Chapter V

Concerning the Absolute Motion of Bodies Acted on by Forces in General.

PROBLEM 13

205. If a corpuscle is acted on by forces so that its motion is completed in the same plane, to define both the distance traversed as well as the position and speed of that at any time.

SOLUTION

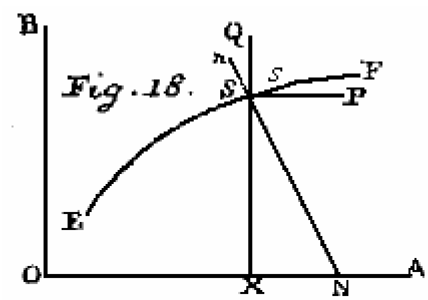


Fig. 18.

As the motion is made in the same plane, it is necessary that as the direction of the forces, upon which the body is continually acted on, as well as the first impressed direction of the motion, are placed in the same plane, which may be referred to as the plane of the table (Fig. 18). In which plane we are able to take freely two directions *OA* et *OB* normal to each other, most convenient to the calculation, and let *ESF* be the interval described by the corpuscle, in which it arrives on the elapsed time *t*, that is expressed in seconds, at the point *S*, thus on sending the perpendicular *SX* to *OA* there are the coordinates $OX = x$ and $XS = y$, on putting the interval itself traversed $ES = s$, in order that $ds = \sqrt{(dx^2 + dy^2)}$. Now let the mass of the corpuscle be equal to *A*, which clearly indicates the same weight, if it is situated in a region of the earth chosen with absolute measures [*i. e.* where the acc. of gravity is 1, in Euler's units], and at *S* it is acted on by any forces, these are allowed through static resolution to be applied by two forces, along the directions *SP* and *SQ* parallel to the directrices. Therefore let the force $SP = P$ and the force $SQ = Q$, both given by weights equal to themselves. With these in place, if the element of time *dt* is assumed constant, and it is understood that the motion is resolved equally along the directions *SP* and *SQ*, then the determination of the motion will be contained in these two formulas :

$$ddx = \frac{2gPdt^2}{A} \text{ and } ddy = \frac{2gQdt^2}{A},$$

where, because it is always to be retained that *g* denotes the height through which a weight drops in a time of one second in a specified region of the earth. Hence the speed of the motion along the side *SP*

$$=\frac{dx}{dt}=\frac{2g}{A}\int Pdt$$

and along *SQ*

$$=\frac{dy}{dt}=\frac{2g}{A}\int Qdt\ .$$

Now if the speed at *S* is put equal to *v*, on account of $v=\frac{ds}{dt}$ and $ds^2=dx^2+dy^2$ then this equation is derived:

$$dxddx+dyddy=dsdds=\frac{2gdt^2}{A}(Pdx+Qdy),$$

from which, since then $ds = vdt$ and $dds = dvdt$, it is elicited :

$$vdv=\frac{2g}{A}(Pdx+Qdy)$$

and hence

$$vv=\frac{4g}{A}\int(Pdx+Qdy).$$

Again on considering $dy = pdx$, so that $ds=dx\sqrt{(1+pp)}$, then

$$ddy=pddx+dpdx=\frac{2gQdt^2}{A}=\frac{2gPdt^2}{A}+dpdx$$

and thus

$$dp=\frac{2gQdt^2}{Adx}(Q-Pp)=\frac{2gdt^2}{Adx^2}(Qdx-Pdy).$$

But on account of

$$ds=vdt=dx\sqrt{(1+pp)}$$

then

$$\frac{dt}{dx}=\frac{\sqrt{(1+pp)}}{v}$$

and hence

$$dp=\frac{2g(1+pp)}{Avv}(Qdx-Pdy).$$

Now the radius of osculation of the curve ESF, in as much as it is seen to be concave towards OA , is equal to

$$-\frac{dx(1+pp)\sqrt{(1+pp)}}{dp}=-\frac{ds(1+pp)}{dp},$$

which if it is called equal to *r*, on account of $dp=-\frac{ds(1+pp)}{r}$ gives :

$$-\frac{ds}{r}=\frac{2g(Qdx-Pdy)}{Avv} \text{ or } \frac{Pdy-Qdx}{ds}=\frac{Avv}{2gr}.$$

COROLLARY 1

206. If therefore in place of the time *t* the speed *v* is introduced, the motion can be expressed from these two equations :

$$Avdv=2g(Pdx+Qdy) \text{ and } Avvds=2gr(Pdy-Qdx),$$

which can be taken more conveniently, if perhaps the forces *P* and *Q* depend on the speed of the body.

COROLLARY 2

207. Here the formula $\frac{Pdx+Qdy}{ds}$ is to be noted that expresses the tangential force, but $\frac{Pdy-Qdx}{ds}$ is the normal force, of which the former is called *T*, and now the latter *N*, we have

$$Avdv=2gTds \text{ and } Avv=2gNr,$$

which agree with the formulas treated in the above books.

COROLLARY 3

208. Moreover with these measures introduced the effect of the tangential force *T* in corresponds to this, as it becomes $T=\frac{Avdv}{2gds}$, moreover the effect of the normal force corresponds to this, as it becomes $N=\frac{Avv}{2gr}$. Or, on putting $dy = pdx$, on account of $r=-\frac{ds(1+pp)}{dp}$ then

$$N=-\frac{Avvdp}{2gds(1+pp)},$$

if indeed we take the normal force to be inclined towards the axis *OA*.

EXAMPLE

209. The body can be acted on continually by a constant force along the direction *BO* and equal to the weight *A* of this, as is had in the case of a body projected above the surface of the earth. Hence there is the force *P* = 0 and the force *Q* = – *A*, thus we have these equations :

$$ddx = 0 \text{ and } ddy = -2gdt^2 .$$

We consider the corpuscle thus to be projected initially from *O*, so that its speed is equal to *c* and the direction makes an angle equal to ζ with the line *OA*, which is considered horizontal, thus so that the initial speed of this along *OA* is equal to $c\cos\zeta$ and along OB = $c\sin\zeta$. With these in place, the first equation gives :

$$\frac{dx}{dt} = c\cos\zeta ,$$

and now the other :

$$\frac{dy}{dt} = c\sin\zeta - 2gt ,$$

since on putting *t* = 0 formulas $\frac{dx}{dt}$ and $\frac{dy}{dt}$ must be given the initial speeds. Again moreover on integrating, since on putting *t* = 0 so *x* as well as *y* must vanish, the equations become :

$$x = ct\cos\zeta \text{ and } y = ct\sin\zeta - gtt$$

or

$$-4gy = 4ggtt - 4cgt\sin\zeta$$

and hence

$$cc\sin^2\zeta - 4gy = (2gt - c\sin\zeta)^2 = (c\sin\zeta - \frac{2gx}{c\cos\zeta})^2 ,$$

thus it is apparent that the curve is the parabola satisfied by this equation :

$$(\frac{cc\sin\zeta\cos\zeta}{2g} - x)^2 = \frac{cc\cos^2\zeta}{g}(\frac{cc\sin^2\zeta}{4g} - y),$$

the parameter of this is equal to $\frac{cc\cos^2\zeta}{g}$ and the distance from the point O to the vertical axis is equal to $\frac{cc\sin\zeta\cos\zeta}{2g}$ and the height of the vertex above *OA* is equal to $\frac{cc\sin^2\zeta}{4g}$. Then on account of

$$\frac{ds}{dt} = \sqrt{(cc - 4cgt\, sin\,\zeta + 4ggtt)} = v$$

the speed at *S* is clearly

$$v = \sqrt{(cc - 4gy)}\,.$$

And hence on making $y = 0$ the length of the throw is found to be equal to

$$\frac{cc\, sin\,\zeta\, cos\,\zeta}{g}.$$

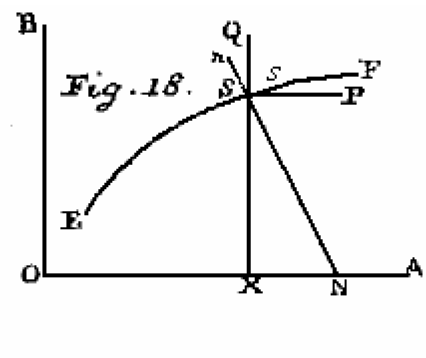
SCHOLIUM

210. I will not tarry here at this point with explanations of other related questions, since I might have pursued this whole argument more fully. But it may be noted that this [question] is concerned with performing absolute and thus free motion ; even if indeed I have deduced the motion of a weight, which as it refers to the earth and certainly is moving relative to the earth, and thus is very much in disagreement with absolute motion, yet in what follows it will be shown that this can be viewed as absolute motion. For since all terrestrial bodies are urged on by like forces, as these are effected by the earth itself, and since these bodies likewise are moving with respect to the earth, then if the earth itself should be at taken at rest then these forces may be absent, which is shown in the following chapter in a most excellent way. Now besides, these are to be understood for free motion, thus so that nothing extrinsic stands in the way, so that nothing hinders the action of external forces on the corpuscle, which is clearly seen to be agreed upon for forced motion, in which the corpuscle moves as if enclosed by a channel, and is unable to move in any other way except along to be led along the channel, I have considered motion of this kind in the second book. Now here I may add a single problem about channels [or tubes or pipes] formed in the same plane, where indeed I have removed from consideration all friction, by which is easier to see, how problems of this kind can be resolved with the aid or the new method and likewise the force of the corpuscle pressing on the sides of the tube can be defined.

PROBLEM 14

211. If a corpuscle should be enclosed in a channel formed in the same plane and likewise acted on by some forces, so to determine the motion of this corpuscle in the channel as well as the forces that it exerts on the channel.

SOLUTION



Hence the figure of the channel *ESR* is given as seen, which is referred to two directions *OA* and *OB* normal to each other as before (Fig. 18). Clearly if in the elapsed time *t* the corpuscle has arrived at *S*, then *OX* = *x*, *XS* = *y*, and the arc *ES* = *s*; moreover the applied forces acting in the same directions are *SP* = *P* and *SQ* = *Q* with the mass of the corpuscle being equal to *A*. Now in as much as the channel changes direction, that the corpuscle is to be travelling through, even if now it exerts unknown forces on that, which can be reduced to the same directions along *SP* = *X* and along *SQ* = *Y*, moreover from which it is agreed that by these forces the motion of the corpuscle is neither to be accelerated or retarded. Since now the forces along *SP* = *P* + *X* and along *SQ* = *Q* + *Y*, on putting the speed at *S* = *v* and the radius of osculation equal to *r*, we will have these equations from §206 :

$$Avdv = 2g((P+X)dx + (Q+Y)dy),$$

$$Avvds = 2gr((P+X)dx - (Q+Y)dx).$$

But since the forces *X* and *Y* bring nothing to the increment of the speed *dv*, then $Xdx + Ydy = 0,$ moreover from the other equation for knowing these forces it is elicited :

$$\frac{Xdy-Ydx}{ds} = \frac{Avv}{2gr} - \frac{Pdy-Qdx}{ds}.$$

Therefore in the first place the motion along the channel is determined by this equation :

$$Avdv = 2g(Pdx + Qdy),$$

thus the speed of the corpuscle *v* is known at some place *S* . Then the channel itself exerts forces of this kind *X* and *Y* along *SP* and *SQ* , in order that :

$$\frac{Xdx+Ydy}{ds}=0 \quad \text{and} \quad \frac{Xdy-Ydx}{ds}=\frac{Avv}{2gr}-\frac{Pdy-Qdx}{ds}.$$

Clearly, if these forces in the direction of the channel *Ss* and along the normal *SN* are reduced, then there arises no force along the direction of the channel, and along the normal *SN* there is a force which is :

$$\frac{Avv}{2gr}+\frac{-Pdy+Qdx}{ds},$$

and by such a force in turn the corpuscle urges the channel along the direction opposite to *Sn*, which is the pressing force sought.

