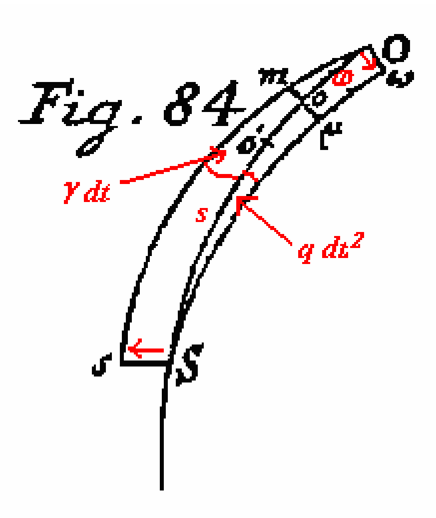
Chapter 10

CONCERNING THE MOMENTARY CHANGE IN THE AXIS OF ROTATION PRODUCED BY FORCES

PROBLEM 62

650. If a rigid body, while it is turning about an axis passing through the centre of inertia, is acted on by forces of such a kind, that if the body itself should be at rest then these forces themselves impress a rotational motion about some other axis : to determine the change of the motion produced in the smallest increment of time.

SOLUTION



Since just as in the motion now in place, so in that motion which may be impressed by the forces, the centre of inertia also remains at rest. Therefore the centre of inertia *I* can be considered as the centre of a sphere on the surface of which is the pole *O*, and the axis *IO* is considered, about which the body now is rotating with an angular speed equal to γ, (Fig. 84) in the sense that the point *S* is carried to *s*. [*Note* : all the points shown are on the surface of this sphere, and the two elemental spherical triangles shown are to be investigated, to which the sine rule is applied, where the two axes of rotation define a plane *ISO* cutting the surface of the sphere in a great circle. Triangle *OsS* is of first order, while $OS\omega$ is of second order in the differentials; essentially the first is a uniform rotation, while the second is an accelerated rotation starting from rest. Initially O is at rest and no rotation takes place about *S*; but at a later increment of time, the opposing motions due to the rotations about *O* and *S* give rise to a position of the axis of rotation instantaneously at rest at some other point *o,* while *O* progresses towards *S* and is at *o*', where *oo'* = *oO*.]
Then the body is acted on now by forces of such a kind that, if the body were at rest, it would rotate about the pole of the axis *IS* and in the increment of time *dt* it is turned through an angle qdt^2, since we have considered this angle to be homogeneous in the square of the time, and it makes this rotation in the sense

that the point O is carried towards ω. Hence the angle is given that these two axes OI and SI constitute at I, or on the spherical surface the arc of the great circle OS is put as $OS = s$; and in the element of time dt this arc OS on account of the motion in place, rotates about the pole O through the angle $SOs = \gamma dt$ arriving at the position Os, in order that the element of arc $Ss = \gamma dt \sin s$ [from the spherical triangle sine rule]. Moreover on account of the impressed motion, the same arc OS rotates about the pole S through the angle $OS\omega = qdt^2$ arriving at the position $S\omega$, so that the arclet shall be $O\omega = qdt^2 \sin s$. Therefore from this motion to both places simultaneously, the point S it transferred to s and the point O to ω, because neither translation disturbs the other ; but all the remaining points participate in each motion. Clearly any point o taken in the arc OS, so that $Oo = \omega$, on account of the motion in place about O is carried to m, so that the distance becomes $om = \gamma dt \sin \omega$, but on account of the motion arising about S it is carred to μ, so that this arc becomes

$$o\mu = qdt^2 \cdot \sin(s - \omega).$$

Since now it shall be the case that either $om > o\mu$ or $o\mu > om$, the point o on account of each motion is carried jointly by the difference of these arclets either towards m or towards μ. Whereby, if it should be that $om = o\mu$, the point o actually remains at rest and therefore is the pole, about which the body is thus agreed now to be rotating, so that the axis of rotation is transferred by the forces acting in the element of time dt from IO to Io. Therefore, we put the momentary change of the axis found to be $om = o\mu$ or

$$\gamma dt \sin \omega = qdt^2 \cdot \sin(s - \omega),$$

then

$$\gamma \sin \omega = qdt \sin s \cos \omega - qdt \cos s \sin \omega,$$

now the arclet present $Oo = \omega$ is indefinitely small, and thus $\sin \omega = \omega$ and $\cos \omega = 1$, and hence

$$\omega = \frac{qdt \sin s}{\gamma + qdt \cos s} = \frac{qdt \sin s}{\gamma}.$$

But the body is rotating about this axis Io with an angular speed so great, that in the time dt the points O and S are transferred to ω and s, thus that is to be understood. Since indeed in that element of time dt, an angle is completed equal to

$$\frac{O\omega}{Oo} = \frac{qdt^2 \sin s}{\omega} = dt(\gamma + qdt \cos s),$$

but in the preceding element of time on account of a similar force acting, as clearly the force cannot be thought to be exerted suddenly, the angle completed must be agreed to be equal to $dt(\gamma - qdt\cos s)$, thus so that the difference in the angles shall be $2qdt^2 \cos s$, and thus the increase in the angular speed itself is taken as $2qdt\cos s$; and by a similar account, because the value q which is introduced to be defined for continuous variations, must be doubled, also the element of distance *Oo* must be agreed to be twice as large. For while in the calculation, the point *O* is assumed to be progressing continually, moreover the point *o* is taken as being at rest, the interval *Oo* found here is different from that small interval, through which the pole of the rotation is carried forwards: for the point *o'* is considered, so that $Oo' = 2Oo$, and I say that *o'* is the pole of the rotation after the time *dt*, as at the start it was *O*. [Thus, the new position of the moving pole *O* is *o'*, twice the angle, as the angular speed is twice as great as for the single motion with no forces; while the point at rest is *o*, at which the two rotary motions cancel.] For at this position it is evident that meanwhile the point *o* remains at rest. Whereby, since here we have found $Oo = \frac{qdt\sin s}{\gamma}$, the small interval *Oo*', through which it has been agreed the pole of the rotation passes through, is twice as great and is equal to $\frac{2qdt\sin s}{\gamma}$. Hence the forces, if the body were at rest, which start the rotational motion about the axis *IS* in the sense $O\omega$, from which in the element of time *dt* an angle $OS\omega = qdt^2$ is completed, thus now disturb the motion of the rotating body in place about the axis *IO* in the sense *Ss* with an initial angular speed equal to γ, so that in the elapsed element of time *dt* the axis of rotation [due to both rotations] becomes the line *Io*, turning from the preceding axis *IO* towards *IS* by an angle

$$OIo = \frac{2qdt\sin s}{\gamma},$$

and likewise the rotary speed γ is taken to increase by an amount equal to $2qdt\cos s$.

COROLLARY 1

651. If the forces were acting in the opposite sense, the quantity *q* must be taken negatively and the point *o* falls in the arc *SO* produced beyond *O* and the rotary speed is diminished.

COROLLARY 2

652. If the arc *OS* either vanishes or is made equal to a semicircle, then the axis of rotation *IO* is not changed, but the total effect is taken up in increasing or decreasing the first rotational motion. Which is the case now treated above, where we have shown the increment or decrement in the speed to be $2qdt$.

COROLLARY 3

653. If the arc *OS* is the quadrant of the circle and thus $\cos s = 0$, then the angular speed γ will undergo no change, but the total effect of the forces will be devoted to the change in the axis of rotation, thus by moving that either closer to or further from *S*.

SCHOLIUM 1

654. Here we taken only forces of this kind to be considered, which for a body if it should be at rest, start a simple rotational motion, with the centre of inertia remaining at rest; any forces produce an effect of this kind, if only if equal and opposite forces to these are applied at the centre of inertia, as has been demonstrated in the above chapter. Now the investigation for other forces is not more difficult, when these forces always produce the same rotary motion, and if equal and opposite forces to these are applied to the centre of inertia ; for progressive motion, that they induce on the body in additon, also here nothing will be changed in the rotational motion in which the body is now involved. But also, if in a body if besides the rotary motion about an axis *IO* now a progressive motion should be involved, this generates no change in the rotation about the axis *IS* ; from which the solution of this problem appears to be the widest, and it can be extended also for progressive motion, that the body either has now or will obtain from the action of forces. Which combination of progressive motion with rotational motion should present no difficulty, this was a particular need, so that we are able to define the rotational motion carefully, however large, on account of other rotational motion arising from disturbing forces.

SCHOLIUM 2

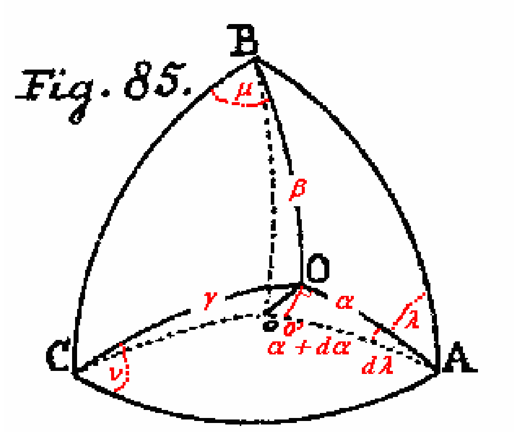
655. If the axis *IO*, about which the body is now assumed to be rotating, should be one of the principal axes of the body, this body motion is always preserved as if acted on by no forces, as we have shown previously. Now, if the axis *IO* is not a principal axis, even if no external forces are acting, yet the motion cannot be conserved, since the motion itself supplies the forces, which tend to change the direction of the axis of rotation ; hence in this case, if we wish to explore some variation arising in the axis of rotation, it is not sufficient to consider only the external forces acting on the body, but to these also must be joined the forces arising from the rotational motion, with which we have shown the above axis to be affected. Which forces, that depend on the position of the axis of gyration *IO* with respect to the principal axis of the body, we will not indeed proceed to investigate further before attending to this, and then we progress to investigate in general, how from any forces the position of the axis of rotation is changed with respect to the principal axis of the body.

PROBLEM 63

656. With the position of the axis of gyration given with respect to the three principal axes of the body, and this to be varied by some forces acting, in order that the body in a minimal elapsed element of time rotates about another axis, to define the position of the variation about the principal axes.

SOLUTION

The spherical surface can be considered again (Fig. 85), at the centre of this is the centre of inertia of the body *I*, and now let the the radii *IA*, *IB*, *IC* be the principal axes of the body, and the body is rotating about an axis *IO* with an angular speed γ, the position of this with respect to the principal axes is given the arc



$$AO=\alpha, BO=\beta \text{ and } CO=\gamma,$$

in order that

$$cos^2\alpha+cos^2\beta+cos^2\gamma=1.$$

Then the angles are put in place :

$$BAO=\lambda, CBO=\mu, ACO=\nu,$$

and on account of the quadrants *AB*, *BC*, and *CA*

[from the cosine rule for sides : $cos\,\beta=cos\,\alpha\, cos\,AB+sin\,\alpha\, sin\,AB\, cos\,\lambda$, as *AB* is a quadrant, etc;]

$$cos\,\beta=sin\,\alpha\, cos\,\lambda,\quad cos\,\gamma=sin\,\beta\, cos\,\mu,\quad cos\,\alpha=sin\,\gamma\, cos\,\nu,$$

thus there arises :

$$cos\,\lambda=\frac{cos\,\beta}{sin\,\alpha},\quad cos\,\mu=\frac{cos\,\gamma}{sin\,\beta},\quad cos\,\nu=\frac{cos\,\alpha}{sin\,\gamma},$$

$$sin\,\lambda=\frac{cos\,\gamma}{sin\,\alpha},\quad sin\,\mu=\frac{cos\,\alpha}{sin\,\beta},\quad sin\,\nu=\frac{cos\,\beta}{sin\,\gamma},$$

hence

$$\text{tang }\lambda=\frac{cos\,\gamma}{cos\,\beta},\quad \text{tang }\mu=\frac{cos\,\alpha}{cos\,\gamma},\quad \text{tang }\nu=\frac{cos\,\beta}{cos\,\alpha}$$

and thus

$$\text{tang}\lambda\cdot\text{tang}\mu\cdot\text{tang}\,\nu=1,$$

which is the relation between the three angles λ, μ, ν, from which the arcs α, β, γ are defined thus, so that :

$$\text{tang}\ \alpha = \frac{\text{tang}\ \nu}{\cos\lambda} = \frac{\cot\mu}{\sin\lambda}, \quad \text{tang}\ \beta = \frac{\text{tang}\ \lambda}{\cos\mu} = \frac{\cot\nu}{\sin\mu}, \quad \text{tang}\ \gamma = \frac{\text{tang}\ \mu}{\cos\nu} = \frac{\cot\lambda}{\sin\nu}.$$

From these relations noted , the remainder may thus be defined from $BAO = \lambda$ and $AO = \alpha$ given, so that

$$\cos\beta = \sin\alpha\cos\lambda, \quad \cos\gamma = \sin\alpha\sin\lambda,$$

$$\text{tang}\ \mu = \frac{\cot\alpha}{\sin\lambda}, \quad \text{tang}\ \nu = \text{tang}\ \alpha\cos\lambda.$$

But if now on account of the forces acting in the element of time *dt*, the axis of rotation goes from *IO* to *Io*, then the whole body, as if meanwhile it can be considered to be going around the axis *Io*, in which motion the points *A*, *B*, *C* can be considered to maintain their distances from the point *o*, thus in the elapsed time *dt*, the pole of rotation *o* now has these distances *Ao*, *Bo*, *Co* from the principal poles *A*, *B*, *C*. Whereby, if the angle of the element $OAo = d\lambda$ and $Ao = \alpha + d\alpha$, the variations of the remainder are accustomed to be found by differentiation :

$$d\beta = \frac{d\lambda\sin\alpha\sin\lambda - d\alpha\cos\alpha\cos\lambda}{\sin\beta},$$

$$d\gamma = \frac{-d\lambda\sin\alpha\cos\lambda - d\alpha\cos\alpha\sin\lambda}{\sin\gamma},$$

$$d\mu = \frac{-d\alpha\sin\lambda - d\lambda\sin\alpha\cos\alpha\cos\lambda}{\cos^2\alpha + \sin^2\alpha\sin^2\lambda},$$

$$d\nu = \frac{d\alpha\cos\lambda - d\lambda\sin\alpha\cos\alpha\sin\lambda}{\cos^2\alpha + \sin^2\alpha\sin^2\lambda}.$$

COROLLARY 1

657. If it is allowed to proceed to the integrations from these differentials, it is possible to assign the position of this axis in the body at some time with respect to the principal axies, about which then it shall be rotating.

COROLLARY 2

658. Clearly here we do not consider the motion of the body itself, but only that it is acted upon, so that the momentary of the axis of rotation with respect to the principal axes is known, and thus the rotational speed is not entered into here in this computation.

COROLLARY 3

659. Since the arclet *Oo* shall have been determined in the preceeding problem, this becomes [The elemental triangle *Ooo'* it treated as a plane rt. angled triangle in Fig. 85, for which] :

$$Oo = \sqrt{\left(d\alpha^2 + d\lambda^2 \sin^2 \alpha\right)},$$

then for the position of this arcet *Oo* with respect to the arc *AO* or *Ao* there is

$$\text{tang } AoO = \frac{d\lambda \sin\alpha}{d\alpha}$$

or

$$\sin AoO = \frac{d\lambda \sin\alpha}{Oo} \text{ et } \cos AoO = \frac{d\alpha}{Oo},$$

thus so that hence we have the elements :

$$d\alpha = Oo \cdot \cos AoO \quad \text{and} \quad d\lambda = \frac{Oo \cdot \sin AoO}{\sin\alpha}.$$

SCHOLIUM

660. Hence with the forces known, by which the body is acted on while it is rotating about some axis, from the preceding chapter this axis can be defined, about which it is initially at rest and begins to rotate, then with the help of the preceding problem the variation made in the axis of rotation is able to be explored. Now, unless the body at first is rotating about some principal axis, besides the external forces by which the body is perhaps acted on, these forces, which arise from rotational motion must be carefully assessed, and of this kind the centrifugal force especially. Which forces even if above we have now assigned in general, yet now it is required to determine the same again, with respect of the principal axes, in as much as the axis of rotation differs from these ; with which known, since it is easy to join on these forces with the external forces, we only consider these henceforth, and investigate accurately how much the position of the axis of rotation is disturbed by these.

