaining are equal to each other, so that there arises $bb = cc$; to which the above formulas can be adapted extremely well [§664]. But if in this case the position of two of the axes *IB* and *IC* is not determined, yet as we are able to consider their position, so that with the aid of these the position of the body at some time can be more easily assigned. But certainly an indefinite number of bodies are given of this kind, and among homogeneous bodies in the first place it is to be extended here to cylinders, cones, and in general all round bodies which arise from the rotation of some figure about a certain fixed axis; thus in order that nearly all bodies, which indeed are accustomed to arise from geometry, can themselves be included in this shapes of this kind. Therefore as we will outline in this chapter an account of the motion that these bodies themselves have, while they are not acted on by any forces, and indeed initially we ask about the position of the axis of rotation in terms of the principal axis at some time, while not being acted on by forces, and what motion these axes themselves shall have, that henceforth we will try to define.

PROBLEM 71

711. If some rotational motion were impressed initially on a said rigid body with two equal principal axes, nor any external forces being present, to assign at some time the position of the axis or rotation in relation to the principal axes.

SOLUTION

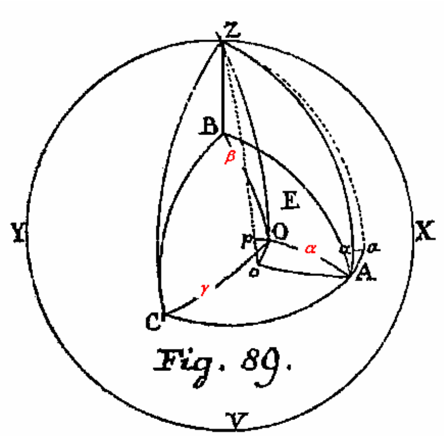


Fig. 89.

With the centre of inertia of the body *I* at the centre of a sphere (Fig. 89), we reduce everything to the surface of this sphere, on putting in place the principal axes of the body *IA*, *IB*, *IC* , and about the first axis *IA* the moment of inertia = *Maa*, but about the remaining two axes *IB* and *IC*, the moments of inertia are equal to each other and equal to *Mcc*, as *bb* = *cc*. But now after a change in time from the initial time *t*, the body may be rotating about the axis *IO* in the sense *ABC* with an angular speed equal to γ , [note the use of a larger font here as in the original text] thus so that the position of the point *O* can be defined relative to the points *A*, *B*, *C*. Hence there are put in place the great circular arcs $OA = \alpha, OB = \beta$ et $OC = \gamma,$ which are to be treated as variables, and by problem 66 in the case in which *bb* = *cc* put in place gives in the first place $d\gamma = 0,$ thus it is apparent that the angular speed remains invariant, and thus at this stage to be equal to that which was impressed on the body had initially. Whereby if initial angular speed is put equal to ε then $\gamma = \varepsilon.$ Then we now have these equations [of motion] from § 674:

$$\text{I.} \quad aac^4 d\alpha \sin\alpha = 0$$

$$\text{II.} \quad aac^4 d\beta \sin\beta = \varepsilon aacc(aa - cc)dt \cos\alpha \cos\gamma$$

$$\text{III.} \quad aac^4 d\gamma \sin\gamma = \varepsilon aacc(cc - aa)dt \cos\alpha \cos\beta,$$

and we understand from the first of these, that $AO = \alpha$ is to be constant, and thus these equations are for that case in which initially the axis of rotation is distant from the singular axis *IA*.
Since therefore

$$\cos\gamma = \sqrt{\left(\sin^2\alpha - \cos^2\beta\right)},$$

the other remaining equation is produced :

$$\frac{d\beta\sin\beta}{\sqrt{\left(\sin^2\alpha - \cos^2\beta\right)}} = \frac{\varepsilon(aa-cc)dt\cos\alpha}{cc},$$

and the integral of this is :

$$A\cos\frac{\cos\beta}{\sin\alpha} = C + \frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

and thus

$$\cos\beta = \sin\alpha\cos\left(C + \frac{\varepsilon(aa-cc)t\cos\alpha}{cc}\right)$$

and

$$\cos\gamma = \sin\alpha\sin\left(C + \frac{\varepsilon(aa-cc)t\cos\alpha}{cc}\right).$$

Whereby if initially, when $t = 0$, there should be

$$AO = \mathfrak{A},\quad BO = \mathfrak{B} \quad\text{and}\quad CO = \mathfrak{C},$$

then $\alpha = \mathfrak{A}$ and

$$\cos\mathfrak{B} = \sin\mathfrak{A}\sin C,$$

thus the constant becomes :

$$\cos C = \frac{\cos\mathfrak{B}}{\sin\mathfrak{A}}$$

and

$$\sin C = \frac{\cos\mathfrak{C}}{\sin\mathfrak{A}}.$$

In which case we have :

$$\cos\beta = \cos\mathfrak{B}\cos\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc} - \cos\mathfrak{C}\sin\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc}$$

$$\cos\gamma = \cos\mathfrak{B}\sin\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc} + \cos\mathfrak{C}\cos\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc},$$

thus if the initial motion were known to us, the position of the axis of rotation about the principal axes, or the arcs $\mathfrak{A}, \mathfrak{B}$ et $\mathfrak{C}$, then we are able to assign the position of the axis of rotation about the principal axes or the arcs α, β, γ for some time elapsed *t*.

COROLLARY 1

712. Therefore if initially there is a rotary motion impressed on the body about the axis *IE* with an angular speed equal to ε in the sense *ABC*, the inclination of the angles being $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ to the principle axes *IA*, *IB*, *IC*; hence the axis of gyration may be changed in some manner, the angular speed always remains the same and equal to ε; and the axis or rotation *IO* is inclined at the same angle $\mathfrak{A}$ to the singular principal axis *IA*.

COROLLARY 2

713. Now if the moment of inertia about the singular axis *IA* is equal to *Maa*, but about the two other axes equal to = *Mcc* for the lapse in the time equal to *t*, because ε denotes the angle, the angle is put in place $\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc} = T$,
which with the time *t* it increases uniformly; and in this time the body rotates about the axis *IO*, so that it becomes

$$AO = AE = \mathfrak{A}$$

and

$$\cos BO = \cos\mathfrak{B}\cos T - \cos\mathfrak{C}\sin T,$$

$$\cos CO = \cos\mathfrak{B}\sin T + \cos\mathfrak{C}\cos T.$$

COROLLARY 3

714. Because the arc *AO* remains equal to the magnitude $\mathfrak{A}$ always, the position of the point *O* can become known most conveniently from the angle *BAO*, since there arises

$$\cos BAO = \frac{\cos BO}{\sin\mathfrak{A}}$$

and

$$\sin BAO = \frac{\cos CO}{\sin\mathfrak{A}},$$

then it becomes

$$\cos BAO = \frac{\cos\mathfrak{B}\cos T - \cos\mathfrak{C}\sin T}{\sin\mathfrak{A}}$$

and

$$\sin BAO = \frac{\cos\mathfrak{B}\sin T + \cos\mathfrak{C}\cos T}{\sin\mathfrak{A}}.$$

COROLLARY 4

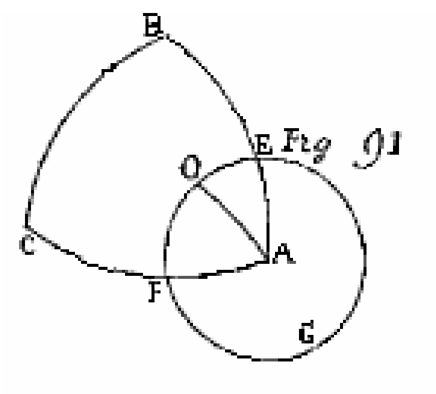
715. If $cc = aa$, which is the case treated before, from which all three moments of inertia are equal to each other, then $T = 0$, and $BO = \mathfrak{B}$, likewise $CO = \mathfrak{C}$, clearly the pole or rotation O about the principal axis remains at rest, as now we have found before.

SCHOLIUM

716. These formulas can be made much simpler, but it is the underlying idea that deserves the merit, as by that rather singular proposition rather than describing the change in detail.

PROBLEM 72

717. With the same items in place, which have been established in the previous problem, to define the forwards motion of the pole of rotation O about the principle axes.



SOLUTION

Everything remains as in the preceding solution, and since the equal poles B and C can be taken as it pleases on the circle PC, thus the quadrant AB is put in place, in order that it is cut by the pole E (Fig. 91). Therefore since this pole of the rotation keeps the same distance from the pole of the principal axis A, the motion of this describes a small circle EFG with A, the distance of this shall be the arc $AE = \alpha$, which we have indicated above by $\mathfrak{A}$. Hence there becomes

$$BE = \mathfrak{B} = 90^o - \alpha$$

and

$$CE = \mathfrak{C} = 90^o.$$

Whereby if in the elapsed time equal to t the pole of the rotation reaches O from E, on account of $\cos\mathfrak{C} = 0$, then

$$\cos BAO = \frac{\cos\mathfrak{B}\cos T}{\sin\alpha} = \cos T$$

and

$$\sin BAO = \frac{\cos\mathfrak{B}\sin T}{\sin\alpha} = \sin T,$$

and thus the angle itself

$$BAO = T.$$

But the angle T is defined thus from the time t, so that it becomes

$T = \frac{\varepsilon(aa-cc)t\cos\alpha}{cc} = BAO,$

thus we achieve this excellent solution. If the moment of inertia about the singular principal axis *IA* were equal to *Maa* and about the remaining two equal axes *IB* and *IC* equal to *Mcc*, moreover the body begins to rotate with an angular speed equal to ε initially about the axis *IE* in the sense *BCA* ; then about the principal axes, which we have considered as if to remain at rest, the pole of the rotation will be advanced through the small circle *EFG* described uniformly about the pole *A*, thus so that in the elapsed time equal to *t*, it completes the angle

$$EAO = \frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

and the motion is made in the sense *BC* conforming to the rotational motion, if indeed it should be that $aa > cc$, but in the opposite sense if $aa < cc$.