COROLLARY 1

212. If therefore the corpuscle, is then moved by the channel, it is acted on by no external forces P and Q, and the motion of this on account of $Avdv=0$ is uniform. Then indeed everywhere the channel is pressed on by the normal force equal to $\frac{Avv}{2gr}$ along the direction *Sn*, opposite to the position of the radius of osculation.

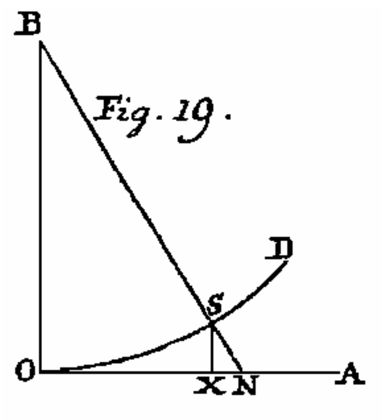
COROLLARY 2

213. This force, by which the channel is pressed, is hence called the *centrifugal force* $\frac{Avv}{2gr}$ that has arisen, because the corpuscle is compelled to progress along the line of the curve against the instigation of inertia, and this force is composed in the direct ratio of the mass *A*, of the square of the speed *v* and with the reciprocal of the radius of osculation *r*.

COROLLARY 3

214. If in addition the corpuscle is acted on by a tangential force along *Ss* equal to *T* and by a normal force along *SN* equal to *N*, in the first place there is $Avdv=2gTds$, then the channel is pressed along the direction *Sn* by a force equal to $\frac{Avv}{2gr}-N$.

EXAMPLE



215. If a corpuscle acted on by gravity is compelled to ascend along a circle *OS* (Fig. 19), the centre of this is *B*, and the radius *OB* = *b*, which is vertical and moreover the horizontal speed at *O* along *OA* is equal to *c*, then the force *P* = 0 and the force *Q* = –*A* and *r* = – *b*; hence for the motion there is given :

$$Avdv = -2Agdy \text{ or } vdv = -2gdy,$$

so that $vv = cc - 4gy$ an the speed vanishes at *D*, where $y = \frac{cc}{4g}$, but the force by which the channel is pressed along SB, is then equal to :

$$-\frac{A(cc-4gy)}{2gb} - \frac{Adx}{ds}.$$

Then on account of $xx + (b-y)^2 = bb$, there is

$$x = \sqrt{(2by - yy)},$$

$$dx = \frac{bdy - ydy}{\sqrt{(2by-yy)}}$$

and

$$ds = \frac{bdy}{\sqrt{(2by-yy)}},$$

and hence the pressing force along *SB* is equal to :

$$-\frac{Acc}{2bg} + \frac{2Ay}{b} - \frac{A(b-y)}{b} = -A + \frac{3Ay}{b} - \frac{Acc}{2gb},$$

which because it is negative, the force acting on the channel along *SN* is equal to

$$A(1 + \frac{cc}{2bg} - \frac{3y}{b}).$$

Moreover since

$$v = \sqrt{(cc - 4gy)},$$

then the element of the time

$$dt = \frac{ds}{v} = \frac{bdy}{\sqrt{(cc-4gy)(2by-yy)}}$$

or on account of

$$y=\frac{cc-vv}{4g} \text{ and } dy=-\frac{vdv}{2g}$$

then

$$dt=-\frac{2bdv}{\sqrt{(cc-vv)(8bg-cc+vv)}}.$$

If the initial speed c is infinitely small besides b, and because v cannot exceed c, then approximately

$$dt=-\frac{dv}{\sqrt{(cc-vv)}}.\sqrt{\frac{b}{2g}}$$

and on integrating

$$t=\frac{\sqrt{b}}{\sqrt{2g}}.A\cos\frac{v}{c}.$$ [A cos means arc cos]

Hence if π is the semi circumference of the circle of which the radius is equal to 1, then the total time of the ascent to D, because the speed v vanishes, is equal to $\frac{\pi\sqrt{b}}{2\sqrt{2g}}$, which is called the time of the *semi–oscillation*. Whereby, in order that the time of a complete oscillation $\frac{\pi\sqrt{b}}{\sqrt{2g}}$ is one second or equal to 1, radius $BO = b$ must be taken equal to $\frac{2g}{\pi\pi}$, which is the length of a simple pendulum with individual oscillations of one second. [*i. e.* the time for a half period] Whereby, if g = 15,625 Rhenish feet, then the length of this pendulum is equal to 3.166287 Rhenish feet.
[Rather like taking g = 32 ft/sec^2 in the old units, where g is now the acceleration of gravity, rather than Euler's 'g', the distance fallen from rest by a body in one second. This problem has of course been transformed into the frictionless bead on the vertical circular wire arc.]

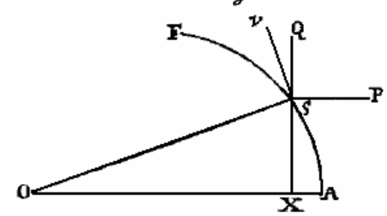
SCHOLIUM

216. There is no need, as I may advise here, that only a channel should thus be considered, in order that the motion is compelled to move along a given line; moreover, just as that can be done in a number of ways by pendulums, a case of this kind has been set out to be considered in the previous example. Again, I have treated at length enough problems of this kind in the second book of the *Mechanicae*. But since here it may be appropriate and worth the effort to set out more carefully a method normally applied to the motions of celestial bodies, that I have not explained previously, and that at last henceforth can be used here. Moreover it is related to problem 13 and only differs from that, in that the motion is not defined by coordinates, but by distances from a fixed point and the angles about that point. Therefore in as much as this motion is solved in a plane, as well as that to be investigated that I treat according to this method, later for the same motion not made in the same plane will be made clear.

PROBLEM 15

217. If a corpuscle can move freely in a plane, in which two forces are acting always, with the one pulling towards a certain fixed point *O*, the other now in a direction normal to that (Fig. 20), at some time to define the distance of the corpuscle *S* from the fixed point *O* and the angle *AOS*.

Fig. 20.



SOLUTION

In the elapsed time *t* the corpuscle, of which the mass is equal to *A*, arrives at *S* from *A*, and the distance *OS* is put equal to *u* and the angle *AOS* = φ. Moreover at *S* in the first place is acted on by a force pushing along *SO*, which shall be equal to *V*, then indeed by a force acting along the direction *SV* normal to *OS*, which shall be equal to *S*. That case we are able to reduce more easily to problem 13, by sending the perpendicular *SX* from *S* to the fixed line *OA* and we introduce the coordinates *OX* = *x* and *XS* = *y*, then $x = u\cos\phi$ and $y = u\sin\phi$. Then we can apply now the two forces *V* and *S* in the same directions *SP* and *SQ* and we will have the force $SP = -V\cos\varphi - S\sin\varphi$ and the force $SQ = -V\sin\varphi + S\cos\varphi$, which we have called *P* and *Q* above. On account of which we find these two equations :

$$ddx = -\frac{2gdt^2}{A}(V\cos\varphi + S\sin\varphi),$$
$$ddy = -\frac{2gdt^2}{A}(V\sin\varphi - S\cos\varphi),$$

and from the combination of these we deduce :

$$ddx\cos\varphi + ddy\sin\varphi = -\frac{2gVdt^2}{A},$$
$$ddx\sin\varphi - ddy\cos\varphi = -\frac{2gSdt^2}{A}.$$

But since $x = u\cos\phi$ and $y = u\sin\phi$, then

$$x\cos\varphi + y\sin\varphi = u \text{ et } x\sin\varphi - y\cos\varphi = 0,$$

thus on different ion :

$$dx\cos\varphi + dy\sin\varphi = du \text{ et } dx\sin\varphi - dy\cos\varphi + ud\varphi = 0$$

or

$$dx\sin\varphi - dy\cos\varphi = -ud\varphi$$

and on differentiation again :

$$ddx\cos\varphi + ddy\sin\varphi + ud\varphi^2 = ddu,$$
$$ddx\sin\varphi - ddy\cos\varphi + dud\varphi = -dud\varphi - udd\varphi.$$

With which values substituted we will arrive at these two equations for the determination of the motion:

$$\text{I.}\ ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0,$$
$$\text{II.}\ udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0.$$

COROLLARY 1

218. The second equation multiplied by u and by integration is reduced to this :

$$uud\varphi = \frac{2gdt}{A}\int Sudt,$$

where it is to be noted that $\frac{1}{2}uud\varphi$ expresses the element of the area *AOS*, thus this area is equal to $\frac{g}{A}\int dt\int Sudt$. Therefore with the vanishing of the lateral force *SV* = *S*, this area *AOS* is proportional to the time *t*, as it will have been compared with the other force *V* acting towards the point *O*.

COROLLARY 2

219. If the first equation is multiplied by *du*, and the second by $ud\varphi$, then the sum becomes

$$duddu + udud\varphi^2 + uud\varphi dd\varphi = -\frac{2gVdt^2du}{A} + \frac{2gSudt^2d\varphi}{A},$$

thus by integration it is elicited that

$$du^2 + uud\varphi^2 = -\frac{4gdt^2}{A}\int(Sud\varphi - Vdu),$$

where $\sqrt{(du^2 + uud\varphi^2)}$ expresses the element of the arc *AS*, thus in order that $\frac{du^2+uud\varphi^2}{dt^2}$ shall be the square of the speed at *S*.

COROLLARY 3

220. If the second equation is multiplied by $2u^3 d\varphi$, on account of *dt* being constant, the integral is found :

$$u^4 d\varphi^2 = \frac{4gdt^2}{A}\int Su^3 d\varphi,$$

thus by the preceding we come upon :

$$uudu^2 = \frac{4gdt^2}{A}(uu\int Sud\varphi - \int Su^3 d\varphi - uu\int Vdu)$$

or

$$uudu^2 = \frac{4gdt^2}{A}(2\int udu\int Sud\varphi - uu\int Vdu),$$

where it is to be noted that the element of the time *dt* is to be found outside the integration sign.

