CHAPTER IV

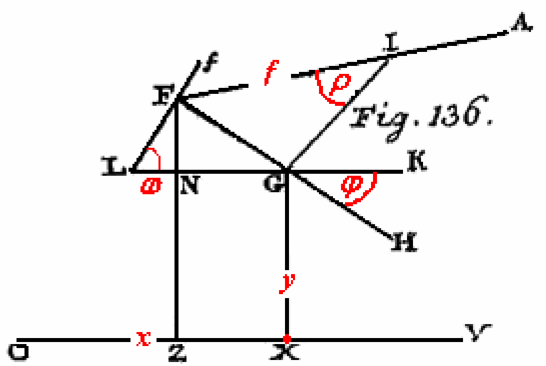
CONCERNING THE MOTION OF TOPS ENDING IN A SHARP POINT ON A HORIZONTAL PLANE WITH FRICTION TAKEN INTO ACCOUNT.

PROBLEM 11

1030. If a top is moving on a horizontal plane in some manner and the pressing force of this on the plane is given with the individual moments, to define the friction and the progressive motion of the top.

SOLUTION

The figure represents the horizontal plane (Fig. 136), upon which the top advances, the axis of this passing through the centre of inertia and the point now at the elapsed time *t* maintains



the position *AIF*, so that *I* is the centre of inertia in the elevated position, and indeed *F* is the point of this, by which it makes contact with the horizontal plane ; and the distance *IF* is put equal to *f*, which is constant. The perpendicular *IG* is sent from *I* to the plane, and on taking the direction *OV* in the plane regarded as in a fixed direction in space [i. e. according to the idea of the universal reference frame of the fixed stars], from *G* and *F* the normals *GX* and *FZ* are drawn to that, and likewise through *G* the line *KL* is drawn parallel to *OV*. The angle $FIG = \rho$, which expresses the declination of the axis of the top *AF* from the vertical position ; and the angle $KGH = \varphi$, which shows the declination of the vertical plane in which the axis of the top is now rotating, from the vertical plane constructed on *OV* or *LK*. Hence let $GI = f\cos\rho$ and $GF = f\sin\rho$; then indeed $GN = f\sin\rho\cos\varphi$ and $FN = f\sin\rho\sin\varphi$. In addition now let $OX = x$ and $XG = y$; from which for the point *F* there becomes

$$OZ = x - f\sin\rho\cos\varphi \text{ and } ZF = y + f\sin\rho\sin\varphi;$$

from which the motion of the point *F* can be deduced :
the speed of this along the direction *OV* or *NG* is equal to

$$\frac{dx - fd\cdot\sin\rho\cos\varphi}{dt},$$

and the speed along the direction *NF* is equal to

$$\frac{dy+fd\cdot\sin\rho\sin\varphi}{dt},$$

each of which, unless it vanishes, is disturbed by the friction of the point F on the surface ; to find the direction of this, let Ff be the direction of the point as it advances, which produced backwards gives the direction of friction FL at L, for which there is put in place the angle $FLG=\omega$ and then

$$\text{tang }\omega=\frac{dy+fd\cdot\sin\rho\sin\varphi}{dx-fd\cdot\sin\rho\cos\varphi}\quad.$$

Now let the pressing force, which the point exerts on the plane be equal to Π , with the whole weight of the top present equal to M, and on account of friction the top is acted on at F by the force equal to $\delta\Pi$ along the direction FL, which resolved gives the force along $XO=\delta\Pi\cos\omega$ and along $FZ=\delta\Pi\sin\omega$. Hence towards defining the progressive motion of the centre of inertia I, in addition to that force acting downwards along IG equal to $M-\Pi$, the forces of friction are considered applied to that, and the principles of motion supply these three equations:

$$\frac{ddx}{2gdt^2}=-\frac{\delta\Pi\cos\omega}{M},\quad \frac{ddy}{2gdt^2}=-\frac{\delta\Pi\sin\omega}{M}$$

and

$$\frac{fdd\cos\rho}{2gdt^2}=-1+\frac{\Pi}{M},$$

from which we deduce at once that $ddx\sin\omega=ddy\cos\omega$. From these equations, if the angles ρ and ω are considered as known at the time t, from which in the first place $\frac{\Pi}{M}$ then indeed the differentials dx and dy are determined, and from these finally the angle φ is found from the formula

$$\text{tang }\omega=\frac{dy+fd\cdot\sin\rho\sin\varphi}{dx-fd\cdot\sin\rho\cos\varphi}$$

COROLLARY 1

1031. If the value $\frac{\Pi}{M}$ found is substituted for in terms of ρ , we have these second order differential equations for determination the quantities x and y

$$ddx=-2\delta gdt^2\cos\omega-\delta f\cos\omega\, dd\cdot\cos\rho,$$
$$ddy=-2\delta gdt^2\sin\omega-\delta f\sin\omega\, dd\cdot\cos\rho.$$

COROLLARY 2

1032. For the direction of friction *FL*, with regard to the line *FH*, since the angle $LGF = \varphi$ and the angle $FLG = \omega$, then the angle $GFL = 180^{\circ} - \varphi - \omega$; but the friction itself is equal to $\delta \Pi$, unless the speed of the point *F* is zero, in which case the friction suddenly vanishes [Euler does not distinguish between static and kinetic friction; one could argue that there is no friction present to give the same result], which comes about if

$$dx = fd \cdot \sin \rho \cos \varphi$$

and

$$dy = -fd \cdot \sin \rho \sin \varphi$$

SCHOLIUM

1033. From these equations it is not possible to conclude anything at this stage, since the relation between the variables ω and Π or ρ at the time *t* is not yet agreed upon, which finally must be elicited from the rotational motion. But with these found, then through the formulas treated here, and thus the progressive motion of the centre of inertia *I* can be defined through the variables *x* and *y*. On account of which we can introduce the angle ω in the determination of the rotational motion, even if the relation between the angles ρ, φ and the time can be assigned. For since then

$$dy \cos \omega + f \cos \omega d \cdot \sin \rho \sin \varphi - dx \sin \omega + f \sin \omega d \cdot \sin \rho \cos \varphi = 0,$$

since we are able to remove *x* and *y* more easily, we put

$$f \cos \omega d \cdot \sin \rho \sin \varphi + f \sin \omega d \cdot \sin \rho \cos \varphi = sdt,$$

in order that

$$dy \cos \omega - dx \sin \omega + sdt = 0;$$

which differential equation on account of $ddy \cos \omega = ddx \sin \omega$ gives

$$-dy \sin \omega - dx \cos \omega + \frac{dsdt}{d\omega} = 0.$$

Again on differentiation, and on account of

$$ddy \sin \omega + ddx \cos \omega = \frac{-2\delta g \Pi dt^2}{M}$$

there is produced

$$\frac{2\delta g \Pi dt^2}{M} - dyd\omega \cos \omega + dxd\omega \sin \omega + dt \cdot \frac{ds}{d\omega} = 0;$$

there is produced in addition first on multiplication by $d\omega$, and then on division by *dt*

$$\frac{2\delta g \Pi dt^2}{M} + sd\omega + d\cdot\frac{ds}{d\omega} = 0,$$

in which equation the relation between s, ω, Π and t is expressed, which perhaps in the following has a use. But s involves the angles $\rho,\ \varphi$ and ω and then

$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos\rho}{2gdt^2}$$

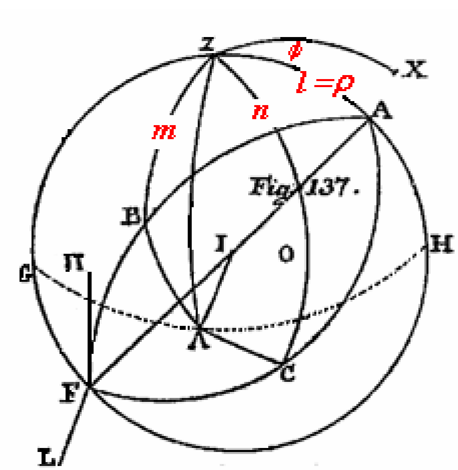
thus so that at this stage there are four variables present $\rho,\ \varphi$ and ω and also t.

PROBLEM 12

1034. While the top is moving in some manner on a horizontal plane and friction is apparent, to determine the moments of the forces acting on the top about the principal axes of the top.

SOLUTION

In a sphere described by the centre of inertia of the top *I* (Fig. 137) the circle *GZH* represents a vertical plane, in which the axis *AIF* of the top now is turning, drawn through the centre of inertia *I* and the sharp point *F*, which likewise is an axis of the top, and the moment of inertia about this is equal to *Maa*, now the two remaining principal axes drawn from *I* touch the sphere at the points *B* and *C*, of which the moments of inertia about these are equal to *Mcc*, thus so that in our general formulas there is *bb* = *cc*, just as we have now assumed above. From the vertical point of the sphere *Z* the arc $ZA = \rho$; moreover we put for the arcs drawn *ZB* and *ZC*, as above, *ZA* = *l*, *ZB* = *m* and *C* = *n* , so that $\rho = l$. With these in place, the forces acting on the top are in the first place the weight of this equal to *M*, which force applied through the centre of inertia *I* presents no moment ; then the pressing force is present, by which the horizontal plane reacts at the point *F*, the direction of this is vertically upwards along $F\Pi$, which force is equal to Π, and we consider it to be the case that



$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\ \cos\rho}{2gdt^2}.$$

And then the top is acted upon at *F* by the friction equal to $\delta\Pi$, unless the point is at rest, and the direction of this *FL* is horizontal; and for the position of this a great horizontal circle $G\Lambda B$ is drawn, in which there is taken the arc, following §1032
$H\Lambda = 180° - \varphi - \ \omega$ or $GA = \varphi + \omega$, where φ denotes the declination of the plane *GZH* from a certain fixed vertical plane ; but the angle ω treated in the preceding problem must be defined and the direction *FL* is parallel to the radius $I\Lambda$. Now regarding the investigation of the moments of these forces about the principal axes, in the first place these forces are to be

resolved along the directions of these axes, which in the end we can consider as applied to the centre of inertia. Hence the force $F\Pi = \Pi$, applied in the direction *IZ*, gives rise to

$$\text{a force along } IA = \Pi \cos ZA = \Pi \cos l \ ,$$
$$\text{a force along } IB = \Pi \cos ZB = \Pi \cos m$$

and

$$\text{a force along } IC = \Pi \cos ZC = \Pi \cos n \, .$$

Then the force $FL = \delta\Pi$ applied at $I\Lambda$ is resolved into the forces

$$1^{\text{st}} \text{ along } IA = \delta\Pi \cos A\Lambda,$$
$$2^{\text{nd}} \text{ along } IB = \delta\Pi \cos B\Lambda,$$
$$3^{\text{rd}} \text{ along } IC = \delta\Pi \cos C\Lambda.$$

Moreover to these set out let *ZX* be that fixed vertical circle and thus the angle $XZA = \varphi$, moreover we put as above the angles $XZA = \lambda$, $XZB = \mu$ and $XZC = \nu$, in order that $\varphi = \lambda$ and, on account of $AZ\Lambda = 180° - \lambda - \omega$, then $XZ\Lambda = 180° - \omega$, and hence $BZ\Lambda = \mu + \omega - 180°$ and $CZ\Lambda = 180° - \nu - \omega$, from which on account of the quadrant *ZA* there is produced

$$\cos A\Lambda = -\sin l \cos(\lambda + \omega) \ ,$$
$$\cos B\Lambda = -\sin m \cos(\mu + \omega)$$

and

$$\cos C\Lambda = -\sin n \cos(\nu + \omega) \ .$$

On account of which we have

$$\text{force along } IA = \Pi \cos l - \delta\Pi \sin l \cos(\lambda + \omega),$$
$$\text{force along } IB = \Pi \cos m - \delta\Pi \sin m \cos(\mu + \omega),$$
$$\text{force along } IC = \Pi \cos n - \delta\Pi \sin n \cos(\nu + \omega),$$

now moreover it is required to consider these forces applied at the point *F* , with *IF* = *f* arising, from which the moments of these about the principal axes, which we have designated above by the letters *P*, *Q*, *R* , are concluded to be

$$P = 0,$$
$$Q = \Pi f \cos n - \delta f \Pi \sin n \cos(\nu + \omega) \ ,$$
$$R = -\Pi f \cos m + \delta\Pi f \sin m \cos(\mu + \omega) \ .$$

PROBLEM 13

1035. From the moments of these forces found to show the equations, in which the motion of the top advancing on a horizontal surface is contained, and which is disturbed by friction.