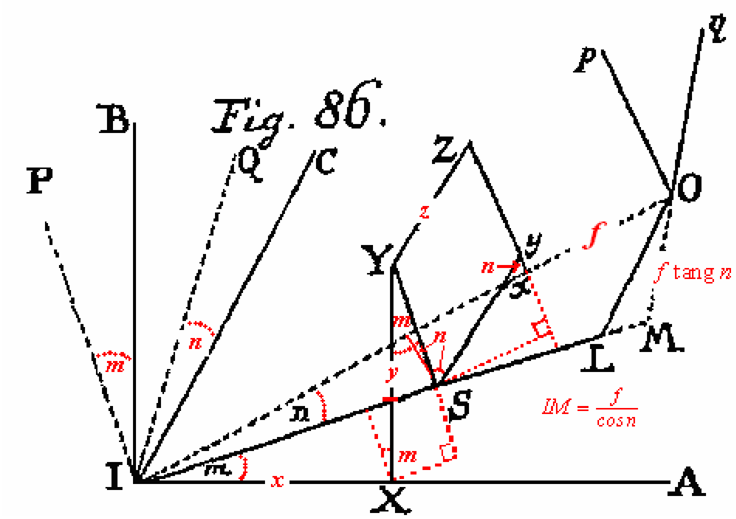
PROBLEMA 64

661. If a rigid body is rotating about some axis passing through the centre of inertia, and the position of this is given with respect to the principal axes, hence to find the [internal] forces arising disturbing the axis of rotation.

SOLUTION

With the centre of inertia present *I*, let *IA*, *IB* and *IC* be the principal axes of this body (Fig. 86) and about these the moments of inertia shall be *Maa*, *Mbb* et *Mcc*. Moreover the body shall be rotating about the axis *IO* with an angular speed equal to γ, from some point *O* of this a perpendicular *OL* is sent to the plane [of the diagram] *AIB*, and with the line *IL* drawn, calling the angles *AIL* = *m* and *LIO* = *n*, so that thus the position of this axis *IO* [*i. e.* the direction cosines] with respect to the principal axes shall be :

$$\cos AIO = \cos m \cos n, \quad \cos BIO = \sin m \cos n \text{ and } \cos CIO = \sin n.$$



Now in the first place with the principal axes taken as directrices, the lines parallel to these constitute the three coordinates *IX* = *x*, *XY* = *y* et *YZ* = *z*, and on taking an element *dM* of the body at *Z* , thus from the nature of the principal axes [*i. e.* no torques are exerted on the principal axes due to the symmetry of the body, as the torques caused by the individual elements of mass cancel in pairs]:

$$\int xydM = 0, \quad \int xzdM = 0 \quad \text{and} \quad \int yzdM = 0,$$

then

$$\int (yy+zz)dM = Maa, \quad \int (xx+zz)dM = Mbb, \quad \text{and} \quad \int (xx+yy)dM = Mcc$$

and thus :

$$\int xxdM = \tfrac{1}{2}M\left(bb+cc-aa\right), \quad \int yydM = \tfrac{1}{2}M\left(aa+cc-bb\right), \quad \int zzdM = \tfrac{1}{2}M\left(aa+bb-cc\right).$$

[Next, a new set of axes *IO*, *IP*, and *IQ* are established about the axis of rotation *IO* as the new *X* axis, based on the polar angles *m* and *n* : note that in this transformation the point Z remains fixed in position.]
Again in the plane *AIB* with *IP* drawn normal to *IL*, and in the plane *LIC* with the line *IQ* drawn normal to *IO*, in order that the lines *IO*, *IP* and *IQ* are normal to each other, and which we may take as the directrices. In the end this is drawn: first *YS* parallel to *IP* in the plane *AIB* , then [note that *y* is put as the sum of two parts in finding *IS*, and as a difference in finding *YS*, and note the angles equal to *m* in the above diagram in red, all in the *AIB* plane of the page. Note also, that $x^2 + y^2 = IX^2 + XY^2 = IS^2 + SY^2 = IY^2$ corresponding to a rotation about *IB* through the angle *m*.]

$$IS = x\cos m + y\sin m \quad \text{and} \quad YS = y\cos m - x\sin m;$$

and from *Z*, *YS* [in the horizontal plane *AIB*] acts parallel to *Zy*, which is drawn normal to the [vertical] plane *LIO*, and $Zy = [YS] = y\cos m - x\sin m$, likewise [in the plane *LIO*] $Sy = YZ = z$. And then from *y* to *IO* there is sent the perpendicular *yx* [which lies in the vertical plane *MIO*], so that now the desired coordinates are : $Ix = X$, $xy = Y$ and $yZ = Z$, and there shall be :

$$X = IS\cos n + Sy\sin n = x\cos m\cos n + y\sin m\cos n + z\sin n,$$
$$Y = yS\cos n - IS\sin n = z\cos n - x\cos m\sin n - y\sin m\sin n,$$
$$Z = y\cos m - x\sin m.$$

[Thus, the second rotation about *IQ* through the angle *n* has the same form as the first about *IC* by the angle *m*. The point *Z* remaining fixed in the body. The first transformation in modern terms is :

$$\begin{pmatrix} IS \\ YS \\ yS \end{pmatrix} = \begin{pmatrix} \cos m & \sin m & 0 \\ -\sin m & \cos m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$; while the second is :

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos n & 0 & \sin n \\ -\sin n & 0 & \cos n \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} IS \\ YS \\ yS \end{pmatrix}$$; giving

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos m\cos n & \sin m\cos n & \sin n \\ -\cos m\sin n & -\sin m\sin n & \cos n \\ -\sin m & \cos m & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$; while the inverse transformation is :

$$\begin{pmatrix} \cos m & -\sin m & 0 \\ \sin m & \cos m & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos n & -\sin n & 0 \\ 0 & 0 & 1 \\ \sin n & \cos n & 0 \end{pmatrix} = \begin{pmatrix} \cos m\cos n & -\cos m\sin n & -\sin m \\ \sin m\cos m & -\sin m\sin n + \cos m\cos n & \cos m \\ \sin n & \cos n & 0 \end{pmatrix}$$

End of note.]

Now since the element *dM* at *Z* on account of the angular speed equal to γ exerts a centrifugal force equal to

$$\frac{\gamma\gamma \cdot xZdM}{2g},$$

thus there arises the following force [on resolution] along *xy* equal to

$$\frac{\gamma\gamma YdM}{2g}$$

and the force applied at *x* along the direction parallel to *yZ* is equal to

$$\frac{\gamma\gamma ZdM}{2g},$$

which forces mutually destroy each other [on summing over the elements], on account of $\int YdM = 0$ and $\int ZdM = 0$, as only the moments of these are to be considered. Therefore taking $IO = f$ the force *Oq* placed at *O* parallel to *IQ* is equivalent to all the forces *yZ* , provided that for these forces equal and opposite forces are applied at the centre of inertia *I*. Therefore since on account of the moment, then [*i.e.* making use again of the integral mean–value theorem :]

$$\text{the force } Oq \cdot IO = \frac{\gamma\gamma}{2g}\int XYdM$$

and

$$\text{the force } Op \cdot IO = \frac{\gamma\gamma}{2g}\int XZdM,$$

then

$$\text{the force } Oq = \frac{\gamma\gamma}{2fg}\int XYdM \quad \text{and the force } Op = \frac{\gamma\gamma}{2fg}\int XZdM.$$

But on returning to the principal axes coordinates, [as the cross products of inertia are zero about the principle axes *x*, *y*, *z*] then

$$\int XYdM = \sin n \cos n\left(\int zzdM - \cos^2 m\int xxdM - \sin^2 m\int yydM\right)$$

and

$$\int XZdM = \sin m \cos m \cos n\left(\int yydM - \int xxdM\right)$$

and thus, from the given moments of inertia ,

$$\int XYdM = M \sin n \cos n\left(aa\cos^2 m + bb\sin^2 m - cc\right)$$

and

$$\int XZdM = M \sin m \cos m \cos n\left(aa - bb\right).$$

Consequently from the given rotational motion, these forces arise :

$$\text{the force } Op = \frac{M\gamma\gamma \sin m \cos m \cos n(aa-bb)}{2fg}$$

and

$$\text{the force } Oq = \frac{M\gamma\gamma \sin n \cos n(aa\cos^2 m + bb\sin^2 m - cc)}{2fg}$$

applied to the point *O* along the directions parallel to the lines *OP* and *OQ*, but to which equal and opposite forces are understood to be applied to the centre of inertia *I* itself.

COROLLARY 1

662. Since the force *Oq* is normal to the axis of rotation *IO* at *O*, that produced crosses the plane *AIB* [*AOB* in the original] at the point *M*, which is situated on *IL* produced, and $IM = \frac{f}{\cos n}$ and $OM = f \text{ tang } n$, on account of the right angle *IOM*.

COROLLARY 2

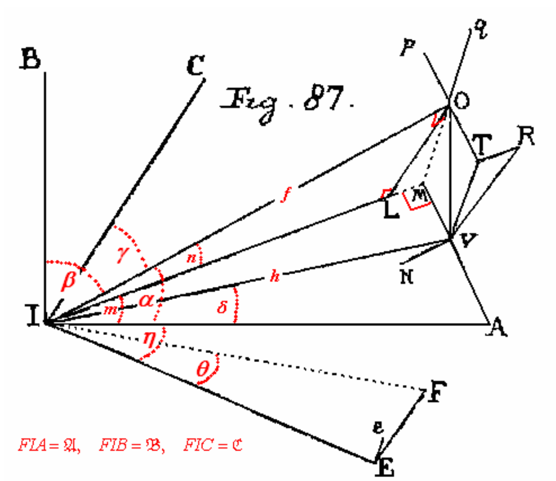
663. But the direction of the other force *Op* is normal to the plane *LIO*, as it is parallel to the line *IP* in the plane *AIB* normal to *IL* ; and the plane *pOq* continued to the plane *AIB* is inclined at an angle equal to $90^o - n$, and that intersects the line *IM* at right angles.

COROLLARY 3

664. Because these forces arising from the rotational motion itself have equal and opposite forces applied at the centre of inertia, these distrurb the rotational motion only and do not induce any progressive motion in the body, thus so that the centre of inertia remains permanently at rest. [These forces produce torques about *I* which cause the body to rotate about the other axes.]

PROBLEM 65

665. With the forces found arising from the rotational motion itself perturbing that motion, to find the axis if the body should be at rest, about which these forces are to set the body rotating.



SOLUTION

With everything remaining, as in the preceding problem, thus so that *IA*, *IB* and *IC* are the principal axes of the body (Fig. 87) and the moments of inertia with respect to these are *Maa*, *Mbb*, *Mcc*, let *IO* be the axis, about which the angular speed of the body is equal to γ, and for the position of this the angles *AIL* =*m* and *LIO* = *n*, with the line *OL* present normal to *AIB*, so that on putting *IO* =*f* then $IL = f\cos n$ and $OM = f\sin n$. Then from *O* the normal *OM* is drawn to *IO*, then

$$IM = \frac{f}{\cos n} \quad \text{and} \quad OM = f \operatorname{tang} n,$$

but with the normal *MA* drawn to *IM* in the plane *AIB*, then

$$IA = \frac{f}{\cos m \cos n} \quad \text{and} \quad MA = \frac{f \operatorname{tang} m}{\cos n}.$$

But now the forces *Op* and *Oq* are in place at *O*, of which *Op* itself is parallel to *AM* and *Oq* has been placed in the direction *OM*; and the forces are these:

$$\text{the force } Op = \frac{M\gamma\gamma \sin m \cos m \cos n(aa-bb)}{2fg}$$

and

$$\text{the force } Oq = \frac{M\gamma\gamma \sin n \cos n(aa\cos^2 m + bb\sin^2 m - cc)}{2fg}$$

of which the mean [*i. e.* resultant] direction cuts the plane *AIB* somewhere at *V* on the line *MA*, in order that $MO : MV = Oq : Op$, thus it is deduced that

$$MV = \frac{f \sin m \cos m(aa-bb)}{\cos n(aa\cos^2 m + bb\sin^2 m - cc)}$$

and hence

$$\text{tang } MIV = \frac{\sin m \cos m(aa-bb)}{aa\cos^2 m + bb\sin^2 m - cc};$$

from which it is concluded, [see §639]

$$\text{tang } AIV = \frac{(bb-cc)\sin m}{(aa-cc)\cos m},$$

we have called that angle above δ, but the distance

$$IV = \frac{f\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc(aa\cos^2 m + bb\sin^2 m)\right)}}{\cos n(aa\cos^2 m + bb\sin^2 m - cc)},$$

which we have called *h* above, so that this length becomes :

$$h = \frac{f(bb-cc)\sin m}{\cos n \sin\delta(aa\cos^2 m + bb\sin^2 m - cc)}$$

or

$$h = \frac{f(aa-cc)\cos m}{\cos n \sin\delta(aa\cos^2 m + bb\sin^2 m - cc)}.$$

Hence it is permitted now to consider these forces to be applied at the point *V*, which are

$$\text{force along } (Op)\ VM = \frac{M\gamma\gamma \sin m \cos m \cos n(aa-bb)}{2fg},$$

$$\text{force along } (Oq)\ VT = \frac{M\gamma\gamma \sin n \cos n(aa\cos^2 m + bb\sin^2 m - cc)}{2fg},$$

of which *VT* is resolved into components along *VR* parallel to *LO* and *VN* parallel to *ML* gives

$$\text{the force along } VR = \frac{M\gamma\gamma \sin n \cos^2 n\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg}$$

and

$$\text{the force along } VN = \frac{M\gamma\gamma \sin^2 n \cos n\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg},$$

of which that [vertical] force *VR* has been indicated above by the letter *R*. But because above there was written $Q\cos\delta - P\sin\delta$, [*i. e.* , the net force perpendicular to the axis *IV*] by which expression the force normal to *IV* in the plane *AIB* is denoted, this becomes [the other components act along *IV* and thus have no moment: see § 639]

$$\text{force } VM \cos MIV - \text{force } VN \sin MIV,$$

thus there is produced :

$$Q\cos\delta - P\sin\delta = \frac{M\gamma\gamma \sin m \cos m \cos^3 n(aa-bb)\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc\left(aa\cos^2 m + bb\sin^2 m\right)\right)}}.$$

Since again

$$\text{tang } \delta = \frac{(bb-cc)\sin m}{(aa-cc)\cos m},$$

then there becomes:

$$\cos\delta = \frac{(aa-cc)\cos m}{\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc\left(aa\cos^2 m + bb\sin^2 m\right)\right)}}$$

[Thus, the body is not subjected to external forces, but is rotating initially about some axis that is not a principal axis, in which case forces derived from the motion, due to the lack of symmetry, act on the axis to change its state of rotation; the point of application of this force is given by the angles *m* and *n* , and this is equivalent to the line *IV* in the plane *AIB* with the plane polar coordinates *h* and δ, and the components of this force are found to calculate the torque . Following this, the orientation of this axis has already been found as that *IF* which minimises the *vis viva*, which is now applied, when the body starts from rest.]