COROLLARY 4

221. If *S* = 0, which is the case of the centripetal force, then there arises $uud\varphi = ffdt$ and $ud\varphi = \frac{ffdt}{u}$, from which with the value in corollary 2 substituted it becomes :

$$du^2 = -\frac{f^4 dt^2}{uu} - \frac{4gdt^2}{A}\int Vdu + ccdt^2$$

and thus

$$dt = \frac{udu}{\sqrt{(ccuu - f^4 \text{-} 4guu\int Vdu : A)}}$$

and

$$d\varphi = \frac{ffdu}{u\sqrt{(ccuu - f^4 \text{-} 4guu\int Vdu : A)}}.$$

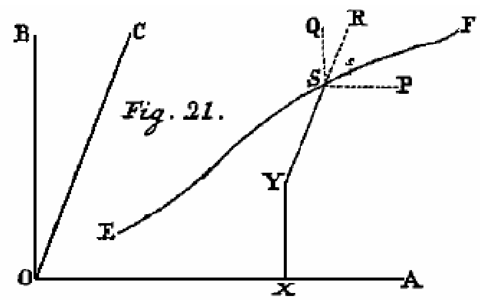
SCHOLIUM

222. The uses of these formulas is greatest in theoretical astronomy, and from these it is customary to determine the longitude, the anomaly, and the distance of a planet to be acted on at a certain point. Now this is not the place to pursue these further, since they are concerned with astronomy. Nevertheless it suffices here that problems of this kind to be treated are to be set forth in general; hence we may progress to the consideration of motion that is not in the same plane.

PROBLEM 16

223. If a corpuscle is freely moving under the action of some forces, to determine the motion of this body by three coordinates normal to each other.

SOLUTION



With the three directrices *OA, OB* and *OC* put in place normal to each other in turn (Fig. 21), a corpuscle may be moved, of which the mass is equal to *A*, on the line *ESF* and in the elapsed time *t* it arrives at *S*, thus with the perpendicular *SY* sent to the plane *AOB*, from *Y* to *OA* the normal *YX* is acting, in order that the three coordinates may be given between the normals and parallel to the directrices, which may be called *OX* = *x*, *XY* = *y* and *YS* = *z*, moreover the distance thus traversed ES is said to be equal to *s*, in order that there arises $ds = \sqrt{(dx^2 + dy^2 + dz^2)}$ and the speed at $S = \frac{ds}{dt}$, which is put equal to *v*. Now the corpuscle at *S* that may be acting on from any forces, these clearly are allowed to be reduced to the same three directions. Hence it is acted on by these forces *SP* = *P*, *SQ* = *Q* and *SR* = *R*, the effect of which will be determined from above by the three following equations :

$$ddx = \frac{2gPdt^2}{A}, \quad ddy = \frac{2gQdt^2}{A} \quad \text{and} \quad ddz = \frac{2gRdt^2}{A},$$

where a certain element *dt* has been taken constant. Therefore as the forces *P, Q, R* depend on the coordinates x, y, z or even on the speed $\frac{ds}{dt} = v$, aids are to be sought for the resolution from analysis. Meanwhile it helps to know, since there is obtained $ds^2 = dx^2 + dy^2 + dz^2$ and $vv = \frac{ds^2}{dt^2}$ and thus

$$vdv = \frac{dsdds}{dt^2} = \frac{dxddx + dyddy + dzddz}{dt^2},$$

to become :

$$vdv = \frac{2g}{A}(Pdx + Qdy + Rdz),$$

from which the acceleration of the corpuscle is defined. Moreover, in order to find the curve there is put in place $dy = pdx$ and $dz = qdx$, in order that

$$ds = dx\sqrt{(1+pp+qq)} \text{ and } v = \frac{dx}{dt}\sqrt{(1+pp+qq)}.$$

Hence on account of $ddy = pddx + dpdx$ and $ddz = qddx + dqdx,$ if in place of ddx the value $\frac{2gPdt^2}{A}$ is substituted, then there is found

$$dpdx = \frac{2gdt^2}{A}(Q-Pp) \text{ and } dqdx = \frac{2gdt^2}{A}(R-Pq).$$

Whereby, if here for dt^2 there is written $\frac{dx^2(1+pp+qq)}{vv}$, then

$$dp = \frac{2gdx(1+pp+qq)}{Avv}(Q-Pp),$$
$$dq = \frac{2gdx(1+pp+qq)}{Avv}(R-Pq)$$

or

$$Qdx - Pdy = \frac{Avvdp}{2g(1+pp+qq)}$$

and

$$Rdx - Pdz = \frac{Avvdq}{2g(1+pp+qq)}.$$

But if for p and q the values $\frac{dy}{dx}$ and $\frac{dz}{dx}$ are reinstated, there arises

$$Qdx - Pdy = \frac{Avv(dxddy - dyddx)}{2gds^2},$$
$$Rdx - Pdz = \frac{Avv(dxddz - dzddx)}{2gds^2},$$

which divided in turn give

$$P(dzddy - dyddz) + Q(dxddz - dzddx) + R(dyddx - dxddy) = 0.$$

COROLLARY 1

224. Therefore the speed at some point on the curve is determined from this differential equation

$$Avdv = 2g(Pdx + Qdy + Rdz),$$

where $\frac{Pdx+Qdy+Rdz}{ds}$ designates the tangential force arising from the forces acting.

COROLLARY 2

225. Moreover for the curve to be defined two from these three equations are sufficient :

$$2gds^2(Qdx-Pdy)=Avv(dxddy-dyddx)=Avvdx^2d.\frac{dy}{dx},$$
$$2gds^2(Pdx-Rdy)=Avv(dzddx-dxddz)=Avvdz^2d.\frac{dx}{dz},$$
$$2gds^2(Rdy-Qdz)=Avv(dyddz-dzddy)=Avvdy^2d.\frac{dz}{dy},$$

for any two likewise involve the third. There is hence an excess in the number of the agreed [independent] differential equations.

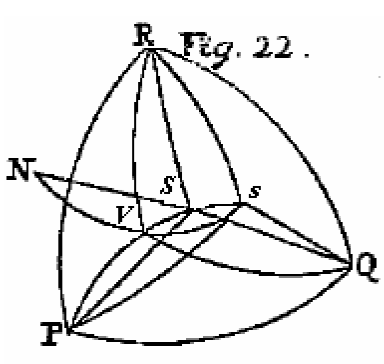
COROLLARY 3

226. With the final equation exempt from the speed, although it retains a second order differential equation, yet it has not been restricted to the assumed constant differential *dt*, thus indeed it can be represented by

$$Pdz^2d.\frac{dy}{dz}+Qdx^2d.\frac{dz}{dx}+Rdy^2d.\frac{dx}{dy}=0.$$

SCHOLIUM

227. The three forces *P*, *Q*, *R*, that we put acting on the corpuscle at *S*, are reduced to one force, which is equal to $\sqrt{(PP+QQ+RR)}$, and if that is put equal to *V*, the direction of this is inclined to *SP* at an angle, the cosine of which is equal to $\frac{P}{V}$, to SQ an angle, the cosine of which is equal to $\frac{Q}{V}$, and to *SR* at an angle, the cosine of which is equal to $\frac{R}{V}$. Then if the direction of the force *V* makes an angle equal to ω with the direction of the motion *Ss*, then the accelerating force or the force acting along *Ss* is equal to $V\cos\omega$, which since it is equal to $\frac{Pdx+Qdy+Rdz}{ds}$, then becomes



$$\cos\omega = \frac{Pdx+Qdy+Rdz}{Vds},$$

thus the normal force is gathered to be equal to $V\sin\omega$, of which the position is best represented with the aid of spherical trigonometry (Fig. 22). *S* is considered as the centre of a sphere, thus the straight lines *SP*, *SQ* and *SR* are extended to the surface, in order that the arcs *PQ*, *PR* and *QR* are quadrants; the direction of the motion passes through *s* and the mean direction of the forces through *V*, and we have :

$$\cos Ps = \frac{dx}{ds},\quad \cos Qs = \frac{dy}{ds},\quad \cos Rs = \frac{dz}{ds},$$

$$\cos PV = \frac{P}{V},\quad \cos QV = \frac{Q}{V},\quad \cos RV = \frac{R}{V}$$

and besides $Vs = \omega$ or

$$\cos\omega = \frac{Pdx+Qdy+Rdz}{Vds}.$$

With the angle ω known, *sVN* is taken equal to a quadrant, the right line drawn from the centre *S* through *N* is in a direction normal to the force ; and the position of the point *N* is thus defined from the distances of this from the points *P*, *Q*, and *R*, in order that :

$$\cos PN = \frac{P}{V\sin\omega} - \frac{dx\cos\omega}{ds\sin\omega},$$

$$\cos QN = \frac{Q}{V\sin\omega} - \frac{dy\cos\omega}{ds\sin\omega}$$

and

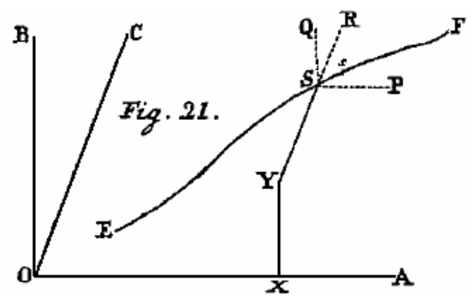
$$\cos RN = \frac{R}{V\sin\omega} - \frac{dz\cos\omega}{ds\sin\omega}.$$

Therefore since an infinitude of right lines are given normal to the direction of motion *Ss*, among these that one is determined, along which the normal force acts, and along which the direction of motion curves, in order that the radius of curvature lies on the same line *SN*, which is equal to $\frac{Avv}{2gV\sin\omega}$ (§207).

PROBLEM 17

228. If the corpuscle, the mass of which is equal to *A*, is moving in a tube or a channel (Fig. 21), and not acted on by any forces, to determine the motion of this and the force that it exerts on the tube everywhere.

SOLUTION



Let *ESF* be the figure of a tube [or pipe], in which the corpuscle is moving, and in the elapsed time it will have reached *S* with the distance $ES = s$ completed. Moreover as the position *S* as before is referred to the three fixed directions *OA*, *OB*, and *OC* normal to each other, for which the parallel coordinates may be called $OX = x$, $XY = y$ and $YS = z$. Now since the corpuscle everywhere is forced to follow the direction of the tube, the tube itself exerts the necessary forces on that, which moreover are thus to be compared, as the speed is not allowed to have any change. Hence the constant speed, which is set equal to *c*, thus becomes $\frac{ds}{dt} = c$ and $s = ct$. The forces exerted by the tube may be reduced to the same three directions, and become $SP = X$, $SQ = Y$ and $SR = Z$, and on account of the unchangeable speed, $Xdx + Ydy + Zdz = 0$. Then, since $dt = \frac{ds}{c}$, in place of the constant *dt* there is the element *ds*, from which the principle formulas are :

$$Accddx = 2gXds^2,\ Accddy = 2gYds^2 \text{ and } Accddz = 2gZds^2$$

with $dx^2 + dy^2 + dz^2 = ds^2$ being present. Hence the total force, that the tub exerts on the corpuscle, becomes :

$$\sqrt{(X^2 + Y^2 + Z^2)} = \frac{Acc\sqrt{(ddx^2 + ddy^2 + ddz^2)}}{2gds^2} = V,$$

the direction of this is inclined to the line *SP* by the angle,

of which the cosine is equal to $\frac{X}{V} = \frac{ddx}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$,

to SQ by the angle,

of which the cosine is equal to $\frac{Y}{V} = \frac{ddy}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$

and to SR by the angle,
of which the cosine is equal to $\frac{Z}{V} = \frac{ddz}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$.

Moreover, equal and opposite to this force is the pressing force [it is better not to use the word pressure here which is intended to be used], that the corpuscle in turn exerts on the tube.