SOLUTION

Initially the top is situated in the figure representing the rotational motion maintained at the elapsed time *t* (Fig.137), where all the denominations made remain in the same manner. And now the top is rotating about the axis *IO* in the sense *ABC* with an angular speed equal to γ', moreover for the point *O* the arcs are $AO=\alpha,\ BO=\beta,\ CO=\gamma$ and there is put in place $\gamma' \cos\alpha = x,\quad \gamma' \cos\beta = y,\quad \gamma' \cos\gamma = z$, which quantities are thus to be determined from the moments just found, so that initially *dx* = 0, and thus *x* = const. Hence putting *x* = *h*, and for *y* and *z* we have these equations :

$$dy + \frac{(aa\text{-}cc)}{cc} hzdt = \frac{2\Pi\, fgdt}{Mcc}\left(\cos n - \delta \sin n \cos(\nu+\omega)\right),$$
$$dz + \frac{(aa\text{-}cc)}{cc} hydt = -\frac{2\Pi\, fgdt}{Mcc}\left(\cos m - \delta \sin m \cos(\mu+\omega)\right).$$

Then truly for the arcs *l*, *m*, *n* and likewise for the angles λ, μ, ν we have shown:

$$dl \sin l = dt(y\cos n - z\cos m),\quad d\lambda \sin^2 l = -dt(y\cos m + z\cos n),$$
$$dm \sin m = dt(z\cos l - h\cos n),\quad d\mu \sin^2 m = -dt(z\cos n + h\cos l),$$
$$dn \sin n = dt(h\cos m - y\cos l),\quad d\nu \sin^2 n = -dt(h\cos l + y\cos m),$$

where in addition these relations are to be noted :

$$\cos(\mu-\lambda) = -\frac{\cos l \cos m}{\sin l \sin m},\qquad \cos(\nu-\lambda) = -\frac{\cos l \cos n}{\sin l \sin n},$$
$$\sin(\mu-\lambda) = -\frac{\cos n}{\sin l \sin m},\qquad \sin(\nu-\lambda) = \frac{\cos m}{\sin l \sin n},$$

from which the angles μ and ν by λ thus are to be defined :

$$\cos\mu = \frac{-\cos\lambda \cos l \cos m + \sin\lambda \cos n}{\sin l \sin m},\qquad \cos\nu = \frac{-\cos\lambda \cos l \cos n - \sin\lambda \cos m}{\sin l \sin n},$$
$$\sin\mu = \frac{-\sin\lambda \cos l \cos m - \cos\lambda \cos n}{\sin l \sin m},\qquad \sin\nu = \frac{-\sin\lambda \cos l \cos n + \cos\lambda \cos m}{\sin l \sin n}.$$

And here there is the equation

$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos l}{2gdt^2}.$$

But the angle ω has entered from the progressive motion, from which because of the distinction, if in fig.136 at the position of the centre of inertia *I* we call the coordinates *OX* = X and *XG* = *Y*, with *GI* = $f\cos l$ present, we must add these to the above equations :

$$\frac{ddX}{2g^2dt} = -\frac{\delta\Pi}{M}\cos\omega\ ,\quad \frac{ddY}{2g^2dt} = -\frac{\delta\Pi}{M}\sin\omega$$

and

$$dY\cos\omega - dX\sin\omega + f\cos\omega d\cdot\sin l\sin\lambda + f\sin\omega d\cdot\sin l\cos\lambda = 0\ .$$

And everything is determined in these equations, both for the progressive motion as well as for the rotational motion considered. If initially we wish to exclude the quantities X from *Y* from the calculation, in place of the last three equations it is sufficient to have present the single equation ; for which if there is put

$$sdt = f\cos\omega d\cdot\sin l\sin\lambda + f\sin\omega d\cdot\sin l\cos\lambda,$$

or with the differentials taken and in place of *dl* and $d\lambda$ with the above values put in place then

$$s = -fy\sin n\sin(\omega+\nu) + fz\sin m\sin(\omega+\mu)$$

and the equation in place of these three taken above becomes

$$\frac{2\delta g\Pi dt}{M} + sd\omega + d\cdot\frac{ds}{d\omega} = 0.$$

SCHOLIUM 1

1036. Many of these equations are unable to be resolved under any circumstances, the reason for this especially is the angle ω entering the first equations. From which it is apparent that the motion of tops is greatly disturbed by friction, thus we are unable to conclude anything at all about which this motion can become known. But if now we consider in passing only the causes of this motion, it is evident that the centre of inertia *I* not only ascends and descends in a straight line, as comes about with the removal of friction, but also gains a horizontal motion, which arises from the force of friction, of which that direction is opposite to the direction of the motion of the sharpened end, and the motion of the centre of inertia is acted on in the same direction, from which the motion is neither uniform nor rectilinear, and as far as it is curved, it is considered convex in that region in which the point is progressing. In a like manner also the rotational motion is greatly disturbed on account of both the speed as well as of the axis of rotation, which can be confirmed with hardly any consideration of friction.

SCHOLIUM 2

1037. Now this great disturbance of the motion only continues until friction stops, but it is evident that this finally must come about by itself, since the motion on account of friction is continually slowed down. But friction is unable to cease, unless the point of the top stays in the same place, by which it is necessary that the motion must thus be tempered, in order that the point finally shall be continuing at the same point in the plane, provided this comes about, before the top falls over. If indeed the rotational motion at first impressed were exceedingly slow, there is no doubt why it should not fall over before that phenomenon arises; from which in turn it is permitted to conclude, if the motion were slow enough, to be the case that as before the top falls over, it is reduced to rotating at the same point of the plane from the friction of the point. Because since that happens, and with the top at this stage involved in rotational motion, from the above it is apparent that the axis of the top must be vertical ; for if it should be inclined, thus in no manner can it be rotating, in order that the cusp remains at the same point. Therefore from these taken together, we can deduce this conclusion : only if enough speed of the rotational motion can be impressed on the top, on account of friction finally the top erects itself into a vertical position and then the rotational motion is to be continuing. Which phenomenon is more noteworthy there, since it is indebted only to friction ; thus in order that with the help of friction it is able to maintain both a vertical line and thus also a horizontal plane ; which may be of great use in navigation, which was pointed out in England some time ago.

[Serson, Phil. Trans. 47 (1752), p. 352., as noted in the O. O. by C. B.]

CHAPTER V

CONCERNING THE MOTION OF GLOBES ON A HORIZONTAL PLANE, HAVING THE CENTRE OF INERTIA PLACED AT THEIR CENTRE

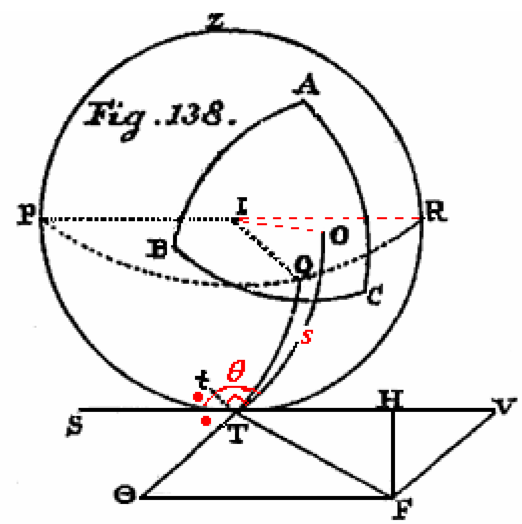
PROBLEM 14

1038. If the globe is moving on a horizontal plane in some manner with both a progressive motion as well as a rotational motion, then to determine the speed and direction, with which the point of contact touches the horizontal surface.

[I have used the word globe throughout for *globus* as it seems more appropriate than ball or sphere]

SOLUTION

Let *I* be the centre and likewise the centre of inertia of the globe (Fig. 138), and the radius of this is equal to *f* and contact is made at the lowest point *T*. Moreover the motion of the globe is thus established so that the centre of inertia *I* is moving along the direction *PIR* with a speed equal to *v*, now likewise it is rotating about some axis *IO* with an angular speed equal to γ' in that sense, so that the point *T* advances along the arclet *Tt* about *O* [normal to *OT*], and from the position of the point *O* we put in place the [spherical] angle $PTO = \vartheta$ and the arc *TO* = *s*, where I take a certain arc thus, as if the radius of the globe were equal to 1. *TV* is drawn parallel to *PIR* itself, and if the rotational motion were absent, the contact point *T* is moving along the horizontal plane with a speed equal to *v* in the direction *TV*. Then if the globe is carried only by rotational motion, because the point *T* is moving along *Tt* with a speed equal to $f\gamma' \sin TO = f\gamma' \sin s$, [Note that the perpendicular distance of *T* from the axis of rotation *IO* is *fsin s*.] since the direction of this is horizontal, it is referred to in the plane by the line $T\Theta$, thus so that the angle $ST\Theta = PTt = \vartheta - 90°$ on account of the right angle *OTt*. [The equality of these angles is a kinematic condition, as indeed is the whole proposition.] Hence then $VT\Theta = 270^{o} - \vartheta$. The lines are taken $TV = v$ and $T\Theta = f\gamma' \sin s$, and because the point *T* is moving from these two motions taken together, now the motion of this point is made along the diagonal line *TF* of the parallelogram $TVF\Theta$. From with the normal *FH* drawn from *F* to *TV*, then



$$VH = f\gamma' \sin s \sin\vartheta \text{ and } FH = -f\gamma' \sin s \cos\vartheta$$

from which there arises

$$TH = v - f\gamma' \sin s \sin\vartheta$$

and the speed of the contact point

$$TF = \sqrt{(vv - 2f\gamma' v \sin s \sin\vartheta + ff\gamma'\gamma' \sin^2 s)}$$

and

$$\text{tang } VTF = -\frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta}.$$

There is drawn *IQ* from the centre *I* parallel to *TF*, then the arc *TQ* is a quadrant and the [spherical] angle *RTQ* = *VTF*. Whereby if *IQ* is parallel to the direction along which the point *T* moves, then

$$\text{tang } PTQ = \frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta},$$

and on putting the speed of the contact point

$$\sqrt{(vv - 2f\gamma' v \sin s \sin\vartheta + ff\gamma'\gamma' \sin^2 s)} = u$$

then

$$\sin PTQ = \frac{-f\gamma' \sin s \cos\vartheta}{u} \quad \text{and} \quad \cos PTQ = \frac{f\gamma' \sin s \sin\vartheta - v}{u}$$

COROLLARY 1

l039. Hence it can happen, that both the speed of the contact point and therefore the grazing [or grinding] action vanish, in which case these two conditions must become instead, for the one $\gamma' \sin s \cos\vartheta = 0$, and for the other $v = f\gamma' \sin s \sin\vartheta$. From which it is apparent at once, if no progressive motion should be present, or *v* = 0, no friction is present, if $\sin s = 0$, that is if the globe is rotating about the vertical *Z T*.

COROLLARY 2

1040. Then the motion of the globe is free from grazing if initially there is put $\cos\vartheta = 0$ or the angle *PTO* is right; then again the progressive speed *v* hence must hold this relation to the angular speed γ', in order that $v = f\gamma' \sin s$ or $TV = T\Theta$ and the angle $ST\Theta = 0$.

COROLLARY 3

1041. Hence when the globe follows motion of this kind, because with all grinding removed also there is no friction present, the sphere constantly maintains the same motion, if indeed the axis of rotation *IO* has the property of a principal axis.

SCHOLIUM 1

1042. Just as here we have established that friction that does not oppose the globe on the horizontal plane any the less from conserving its pure motion, which yet we observe to be reduced, since a globe placed on a table with such a motion soon loses all the motion, and the reason for this cannot be attributed to the resistance of the air. Now here initially I note that experiments do not agree perfectly with theory ; just as while in the case treated here we have assumed the contact to be made at a single point, that always in practice turns out otherwise. Yet meanwhile if the arc *TO* is a quadrant and *PTO* = 90°, with $v = f\gamma'$ arising, even if the contact is not made at a single point, yet the grinding action vanishes, and therefore here the extinction of the motion cannot be ascribed by any means to friction. From which we must conclude that in addition to friction, as we have defined this here, another impediment to the motion must be given at this point, while bodies proceed on surfaces, to be distinguished properly from friction, an account of this should be provided in some manner, it is agreed rather to investigate the effect of this on its own, than to change the innate established nature of friction. And just as here we abstract the resistance of the air, thus also it is permitted to separate this obstacle attending friction from the present argument.

[People at this time did not fully appreciate the nature of the interactions of a ball rolling on a surface, a sort of continual slightly inelastic collision process of parts of the ball within a finite area or contact patch involving deformation of the surfaces, leading to rolling friction.]

SCHOLION 2

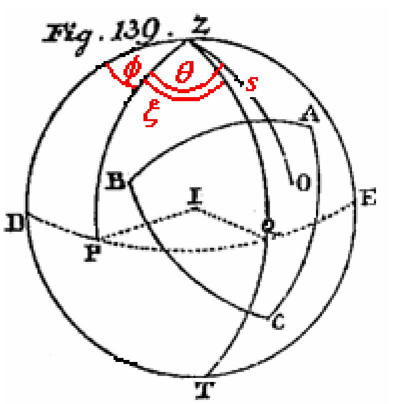
1043. Here I examine spherical bodies, in which the centre of inertia *I* of these is placed at the centre, hence as also it is moving in a horizontal plane then it is always vertically above the point of contact *T*. From which it is apparent that the contact force at contact is always equal to the weight of the body *M*. Hence, if the body should consist of some uniform material, then all the diameters of this share the characteristic of principal axes, but we can consider the distribution of the material to be unequal in some manner, yet thus in order that the centre of inertia falls on the centre of the figure. Whereby it is required that three principal axes are to be considered in the globe, which extend from the centre *I* to the points *A, B, C,* with the distance from each other in quadrants in turn, about which as we have put in place above, the moments of inertia are *Maa, Mbb, Mcc.* But then nevertheless we may decide that two or all of these moments are equal to each other, yet it is appropriate to observe three points of this kind fixed on the surface, because from the relation of these to absolute space, the motion of the globe can be more easily defined. Moreover with these three points in place *A*, *B,* and *C,* because the rotational motion about *O,* which we assume to be directed along *Tt,* has the opposite sense *CBA* to that, which we have put in place above, in the application of the general formulas in this case, we must consider the angular speed γ' as negative.

PROBLEM 15

1044. If a sphere is moving in some manner on a horizontal surface, to define the forces by which it is acted on, and the moments of these about the three principal axes.

SOLUTION

The included globe is considered to be either for a fixed sphere or with that having a constant progressive motion (Fig. 139), in which *Z* is the vertical point and the opposite of this is the point of contact *T*, now *DE* is a horizontal diameter tending towards a certain direction in the world [*i. e.* the fixed stars], and *DPQE* a horizontal great circle. But now in the elapsed time *t*, the globe is moved in a progressive motion along the direction *PI* with a speed equal to *v*, and the arc *DP* is put in place or the angle $DZP = \varphi$; moreover the principal axes now are at $A, B, \& C$ .Then indeed the glove now rotates about the axis *IO* with an angular speed equal to γ' in the sense *ACB;* and for the position of the point *O* let the angle be PTO or $PZO = \vartheta$ and the arc *ZO* = *s*. And if indeed before we put the arc *TO* equal to *s*, likewise it is the case that only the sine of this enters into the calculation. Hence the sine of



the angle $DZO = \vartheta + \varphi$ and $EZO = 180^{\circ} - \vartheta - \varphi.$ Then if from the points *A*, *B*, *C* both to *O* as well as to *Z* arcs of great circles are considered to be drawn, in order that the arcs so far are :

$$AO = \alpha, \quad BO = \beta \; , CO = \gamma,$$
$$ZA = l, \quad ZB = m, \quad ZC = n$$

and the angles

$$EZA = \lambda, \quad EZB = \mu, \quad EZC = \nu .$$

[No attempt has been made to put these well–know quantities on the diagram, to avoid undue clutter.]