From these definitions now there shall be that axis *IF*, about which the body if at rest, these forces are about to set in rotation, and with the perpendicular *FE* drawn from *F* in the plane *AIB* the angles are called $AIE = \eta$ and $EIF = \vartheta$, and by problem 60 we can continue :

$$\text{tang } \eta = \frac{aa\cos\delta}{bb\sin\delta} = \frac{aa(aa-cc)\cos m}{bb(bb-cc)\sin m}$$

and

$$\text{tang } \vartheta = \frac{Q\cos\delta - P\sin\delta}{Rcc\cos\delta}\cdot bb\sin\eta = \frac{\sin m\cos n(aa-bb)bb\sin\eta}{cc(aa-cc)\sin n}.$$

And then in the element of time *dt* there is generated about this axis *IF* the angle $d\omega$, in order that it becomes :

$$d\omega = \frac{\gamma\gamma dt^2 \sin n\cos n\sqrt{\left(a^4(aa-cc)^2\cos^2 m + b^4(bb-cc)^2\sin^2 m\right)}}{2aabb\cos\vartheta},$$

or

$$d\omega = \frac{\gamma\gamma(aa-cc)dt^2\cos m\sin n\cos n}{2bb\sin\eta\cos\vartheta} = \frac{\gamma\gamma(bb-cc)dt^2\sin m\sin n\cos n}{2aa\cos\eta\cos\vartheta}.$$

COROLLARY 1

666. If for the proposed axis of rotation *IO* [on the spherical surface with origin *I*], the angles are put in place [for the principal axes] :

$$OIA = \alpha, \quad OIB = \beta, \quad OIC = \gamma,$$

but for the axis of rotation of the element *IF* the [spherical] angles are :

$$FIA = \mathfrak{A}, \quad FIB = \mathfrak{B}, \quad FIC = \mathfrak{C},$$

then there is

$$\cos\alpha = \cos m\cos n, \quad \cos\beta = \sin m\cos n, \quad \cos\gamma = \sin n$$

and

$$\cos\mathfrak{A} = \cos\eta\cos\vartheta, \quad \cos\mathfrak{B} = \sin\eta\cos\vartheta, \quad \cos\mathfrak{C} = \sin\vartheta.$$

COROLLARY 2

667. Hence on account of

$$\text{tang } \eta = \frac{aa(aa-cc)\cos\alpha}{bb(bb-cc)\cos\beta},$$

if for the sake of brevity there is put

$$\sqrt{\left(a^4(aa-cc)^2\cos^2\alpha + b^4(bb-cc)^2\cos^2\beta\right)} = W,$$

then there will be

$$sin\,\eta = \frac{aa(aa-cc)cos\,\alpha}{W} \quad \text{and} \quad cos\,\eta = \frac{bb(bb-cc)cos\,\beta}{W}.$$

But again on putting

$$\sqrt{\left(a^4b^4(aa-bb)^2 cos^2\alpha\, cos^2\beta + a^4c^4(aa-cc)^2 cos^2\alpha\, cos^2\gamma + b^4c^4(bb-cc)^2 cos^2\beta\, cos^2\gamma\right)} = \Omega$$

there is found :

$$cos\,\mathfrak{A} = \frac{bbcc(bb-cc)cos\,\beta\, cos\,\gamma}{\Omega},$$

$$cos\,\mathfrak{B} = \frac{aacc(cc-aa)cos\,\alpha\, cos\,\gamma}{\Omega},$$

$$cos\,\mathfrak{C} = \frac{aabb(aa-bb)cos\,\alpha\, cos\,\beta}{\Omega}$$

and

$$d\omega = \frac{\gamma\gamma\Omega dt^2}{2aabbcc}.$$

[Finally, the equation of motion is written in terms of coordinates on the sphere]

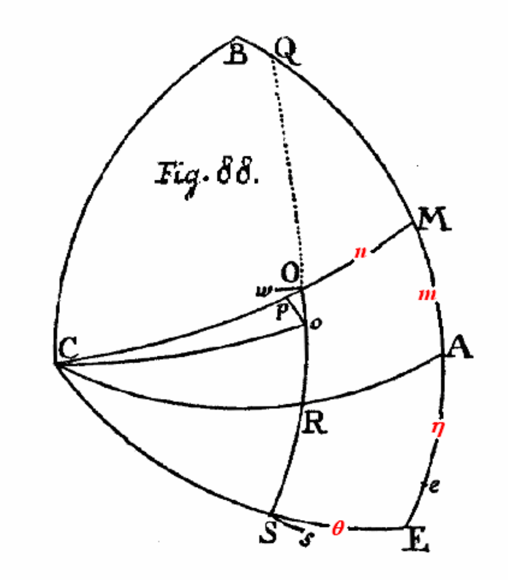
SCHOLIUM

668. Because the sense is kept, in which the rotation about the axis *IF* is made, the angle of the element $d\omega = \frac{\gamma\gamma\Omega dt^2}{2aabbcc}$ is always positive, and it is to be noted in the investigation of this value that the force *VR* it considered as positive, hence following the figure the point *E* is carried in the in sense *Ee* towards *A* by the rotary motion. For even if this account only has a place in the figure, where the angles m, n, η, ϑ are positive and less than a right angle, yet the account of the sense hence can be correctly concluded ; in which once introduced into the calculation henceforth generally we cling to the truth. Furthermore it is evident, if the axis *IO* falls on some principal axis, to be $d\omega = 0$; for in fact if $\alpha = 0$, this makes $\beta = \gamma = 90^o$ and thus $cos\,\beta = cos\,\gamma = 0$, in which case the quantity Ω disappears everywhere ; now likewise it is evident that in no other case is it possible for the perturbation $d\omega$ to vanish, and thus no more than three axes are able to be free from rotations, unless perhaps two moments of inertia should be equal.

PROBLEM 66

669. If a body is rotating about some axis passing through the centre of inertia different from the principal axes, to define the momentary variation, which both the axis of rotation as well as the angular speed experience.

SOLUTION



Everything is to be transferred, which have been found in the preceding problem, to the spherical surface described with the centre of inertia *I*, in which *A, B,* and *C* are the poles of the principal axes (Fig. 88), with respect to which the moments of inertia are *Maa, Mbb, Mcc*. Now let *O* then be the pole of that axis, about which the body now rotates with an angular speed equal to γ' in the sense *ABC*. [Note that the angulal speed has been designated by γ up to this point; clearly this has to be distinguished from the angle γ on the sphere to be used. Euler used a larger font which is not convenient, while the *O. O.* invents its own symbol which is not availabe to me, and is also inconvenient.]From *C* through *O* with the great circle *COM* drawn, which is a quadrant, there are the arcs *AM* = *m* and *MO* = *n*; then in the quadrant *BA* produced there is taken $AE = \eta$ and with the quadrant *CE* drawn the arc is $ES = \vartheta$, in order that

$$\text{tang } \eta = \frac{aa(aa-cc)\cos m}{bb(bb-cc)\sin m},$$

or

$$\frac{bb\sin m\sin\eta}{aa-cc} = \frac{aa\cos m\cos\eta}{bb-cc}$$

and

$$\text{tang } \vartheta = \frac{bb\sin m\sin\eta\cdot(aa-bb)\cos n}{cc(aa-cc)\sin n}$$
$$= \frac{aa\cos m\cos\eta\cdot(aa-bb)\cos n}{cc(bb-cc)\sin n}.$$

Thus with these defined on account of the centrifugal forces of the body, the body tries to rotate about the pole *S* in the sense *Ee*, thus so that in the element of time *dt* an angle will be described

$$d\omega = \frac{\gamma'\gamma'(aa-cc)dt^2 \cos m \sin n \cos n}{2bb \sin\eta \cos\vartheta} = \frac{\gamma'\gamma'(bb-cc)dt^2 \sin m \sin n \cos n}{2aa \cos\eta \cos\vartheta}$$

or

$$d\omega = \frac{\gamma'\gamma'(aa-bb)dt^2 \sin m \cos m \cos^2 n}{2cc \sin\vartheta}.$$

Therefore the great arc *OS* is drawn, which is equal to *s*, which we can determine next, and in problem 62 it becomes

$$q = \frac{\gamma'\gamma'(aa-bb)\sin m \cos m \cos^2 n}{2cc \sin\vartheta},$$

and hence on account of the elemental rotational motion the body rotates about the pole *o*, in order that

$$\text{the arclet } Oo = \frac{\gamma'\gamma'(aa-bb)dt \sin m \cos m \cos^2 n \sin s}{cc \sin\vartheta};$$

but an increase of the angular speed γ' is taken $d\gamma'$, in order that

$$d\gamma' = \frac{\gamma'\gamma'(aa-bb)dt \sin m \cos m \cos^2 n \cos s}{cc \sin\vartheta}.$$

Now therefore in the first place the position of the arc *OS* must be found, or the angle *COS*, by which it is inclined to the arc *CO* ; which in the end the triangle *OCS* is considered, in which

$OC = 90^0 - n$, $CS = 90^0 - \vartheta$, and the angle $OCS = m + \eta$, hence there is found :

$$\cot COS = \frac{\cos n \tan\vartheta}{\sin(m+\eta)} - \frac{\sin n \cos(m+\eta)}{\sin(m+\eta)}.$$

Now indeed,

$$\tan(m+\eta) = \frac{aa(aa-cc)\cos^2 m + bb(bb-cc)\sin^2 m}{(aa-bb)(cc-aa-bb)\sin m \cos m}$$

and

$$\frac{\cos n \tan\vartheta}{\sin(m+\eta)} = \frac{aabb(aa-bb)\sin m \cos m \cos^2 n}{cc \sin n\left(bb(bb-cc)\sin^2 m + aa(aa-cc)\cos^2 m\right)},$$

thus giving

$$\tan COS = \frac{cc \sin n\left(aa(aa-cc)\cos^2 m + bb(bb-cc)\sin^2 m\right)}{(aa-bb)\sin m \cos m\left(aabb \cos^2 n + cc(aa+bb)\sin^2 n - c^4 \sin^2 n\right)}.$$

Again from the same triangle *OCS* it is deduced that

$$\cos s = \cos(m+\eta)\cos n\cos\vartheta + \sin n\sin\vartheta = \sin\vartheta\left(\sin n + \frac{\cos n\cos(m+\eta)}{\operatorname{tang}\vartheta}\right)$$

or

$$\cos s = \frac{\sin n\sin\vartheta\left(aabb-(aa+bb)cc+c^4\right)}{aabb} = \frac{(aa-cc)(bb-cc)\sin n\sin\vartheta}{aabb},$$

thus there arises

$$d\gamma' = \frac{\gamma'\gamma'(aa-bb)(aa-cc)(bb-cc)\sin m\cos m\sin n\cos^2 n}{aabbcc}\cdot dt.$$

And from this on putting $OA=\alpha,\quad OB=\beta,\quad OC=\gamma$ the arc is :

$$Oo = \frac{\gamma dt}{aabbcc}\sqrt{\begin{pmatrix} a^4b^4(aa-bb)^2\cos^2\alpha\cos^2\beta + a^4c^4(aa-cc)^2\cos^2\alpha\cos^2\gamma \\ +b^4c^4(bb-cc)^2\cos^2\beta\cos^2\gamma \\ -(aa-bb)^2(aa-cc)^2(bb-cc)^2\cos^2\alpha\cos^2\beta\cos^2\gamma \end{pmatrix}}$$

and

$$d\gamma' = \frac{\gamma'\gamma'(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc}\cdot dt.$$

Now if from *o* to *CO* the perpendicular *op* is drawn, by the rules of spherical trigonometry the elemental arcs *Op* and *op* thus can be expressed rationally, in order that :

$$Op = \frac{\gamma'(aa-bb)dt\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)}{aabbcc\sin\gamma},$$

$$op = \frac{\gamma' dt\cos\gamma\left(aa(aa-cc)\cos^2\alpha + bb(bb-cc)\cos^2\beta\right)}{aabbcc\sin\gamma}.$$

COROLLARY 1

670. Since it is the case that

$$d\gamma' = \frac{\gamma'\gamma'(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc}\cdot dt,$$

it is apparent, if of the three principal moments two were equal to each other, then clearly the angular speed does not change.

COROLLARY 2

671. With the distances introduced α, β, γ of the pole *O* from the principal poles *A*, *B*, *C* there is

$$\text{tang } COS = \frac{cc \cos\gamma\left(aa(aa-cc)\cos^2\alpha + bb(bb-cc)\cos^2\beta\right)}{(aa-bb)\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)},$$

but with the arc drawn *AO* then it becomes

$$\text{tang } AOC = \frac{-\cos\beta}{\cos\alpha\cos\gamma},$$

thus it is concluded

$$\text{tang } AOS = \frac{aa\cos\alpha\left(bb(bb-aa)\cos^2\beta + cc(cc-aa)\cos^2\gamma\right)}{(bb-cc)\cos\beta\cos\gamma\left((bb-aa)(cc-aa)\cos^2\alpha - bbcc\right)}.$$

COROLLARY 3

672. This formula for the angle *AOS* is analogous to that for the angle *COS* and thus it arises, if the letters *a*, *b*, *c*, likewise α, β, γ are advanced one place in order ; but in this way a negative sign is produced, which is consistent with the nature of the thing, when the angle *AOS* falls in the opposite sense with respect to the first.

COROLLARY 4

673. If the arc OS cuts the quadrant AC at the point R, it is deduced :

$$\text{tang } AR = \frac{aa\cos\alpha\left(bb(aa-bb)\cos^2\beta + cc(aa-cc)\cos^2\gamma\right)}{cc\cos\gamma\left(aa(aa-cc)\cos^2\alpha + bb(bb-cc)\cos^2\beta\right)},$$

and if likewise the arc *SO* produced cuts the quadrant *BA* at *Q*, then by analogy :

$$\text{tang } BQ = \frac{bb\cos\beta\left(cc(bb-cc)\cos^2\gamma + aa(bb-aa)\cos^2\alpha\right)}{aa\cos\alpha\left(bb(bb-aa)\cos^2\beta + bb(cc-aa)\cos^2\gamma\right)} = \cot AQ.$$

COROLLARY 5

674. Since in the element of time *dt* the arc $CO = \gamma$ is diminished by the small amount *Op*, then by differentiation it becomes :

$$aabbccd\gamma \sin\gamma = \gamma'\left(bb-aa\right)dt \cos\alpha \cos\beta\left(aabb-\left(aa-cc\right)\left(bb-cc\right)\cos^2\gamma\right)$$

and hence by analogy :

$$aabbccd\beta \sin\beta = \gamma'\left(aa-cc\right)dt \cos\gamma \cos\alpha\left(aacc-\left(cc-bb\right)\left(aa-bb\right)\cos^2\beta\right),$$

$$aabbccd\beta \sin\beta = \gamma'\left(cc-bb\right)dt \cos\beta \cos\gamma\left(bbcc-\left(bb-aa\right)\left(cc-aa\right)\cos^2\alpha\right).$$

SCHOLIUM

675. We have assumed in the solution, because properly it is to be noted that the body is rotating about the axis *IO* in the sense *ABC*, hence to which case the formulas found are applied ; but if the body should be rotating in the opposite sense, the formulas easily put in place refer to a negative angular speed γ . And thus this most difficult problem, in which the momentary variation is being sought, while the body is rotating about a non-principal axis, we can resolve well enough with the latter formulas, to which only the solution has been produced, they are thus not complex as it is permitted to expect them to be simpler. Also lest the suspicion of any error in the calculation is present in the calculation, with the formula by which the increment in the angular speed $d\gamma$ is expressed, to all three principal axes it is referred equally, then the derivative of the angle *AOS* from the angle *COS* brings about the result most firmly and finally the equations in the last corollary show that they have adopted this property, so that it becomes

$$d\alpha \sin\alpha \cos\alpha + d\beta \sin\beta \cos\beta + d\gamma \sin\gamma \cos\gamma = 0,$$

as the condition of the principal axes prevails :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

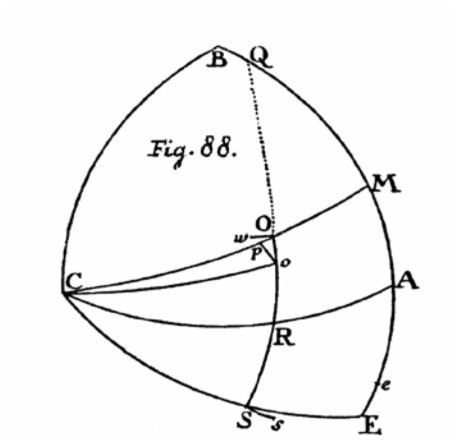
But the three last equations with this, which defines the differential $d\gamma$, contain a complete solution of the problem, where indeed any of these three can be omitted. If the above body is acted on by external forces, a solution prevails that is not much more difficult, as is shown in the following problem.

PROBLEM 67

676. If a rigid body, while it is rotating about some axis through the centre of mass, is acted on by some forces, to define the momentary variation arising both in the axis as well as in the angular speed.