COROLLARY 1

229. If the radius of osculation of the curve at *S* is put equal to *r*, on account of the normal force equal to *V* and the speed equal to *c*, then $r = \frac{Acc}{2gV}$, and thus

$$r = \frac{ds^2}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$$ on taking *ds* constant.

COROLLARY 2

230. Moreover the position of the radius of osculation with the direction of the force *V*, by which the corpuscle is pressed on by the tube, it is agreed, therefore that is inclined to the line *SP* by the angle,
of which the cosine is equal to $\frac{ddx}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$,
to *SQ* by the angle,
of which the cosine is equal to $\frac{ddy}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$,
and to *SR* by the angle,

of which the cosine is equal to $\frac{ddz}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$.

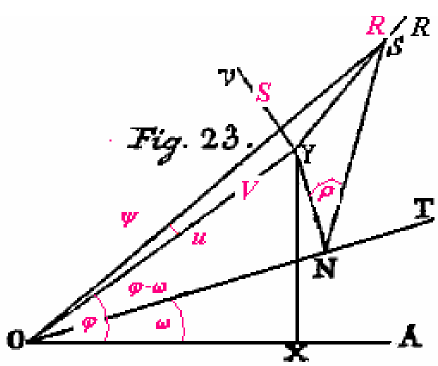
SCHOLIUM

231. It is also possible for the motion to be considered, when the corpuscle is not forced along a given curve but yet forced to progress on a surface, but since this argument has been treated at great length in the second book of the *Mechanica*, lest here I should be exceedingly verbose, I will not touch on that. Especially since it is apparent that the whole calculation here corresponds to that, since the direction of the force that the surface exerts on the corpuscle is normal to the surface itself. Whereby from the equation of the proposed surface the position of the proposed normal is determined, or the inclination of this to the three directions *SP*, *SQ* and *SR*, as it must agree with the position of the force *V* defined before. And hence a new equation is gathered between the coordinates x, y, z, which on being joined with the former given equation, defines the path traversed on the surface that has been seen to be the shortest between the two terminal points themselves. Hence I shall return to the free motion, and I will show how motion not made in the same plane can be defined in agreement with the angles referred to certain fixed points, clearly following from that reasoning that I have set out above (problem 5, §70). Since this has the greatest use in theoretical astronomy, and neither has this evolution of solving motion problems been set out in the previous books, so we dedicate the following problem to this.

PROBLEM 18

232. If a corpuscle is acted on by forces in part towards a fixed point O and in part by some other forces, to define the motion of this body with respect to this point.

SOLUTION



With the plane put in place, which shall be the plane of the table (Fig. 23; the red variables have been added in the translation), passing through the fixed point O, to which the motion is referred, and in that same with the fixed directrix OA taken, the corpuscle arrives at S in the elapsed time t, thus first the perpendicular SY is sent to the plane, and from Y the normal YX is sent to the line OA, in order that the orthogonal coordinates $OX = x$, $XY = y$, and $YS = z$ are obtained. Now since in the first place the corpuscle at S is acted on by a force along SO, that is resolved along the directions YO and SY ; now these remaining forces can then be reduced into forces in the same direction as these and along YV normal to OY in the plane of the table, thus so that in general three forces may be given, of which the first along $YO = V$, the second along $YV = S$ and the third along $SR = R$. Since these forces are known, they can be resolved in the directions of the coordinates, and thus on putting the angle $AOY = \varphi$, these forces are obtained :

force along $XO = V\cos\varphi + S\sin\varphi = -P$,

force along $YX = V\sin\varphi - S\cos\varphi = -Q$,

force along SR= R,

the effect of which is expressed by the three following formulas :

$$Addx = -2gdt^2(\mathrm{V}\cos\varphi + S\sin\varphi),$$
$$Addy = -2gdt^2(\mathrm{V}\sin\varphi - S\cos\varphi),$$
$Addz = \ 2gRdt^2$ with the mass of the body put as A.

Again calling the distance $OY = u$, and on account of $x = u\cos\varphi$ and $y = u\sin\varphi$ the two first equations used above (§217) are reduced to these two :

$$\text{I.}\quad ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0,$$
$$\text{II.}\quad udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0.$$

Now the angle $SOY = \psi$ is put in place, which is called the latitude of the corpuscle, then the angle $AOY = \varphi$ is the longitude of this corpuscle; then $SY = z = u$ tang ψ . But for this angle ψ, it shall be more convenient to find the nodal line OT, the angle $AOT = \omega$, and the inclination of the plane through O and the direction of the motion at S drawn to the plane through O is assumed equal to ρ, then $TOY = \varphi - \omega$, hence with the lines YN and SN drawn normal to OT, it becomes $ON = u\cos(\varphi - \omega)$ and $YN = u\sin(\varphi - \omega)$ and thus $YS = u\sin(\varphi - \omega)\text{tang }\rho = z$, and hence tang $\psi = \sin(\varphi - \omega)\text{tang }\rho$ and as used above (§70):

$$\frac{d\omega}{\text{tang}(\varphi\text{-}\omega)} = \frac{d\rho}{\sin\rho\cos\rho} = d.l\ \text{tang }\rho.$$

Whereby since then $d\rho = \frac{d\omega\sin\rho\cos\rho}{\text{tang}(\varphi\text{-}\omega)}$, it follows that

$$dz = du\sin(\varphi\text{-}\omega)\text{tang }\rho + u(d\varphi - d\omega)\cos(\varphi\text{-}\omega)\ \text{tang}\rho + u\sin(\varphi\text{-}\omega).\frac{d\omega\ \text{tang }\rho}{\text{tang}(\varphi\text{-}\omega)}$$
$$\text{or }\ dz = (du\sin(\varphi\text{-}\omega) + ud\varphi\cos(\varphi\text{-}\omega))\ \text{tang}\rho,$$

which value again differentiated gives :

$$ddz = (ddu\sin(\varphi\text{-}\omega) + du(2d\varphi - d\omega)\cos(\varphi\text{-}\omega) + udd\varphi\cos(\varphi\text{-}\omega) - ud\varphi(d\varphi - d\omega)\sin(\varphi\text{-}\omega))\text{tang }\rho$$
$$+ (du\sin(\varphi - \omega) + ud\varphi\cos(\varphi\text{-}\omega)).\frac{d\omega\ \text{tang }\rho}{\text{tang}(\varphi\text{-}\omega)}$$

or

$$ddz = \left(ddu\sin(\varphi\text{-}\omega) + 2dud\varphi\cos(\varphi\text{-}\omega) + udd\varphi\cos(\varphi\text{-}\omega) - ud\varphi^2\sin(\varphi\text{-}\omega) + \frac{ud\varphi d\omega}{\sin(\varphi\text{-}\omega)}\right)\text{tang }\rho.$$

Since therefore there arises

$$ddu - ud\varphi^2 = -\frac{2gVdt^2}{A} \text{ and } udd\varphi + 2dud\varphi = \frac{2gSdt^2}{A},$$

there is obtained :

$$ddz = \left(\frac{-2gVdt^2}{A} sin(\varphi\text{-}\omega) + \frac{2gSdt^2}{A} cos(\varphi\text{-}\omega) + \frac{ud\varphi d\omega}{sin(\varphi\text{-}\omega)}\right) \text{tang } \rho.$$

Whereby on account of $ddz = \frac{2gRdt^2}{A}$ it follows that

$$\frac{ud\varphi d\omega}{sin(\varphi\text{-}\omega)} = \frac{2gdt^2}{A}\left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho\right)$$

or

$$d\omega = \frac{2gdt^2\, sin(\varphi\text{-}\omega)}{Aud\varphi}\left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho\right)$$

and

$$d.l \text{ tang } \rho = \frac{2gdt^2\, cos(\varphi\text{-}\omega)}{Aud\varphi}\left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho\right).$$

Therefore four equations have been found, in which the solution of the problem is contained.

COROLLARY 1

233. When therefore for the given time t , these four quantities have be assigned $u, \varphi, \omega,$ and $\rho,$ and we have obtained in the first place these two differentio-differential equations :

$$ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0 \text{ et } udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0,$$

then these two simpler differentials :

$$d\omega = \frac{2gdt^2\, sin(\varphi\text{-}\omega)}{Aud\varphi}\left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho\right)$$

and

$$d.l \text{ tang } \rho = \frac{d\omega}{\text{tang}(\varphi\text{-}\omega)} \text{ or } d.\text{tang } \rho = \frac{d\omega \text{ tang } \rho}{\text{tang}(\varphi\text{-}\omega)}.$$

COROLLARY 2

234. Moreover from these values found it may be gathered that both the angle $SOY = \psi$ called the *latitude,* as well as the true distance *SO* can be found from these formulas

$$\text{tang } \psi = sin(\varphi\text{-}\omega)\text{tang } \rho \text{ and } OS = \frac{u}{cos\,\omega},$$

where *u* is accustomed to be called the shortened distance.

COROLLARY 3

235. If it should be the case that $sin(\varphi\text{-}\omega) = 0,$ that is, if the corpuscle is assumed to cross through the plane, we now see from above that $d\omega = 0$; but now it is apparent that both the nodal line and the inclination suffer no change, if it should be the case that

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) + R\,cot\,\rho = 0.$$

COROLLARY 4

236. Now it is the case that

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) = -Q\,cos\,\omega + P\,sin\,\omega,$$

and with the first forces *P, Q, R* introduced then

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) + R\,cot\,\rho = P\,sin\,\omega - Q\,cos\,\omega + R\,cot\,\rho,$$

and this is as if it is the force is changing the position of the nodal line as well as the inclination.

SCHOLIUM

237. It deserves to be noted here first, since the momentary variation in the position of the nodal line and the inclination can be expressed in a satisfactory manner by this method, that thus there are great benefits to theoretical astronomy. From this source the most excellent Lunar Tables have been deduced with incredible enthusiasm by the most Celebrated Prof. Mayer of Gottingen, by which it must be agreed that astronomy has risen to the most exalted peak [Johan Tobias Mayer, Commentarii Societatis regiae scientiarum 3, Gottingae 1753. Noted in *O. O.*] Moreover since the motion of the moon, which is defined by this method, is by no means absolute, but related to the centre of the earth, in this investigation likewise the motion of the earth must be given; whereby, in order that we are able to use this method, it is fitting to consider in advance, with the aid of which the relative motion can be applied to the calculation, if indeed the motion of this body can be known, with respect to which the motion of the other body is to be reckoned. Since this argument has not been set out with sufficient clarity in the above *Mechanicae* books, here I will set this out with more care; with which done for the motion of finite bodies, which I have not yet touched on, one may be able to progress with a more happy outcome.

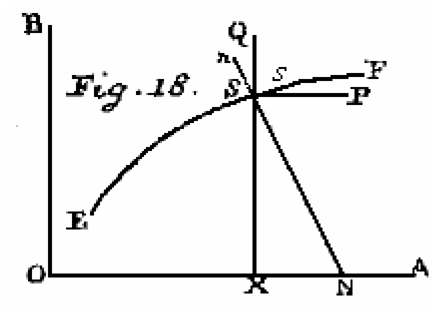
CAPUT V

DE MOTU ABSOLUTO CORPUSCULORUM A VIRIBUS QUIBUSCUNQUE ACTORUM

PROBLEMA 13

205. Si corpusculum a viribus ita sollicitetur, ut motum suum in eodem plano absolvat, definire tam spatium percursum quam ad quodvis tempus eius locum et celeritatem.