Moreover in the preceding problem we have shown that the plane contact point *T*, subject to abrasion, progresses along the direction parallel to the radius *IQ* with a speed equal to

$$\sqrt{(vv - 2f\gamma' v \sin s \sin\vartheta + ff\gamma'\gamma' \sin^2 s)} .$$

and there is present

$$\text{tang } PTQ = \text{tang } PZQ = \frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta}$$

with *f* denoting the radius of the globe. Therefore since the pressing force at *T* is equal to *M*, then the friction is equal to δM , at which point *T* it has been applied along the direction parallel to *QI*. Hence with this force resolved along the directions of the principal axes *IA, IB, IC,* there arises :

$$\text{the force along } IA = -\delta M\, cos\, AQ,$$
$$\text{the force along } IB = -\delta M\, cos\, BQ,$$

and

$$\text{the force along } IC = -\delta M\, cos\, CQ\,,$$

which threefold forces are to be considered to be applied at the point T, from which the moments are deduced :

About the axis IA in the sense $BC = \; -\delta Mf\; cos\, CQ\, cos\, BT + \delta Mf\; cos\, BQ\, cos\, OT = P,$
About the axis IB in the sense $CA = \; -\delta Mf\; cos\, AQ\, cos\, CT + \delta Mf\; cos\, CQ\, cos\, AT = Q,$
About the axis IC in the sense $AB = \; -\delta Mf\; cos\, BQ\, cos\, AT + \delta Mf\; cos\, AQ\, cos\, BT = R\;.$

Hence there are these three moments:

$$P = \delta Mf\,(\,cos\, m\, cos\, CQ - cos\, n\, cos\, BQ),$$
$$Q = \delta Mf\,(\,cos\, n\, cos\, AQ - cos\, l\, cos\, CQ),$$
$$R = \delta Mf\,(\,cos\, l\, cos\, BQ - cos\, m\, cos\, AQ).$$

Moreover for the point Q we put the angle $PZQ = \xi$; in order that

$$\text{tang } \xi = \frac{f\gamma'\, sin\, s\, cos\, \vartheta}{v - f\gamma'\, sin\, s\, sin\, \vartheta}$$

and with the speed of the abrasion

$$\sqrt{(vv - 2f\gamma' v\, sin\, s\, sin\, \vartheta + ff\gamma'\gamma'\, sin^2\, s)} = u\,,$$

then

$$sin\, \xi = \frac{-f\gamma'\, sin\, s\, cos\, \vartheta}{u} \quad \text{and} \quad cos\, \xi = \frac{f\gamma'\, sin\, s\, sin\, \vartheta - v}{u}\,.$$

Hence there becomes $DZQ = \varphi + \xi$ and $EZQ = \; 180° - \xi - \varphi$; and hence

$$AZQ = 180^{o} - \xi - \varphi - \lambda, \quad \text{BZQ} = \mu + \xi + \varphi - 180^{o}, \; \text{CZQ} = 180^{o} - \xi - \varphi - \nu$$

hence

$$cos\, AQ = -cos(\,\xi + \varphi + \lambda\,)\, sin\, l\;,$$
$$cos\, BQ = -cos(\,\xi + \varphi + \mu\,)\, sin\, m,$$
$$cos\, CQ = -cos(\,\xi + \varphi + \nu\,)\, sin\, n\;.$$

Therefore from the relation, which exists between the angles λ, μ and ν, we conclude that the moments of the forces:

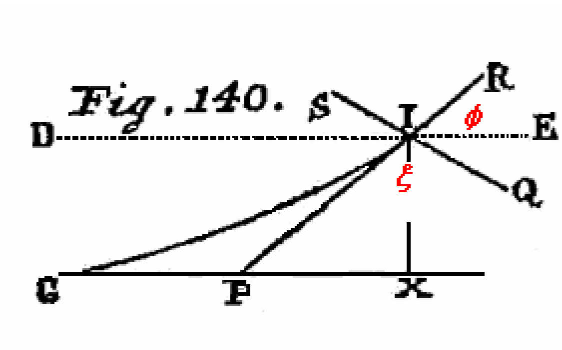
$$P = \delta Mf \sin l \sin(\lambda + \varphi + \xi),$$
$$Q = \delta Mf \sin m \sin(\mu + \varphi + \xi),$$
$$R = \delta Mf \sin n \sin(\nu + \varphi + \xi).$$

PROBLEM 16

1045. If we consider the rotational motion as given at some time, to define the progressive motion of the sphere.

SOLUTION

Because the centre of the globe is moving on the horizontal plane, it describes the line *GI* in the time *t*, which refers to the above direction *GX* parallel to the fixed direction *DE*, and with *IX* drawn normal to *GX*, the coordinates are *GX* = *X*, *XI* = *Y* (Fig. 140). The line *DE* is drawn through *I* parallel to *GX*, which is the diameter itself *DE* in figure 139. *IP* is drawn, in order that $DIP = EIR = \varphi$, and the centre *I* by the hypothesis is progressing in the direction *I R* with a speed equal to *v*, thus in order that the speed along $XI = v \sin\varphi$ and thus $dX = vdt \cos\varphi$ and $dY = vdt \sin\varphi$. The line *QIS* is drawn, in order that *IQ* is parallel to the direction in which the point of contact rubs against the surface, then the angle $EIQ = DIS = 180° - \xi - \varphi$; indeed it is equal to the angle *EZQ* in the preceding figure, from which the globe is agreed to be acted on by the force equal to δM in the direction *IS.* Hence therefore there arises the force along $ID = -\delta M \cos(\xi + \varphi)$ and the force along $XI = -\delta M \sin(\xi + \varphi)$. From which it is deduced that :



$$\frac{d \cdot v \cos\varphi}{2gdt} = \frac{dv \cos\varphi - vd\varphi v \sin\varphi}{2gdt} = \delta \cos(\xi + \varphi),$$
$$\frac{d \cdot v \sin\varphi}{2gdt} = \frac{dv \sin\varphi + vd\varphi v \cos\varphi}{2gdt} = \delta \sin(\xi + \varphi)$$

and hence again

$$\frac{dv}{2gdt} = \delta \cos\xi \quad \text{and} \quad \frac{vd\varphi}{2gdt} = \delta \sin\xi$$

thus in order that

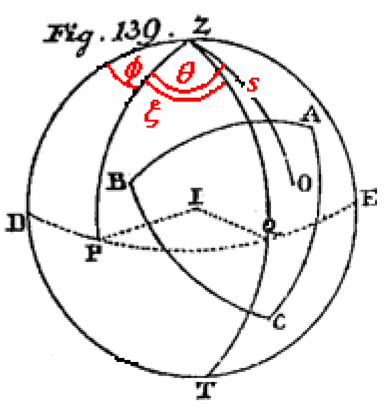
$$\frac{vd\varphi}{dv} = \text{tang } \xi = \frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta}$$

PROBLEM 17

1046. With the progressive motion of the globe defined, to determine the rotational motion of the globe.

SOLUTION

Now the centre of the globe *I* can be considered at rest, and all the denominations applied in problem 15 remain, and let *Maa*, *Mbb*, *Mcc* be the moments of inertia about the principal axes *IA*, *IB*, *IC*, which initially we consider as unequal (Fig. 139). Now since here we must consider the angular speed γ' as negative, because it tends in the sense *ACB*, if we put



$$\gamma' \cos\alpha = x, \quad \gamma' \cos\beta = y, \quad \text{et} \quad \gamma' \cos\gamma = z,$$

into the general formulas it is required to take these letters *x*, *y*, *z* as negative, from § 810 we have these equations determining the motion :

$$dx + \frac{bb-cc}{aa} yzdt + \frac{2\delta fg}{aa} dt \sin l \sin(\lambda+\varphi+\xi) = 0,$$
$$dy + \frac{cc-aa}{bb} xzdt + \frac{2\delta fg}{bb} dt \sin m \sin(\mu+\varphi+\xi) = 0,$$
$$dz + \frac{aa-bb}{cc} xydt + \frac{2\delta fg}{cc} dt \sin n \sin(\nu+\varphi+\xi) = 0,$$
$$dl \sin l = dt(z\cos m - y\cos n), \quad d\lambda \sin^2 l = dt(y\cos m + z\cos n),$$
$$dm \sin m = dt(x\cos n - z\cos l), \quad d\mu \sin^2 m = dt(z\cos n + x\cos l),$$
$$dn \sin n = dt(y\cos l - x\cos m), \quad d\nu \sin^2 n = dt(x\cos l + y\cos m).$$

Then indeed from the progressive motion we have:

$$dv = 2\delta gdt \cos\xi, \quad vd\varphi = 2f\delta gdt \sin\xi$$

and

$$\text{tang } \xi = \frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta},$$

where $PZO = \vartheta$ and $ZO = s$. Hence since $EZO = 180^{\circ} - \vartheta - \varphi$, then $AZO = 180^{\circ} - \lambda - \vartheta - \varphi$ and hence

$$\cos\alpha = \cos l \cos s - \sin l \sin s \cos(\lambda+\vartheta+\varphi),$$
$$\cos\beta = \cos m \cos s - \sin m \sin s \cos(\mu+\vartheta+\varphi),$$
$$\cos\gamma = \cos n \cos s - \sin n \sin s \cos(\nu+\vartheta+\varphi),$$

with
$$\cos s = \cos l \cos\alpha + \cos m \cos\vartheta + \cos n \cos\gamma,$$
from which it follows that

$$\sin l \cos l \cos(\lambda+\vartheta+\varphi) + \sin m \cos m \cos(\mu+\vartheta+\varphi) + \sin n \cos n \cos(\nu+\vartheta+\varphi) = 0 .$$

We put $\gamma' \cos s = p$ and $\gamma' \sin s = q$, in order that
$$\text{tang } \xi = \frac{fq\cos\vartheta}{v - fq\sin\vartheta} = \frac{vd\varphi}{dv};$$

and then
$$x = p\cos l - q\sin l \cos(\lambda+\vartheta+\varphi),$$
$$y = p\cos m - q\sin m \cos(\mu+\vartheta+\varphi),$$
$$z = p\cos n - q\sin n \cos(\nu+\vartheta+\varphi),$$

from which values there becomes

$$dl = qdt\sin(\lambda+\vartheta+\varphi), \quad d\lambda = pdt + qdt\cot l\cos(\lambda+\vartheta+\varphi),$$
$$dm = qdt\sin(\mu+\vartheta+\varphi), \quad d\mu = pdt + qdt\cot m\cos(\mu+\vartheta+\varphi),$$
$$dn = qdt\sin(\nu+\vartheta+\varphi), \quad d\nu = pdt + qdt\cot n\cos(\nu+\vartheta+\varphi)$$

and thus again

$$dx = dp\cos l - dq\sin l\cos(\lambda+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin l\sin(\lambda+\vartheta+\varphi),$$
$$dy = dp\cos m - dq\sin m\cos(\mu+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin m\sin(\mu+\vartheta+\varphi),$$
$$dz = dp\cos n - dq\sin n\cos(\nu+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin n\sin(\nu+\vartheta+\varphi).$$

But without the help of these substitutions from the three equations, since in general then

$$\sin l\cos l\sin(\lambda+A) + \sin m\cos m\sin(\mu+A) + \sin n\cos n\sin(\nu+A) = 0,$$

we elicit this equation

$$aadx\cos l + bbdy\cos m + ccdz\cos n - aaxdl\sin l - bbydm\sin m - cczdn\sin n = 0$$

and the integral of this is
$$aax\cos l + bby\cos m + ccz\cos n = C,$$

which with the substitutions applied goes over into this form :

$$p(aa\cos^2 l+bb\cos^2 m+cc\cos^2 n)-q(aa\sin l\cos l\cos(\lambda+\vartheta+\varphi)$$
$$+bb\sin m\cos m\cos(\mu+\vartheta+\varphi)+cc\sin n\cos n\cos(\nu+\vartheta+\varphi))=\text{ Const.}$$

Then also by the reductions treated in § 934 this differential equation is deduced for the *vis viva* :

$$aaxdx+bbydy+cczdz=2\delta\, fgqdt\sin(\xi-\vartheta).$$

SCHOLIUM

1047. Towards understanding the reductions made here from the formulas treated above, where we express the angles Π and v by $\lambda,\ l,\ m,\ n$, it is appropriate to be made aware that:

$$\cos(\mu+\vartheta+\varphi)=\frac{-\cos l\cos m\cos(\lambda+\vartheta+\varphi)+\cos n\sin(\lambda+\vartheta+\varphi)}{\sin l\sin m},$$
$$\cos(\nu+\vartheta+\varphi)=\frac{-\cos l\cos n\cos(\lambda+\vartheta+\varphi)-\cos m\sin(\lambda+\vartheta+\varphi)}{\sin l\sin n},$$
$$\sin(\mu+\vartheta+\varphi)=\frac{-\cos l\cos m\sin(\lambda+\vartheta+\varphi)-\cos n\cos(\lambda+\vartheta+\varphi)}{\sin l\sin m},$$
$$\sin(\nu+\vartheta+\varphi)=\frac{-\cos l\cos n\sin(\lambda+\vartheta+\varphi)+\cos m\cos(\lambda+\vartheta+\varphi)}{\sin l\sin n}.$$

And in a similar manner the angles $\mu+\varphi+\xi$ and $\nu+\varphi+\xi$ can be recalled to the angle $\lambda+\varphi+\xi$. Then also for the following reductions this form especially is to be noted :

$$\sin(\mu+B)\cos(v+C)-\sin(v+B)\cos(\mu+C)$$

which on account of

$$\sin M\cos N=\tfrac{1}{2}\sin(M+N)+\tfrac{1}{2}\sin(M-N)$$

is reduced to

$$\sin(\mu-v)\cos(B-C);$$

and in this way we can find a reduction to be put in place for all the formulas :

$$\sin(\mu+B)\cos(v+C)-\sin(v+B)\cos(\mu+C)=\sin(\mu-v)\cos(B-C),$$
$$\sin(\mu+B)\sin(v+C)-\sin(v+B)\sin(\mu+C)=-\sin(\mu-v)\sin(B-C),$$
$$\cos(\mu+B)\cos(v+C)-\cos(v+B)\cos(\mu+C)=-\sin(\mu-v)\sin(B-C),$$

where $\sin(\mu-v)$ given for all the formulas used, is indeed

$$\sin(\mu-v)=\frac{\cos l}{\sin m\sin n}.$$

PROBLEM 18

1048. If the globe is established from a uniform material, or perhaps it is assemble thus, so that all the moments of inertia are equal to each other and that is initially impressed with some motion, the determine the continuation of this motion.