COROLLARY 1

718. The pole of rotation remains at rest in these cases :
1: if $AE = 0$ or the body begins to rotate about the principal axis *IA*.
2: if $AE = 90^{\circ}$ or if the body begins to rotate about some axis normal to IA, and
3: if $aa = cc$, that is, if the body has all three principal axes equal.

COROLLARY 2

719. If $aa > cc$, then the pole of rotation *E* about *A* in the same sense *BC*, in which the rotation is made, is carried round with an angular speed equal to $\frac{\varepsilon(aa-cc)\cos AE}{cc}$; but if $aa < cc$, it is carried around in the opposite sense with an angular speed equal to $\frac{\varepsilon(cc-aa)\cos AE}{cc}$.

COROLLARY 3

720. But the arc of the small circle itself *EO*, through which the axis of rotation progresses in the time t is equal to $\frac{\varepsilon(aa-cc)t\sin AE\cos AE}{cc} = \frac{\varepsilon(aa-cc)t\sin 2AE}{2cc}$, because therefore the distance from the other equal parts is a maximum if $AE = \frac{1}{2}AB = 45^{\circ}$, that is if the axis of rotation is equally distant from the principal axes.

COROLLARY 4

721. With the ratio of the diameter to the periphery put equal to $1:\pi$, the pole of the rotation traverses the whole circumference *EFGE* in a time equal to $\frac{2\pi cc}{\varepsilon(aa-cc)\cos AE}$ sec. and thus the uniform motion is conserved always.

SCHOLIUM

722. Here we are not yet acting on the motion of the body, for since it is to be noted properly, if the body should be at rest or we are considering another body equal to it and at rest, and for that body we have shown how to define the axis of rotation *IO* at some time, about which a moving body should rotate, without this body being disturbed, for in which case this axis of rotation would then be obtained with respect to absolute space. Therefore now we proceed to an examination of the complete motion.

PROBLEM 73

723. If some rotary motion were impressed on a rigid body with the two said equal principal axes, at a given time both the position of the principal axis as well as the axis of rotation are to be assigned with respect to absolute space.

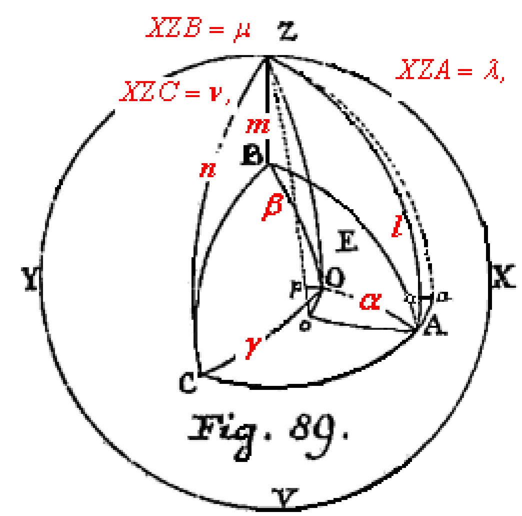


Fig. 89.

SOLUTION

The sphere circumscribed from the centre of inertia of the body may be enclosed by the surface of a fixed sphere *ZXVY* (Fig. 89), and in the elapsed time *t* the mobile sphere keeps that position with the body, in order that the poles or the three principal axes are at *A, B, C*, about the first of which *IA* the moment of inertia is equal to *Maa*, but about the remaining two equal to *Mcc*. Thus with the arcs *AZ, BZ* and *CZ* drawn to a certain fixed point *Z*, as in problem 68 we put

$$AZ = l,\quad BZ = m \quad \text{et} \quad CZ = n,$$

in order that

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

then let the angles be

$$XZA = \lambda,\quad XZB = \mu \quad \text{et} \quad XZC = \nu,$$

and because the rotational motion, as we have now shown, remains uniform, let the angular speed of this be equal to ε directed in the sense *ABC*. Again since the axis of rotation equally always remains distant from the axis *IA*. Let the arc $AO = \alpha$, and equal initially to AE, where we assume initially that the pole of the rotation *E* to have been placed on the arc *AB*. Therefore from the preceding, if we put

$$\frac{\varepsilon(aa-cc)t\cos\alpha}{cc}=T,$$

then in the elapsed time equal to *t* the angle *BAO* = *T*; thus if we put the arc $BO=\beta$ and $CO=\gamma$, then

$$\cos\beta=\sin\alpha\cos T \quad \text{and} \quad \cos\gamma=\sin\alpha\sin T$$

on account of the right angle BAC. With these in place, [note the large type γ] $\gamma=\ \varepsilon$ from § 678 we will have these equations [where we have put γ' in place of γ]:

$$dl\sin l=\varepsilon dt\sin\alpha(\cos n\cos T-\cos m\sin T)$$
$$-d\lambda\sin^2 l=\varepsilon dt\sin\alpha(\cos m\cos T+\cos n\sin T)$$
$$dm\sin m=\varepsilon dt\sin\alpha(\cos l\sin T-\cos n\cot\alpha)$$
$$-d\mu\sin^2 m=\varepsilon dt\sin\alpha(\cos n\sin T+\cos l\cot\alpha)$$
$$dn\sin n=\varepsilon dt\sin\alpha(\cos m\cot\alpha-\cos l\cos T)$$
$$-d\nu\sin^2 n=\varepsilon dt\sin\alpha(\cos l\cot\alpha+\cos m\cos T),$$

which since they lead easier to integration, we consider the arc $ZO=v$, and since there arises

$$\cos v=\cos\alpha\cos l+\sin\alpha(\cos m\cos T+\cos n\sin T),$$

on differentiation this becomes

$$dv\sin v=dl\sin l\cos\alpha+dm\sin m\sin\alpha\cos T+dn\sin n\sin\alpha\sin T$$
$$+dT\sin\alpha\cos m\sin T-dT\sin\alpha\cos n\cos T,$$

but with these values put in place for $dl\sin l, dm\sin m, dn\sin n$ there becomes

$$dv\sin v=-dT\sin\alpha(\cos n\cos T-\cos m\sin T)=-dT\sin\alpha\cdot\frac{dl\sin l}{\varepsilon dt\sin\alpha}.$$

Therefore since then $dT=\frac{\varepsilon(aa-cc)dt\cos\alpha}{cc}$, there arises

$$dv\sin v=\frac{-(aa-cc)\cos\alpha}{cc}\cdot dl\sin l$$

and on integration,

$$\cos v=C-\frac{(aa-cc)\cos\alpha\cos l}{cc}\cdot=\cos\alpha\cos l+\sin\alpha(\cos m\cos T+\cos n\sin T),$$

Hence moreover it is permitted to conclude this particular integration, by putting the arc *l* constant, and

$$\cos m = \sin l \cos T$$

and

$$\cos n = \sin l \sin T,$$

in order that it becomes

$$\cos^2 l + \cos^2 m + \cos^2 n = 1,$$

and in a like manner $dl \sin l = 0$ satisfies the first equation, while the rest give :

$$dm \sin m = dT \sin l \sin T = \varepsilon dt \sin\alpha\left(\cos l \sin T - \cot\alpha \sin l \sin T\right),$$

$$dn \sin n = -dT \sin l \cos T = \varepsilon dt \sin\alpha\left(\cot\alpha \sin l \cos T - \cos l \cos T\right),$$

from each of these there is produced

$$dT \sin l = \varepsilon dt \sin\alpha\left(\cos l - \cot\alpha \sin l\right) = \frac{\varepsilon(aa-cc)dt \cos\alpha \sin l}{cc},$$

or

$$\sin\alpha \cos l - \cos\alpha \sin l = \frac{(aa-cc)\cos\alpha \sin l}{cc},$$

and hence

$$\text{tang } l = \frac{cc \text{ tang } \alpha}{aa},$$

but likewise the arc $ZO = v$ may be made constant, clearly

$$\cos v = \cos\alpha \cos l + \sin\alpha \sin l$$

consequently

$$ZO = v = \alpha - l$$

and

$$\text{tang } AZO = 0,$$

thus so that the points *A*, *Z* and *O* shall always be on the same great circle. Then for the position of the arc *ZA* now there is had

$$-d\lambda \sin^2 l = \varepsilon dt \sin\alpha \sin l,$$

and hence

$$\lambda = \frac{-\varepsilon t \sin\alpha}{\sin l}.$$

But on knowing the angle $XZA = \lambda$ the others $XZB = \mu$ and $XZC = v$ can be defined from these formulas :

$$\sin(\mu - \lambda) = \frac{-\cos n}{\sin l \sin m}, \quad \sin(v - \lambda) = \frac{\cos m}{\sin l \sin n}$$

or

$$\cos(\mu - \lambda) = \frac{-\cos l \cos m}{\sin l \sin m}, \quad \cos(v - \lambda) = \frac{-\cos l \cos n}{\sin l \sin n}$$

or

$$\text{tang}(\mu - \lambda) = \frac{\cos n}{\cos l \cos m} = \frac{\text{tang } T}{\cos l}$$

and

$$\mathrm{tang}(\nu-\lambda)=\frac{-\cot T}{\cos l}.$$

Moreover as this is a particular solution, we can elicit the general solution in the following manner.