SOLUTION

Let *IO* be the axis, about which the body is rotating now with an angular speed equal to γ' in the sense *ABC* (Fig. 88),



and at first the situation of this with respect of the principal axis *IA*, *IB*, *IC* can be considered, of which the moments of inertia respectively are *Maa*, *Mbb*, *Mcc*, and with the arcs put in place $OA = \alpha, OB = \beta, OC = \gamma$ there is sought by the previous problem, how great in the element of time *dt* both the axis of rotation *IO* as well as the angular speed must change on account of the rotary motion alone. Evidently, if the pole of rotation goes from *O* to *o*, we have seen that the increment in the distance $CO = \gamma$:

$$Co - CO = \frac{\gamma'(aa-bb)dt\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)}{aabbcc\sin\gamma}$$

and the increment of the angle *BCO*

$$OCo = \frac{\gamma' dt\cos\gamma\left(aa(aa-cc)\cos^2\alpha+bb(bb-cc)\cos^2\beta\right)}{aabb\sin^2\gamma}$$

from which elements the position of the point *o* is defined without ambiguity. But besides this change of the axis of rotation, the angular speed γ takes an increment equal to

$$\frac{\gamma'\gamma'(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc}dt\,.$$

Then the forces acting can be assessed, whether they impress a progressive motion to the body : and this has been easily judged, while all the forces along each of their directions are considered to be applied to the centre of inertia. For indeed if they hold the body in equilibrium between themselves, then no progressive motion is imparted to the body, but if a force equivalent to these should be given, from that the progressive motion is generated in the body that we can easily define from first principles. Then to that force an equal and opposite equivalent force can be applied to the centre of the body, so that now this centre is held at rest, and with this force with these, by which the body is actually disturbed, all taken together can be reduced to two, of which one is applied at the centre and the other is applied at a certain other point, which two forces are equal and opposite. Again from the preceding chapter the axis is sought, about which the body begins to be turned by these forces, and likewise the angle of the momentary conversion, thus by problem 62 without any change had with respect to the change now found since this is infinitely small, as if the body at this stage should be rotating about the axis *Oo*, the change in the axis is sought and with the change in the angle arising, of which these are reduced to the increment or decrement both in the *CO* as well as in the angle *BCO* arising. Finally these three elements with these, which now have been defined before from rotational motion, are joined together, and thus there will be obtained the true variation both in the axis *IO* as in the angular speed produced at the same time from each cause.

SCHOLIUM

677. While the effect of the forces acting is explored, the variation of the axis thus produced can be expressed by the elementary angle *OCo* and the differential of the arc *CO* and *Co*. Clearly in the first place the axis is sought, about which the body, if it should be at rest, may be turned by the forces, which shall be *IS*, and let qdt^2 be the angle of rotation produced about *S* in the sense $O\omega$ in the element of time *dt* and for the point *S* there is put in place the arc $AE = \eta$ and $ES = \vartheta$, which values are to be properly distinguished from the preceding rotational motion itself arising. Therefore since *AM* = *ACM* = *m*, in order that

$$\cos m = \frac{\cos\alpha}{\sin\gamma} \quad \text{and} \quad \sin m = \frac{\cos\beta}{\sin\gamma},$$

then $MCE = m + \eta$ and from triangle *OCS* there is found :

$$\cos OS = \cos s = \cos(m+\eta)\sin\gamma\cos\vartheta + \cos\gamma\sin\vartheta$$

and

$$\cot COS = \frac{\sin\gamma\,\mathrm{tang}\,\vartheta}{\sin(m+\eta)} - \frac{\cos\gamma\cos(m+\eta)}{\sin(m+\eta)}.$$

But now from problem 62 the pole of the rotation *O* is transferred to *o*, in order that there becomes

$$Oo = \frac{2qdt \sin s}{\gamma'}$$

and the increment of the angular speed

$$2qdt \cos s = 2qdt\left(\sin\gamma \cos\vartheta \cos\left(m+\eta\right) + \cos\gamma \sin\vartheta\right).$$

Hence form *Oo* there is elicited

$$Op = Oo \cos COS = \frac{2qdt \cos s}{\gamma'} \cos COS$$

and

$$op = Oo \sin COS = \frac{2qdt \sin s}{\gamma'} \sin COS = \frac{2qdt}{\gamma'} \cos\vartheta \sin\left(m+\eta\right)$$

and hence

$$\text{the angle } OCo = \frac{2qdt \cos\vartheta \sin(m+\eta)}{\gamma' \sin\gamma}.$$

Now again hence there is deduced

$$CO - Co = Op = op \cot COS = \frac{2qdt}{\gamma'}\left(\sin\gamma \sin\vartheta - \cos\gamma \cos\vartheta \cos\left(m+\eta\right)\right).$$

Therefore there remains only, that these elements should be combined with these which have been destroyed by the rotational motion, in order that there should be obtained the variation of the axis of gyration with the increment or decrement of the angular speed.

PROBLEM 68

678. If at some time the position of a rigid body should be given rotating about a certain axis passing through the centre of inertia, and the axis of rotation as well as the rotational speed can be varied in some manner, to find the momentary change in the position of the body arising [relative to absolute space].

SOLUTION

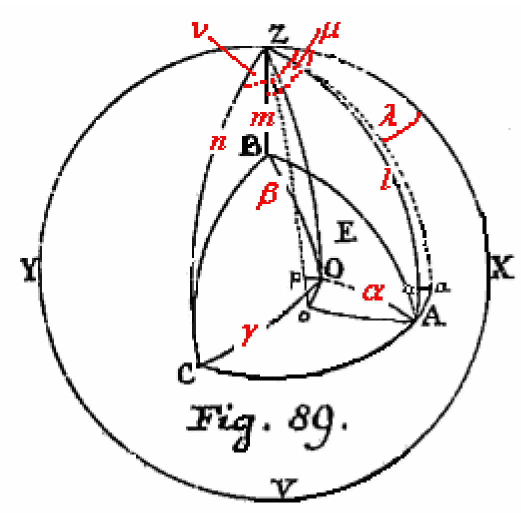


Fig. 89.

Since the centre of inertia of the body is at rest, the position of the body is referred to a fixed sphere described with the same centre, within which the body completes its own motion (Fig. 89). On this sphere there is taken the great circle *VXZY* and on that the fixed point *Z*; and at the given time equal to *t* the principal axes of the body correspond to the points on the spherical surface *A*, *B*, *C*, in order that *AB*, *BC*, and *CA* are quadrants; to the position of which the arcs of the great circles are to be represented by the symbols

$$ZA = l, \quad ZB = m, \quad ZC = n,$$

then

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

and the angles are put in place

$$XZA = \lambda, \quad XZB = \mu, \quad XZC = \nu,$$

and from the spherical [triangles]

$$cos(\mu-\lambda) = -cot\, l\, cot\, m,$$
$$cos(\nu-\mu) = -cot\, m\, cot\, n,$$
$$cos(\nu-\lambda) = -cot\, l\, cot\, n,$$

hence

$$cos(\mu-\lambda)cos(\nu-\lambda) = cot^2 l\, cot\, m\, cot\, n = -cot^2 l\, cos(\nu-\mu),$$

thus there becomes

$$cot^2 l = -\frac{cos(\mu-\lambda)cos(\nu-\lambda)}{cos(\nu-\mu)},$$
$$cot^2 m = -\frac{cos(\lambda-\mu)cos(\nu-\mu)}{cos(\nu-\lambda)},$$
$$cot^2 n = -\frac{cos(\lambda-\nu)cos(\mu-\nu)}{cos(\mu-\lambda)}.$$

Now since
[$\nu-\mu=(\nu-\lambda)-(\mu-\lambda)$,]

$$cos(\nu-\mu)-cos(\mu-\lambda)cos(\nu-\lambda) = sin(\mu-\lambda)sin(\nu-\lambda),$$
$$cos^2 l = -cot(\mu-\lambda)cot(\nu-\lambda),$$
$$cos^2 m = -cot(\lambda-\mu)cot(\nu-\mu),$$
$$cos^2 n = -cot(\lambda-\nu)cot(\mu-\nu).$$

From this relation between the quantities $l, m, n, \lambda, \mu, \nu,$ which in the element of time *dt* are agreed to increase by their own differentials, it is to be noted now that *O* is the pole of the rotation and the arcs $AO=\alpha, \quad BO=\beta, \quad CO=\gamma$, in order that :

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1;$$

then

$$cos\, BAO = \frac{cos\,\beta}{sin\,\alpha}$$

and [since the angles *BAC*, etc on the surface are right,]

$$sin\, BAO = \frac{cos\,\gamma}{sin\,\alpha},$$

but in the triangle *ZAB* there is present

$$cos\, ZAB = \frac{cos\, m}{sin\, l} \quad \text{and} \quad sin\, ZAB = -cos\, ZAC = -\frac{cos\, n}{sin\, l},$$

thus in order that for triangle *ZAO* there shall be [from $sin(ZAO) = sin(ZAB+BAO)$, etc.]

$$\sin ZAO = \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin\alpha\sin l}, \quad \cos ZAO = \frac{\cos\beta\cos m + \cos\gamma\cos n}{\sin\alpha\sin l},$$

thus on deducing [for us the scalar product of the unit vectors *IO* and *IZ*]

$$\cos ZO = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n$$

and

$$\cot AZO = \frac{\cos\alpha - \cos l(\cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n)}{\cos\gamma\cos m - \cos\beta\cos n},$$

and hence at any time the pole *O* of the rotation is known.

Then with the angular speed put equal to γ' in the sense *ABC*, in the element of time *dt* the point *A* describes an arclet about *O*, [about the axis *IO*; note that *OAa* is a rt. angle, so that *ZAO* is the complement of *ZAa*.] $Aa = \gamma' \, dt \sin\alpha$, whereby with $a\alpha$ drawn normal to *ZA* then

$$A\alpha = [Aa\sin ZAO] = \gamma' \, dt \cdot \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin l},$$

$$a\alpha = [Aa\cos ZAO] = \gamma' \, dt \cdot \frac{\cos\beta\cos m + \cos\gamma\cos n}{\sin l},$$

thus, from the differences of the quantities l and λ [: note that $dl = -A\alpha$, and in the linear elemental triangle $Za\alpha$, $d\lambda = -\frac{a\alpha}{\sin l}$,] there are deduced

$$dl\sin l = \gamma' \, dt(\cos\beta\cos n - \cos\gamma\cos m)$$

and

$$-d\lambda\sin^2 l = \gamma' \, dt(\cos\beta\cos m + \cos\gamma\cos n)$$

and in a like manner there may be found :

$$dm\sin m = \gamma' \, dt(\cos\gamma\cos l - \cos\alpha\cos n),$$

$$-d\mu\sin^2 m = \gamma' \, dt(\cos\gamma\cos n + \cos\alpha\cos l)$$

$$dn\sin n = \gamma' \, dt(\cos\alpha\cos m - \cos\beta\cos l),$$

$$-d\nu\sin^2 n = \gamma' \, dt(\cos\alpha\cos l + \cos\beta\cos m).$$

Concerning which, if at a certain time α, β, γ and γ' are given quantities and likewise the differentials of these arise in the same element of time *dt*, hence the variations in the same element of time are deduced for the arcs *l*, *m*, *n* and the angles λ, μ, ν produced. Now in addition the variation in the pole *O* of the rotation is easily inferred, since finally it is necessary that the arc *ZO* and the angle *AZO* should be differentiated, but only with the arcs α, β, γ variable, because in this manner the pole *O* is transferred to the following place *o*. Hence there is produced [for the elemental triangle *ZOo* on differentiating *cos ZO*]:

$$(Zo - ZO)\sin ZO = d\alpha \sin\alpha \cos l + d\beta \sin\beta \cos m + d\gamma \sin\gamma \cos n$$

and, since

$$\cot AZO(\cos\gamma \cos m - \cos\beta \cos n) = \cos\alpha - \cos l \cos ZO,$$

then the equation arises [also from differentiation, but watch the signs]

$$\frac{OZo}{\sin^2 AZO}(\cos\gamma \cos m - \cos\beta \cos n) + \cot AZO(d\gamma \sin\gamma \cos m - d\beta \sin\beta \cos n)$$

$$=[\sin\alpha d\alpha - d\alpha \sin\alpha \cos^2 l - d\beta \sin\beta \cos l \cos m - d\gamma \sin\gamma \cos l \cos n]$$

$$= d\alpha \sin\alpha \sin^2 l - d\beta \sin\beta \cos l \cos m - d\gamma \sin\gamma \cos l \cos n$$

and hence on reduction :

$$\frac{OZo}{\sin^2 AZO}$$

$$= \frac{\sin^2 l\big(d\alpha \sin\alpha(\cos\gamma \cos m - \cos\beta \cos n) + d\beta \sin\beta(\cos\alpha \cos n - \cos\gamma \cos l) + d\gamma \sin\gamma(\cos\beta \cos l - \cos\alpha \cos m)\big)}{(\cos\gamma \cos m - \cos\beta \cos n)^2}$$

and hence the angle of the element *OZo* =

$$\frac{d\alpha \sin\alpha(\cos\gamma \cos m - \cos\beta \cos n) + d\beta \sin\beta(\cos\alpha \cos n - \cos\gamma \cos l) + d\gamma \sin\gamma(\cos\beta \cos l - \cos\alpha \cos m)}{1 - (\cos\alpha \cos l + \cos\beta \cos m + \cos\gamma \cos n)^2},$$

in which formula two of the three letters α, β, γ, et l, m, n are increased equally, as the nature of the problem demands.

COROLLARY 1

679. If from *O* to *Zo* the elemental arc *Op* is drawn perpendicularly, then it becomes

$$po = \frac{d\alpha \sin\alpha \cos l + d\beta \sin\beta \cos m + d\gamma \sin\gamma \cos n}{\sin ZO}$$

and

$$Op = \frac{d\alpha \sin\alpha(\cos\gamma \cos m - \cos\beta \cos n) + d\beta \sin\beta(\cos\alpha \cos n - \cos\gamma \cos l) + d\gamma \sin\gamma(\cos\beta \cos l - \cos\alpha \cos m)}{\sin ZO}.$$

COROLLARY 2

680. Again from $\sin BAO = \frac{\cos\gamma}{\sin\alpha}$ and $\cos BAO = \frac{\cos\beta}{\sin\alpha}$ the angle is deduced

$$OAo = \frac{-d\alpha \cos\alpha \cos\gamma - d\gamma \sin\alpha \sin\gamma}{\sin\alpha \cos\beta}$$

and hence the element

$$Oo = \frac{\sqrt{\left(\left(d\alpha^2 + d\gamma^2\right)\sin^2\alpha \sin^2\gamma + 2d\alpha d\gamma \sin\alpha \cos\alpha \sin\gamma \cos\gamma\right)}}{\cos\beta},$$

which since it is equally referred to $\alpha, \beta, \gamma,$ on account of

$$d\alpha \sin\alpha \cos\alpha + d\beta \sin\beta \cos\beta + d\gamma \sin\gamma \cos\gamma = 0$$

it can be reduced to

$$Oo = \sqrt{\left(d\alpha^2 \sin^2\alpha + d\beta^2 \sin^2\beta + d\gamma^2 \sin^2\gamma\right)}.$$

COROLLARY 3

681. We put $ZO = v$ and, since there arises

$$\cos v = \cos\alpha \cos l + \cos\beta \cos m + \cos\gamma \cos n,$$

then

$$\text{tang } AZO = \frac{\cos\gamma \cos m - \cos\beta \cos n}{\cos\alpha - \cos l \cos v}$$

and on account of the analogy, since *B* falls on the other part of *ZO* in the figure,

$$-\text{tang } BZO = \frac{\cos\alpha \cos n - \cos\gamma \cos l}{\cos\beta - \cos m \cos v},$$

thus it becomes

$$\text{tang } AZB = \text{tang}\left(\mu - \lambda\right) = \frac{\cos n}{\cos l \cos m};$$

which value with that found above

$$\cos\left(\mu - \lambda\right) = -\frac{\cos l \cos m}{\sin l \sin m}$$

is in complete agreement, and then

$$\sin\left(\mu - \lambda\right) = -\frac{\cos n}{\sin l \sin m}.$$

COROLLARY 4

682. Hence therefore from the differentials of the three angles λ, μ, v we arrive at these conclusions

$$sin(\mu-\lambda) = -\frac{cos\,n}{sin\,l\,sin\,m},$$
$$sin(\nu-\mu) = -\frac{cos\,l}{sin\,m\,sin\,n},$$
$$sin(\lambda-\nu) = -\frac{cos\,m}{sin\,l\,sin\,n},$$
$$cos(\mu-\lambda) = -\frac{cos\,l\,cos\,m}{sin\,l\,sin\,m},$$
$$cos(\nu-\mu) = -\frac{cos\,m\,cos\,n}{sin\,m\,sin\,n},$$
$$cos(\lambda-\nu) = -\frac{cos\,l\,cos\,n}{sin\,l\,sin\,n}.$$

SCHOLIUM

683. Up to the present we have set out the momentary change undergone, both by the action of the rotational motion itself as well as by external forces acting, which constitute the foundation of the general theory of the motion of rigid bodies, since the transition from the known elemental change to the determination of the motion itself is apparent from integral calculus. Therefore we undertake the free motion of bodies of this kind, by which as if either by their own instigation or by forces acting are able to comply freely, and in the first place indeed we shall remove the external forces acting, leaving only the bodies to be themselves contemplated, so that nothing from without is added, that brings something to the motion. But because there is an innate property of the principal axes, with which the body is provided, this especially enters into the calculation, thus as if from the nature it is convenient to set up a distinction between the bodies, as the moments of inertia with respect to these may be compared. Therefore we set up three classes of bodies, to the first of which we refer to these bodies, of which the moments with respect to the principal axes are equal between themselves; now to the second class these bodies, in which two moments with respect to the principal axes are equal, and now the third for these which are unequal. Now the third class in general embraces all these bodies, of which the moments with respect to the principal axes are all unequal between themselves.