SOLUTION

Since here there is $aa = bb = cc$ or the moments of inertia about all of the diameters are equal to Maa, the first equation integrated gives aap = Const., from which p is a constant quantity. Hence there is put in place $p = h$, and the three differential equations adopt these forms :

$$\text{I. } -dq\cos(\lambda+\vartheta+\varphi)+q(d\vartheta+d\varphi)\sin(\lambda+\vartheta+\varphi)+\frac{2\delta fg}{aa}dt\sin(\lambda+\varphi+\xi)=0,$$

$$\text{II. } -dq\cos(\lambda+\vartheta+\varphi)+q(d\vartheta+d\varphi)\sin(\mu+\vartheta+\varphi)+\frac{2\delta fg}{aa}dt\sin(\mu+\varphi+\xi)=0,$$

$$\text{III.} -dq\cos(\lambda+\vartheta+\varphi)+q(d\vartheta+d\varphi)\sin(\nu+\vartheta+\varphi)+\frac{2\delta fg}{aa}dt\sin(\nu+\varphi+\xi)=0,$$

of which moreover two suffice to be considered, because now the conclusion $p = h$ has been noted . Now from the above reductions the two final equations thus can be combined :

$$\text{II. } \cos(\nu+\vartheta+\varphi)-\text{ III. } \cos(\mu+\vartheta+\varphi)$$

giving

$$q(d\vartheta+d\varphi)\sin(\mu-\nu)+\frac{2\delta fg}{aa}dt\sin(\mu-\nu)\cos(\xi-\vartheta)=0$$

or

$$q(d\vartheta+d\varphi)+\frac{2\delta fg}{aa}dt\cos(\xi-\vartheta)=0.$$

Then

$$\text{II. } \sin(\nu+\vartheta+\varphi)\ -\text{ III. } \sin(\mu+\vartheta+\varphi)$$

gives

$$q\sin(\mu-\nu)-\frac{2\delta fg}{aa}dt\sin(\mu-\nu)\sin(\xi-\vartheta)=0$$

or

$$q=\frac{2\delta fg}{aa}dt\sin(\xi-\vartheta)$$

which value substituted in the last equation for the vis viva produces

$$xdx+ydy+zdz=qdq$$

and hence

$$xx+yy+zz=\gamma'\gamma'=\text{ Const.}+qq=\text{ Const.}+\gamma'\gamma'\sin^2 s$$

thus so that $\gamma'\gamma' cos^2 s = \text{const}$., as we find now on account of $\gamma' cos\, s = p = h$. Hence we have these equations exempt from the letters $l,\ m,\ n,\ \lambda,\ \mu,\ \nu$:

$$\text{1. } q(d\vartheta + d\varphi) + \frac{2\delta fg}{aa}\ dt\, cos\,(\xi - \vartheta) = 0\ ,$$

$$\text{II. } dq = \frac{2\delta fg}{aa} dt\, sin\,(\xi - \vartheta)\ ,$$

$$\text{III. } dv = 2\delta gdtcos\xi,$$

$$\text{IV.} vd\varphi = 2\delta gdt\, sin\,\xi,$$

to which this finite equation can be adjoined

$$\text{tang}\xi = \frac{fq\, cos\,\vartheta}{v - fq\, sin\,\vartheta},$$

which transformed into this

$$v\, sin\,\xi - fq\, cos\,(\xi - \vartheta) = 0,$$

is differentiated :

$$dv\, sin\,\xi + vdv\, cos\,\xi - fdq\, cos\,(\xi - \vartheta) + fqd\xi\, sin\,(\xi - \vartheta) - fqd\vartheta\, sin\,(\xi - \vartheta) = 0.$$

Now

$$\text{I. } sin\,(\xi - \vartheta) + \text{II. } cos\,(\xi - \vartheta)$$

gives

$$q(d\vartheta + d\varphi)\, sin\,(\xi - \vartheta) + dq\, cos\,(\xi - \vartheta) = 0,$$

which multiplied by *f* is added to that:

$$dv\, sin\,\xi + vd\xi\, cos\,\xi + fq(d\xi + d\varphi)\, sin\left(\xi - \vartheta\right) = 0\ .$$

Again on account of

$$\frac{dv}{vd\varphi} = \frac{cos\,\xi}{sin\,\xi}$$

there then arises

$$v(d\varphi + d\xi)\, cos\,\xi + fq(d\xi + d\varphi)\, sin\,(\xi - \vartheta) = 0$$

or

$$(d\varphi + d\xi)\left(v\, cos\,\xi + fq\, sin\,(\xi - \vartheta)\right) = 0\ ,$$

of which the finite factor

$$v\, cos\,\xi + fq\, sin\,(\xi - \vartheta)$$

is unable to vanish on account of [from above]

$$v\, sin\,\xi - fq\, cos\,(\xi - \vartheta) = 0\ ;$$

for then it follows that

$$v\cos\vartheta = 0 \text{ and } fq\cos\vartheta = 0,$$

which cannot have a place unless $\vartheta = 90°$. Hence it is left that $d\varphi + d\xi = 0$ and thus

$$\varphi + \xi = \text{Const.}$$

With this explained the remainder can be extricated without difficulty ; but towards determining the integrations, for the initial position $t = 0$ we can put the progressive speed $v = e$, the angle $\varphi = 0$, the angle $PZO = \vartheta = \mathfrak{h}$, the arc $ZO = s = \mathfrak{f}$ and the angular speed $\gamma' = \varepsilon$ in the sense *ACB;* and hence

$$p = h = \gamma'\cos s = \varepsilon\cos\mathfrak{f} \text{ and } q = \varepsilon\sin\mathfrak{f};$$

again

$$\text{tang } \xi = \frac{\varepsilon f \sin\mathfrak{f}\cos\mathfrak{h}}{\varepsilon - \varepsilon f \sin\mathfrak{f}\sin\mathfrak{h}}$$

There is established

$$\frac{\varepsilon f \sin\mathfrak{f}\cos\mathfrak{h}}{\varepsilon - \varepsilon f \sin\mathfrak{f}\sin\mathfrak{h}} = \text{tang } \zeta,$$

so that initially $\xi = \zeta$, and always there shall be $\xi + \varphi = \zeta$ thus so that the angle $DZQ = \zeta$ remains constant. Whereby since $\xi = \zeta - \varphi$, then

$$v\sin(\zeta - \varphi) = fq\cos(\zeta - \vartheta - \varphi).$$

But above we have found :

$$\frac{d\cdot v\cos\varphi}{2gdt} = \delta\cos(\xi + \varphi) = \delta\cos\zeta \quad \text{et} \quad \frac{d\cdot v\sin\varphi}{2gdt} = \delta\sin(\xi + \varphi) = \delta\sin\zeta,$$

from which on integration we deduce

$$v\cos\varphi = e + 2\delta gt\cos\zeta \quad \text{and} \quad v\sin\varphi = e + 2\delta gt\sin\zeta$$

and hence

$$v = \sqrt{(ee + 4\delta egt\cos\zeta + 4\delta\delta ggtt)} \text{ and tang } \varphi = \frac{2\delta gt\sin\zeta}{e + 2\delta gt\cos\zeta}$$

and

$$\text{tang}(\zeta - \varphi) = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt} = \frac{fq\cos\vartheta}{v - fq\sin\vartheta} = \text{tang}\xi.$$

Then on account of $d\varphi = -d\xi$ the two first equations become

$$\text{I. } q(d\xi - d\vartheta) = \frac{2\delta fg}{aa}\, dt\cos(\xi - \vartheta),$$

$$\text{II. } dq = \frac{2\delta fg}{aa} dt\sin(\xi - \vartheta),$$

of which the one divided by the other gives :

$$\frac{dq}{q(d\xi - d\vartheta)} = \frac{\sin(\xi-\vartheta)}{\cos(\xi-\vartheta)}$$

which integrated gives $q\cos(\xi-\vartheta)$ = Const. : and thus

$$q\cos(\xi-\vartheta) = \varepsilon\sin\mathfrak{f}\cos(\zeta-\mathfrak{h}),$$

from which the value of q substituted in the first gives :

$$\frac{\varepsilon(d\xi-d\vartheta)\sin\mathfrak{f}\cos(\zeta-\mathfrak{h})}{\cos^2(\xi-\vartheta)} = \frac{2\delta fg}{aa}dt$$

and on integrating

$$\varepsilon\sin\mathfrak{f}\cos(\zeta-\mathfrak{h})\,\text{tang}(\xi-\vartheta) = C + \frac{2\delta fg}{aa}t,$$

where $C = \varepsilon\sin\mathfrak{f}\sin(\zeta-\mathfrak{h})$. But

$$\text{tang}(\xi-\vartheta) = \text{tang}\,(\zeta-\varphi-\vartheta) = \frac{\text{tang}\,(\zeta-\varphi)-\text{tang}\,\vartheta}{1+\text{tang}\,\vartheta\,\text{tang}\,(\zeta-\varphi)}$$

and

$$\text{tang}\,\vartheta = \frac{\text{tang}\,\xi-\text{tang}\,(\xi-\vartheta)}{1+\text{tang}\,\xi\,\text{tang}\,(\xi-\vartheta)}$$

But by the hypothesis there is :

$$\varepsilon\sin\mathfrak{f} = \frac{e\sin\zeta}{f\cos(\zeta-\mathfrak{h})}$$

from which there becomes

$$\text{tang}\left(\xi-\vartheta\right) = \text{tang}\left(\zeta-\mathfrak{h}\right) + \frac{2\delta ffgt}{eaa\sin\zeta},$$

but

$$\text{tang}\,\xi = \frac{e\sin\zeta}{e\cos\zeta+2\delta gt}$$

and hence the angle ϑ is easily determined, and thus

$$q = \frac{e\sin\zeta}{f\cos\left(\xi-\vartheta\right)}$$

Now this is required to be noted, since then

$$\text{tang}\,\zeta = \frac{\varepsilon f\sin\mathfrak{f}\cos\mathfrak{h}}{e-\varepsilon f\sin\mathfrak{f}\sin\mathfrak{h}},$$

to be as we have shown above for the angle ξ,

$$sin\zeta = -\frac{\varepsilon f\ sin\mathfrak{f} cos\mathfrak{h}}{\sqrt{(ee-2\varepsilon ef\ sin\mathfrak{f}\, sin\mathfrak{h}+eeff\ sin^2\mathfrak{f})}}$$

and

$$cos\zeta = \frac{-e+\varepsilon f\ sin\mathfrak{f}\, sin\mathfrak{h}}{\sqrt{(ee-2\varepsilon ef\ sin\mathfrak{f}\, sin\mathfrak{h}+eeff\ sin^2\mathfrak{f})}}$$

from which

$$cos(\zeta-\mathfrak{h}) = -\frac{e\, cos\mathfrak{h}}{\sqrt{(ee-2\varepsilon ef\ sin\mathfrak{f}\, sin\mathfrak{h}+eeff\ sin^2\mathfrak{f})}}.$$

With these found since $\gamma' cos\, s = \varepsilon\, cos\mathfrak{f}$ and $\gamma' sin\, s = q$, then

$$\gamma' = \sqrt{(qq+\varepsilon\varepsilon\, cos^2\mathfrak{f})} \text{ and tang } s = \frac{q}{\varepsilon\, cos\mathfrak{f}}$$

And thus now both the progressive motion, as well as the angular speed γ' can be assigned to the axis of rotation *O* at whatever time, which is sufficient to recognise the motion. But the determination of the position of the points *A*, *B*, *C* at some time is exceedingly hard, so as that that can be brought about.

COROLLARY I

1049. Since the angular speed $\gamma' = \frac{\varepsilon\, cos\mathfrak{f}}{cos\, s}$ or inversely proportional to the cosine of the arc *SO*, it follows, if the pole of the rotation *O* should be in the upper hemisphere *DZE* initially, at no time is it possible for that to arrive in the lower hemisphere ; for in crossing through the horizontal circle *DE* an infinite angular speed γ' is produced.

COROLLARY 2

1050. On account of the same reason, if the pole of rotation *O* initially were in the lower hemisphere *DTE*, this at no time rises to the upper. But if initially it should be in the horizontal circle *DE*, it always remains in the same. Obviously if initially the axis of rotation were horizontal, it always remains horizontal.

COROLLARY 3

1051. If the initial angle $DZO = \mathfrak{h}$ is right, this makes $sin\zeta = 0$ and on account of

$$\text{tang}(\zeta-\mathfrak{h}) = \frac{\varepsilon f\ sin\mathfrak{f} - e\, sin\mathfrak{h}}{e\, cos\mathfrak{h}}$$

also $\xi - \vartheta$ is right. But on account of

$$\text{tang}\,\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt}$$

the angle ξ vanishes, from which the angle $\vartheta = PZO$ becomes right. Therefore as soon as the angle *PZO* is made right, it always remains right.

COROLLARY 4

1052. Also remarkable is the property that the angle $\xi + \varphi$ or *DZQ* remains constant and in Fig. 140 the angle *DIQ* remains constant. For the line *QIS* always remains parallel to itself, an because the globe in progressive motion is acted on by the constant force δM along the same direction *IS*, it is necessary that curve described from that that *GI* is a parabola.

SCHOLION 1

1053. But this motion of the globe, as defined from our formulas, does not last for ever, as actually friction is present, or it is being acted on by the friction of the horizontal plane at the point of contact T. For if it comes about that the friction ceases or the speed of the grazing action vanishes at *T*, the friction vanishes at once and the formulas found no more have a place. Therefore then the globe moves forwards both with a progressive motion as well as with a uniform rotation, and it is apparent that the axis of rotation is not allowed any more change. And if at once the initial motion impressed on the globe was put in place, so that there was no friction, which comes about both $\varepsilon f \sin\mathfrak{f}\cos\mathfrak{h} = 0$ as well as $e = \varepsilon f \sin\mathfrak{f}\sin\mathfrak{h}$, then also the globe experiences no friction ; and at once from the start it pursues the uniform progressive motion in direction and likewise it rotates uniformly about the same axis. Now if initially some other motion were impressed on the body , in any elapsed time that is always reduced, in order that friction vanishes and then it pursues its own uniform motion ; which remarkable point of time we investigate in the following problem.