GENERAL SOLUTION

We put

$$\cos m=\sin l\cos\Theta$$

and

$$\cos n=\sin l\sin\Theta,$$

in order that

$$\cos^2 l+\cos^2 m+\cos^2 n=1,$$

and then

$$dl\sin l=\varepsilon dt\sin\alpha\left(\sin l\sin\Theta\cos T-\sin l\cos\Theta\sin T\right)$$

or

$$dl=\varepsilon dt\sin\alpha\sin\left(\Theta-T\right)$$

then indeed there is obtained

$$d\Theta\sin l\sin\Theta=\varepsilon dt\sin\alpha\left(\cos l\cos\Theta\sin\left(\Theta-T\right)+\cos l\sin T-\cot\alpha\sin l\sin\Theta\right);$$

but because

$$T=\Theta-\left(\Theta-T\right)$$

then

$$\sin T=\sin\Theta\cos\left(\Theta-T\right)-\cos\Theta\sin\left(\Theta-T\right),$$

thus on being divided by $\sin\Theta$ there is

$$d\Theta\sin l=\varepsilon dt\sin\alpha\left(\cos l\cos\left(\Theta-T\right)-\cot\alpha\sin l\right).$$

Now we put in place

$$\Theta-T=\varphi,$$

then

$$d\Theta=d\varphi+\frac{\varepsilon\left(aa-cc\right)dt\cos\alpha}{cc}$$

and

$$d\varphi\sin l+\frac{\varepsilon\left(aa-cc\right)dt\cos\alpha\sin l}{cc}=\varepsilon dt\sin\alpha\cos l\cos\varphi-\varepsilon dt\cos\alpha\sin l$$

or

$$d\varphi\sin l+=\varepsilon dt\sin\alpha\cos l\cos\varphi-\frac{\varepsilon aadt\cos\alpha\sin l}{cc},$$

which equation must be joined with the preceding

$$dl=\varepsilon dt\sin\alpha\sin\varphi$$

and resolved, which indeed consist of the three variables *l*, *t* et φ, of which the middle one because of

$$\varepsilon dt = \frac{dl}{\sin\alpha \sin\varphi}$$

is easily eliminated ; for there arises

$$d\varphi \sin l = \frac{dl \cos l \cos\varphi}{\sin\varphi} - \frac{aadl \cos\alpha \sin l}{cc \sin\alpha \sin\varphi}$$

or
and the integral of this is

$$\sin l \cos\varphi = C - \frac{aa \cos\alpha \sin l}{cc \sin\alpha}$$

For the sake of brevity we put $\frac{aa \cos\alpha \sin l}{cc \sin\alpha} = D$, so that

$$\cos\varphi = \frac{C - D\cos l}{\sin l}$$

and

$$\sin\varphi = \frac{1}{\sin l}\sqrt{\left(1 - CC + 2CD\cos l - (1+DD)\cos^2 l\right)},$$

with which value substituted in the other equation there arises

$$\varepsilon dt = \frac{dl \sin l}{\sin\alpha\sqrt{\left(1 - CC + 2CD\cos l - (1+DD)\cos^2 l\right)}},$$

and the integral of this is :

$$\varepsilon t + E = \frac{1}{\sin\alpha\sqrt{(1+DD)}} A\sin\frac{CD - (1+DD)\cos l}{\sqrt{(1-CC+DD)}},$$

or

$$\frac{CD - (1+DD)\cos l}{\sqrt{(1-CC+DD)}} = \sin\left((\varepsilon t + E)\sin\alpha\sqrt{(1+DD)}\right)$$

thus at some time the arc $ZA = l$, and thus the angle $\varphi = \Theta - T$, and hence the angle $\Theta = \varphi + T$ becomes known, with which found then

$$\cos m = \sin l \cos\Theta \text{ et } \cos n = \sin l \sin\Theta .$$

Again there can be put in place

$$\cos ZO = \cos\alpha \cos l + \sin\alpha \sin l \cos\varphi = \cos\alpha \cos l + C\sin\alpha - D\sin\alpha \cos l,$$

or

$$\cos ZO = C\sin\alpha - \frac{(aa-cc)\cos\alpha \cos l}{cc}.$$

And hence we obtain for the angle $XZA = \lambda$:

$$-d\lambda \sin^2 l = \varepsilon dt \sin\alpha \sin l \cos\varphi$$

or

$$d\lambda = \frac{-\varepsilon dt \sin\alpha (C - D\cos l)}{\sin^2 l},$$

where if in place of εdt the above value is substituted, then there arises

$$d\lambda = \frac{-dl(1-\cos l)}{\sin l\sqrt{(1-CC+2CD\cos l-(1+DD)\cos^2 l)}},$$

the integral of this is elicited

$$\lambda = E + A\sin\frac{-D+C\cos l}{\sin l\sqrt{(1-CC+DD)}}$$

and thus everything has been determined in general.

COROLLARY 1

724. From the general solution there arises the particular solution elicited before, if the constant is put in place

$$C = \sqrt{(1+DD)};$$

for then in the equation

$$\varepsilon t + E = \frac{1}{\sin\alpha\sqrt{(1+DD)}} A\sin\frac{CD-(1+DD)\cos l}{\sqrt{(1-CC+DD)}},$$

on account of the denominator

$$\sqrt{(1-CC+DD)} = 0,$$

also the numerator

$$CD - (1+DD)\cos l$$

must vanish, thus it becomes

$$\cos l = \frac{D}{\sqrt{(1+DD)}}$$

and

$$\sin l = \frac{1}{\sqrt{(1+DD)}},$$

and thus

$$\text{tang } l = \frac{1}{D} = \frac{cc}{aa}\text{tang } \alpha.$$

COROLLARY 2

725. But with the constant taken as $C = \sqrt{(1+DD)}$
it becomes

$$\cos\varphi = \frac{\sqrt{(1+DD)} - D\cos l}{\sin l} = 1,$$

and thus $\varphi = 0$ and $\Theta = T$, thus there is deduced

$$\cos m = \sin l \cos T$$

and

$$\cos n = \sin l \sin T$$

and hence

$$\lambda = E + A\sin\frac{-D + \cos l\sqrt{(1+DD)}}{\sin l\sqrt{(1-CC+DD)}} = E + A\sin\frac{0}{0}.$$

Truly on account of $\varphi = 0$, so that this inconvenience can be avoided, the equation is taken

$$d\lambda \sin l = -\varepsilon dt \sin\alpha,$$

thus there becomes as before :

$$\lambda = E - \frac{\varepsilon t \sin\alpha}{\sin l}.$$

SCHOLIUM

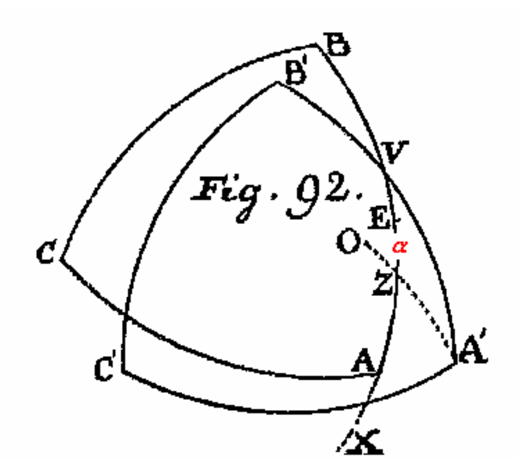
726. Thus the general solution involves arbitrary constants, in order that wherever the fixed point *Z* on the sphere may be taken on the fixed sphere, the solution can be adapted to that point. But since the point *Z* depends on our choice, that always can thus be accepted, so that the location for a particular solution can be had; which because it should be especially evident that the most simple motion provided is understood by us, as likewise the motion, if it is referred to other fixed points, must be seen to be greatly disturbed. Whereby we have not assumed this fixed point *Z* as it pleases but thus we have assumed it so that the solution comes about for that particular place.

PROBLEM 74

727. If initially a rotational motion were impressed on a body with two said equal principal axes about some axis passing through the centre of inertia, to determine the continued motion of this body.

SOLUTION

The centre of the sphere is considered to be at the centre of inertia of the body, which also remains at rest (Fig. 92); and in the beginning the principal axes of the body should be at *A*, *B*, *C*, of which the first moment of inertia about *IA* shall be equal to Maa, now about the remaining two axes it shall be equal to *Mcc* ; but then the rotational motion of the body can be taken about some axis *IE* in the sense *BCA*, with the angular speed equal to ε and let the arc $AE = \alpha$. Now so that we can investigate the continuation of the impressed motion of this, using in the particular solution, in the arc *AB*, that we regard as a fixed meridian on the sphere, *AZ* thus is taken, so that there arises



$$\text{tang } AZ = \frac{cc \text{ tang } \alpha}{aa}$$

and *Z* is taken for that fixed point, to which henceforth we will refer the position of the body, moreover we put $AZ = l$, so that $ZE = \alpha - l$. Now in an elapsed time equal to *t* the poles of the principal axes arrive at *A'*, *B'*, *C'* ; and at this stage we see that *ZA'* = *ZA* = *l*, and the point *O* is to be found in that same arc *A'Z*, about which the body now rotates with the angular speed equal to ε as the pole in the sense *B'C'A'*. But from the proceeding, where we put the angle *XZA* equal to λ , because the negative of this here denotes the angle *AZA'*, which at the start was equal to 0, then this angle

$$AZA' = \frac{\varepsilon t \sin \alpha}{\sin l},$$

thus at some time the position of the principal axis *IA'* becomes known. The two remaining poles shall be at *B'* and *C'*, and we have found in §717, to be the angle

$$ZA'B' = \frac{\varepsilon(aa-cc)t \cos \alpha}{cc},$$

and on taking the arc *A'B'* for a quadrant, *B'* will be one of the two remaining principal poles, thus at once the third pole *C'* is apparent.

COROLLARY 1

728. Therefore the principal axes *IA* is rotating uniformly about the fixed line *IZ*, but not related to the body ; thus in order that the arc $AZ = A'Z = l$ with being present

$$\text{tang } l = \frac{cc \text{ tang } \alpha}{aa},$$

and in the time *t* the angle is completed :

$$AZA' = \frac{\varepsilon t \sin\alpha}{\sin l},$$

therefore the angular speed of this motion in the sense *AA'* or *BCA* is equal to $\frac{\varepsilon \sin\alpha}{\sin l}$.

COROLLARY 2

729. But meanwhile the arc in the body *AB*, which in the beginning fell on *AZ* , thus is rotating about *ZA*, and while in the time *t* that it proceeds to *ZA'* , thus completes the angle

$$ZA'B' = \frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

hence the angular speed of this motion is equal to $\frac{\varepsilon(aa-cc)\cos\alpha}{cc}$.