SOLUTIO



Ut motus fiat in eodem plano, tam directiones virium, quibus continuo sollicitatur, quam directio motus primo impressi in eodem plano sitae sint necesse est, quod planum ipsa tabula referatur (Fig. 18). In quo ad lubitum assumantur binae directiones *OA* et *OB* ad calculi commoditatem inter se normales sitque *ESF* spatium a corpusculo descriptum, in quo pervenerit elapso tempore *t*, quod in minutis secundis exprimatur, in punctum *S*, unde ad *OA* demisso perpendiculo *SX* sint coordinatae $OX = x$ et $XS = y$, posito ipso spatio percurso $ES = s$, ut sit $ds = \sqrt{(dx^2 + dy^2)}$. Sit iam massa corpusculi = *A*, quae scilicet eius pondus indicaret, si in regione terrae ad mensuras absolutas electa versaretur, et quibuscunque viribus in *S* sollicitetur, eas per resolutionem staticam ad duas revocare licet, secundum directiones *SP* et *SQ* directricibus parallelas. Sit ergo vis $SP = P$ et vis $SQ = Q$, ambae in ponderibus ipsis aequalibus datae. His positis, si temporis elementum *dt* constans assumatur motusque pariter secundum directiones *SP* et *SQ* resolutus intelligatur, determinatio motus his duabus formulis continebitur :

$$ddx = \frac{2gPdt^2}{A} \text{ et } ddy = \frac{2gQdt^2}{A},$$

ubi, quod perpetuo tenendum, *g* denotat altitudinem, per quam grave in regione terrae memorata uno minuto secundo delabitur. Hinc erit celeritas motus lateralis = *SP*

$$=\frac{dx}{dt}=\frac{2g}{A}\int Pdt$$

et secundum *SQ*

$$=\frac{dy}{dt}=\frac{2g}{A}\int Qdt\ .$$

Quodsi iam celeritas vera in *S* ponatur = *v*, ob $v=\frac{ds}{dt}$ et $ds^2=dx^2+dy^2$ derivabitur inde haec aequatio :

$$dxddx+dyddy=dsdds=\frac{2gdt^2}{A}(Pdx+Qdy),$$

ex qua, cum sit $ds = vdt$ et $dds = dvdt$, elicitur :

$$vdv=\frac{2g}{A}(Pdx+Qdy)$$

hincque

$$vv=\frac{4g}{A}\int(Pdx+Qdy).$$

Porro posito $dy = pdx$, ut sit $ds=dx\sqrt{(1+pp)}$, erit

$$ddy=pddx+dpdx=\frac{2gQdt^2}{A}=\frac{2gPdt^2}{A}+dpdx$$

ideoque

$$dp=\frac{2gQdt^2}{Adx}(Q-Pp)=\frac{2gdt^2}{Adx^2}(Qdx-Pdy).$$

At ob

$$ds=vdt=dx\sqrt{(1+pp)}$$

erit

$$\frac{dt}{dx}=\frac{\sqrt{(1+pp)}}{v}$$

hincque

$$dp=\frac{2g(1+pp)}{Avv}(Qdx-Pdy).$$

Verum curvae ESF, quatenus versus OA concava spectatur, radius osculi est =

$$-\frac{dx(1+pp)\sqrt{(1+pp)}}{dp}=-\frac{ds(1+pp)}{dp},$$

qui si vocetur = *r*, ob $dp=-\frac{ds(1+pp)}{r}$ habebitur :

$$-\frac{ds}{r} = \frac{2g(Qdx - Pdy)}{Avv} \text{ seu } \frac{Pdy - Qdx}{ds} = \frac{Avv}{2gr}.$$

COROLLARIUM 1

206. Si ergo loco temporis t introducatur celeritas v, motus his duabus aequationibus exprimetur :

$$Avdv = 2g(Pdx + Qdy) \text{ et } Avvds = 2gr(Pdy - Qdx),$$

quae commodius adhibentur, si forte vires P et Q a celeritate corporis pendeant.

COROLLARIUM 2

207. Hic notandum est formulam $\frac{Pdx+Qdy}{ds}$ exprimere vim tangentialem, at $\frac{Pdy-Qdx}{ds}$ vim normalem, quarum illa si dicatur = T, haec vero = N, habebimus

$$Avdv = 2gTds \text{ et } Avv = 2gNr,$$

quae conveniunt cum formulis superiori libro traditis.

COROLLARIUM 3

208. His autem introductis mensuris effectus vis tangentialis T in hoc consistit, ut sit $T = \frac{Avdv}{2gds}$, vis autem normalis effectus in hoc, ut sit $N = \frac{Avv}{2gr}$. Seu posito $dy = pdx$ ob $r = -\frac{ds(1+pp)}{dp}$ erit

$$N = -\frac{Avvdp}{2gds(1+pp)},$$

siquidem vim normalem versus axem OA vergere sumamus.

EXEMPLUM

209. Sollicitetur corpusculum continuo secundum directionem BO vi constante et eius ponderi A aequali, ut habeatur casus corporis supra terram proiecti. Erit ergo vis $P = 0$ et vis $Q = -A$, unde habemus has aequationes :

$$ddx = 0 \text{ et } ddy = -2gdt^2.$$

Ponamus corpusulum initio in O ita esse proiectum, ut fuerit eius celeritas = c et directio fecerit cum recta OA, quae horizontalis fingatur, angulum = ζ, ita ut initio eius celeritas secundum *OA* fuerit = $c\cos\zeta$ et secundum OB = $c\sin\zeta$. His positis, prior aequatio dabit

$$\frac{dx}{dt} = c\cos\zeta,$$

altera vero

$$\frac{dy}{dt} = c\sin\zeta - 2gt,$$

quoniam posito $t = 0$ formulae $\frac{dx}{dt}$ et $\frac{dy}{dt}$ dare debent celeritates initiales. Porro autem integrando, quia posito $t = 0$ tam x quam y evanescere debet, fiet

$$x = ct\cos\zeta \text{ et } y = ct\sin\zeta - gtt$$

seu

$$-4gy = 4ggtt - 4cgt\sin\zeta$$

hincque

$$cc\sin^2\zeta - 4gy = (2gt - c\sin\zeta)^2 = (c\sin\zeta - \frac{2gx}{c\cos\zeta})^2,$$

unde patet curvam esse parabolam hac aequatione contentam

$$(\frac{cc\sin\zeta\cos\zeta}{2g} - x)^2 = \frac{cc\cos^2\zeta}{g}(\frac{cc\sin^2\zeta}{4g} - y),$$

cuius parameter = $\frac{cc\cos^2\zeta}{g}$ et axis verticalis a puncto *O* distans intervallo = $\frac{cc\sin\zeta\cos\zeta}{2g}$ atque vertices supra *OA* elevatio = $\frac{cc\sin^2\zeta}{4g}$. Deinde ob

$$\frac{ds}{dt} = \sqrt{(cc - 4cgt\sin\zeta + 4ggtt)} = v$$

fiet celeritas in *S* nempe

$$v = \sqrt{(cc - 4gy)}\,.$$

Ac denique facto $y = 0$ reperitur longitudo iactus =

$$\frac{cc\sin\zeta\cos\zeta}{g}.$$

SCHOLION

210. Aliis quaestionibus huc pertinentibus evolvendis hic non immoror, cum totum hoc argumentum iam fusius sim persecutus. Notetur autem hic agi de motu absoluto eoque libero; etsi enim motum gravium hinc deduxi, qui cum ad terram referatur utique est respectivus atque a motu absoluto plurimum discrepans, tamen in sequentibus ostendetur eum tanquam absolutum spectari posse. Cum enim omnia corpora terrestria similibus viribus urgeantur atque ipsa terra, his efficitur, ut ea respectu terrae perinde moveantur, ac si terra quiesceret eaeque vires abessent, id quod capite sequente luculenter ostendetur. Praeterea vero haec intelligenda sunt de motu libero, ita ut extrinsecus nihil obstet, quominus corpusculum actioni virium obsequatur, quem motum probe discerni convenit a motu coacto, quo corpusculum quasi canali inclusum aliter nisi secundum ductum canalis moveri nequit, cuiusmodi motus in libro secundo sum contemplatus. Hic vero unicum adiiciam problema circa canalem in eodem plano formatum, ubi quidem ab omni frictione mentem abstraho, quo facilius perspiciatur, quomodo huiusmodi problemata ope huius novae methodi resolvi simulque pressio corpusculi in latera tube definiri debeat.

PROBLEMA 14

211. Si corpusculum canali in eodem plano formato fuerit inclusum simulque a viribus quibuscunque sollicetur, determinare tam eius motum in canali quam pressionem, quam in canalem exerit.

SOLUTIO

Figura ergo canalis *ESR* ut data spectatur, quae ad binas directiones *OA* et *OB* inter se normales referatur ut ante (Fig. 18). Scilicet si elapso tempore t corpusculum pervenerit in *S*, sit $OX = x$, $XS = y$, arcus $ES = s$; vires autem sollicitantes ad easdem directiones revocatae sint $SP = P$ et $SQ = Q$ existente corpusculi massa = *A*. Iam quatenus canalis inflectit directionem, quam corpusculum per se esset secuturum, in id vires exerit etiamnum incognitas, quae ad easdem directiones reductae sint secundum $SP = X$ et secundum $SQ = Y$, de quibus autem hoc constat motum corpusculi ab iis neque accelerari neque retardari. Cum nunc sint vires secuncum $SP = P + X$ et secundum $SQ = Q + Y$, posita celeritate in $S = v$ et radio osculi = *r* habebimus ex §206 has aequationes :

$$Avdv = 2g((P+X)dx+(Q+Y)dy),$$
$$Avvds = 2gr((P+X)dx-(Q+Y)dx).$$

Sed quia vires *X* et *Y* nihil conferunt ad celeritatis incrementum *dv*, erit $Xdx+Ydy=0,$ ex altera autem aequatione pro harum virium cognitione elicitur

$$\frac{Xdy-Ydx}{ds}=\frac{Avv}{2gr}-\frac{Pdy-Qdx}{ds}.$$

Primo igitur motus per canalem determinatur hac aequatione :

$$Avdv=2g(Pdx+Qdy),$$

unde cereritas corpusculi v in quovis loco S cognoscitur. Deinde ipse canalis eiusmodi vires X et Y secundum directiones SP et SQ exerit, ut sit

$$\frac{Xdx+Ydy}{ds}=0 \text{ et } \frac{Xdy-Ydx}{ds}=\frac{Avv}{2gr}-\frac{Pdy-Qdx}{ds}.$$

Scilicet, si hae vires ad directionem canalis Ss et normalis SN reducantur, inde oritur secundum directionem canalis vis nulla et secundum normalem SN vis, quae est =

$$\frac{Avv}{2gr}+\frac{-Pdy+Qdx}{ds},$$

atque tanta vi vicissim corpusculum urget canalem secundum directionem oppositam Sn, quae est pressio quaesita.