SCHOLIUM 2

1054. Which matters we elicited in the solution of the problem, here they are presented again : From the first motion impressed we have the speed of the progressive motion equal to *e* along the direction *DI;* and if the globe is rotating about the axis *IO* with an angular speed ε in the sense *ACB* or *ZETD* (Fig. 139), which usually said to be *tending clockwise* [*tendens antrorsum* in the original Latin text, a turning forwards or upwards (shorter Oxford English dictionary): there is a problem as to which way one is looking along the axis, here it has been assumed by the translator that one is looking along *OI*, *i. e.* towards the origin. There is of course no equivalent word in classic Latin, as such an idea was not encountered with water clocks!], the arc $ZO = \mathfrak{f}$ and the angle $DZO = \mathfrak{h}$; then the radius of the globe is now equal to f and the moment of inertia of this is equal to *Maa* about all the diameters, with *M* present as the mass; with these given, the speed of the grazing contact point is deduced

$$\sqrt{(ee - 2\varepsilon ef\,\sin\mathfrak{f}\sin\mathfrak{h} + \varepsilon\varepsilon ff\,\sin^2\mathfrak{f})}\,,$$

which if it is put equal to *k*, the angle ζ is sought so that

$$sin\,\zeta = -\frac{\varepsilon f\ sin\mathfrak{f}\,cos\mathfrak{h}}{k} \quad \text{and} \quad cos\,\zeta = \frac{\varepsilon f\ sin\mathfrak{f}\,sin\mathfrak{h} - e}{k},$$

which is $DZQ = \zeta$, and then *IQ* is the direction of the grazing motion. Then if in the elapsed time *t* the centre of the globe is carried forwards with a speed *v* along the direction *PI* and is rotating with an angular speed equal to γ' in the sense *ZETD* about the pole *O* and there is put $DZP = \varphi$, $PZO = \vartheta$ and $ZO = s$, initially we find :

$$\text{tang}\,\varphi = \frac{2\delta gt\,sin\,\zeta}{e + 2\delta gt\,cos\,\zeta}$$

and the speed of the centre

$$D = \sqrt{(ee + 4\delta egt\,cos\,\zeta \ + 4\delta\delta ggtt)},$$

but the grazing speed even now is made in the direction *IQ*, with $DZQ = \zeta$ present: from which on putting $PZQ = \xi$ then

$$\text{tang}\,\xi = \frac{e\,sin\,\zeta}{e\,cos\,\zeta + 2\delta gt}.$$

Again there is

$$\text{tang}\left(\xi - \vartheta\right) = \text{tang}\left(\zeta - \mathfrak{h}\right) + \frac{2\delta\,ffgt}{eaa\,sin\,\zeta}$$

with the equation arising

$$\text{tang}\left(\zeta - \mathfrak{h}\right) = \frac{ef\ sin\mathfrak{f} - e\,sin\mathfrak{h}}{e\,cos\mathfrak{h}},$$

from which the angle ϑ becomes known, and hence on account of $DZ0 = \varphi + \vartheta = \zeta - \xi + \vartheta$ there is deduced

$$\text{tang}\ DZO = \ \text{tang}\left(\varphi + \vartheta\right) = \frac{\varepsilon aak\ sin\mathfrak{f}\,sin\mathfrak{h} + 2\delta\ fgt(e\text{-}\varepsilon\ f\ sin\mathfrak{f}\,sin\mathfrak{h})}{\varepsilon aak\ sin\mathfrak{f}\,cos\mathfrak{h} - 2\delta\varepsilon\ ffgt\,sin\mathfrak{f}\,cos\mathfrak{h}}.$$

And from these finally obtained we have :

$$\gamma'\,cos\,s = \varepsilon\,cos\,\mathfrak{f} \ \text{and} \ \gamma'\,sin\,s = \frac{e\,sin\,\zeta}{f\,cos\left(\xi - \vartheta\right)}.$$

And then for the grazing speed along *IQ* this is equal to

$$\sqrt{(vv - 2\gamma' \; fv \sin s \sin \vartheta + \gamma' \gamma' \; ff \; \sin^2 s)}\,;$$

which if it is called equal to *w*, we have shown above to be

$$\sin \xi = \frac{-\gamma' \; f \sin s \cos \vartheta}{w} \quad \text{and} \quad \cos \xi = \frac{\gamma' \; f \sin s \sin \vartheta - v}{w},$$

from which γ' and s are defined. But for the position of the points *A*, *B*, *C* in the fixed globe at some time to be determined the formulas thus become intricate, so that nothing can be concluded. Meanwhile if for the point *A* on calling $ZA = l$ and $EZA = \lambda$, the whole calculation is reduced to these two equations :

$$\text{I. } dl = dt\left(\varepsilon \sin \mathfrak{f} \sin(\mathfrak{h} + \lambda) - \frac{2\delta fgt}{aa} \cos(\zeta + \lambda)\right),$$

$$\text{II. } d\lambda \sin l = \varepsilon dt \cos \mathfrak{f} \sin l + dt \cos l\left(e \sin \mathfrak{f} \cos\left(\mathfrak{h} + \lambda\right) + \frac{2\delta fgt}{aa} \sin(\zeta + \lambda)\right),$$

the resolution of which can be undertaken without disappointment I fear. But since at some time we are able to assign the axis of rotation with the angular speed, so that in recognising the motion, such as generally is desired, it suffices to be able, and with that the more the wonder it is that, as it were, analytical forces can overcome the motions of the single points of the globes. It is much less permissible therefore to define any motion concerned with globes, in which the moments of inertia are unequal.

PROBLEM 19

1055. If some motion should be impressed on a globe, all the moments of inertia of which are equal to each other, then to assign the point in time when the grazing speed and thus the friction vanish, and thus the globe goes on to progress with a uniform motion.

SOLUTION

Above in § 1039 we saw, that these two conditions are required in order that the grazing motion vanishes : the one that $\gamma' \sin s \cos \vartheta = 0$ and the other that $v = f\gamma' \sin s \sin \vartheta$, or in the expression

$$\text{tang } \zeta = \frac{f\gamma' \sin s \cos \vartheta}{v - f\gamma' \sin s \sin \vartheta}$$

so the numerator as well as the denominator likewise must vanish. But since we have found

$$\text{tang } \zeta = \frac{e \sin \zeta}{e \cos \zeta + 2\delta gt},$$

where the numerator $e\,sin\,\zeta$ is constant, if in that formula the numerator vanishes, it is necessary that likewise the denominator vanishes, because otherwise the equality between these two fractions cannot stand. From which on putting $cos\vartheta = 0$ the time sought is indicated. Now we can determine the same more clearly, if we investigate the speed of grazing *w* at some elapsed time *t*. Therefore since from the formula

$$sin\,\xi = -\frac{\gamma'\,f\,sin\,s\,cos\,\vartheta}{w}$$

then

$$w = -\frac{\gamma'\,f\,sin\,s\,cos\,\vartheta}{sin\,\xi},$$

which expression on account of

$$\gamma'\,sin\,s = \frac{e\,sin\,\zeta}{f\,cos(\xi-\vartheta)},$$

turns into

$$w = \frac{-e\,sin\,\zeta\,cos\,\vartheta}{sin\,\xi\,cos(\xi-\vartheta)},$$

and on account of $\vartheta = \xi - (\xi - \vartheta)$,

into this

$$w = -e\,sin\,\zeta\left(\cot\xi + \mathrm{tang}\left(\xi - \vartheta\right)\right) \ ;$$

if here for tang ξ and tang $(\xi - \vartheta)$ we substitute the above values found, then we find:

$$w = -\left(e\,cos\,\zeta + 2\delta gt + e\,sin\,\zeta\,\mathrm{tang}\left(\zeta - \mathfrak{h}\right) + \frac{2\delta\,ffgt}{aa}\right)$$

But

$$cos\,\zeta + sin\,\zeta\,\mathrm{tang}\ (\zeta - \mathfrak{h}) = \frac{cos\,\mathfrak{h}}{sin\,(\zeta - \mathfrak{h})} \quad \text{et} \quad cos\,(\zeta - \mathfrak{h}) = -\frac{e\,cos\,\mathfrak{h}}{k};$$

from which

$$e\,cos\,\zeta + e\,sin\,\zeta\,\mathrm{tang}\ (\zeta - \mathfrak{h}) = -k\,,$$

where k denotes the initial grazing speed. On account of which in the elapsed time *t* we have the grazing speed

$$w = k - 2\delta g\left(1 + \frac{ff}{aa}\right)t\,,$$

thus so that this decreases uniformly with the passage of time. Hence finally it certainly disappears, and that happens in the elapsed time

$$t = \frac{aak}{2\delta g\left(aa + ff\right)};$$

then at this time $\cos\vartheta = 0$ and $\vartheta = 90^{\circ} = PZO$. Which hence as it comes about, we can consider how the determination of the remaining motions themselves can be obtained ; and since $2\delta gt = \frac{aak}{aa+ff}$, then

$$\text{tang } \varphi = \frac{aak \sin\zeta}{e(aa+ff)+aak\cos\zeta} \quad \text{and} \quad \text{tang } \xi = \frac{e(aa+ff)\sin\zeta}{e(aa+ff)\cos\zeta+aak}.$$

[Corrected from the first edition by C. B. in the O. O. edition.]
Hence there becomes

$$\gamma' \sin s = \frac{e\sin\zeta}{f\sin\xi}.$$

But as there is

$$v = \sqrt{\left(ee + \frac{2aaek\cos\zeta}{aa+ff} + \frac{a^4kk}{(aa+ff)^2}\right)},$$

then

$$\sin\varphi = \frac{aak\sin\zeta}{(aa+ff)v} \quad \text{and} \quad \cos\varphi = \frac{e(aa+ff)+aak\cos\zeta}{(aa+ff)v}$$

and

$$\sin\xi = \frac{e\sin\zeta}{v} \quad \text{and thus} \quad \gamma' \sin s = \frac{v}{f}.$$

Again since $\gamma' \cos s = \varepsilon \cos f$, then

$$\text{tang } s = \frac{v}{\varepsilon f \cos\mathfrak{f}} \quad \text{and} \quad \gamma' = \sqrt{\left(\frac{vv}{ff} + \varepsilon\varepsilon\cos^2\mathfrak{f}\right)}$$

or

$$\gamma' = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf\sin\mathfrak{f}\sin\mathfrak{h} + \varepsilon\varepsilon a^4\sin^2\mathfrak{f} + \varepsilon\varepsilon(aa+ff)^2\cos^2\mathfrak{f}\right)}}{aa+ff}$$

on account of $kk = ee - 2\varepsilon ef\sin\mathfrak{f}\sin\mathfrak{h} + eeff\sin^2\mathfrak{f}$.

COROLLARY 1

1056. So that hence the greater the initial speed of grazing k should be, for that a longer motion endures, before with friction ceasing it is reduced to uniform motion. And if the globe consists of a homogeneous material, there is made $aa = \frac{2}{5} ff$, and thus the uniform motion starts after the time $t = \frac{k}{7\delta g}$ sec. hence in the hypothesis $\delta = \frac{1}{3}$ there comes about $t = \frac{3k}{7g}$, on taking $g = 15\frac{5}{8}$ Rhen.ft.

[Recall that a Rhenish foot is around 3% greater than the Imperial foot.]

COROLLARY 2

1057. In order that the centre of the globe is reduced to rest in the same time, the initial position must be prepared thus, in order that $\cos\zeta = -1$ and $e = \frac{aak}{aa+ff}$. Hence there arises

$$k = e - \varepsilon f \sin\mathfrak{f} \sin\mathfrak{h} \quad \text{and} \quad \sin\mathfrak{h} = 1 \quad \text{or} \quad \mathfrak{h} = 90° \quad \text{and} \quad k = e - \varepsilon f \sin\mathfrak{f};$$

and hence

$$\varepsilon \sin\mathfrak{f} = -\frac{ef}{aa}.$$

Again on account of $v = 0$ there arises $s = 0$ and $\gamma' = \varepsilon\cos\mathfrak{f}$, from which angular speed the globe now rotates about a vertical axis at rest after an elapsed time from the beginning $t = \frac{e}{2\delta g}$ min. sec.

COROLLARY 3

1058. Moreover in this case, in which initially $\mathfrak{h} = 90°$ and $\varepsilon = -\frac{ef}{aa\sin\mathfrak{f}}$, there becomes $\zeta = 180°, \quad \varphi = 0, \quad \xi = 180°, \quad \vartheta = 90°, \quad v = e - 2\delta gt$; then now

$$\gamma' \cos s = \frac{-ef\cos\mathfrak{f}}{aa\sin\mathfrak{f}},$$

$$\gamma' \sin s = \frac{-ef}{aa}\left(1 - \frac{2\delta gt}{e}\right),$$

and hence

$$\text{tang } s = \left(1 - \frac{2\delta gt}{e}\right) \text{tang } \mathfrak{f}$$

and

$$\gamma' = \frac{-ef}{aa\sin\mathfrak{f}} \sqrt{\left(1 - \frac{4\delta gt}{e}\sin^2\mathfrak{f} + \frac{4\delta\delta ggtt}{ee}\sin^2\mathfrak{f}\right)}.$$

But initially the grazing speed was

$$k = e\left(1 + \frac{ff}{aa}\right),$$

and moreover after the t this is

$$w = e\left(1 + \frac{ff}{aa}\right)(e - 2\delta gt),$$

and thus on putting $t = -\frac{e}{2\delta g}$ likewise there becomes $w = 0$, $v = 0$ and $s = 0$, as before.

COROLLARIUM 4

1059. Lest the value

$$\gamma' \sin s = \frac{e \sin \zeta}{f \cos(\xi - \vartheta)}$$

is considered to be indefinite, which happens if the numerator and the denominator vanish or $\zeta = 0$, it is agreed to substitute the values from above in place of $\sin\zeta$ and $\cos(\xi - \vartheta)$, and hence there is found:

$$\gamma' \sin s = \sqrt{\left(\varepsilon\varepsilon \sin^2 \mathfrak{f} - \frac{4\delta\varepsilon fgt(\varepsilon f \sin \mathfrak{f} - e \sin \mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)},$$

from which on account of $\gamma' \cos s = e \cos \mathfrak{f}$ there is produced:

$$\gamma'\gamma' = \varepsilon\varepsilon - \frac{4\delta\varepsilon fgt \sin \mathfrak{f}(\varepsilon f \sin \mathfrak{f} - e \sin \mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}.$$

COROLLARY 5

1060. Since the *vis viva* of the globe is equal to $M(vv + aa\gamma'\gamma')$, that initially is equal to $M(ee + \varepsilon\varepsilon aa)$, but after the time t that becomes equal to

$$M\left(ee + \varepsilon\varepsilon aa - 4\delta gkt + 4(1 + \frac{ff}{aa})\delta\delta ggtt\right).$$

But after the time

$$t = \frac{aak}{2\delta g(aa + ff)}$$

the *vis viva* becomes

$$\frac{M(eeff + 2\varepsilon eaaf \sin \mathfrak{f} \sin \mathfrak{h}) + \varepsilon\varepsilon aa(aa + ff\cos^2 \mathfrak{f})}{aa + ff},$$

and the deficiency of this from the beginning is

$$\frac{\left(Maaee-2\varepsilon ef\ \sin\mathfrak{f}\sin\mathfrak{h}+\varepsilon\varepsilon ff\cos^2\mathfrak{f}\right)}{aa+ff}=\frac{Maakk}{aa+ff},$$

thus so that this *vis viva* is equal to

$$M\left(ee+\varepsilon\varepsilon aa-\frac{aakk}{aa+ff}\right)$$

SCHOLlUM

1061. Hence the whole motion of the globe can be assigned from these formulas , whatever motion initially should be impressed on that ; yet meanwhile these formulas are not a little complex, from which towards presenting a clearer explanation independent of these, I have set out certain more notable cases. They are of this kind, as we have now intimated above, two in particular, the one in which the arc *ZO* is a quadrant initially, and the other now in which the angle $DZO=\mathfrak{h}$ is right; therefore we shall explain each in turn.