COROLLARY 3

730. Hence the motion of the body can be represented as composed from two rotations. Clearly in the first the body is rotating about its own singular principal pole with an angular speed equal to $\frac{\varepsilon(aa-cc)\cos\alpha}{cc}$ in the sense *CB*; then meanwhile this pole itself *A* is rotating about the point *Z* in absolute space with an angular speed equal to $\frac{\varepsilon \sin\alpha}{\sin l}$.

COROLLARY 4

731. With the arc put as $ZA = l$, let the angular speed, by which the point *A* is rotating about the fixed point *Z* be equal to ζ , in the sense *AA'*, which two elements can be considered as given, then

$$\text{tang } \alpha = \frac{aa}{cc} \text{tang } l$$

and

$$\varepsilon = \frac{\zeta \sin l}{\sin\alpha}.$$

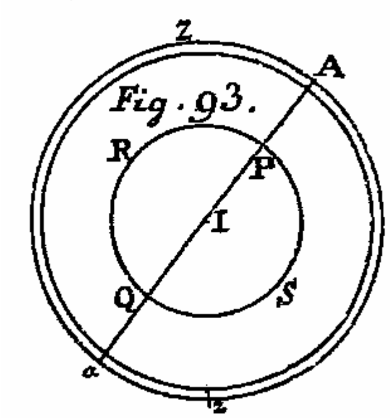
Hence the angular speed, by which meanwhile the arc *AB* rotates about *A* in the opposite sense, is given by $\frac{\zeta(aa-cc)\sin l}{cc \text{ tang } \alpha} = \frac{\zeta(aa-cc)\cos l}{aa}$.

SCHOLIUM 1

732. This motion of the body can be represented most conveniently in the same manner as we consider the rotational motion of the earth, in as much as the axes or poles progress in the heavens. Clearly the body is seen as the earth, one pole of which is *A*, but in the heavens the point *Z* is the pole of the ecliptic [*i. e*. the zenith], from which the pole of the earth maintains the same distance *ZA = l*, and about which it rotates with an angular speed equal to ζ in the sense *AA'*, which motion corresponds to the procession of the poles of the earth in the heavens. But meanwhile provided that the arc *AB* or *A'B'* is rotating about *A* or *A'*, from the arc *ZA* there is returned the angular speed equal to $\frac{\zeta(aa-cc)\cos l}{aa}$ in the sense *CB* , and this motion corresponds to the diurnal motion of the earth. But actually such a motion is in strong disagreement with the rotational motion of the earth, since here the motion of the meridian *AB* about the pole A shall be very slow sit with respect to the angular motion of the pole *A* about the fixed point *Z*, since contrarily on the earth the diurnal motion is much faster then the motion of the pole about the pole of the ecliptic. Therefore because if the motion of the poles of the earth about poles of the ecliptic should be faster, on the other hand now the motion of the earth about the poles would be much slower, the cause of this motion by no means can be agreed upon to be sought from external forces, since the earth may be able to be moved in such a motion on account of its own inertia. But now since the opposite may eventuate, the cause of this phenomena evidently is situated in external forces.

SCHOLIUM 2

733. This is certainly worth remembering, that the motion of the body, which is actually taking place about a variable axis *IO*, as if freely can be reduced to two rotational motions, but which are to be properly distinguished from each other in turn, while one is made about an axis present in the body, and the other about an axis as if present outside the body and related to absolute space. In order to set out that motion more clearly to the mind, the body *PRQS* is considered to be transfixed by a shaft *APQa* [really a spear in the original text], which passes through the centre of inertia *I* of this body, and refers to the singular principal axis of this (Fig. 93) : then the shaft with the ends *A* and *a* of this thus inserted into the annulus *ZAza*, so that the body can turn freely about that axis ; but the annulus may have pivots at the opposite poles *Z* and *z*, which are thus firmly attached to the outside, so that the annulus is equally free to rotate about these. [Such as the mountings of



a modern gyro' for a fixed angle.] Because if now the body *PRQS* is set in motion about the shaft *Aa* and likewise the annulus *AzaZ* rotates about the pivots *Z* and *z*, a motion of this kind arises, such as we have described here, where the shaft refers to the axis now present in the body and moving with the motion of the body, and truly the pivots *Z* and *z* are fixed to another axis outside the body. But these two rotational motions are in agreement with this, because each can be taken away from the other resulting in a true rotation about a fixed axis; for if the annulus is at rest, the body is rotating about the fixed axis PQ, about the axis at rest *Aa* ; but if the motion about the shaft should be taken away, and the body is only rotating about the pivots *Z* and *z*, a simple rotational motion arises in the body about the fixed axis pertaining to the pivots *Z* and *z*.

SCHOLIUM 3

734. Such motion is said to happen about a moveable axis, which is to be distinguished properly from the motion about a variable axis, such as we have considered in the preceding. For a body is said to be rotating about a variable axis, when it is lead continually to rotate about another line through the centre of inertia of this, which also at that instant is actually at rest; and everything is known about such an axis, which have been set out above concerning rotational motion. But when we say a body is rotating about a mobile axis, which idea now at last has arisen to be considered by us, indeed the axis is a certain invariable line present in the body, but which itself may be moving with the body; thus so that this mobile axes is never at rest. Thus the axis of the earth, which this name it is accustomed to bear, is not a variable axis but a mobile one, since there is a certain fixed line in the earth, but with time passing it points towards other and yet other points of the heavens; which therefore with the abstraction made from the motion of the earth for a year at no point of time does it remain at rest, even if the motion should be most slow. Now at some time a certain other line can be assigned in the earth, which then is actually at rest, but it shall be changing continually at successive times : and with respect of this the earth may be said to be rotating about a variable axis. But on account of the most slow motion of the equinoxes the difference between the diurnal motion from the true axis of the earth and the variable axis having been located at some particular time is nearly completely imperceptible; but which if it should be noteworthy , it would demand the greatest attention in astronomy, as the observations for the elevation of the pole put in place truly do not show a true position for the axis, but a variable axis in that time, about which clearly the rotating earth shall then be at rest.

PROBLEM 75

735. If some motion is impressed on a rigid body with the two said equal principal axes, and the body is not acted on by any external forces, so that neither is it retained anywhere nor unable to pursue its own free motion, to determine the motion by which it begins to move.

SOLUTION

First it is to be discerned, whether on account of the impressed force, the centre of inertia should move or not ? for if it should be moving, then the progressive motion of the body itself has to be considered, by which it is to progress uniformly along a line, and at least in the mind this motion can be removed, while clearly by a motion considered to be carried out in space in the opposite direction. Hence with the progressive motion removed, an account of this likewise has been provided, and if no other motion in addition should be present on the body, then the centre of inertia can be considered as being at rest; about which in some manner the body may be disturbed, a certain line drawn through that somehow initially will be at rest, which is the axis of rotation of this body. Then if this axis agrees with some principal axis, that is, if either it falls on the singular principal axis or it shall be normal to that, also here the motion remains uniform, and the axis is at rest, or by being joined to a progressive motion it moves away it moves away remaining parallel to itself. But if that axis, about which the body initially begins to rotate, neither agrees with the singular principal axis nor is normal to that, then the body begins to rotate about a variable axis, which continues to change in a manner that we have made abundantly clear. Also this motion is seen to be more clear by the reduction of that motion to a moving axes, by which the body is rotating uniformly about the principal singular axis, while this axis itself about some poles fixed beyond the body are carried around equally in a uniform motion.

SCHOLIUM

736. By this problem the general argument has been drawn out, that we have undertaken to be treated in this chapter, thus in order that in general the free motion is to be determined of rigid bodies with the two said equal principal axes and acted on by no forces and to be adjusted to whatever case we may wish to prevail. Hence bodies of the third class remain, the moments of inertia of which are unequal, which is to be resolved in the following chapter.

CAPUT XII

DE MOTU LIBERO CORPORUM RIGIDORUM DUABUS AXIBUS PRINCIPALIBUS PARIBUS PRAEDITORUM ET A NULLIS VIRIBUS SOLLICITATORUM

DEFINITIO 12

706. Corpus rigidum duos *axes principales pares* habere dicitur, quando inter se eius momenta inertiae respectu axium principalium duo sunt aequalia.

COROLLARIUM 1

707. Huius generis ergo corpora innumerabiles habent axes principales; statim enim ac duo axes principales aequalia habent momenta inertiae, omnes rectae in eorum plano per centrum inertiae ductae aeque pro axibus principalibus haberi possunt, eodemque momento inertiae sunt praeditae.

COROLLARIUM 2

708. Hic igitur axis ille principalis, cuius momentum ineritae reliquis est inaequale, erit singularis, atque omnes rectae per centrum inertiae ad eum normaler ductae paria habebunt momenta inertia, et tanquam axes principales spectare poterunt.

COROLLARIUM 3

709. Cognito itaque axe singulari, positio binorum reliquorum non determinatur, sed eorum loco pro lubitu binae rectae quacunque tam inter se quam ad illum normales accipi possunt, dummodo per centrum inertiae transeant.

SCHOLION

710. Cum igitur supra in genere pro axibus principalibus *IA*, *IB*, *IC* posuerimus momenta inertiae *Maa*, *Mbb*, *Mcc*, horum duo in isthoc capite aequalia statuamus. Sit igitur primus axis *IA* singularis, reliquorumque momenta inertiae inter se aequalia, ut sit $bb = cc$; ex quo formulae supra inventae [§664] mirifice contrahentur. Etsi autem hoc casu situs binorum axium *IB* et *IC* non determinatur, tamen eos tanquam determinatos spectabimus, ut eorum ope situs corporis ad quodvis tempus facilius assignari possit. Huius autem generis utique inifinita dantur corpora, atque inter homogenea imprimis huc pertinent cylindri, coni, atque in genere omnia corpora rotunda, quae conversione figurae cuiuscunque circa quempiam axem fixum nascuntur; ita ut hoc genus fere omnia corpora, quae quidem a geometris considerari solent, in se complectatur. Quemadmodum ergo haec corpora ratione motus se sint habitura, dum a nullis viribus sollicitantur, in hoc capite investigabimus, ac primo quidem ad quodvis tempus in positionem axis gyrationis ratione axium principalium inquiramus, nondum solliciti quemnam motum hi ipsi axes sint habituri, quem deinceps definire conabimur.