CAPUT X

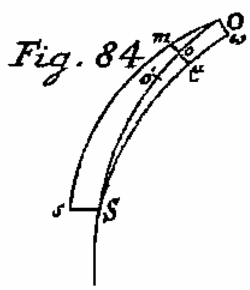
DE VARIATIONE MOMENTANEA AXIS GYRATIONIS A VIRIBUS PRODUCTA

PROBLEMA 62

650. Si corpus rigidum, dum circa axem per centrum inertiae transeuntem gyratur, ab eiusmodi viribus sollicitetur, quae ipsi, si quiesceret, motum gyratorium circa alium axem essent impressurae, determinare motus mutationem tempusculo minimo productam.

SOLUTIO

Cum tam in motu iam insito quam in eo, qui a viribus imprimeretur, centrum inertiae quiescat, id etiam coniunctim in quiete perseverabit.



Consideretur ergo centrum inertiae *I* tanquam centrum sphaerae, in cuius superficie sit *O* polus, et *IO* axis, circa quem corpus iam gyratur celeritate angulari = γ (Fig. 84) idque in eum sensum, quo punctum *S* feratur in *s*. Tum vero corpus ab eiusmodi viribus sollicitetur, ut, si quiesceret, gyratetur circa polum *S* seu axis *IS* tempusculoque *dt* verteretur per angulum qdt^2, quandoquidem vidimus hunc angulum quadrato tempusculi *dt* esse homogeneum, fiatque hac conversio in eum sensum, quo punctum *O* versus ω ferretur. Datur ergo angulus, quem hi duo axes *OI* et *SI* in *I* constituunt, seu in superficie sphaerica arcus circuli maximi *OS*, qui ponatur *OS* = *s*; ac tempusculo *dt* hic arcus *OS* ob motum insitum circa polum *O* gyrabitur per angulum $SOs = \gamma dt$ perventurus in situm *Os*, ut esset arculus $Ss = \gamma dt \sin s$. Ob motum autem impressum idem arcus *OS* circa polum *S* gyrabitur per angulum $OS\omega = qdt^2$ perventurus in situm $S\omega$, ut esset arculus

$O\omega = qdt^2 \sin s$. Utroque igitur hoc motu simul punctum *S* in *s* et punctum *O* in ω transferetur, quia neutra translatio alteram turbat; reliqua autem puncta omnia utrumque motum percipient. Scilicet punctum quodvis *o* in ipso arcu *OS* assumtum, ut sit $Oo = \omega$, ob motum insitum circa *O* transferetur in *m*, ut sit $om = \gamma dt \sin \omega$, at ob motum genitum circa S transferetur in μ , ut sit

$$o\mu = qdt^2 \cdot \sin(s-\omega).$$

Prout iam fuerit vel $om > o\mu$ vel $o\mu > om,$ punctum *o* ob utrumque motum coniunctim vel *m* versus vel μ versus per differentiam istorum arculorum feretur. Quare, si fuerit $om = o\mu,$ punctum *o* revera quiescet eritque propterea polus, circa quem corpus iam gyrati est censendum, ita ut ob vires sollicitantes axis gyrationis *IO* tempusculo *dt* in *Io* transferetur. Ad hanc igitur axis variationem momentaneam inveniendam ponamus $om = o\mu$ seu

$$\gamma dt \sin \omega = qdt^2 \cdot \sin(s-\omega),$$

erit

$$\gamma \sin \omega = qdt \sin s \cos \omega - qdt \cos s \sin \omega,$$

unde evidens est arculum $Oo = \omega$ esse infinite parvum ideoque $\sin \omega = \omega$ et $\cos \omega = 1$ hincque

$$\omega = \frac{qdt \sin s}{\gamma + qdt \cos s} = \frac{qdt \sin s}{\gamma}.$$

Circa hunc autem axem *Io* corpus tanta celeritate angulari gyratur, qua tempusculo *dt* puncta *O* et *S* in ω et *s* transferantur, unde ea cognosci poterit. Cum enim ea tempusculo *dt* conficiatur

$$\text{angulus} = \frac{O\omega}{Oo} = \frac{qdt^2 \sin s}{\omega} = dt(\gamma + qdt \cos s),$$

praecedente autem tempusculo ob similem vim, quippe quae nunc non subito exorta est putanda, angulus confectus censeri debeat $= dt(\gamma - qdt \cos s)$, ita ut differentia sit $2qdt^2 \cos s$, ipsa celeritas angularis augmentum accepit $2qdt \cos s$; atque ob similem rationem, quia valor *q*, dum ad variationes continuas definiendas inducitur, duplicari debet, etiam spatiolum *Oo* duplo maius est censendum. Dum enim in calculo punctum *O* continuo progredi assumitur, hic autem in o quiescens assumatur, intervallum *Oo* hic inventum diversum est a spatiolo, per quod polus gyrationis profertur: concipiatur enim punctum *o'*, ut sit $Oo' = 2Oo,$ ac dico fore *o'* polum gyrationis post tempus *dt*, cum initio esset *O*. Hoc enim posito manifestum est interea punctum *o* manere immotum. Quare, cum hic invenissemus $Oo = \frac{qdt \sin s}{\gamma}$, spatiolum *Oo'*, per quod polus gyrationis transiisse est consendus, erit duplo maius = $\frac{2qdt\sin s}{\gamma}$. Vires ergo, quae corpori, si quiesceret, imprimerent motum gyratorium circa axem *IS* in sensum $O\omega$, quo tempusculo *dt* absolveretur angulus $OS\omega = qdt^2$, motum corporis gyratorium iam insitum circa axem *IO* in sensum *Ss* celeritate angulari $=\gamma$ ita turbant, ut elapso tempusculo *dt* axis gyrationis sit recta *Io,* a preacedente *IO* versus *IS* vergens angulo

$$OIo = \frac{2qdt\sin s}{\gamma},$$

simulque celeritas gyratoria γ augmentum capiat $= 2qdt\cos s$.

COROLLARIUM 1

651. Si vires sollicitantes in sensum oppositum tenderent, quantitas q negative accipi deberet et punctum *o* in arcum *SO* ultra *O* productum caderet celeritasque gyratoria minueretur.

COROLLARIUM 2

652. Si arcus *OS* vel evanesceret vel semicirculo esset aequalis, axis gyrationis *IO* non mutaretur, sed totus effectus in priori motu gyratorio vel accelendo vel retardando consumeretur. Qui est casus iam supra pertractatus, ubi ostendimus incrementum vel decrementum celeritatis esse $2qdt$.

COROLLARIUM 3

653. Si arcus *OS* est quadrans circuli ideoque $\cos s = 0$, celeritas angularis γ nullam mutationem patietur, sed totus effectus virium in axe gyationis mutando insumetur, eum vel propius ad *S* vel longius inde removendo.

SCHOLION 1

654. Hic eiusmodi tantum vires sumus contemplati, quae corpori, si quiesceret, motum gyratorium simplicem imprimerent, centro inertiae manente immoto; cuiusmodi effectum producant vires quaecunque, si modo ipsis aequales et contrariae in centro inertiae applicentur, quemadmodum in superiori capite fusius est ostensum. Neque vero pro aliis viribus indagatio erit difficilior, cum eae eundem motum gyratorium semper producant, ac si ipsis aequales et contrariae centro inertiae essent applicatae; motus enim progressivus, quem corpori praeterea inducant, etiam hic nihil in motu gyratorio, qui corpori iam inest, esset mutarurus. Quin etiam, si in corpore praeter motum gyratorium circa axem *IO* iam inesset motus progressivus, is nihil a gyratione circa axem *IS* genita mutaretur; ex quo solutio huius problematis latissime patet atque etiam ad motum progressivum, quem corpus vel iam habet vel a viribus sollicitantibus nancisceretur, extendi potest. Quae combinatio motus progressivi cum gyratorius nihil habeat difficultatis, hic erat praecipuum opus, ut, quantum motus gyratorius ob alium motum gyratorium a viribus oriundum perturbetur, sollicite defineremus.

SCHOLION 2

655. Si axis *IO*, circa quem corpus iam gyrari assumitur, esset corporis axis principalis, corpus hunc motum, si a nullis viribus sollicitaretur, perpetuo esset conservaturum, uti in anticedentibus demonstravimus. Verum, si axis *IO* non sit principalis, etiamsi nullae vires extrinsecus urgerent, tamen motus conservari non posset, quoniam ipse motus vires suppeditat, quae ad axem gyrationis deflectendum tendunt; hoc ergo casu, si, quanta variato in axe gyrationis gignantur, explorare velimus, non sufficit vires extrinsecus in corpus agentes contempari, sed cum iis etiam coniungi debent vires ex ipso motu gyratorio natae, quibus axem supra affici ostendimus. Quae vires cum pendeant a positione axis gyrationis *IO* respectu axium principalium corporis, haud abs re erit, antequam ulterius progrediamur, in genere investigare, quomodo a viribus quibusque positio axis gyrationis respectu axium principalium corporis immutetur.

PROBLEMA 63

656. Data positione axis gyrationis respectu trium axium principalium corporis, isque a viribus sollicitantibus varietur, ut corpus elapso tempusculo minimo circa alium axem gyretur, definire positionem huius axis variati respectu axium principalium.

SOLUTIO

Consideretur iterum superficies sphaerica (Fig. 85), in cuius centro sit corporis centrum inertiae *I*, sintque nunc radii *IA*, *IB*, *IC* axes principales corporis corpusque circa axem *IO* gyretur celeritate angulari γ, cuius positio cum detur respectu axium principalium, ponatur arcus

$$AO = \alpha, BO = \beta \text{ et } CO = \gamma,$$

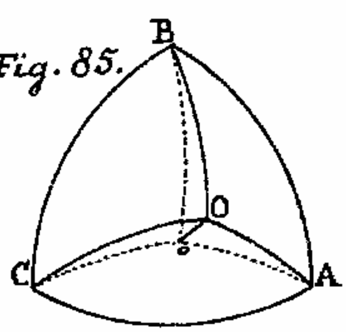
ut sit

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Tum vero ponatur anguli

$$BAO = \lambda, CBO = \mu, ACO = \nu,$$

Fig. 85.



erit ob quadrantes *AB*, *BC*, et *CA*

$$\cos\beta = \sin\alpha\cos\lambda, \quad \cos\gamma = \sin\beta\cos\mu, \quad \cos\alpha = \sin\gamma\cos\nu,$$

unde fit

$$\cos\lambda = \frac{\cos\beta}{\sin\alpha}, \quad \cos\mu = \frac{\cos\gamma}{\sin\beta}, \quad \cos\nu = \frac{\cos\alpha}{\sin\gamma},$$
$$\sin\lambda = \frac{\cos\gamma}{\sin\alpha}, \quad \sin\mu = \frac{\cos\alpha}{\sin\beta}, \quad \sin\nu = \frac{\cos\beta}{\sin\gamma},$$

ergo

$$\text{tang } \lambda = \frac{\cos\gamma}{\cos\beta}, \quad \text{tang } \mu = \frac{\cos\alpha}{\cos\gamma}, \quad \text{tang } \nu = \frac{\cos\beta}{\cos\alpha}$$

ideoque

$$\text{tang } \lambda \, \text{tang } \mu \, \text{tang } \nu = 1,$$

quae est relatio inter ternos angulos λ, μ, ν, ex quibus arcus α, β, γ ita definiuntur, ut sit :

$$\text{tang } \alpha = \frac{\text{tang } \nu}{\cos\lambda} = \frac{\cot\mu}{\sin\lambda}, \quad \text{tang } \beta = \frac{\text{tang } \lambda}{\cos\mu} = \frac{\cot\nu}{\sin\mu}, \quad \text{tang } \gamma = \frac{\text{tang } \mu}{\cos\nu} = \frac{\cot\lambda}{\sin\nu}.$$

His relationibus notatis ex datis $BAO = \lambda$ et $AO = \alpha$ reliqua sic definiuntur, ut sit

$$\cos\beta = \sin\alpha\cos\lambda, \quad \cos\gamma = \sin\alpha\sin\lambda,$$
$$\text{tang } \mu = \frac{\cot\alpha}{\sin\lambda}, \quad \text{tang}\,\nu = \text{tang } \alpha\cos\lambda.$$

Quodsi iam ob vires sollicitantes tempusculo *dt* axis gyrationis *IO* abeat in *Io*, totum corpus, quasi interea circa axem *Io* esset gyratum, considerari potest, quo motu puncta *A, B, C* suas distantias a puncto *o* conservabunt, ita ut elapso tempusculo dt polus gyrationis *o* a polis principalibus *A*, *B*, *C* habiturus sit distantias *Ao*, *Bo*, *Co*. Quare, si detur angulus elementaris $OAo = d\lambda$ et $Ao = \alpha + d\alpha$, variatio reliquorum per differentiationem consuetam elicietur :

$$d\beta = \frac{d\lambda\sin\alpha\sin\lambda - d\alpha\cos\alpha\cos\lambda}{\sin\beta},$$
$$d\gamma = \frac{-d\lambda\sin\alpha\cos\lambda - d\alpha\cos\alpha\sin\lambda}{\sin\gamma},$$
$$d\mu = \frac{-d\alpha\sin\lambda - d\lambda\sin\alpha\cos\alpha\cos\lambda}{\cos^2\alpha + \sin^2\alpha\sin^2\lambda},$$
$$d\nu = \frac{d\alpha\cos\lambda - d\lambda\sin\alpha\cos\alpha\sin\lambda}{\cos^2\alpha + \sin^2\alpha\sin^2\lambda}.$$

COROLLARIUM 1

657. Si ab his differentialibus ad integralia progredi liceret, in corpore ad quodvis tempus ille axis, circa quem tum sit gyraturum, eiusque positio respectu axium principalium assignari posset.

COROLLARIUM 2

658. Hic scilicet non ad ipsum motum corporis respicimus, sed tantum id agitur, ut variatio momentanea axis gyrationis respectu axium principalium cognoscatur, ideoque ipsa celeritas gyratoria hic in computum non est ingressa.

COROLLARIUM 3

659. Cum in praecedente problemate arculus *Oo* sit determinatus, hic erit

$$Oo = \sqrt{\left(d\alpha^2 + d\lambda^2 \sin^2 \alpha\right)},$$

tum vero pro positione huius arculi *Oo* respectu arcus *AO* seu *Ao* est

$$\text{tang } AoO = \frac{d\lambda \sin\alpha}{d\alpha}$$

seu

$$\sin AoO = \frac{d\lambda \sin\alpha}{Oo} \text{ et } \cos AoO = \frac{d\alpha}{Oo},$$

ita ut hinc habeamus elementa

$$d\alpha = Oo \cdot \cos AoO \quad \text{et} \quad d\lambda = \frac{Oo \cdot \sin AoO}{\sin\alpha}.$$

SCHOLION

660. Cognitis ergo viribus, quibus corpus, dum circa quempiam axem gyratur, sollicitatur, per caput praecedens is axis, circa quem quiescens gyrari inciperet, definiri, tum vero ope praecedentis problematis variatio in axe gyrationis facta explorari poterit. Verum, nisi corpus primo circa axem quempiam principalem gyratur, praeter vires externas, quibus corpus forte sollicitatur, imprimis eae vires, quae ex ipso motu gyratorio eiusque vi centrifuga nascuntur, perpendi debent. Quas vires etiamsi supra iam in genere assignavimus, tamen easdem nunc denuo respectu axium principalium, quatenus axis gyrationis ab iis discrepat, determinari oportet; quibus cognitis, cum facile sit eas cum viribus externis coniungere, eas deinceps solas contemplemur et, quantum positio axis gyrationis iis turbetur, accurate investigemus.