PROBLEM 20

1062. If initially a rotational motion about a horizontal axis in addition to a progressive motion were impressed on a globe, to define the continuation of this motion.

SOLUTION

Since the initial axis of rotation is horizontal, then $\mathfrak{f}=ZO=90°.$ Hence on denoting by *e* the progressive motion along the direction *DIE* (Fig. 139) et ε the angular speed about the axis *IO* in the sense *ZETD*, then for the point *O* the angle $DZO=\mathfrak{h}$, with f remaining equal to the radius of the globe and Maa equal to the moment of inertia. From these initially the grazing speed

$$k=\sqrt{\left(ee-2e\varepsilon f\ \sin\mathfrak{h}+\varepsilon\varepsilon ff\right)}$$

and for the direction *1Q* of this the angle $DZQ=\zeta$, so that then (Fig. 140).

$$\sin\zeta=-\frac{\varepsilon f\cos\mathfrak{h}}{k}\quad\text{and}\quad\cos\zeta=\frac{\varepsilon f\sin\mathfrak{h}-e}{k}.$$

With these in place for the initial position, after a time *t* the centre of the globe has described a path *GI*, so that now it is at *I*, where the speed along *1R* is equal to

$$v=\sqrt{\left(ee+\frac{4\delta egt\left(\varepsilon f\sin\mathfrak{h}-e\right)}{k}+4\delta\delta ggtt\right)};$$

from which with the coordinates in place *GX* = *X* and *XI* = *Y* on account of

$$\text{tang } EIR=\text{ tang}\varphi=-\frac{2\delta\varepsilon fgt\cos\mathfrak{h}}{ek+2\delta gt\left(\varepsilon f\sin\mathfrak{h}-e\right)}$$

then there is

$$dX = edt + \frac{2\delta gtdt}{k}\left(\varepsilon f \sin\mathfrak{h} - e\right)$$

and

$$dY = \frac{-2\delta\varepsilon fgtdt \cos\mathfrak{h}}{k}$$

and thus

$$GX = X = et + \frac{\delta gtt}{k}\left(\varepsilon f \sin\mathfrak{h} - e\right) \quad \text{et} \quad XI = Y = -\frac{\delta gtt}{k}\cos\mathfrak{h}.$$

Then for the rotational motion, which now is made in the sense *ZETD* with an angular speed equal to γ' about the point *O* with $ZO = s$ arising, $PZO = \vartheta$ and $DZQ = \varphi + \xi$, where *IQ* refers to the direction of the grazing speed, since that constantly is $\varphi + \xi = \zeta$, or along the fixed direction *IQ*, then we have

$$\text{tang}\ \xi = -\frac{\varepsilon ef \cos\mathfrak{h}}{\varepsilon ef \sin\mathfrak{h} - ee + 2\delta gkt}$$

and

$$\text{tang}\left(\xi - \vartheta\right) = \frac{\varepsilon f - e\sin\mathfrak{h}}{a\cos\mathfrak{h}} - \frac{2\delta gkt}{\varepsilon eaa\cos\mathfrak{h}},$$

from which both the angles ξ et ϑ are defined. Or

$$\text{tang}\left(\varphi + \vartheta\right) = \frac{\varepsilon aak \sin\mathfrak{h} + 2\delta gkt\left(e - \varepsilon f \sin\mathfrak{h}\right)}{\varepsilon aak \cos\mathfrak{h} - 2\delta\varepsilon ffgt \cos\mathfrak{h}}.$$

But the grazing speed along the direction *IQ* is

$$w = k - 2\delta g\left(1 + \frac{ff}{aa}\right)t \ .$$

Then now on account of $\gamma' \cos s = 0$ the arc $ZO = s$ is a quadrant and

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt\left(\varepsilon f - e\sin\mathfrak{h}\right)}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)}$$

But this non-uniform motion only lasts for the time

$$t = \frac{aak}{2\delta g\left(aa + ff\right)} \ ,$$

at the end of which

$$\mathrm{tang}\,\varphi = -\frac{\varepsilon aaf\cos\mathfrak{h}}{e(aa+ff)+aa(\varepsilon f\sin\mathfrak{h}-e)} = -\frac{\varepsilon aa\cos\mathfrak{h}}{ef+\varepsilon aa\sin\mathfrak{h}},$$

$$v = \sqrt{\left(ee+\frac{2aae(\varepsilon f\sin\mathfrak{h}-e)}{aa+ff}+\frac{a^4kk}{(aa+ff)^2}\right)},$$

$$s = 90^{\mathrm{o}} \text{ and } \gamma' = \frac{v}{f} = \frac{\sqrt{(eeff+2\varepsilon eaaf\sin\mathfrak{h}+\varepsilon\varepsilon a^4)}}{aa+ff},$$

substituted for the value *kk*. But then the angle ϑ becomes 90^{o} and

$$\sin\xi = \frac{e\sin\zeta}{v}.$$

COROLLARY 1

1063. If initially the angle $DZO = \mathfrak{h} = 0$, then

$$k = \sqrt{(ee+\varepsilon\varepsilon ff)};$$

and for the angle $DZQ = \zeta$ there becomes

$$\sin\zeta = -\frac{\varepsilon f}{k}, \quad \cos\zeta = -\frac{e}{k};$$

then after the time *t* there is produced

$$v = \sqrt{\left(ee-\frac{4\delta eegt}{k}+4\delta\delta ggtt\right)}, \quad \mathrm{tang}\,\varphi = -\frac{2\delta\varepsilon fgt}{e(k-2\delta gt)},$$

$$X = et\left(1-\frac{\delta gt}{k}\right), \quad Y = -\frac{\delta\varepsilon fgtt}{k}.$$

Again

$$\mathrm{tang}\,\xi = -\frac{eef}{eef-2\delta gkt}, \quad \mathrm{tang}(\xi-\vartheta) = \frac{\varepsilon f}{a} - \frac{2\delta gkt}{\varepsilon eaa};$$

$$\mathrm{tang}(\varphi+\vartheta) = \frac{2\delta efgt}{\varepsilon aak-2\delta\varepsilon ffgt},$$

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon-\frac{4\delta\varepsilon\varepsilon ffgt}{aak}+\frac{4\delta\delta ffggtt}{a^4}\right)} \quad \text{et} \quad w = k-2\delta g\left(1+\frac{ff}{aa}\right)t$$

but in the elapsed time $t=\frac{aak}{2\delta g(aa+ff)}$ then

$$\text{tang}\varphi=-\frac{\varepsilon aa}{ef}, \quad v=\frac{f\sqrt{\left(eeff+\varepsilon\varepsilon a^4\right)}}{aa+ff}=f\gamma',$$

$$\vartheta=90^{\text{o}} \text{ and } \tan\xi=\frac{\varepsilon ef(aa+ff)}{ee(aa+ff)-aakk}=\frac{\varepsilon e(aa+ff)}{f(ee-\varepsilon\varepsilon aa)}$$

COROLLARY 2

1064. If the angle $DZO=\mathfrak{h}$ is equal to 180°, the same formulas indicate the motion, with the angular speed ε taken negative or in a rotational motion in the opposite sense. But if $\varepsilon=0$ or only a progressive motion were impressed on the globe, then there arises *k*=*e*, $\zeta=180°, v=e-2\delta gt, \ \varphi=0, \ X=t\left(e-\delta gt\right), Y=0, \ \xi=180°; \ \vartheta=90^{\text{o}}, \ \gamma'=\frac{2\delta fgt}{aa}$ and in the elapsed time $t=\frac{aae}{2\delta g\left(gaa+ff\right)}$,
there arises

$$v=\frac{eff}{aa+ff}, \quad \gamma'=\frac{ef}{aa+ff} \quad \text{et} \quad X=\frac{et\left(aa+2ff\right)}{2(aa+ff)}=\frac{aaee(aa+2ft)}{4\delta g(aa+ff)^2}.$$

[Small corrections have been made in the O. O. to these formulas by C. B.]

SCHOLIUM

1065. Here the case prevails in general, in which the globe has not been given any rotational motion initially, and neither is it restricted to any hypothesis of the angles $\mathfrak{f}$ and $\mathfrak{h}$. Therefore then the globe advances in the direction of the motion with uniform retardation, and the motion gradually acquires rotational motion, then in the elapsed time $t=\frac{aae}{2\delta g(aa+ff)}$ it has acquired a uniform motion, which henceforth it progresses continually. Hence we can deduce that case, in which the globe initially has only taken rotational motion without any progressive motion, and this is easily set out. For on putting *e* = 0 then $k=\varepsilon f \sin\mathfrak{f}$ and hence there arises

$$\sin\zeta=-\cos\mathfrak{h} \text{ and } \cos\zeta=\sin\mathfrak{h},$$

hence $\zeta=\mathfrak{h}-90°$; where for the initial axis of rotation *IO* impressed $ZO=\mathfrak{f}$ and $DZO=\mathfrak{h}$, with the angular speed arising in the sense $ZETD=\varepsilon$. Hence in the elapsed time *t* there becomes $\varphi=\zeta$, clearly with the right angle *PZO* removed from the angle $DZO=\mathfrak{h}$ then *PI* is the direction of the progressive motion which the globe acquires, and the speed of this is then $v=2\delta gt$ and thus proportional to the time. Then there arises $\text{tang}\xi=0$ and $\text{tang}\left(\xi-\vartheta\right)=\infty$, hence on account ot $\varphi+\xi=\zeta=\mathfrak{h}-90°$ then $\xi=0$ and $\vartheta=90°$, hence $DZO = \zeta + 90^\circ = \mathfrak{h}$, thus so that the pole or the rotation O is always found in the same vertical circle. And then from §1059 we have

$$\gamma' \sin s = \sqrt{\left(\varepsilon\varepsilon \sin^2 \mathfrak{f} - \frac{4\delta\varepsilon\, fgt \sin \mathfrak{f}}{aa} + \frac{4\delta\delta\, ffggtt}{a^4}\right)} = \varepsilon \sin \mathfrak{f} - \frac{2\delta\, fgt}{aa}$$

and

$$\gamma' \cos s = \varepsilon \cos \mathfrak{f},$$

from which there becomes

$$\text{tang } s = \text{tang } \mathfrak{f} - \frac{2\delta\, fgt}{\varepsilon aa \cos \mathfrak{f}},$$

thus in order that the arc ZO is diminished, unless it should be a quadrant or greater, and

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon\, fgt \sin \mathfrak{f}}{aa} + \frac{4\delta\delta\, ffggtt}{a^4}\right)}.$$

But the motion is reduced to uniformity in the elapsed time

$$t = \frac{\varepsilon aaf \sin \mathfrak{f}}{aa + ff};$$

and then it becomes

$$\gamma' = \frac{\varepsilon\sqrt{\left(a^4 \sin^2 \mathfrak{f} + (aa + ff)^2 \cos^2 \mathfrak{f}\right)}}{aa + ff}, \quad v = \frac{\varepsilon aaf \sin \mathfrak{f}}{aa + ff} \quad \text{and} \quad \text{tang } s = \frac{aa \text{ tang } \mathfrak{f}}{aa + ff}.$$

Hence if it should be that $\mathfrak{f} = 0$ or the motion should be impressed on the globe about a vertical axis without any progressive motion, then the same motion would be conserved without any change.

[This non-physical event occurs in the derivation as the body rests on a point, about which frictional forces cannot exert any moment.]

PROBLEM 21

1066. If the rotational motion were impressed on a globe about an axis normal to the direction of progression, in which all the moments of inertia are equal, to define the continuation of the motion.

SOLUTION

Since the initial direction of the progressive motion impressed is the line *DIE* (Fig. 139) and the speed equals e, the angle $DZO = \mathfrak{h}$ is right, and on taking $ZO = \mathfrak{f}$ then O is the pole, about which the globe initially accepts the angular speed equal to ε in the sense *ZETD*. Hence we have $k = \pm(e-\varepsilon f\ sin\mathfrak{f})$, as the positive value for *k* it is required to take, thus so that here two cases arise to be treated.

Case I. Let $e > \varepsilon f\ sin\mathfrak{f}$, then $k = e-\varepsilon f\ sin\mathfrak{f}$, which is the initial grazing speed, and *IQ* is the direction of this, as then *sin DQ* = 0 and *cos DQ* = –1 and thus $DQ = \zeta = 180^{\circ}$ and *Q* falls on *E;* and the globe is held back constantly by the friction δM along *ID*, from which it is deduced at once that the centre *I* of the globe is always on the same line *DE*. In the elapsed time *t* hence on account of $cos\,\zeta = -1$ the speed of the centre $v = e-2\delta gt$ and the grazing speed

$$w = e-\varepsilon\ fsin\mathfrak{f} - 2\delta \mathrm{g}\left(1+\frac{f}{aa}\right)t\ ;$$

then $\varphi = 0$ and $\xi = 180^{\circ}$ and also $\vartheta = 90^{\circ}$. Whereby for the present axis of rotation *IO* there is *DIO* = 90° and on putting the arc *ZO* = *s* and with the angular speed equal to γ' we have

$$\gamma' cos\,s = \varepsilon\,cos\,\mathfrak{f}$$

and from § 1059

$$\gamma' sin\,s = \varepsilon\,sin\,\mathfrak{f} + \frac{2\delta\,fgt}{aa},$$

from which it is deduced,

$$\text{tang } s = \text{tang } \mathfrak{f} + \frac{2\delta\,fgt}{\varepsilon aa\,cos\,\mathfrak{f}} \quad \text{and} \quad \gamma' = \sqrt{\left(\varepsilon\varepsilon + \frac{4\delta\varepsilon\,fgt\,sin\,\mathfrak{f}}{aa} + \frac{4\delta\delta\,ffggtt}{a^4}\right)}.$$

And in this time *t* the centre *I* traverses the straight line $GX = \mathrm{X} = t\left(e-\delta gt\right)$.