PROBLEMA 71

711. Si corpori rigido duobus axibus principalibus paribus praedito motus quicunque gyratorius initio fuerit impressus, neque ullae adsint vires externae, ad quodvis tempus positionem axis gyrationis ratione axium principalium assignare.

SOLUTIO

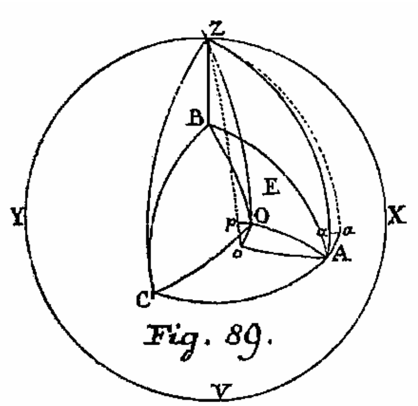


Fig. 89.

Centro inertiae corporis *I* in centro sphaerica (Fig. 89), ad cuius superficiem omnia reducamus, constituto sint *IA, IB, IC* axes corporis principales, ac respectu primi *IA* momentum inertiae = *Maa*, respectu binorum reliquorum autem *IB* et *IC* sint momenta inertiae inter se aequalia = *Mcc*, ut sit *bb = cc*. Nunc autem elapso ab initio tempore = *t*, corpus gyretur circa axem *IO* in sensum *ABC* celeritate angulari = γ', ita ut situs puncti *O* respectu punctorum *A, B, C* definiri debeat. Ponantur ergo arcus circulorum maximorum

$$OA=\alpha, OB=\beta \text{ et } OC=\gamma,$$

qui tanquam variabiles sunt tractandi, atque problema 66 ad hunc casum quo *bb = cc* translatum dabit primo $d\gamma'=0$, unde patet celeritatem angularem manere invariabilem, ideoque adhuc esse aequalem ei, quae initio corpori fuerit impessa. Quare si haec prima celeritas angularis ponatur = ε erit $\gamma'=\varepsilon$. Deinde vero ex § 674 habebimus has aequationes :

$$\text{I.}\quad aac^4 d\alpha \sin\alpha = 0$$

$$\text{II.}\quad aac^4 d\beta \sin\beta = \varepsilon aacc(aa-cc)\,dt\cos\alpha\cos\gamma$$

$$\text{III.}\quad aac^4 d\gamma \sin\gamma = \varepsilon aacc(cc-aa)\,dt\cos\alpha\cos\beta,$$

ex quarum prima discimus, arcum $AO=\alpha$ esse constantem, ideoque aequationes illi, quo initio illi, quo intio axis gyrationis distabit ab axe singulari *IA*. Cum igitur sit

$$\cos\gamma=\sqrt{(\sin^2\alpha-\cos^2\beta)},$$

reliquarum aequationum altera praebet :

$$\frac{d\beta\sin\beta}{\sqrt{(\sin^2\alpha-\cos^2\beta)}}=\frac{\varepsilon(aa-cc)dt\cos\alpha}{cc},$$

cuius integrale est

$$A\cos\frac{\cos\beta}{\sin\alpha}=C+\frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

ideoque

$$\cos\beta=\sin\alpha\cos\left(C+\frac{\varepsilon(aa-cc)t\cos\alpha}{cc}\right)$$

et

$$\cos\gamma=\sin\alpha\sin\left(C+\frac{\varepsilon(aa-cc)t\cos\alpha}{cc}\right).$$

Quare si initio, ubi t = 0, fuerit

$$AO=\mathfrak{A},\quad BO=\mathfrak{B}\quad \text{et}\quad CO=\mathfrak{C},$$

erit $\alpha=\mathfrak{A}$ et

$$\cos\mathfrak{B}=\sin\mathfrak{A}\sin C,$$

unde fit constans

$$\cos C=\frac{\cos\mathfrak{B}}{\sin\mathfrak{A}}$$

et

$$\sin C=\frac{\cos\mathfrak{C}}{\sin\mathfrak{A}}.$$

Quocirca habebimus

$$\cos\beta=\cos\mathfrak{B}\cos\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc}-\cos\mathfrak{C}\sin\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc}$$
$$\cos\gamma=\cos\mathfrak{B}\sin\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc}+\cos\mathfrak{C}\cos\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc},$$

unde si initio motus cognoverimus situm axis gyrationis respectu axium principalium, seu arcus $\mathfrak{A},\mathfrak{B}$ et $\mathfrak{C}$, pro quovis tempore elapso t situm axis gyrationis respectu eorumdem axium principalium seu arcus α,β,γ assignare valemus.

COROLLARIUM 1

712. Si igitur initio corpori impressus fuerit motus gyratorius circa axem *IE*, ad axes principales *IA, IB, IC* inclinatatum angulis $\mathfrak{A},\mathfrak{B},\mathfrak{C}$, celeritate angulari = ε in sensum *ABC* ; quomodocunque deinceps axis gyrationis varietur, celeritas angularis perpetuo manebit eadem = ε ; et axis gyrationis *IO* eodem angulo $\mathfrak{A}$ ad axem principalem singularum *IA* inclinabitur.

COROLLARIUM 2

713. Tum vero si momentum inertiae respectu axis singularis *IA* sit = *Maa*, respectu binorum reliquorum autem = *Mcc* pro tempore elapso = *t*, quia ε angulum denotat, ponatur angulus

$$\frac{\varepsilon(aa-cc)t\cos\mathfrak{A}}{cc}=T,$$

qui cum tempore t uniformite crescit; atque hoc tempore corpus gyrabitur circa axem IO, ut sit

$$AO = AE = \mathfrak{A}$$

et

$$\cos BO = \cos\mathfrak{B}\cos T - \cos\mathfrak{C}\sin T,$$

$$\cos CO = \cos\mathfrak{B}\sin T + \cos\mathfrak{C}\cos T.$$

COROLLARIUM 3

714. Quia arcus AO perpetuo manet aeque magnus = $\mathfrak{A}$, situs puncti O commodissime ex angulo BAO innotescet, et cum sit

$$\cos BAO = \frac{\cos BO}{\sin\mathfrak{A}}$$

et

$$\sin BAO = \frac{\cos CO}{\sin\mathfrak{A}},$$

erit

$$\cos BAO = \frac{\cos\mathfrak{B}\cos T - \cos\mathfrak{C}\sin T}{\sin\mathfrak{A}}$$

et

$$\sin BAO = \frac{\cos\mathfrak{B}\sin T + \cos\mathfrak{C}\cos T}{\sin\mathfrak{A}}.$$

COROLLARIUM 4

715. Si fuerit $cc = aa$, qui est casus ante tractatus, quo omnia tria momenta inertiae sunt inter se aequalia, erit $T = 0$, et $BO = \mathfrak{B}$, item $CO = \mathfrak{C}$, polus scilicet gyrationis O respectu axium principalium maneret immotus, uti iam ante invenimus.

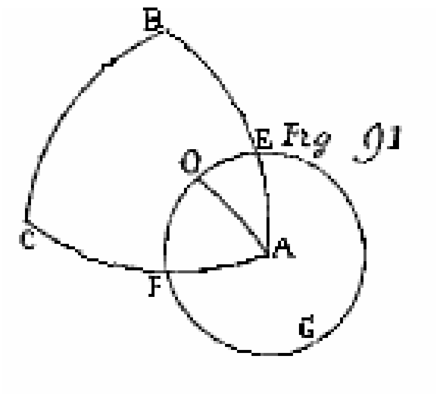
SCHOLION

716. Formulae hae multo simpliciores reddi possunt, sed rei dignitas mereretur, ut id potius singulari propositione quam in transitu prosequamur.

PROBLEMA 72

717. Iisdem positis, quae in praecedente problemate sunt constituta, definire promotionem poli gyrationis O respectu axium principalium.

SOLUTIO



Maneant omnia uti in praecedente solutione, et cum poli pares *B* et *C* in circulo *PC* pro lubitu accipi queant, quadrans *AB* ita constituatur, ut per polum *E*, circa quem corpus primum gyrati incipit, transeat (Fig. 91). Cum igitur hic polus gyratorius perpetuo eandem a polo principali *A* servet distantiam, eius motus fiet per circulum minorem *EFG* centro *A* descriptum, cuius distantia sit arcus $AE = \alpha$, quem supra per $\mathfrak{A}$ indicavimus. Erit ergo

$$BE = \mathfrak{B} = 90^o - \alpha$$

et

$$CE = \mathfrak{C} = 90^o.$$

Quare si elapso tempore = *t* polus gyrationis ex *E* pervenerit in *O*, ob $\cos\mathfrak{C} = 0,$ erit

$$\cos BAO = \frac{\cos\mathfrak{B}\cos T}{\sin\alpha} = \cos T$$

et

$$\sin BAO = \frac{\cos\mathfrak{B}\sin T}{\sin\alpha} = \sin T,$$

ideoque ipse

angulus *BAO* = *T*.

At angulus *T* ita ex tempore *t* definitur, ut sit

$$T = \frac{\varepsilon(aa-cc)t\cos\alpha}{cc} = BAO,$$

unde hanc egregiam solutionem consequimur. Si momentum inertiae respectu axis principalis singularis *IA* fuerit = *Maa* et respectu binorum reliquorum parium *IB* et *IC* = *Mcc*, corpus autem initio circa axem *IE* in sensum *BCA* celeritate angulari = ε gyrari coeperit; tum respectu axium principalium, quos quasi in quiete spectamus, polus gyrationis per circulum minorem *EFG* circa polum *A* descriptum uniformiter proferetur, ita ut elapso tempore = *t* conficiat angulum

$$EAO = \frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

motusque fiat in sensum *BC* conformem motui gyratorio, siquidem fuerit $aa > cc,$ in contrarium autem, si $aa < cc$.