COROLLARIUM 1

212. Si ergo corpusculum, dum per canalem movetur, a nullis viribus externis P et Q sollicitatur, motus eius ob $Avdv=0$ erit uniformis. Tum vero ubique canalem premet normaliter vi = $\frac{Avv}{2gr}$ secundum directionem Sn positioni radii osculi oppositam.

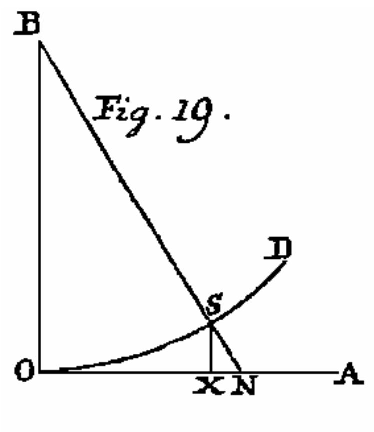
COROLLARIUM 2

213. Vis haec, qua canalis premitur, $\frac{Avv}{2gr}$ vocatur *vis centrifuga* inde orta, quod corpusculum contra instinctum inertiae in linea curva progredi cogitur, estque in ratione composita directa massae A, quadrati celeritatis v et reciproca radii osculi r.

COROLLARIUM 3

214. Si corpusculum praeterea sollicitetur a vi tangentiali secundum $Ss = T$ et normali secundum $SN = N$, erit primo $Avdv=2gTds$, deinde canalis premitur secundum directionem Sn vi = $\frac{Avv}{2gr}-N$.

EXEMPLUM



215. Si corpusculum a gravitate sollicitatum per circularem *OS* ascendere cogatur (Fig. 19), cuius centrum *B*, radius *OB* = *b*, qui sit verticalis et recta *OA* horizontalis celeritas autem in *O* fuerit = *c*, erit vis *P* = 0 et vis *Q* = –*A* atque *r* = – *b*; unde pro motu corporis habetur:

$$Avdv = -2Agdy \text{ seu } vdv = -2gdy,$$

ut sit $vv = cc - 4gy$ et celeritas evanescat in *D*, ubi $y = \frac{cc}{4g}$, vis autem, qua canalis premitur secundum *SB*, erit =

$$-\frac{A(cc-4gy)}{2gb} - \frac{Adx}{ds}.$$

Tum vero ob $xx + (b-y)^2 = bb$ erit

$$x = \sqrt{(2by - yy)},$$

$$dx = \frac{bdy - ydy}{\sqrt{(2by-yy)}}$$

et

$$ds = \frac{bdy}{\sqrt{(2by-yy)}},$$

hincque pressio secundum *SB* =

$$-\frac{Acc}{2bg} + \frac{2Ay}{b} - \frac{A(b-y)}{b} = -A + \frac{3Ay}{b} - \frac{Acc}{2gb},$$

quae quia est negativa, pressio in canalem aget secundum *SN* eritque =

$$A(1 + \frac{cc}{2bg} - \frac{3y}{b}).$$

Cum autem sit

$$v = \sqrt{(cc - 4gy)},$$

erit elementum temporis

$$dt = \frac{ds}{v} = \frac{bdy}{\sqrt{(cc-4gy)(2by-yy)}}$$

vel ob

$$y = \frac{cc-vv}{4g} \text{ et } dy = -\frac{vdv}{2g}$$

erit

$$dt = -\frac{2bdv}{\sqrt{(cc-vv)(8bg-cc+vv)}}.$$

Si celeritas initialis c sit quasi infinite parva prae b, quia v excedere nequit c, erit proxime

$$dt = -\frac{dv}{\sqrt{(cc-vv)}}.\sqrt{\frac{b}{2g}}$$

et integrando

$$t = \frac{\sqrt{b}}{\sqrt{2g}}.A\cos\frac{v}{c}.$$

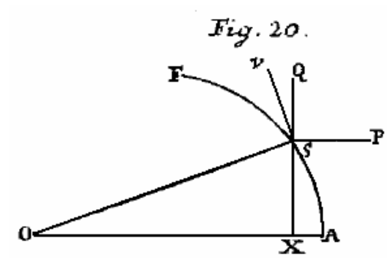
Unde si π sit semicircumferentia circuli, cuius radius = 1, erit tempus totius ascensus in D, quoad celeritas v evanescat, = $\frac{\pi\sqrt{b}}{2\sqrt{2g}}$, quod tempus *semioscillatio* vocatur. Quare, ut tempus integrae oscillationis $\frac{\pi\sqrt{b}}{\sqrt{2g}}$ sit unius minuti secundi seu = 1, radius $BO = b$ capi debet = $\frac{2g}{\pi\pi}$, quae est longitudo penduli simplicis singulis minutis secundis oscillantis. Quare, si g = 15,625 ped. Rhen., erit longitudo istius penduli = 3,166287 ped. Rhen.

SCHOLION

216. Non opus est, ut moneam canalem ideo hic tantum esse assumtum, ut motus secundum datam lineam cogatur; id autem pluribus modis veluti pendulis effici potest, cuiusmodi casum in preacedente examplo evolvere visum est. Ceterum problemata huc pertinentia in secundo Mechanicae Libro satis prolixe pertractavi. Cum autem ibi hoc desiderari possit, quod methodum, qua nunc quidem corporum coelestium motus ad calculum revocari solent et quam deinceps demum usurpare coepi, non exposuerim, operae pretium erit eam hic accuratius explicare. Pertinet autem ad problema 13 ab eoque tantum hoc differt, quod motus non per coordinatas, sed per distantias a puncto fixo et angulos circa id descriptos definiatur. Quatenus ergo hic motus in plano absolvitur, praecepta eum secundum hanc methodum investigandi tradam, postea idem pro motu non in eodem plano facto ostensurus.

PROBLEMA 15

217. Si corpusculum libere moveatur in plano, in quo perpetuo duabus sollicitetur viribus, altera ad punctum quoddam fixum *O* tendente, alterius vero directione ad illam existente normali (Fig. 20), ad quodvis tempus distantiam corpusculi *S* a puncto fixo *O* et angulum *AOS* definire.



SOLUTIO

Elapso tempore *t* corpusculum, cuius massa = *A*, pervenerit ex *A* in *S*, ponaturque distantia *OS* = *u* et angulus *AOS* = φ. In *S* autem sollicitetur primo a vi secundum *SO* pellente, quae sit = *V*, deinde vero a vi secundum directionem *SV* ad *OS* normali urgente, quae sit = *S*. Quem casum quo facilius ad problema 13 reducere possimus, demisso ex *S* ad fixam *OA* perpendiculo *SX* introducamus coordinatas *OX* = *x* et *XS* = *y*, erit $x = u\cos\varphi$ et $y = u\sin\varphi$. Tum vero binas vires *V* et *S* ad easdem directiones *SP* et *SQ* revocemus habebimusque vim $SP = -V\cos\varphi - S\sin\varphi$ et vim $SQ = -V\sin\varphi + S\cos\varphi$, quas supra vocavimus *P* et *Q*. Quocirca nanciscemur has duo aequationes :

$$ddx = -\frac{2gdt^2}{A}(V\cos\varphi + S\sin\varphi),$$

$$ddy = -\frac{2gdt^2}{A}(V\sin\varphi - S\cos\varphi),$$

ex quarum combinatione deducimus

$$ddx\cos\varphi + ddy\sin\varphi = -\frac{2gVdt^2}{A},$$

$$ddx\sin\varphi - ddy\cos\varphi = -\frac{2gSdt^2}{A}.$$

Cum autem sit $x = u\cos\varphi$ et $y = u\sin\varphi$, erit

$$x\cos\varphi + y\sin\varphi = u \text{ et } x\sin\varphi - y\cos\varphi = 0,$$

unde differentiando :

$$dx\cos\varphi + dy\sin\varphi = du \text{ et } dx\sin\varphi - dy\cos\varphi + ud\varphi = 0$$

seu

$$dx\sin\varphi - dy\cos\varphi = -ud\varphi$$

denuoque differentiando :

$$ddx\cos\varphi + ddy\sin\varphi + ud\varphi^2 = ddu,$$
$$ddx\sin\varphi - ddy\cos\varphi + dud\varphi = -dud\varphi - udd\varphi.$$

Quibus valoribus substitutis adipiscemur pro motus determinatione has duas aequationes

$$\text{I.}\ ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0,$$
$$\text{II.}\ udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0.$$

COROLLARIUM 1

218. Posterior aequatio per u multiplicata per integrationem reducitur ad hanc

$$uud\varphi = \frac{2gdt}{A}\int Sudt,$$

ubi notandum est $\frac{1}{2}uud\varphi$ exprimere elementum areae *AOS*, unde haec area erit $= \frac{g}{A}\int dt\int Sudt$. Evanescente ergo vi laterali *SV* = *S*, haec area *AOS* est ipsi tempori t proportionalis, quomodocunque fuerit comparata altera vis *V* versus punctum *O* sollicitans.

COROLLARIUM 2

219. Si prior aequatio per *du*, posterior per $ud\varphi$ multiplicetur, aggregatum fiet

$$duddu + ududu\varphi^2 + uud\varphi dd\varphi = -\frac{2gVdt^2du}{A} + \frac{2gSudt^2d\varphi}{A},$$

unde integrando elicitur

$$du^2 + uud\varphi^2 = -\frac{4gdt^2}{A}\int (Sud\varphi - Vdu),$$

ubi $\sqrt{(du^2 + uud\varphi^2)}$ exprimit elementum arcus *AS*, ita ut $\frac{du^2+uud\varphi^2}{dt^2}$ sit quadratum celeritatis in *S*.

COROLLARIUM 3

220. Si secunda multiplicetur per $2u^3d\varphi$, ob *dt* constans reperitur integrale :

$$u^4d\varphi^2 = \frac{4gdt^2}{A}\int Su^3d\varphi,$$

unde per praecedentem eruimus :

$$uudu^2 = \frac{4gdt^2}{A}(uu\int Sud\varphi - \int Su^3 d\varphi - uu\int Vdu)$$

seu

$$uudu^2 = \frac{4gdt^2}{A}(2\int udu\int Sud\varphi - uu\int Vdu),$$

ubi notandum, quod elementum temporis *dt* extra signa integralia reperiatur.

COROLLARIUM 4

221. Si $S = 0$, qui est casus virium centripetarium, erit $uud\varphi = ffdt$ et $ud\varphi = \frac{ffdt}{u}$, quo valore in corollario 2 substituto fit

$$du^2 = -\frac{f^4dt^2}{uu} - \frac{4gdt^2}{A}\int Vdu + ccdt^2$$

ideoque

$$dt = \frac{udu}{\sqrt{(ccuu - f^4 \text{-} 4guu\int Vdu : A)}}$$

et

$$d\varphi = \frac{ffdu}{u\sqrt{(ccuu - f^4 \text{-} 4guu\int Vdu : A)}}.$$

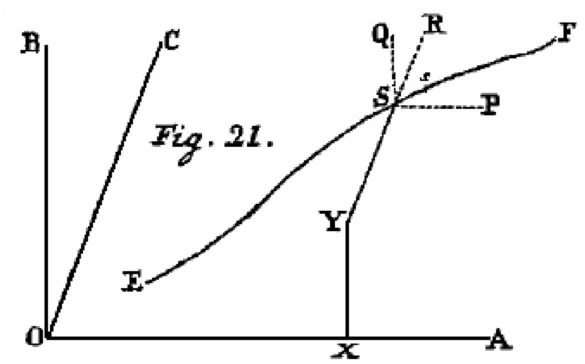
SCHOLION

222. Usus harum formularum est amplissimus in Theoria Astronomiae ex iisque determinari solent longitudo, anomalia et distantia planetae ad certum punctum sollicitati. Verum hic non est locus haec fusius prosequi, cum ad Astronomiam pertineant. Sufficiat nimirum hic methodum eiusmodi problemata tractandi in genere explicasse ; progrediamur ergo ad motus non in eodem plano factos expendendos.