But here the uneven motion lasts for the time

$$t = \frac{aa\left(e-\varepsilon f\ sin\,\mathfrak{f}\right)}{2\delta g\left(aa+ff\right)},$$

with which elapsed the distance is

$$X = \frac{aa(e-\varepsilon f\ sin\,\mathfrak{f})(e(aa+2ff)+\varepsilon aaf\ sin\,\mathfrak{f})}{2\delta g(aa+ff)^2}$$

and the speed

$$v = \frac{f(ef + \varepsilon aa \sin \mathfrak{f})}{aa + ff}.$$

But for the rotational motion

$$\text{tang } s = \text{ tang } ZO = \text{ tang } \mathfrak{f} = \frac{f(e - \varepsilon f \sin \mathfrak{f})}{\varepsilon(aa + ff)\cos \mathfrak{f}} = \frac{ef + \varepsilon aa \sin \mathfrak{f}}{\varepsilon(aa + ff)\cos \mathfrak{f}}.$$

with $DIO = 90^{\text{o}}$ always arising and the angular speed

$$\gamma' = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf \sin \mathfrak{f} + \varepsilon\varepsilon a^4 \sin^2 \mathfrak{f} + \varepsilon\varepsilon\left(aa + ff\right)^2 \cos^2 \mathfrak{f}\right)}}{aa + ff}.$$

Case II. Let $e < \varepsilon f \sin \mathfrak{f}$ or $k = \varepsilon f \sin \mathfrak{f} - e$, which is the initial grazing speed, and the direction of this *IQ* such that, as $\sin DQ = 0$ and $\cos DQ = 1$, hence $DQ = \zeta = 0$ and *Q* falls in *D*. Hence the globe is constantly accelerated by the friction δM along the direction *IE* and the centre *I* progresses along the same line *IE* ; and in the elapsed time *t* then the speed of this $v = e + 2\delta gt$ and the grazing speed

$$w = \varepsilon f \sin \mathfrak{f} - e - 2\delta \text{g}\left(1 + \frac{f}{aa}\right)t$$

Then let $\varphi = 0$ and $\xi = 0$ and also $\vartheta = 90°$. Whereby for the present axis of rotation *IO* there is $DIO = 90°$ and on putting the arc *ZO* = *s* and with the speed of rotation γ' we have

$$\gamma' \cos s = \varepsilon \sin \mathfrak{f} \text{ and } \gamma' \sin s = \varepsilon \sin \mathfrak{f} - \frac{2\delta fgt}{aa}$$

from which there arises

$$\text{tang } s = \text{tang } \mathfrak{f} - \frac{2\delta fgt}{\varepsilon aa \cos \mathfrak{f}} \quad \text{and} \quad \gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt \sin \mathfrak{f}}{aa} + \frac{4\delta\delta ffggtt}{a^4}\right)}$$

and thus in the time *t* the centre of the globe traverses the line

$$GX = X = t\left(e + \delta gt\right)$$

Moreover here the unequal motion lasts for the time

$$t = \frac{aa\left(\varepsilon f \sin \mathfrak{f} - e\right)}{2\delta g\left(aa + ff\right)},$$

from which in the elapsed time

$$v = \frac{f(ef - \varepsilon aa \sin \mathfrak{f})}{aa + ff}$$

and the distance

$$X=\frac{aa(\varepsilon f\,sin\,\mathfrak{f}-e)(e(aa+2ff)+\varepsilon aaf\,sin\,\mathfrak{f})}{2\delta g(aa+ff)^2}$$

But for the rotational motion there is found

$$\text{tang } s=\text{ tang } ZO=\frac{ef+\varepsilon aa\,sin\,\mathfrak{f}}{\varepsilon(aa+ff)\cos\mathfrak{f}}$$

with DIO = 90° always arising and the angular speed

$$\gamma'=\frac{\sqrt{\left(eeff+2\varepsilon eaaf\,sin\,\mathfrak{f}+\varepsilon\varepsilon a^4\,sin^2\,\mathfrak{f}+\varepsilon\varepsilon\left(aa+ff\right)^2\cos^2\mathfrak{f}\right)}}{(aa+ff)}$$

COROLLARY 1

1067. If it were that $e=\varepsilon f\,sin\,\mathfrak{f}$, then the globe at once proceeds uniformly with its initial speed, both progressive as well as rotational, which case constitutes the boundary between the two cases treated.

COROLLARY 2

1068. These situations in which ε has a negative value or the initial motion impressed on the globe should be in the sense *ZDTE*, are to be referred to the first case in which $e>\varepsilon f\,sin\,\mathfrak{f}$. For on putting $-\varepsilon$ in place of ε it can happen, so that on reverting the globe, that as before it reached uniform motion, clearly as if it were the case that $\varepsilon>\frac{ef}{aa\,sin\,\mathfrak{f}}$.

COROLLARY 3

1069. In this case, in which ε is taken as negative, we have at the time *t:*

$$\varphi=0,\quad \vartheta=90^{\text{o}}\quad \xi=180^{\text{o}},\quad v=e-2g\delta gt,\quad w=e+\varepsilon f\,sin\,\mathfrak{f}-2\delta g\left(1+\frac{ff}{aa}\right)t,$$

$$\text{tang } s=\text{tang }\mathfrak{f}-\frac{2\delta fgt}{\varepsilon aa\cos\mathfrak{f}}\quad\text{and}\quad \gamma'=\sqrt{\left(\varepsilon\varepsilon-\frac{4\delta\varepsilon\,fgt\,sin\,\mathfrak{f}}{aa}+\frac{4\delta\delta\,ffggtt}{a^4}\right)}$$

But after the time

$$t=\frac{aa\left(e+\varepsilon f\,sin\,\mathfrak{f}\right)}{2\delta g\left(aa+ff\right)}$$

in the distance traversed

$$X=\frac{aa(e+\varepsilon f\,sin\,\mathfrak{f})(e(aa+2ff)-\varepsilon aaf\,sin\,\mathfrak{f})}{2\delta g(aa+ff)^2}$$

it can reach uniformity and then

$$v=\frac{f(ef-\varepsilon aa\,sin\,\mathfrak{f})}{aa+ff},\quad \text{tang } s=\frac{\varepsilon aa\,sin\,\mathfrak{f}-ef}{\varepsilon(aa+ff)\cos\mathfrak{f}}$$

and

$$\gamma' = \frac{\sqrt{\left(eeff - 2\varepsilon eaaf \sin \mathfrak{f} + \varepsilon\varepsilon a^4 \sin^2 \mathfrak{f} + \varepsilon\varepsilon (aa+ff)^2 \cos^2 \mathfrak{f}\right)}}{aa+ff}$$

SCHOLIUM

1070. Here the case is especially remarkable, in which motion of this kind is impressed on the globe, so that in the first place it recedes, but soon it reverts back again, as it is accustomed to show by experiment, while with a finger applied to the globe around *D* and on pressing downwards a two-fold motion is impressed on the globe, the one progressive in the direction *DIE*, the other of rotation in the sense *ZDTE*. But in order that the phenomenon succeeds, it is required that the rotational speed must exceed the progressive speed by a certain limit ; which in order that we may more easily understand this , we adapt the calculation to this case, in which in which the rotational motion is impressed on the globe about a horizontal axis and normal to the direction of the progressive motion. Hence if *e* denotes the progressive speed along the direction *DIE* and γ' the backwards angular speed in the sense *ZNTE*, with the radius of the globe *f* arising and *Maa* the moment of inertia of this, and with friction equal to δM ; in the first place globe proceeds in the direction *DIE*, and in the elapsed time *t* the speed of this in the same direction is given by $v = e - 2\delta gt$, with the interval completed $X = t(e - \delta gt)$; then even now it is rotating backwards about the same axis with an angular speed

$$\gamma' = \varepsilon - \frac{2\delta fgt}{aa}$$

But uniform motion eventuates after the elapsed time

$$t = \frac{aa(e+\varepsilon f)}{2\delta g(aa+ff)}$$

and then

the progressive speed $v = \frac{f(ef - \varepsilon aa)}{aa+ff}$ and the angular speed $\gamma' = \frac{\varepsilon aaf - ef}{aa+ff}$.

Whereby if it is the case that $\varepsilon > \frac{ef}{aa}$, the globe now moves backwards, at this stage with a backwards rotation ; but if it should be that $\varepsilon < \frac{ef}{aa}$, the globe at this point proceeds and with the rotation in the contrary sense reversed. In the case that the globe begins to return, and in the elapsed time $t = \frac{e}{2\delta g}$ and with the interval traversed $X = \frac{ee}{4\delta g}$.

If the globe is homogeneous, then $aa = \frac{2}{5} ff$ and εf expresses the speed of rotation at the contact point, which if it is called equal to *h*, then after the time *t* the progressive speed $v = e - 2\delta gt$ and the rotational speed at the point of contact, which is $u = h - 5\delta gt$, and the interval traversed is equal to $t(e - \delta gt)$; now the uniform motion arises after the elapsed time $t = \frac{e+h}{7\delta g}$ and on completing an interval equal to $\frac{(6e-h)(e+h)}{49\delta g}$; when there is

$$v = \frac{5e-2h}{7} \quad \text{and} \quad u = \frac{2h-5e}{7}.$$

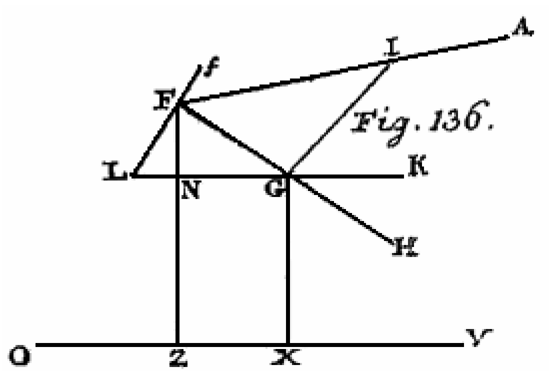
Hence in order that the remarkable phenomenon succeeds, initially there must be $h > \frac{5}{2}e$. But if it should be that $h = \frac{5}{2}e$ each motion is likewise destroyed, in an elapsed time equal to $\frac{e}{2\delta g}$ sec., and on completing a distance equal to $\frac{ee}{4\delta g}$.

CAPUT IV
DE MOTU TURBINUM IN CUSPIDEM DESINENTIUM SUPER PLANO HORIZONTALI FRICTIONIS HABITA RATIONE

PROBLEMA 11

1030. Si turbo super plano horizontali moveatur utcunque deturque singulis momentis eius pressio in planum, definire frictionem motumque turbinis progressivum.

SOLUTIO



Repraesentet tabula planum horizontale (Fig. 136), super quo turbo incedit, cuius axis transiens per centrum inertiae et cuspidem nunc elapso tempore *t* situm teneat *AIF*, ut *I* sit centrum inertiae in sublimi situm, *F* vero cuspis, qua fit contactus in plano horizontali; voceturque intervalium *IF* = *f*, quod est constans. Ex *I* in planum demittatur perpendiculum *IG*, et sumta in plano recta directrice *OV* ad fixam mundi plagam spectante, ad eam ex *G* et *F* ducantur normales *GX* et *FZ*, itemque per *G* recta *KL* ipsi *OV* parallela. Ponatur angulus $FIG = \rho$, qui exprimit declinationem axis turbinis *AF* a situ verticali; et angulus $KGH = \varphi$, qui praebet declinationem plani verticalis, in quo iam axis turbinis versatur, a plano verticali super *OV* vel *LK* exstructo. Erit ergo $GI = f\cos\rho$ et $GF = f\sin\rho$; tum vero $GN = f\sin\rho\cos\varphi$ et $FN = f\sin\rho\sin\varphi$. Praeterea vero sit *OX* = *x* et *XG* = *y;* unde pro puncto *F* fit

$$OZ = x - f\sin\rho\cos\varphi \text{ et } ZF = y + f\sin\rho\sin\varphi;$$

ex quibus motus cuspidis *F* colligi potest, cuius celeritas secundum directionem *OV* vel *NG* est

$$= \frac{dx - fd\cdot\sin\rho\cos\varphi}{dt}$$

et celeritas secundum directionem *NF*

$$= \frac{dy + fd\cdot\sin\rho\sin\varphi}{dt} ,$$

quarum utraque nisi evanescat, cuspis *F* super plano movebitur frictionemque excitabit; ad cuius directionem inveniendam sit *Ff* directio secundum quam cuspis progreditur, quae retro in *L* producta dabit directionem frictionis *FL*, pro qua ponatur angulus $FLG = \omega$ eritque

$$\operatorname{tang}\,\omega = \frac{dy + fd\cdot \sin\rho \sin\varphi}{dx - fd\cdot \sin\rho \cos\varphi} \quad .$$

Sit iam pressio, quam cuspis in planum exerit = Π , pondere totius turbinis existente = *M*, atque ob frictionem turbo in *F* sollicitatur secundum directionem *FL* vi $= \delta\Pi$, quae resoluta dat vim sec. $XO = \delta\Pi \cos\omega$ et sec. $FZ = \delta\Pi \sin\omega$. Ad motum ergo progressivum centri inertiae *I* definiendum praeter has vires frictionis ei applicata concipiatur vis deorsum urgens secundum $IG = M - \Pi$, ac principia motus suppeditabunt has ternas aequationes:

$$\frac{ddx}{2gdt^2} = -\frac{\delta\Pi \cos\omega}{M}, \quad \frac{ddy}{2gdt^2} = -\frac{\delta\Pi \sin\omega}{M}$$

et

$$\frac{fdd\cos\rho}{2gdt^2} = -1 + \frac{\Pi}{M},$$

unde statim colligimus $ddx \sin\omega = ddy \cos\omega$. Ex his aequationibus, si anguli ρ et ω ad tempus *t* ut cogniti spectentur, inde primo $\frac{\Pi}{M}$ tum vero differentialia *dx* et *dy* determinantur ex iisque tandem angulus φ ex formula

$$\operatorname{tang}\,\omega = \frac{dy + fd\cdot \sin\rho \sin\varphi}{dx - fd\cdot \sin\rho \cos\varphi}$$