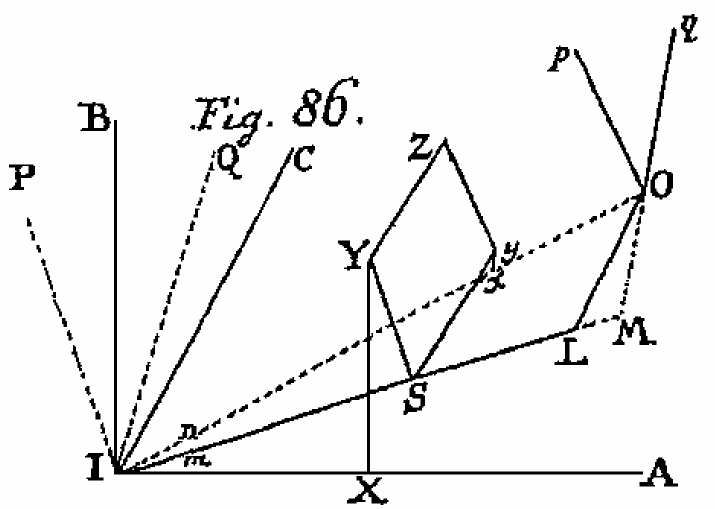
PROBLEMA 64

661. Si corpus rigidum gyretur circa axem quemcunque per eius centrum inertiae transeuntem, cuius positio respectu axium principalium detur, invenire vires hinc ad axem gyrationis turbandum natas.

SOLUTIO

Existente I centro inertiae sint *IA*, *IB* et *IC* eius axes principales (Fig. 86) eorumque respectu momenta inertiae *Maa*, *Mbb* et *Mcc*. Gyretur autem corpus circa axem *IO* celeritate angulari = γ, ex cuius quovis puncto *O* demittatur ad planum *AIB* perpendiculum *OL*, ductaque recta *IL* vocentur anguli *AIL* = *m* et *LIO* = *n*, ita ut pro situ huius axis *IO* respectu axium principalium sit

$$\cos AIO = \cos m \cos n, \quad \cos BIO = \sin m \cos n \text{ et } \cos CIO = \sin n.$$



Iam sumtis primo axibus principalibus pro directricibus iis parallelae constituantur ternae coordinate *IX* = *x*, *XY* = *y* et *YZ* = *z*,et in *Z* sumto corporis elemento *dM* erit ex natura axium principalium
ideoque :

$$\int xydM = 0, \quad \int xzdM = 0 \quad \text{et} \quad \int yzdM = 0,$$

tum vero

$$\int (yy + zz)dM = Maa, \quad \int (xx + zz)dM = Mbb, \quad \text{et} \quad \int (xx + yy)dM = Mcc$$

ideoque:

$$\int xxdM = \tfrac{1}{2}M(bb + cc - aa), \quad \int yydM = \tfrac{1}{2}M(aa + cc - bb), \quad \int zzdM = \tfrac{1}{2}M(aa + bb - cc).$$

Porro in plano *AIB* ducta *IP* ad *IL*, et in plano *LIC* recta *IQ* ad *IO* normali, ut rectae *IO*, *IP* et *IQ* sint inter se normales, quas tanquam directrices adhibeamus. Hunc in finem ducatur primo *YS* ipsi *IP* in plano *AIB* parallela, erit

$$IS = x\cos m + y\sin m \quad \text{et} \quad YS = y\cos m - x\sin m;$$

atque ex *Z* ipsi *YS* agatur parallela *Zy*, quae erit in planum *LIO* normalis, et $Zy = y\cos m - x\sin m,$ item $Sy = YZ = z.$ Denique ex *y* ad *IO* demittatur perpendiculum *yx*, ut iam desideratae coordinatae sint $Ix = X$, $xy = Y$ et $yZ = Z$, fietque

$$X = IS\cos n + Sy\sin n = x\cos m\cos n + y\sin m\cos n + z\sin n,$$
$$Y = yS\cos n - IS\sin n = z\cos n - x\cos m\sin n - y\sin m\sin n,$$
$$Z = y\cos m - x\sin m.$$

Cum iam elementum *dM* in *Z* ob celeritatem angularem = γ exerat vim centrifugam =

$$\frac{\gamma\gamma\cdot xZdM}{2g},$$

nascetur inde vis secundum *xy* =

$$\frac{\gamma\gamma YdM}{2g}$$

et vis secundum directionem ipsi *yZ* parallelam in *x* applicata =

$$\frac{\gamma\gamma ZdM}{2g},$$

quae vires ipsae cum se mutuo destruant, ob $\int YdM = 0$ et $\int ZdM = 0$ earum momenta tantum erunt spectanda. Sumta ergo $IO = f$ dabitur in *O* vis *Oq* ipsi *IQ* parallela omnibus viribus *yZ* aequivalens, si modo his viribus aequales et contrariae ipsi centro inertiae *I* applicentur. Cum igitur ob momenta sit

$$\text{vis } Oq\cdot IO = \frac{\gamma\gamma}{2g}\int XYdM$$

et

$$\text{vis } Op\cdot IO = \frac{\gamma\gamma}{2g}\int XZdM,$$

erit

$$\text{vis } Oq = \frac{\gamma\gamma}{2fg}\int XYdM \quad \text{et} \quad \text{vis } Op = \frac{\gamma\gamma}{2fg}\int XZdM.$$

At regrediendo ad coordinatas principales est

$$\int XYdM = \sin n \cos n \left(\int zzdM - \cos^2 m \int xxdM - \sin^2 m \int yydM \right)$$

et

$$\int XZdM = \sin m \cos m \cos n \left(\int yydM - \int xxdM \right)$$

ideoque per momenta inertiae data

$$\int XYdM = M \sin n \cos n \left(aa \cos^2 m + bb \sin^2 m - cc \right)$$

et

$$\int XZdM = M \sin m \cos m \cos n (aa - bb).$$

Consequenter ex motu gyatorio nascuntur hae vires

$$\text{vis } Op = \frac{M\gamma\gamma \sin m \cos m \cos n (aa-bb)}{2fg}$$

et

$$\text{vis } Oq = \frac{M\gamma\gamma \sin n \cos n \left(aa \cos^2 m + bb \sin^2 m - cc \right)}{2fg}$$

puncto *O* secundum directiones rectis *OP* et *OQ* parallelas applicatae, quibus autem aequales et contrariae in ipso centro inertiae I applicatae sunt intelligendae.

COROLLARIUM 1

662. Cum vis *Oq* sit ad axem gyrationis *IO* in O normalis, ea producta plano *AOB* in puncto *M* occurret, quod in *IL* producta erit situm, eritque

$IM = \frac{f}{\cos n}$ et $OM = f \tan n$ ob IOM angulum rectum.

COROLLARIUM 2

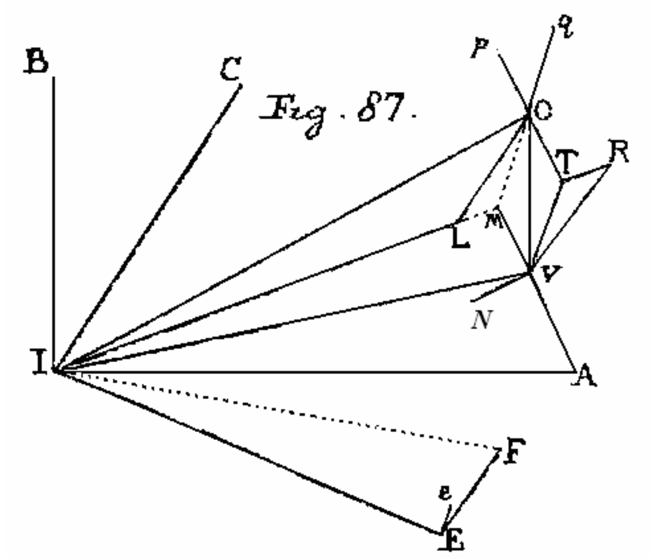
663. Directio autem alterius vis *Op* est ad planum *LIO* normalis, utpote rectae *IP* in plano *AIB* ad *IL* normali parallela; atque planum *pOq* continuatum ad planum *AIB* inclinatur angulo = $90^o - n$ idque intersecat recta ad *IM* normali.

COROLLARIUM 3

664. Quoniam hae vires ex motu gyratorio ipso natae sibi aequales et contrarias in centro inertiae applicatas habent, eae solum motum gyratorium perturbabunt neque corpori ullum motum progressivum inducent, ita ut centrum inertiae in quiete sit permansurum.

PROBLEMA 65

665. Inventis viribus ex motu gyratorio ipso natis ad eum perturbandum invenire axem, circa quem hae vires corpus, si esset in quiete, gyraturae essent.



SOLUTIO

Manentibus omnibus, ut in problemate praecedente, ita ut *IA*, *IB* et *IC* sint axes corporis principales (Fig. 87) eorumque respectu momenta inertiae *Maa, Mbb, Mcc,* sit *IO* axis, circa quem iam copus gyratur celeritas = γ, et pro eius situ anguli *AIL* =*m* et *LIO* = *n*, existente recta *OL* ad planum *AIB* normali, ut posita *IO* =*f* sit $IL = f\cos n$ et $OM = f\sin n.$ Tum vero ex *O* ad *IO* ducatur normalis *OM*, erit

$$IM = \frac{f}{\cos n} \quad \text{et} \quad OM = f \operatorname{tang} n,$$

ducta autem ad *IM* in plano *AIB* normali *MA*, erit

$$IA = \frac{f}{\cos m \cos n} \quad \text{et} \quad MA = \frac{f \operatorname{tang} m}{\cos n}.$$

Nunc autem in *O* habentur vires *Op* et *Oq*, quarum *Op* ipsi *AM* parallela et *Oq* cum *OM* in directum est sita; suntque hae vires :

$$\text{vis } Op = \frac{M\gamma\gamma \sin m \cos m \cos n(aa-bb)}{2fg}$$

et

$$\text{vis } Oq = \frac{M\gamma\gamma \sin n \cos n(aa\cos^2 m + bb\sin^2 m - cc)}{2fg}$$

quarum media directio planum *AIB* alicubi in *V* in recta *MA* secabit, ut sit $MO : MV = Oq : Op$, unde colligitur

$$MV = \frac{f \sin m \cos m(aa-bb)}{\cos n(aa\cos^2 m + bb\sin^2 m - cc)}$$

hincque

$$\text{tang } MIV = \frac{\sin m \cos m(aa-bb)}{aa\cos^2 m + bb\sin^2 m - cc};$$

ex quo concluditur

$$\text{tang } AIV = \frac{(bb-cc)\sin m}{(aa-cc)\cos m},$$

quem angulum supra [§639] vocavimus δ, at distantia

$$IV = \frac{f\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc(aa\cos^2 m + bb\sin^2 m)\right)}}{\cos n(aa\cos^2 m + bb\sin^2 m - cc)},$$

quam supra vocavimus = *h*, ut sit

$$h = \frac{f(bb-cc)\sin m}{\cos n \sin\delta(aa\cos^2 m + bb\sin^2 m - cc)}$$

seu

$$h = \frac{f(aa-cc)\cos m}{\cos n \sin\delta(aa\cos^2 m + bb\sin^2 m - cc)}.$$

Nunc igitur in puncto *V* illas vires applicatas concipere licet, quae sunt

$$\text{vis sec. } VM = \frac{M\gamma\gamma \sin m \cos m \cos n(aa-bb)}{2fg},$$

$$\text{vis sec. } VT = \frac{M\gamma\gamma \sin n \cos n(aa\cos^2 m + bb\sin^2 m - cc)}{2fg},$$

quarum haec secundum *VR* ipsi *LO* et *VN* ipsi *ML* parallelam resoluta dat

$$\text{vim sec. } VR = \frac{M\gamma\gamma \sin n \cos^2 n\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg}$$

et

$$\text{vim sec. } VN = \frac{M\gamma\gamma \sin^2 n \cos n\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg},$$

quarum illa *VR* supra littera *R* est indicata. At quod supra erat $Q\cos\delta - P\sin\delta$, qua expressione vis ad *IV* in plano *AIB* normalis denotatur, hic est

$$\text{vis } VM\cos MIV - \text{vis } VN\sin MIV,$$

unde prodit

$$Q\cos\delta - P\sin\delta = \frac{M\gamma\gamma \sin m\cos m\cos^3 n(aa-bb)\left(aa\cos^2 m + bb\sin^2 m - cc\right)}{2fg\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc\left(aa\cos^2 m + bb\sin^2 m\right)\right)}}.$$

Cum porro sit

$$\text{tang } \delta = \frac{(bb-cc)\sin m}{(aa-cc)\cos m},$$

erit

$$\cos\delta = \frac{(aa-cc)\cos m}{\sqrt{\left(a^4\cos^2 m + b^4\sin^2 m + c^4 - 2cc\left(aa\cos^2 m + bb\sin^2 m\right)\right)}}$$

His definitis sit iam *IF* axis ille, circa quem istae vires corpus, si quiesceret, essent gyraturae, ductoque ex *F* ex planum *AIB* perpendiculo *FE* vocentur anguli $AIE = \eta$ et $EIF = \vartheta$, ac per problema 60 consequimur :

$$\text{tang } \eta = \frac{aa\cos\delta}{bb\sin\delta} = \frac{aa(aa-cc)\cos m}{bb(bb-cc)\sin m}$$

et

$$\text{tang } \vartheta = \frac{Q\cos\delta - P\sin\delta}{Rcc\cos\delta}\cdot bb\sin\eta = \frac{\sin m\cos n(aa-bb)bb\sin\eta}{cc(aa-cc)\sin n}.$$

Denique tempusculo *dt* circa hunc axem *IF* angulus $d\omega$ generabitur, ut sit :

$$d\omega = \frac{\gamma\gamma dt^2 \sin n\cos n\sqrt{\left(a^4(aa-cc)^2\cos^2 m + b^4(bb-cc)^2\sin^2 m\right)}}{2aabb\cos\vartheta},$$

seu

$$d\omega = \frac{\gamma\gamma(aa-cc)dt^2\cos m\sin n\cos n}{2bb\sin\eta\cos\vartheta} = \frac{\gamma\gamma(bb-cc)dt^2\sin m\sin n\cos n}{2aa\cos\eta\cos\vartheta}.$$

COROLLARIUM 1

666. Si pro axe gyrationis proposita *IO* ponantur anguli

$$OIA=\alpha,\quad OIB=\beta,\quad OIC=\gamma,$$

at pro axe gyrationis elementaris *IF* anguli

$$FIA=\mathfrak{A},\quad FIB=\mathfrak{B},\quad FIC=\mathfrak{C},$$

erit

$$\cos\alpha=\cos m\cos n,\quad \cos\beta=\sin m\cos n,\quad \cos\gamma=\sin n$$

atque

$$\cos\mathfrak{A}=\cos\eta\cos\vartheta,\quad \cos\mathfrak{B}=\sin\eta\cos\vartheta,\quad \cos\mathfrak{C}=\sin\vartheta.$$

COROLLARIUM 2

667. Deinde ob

$$\text{tang }\eta=\frac{aa(aa-cc)\cos\alpha}{bb(bb-cc)\cos\beta},$$

si ponatur brevitatis gratia

$$\sqrt{\left(a^4(aa-cc)^2\cos^2\alpha+b^4(bb-cc)^2\cos^2\beta\right)}=W,$$

erit

$$\sin\eta=\frac{aa(aa-cc)\cos\alpha}{W}\quad \text{et}\quad \cos\eta=\frac{bb(bb-cc)\cos\beta}{W}.$$