PROBLEMA 16

223. Si corpusculum libere moveatur a viribus quibuscunque sollicitatum, determinare eius motum per ternas coordinatas inter se normales.

SOLUTIO



Constitutis ternis directricibus *OA*, *OB* et *OC* ad se invicem normalibus (Fig. 21), moveatur corpusculum, cuius massa = *A*, in linea *ESF* et elapso tempore *t* pervenerit in *S*, unde ad planum *AOB* demisso perpendiculo *SY*, ex *Y* ad *OA* agatur normalis *YX*, ut habeantur tres coordinatae inter se normales et directricibus parallelae, quae vocentur *OX* = *x*, *XY* = *y* et *YS* = *z*, spatium autem iam percursuram ES dicatur = *s*, ut sit $ds=\sqrt{(dx^2+dy^2+dz^2)}$ et celeritas in $S=\frac{ds}{dt}$, quae ponatur = *v*. Iam a quibuscunque viribus corpusculum in *S* sollicitetur, eas reducere licet ad easdem ternas directiones. Sollicitetur ergo ab his viribus *SP* = *P*, *SQ* = *Q* et *SR* = *R*, quarum effectus per superiora determinabuntur per tres sequentes aequationes :

$$ddx=\frac{2gPdt^2}{A},\ ddy=\frac{2gQdt^2}{A}\quad \text{et}\quad ddz=\frac{2gRdt^2}{A},$$

ubi quidem elementum *dt* sumtum est constant. Prout ergo vires *P*, *Q*, *R* a coordinatis *x, y, z* vel etiam a celeritate $\frac{ds}{dt}=v$ pendeant, ex Analysi subsidia resolutionis erunt petenda. Interim notasse iuvabit, cum fit $ds^2=dx^2+dy^2+dz^2$ et $vv=\frac{ds^2}{dt^2}$ ideoque

$$vdv=\frac{dsdds}{dt^2}=\frac{dxddx+dyddy+dzddz}{dt^2},$$

fore :

$$vdv=\frac{2g}{A}(Pdx+Qdy+Rdz),$$

qua acceleratio corpusculi definitur. Pro curva autem invenienda ponatur *dy* = *pdx* et *dz* = *qdx*, ut sit

$$ds=dx\sqrt{(1+pp+qq)}\ \text{ et }\ v=\frac{dx}{dt}\sqrt{(1+pp+qq)}.$$

Hinc ob $ddy = pddx + dpdx$ et $ddz = qddx + dqdx$, si loco ddx valor $\frac{2gPdt^2}{A}$ substituantur, reperitur

$$dpdx = \frac{2gdt^2}{A}(Q - Pp) \text{ et } dqdx = \frac{2gdt^2}{A}(R - Pq).$$

Quare, si hic pro dt^2 scribatur $\frac{dx^2(1+pp+qq)}{vv}$, erit

$$dp = \frac{2gdx(1+pp+qq)}{Avv}(Q - Pp),$$
$$dq = \frac{2gdx(1+pp+qq)}{Avv}(R - Pq)$$

seu

$$Qdx - Pdy = \frac{Avvdp}{2g(1+pp+qq)}$$

et

$$Rdx - Pdz = \frac{Avvdq}{2g(1+pp+qq)}.$$

At si pro p et q restituantur valores $\frac{dy}{dx}$ et $\frac{dz}{dx}$, fiet

$$Qdx - Pdy = \frac{Avv(dxddy - dyddx)}{2gds^2},$$
$$Rdx - Pdz = \frac{Avv(dxddz - dzddx)}{2gds^2},$$

quae invicem divisae praebent

$$P(dzddy - dyddz) + Q(dxddz - dzddx) + R(dyddx - dxddy) = 0.$$

COROLLARIUM 1

224. Celeritas igitur in quovis curvae puncto determinatur hac aequatione differentiali

$$Avdv = 2g(Pdx + Qdy + Rdz),$$

ubi $\frac{Pdx+Qdy+Rdz}{ds}$ designat vim tangentialem ex viribus sollicitantibus ortam.

COROLLARIUM 2

225. Pro curva autem definienda binae ex his tribus aequationibus sufficiunt :

$$2gds^2(Qdx-Pdy) = Avv(dxddy-dyddx) = Avvdx^2d.\frac{dy}{dx},$$
$$2gds^2(Pdx-Rdy) = Avv(dzddx-dxddz) = Avvdz^2d.\frac{dx}{dz},$$
$$2gds^2(Rdy-Qdz) = Avv(dyddz-dzddy) = Avvdy^2d.\frac{dz}{dy},$$

binae enim simul tertiam involvunt. Tum vero hinc consideratio differentialis constantis excessit.

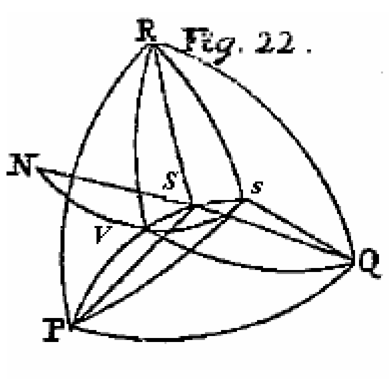
COROLLARIUM 3

226. Ultimo aequatio a celeritate immunis, etsi differentialia secundi gradus continet, tamen non ad differentiale *dt* constans assumtum est adstricta, ita enim potest repraesentari

$$Pdz^2d.\frac{dy}{dz}+Qdx^2d.\frac{dz}{dx}+Rdy^2d.\frac{dx}{dy}=0.$$

SCHOLIUM

227. Ternae vires *P, Q, R*, quibus corpusculum in *S* sollicitari ponimus, reducuntur ad unam, quae est = $\sqrt{(PP+QQ+RR)}$, ac si ea ponatur = *V*, eius directio inclinatur ad *SP* angulo, cuius cosinus est = $\frac{P}{V}$, ad SQ angulo, cuius cosinus est = $\frac{Q}{V}$, et ad *SR* angulo,



cuius cosinus est = $\frac{R}{V}$. Tum si directio istius vis *V* cum directione motus *Ss* faciat angulum = ω, erit vis accelerans seu secundum *Ss* sollicitans $=V\cos\omega$, quae cum sit = $\frac{Pdx+Qdy+Rdz}{ds}$, erit

$$\cos\omega=\frac{Pdx+Qdy+Rdz}{Vds},$$

unde vis normalis colligitur = $V\sin\omega$, cuius positio ope trigonometriae sphaericae commodissime repraesentatur (Fig. 22). Concipiatur *S* ut centrum sphaericae, unde ad superficemem porrigantur rectae *SP, SQ* et *SR*, ut sint arcus *PQ, PR* et *QR* quadrantes; directio motus transeat per *s* et media directio virium per *V*, eritque

$$\cos Ps = \frac{dx}{ds}, \quad \cos Qs = \frac{dy}{ds}, \quad \cos Rs = \frac{dz}{ds},$$
$$\cos PV = \frac{P}{V}, \quad \cos QV = \frac{Q}{V}, \quad \cos RV = \frac{R}{V}$$

ac praeterea $Vs = \omega$ seu

$$\cos \omega = \frac{Pdx + Qdy + Rdz}{Vds}.$$

Cognito angulo ω capiatur *sVN* = quadranti, erit recta ex centro *S* per *N* ducta directio vis normalis; et puncti *N* positio ita ex eius distantiis a punctis *P*, *Q*, *R* definitur, ut sit

$$\cos PN = \frac{P}{V \sin \omega} - \frac{dx \cos \omega}{ds \sin \omega},$$
$$\cos QN = \frac{Q}{V \sin \omega} - \frac{dy \cos \omega}{ds \sin \omega}$$

et

$$\cos RN = \frac{R}{V \sin \omega} - \frac{dz \cos \omega}{ds \sin \omega}.$$

Hinc igitur cum infinitae dentur rectae normales ad directionem motus *Ss*, inter eas determinatur illa, secundum quam agit vis normalis et quae directionem motus incurvat, ita ut radius curvidininis in ipsam rectam SN incidat, qui erit $= \frac{Avv}{2gV \sin \omega}$ (§207).

PROBLEMA 17

228. Si corpusculum, cuius massa = *A*, in tubo seu canali moveatur (Fig. 21), neque ab ullius viribus sollicitetur, determinare eius motum et pressionem, quam ubique in tubum exeret.

SOLUTIO

Sit *ESF* figura tubi, in quo corpusculum moveatur, in quo elapso tempore pertetigit ['pertigerit' in O. O.] ad *S* confecto spatio *ES* = *s*. Locus autem *S* ut ante referatur ad ternas directiones fixas *OA*, *OB*, *OC* inter se normales, quibus coordinatae parallelae vocentur *OX* = *x*, *XY* = *y* et *YS* = *z*. Iam quia corpusculum cogitur ubique tubi directionem sequi, ipse tubus vires in id necessarias exeret, quae autem ita erunt comparatae, ut inde celeritas nullam mutationem patiatur. Erit ergo celeritas constans, quae sit = c, unde fit $\frac{ds}{dt} = c$ et *s* = *ct*. Revocentur vires a tubo exertae ad easdem ternas directiones sintque *SP* = *X*, *SQ* = *Y* et *SR* = *Z*, et ob celeritatem immutatibilem $Xdx + Ydy + Zdz = 0$. Deinde, quia $dt = \frac{ds}{c}$, loco *dt* constans erit elementum *ds*, ex quo formulae principales erunt

$$Accddx = 2gXds^2,\ Accddy = 2gYds^2 \text{ et } Accddz = 2gZds^2$$

existente $dx^2 + dy^2 + dz^2 = ds^2$. Tota ergo vis, quam tubus in corpusculum exercet, fiet

$$\sqrt{(X^2 + Y^2 + Z^2)} = \frac{Acc\sqrt{(ddx^2 + ddy^2 + ddz^2)}}{2gds^2} = V,$$

cuius directio inclinata erit ad rectam *SP* angulo,

cuius cosinus $= \frac{X}{V} = \frac{ddx}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$,

ad SQ angulo,

cuius cosinus $= \frac{Y}{V} = \frac{ddy}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$

et ad SR angulo,

cuius cosinus $= \frac{Z}{V} = \frac{ddz}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$.

Huic autem vi aequalis et contraria est pressio, quam corpusculum vicissim in tubum exerit.