COROLLARIUM 1

718. Polus gyrationis his casibus quiescet : 1^o si $AE = 0$ seu corpus circa axem principalem *IA* gyratari inceperit. 2^o si $AE = 90^o$ seu si corpus circa quemcunque axem ad *IA* normalem gyrari inceperit, ac 3^o si $aa = cc$, hoc est si corpus habuerit omnes tres axes principales pares.

COROLLARIUM 2

719. Si fuerit $aa > cc$, polus gyrationis *E* circa *A* in eundem sensum *BC*, in quem fit gyratio, circumferetur celeritate angulari = $\frac{\varepsilon(aa-cc)\cos AE}{cc}$; sin autem fuerit $aa < cc$, in sensum contrarium circumferetur celeritate angulari = $\frac{\varepsilon(cc-aa)\cos AE}{cc}$.

COROLLARIUM 3

720. Ipse autem arcus circuli minoris *EO*, per quem axis gyrationis tempore *t* procedit, est = $\frac{\varepsilon(aa-cc)t\sin AE\cos AE}{cc} = \frac{\varepsilon(aa-cc)t\sin 2AE}{2cc}$, quod ergo spatium ceteris paribus est maximus, si $AE = \frac{1}{2}AB = 45^o$, hoc est si axis gyrationis aequaliter distet ab axibus principalibus.

COROLLARIUM 4

721. Posita ratione diametri ad peripheram = $1 : \pi$, polus gyrationis totam circumferentiam *EFGE* percurret tempore = $\frac{2\pi cc}{\varepsilon(aa-cc)\cos AE}$ min.sec. huncque motum perpetuo uniformem conservabit.

SCHOLION

722. Hic nondum de ipso corporis motu agimus, sed quod probe est notandum, corpus, quasi quiesceret, vel aliud ipsi aequale in quiete contemplamur, in eoque ad quodvis tempus axem gyrationis *IO* definire docuimus, circa quem corpus motum tum sit gyraturum, neque hic sumus solliciti, quemnam situm hic axis gyrationis tum respectu spatii absoluti sit habiturus. Nunc igitur istam completam motus cognitionem aggrediamur.

PROBLEMA 73

723. Si corpori rigido duobus axibus principalibus praedito impressus fuerit initio motus gyratorius quicunque, ad datum tempus tam situm axium principalium quam gyrationis respectu spatii absoluti assignare.

SOLUTIO

Sphaera ex centro inertiae corpori circumscripta cingatur superficie sphaerica immobili *ZXVY* (Fig. 89), atque elapso tempore *t* sphaera mobilis cum corpore eum teneat situm, utaxium ternorum principalim poli sint in *A*, *B*, *C* , respectu quorum primi *IA* momentum inertiae sit = *Maa*, respectu autem binorum reliquorum = *Mcc*. Ductis inde ad punctum quoddam fixum *Z* arcubus *AZ*, *BZ* et *CZ*, ponamus ut in problemate 68

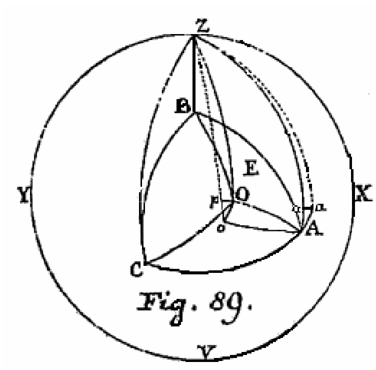


Fig. 89.

$$AZ = l, \quad BZ = m \quad \text{et} \quad CZ = n,$$

ut sit

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

tum vero sint anguli

$$XZA = \lambda, \quad XZB = \mu \quad \text{et} \quad XZC = \nu,$$

et quia motus gyratorius, uti iam ostendimus, manet aequabilis, sit eius celeritas angularis = ε in sensum *ABC* directa. Porro quoniam axis gyrationis perpetuo ab axis *IA* aeque maneat remotus. Sit arcus $AO = \alpha$, et aequalis initiali *AE*, ubi assumamus initio polum gyrationis *E* in ipso arcus *AB* positum fuisse. Ex praecedentibus ergo si ponamus

$$\frac{\varepsilon(aa - cc)t\cos\alpha}{cc} = T,$$

erit nunc elapso tempore = *t* angulus *BAO* = *T*; unde si ponamus arcus $BO = \beta$ et $CO = \gamma$, erit

$$\cos\beta = \sin\alpha\cos T \quad \text{et} \quad \cos\gamma = \sin\alpha\sin T$$

ob *BAC* angulum rectum. His positis ob $\gamma' = \varepsilon$ ex § 678 habemus has aequationes :

$$dl\sin l = \varepsilon dt\sin\alpha(\cos n\cos T - \cos m\sin T)$$

$$-d\lambda\sin^2 l = \varepsilon dt\sin\alpha(\cos m\cos T + \cos n\sin T)$$

$$dm\sin m = \varepsilon dt\sin\alpha(\cos l\sin T - \cos n\cot\alpha)$$

$$-d\mu\sin^2 m = \varepsilon dt\sin\alpha(\cos n\sin T + \cos l\cot\alpha)$$

$$dn\sin n = \varepsilon dt\sin\alpha(\cos m\cot\alpha - \cos l\cos T)$$

$$-d\nu\sin^2 n = \varepsilon dt\sin\alpha(\cos l\cot\alpha + \cos m\cos T),$$

quae quo facilius ad integrationem perduci queant, consideremus arcum $ZO = v$, et cum sit

$$\cos v = \cos\alpha \cos l + \sin\alpha\left(\cos m \cos T + \cos n \sin T\right),$$

erit differentiando

$$dv \sin v = dl \sin l \cos\alpha + dm \sin m \sin\alpha \cos T + dn \sin n \sin\alpha \sin T$$
$$dT \sin\alpha \cos m \sin T - dT \sin\alpha \cos n \cos T,$$

substitutis autem pro $dl \sin l, dm \sin m, dn \sin n$ illis valoribus fit

$$dv \sin v = -dT \sin\alpha\left(\cos n \cos T - \cos m \sin T\right) = -dT \sin\alpha \cdot \frac{dl \sin l}{\varepsilon dt \sin\alpha}.$$

Cum igitur sit $dT = \frac{\varepsilon(aa-cc)dt \cos\alpha}{cc}$, oritur

$$dv \sin v = \frac{-(aa-cc)\cos\alpha}{cc} \cdot dl \sin l$$

et integrando

$$\cos v = C - \frac{(aa-cc)\cos\alpha \cos l}{cc} \cdot = \cos\alpha \cos l + \sin\alpha\left(\cos m \cos T + \cos n \sin T\right),$$

Hinc autem concludere licet integrationem particularem, ponendo arcum l constantem, et

$$\cos m = \sin l \cos T$$

atque

$$\cos n = \sin l \sin T,$$

ut fiat

$$\cos^2 l + \cos^2 m + \cos^2 n = 1,$$

simulque primae aequationi $dl \sin l = 0$ satisfiat, reliquae vero dabunt :

$$dm \sin m = dT \sin l \sin T = \varepsilon dt \sin\alpha\left(\cos l \sin T - \cot\alpha \sin l \sin T\right),$$
$$dn \sin n = -dT \sin l \cos T = \varepsilon dt \sin\alpha\left(\cot\alpha \sin l \cos T - \cos l \cos T\right),$$

ex quarum utraque prodit

$$dT \sin l = \varepsilon dt \sin\alpha\left(\cos l - \cot\alpha \sin l\right) = \frac{\varepsilon(aa-cc)dt \cos\alpha \sin l}{cc},$$

seu

$$\sin\alpha \cos l - \cos\alpha \sin l = \frac{(aa-cc)\cos\alpha \sin l}{cc},$$

hincque

$$\text{tang } l = \frac{cc \text{ tang } \alpha}{aa},$$

simul autem arcus $ZO = v$ fiet constans, nempe

$$\cos v = \cos\alpha \cos l + \sin\alpha \sin l$$

consequenter

$$ZO = \nu = \alpha - l$$

et

$$\text{tang } AZO = 0,$$

ita ut puncta *A*, *Z* et *O* semper sint in eodem circulo maximo. Deinde vero pro situ arcus *ZA* habebitur

$$-d\lambda \sin^2 l = \varepsilon dt \sin\alpha \sin l,$$

hincque

$$\lambda = \frac{-\varepsilon t \sin\alpha}{\sin l}.$$

Cognito autem angulo $XZA = \lambda$ reliqui $XZB = \mu$ et $XZC = \nu$ ex his formulis definientur :

$$\sin(\mu - \lambda) = \frac{-\cos n}{\sin l \sin m}, \quad \sin(\nu - \lambda) = \frac{\cos m}{\sin l \sin n}$$

seu

$$\cos(\mu - \lambda) = \frac{-\cos l \cos m}{\sin l \sin m}, \quad \cos(\nu - \lambda) = \frac{-\cos l \cos n}{\sin l \sin n}$$

seu

$$\text{tang}(\mu - \lambda) = \frac{\cos n}{\cos l \cos m} = \frac{\tan\text{g } T}{\cos l}$$

et

$$\text{tang}(\nu - \lambda) = \frac{-\cot T}{\cos l}.$$

Cum autem haec solutio sit particularis, generalem sequenti modo eliciemus.