Porro autem posito

$$\sqrt{\left(a^4b^4(aa-bb)^2\cos^2\alpha\cos^2\beta+a^4c^4(aa-cc)^2\cos^2\alpha\cos^2\gamma+b^4c^4(bb-cc)^2\cos^2\beta\cos^2\gamma\right)}=\Omega$$

habebitur:

$$\cos\mathfrak{A}=\frac{bbcc(bb-cc)\cos\beta\cos\gamma}{\Omega},$$

$$\cos\mathfrak{B}=\frac{aacc(cc-aa)\cos\alpha\cos\gamma}{\Omega},$$

$$\cos\mathfrak{C}=\frac{aabb(aa-bb)\cos\alpha\cos\beta}{\Omega}$$

et

$$d\omega=\frac{\gamma\gamma\Omega dt^2}{2aabbcc}.$$

SCHOLION

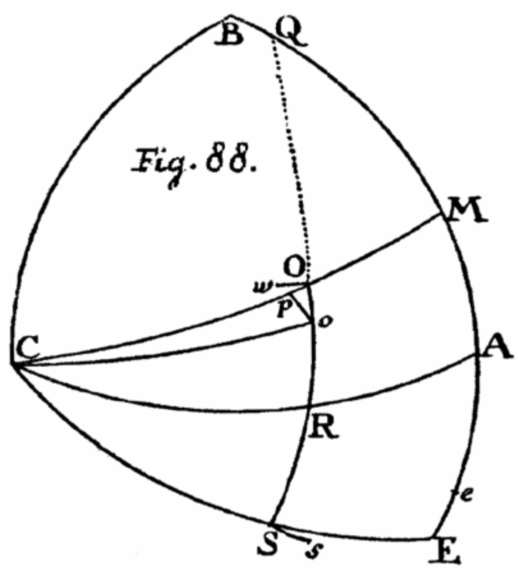
668. Quod ad sensum attinet, in quem gyratio circa axem *IF* fiet, quoniam angulus elementaris $d\omega = \frac{\gamma\gamma\Omega dt^2}{2aabbcc}$ semper est positivus, notandum est in indagatione huius valoris vim *VR* ut positivam esse spectatam, unde secundum figuram punctum *E* in sensum *Ee* versus *A* motu gyratorio feretur. Etsi enim haec ratio tantum in figura, ubi anguli m, n, η, ϑ sunt positivi et recto minores, locum habet, tamen hinc ratio sensus recte concludi potest; quo semel in calculum introducto deinceps generatim veritati inhaerebimus. Ceterum evidens est, si axis *IO* in quempiam principalium cadat, fore $d\omega = 0$; namque si $\alpha = 0$, fit $\beta = \gamma = 90^o$ ideoque $\cos\beta = \cos\gamma = 0$, quo casu utique quantitas Ω evanescit; simul vero perspicuum est nullo alio casu hanc perturbationem $d\omega$ evanescere posse ideoque plures tribus non dari axes gyrationis liberos, nisi forte duo momenta principalia fuerint aequalia.

PROBLEMA 66

669. Si corpus gyretur circa axem quemcunque per eius centrum inertiae transeuntem ab axibus principalibus diversum, definire variationem momentaneam, quam cum ipse axis gyrationis tum celeritas angularis patietur.

SOLUTIO

Transferantur omnia, quae in praecedente problemate sunt inventa, ad



superficiem sphaericam centro inertiae *I* descriptam, in qua *A*, *B*, *C* sint poli axium principalium (Fig. 88) eorumque respectu momenta inertiae *Maa*, *Mbb*, *Mcc*. Tum vero sit *O* polus axis illius, circa quem corpus iam gyratur celeritate angulari = γ in sensum *ABC*. Ex *C* per *O* ducto circulo maximo *COM*, qui est quadrans, erunt arcus $AM = m$ et $MO = n$; tum in quadrante *BA* producto capiatur $AE = \eta$ et ducto quadrante *CE* arcus $ES = \vartheta$, ut sit

$$\text{tang } \eta = \frac{aa(aa-cc)\cos m}{bb(bb-cc)\sin m},$$

seu

$$\frac{bb\sin m\sin\eta}{aa-cc} = \frac{aa\cos m\cos\eta}{bb-cc}$$

atque

$$\text{tang } \vartheta = \frac{bb\sin m\sin\eta\cdot(aa-bb)\cos n}{cc(aa-cc)\sin n}$$
$$= \frac{aa\cos m\cos\eta\cdot(aa-bb)\cos n}{cc(bb-cc)\sin n}.$$

His ita definitis ob vires corporis centrifugas corpus conabitur circa polum *S* gyrari in sensum *Ee*, ita ut tempusculo *dt* descripturum esset angulum

$$d\omega = \frac{\gamma\gamma(aa-cc)dt^2\cos m\sin n\cos n}{2bb\sin\eta\cos\vartheta} = \frac{\gamma\gamma(bb-cc)dt^2\sin m\sin n\cos n}{2aa\cos\eta\cos\vartheta}$$

seu

$$d\omega = \frac{\gamma\gamma(aa-bb)dt^2\sin m\cos m\cos^2 n}{2cc\sin\vartheta}.$$

Ducatur ergo arcus circuli maximi *OS*, qui sit = *s*, quem deinceps determinemus, atque in problemate 62 erit

$$q = \frac{\gamma\gamma(aa-bb)\sin m\cos m\cos^2 n}{2cc\sin\vartheta},$$

hincque ob motum gyratorium elementarem corpus gyrabitur circa polum o, ut sit

$$\text{arculus } Oo = \frac{\gamma\gamma(aa-bb)dt\sin m\cos m\cos^2 n\sin s}{cc\sin\vartheta};$$

celeritas autem angularis γ augmentum accipiet $d\gamma$, ut sit

$$d\gamma = \frac{\gamma\gamma(aa-bb)dt\sin m\cos m\cos^2 n\cos s}{cc\sin\vartheta}.$$

Nunc igitur primo quaeri debet positio arcus *OS* seu angulus *COS*, quo ad arcum *CO* inclinatur; quem in finem consideretur triangulum *OCS*, in quo est $OC = 90^0 - n, \quad CS = 90^0 - \vartheta,$ et angulus $OCS = m + \eta,$ unde reperitur :

$$\cot COS = \frac{\cos n \operatorname{tang} \vartheta}{\sin(m+\eta)} - \frac{\sin n \cos(m+\eta)}{\sin(m+\eta)}.$$

Est vero

$$\operatorname{tang}(m+\eta) = \frac{aa(aa-cc)\cos^2 m + bb(bb-cc)\sin^2 m}{(aa-bb)(cc-aa-bb)\sin m \cos m}$$

and

$$\frac{\cos n \operatorname{tang} \vartheta}{\sin(m+\eta)} = \frac{aabb(aa-bb)\sin m \cos m \cos^2 n}{cc \sin n\big(bb(bb-cc)\sin^2 m + aa(aa-cc)\cos^2 m\big)},$$

thus giving

$$\operatorname{tang} COS = \frac{cc \sin n\big(aa(aa-cc)\cos^2 m + bb(bb-cc)\sin^2 m\big)}{(aa-bb)\sin m \cos m\big(aabb\cos^2 n + cc(aa+bb)\sin^2 n - c^4 \sin^2 n\big)}.$$

Porro ex eodem triangulo *OCS* colligitur

$$\cos s = \cos(m+\eta)\cos n \cos \vartheta + \sin n \sin \vartheta = \sin \vartheta\left(\sin n + \frac{\cos n \cos(m+\eta)}{\operatorname{tang} \vartheta}\right)$$

seu

$$\cos s = \frac{\sin n \sin \vartheta\big(aabb - (aa+bb)cc + c^4\big)}{aabb} = \frac{(aa-cc)(bb-cc)\sin n \sin \vartheta}{aabb},$$

unde fit

$$d\gamma = \frac{\gamma\gamma(aa-bb)(aa-cc)(bb-cc)\sin m \cos m \sin n \cos^2 n}{aabbcc} \cdot dt.$$

Denique positis $OA = \alpha, \quad OB = \beta, \quad OC = \gamma$ erit arculus

$$Oo = \frac{\gamma dt}{aabbcc}\sqrt{\begin{pmatrix} a^4b^4(aa-bb)^2\cos^2\alpha\cos^2\beta + a^4c^4(aa-cc)^2\cos^2\alpha\cos^2\gamma \\ +b^4c^4(bb-cc)^2\cos^2\beta\cos^2\gamma \\ -(aa-bb)^2(aa-cc)^2(bb-cc)^2\cos^2\alpha\cos^2\beta\cos^2\gamma \end{pmatrix}}$$

et

$$d\gamma = \frac{\gamma\gamma(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc} \cdot dt.$$

Verum si ex o ad *CO* perpendiculum ducatur *op*, per regulas trigonometriae sphaericae arculi elementares *Op* et *op* ita rationaliter exprimuntur, ut sit :

$$Op = \frac{\gamma(aa-bb)dt\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)}{aabbcc\sin\gamma},$$

$$op = \frac{\gamma dt\cos\gamma\left(aa(aa-cc)\cos^2\alpha+bb(bb-cc)\cos^2\beta\right)}{aabbcc\sin\gamma}.$$

COROLLARIUM 1

670. Cum sit

$$d\gamma = \frac{\gamma\gamma(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc}\cdot dt,$$

patet, si trium momentorum principalium duo fuerint inter se aequalia, tum celeritatem angularem plane non immutari.

COROLLARIUM 2

671. Introductis distantiis α, β, γ poli *O* a polis principalibus *A*, *B*, *C* erit

$$\text{tang } COS = \frac{cc\cos\gamma\left(aa(aa-cc)\cos^2\alpha+bb(bb-cc)\cos^2\beta\right)}{(aa-bb)\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)},$$

ducto autem arcu *AO* erit

$$\text{tang } AOC = \frac{-\cos\beta}{\cos\alpha\cos\gamma},$$

unde concluditur

$$\text{tang } AOS = \frac{aa\cos\alpha\left(bb(bb-aa)\cos^2\beta+cc(cc-aa)\cos^2\gamma\right)}{(bb-cc)\cos\beta\cos\gamma\left((bb-aa)(cc-aa)\cos^2\alpha-bbcc\right)}.$$

COROLLARIUM 3

672. Haec formula pro angula *AOS* analoga est illi pro angulo *COS* indeque oritur, si litterae *a*, *b*, *c*, item α, β, γ in ordine uno loco promoveantur; hoc autem modo signum prodiret negativum, id quod rei naturae est consentaneum, cum angulus *AOS* in sensum contrarium cadat respectu prioris.

COROLLARIUM 4

673. Si arcus OS quadrantem AC secet in puncto R, collititur :

$$\text{tang } AR = \frac{aa\cos\alpha\left(bb(aa-bb)\cos^2\beta+cc(aa-cc)\cos^2\gamma\right)}{cc\cos\gamma\left(aa(aa-cc)\cos^2\alpha+bb(bb-cc)\cos^2\beta\right)},$$

ac si idem arcus *SO* productus occurrat quadranti *BA* in *Q*, erit per analogiam :

$$\text{tang } BQ = \frac{bb\cos\beta\left(cc(bb-cc)\cos^2\gamma+aa(bb-aa)\cos^2\alpha\right)}{aa\cos\alpha\left(bb(bb-aa)\cos^2\beta+bb(cc-aa)\cos^2\gamma\right)} = cot\ AQ.$$

COROLLARIUM 5

674. Cum tempusculo *dt* arcus $CO = \gamma$ minuatur particula *Op*, erit per differentialia

$$aabbccd\gamma\sin\gamma = \gamma(bb-aa)dt\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)$$

hincque per analogiam :

$$aabbccd\beta\sin\beta = \gamma(aa-cc)dt\cos\gamma\cos\alpha\left(aacc-(cc-bb)(aa-bb)\cos^2\beta\right),$$

$$aabbccd\beta\sin\alpha = \gamma(cc-bb)dt\cos\beta\cos\gamma\left(bbcc-(bb-aa)(cc-aa)\cos^2\alpha\right).$$

SCHOLION

675. Assumsimus in solutione, quod probe est notandum corpus circa axem IO in sensum *ABC* gyrari, ad quem ergo casum formulae inventae sunt accommodatae; sin autem corpus gyraretur in sensum contrarium, formulae facillime eo referentur statuendo celeritatem gyratoriam γ negativam. Atque sic problema hoc difficillimum, quo variatio momentanea quaeritur, dum corpus circa axem non-principalem gyratur, satis commodo resolvimus, cum formulae postremae, ad quas tandem solutio est perducta, non adeo sint intricatae, ut simpliciores expectare licuisset. Neque etiam suspicio ullius erroris in calculo commisi locum habet, cum formual, qua incrementum celeritatis angularis $d\gamma$ exprimitur, ad omnes tres axes principales aeque referatur, tum derivatio anguli *AOS* ex angulo *COS* rem firmissime evincit ac tandem aequationes in postremo corollario exhibitae hanc proprietatem habere deprehenduntur, ut sit

$$d\alpha\sin\alpha\cos\alpha + d\beta\sin\beta\cos\beta + d\gamma\sin\gamma\cos\gamma = 0,$$

uti conditio principalis

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

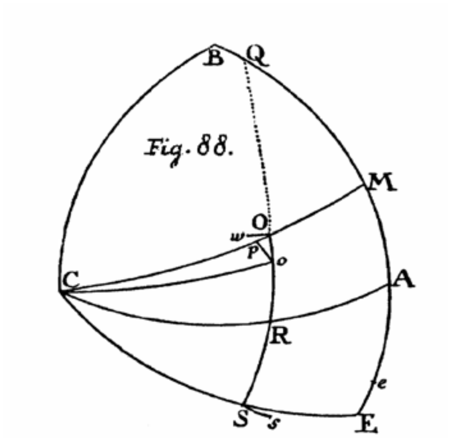
exigit. Ternae autem postremae aequationes cum ea, quae differentiale $d\gamma$ definit, plenam problematis solutionem continent, ubi quidem quaelibet trium illarum omitti potest. Si corpus insuper a viribus externis sollicitaretur, solutio non multo difficilior evaderet, quemadmodum in sequentem probleme ostendetur.

PROBLEMA 67

676. Si corpus rigidum, dum circa axem quemcunque per eius centrum inertiae transeuntem gyretur, a viribus quibuscunque sollicitetur, definire variationem momentaneam tam in ipso axe quam in celeritate angulari inde ortam.

SOLUTIO

Sit *IO* axis, circa quem corpus nunc gyretur celeritate angulari = γ in sensum *ABC* (Fig. 88), ac primo dispiciatur eius situs respectu axium principalium *IA*, *IB*, *IC*, quorum respectu momentia inertiae sint *Maa*, *Mbb*, *Mcc*, positisque arcubus $OA=\alpha, OB=\beta, OC=\gamma$ per problema praecedens quaeratur, quantum tempusculo *dt* tam axis gyrationis *IO* quam celeritas angularis ob solum motum gyatorium mutari debeat. Scilicet, si polus gyrationis ex *O* abeat in *o*, vidimus fore incrementum distantiae $CO=\gamma$:



$$Co-CO=\frac{\gamma(aa-bb)dt\cos\alpha\cos\beta\left(aabb-(aa-cc)(bb-cc)\cos^2\gamma\right)}{aabbcc\sin\gamma}$$

atque incrementum anguli *BCO*

$$OCo=\frac{\gamma dt\cos\gamma\left(aa(aa-cc)\cos^2\alpha+bb(bb-cc)\cos^2\beta\right)}{aabb\sin^2\gamma}$$

quibus elementis situs puncti *o* sine ambiguitate definitur. Praeter hanc autem axis gyrationis mutationem celeritas angularis γ capiet incrementum =

$$\frac{\gamma\gamma(aa-bb)(aa-cc)(bb-cc)\cos\alpha\cos\beta\cos\gamma}{aabbcc}dt\,.$$

Deinde perpendantur vires sollicitantes, utrum corpori motum progressivum imprimant : cuius rei facillimum est iudicium, dum omnes vires secundum suas quasque directiones ipsi centro inertiae applicatae concipiantur. Si enim se mutuo in aequilibrio teneant, corpori nullus motus progressivus imprimetur, sin autem detur vis illis aequivalens, ab hac motus progressivus in corpore generabitur ex primis principaliis facile definiendus. Tum isti vi aequivalenti aequalis et contraria ipsi centro inertiae applicetur, ut iam hoc centrum in quiete teneatur, atque hac vi cum iis, quibus corpus actu sollicitatur, coniuncta omnes revocentur ad duas, quarum altera in centro inertiae, altera in alio quodam puncto sit applicata, quae duae vires erunt aequales sed contrariae. Porro ex praecedente capite quaeritur axis, circa quem corpus ab istis viribus converti incipiet, simulque angulus conversionis momentaneae, unde per problema 62 sine ullo respectu ad mutationem iam inventam habito quoniam haec est infinite parva, quasi corpus adhunc circa axem *Oo* gyraretur, quaeratur variatio in axe et celeritate angulari inde orta, quarum illa ad incrementa vel decrementa tam in arcu *CO* quam in angulo *BCO* nata reducantur. Denique haec terna elementa cum iis, quae iam ante ex motu gyratorio sunt definita, coniungantur, sicque obtinebitur vera variatio tam in axe *IO* quam in celeritate angulari ab utraque causa simul producta.