COROLLARIUM 1

229. Si radius osculi curvae in *S* ponatur = *r*, ob vim normalem = *V* et celeritatem = *c* erit $r = \frac{Acc}{2gV}$ ideoque

$$r = \frac{ds^2}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}} \text{ sumto } ds \text{ constante.}$$

COROLLARIUM 2

230. Radii osculi autem positio cum directione vis *V*, qua corpusculum a tubo urgetur, congruit, inclinabitur igitur is ad rectam *SP* angulo,

cuius cosinus $= \frac{ddx}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$,

ad *SQ* angulo,

cuius cosinus $= \frac{ddy}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$

et ad *SR* angulo,

cuius cosinus $= \frac{ddz}{\sqrt{(ddx^2 + ddy^2 + ddz^2)}}$.

SCHOLION

231. Posset hic etiam motus expendi, quando corpusculum non in linea data sed tantum in dat superficie progredi cogitur, sed quia hoc argumentum copiose iam est tractatum in secundo Libro Mechanicae, ne hic nimis sim prolixus, id non attingam. Praesertim cum pateat totum negotium huc redire, ut directio vis, quam superficies in corpusculum exerit, sit ad ipsam superficiem normalis. Quare ex aequatione superficiei proposita determinatur positio normalis, seu eius inclinatio ad ternas directiones *SP*, *SQ* et *SR*, quae cum posititione vis *V* ante definita congruere debebit. Atque hinc nova colligetur aequatio inter coordinatas *x, y, z*, quae cum priori data coniuncta definiet viam in superficie percursam, quam esse inter suos terminos brevissimam per se est perspicuum. Revertor ergo ad motum liberum ac docebo, quomodo motus non in eodem plano factos per angulos ad certum punctum fixum relatos definiri conveniat, ea scilicet ratione, quam supra (problema 5, §70) exposui. Quod quia in Astronomia Theoretica maximam affert utilitatem, neque haec motuum evolutio in praecedentibus libris est explicata, ei sequens problema destinemus.

PROBLEMA 18

232. Si corpusculum partim ad punctum fixum *O* partim aliis quibuscunque viribus sollicitetur, definire eius motum respectu eius puncti.

SOLUTIO

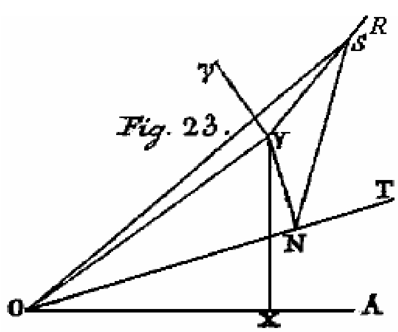


Fig. 23.

Constituto plano, quod sit planum tabulae (Fig. 23), per punctum fixum *O* transeunte, ad quod motus referatur, in eoque sumta directrice fixa *OA*, pervenerit corpusculum elapso tempore *t* in *S*, unde primo in planum demittatur perpendiculum *SY*, et ex *Y* in rectam *OA* normalis *YX*, ut habeantur coordinatae orthogonales $OX = x$, $XY = y$, $YS = z$. Cum iam corpusculum in *S* primo a vi secundum *SO* sollicitetur, ea resolvatur in directiones *YO* et *SY*; reliquae vero vires cum ad easdem directiones tum ad *YV* in plano tabulae ad *OY* normalem revocentur, ita ut omnino tres habeantur vires, quarum prima sit secundum $YO = V$, altera secundum $YV = S$ et tertia secundum $SR = R$. Quae vires cum sint cognitae, ad directiones coordinatarum reducantur, sicque posito angulo $AOY = \varphi$ obtinebuntur hae vires :

vis secundum $XO = V\cos\varphi + S\sin\varphi = -P,$
vis secundum $YX = V\sin\varphi - S\cos\varphi = -Q,$
vis secundum SR= $R,$
quarum effectus per tres sequentes formulas exprimetur :

$$Addx = -2gdt^2(\mathrm{V}\cos\varphi + S\sin\varphi),$$
$$Addy = -2gdt^2(\mathrm{V}\sin\varphi - S\cos\varphi),$$
$$Addz = 2gRdt^2 \text{ posita massa corpusculi} = A.$$

Vocetur porro distantia $OY = u$, et ob $x = u\cos\varphi$ et $y = u\sin\varphi$ binae priores aequationes uti supra (§217) ad has duas redigentur :

$$\text{I.}\quad ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0,$$
$$\text{II.}\quad udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0.$$

Ponatur nunc angulus $SOY = \psi$, qui corpusculi latitudo vocatur, dum angulus $AOY = \varphi$ est eius longitudino; erit $SY = z = u$ tang ψ. At pro hoc angulo ψ commodius inveniendo sit OT linea nodorum, angulus $AOT = \omega$ et inclinatio plani per O et directionem motus in S ducti ad planum assumtum = ρ, erit $TOY = \varphi - \omega$, hinc ductis YN et SN ad OT normalibus fiet
$ON = u\cos(\varphi - \omega)$ et $YN = u\sin(\varphi - \omega)$ adeoque $YS = u\sin(\varphi - \omega)\text{tang }\rho = z,$ hincque tang $\psi = \sin(\varphi - \omega)\text{tang }\rho$ et uti supra (§70)

$$\frac{d\omega}{\text{tang}(\varphi\text{-}\omega)} = \frac{d\rho}{\sin\rho\cos\rho} = d.l\text{ tang }\rho.$$

Quare cum sit $d\rho = \frac{d\omega\sin\rho\cos\rho}{\text{tang}(\varphi\text{-}\omega)}$,erit

$$dz = du\sin(\varphi\text{-}\omega)\text{tang }\rho + u(d\varphi - d\omega)\cos(\varphi\text{-}\omega)\text{ tang}\rho + u\sin(\varphi\text{-}\omega).\frac{d\omega\text{ tang }\rho}{\text{tang}(\varphi\text{-}\omega)}$$
$$\text{seu }\ dz = (du\sin(\varphi\text{-}\omega) + ud\varphi\cos(\varphi\text{-}\omega))\text{ tang}\rho,$$

qui valor denuo differentiatus dat :

$$ddz = (ddu\sin(\varphi\text{-}\omega) + du(2d\varphi - d\omega)\cos(\varphi\text{-}\omega) + udd\varphi\cos(\varphi\text{-}\omega) - ud\varphi(d\varphi - d\omega)\sin(\varphi\text{-}\omega))\text{tang }\rho$$
$$+ (du\sin(\varphi - \omega) + ud\varphi\cos(\varphi\text{-}\omega)).\frac{d\omega\text{ tang }\rho}{\text{tang}(\varphi\text{-}\omega)}$$

sive

$$ddz = \left(ddu\, sin(\varphi\text{-}\omega) + 2dud\varphi\, cos(\varphi\text{-}\omega) + udd\varphi\, cos(\varphi\text{-}\omega) - ud\varphi^2\, sin(\varphi\text{-}\omega) \quad + \frac{ud\varphi d\omega}{sin(\varphi\text{-}\omega)} \right) \text{tang } \rho.$$

Cum igitur sit

$$ddu - ud\varphi^2 = -\frac{2gVdt^2}{A} \text{ et } udd\varphi + 2dud\varphi = \frac{2gSdt^2}{A},$$

obtinebitur

$$ddz = \left(\frac{-2gVdt^2}{A} sin(\varphi\text{-}\omega) + \frac{2gSdt^2}{A} cos(\varphi\text{-}\omega) + \frac{ud\varphi d\omega}{sin(\varphi\text{-}\omega)} \right) \text{tang } \rho.$$

Quare ob $ddz = \frac{2gRdt^2}{A}$ erit

$$\frac{ud\varphi d\omega}{sin(\varphi\text{-}\omega)} = \frac{2gdt^2}{A} \left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho \right)$$

seu

$$d\omega = \frac{2gdt^2\, sin(\varphi\text{-}\omega)}{Aud\varphi} \left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho \right)$$

et

$$d.l \text{ tang } \rho = \frac{2gdt^2\, cos(\varphi\text{-}\omega)}{Aud\varphi} \left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho \right).$$

Inventae ergo sunt quatuor aequationes, quibus problematis solutio continetur.

COROLLARIUM 1

233. Cum igitur ad datum tempus *t* assignari debeant hae quatuor quantitates $u, \varphi, \omega,$ et ρ, nacti sumus primo has duas aequationes differentio-differentiales

$$ddu - ud\varphi^2 + \frac{2gVdt^2}{A} = 0 \text{ et } udd\varphi + 2dud\varphi - \frac{2gSdt^2}{A} = 0,$$

deinde has duas simpliciter differentiales

$$d\omega = \frac{2gdt^2\, sin(\varphi\text{-}\omega)}{Aud\varphi} \left(V\, sin(\varphi\text{-}\omega) - S\, cos(\varphi\text{-}\omega) + R\, cot\, \rho \right)$$

et

$$d.l \text{ tang } \rho = \frac{d\omega}{\text{tang}(\varphi\text{-}\omega)} \text{ seu } d.\text{tang } \rho = \frac{d\omega \text{ tang } \rho}{\text{tang}(\varphi\text{-}\omega)}.$$

COROLLARIUM 2

234. Inventis autem his valoribus colligetur tam angulus $SOY = \psi$ *latitudo* dictus quam distantia vera *SO* ex his formulis

$$\text{tang } \psi = sin(\varphi\text{-}\omega) \text{tang } \rho \text{ et } OS = \frac{u}{cos\, \omega},$$

ubi *u* dici solet *distantia curtata*.

COROLLARIUM 3

235. Si fuerit $sin(\varphi\text{-}\omega) = 0,$ hoc est, si corpusculum per planum assumtum transit, supra iam vidimus fore $d\omega = 0$; at nunc patet tam lineam nodorum quam inclinatinem nullam pati, si fuerit :

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) + R\,cot\,\rho = 0.$$

COROLLARIUM 4

236. Est vero

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) = -Q\,cos\,\omega + P\,sin\,\omega,$$

atque introducis primitivis viribus *P*, *Q*, *R* erit

$$V\,sin(\varphi\text{-}\omega) - S\,cos(\varphi\text{-}\omega) + R\,cot\,\rho = P\,sin\,\omega - Q\,cos\,\omega + R\,cot\,\rho,$$

atque haec est quasi vis tam locum lineae nodorum quam inclinationem immutans.

SCHOLIUM

237. Imprimis hic notari meretur, quod variatio momentanea in situ lineae nodorum et inclinatione satis concinna hac methodo exprimi potuerit, unde in Astronomiam Theoreticam insignia commoda redundant. Ex hoc fonte a Celeberrimo Mayero Prof. Goetting. incredibili studio deductae sunt Tabulae Lunares excellentissimae, quibus Astronomia fere ad summum fastigium evecta est censenda. [Johan Tobias Mayer, Commentarii Societatis regiae scientiarum 3, Gottingae 1753. Cum nota in O. O.] Cum autem motus lunae, qui hac methodo definitur, neutiquam sit absolutus, sed ad centrum terrae relatus, in hac investigatione simul motus terrae est habenda; quare, ut hac methodo uti queamus, praecepta tradi conveniet, quorum ope motus respectivos ad calculum revocare liceat, siquidem motus eius corporis, cuius respectu aliorum corporum motus aestimantur, fuerit cognitus. Quod argumentum cum non satis dilucide in superioribus Mechanicae Libris sit expositum, hic maiori cura illud pertractabo; quo facto ad motus corporum finitorum, quos ibi nondum attigeram, feliciori cum successu progredi licebit.