SOLUTIO GENERALIS

Ponamus

$$\cos m = \sin l \cos\Theta$$

et

$$\cos n = \sin l \sin\Theta,$$

ut sit

$$\cos^2 l + \cos^2 m + \cos^2 n = 1,$$

eritque

$$dl \sin l = \varepsilon dt \sin\alpha (\sin l \sin\Theta \cos T - \sin l \cos\Theta \sin T)$$

sive

$$dl = \varepsilon dt \sin\alpha \sin(\Theta - T)$$

tum vero habebitur

$$d\Theta \sin l \sin\Theta = \varepsilon dt \sin\alpha \left(\cos l \cos\Theta \sin(\Theta - T) + \cos l \sin T - \cot\alpha \sin l \sin\Theta\right);$$

at ob

$$T = \Theta - (\Theta - T)$$

est

$$\sin T = \sin\Theta \cos(\Theta - T) - \cos\Theta \sin(\Theta - T),$$

unde per $\sin\Theta$ dividendo erit

$$d\Theta \sin l = \varepsilon dt \sin\alpha\left(\cos l \cos\left(\Theta - T\right) - \cot\alpha \sin l\right).$$

Statuamus iam

$$\Theta - T = \varphi,$$

erit

$$d\Theta = d\varphi + \frac{\varepsilon\left(aa - cc\right)dt \cos\alpha}{cc}$$

et

$$d\varphi \sin l + \frac{\varepsilon\left(aa - cc\right)dt \cos\alpha \sin l}{cc} = \varepsilon dt \sin\alpha \cos l \cos\varphi - \varepsilon dt \cos\alpha \sin l$$

seu

$$d\varphi \sin l + = \varepsilon dt \sin\alpha \cos l \cos\varphi - \frac{\varepsilon aadt \cos\alpha \sin l}{cc},$$

quae aequatio cum praecedente

$$dl = \varepsilon dt \sin\alpha \sin\varphi$$

est coniungenta et resolvenda, quae quidem continent tres variabiles *l, t* et φ, quarum media ob

$$\varepsilon dt = \frac{dl}{\sin\alpha \sin\varphi}$$

facile eliminatur; oritur enim

$$d\varphi \sin l = \frac{dl \cos l \cos\varphi}{\sin\varphi} - \frac{aadl \cos\alpha \sin l}{cc \sin\alpha \sin\varphi}$$

seu
cuius integrale est

$$\sin l \cos\varphi = C - \frac{aa \cos\alpha \sin l}{cc \sin\alpha}$$

Statuamus brevitatis gratia $\frac{aa \cos\alpha \sin l}{cc \sin\alpha} = D$, ut sit

$$\cos\varphi = \frac{C - D\cos l}{\sin l}$$

et

$$\sin\varphi = \frac{1}{\sin l}\sqrt{\left(1 - CC + 2CD \cos l - \left(1 + DD\right)\cos^2 l\right)},$$

quo valore in altera aequatione substituto oritur

$$\varepsilon dt = \frac{dl \sin l}{\sin\alpha\sqrt{\left(1 - CC + 2CD \cos l - \left(1 + DD\right)\cos^2 l\right)}},$$

cuius integrale est

$$\varepsilon t + E = \frac{1}{\sin\alpha\sqrt{(1+DD)}} A\sin\frac{CD-(1+DD)\cos l}{\sqrt{(1-CC+DD)}},$$

seu

$$\frac{CD-(1+DD)\cos l}{\sqrt{(1-CC+DD)}} = \sin\left((\varepsilon t + E)\sin\alpha\sqrt{(1+DD)}\right)$$

unde ad quodvis tempus arcus $ZA = l$, indeque angulus $\varphi = \Theta - T$, hincque angulus $\Theta = \varphi + T$ innotescit, quo invento erit

$$\cos m = \sin l \cos\Theta \text{ et } \cos n = \sin l \sin\Theta .$$

Porro fiet

$$\cos ZO = \cos\alpha\cos l + \sin\alpha\sin l\cos\varphi = \cos\alpha\cos l + C\sin\alpha - D\sin\alpha\cos l,$$

seu

$$\cos ZO = C\sin\alpha - \frac{(aa-cc)\cos\alpha\cos l}{cc}.$$

Denique pro angulo $XZA = \lambda$ obtinemus :

$$-d\lambda\sin^2 l = \varepsilon dt\sin\alpha\sin l\cos\varphi$$

seu

$$d\lambda = \frac{-\varepsilon dt\sin\alpha(C-D\cos l)}{\sin^2 l},$$

ubi si loco εdt superior valor substituatur, provenit

$$d\lambda = \frac{-dl(1-\cos l)}{\sin l\sqrt{(1-CC+2CD\cos l-(1+DD)\cos^2 l)}},$$

cuius integrale elicitur

$$\lambda = E + A\sin\frac{-D+C\cos l}{\sin l\sqrt{(1-CC+DD)}}$$

sicque omnia in genere sunt determinata.

COROLLARIUM 1

724. Ex solutione generali nascitur solutio particularis prius eruta, si ponatur constans

$$C = \sqrt{(1+DD)};$$

tum enim in aequatione

$$\varepsilon t + E = \frac{1}{\sin\alpha\sqrt{(1+DD)}} A\sin\frac{CD-(1+DD)\cos l}{\sqrt{(1-CC+DD)}},$$

ob denominatorem

$$\sqrt{(1-CC+DD)} = 0,$$

etiam numerator

$$CD-(1+DD)\cos l$$

evanescere debet, unde fit

$$\cos l=\frac{D}{\sqrt{(1+DD)}}$$

et

$$\sin l=\frac{1}{\sqrt{(1+DD)}},$$

ideoque

$$\text{tang } l=\frac{1}{D}=\frac{cc}{aa}\text{tang } \alpha.$$

COROLLARIUM 2

725. Sumta autem constante

$$C=\sqrt{(1+DD)}$$

fit

$$\cos\varphi=\frac{\sqrt{(1+DD)}-D\cos l}{\sin l}=1,$$

ideoque $\varphi=0$ et $\Theta=T$, unde collititur

$$\cos m=\sin l\cos T$$

et

$$\cos n=\sin l\sin T$$

ac denique

$$\lambda=E+A\sin\frac{-D+\cos l\sqrt{(1+DD)}}{\sin l\sqrt{(1-CC+DD)}}=E+A\sin\frac{0}{0}.$$

Verum ob $\varphi=0$, ad hoc incommodum evitandum, sumatur aequatio

$$d\lambda\sin l=-\varepsilon dt\sin\alpha,$$

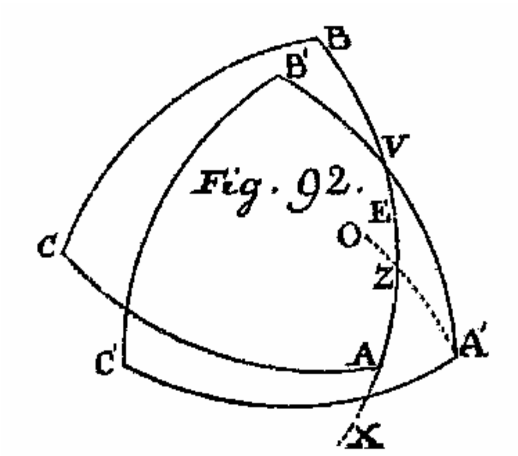
unde fit ut ante

$$\lambda=E-\frac{\varepsilon t\sin\alpha}{\sin l}.$$

SCHOLION

726. Solutio generalis ideo tot involvit constantes arbitrarias, ut ubicunque punctum fixum Z in sphaera immobili accipiatur, ad id possit accommodari. Cum autem punctum Z ab arbitrio nostro pendeat, id semper ita accipere licebit, ut pro eo solutio particularis locum sit habitura; quae cum sit simplicissima maxime nobis perspicuam cognitionem motus largietur, cum idem motus, si ad alia puncta fixa referretur, vehementer perturbatus videri debeat. Quare punctum hoc fixum Z non pro lubitu sed ita assumamus, ut solutio illa particularis locum inveniat.

PROBLEMA 74



727. Si corpori rigido duobus axibus principalibus paribus praedito impressus fuerit initio motus gyratorius circa axem quemcunque per centrum inertiae transeuntem, motus huius continuationem determinare.

SOLUTIO

In centro sphaero immobilis concipiatur centrum inertiae corporis, quod etiam quiescit (Fig. 92); atque initio axes corporis principales fuerint in *A*, *B*, *C*, quorum primi *IA* respectu momentum inertiae sit = *Maa*, respectu vero binorum reliquorum = *Mcc* ; tum autem acceperit corpus motum gyratorium circa axem *IE* in sensum *BCA*, celeritate angulari $=\varepsilon$ sitque arcus $AE=\alpha$. Quo nunc huius motus impressi continuationem investigemus, solutione particulari utentes, in arcu *AB*, quem in sphaerica immobili tanquam meridianum fixum spectemus, capiatur *AZ* ita, ut sit

$$\text{tang } AZ=\frac{cc \text{ tang } \alpha}{aa}$$

sumaturque *Z* pro puncto illo fixo, ad quod deinceps situm corporis perpetuo referamus, ponamus autem $AZ=l$, ut sit $ZE=\alpha-l$. Iam elapso tempore = *t* pervenerint poli axium principalium in *A'*, *B'*, *C'* ; et vidimus fore adhuc *ZA'* = *ZA* = *l*, et in eodem arcu *A'Z* reperiri punctum *O*, circa quod tanquam polum corpus nunc gyretur celeritate angulari $=\varepsilon$ in sensum *B'C'A'*. Ex praecedintibus autem, ubi angulum *XZA* posuimus $=\lambda$, quoniam eius negativum hic angulum *AZA'* denotat, qui initio erat = 0, erit hic angulus

$$AZA'=\frac{\varepsilon t \sin\alpha}{\sin l},$$

unde ad quodvis tempus posito axis principalis *IA'* cognoscitur. Sint bini reliqui in *B'* et *C'*, atque §717 invenimus, fore angulus

$$ZA'B'=\frac{\varepsilon(aa-cc)t \cos\alpha}{cc},$$

et sumto arcu *A'B'* quadrante, erit *B'* alter duorum reliquorum polorum principalium, unde sponte tertius *C'* patet.