SCHOLION

677. Dum virium sollicitantium effectus exploratur, variatio axis inde orta eodem modo per angulum elementarem *OCo* et differentiam arcuum *CO* et *Co* exprimi potest, quo hic usi sumus. Scilicet quaeratur primo axis, circa quem corpus, si quiesceret, a viribus vertetetur, qui sit *IS*, sitque qdt^2 angulus conversionis tempusclo *dt* productus circa *S* in sensum $O\omega$, ac pro puncto *S* ponatur arcus $AE=\eta$ et $ES=\vartheta$, qui valores a praecedentibus ex ipso motu gyratorio ortis probe sunt distinguendi. Cum ergo sit $AM = ACM = m$, ut sit

$$cos\, m=\frac{cos\,\alpha}{sin\,\gamma} \quad et \quad sin\, m=\frac{cos\,\beta}{sin\,\gamma},$$

erit $MCE=m+\eta$ et ex triangulo *OCS* reperitur :

$$cos\, OS=cos\, s=cos(m+\eta)\, sin\,\gamma\, cos\,\vartheta+cos\,\gamma\, sin\,\vartheta$$

et

$$cot\, COS=\frac{sin\,\gamma\,\mathrm{tang}\,\vartheta}{sin(m+\eta)}-\frac{cos\,\gamma\, cos(m+\eta)}{sin(m+\eta)}.$$

Nunc autem ex problemate 62 polus gyrationis *O* transfertur in *o*, ut sit

$$Oo=\frac{2qdt\, sin\, s}{\gamma}$$

et incrementum celeritatis angularis

$$2qdt\, cos\, s=2qdt\left(sin\,\gamma\, cos\,\vartheta\, cos(m+\eta)+cos\,\gamma\, sin\,\vartheta\right).$$

Deinde ex *Oo* elicitur

$$Op = Oo\cos COS = \frac{2qdt\cos s}{\gamma}\cos COS$$

et

$$op = Oo\sin COS = \frac{2qdt\sin s}{\gamma}\sin COS = \frac{2qdt}{\gamma}\cos\vartheta\sin(m+\eta)$$

ideoque

$$\text{angulus } OCo = \frac{2qdt\cos\vartheta\sin(m+\eta)}{\gamma\sin\gamma}.$$

Hinc vero porro deducitur

$$CO - Co = Op = op\cot COS = \frac{2qdt}{\gamma}\left(\sin\gamma\sin\vartheta - \cos\gamma\cos\vartheta\cos(m+\eta)\right).$$

Tantum ergo superest, ut haec elementa cum illis, quae ex motu gyratorio sunt eruta, combinentur, ut obtineatur axis gyrationis variatus cum incremento vel decremento celeritatis angularis.

PROBLEMA 68

678. Si ad aliquod tempus detur situs corporis rigidi circa quempiam axem per eius centrum inertiae transeuntem gyrantis atque tam axis gyrationis quam celeritas angularis utcunque varietur, invenire mutationem momentaneam in corporis situ ortam.

SOLUTIO

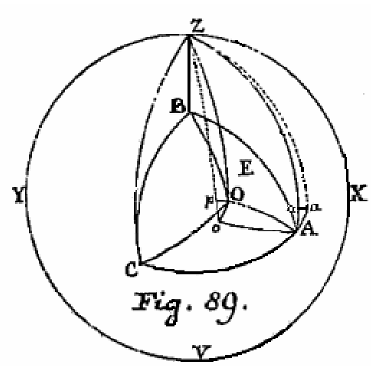


Fig. 89.

Cum centrum inertiae corporis quiescat, situs corporis referatur ad sphaeram fixam eodem centro descriptam, intra quam fixam corpus motum suum absolvat (Fig. 89). In hac sphaera capiatur circulus magnus *VXZY* in eoque punctum fixum *Z*; atque ad datum tempus = *t* axes corporis principales in superficie sphaerica respondeant punctis *A*, *B*, *C*, ut *AB*, *BC*, *CA* sint quadrants; ad quorum situm symbolis repraesentandum sint arcus circulorum maximorum

$$ZA = l, \quad ZB = m, \quad ZC = n,$$

erit

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

ac ponatur anguli

$$XZA = \lambda, \quad XZB = \mu, \quad XZC = \nu,$$

erit ex sphaericis

$$\cos(\mu-\lambda) = -\cot l \cot m,$$
$$\cos(\nu-\mu) = -\cot m \cot n,$$
$$\cos(\nu-\lambda) = -\cot l \cot n,$$

ergo

$$\cos(\mu-\lambda)\cos(\nu-\lambda) = \cot^2 l \cot m \cot n = -\cot^2 l \cos(\nu-\mu),$$

unde fit

$$\cot^2 l = -\frac{\cos(\mu-\lambda)\cos(\nu-\lambda)}{\cos(\nu-\mu)},$$
$$\cot^2 m = -\frac{\cos(\lambda-\mu)\cos(\nu-\mu)}{\cos(\nu-\lambda)},$$
$$\cot^2 n = -\frac{\cos(\lambda-\nu)\cos(\mu-\nu)}{\cos(\mu-\lambda)}.$$

Cum vero sit

$$\cos(\nu-\mu) - \cos(\mu-\lambda)\cos(\nu-\lambda) = \sin(\mu-\lambda)\sin(\nu-\lambda),$$
$$\cos^2 l = -\cot(\mu-\lambda)\cot(\nu-\lambda),$$
$$\cos^2 m = -\cot(\lambda-\mu)\cot(\nu-\mu),$$
$$\cos^2 n = -\cot(\lambda-\nu)\cot(\mu-\nu).$$

Hac relatione inter quantities $l, m, n, \lambda, \mu, \nu,$ quae tempusculo dt suis differentialibus crescere sunt censendae, notata sit nunc O polus gyrationis arcusque $AO = \alpha, \quad BO = \beta, \quad CO = \gamma$, ut sit :

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1;$$

erit

$$\cos BAO = \frac{\cos \beta}{\sin \alpha}$$

et

$$\sin BAO = \frac{\cos \gamma}{\sin \alpha},$$

at in triangulo *ZAB* est

$$\cos ZAB = \frac{\cos m}{\sin l} \quad et \quad \sin ZAB = -\cos ZAC = -\frac{\cos n}{\sin l},$$

ita ut sit pro triangulo *ZAO*

$$\sin ZAO = \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin\alpha\sin l}, \quad \cos ZAO = \frac{\cos\beta\cos m + \cos\gamma\cos n}{\sin\alpha\sin l},$$

unde colligitur

$$\cos ZO = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n$$

et

$$\cot AZO = \frac{\cos\alpha - \cos l(\cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n)}{\cos\gamma\cos m - \cos\beta\cos n},$$

hincque ad quodvis tempus polus gyrationi *O* innotescit.

Deinde posita celeritate angulari = γ' in sensum *ABC*, tempusculo *dt* punctum *A* circa *O* describit arculum $Aa = \gamma' dt \sin\alpha$, quare ducta $a\alpha$ ad *ZA* normali erit

$$A\alpha = \gamma' dt \cdot \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin l},$$

$$a\alpha = \gamma' dt \cdot \frac{\cos\beta\cos m + \cos\gamma\cos n}{\sin l},$$

unde differentialia quantitatum l et λ deducantur

$$dl \sin l = \gamma' dt(\cos\beta\cos n - \cos\gamma\cos m)$$

et

$$-d\lambda \sin^2\lambda = \gamma' dt(\cos\beta\cos m + \cos\gamma\cos n)$$

similique modo reperietur :

$$dm \sin m = \gamma' dt(\cos\gamma\cos l - \cos\alpha\cos n),$$

$$-d\lambda \sin^2 m = \gamma' dt(\cos\gamma\cos n + \cos\alpha\cos l)$$

$$dn \sin n = \gamma' dt(\cos\alpha\cos m - \cos\beta\cos l),$$

$$-d\nu \sin^2 n = \gamma' dt(\cos\alpha\cos l + \cos\beta\cos m).$$

Quodcirca, si ad quodvis tempus α, β, γ et $\gamma' t$ dentur quantitates ideoque earum differentialia tempusculo *dt* nata, hinc colliguntur variationes eodem tempusculo in arcibus *l*, *m*, *n* et angulis λ, μ, ν productae. Praeterea vero variatio in polo gyrationis *O* facta facile concluditur, quia tantum opus est, ut arcus *ZO* et angulus *AZO* differentientur, ponendo solum arcus α, β, γ variabiles, quia hoc modo polus *O* in situm sequentem *o* transfertur. Erit ergo

$$(Zo - ZO)\sin ZO = d\alpha\sin\alpha\cos l + d\beta\sin\beta\cos m + d\gamma\sin\gamma\cos n$$

et, cum sit

$$\cot AZO(\cos\gamma\cos m - \cos\beta\cos n) = \cos\alpha - \cos l\cos ZO,$$

erit

$$\frac{OZo}{sin^2 AZO}\left(cos\gamma\, cos\, m - cos\beta\, cos\, n\right) + cot\, AZO\left(d\gamma\, sin\gamma\, cos\, m - d\beta\, sin\beta\, cos\, n\right)$$
$$= d\alpha\, sin\alpha\, sin^2 l - d\beta\, sin\beta\, cos\, l\, cos\, m - d\gamma\, sin\gamma\, cos\, l\, cos\, n$$

hincque reducendo :

$$\frac{OZo}{sin^2 AZO}$$
$$= \frac{sin^2 l\left(d\alpha\, sin\alpha\left(cos\gamma\, cos\, m - cos\beta\, cos\, n\right) + d\beta\, sin\beta\left(cos\alpha\, cos\, n - cos\gamma\, cos\, l\right) + d\gamma\, sin\gamma\left(cos\beta\, cos\, l - cos\alpha\, cos\, m\right)\right)}{\left(cos\gamma\, cos\, m - cos\beta\, cos\, n\right)^2}$$

ac denique angulus elementaris OZo =

$$\frac{d\alpha\, sin\alpha\left(cos\gamma\, cos\, m - cos\beta\, cos\, n\right) + d\beta\, sin\beta\left(cos\alpha\, cos\, n - cos\gamma\, cos\, l\right) + d\gamma\, sin\gamma\left(cos\beta\, cos\, l - cos\alpha\, cos\, m\right)}{1 - \left(cos\alpha\, cos\, l + cos\beta\, cos\, m + cos\gamma\, cos\, n\right)^2},$$

in quam formulam bis ternae litterae α, β, γ, et l, m, n aequaliter ingrediuntur, ut natura rei postulat.

COROLLARIUM 1

679. Si ex O in Zo arculus Op perpendiculariter ducatur, erit

$$po = \frac{d\alpha\, sin\alpha\, cos\, l + d\beta\, sin\beta\, cos\, m + d\gamma\, sin\gamma\, cos\, n}{sin\, ZO}$$

et

$$Op = \frac{d\alpha\, sin\alpha\left(cos\gamma\, cos\, m - cos\beta\, cos\, n\right) + d\beta\, sin\beta\left(cos\alpha\, cos\, n - cos\gamma\, cos\, l\right) + d\gamma\, sin\gamma\left(cos\beta\, cos\, l - cos\alpha\, cos\, m\right)}{sin\, ZO}.$$

COROLLARIUM 2

680. Porro ex $sin\, BAO = \frac{cos\gamma}{sin\alpha}$ et $cos\, BAO = \frac{cos\beta}{sin\alpha}$ colgititur angulus

$$OAo = \frac{-d\alpha\, cos\alpha\, cos\gamma - d\gamma\, sin\alpha\, sin\gamma}{sin\alpha\, cos\beta}$$

hincque elementum

$$Oo = \frac{\sqrt{\left(\left(d\alpha^2 + d\gamma^2\right) sin^2\alpha\, sin^2\gamma + 2d\alpha d\gamma\, sin\alpha\, cos\alpha\, sin\gamma\, cos\gamma\right)}}{cos\beta},$$

quod cum aeque referatur ad α, β, γ, ob

$$d\alpha\, sin\alpha\, cos\alpha + d\beta\, sin\beta\, cos\beta + d\gamma\, sin\gamma\, cos\gamma = 0$$

reducitur ad

$$Oo = \sqrt{\left(d\alpha^2 \sin^2 \alpha + d\beta^2 \sin^2 \beta + d\gamma^2 \sin^2 \gamma\right)}.$$

COROLLARIUM 3

681. Ponamus $ZO = v$ et, cum sit

$$\cos v = \cos\alpha \cos l + \cos\beta \cos m + \cos\gamma \cos n,$$

erit

$$\text{tang } AZO = \frac{\cos\gamma \cos m - \cos\beta \cos n}{\cos\alpha - \cos l \cos v}$$

et ob analogiam, quia B alteram partem ipsius ZO in figura cadit,

$$-\text{tang } AZO = \frac{\cos\alpha \cos n - \cos\gamma \cos l}{\cos\beta - \cos m \cos v},$$

unde fit

$$\text{tang } AZB = \text{tang}\left(\mu - \lambda\right) = \frac{\cos n}{\cos l \cos m};$$

qui valor cum supra invento

$$\cos\left(\mu - \lambda\right) = -\frac{\cos l \cos m}{\sin l \sin m}$$

egregie conspirat, eritque

$$\sin\left(\mu - \lambda\right) = -\frac{\cos n}{\sin l \sin m}.$$

COROLLARIUM 4

682. Hinc ergo pro differentiis ternorum angulorum λ, μ, v has adipiscimur determinationes

$$\sin\left(\mu - \lambda\right) = -\frac{\cos n}{\sin l \sin m},$$

$$\sin\left(v - \mu\right) = -\frac{\cos l}{\sin m \sin n},$$

$$\sin\left(\lambda - v\right) = -\frac{\cos m}{\sin l \sin n},$$

$$\cos\left(\mu - \lambda\right) = -\frac{\cos l \cos m}{\sin l \sin m},$$

$$\cos\left(v - \mu\right) = -\frac{\cos m \cos n}{\sin m \sin n},$$

$$\cos\left(\lambda - v\right) = -\frac{\cos l \cos n}{\sin l \sin n}.$$

SCHOLION

683. Quae hactenus de mutatione momentanea, quam motus gyratorius tam per se quam ob vires sollicitantes subit, exposuimus, fundamentum constituunt universae Theoriae de motu corporum rigidorum, quandoquidem ex cognita mutatione elementari ad ipsam motus determinationem transitus per calculum integralem patet. Aggrediamur ergo motum liberum huiusmodi corporum, quo sive proprio quasi instinctui sive viribus sollicitantibus libere obsequi possunt, ac primo quidem vires sollicitantes externas removeamus, corpora sibi tantum relicta contemplaturi, ut extrinsecus nihil accedat, quod ad motum quicquem conferat. Quoniam autem indoles axium principalium, quibus corpus est praeditum, hic imprimis in computum ingreditur, inde naturale quasi discrimen in corporibus constitui conveniet, prout momenta inertiae eorum respectu fuerint comparata. Tres igitur corporum classes constituamus, ad quarum primum ea referamus corpora, quorum momenta respectu axium principalium sint inter se aequalia; ad secundum vero classem ea corpora, in quibus duo momenta respectu axium principalium sint aequalia, tertium vero illis inaequale. Tertia vero classis in genere omnia ea corpora complectatur, quorum momenta respectu axium principalium inter se sint inaequalia.