COROLLARIUM 1

728. Axis ergo principalis *IA* uniformiter gyratur circa lineam *IZ* fixam, sed non ad corpus pertinentem; ita ut sit arcus *AZ* = *A'Z* = *l* existente

$$\text{tang } l=\frac{cc \text{ tang } \alpha}{aa},$$

et tempore *t* absolvatur angulus

$$AZA'=\frac{\varepsilon t \sin\alpha}{\sin l},$$

cuius ergo motus celeritas angularis in sensum AA' seu BCA erit $=\frac{\varepsilon \sin\alpha}{\sin l}$.

COROLLARIUM 2

729. Interea autem arcus in corpore *AB*, qui initio in *AZ* cadebat, ita circa *ZA*, dum tempore *t* in *ZA'* procedit, gyratur, ut conficiat angulum

$$ZA'B'=\frac{\varepsilon(aa-cc)t\cos\alpha}{cc},$$

cuius ergo motus celeritas angularis est

$$=\frac{\varepsilon(aa-cc)\cos\alpha}{cc}.$$

COROLLARIUM 3

730. Motus ergo corporis potest repraesentari tanquam compositus ex duplici gyratorio. Primo scilicet corpus gyrabitur circa suum polum principalem singularem *A* celeritate angulari

$$=\frac{\varepsilon(aa-cc)\cos\alpha}{cc}$$

in sensum *CB*; tum vero interea ipse hic polus *A* gyrabitur circa punctum *Z* in spatio absoluto fixum celeritate angulari

$$=\frac{\varepsilon\sin\alpha}{\sin l}.$$

COROLLARIUM 4

731. Posito arcu *ZA* = *l*, sit celeritas angularis, qua punctum *A* circa punctum fixum *Z* gyratur $=\zeta$, in sensum *AA'*, quae duo elementa ut data considerentur, erit

$$\text{tang }\alpha=\frac{aa}{cc}\text{tang }l$$

et

$$\varepsilon=\frac{\zeta\sin l}{\sin\alpha}.$$

Hinc celeritas angularis, qua interea arcus *AB* circa *A* gyratur in sensum contrarium, erit

$$=\frac{\zeta(aa-cc)\sin l}{cc\text{ tang }\alpha}=\frac{\zeta(aa-cc)\cos l}{aa}.$$

SCHOLION 1

732. Hic corporis motus commodissime eodem modo repraesentari potest, quo motum vertiginis terrae concipimus, quatenus axis seu poli in coelo progrediuntur. Corpus nempe tanquam terra spectetur, cuius alter polus sit *A*, in coelo autem punctum *Z* polus eclipticae, a quo polus terrae constanter eandem servet distantiam *ZA* = *l*, et circa quem gyretur celeritate angulari $=\zeta$ in sensum *AA'*, qui motus respondet processui poli terrestis in coelo. Interea autem dum arcus *AB* vel *A'B'* gyratur circa *A* vel *A',* ab arcu *ZA* rededens in sensum *CB* celeritate angulari

$$=\frac{\zeta(aa-cc)\cos l}{aa},$$

hic motus respondebit motui diurno terrae. Revera autem talis motus maxime discrepat a motu vertiginis terrae, cum hic motus meridiani *AB* circa polum A sit admodum lentus respectu motus angularis poli *A* circa punctum fixum *Z*, cum contra in terra motus diurnus sit velocissimus prae motu poli circa polum eclipticae. Quodsi ergo motus polorum terrae circa polos eclipticae esset velocissimus, contra vero motus vertiginis circa polos terrae tardissimus, causam huius motus neutiquam in viribus externis quaeri conveniret, cum terra per se ob inertiam tali motu cieri posset. Nunc autem cum contrarium eveniat, huius phoenomeni causa manifesto in viribus externis, quibus terra sollicitatur, est sita.

SCHOLION 2

733. Memoratu hic omnino dignum est, quod motus corporis, qui revera circa axem variabilem *IO* fiebat, quasi sponte reductus fuerit ad binos motus gyratorios, qui autem probe a se invicem sunt distinguendi, dum alter fit circa axem verum et in corpore existentem, alter vero circa axem quasi extra corpus existentem et ad spatium absolutum relatum. Ad quem motum clarius menti exponendum, corpus *PRQS* hasta *APQa* transfixum concipiatur, quae per eius centrum inertiae *I* transeat, eiusque axem principalem singularem referat (Fig. 93) : tum vero hasta terminis suius *A* et *a* ita annulo *ZAza* inseratur, ut corpus libere circa eam gyrari queat; annulus autem in punctis oppositis *Z* et *z* habeat cardines, qui extrinsecus ita firmiter retineantur, ut annulus circa eos pariter libere circumferri possit. Quod si iam corpus *PRQS* circa hastam *Aa* in gyrum agatur simulque annulus *AzaZ* circa cardines *Z* et *z* circumferatur, eiusmodi motus orietur, qualem hic descripsimus, ubi hasta refert axem verum in corpore existentem et cum corpore motum, cardines vero *Z* et *z* axem alterum extra corpus fixum. Bini autem hi motus gyratorii in hoc conveniunt, quod uterque altero sublato abeat in verum motum gyratorium circa axem fixum; si enim annulus quiescat, corpus circa hastam quiescentem Aa seu axem *PQ* fixum gyrabitur; sin autem dematur motus circa hastam, solusque annulus circa cardines *Z* et *z* gyretur, in corpore orietur motus gyratorius simplex circa axem fixum ad cardines *Z* et *z* pertingentem.

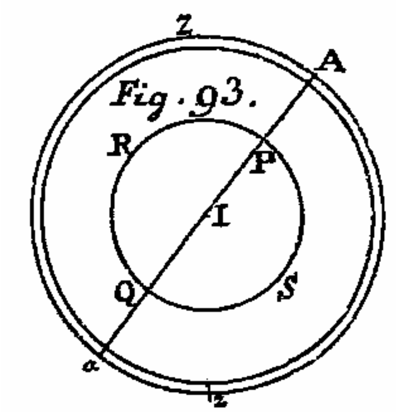


Fig. 93.

SCHOLION 3

734. Talis motus fieri dicitur circa axem mobilem, qui probe distinguendus est a motu circa axem variabilem, qualem in praecedentibus consideravimus. Corpus enim circa axem variabilem gyrari dicitur, quando continuo circa aliam lineam per eius centrum inertiae ductam gyratur, quae etiam eo instanti revera quiescat; atque de tali axe omnia sunt intellegenda, quae supra de motu gyratorio sunt exposita. Quando autem dicimus corpus circa axem mobilem gyrari, quae idea nunc demum nobis nata est censenda, axis quidem erit certa quaedam linea in corpore existens invariabilis, quae autem ipsa cum corpore moveatur. ita ut iste axis mobilis nunquam quiescat. Ita axis terrae, qui hoc nomen gerere solet, non est axis variabilis sed mobilis, cum in terrae sit linea quaedam fixa, sed labente tempore ad alia atque alia coeli puncta dirigatur; qui ergo etiam abstratione facta a motu terrae annuo nullo temporis puncto quiescit, etiamsi eius motus sit tardissimus. Verum quovis tempore alia quaedam linea in terra assignari potest, quae tum revera quiescat, successu temporis autem continuo mutetur : huiusque respectu terra circa axem variabilem gyrati est dicenda. Ob motum autem aequinoctiorum tardissimum prae motu diurno differentia inter verum terrae axem et axem variabilem quovis tempore locum habentem fere penitus est imperceptibilis; quae autem si esset notabilis, in Astronomia summam attentionem postularet, cum observationes pro elevatione poli institutae non situm axis veri, sed axis variabilis eo tempore ostendant, circa quem scilicet tum quiescentem terra gyretur.

PROBLEMA 75

735. Si corpori rigido duobus axibus principalibus paribus praedito motus quicunque imprimatur, corpusque a nullis viribus externis sollicetur, neque usquam retineatur, quominus motum suum libere prosequi possit, determinare motum, quo moveri perget.

SOLUTIO

Primum dispiciatur, utrum ob motum impressum centrum inertiae moveatur nec ne ? si enim moveatur, corpus habebit motum progressivum seorsim considerandum, quo uniformiter in directum progredietur, atque mente saltem hunc motum tollere licebit, dum scilicet ipsum spatium motu contrario proferri concipiatur. Sublato ergo motu progressivo, cuius ratio perinde est comparata, ac si praeterea nullus alius motus in corpore inesset, centrum corporis inertiae tanquam quiescens considerari poterit; circa quod quomodocunque corpus agitetur, linea quaepiam recta per id ducta primo saltem initio quiescet, quae eius erit axis gyrationis. Tum si iste axis conveniat cum aliquo axium principalium, hoc est, si vel incidat in axem principalem singularem vel ad eum sit normalis, etiam hic motus manebit aequabilis, axisque quiescet, vel adiuncto motu progressivo sibi fugiter manebit parallelus. At si axis ille, circa quem corpus primum gyrari coepit, neque cum axe principali singulari congruat, neque ad eum sit normalis, corpus circa axem variabilem gyrabitur, qui modo continuo varietur, in praecedentibus abunde ostendimus. Clarius etiam hic motus perspicietur per reductionem illam ad axem mobilem, qua corpus circa axem principalem singularem aequabiliter gyratur, dum ipse hic axis circa quosdam polos extra corpus fixos circumfertur pariter motu uniformi.

SCHOLION

736. Hoc problemate universum argumentum, quod hoc capite tractandum suscepimus, exhauritur, ita ut corporum rigidorum duobus axibus principalibus paribus praeditorum et a nullis viribus sollicitatorum motus liberos in genere determinare atque ad quosvis casus accommodare valeamus. Supersunt ergo corpora tertiae classis, quorum momenta inertiae principalia sunt inaequalia, quibus sequens caput destinatur.