COROLLARIUM 1

1031. Si pro $\frac{\Pi}{M}$ valor inventus per ρ substituatur, pro quantitatibus *x* et *y* determinandis habebimus has aequationes differentiales secundi gradus

$$ddx = -2\delta gdt^2 \cos\omega - \delta f \cos\omega\, dd\cdot \cos\rho,$$

$$ddy = -2\delta gdt^2 \sin\omega - \delta f \sin\omega\, dd\cdot \cos\rho.$$

COROLLARIUM 2

1032. Pro directione frictionis *FL*, ratione rectae *FH, cum sit angulus* $LGF = \varphi$ et angulus $FLG = \omega$, erit angulus $GFL = 180^{\circ} - \varphi - \omega$; ipsa autem frictio est = $\delta\Pi$, nisi sit celeritas cuspidis *F* nulla, quo casu frictio subito evanescit, id quod evenit, si fuerit

$$dx = fd\cdot \sin\rho \cos\varphi$$

et

$$dy = -fd\cdot \sin\rho \sin\varphi$$

SCHOLION

1033. Ex his aequationibus nihil adhuc concludere licet, cum relatio variabilium ω et Π seu ρ ad tempus *t* nondum constet, quae demum ex motu gyratorio erui debet. Iis autem inventis, per formulas hic traditas variabiles *x* et *y*, sicque motus progressivus centri inertiae *I* definiri poterit. Quamobrem angulum ω in determinationem motus gyratorii introducamus, etiamsi eius relatio ad angulos ρ, φ et tempus assignari possit. Cum enim sit

$$dy\cos\omega + f\cos\omega d\cdot\sin\rho\sin\varphi - dx\sin\omega + f\sin\omega d\cdot\sin\rho\cos\varphi = 0,$$

quo *x* et *y* facilius elidere queamus, ponamus

$$f\cos\omega d\cdot\sin\rho\sin\varphi + f\sin\omega d\cdot\sin\rho\cos\varphi = sdt,$$

ut sit

$$dy\cos\omega - dx\sin\omega + sdt = 0;$$

quae aequatio differentiata ob $ddy\cos\omega = ddx\sin\omega$ dat

$$-dy\sin\omega - dx\cos\omega + \frac{dsdt}{d\omega} = 0.$$

Differentietur porro, et ob

$$ddy\sin\omega + ddx\cos\omega = \frac{-2\delta g\Pi dt^2}{M}$$

prodit

$$\frac{2\delta g\Pi dt^2}{M} - dyd\omega\cos\omega + dxd\omega\sin\omega + dt\cdot\frac{ds}{d\omega} = 0$$

addatur prima per $d\omega$ multiplicata fietque per *dt* dividendo

$$\frac{2\delta g\Pi dt^2}{M} + sd\omega + d\cdot\frac{ds}{d\omega} = 0$$

qua aequatione relatio inter s, ω, Π et *t* exprimitur, quae forte in sequentibus usum habebit. Involvit autem *s* angulos ρ, φ et ω estque

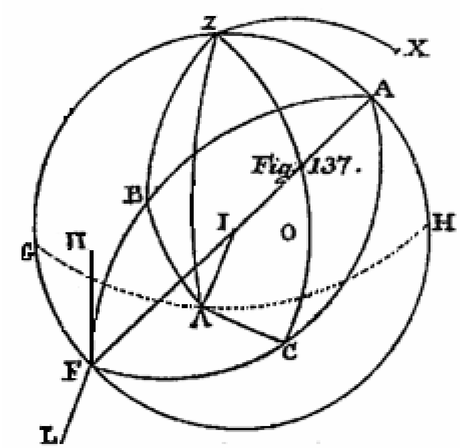
$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos\rho}{2gdt^2}$$

ita ut hic adhuc insint quatuor variabiles ρ, φ et ω et t.

PROBLEMA 12

1034. Dum turbo utcunque super plano horizontali movetur et frictionem patitur, determinare virium, quibus sollicitatur, momenta respectu axium principalium turbinis.

SOLUTIO



In sphaera centro inertiae turbinis *I* descripta (Fig. 137) repraesentet circulus *GZH* planum verticale, in quo iam axis turbinis per centrum inertiae *I* et cuspidem *F* ductus *AIF* versetur, qui simul sit axis principalis turbinis, eiusque respectu momentum inertiae $= Maa$, bini reliqui vero axes principales ex *I* ad sphaerae puncta *B* et *C* pertingant, quorum respectu sint momenta inertiae aequalia $= Mcc$, ita ut in formulis nostris generalibus sit $bb = cc$, quemadmodum iam supra assumsimus. Posito *Z* puncto sphaerae verticali, erit arcus $ZA = \rho$; ponamus autem ductis arcubus *ZB* et *ZC*, ut supra $ZA = l$, $ZB = m$ et $ZC = n$, ut sit $\rho = l$. His positis vires, quibus turbo sollicitatur, sunt primo eius pondus $= M$, quae vis centro inertiae *I* applicata nulla praebet momenta; deinde adest pressio, qua planum horizontale in cuspidem *F* reagit, cuius directio est verticalis sursum directa $F\Pi$, quae vis sit $= \Pi$, vidimusque esse

$$\frac{\Pi}{M} = 1 + \frac{fdd \cdot \cos\rho}{2gdt^2}.$$

Denique sollicitatur turbo in *F* a frictione $= \delta\Pi$, nisi cuspis qu iescat, cuius directio *FL* est horizontalis; ac pro eius situ ducatur circulus maximus horizontalis $G\Lambda B$, in quo capiatur secundum paragraphum 1032 arcus $H\Lambda = 180° - \varphi - \omega$ seu $GA = \varphi + \omega$, ubi φ denotat declinationem plani *GZH* a plano quodam verticali fixo; angulus ω autem ex formulis in praecedente problemate traditis definiri debet eritque directio *FL* radio $I\Lambda$ parallela. Iam ad virium harum momenta respectu axium principalium investiganda primum ipsae vires secundum directiones horum axium resolvantur, quem in finem ut in centro inertiae applicatae considerentur. Vis ergo $F\Pi = \Pi$, in directione *IZ* applicata praebet

$$\text{vim sec.} IA = \Pi \cos ZA = \Pi \cos l ,$$

$$\text{vim sec.} IB = \Pi \cos ZB = \Pi \cos m$$

et

$$\text{vim sec.} IC = \Pi \cos ZC = \Pi \cos n .$$

Deinde vis $FL = \delta\Pi$ in $I\Lambda$ applicata resolvitur in vires

$$1° \text{ sec.} IA = \delta\Pi \cos A\Lambda,$$

$$2° \text{ sec.} IB = \delta\Pi \cos B\Lambda,$$

$$3° \text{ sec.} IC = \delta\Pi \cos C\Lambda.$$

Ad has autem evolvendas sit *ZX* ille circulus verticalis fixus ideoque angulus

$XZA = \varphi$, ponamus autem ut supra angulos $XZA = \lambda$, $XZB = \mu$ et $XZC = \nu$, ut sit $\varphi = \lambda$ et, ob $AZ\Lambda = 180° - \lambda - \omega$, erit $XZ\Lambda = 180° - \omega$, hincque $BZ\Lambda = \mu + \omega - 180°$ et $CZ\Lambda = 180° - \nu - \omega$, unde ob *ZA* quadrantem prodit

$$\cos A\Lambda = -\sin l \cos(\lambda + \omega),$$

$$\cos B\Lambda = -\sin m \cos(\mu + \omega)$$

et

$$\cos C\Lambda = -\sin n \cos(\nu + \omega).$$

Quocirca habebimus

$$\text{vim sec. } IA = \Pi \cos l - \delta\Pi \sin l \cos(\lambda + \omega),$$

$$\text{vim sec. } IB = \Pi \cos m - \delta\Pi \sin m \cos(\mu + \omega),$$

$$\text{vim sec. } IC = \Pi \cos n - \delta\Pi \sin n \cos(\nu + \omega),$$

has autem vires nunc in puncto *F* applicatas concipi oportet, existente *IF* = *f*, unde momenta earum respectu axium principalium, quae supra litteris *P*, *Q*, *R* designavimus, concluduntur

$$P = 0,$$

$$Q = \Pi f \cos n - \delta f \Pi \sin n \cos(\nu + \omega),$$

$$R = -\Pi f \cos m + \delta\Pi f \sin m \cos(\mu + \omega).$$

PROBLEMA 13

1035. His virium momentis inventis exhibere aequationes, quibus motus turbinis super plano horizontali incedentis et a frictione perturbatus contineatur.

SOLUTIO

Primo pro motu gyratorio teneat elapso tempore *t* turbo situm in figura repraesentatum (Fig.137), ubi omnes denominationes modo factae maneant. Ac nunc gyretur turbo circa axem *IO* in sensum *ABC* celeritate angulari = γ', pro puncto *O* autem sint arcus $AO = \alpha$, $BO = \beta$, $CO = \gamma$ ponaturque $\gamma' \cos\alpha = x$, $\gamma' \cos\beta = y$, $\gamma' \cos\gamma = z$, quae quantitates per momenta modo inventa ita determinantur, ut primo sit $dx = 0$, ideoque x = const. Ponatur ergo $x = h$, et pro *y* et *z* has habebimus aequationes

$$dy + \frac{(aa\text{-}cc)}{cc} hzdt = \frac{2\Pi\, fgdt}{Mcc}\left(\cos n - \delta \sin n \cos(\nu + \omega)\right),$$

$$dz + \frac{(aa\text{-}cc)}{cc} hydt = -\frac{2\Pi\, fgdt}{Mcc}\left(\cos m - \delta \sin m \cos(\mu + \omega)\right).$$

Tum vero pro arcubus *l*, *m*, *n* itemque angulis λ, μ, ν ostendimus esse:

$$dl\sin l = dt(y\cos n - z\cos m),\quad d\lambda\sin^2 l = -dt(y\cos m + z\cos n),$$
$$dm\sin m = dt(z\cos l - h\cos n),\quad d\mu\sin^2 m = -dt(z\cos n + h\cos l),$$
$$dn\sin n = dt(h\cos m - y\cos l),\quad dv\sin^2 n = -dt(h\cos l + y\cos m)\ ,$$

ubi praeterea hae relationes sunt notandae:

$$\cos(\mu-\lambda) = -\frac{\cos l\cos m}{\sin l\sin m},\quad \cos(\nu-\lambda) = -\frac{\cos l\cos n}{\sin l\sin n},$$
$$\sin(\mu-\lambda) = -\frac{\cos n}{\sin l\sin m},\quad \sin(\nu-\lambda) = \frac{\cos m}{\sin l\sin n},$$

unde anguli μ et v per λ ita definiuntur :

$$\cos\mu = \frac{-\cos\lambda\cos l\cos m + \sin\lambda\cos n}{\sin l\sin m},\quad \cos\nu = \frac{-\cos\lambda\cos l\cos n - \sin\lambda\cos m}{\sin l\sin n},$$
$$\sin\mu = \frac{-\sin\lambda\cos l\cos m - \cos\lambda\cos n}{\sin l\sin m},\quad \sin\nu = \frac{-\sin\lambda\cos l\cos n + \cos\lambda\cos m}{\sin l\sin n}.$$

Hicque est

$$\frac{\Pi}{M} = 1 + \frac{fdd\cdot\cos l}{2gdt^2}\ .$$

At angulus ω ex motu progressivo est ingressus, pro quo si in fig.136 ad situm centri inertiae *I* definiendum distinctionis causa vocemus coordinatas
OX = X et *XG* = *Y*, existente *GI* = $f\cos l$, ad superiores aequationes insuper has adiungere debemus:

$$\frac{ddX}{2g^2dt} = -\frac{\delta\Pi}{M}\cos\omega\ ,\quad \frac{ddY}{2g^2dt} = -\frac{\delta\Pi}{M}\sin\omega$$

et

$$dY\cos\omega - dX\sin\omega + f\cos\omega d\cdot\sin l\sin\lambda + f\sin\omega d\cdot\sin l\cos\lambda = 0\ .$$

Atque in his aequationibus omnia, quae tam ad motum progressivum quam gyratorium spectant, determinantur. Si primo quantitates X et *Y* e calculo excludere velimus, loco harum trium postremarum aequationum sequentem unicam adhibuisse sufficit; pro qua si ponatur

$$sdt = f\cos\omega d\cdot\sin l\sin\lambda + f\sin\omega d\cdot\sin l\cos\lambda,$$

seu sumtis his differentialibus locoque *dl* et $d\lambda$ valoribus superioribus substitutis erit

$$s = -fy\sin n\sin(\omega+\nu) + fz\sin m\sin(\omega+\mu)$$

et aequatio loco illarum trium usurpanda supra inventa est

$$\frac{2\delta g\Pi dt}{M} + sd\omega + d\cdot\frac{ds}{d\omega} = 0.$$

SCHOLION 1

1036. Multitudo harum aequationum, praecipue autem angulus ω primas aequationes ingrediens in causa est, quod earum resolutionem nullo modo suscipere liceat. Unde patet motum turbinum ob frictionem maxime fore peturbatum, ita ut ex his aequationibus nihil omnino, unde hic motus cognosci posset, concludere valeamus. Quodsi vero huius motus causas obiter tantum contemplemur, evidens est centrum inertiae *I* non in recta tantum verticali, ut remota frictione eveniebat, ascendere vel descendere, sed etiam motum horizontalem adipisci, qui oritur a vi frictionis, euius directio eum sit contraria motui cuspidis, motus centri inertiae secundum eandem directionem incitatur, unde neque uniformis neque rectilineus erit, et quatenus incurvatur, eius convexitas in eam regionen spectabit, in quam cuspis progreditur. Simili modo etiam motus gyratorius tam ratione celeritatis quam ratione axis gyrationis maxime perturbabitur, de quo vix quicquam ex consideratione frictionis affirmare licet.

SCHOLION 2

1037. Verum haec tanta motus perturbatio tamdiu duntaxat durat, donec frictio cessat, hoc autem tandem evenire debere per se est evidens, quandoquidem motus ob frictionem continuo retardatur. At frictio cessare nequit, nisi cuspis turbinis in eodem loco persistat, ex quo necesse est motum ita temperari debere, ut cuspis tandem in eodem plani puncto sit perseveraturus, dummodo hoc eveniat, antequam turbo procumbat. Si enim turbini primo motus gyratorius nimis tardus fuerit impressus, nullum est dubium, quin procumbat, antequam illud phaenomenon oriatur; ex quo vicissim concludere licet, si motus satis fuerit celer, fore, ut antequam turbo procumbat, cuspis a frictione ad idem plani horizontalis punctum redigatur. Quod cum evenerit, atque in turbine adhuc motus insit gyratorius, ex superioribus patet axem turbinis verticalem esse debere; si enim esset inclinatus, nullo modo ita gyrari posset, ut cuspis eidem puncto insisteret. Ex his igitur coniunctis hanc conclusionem deducimus: turbinem, si modo ei satis celer motus gyratorius fuerit impressus, ob frictionem se tandem in situm verticalem erigere et tum circa axem verticalem motum gyratorium esse continuaturum. Quod phaenomenon eo magis est notatu dignum, quod soli frictioni debeatur; ita ut ope frictionis linea verticalis ideoque etiam planum horizontale obtineri queat; id quod in navigatione magnum usum habere potest, ad quem etiam in Anglia olim fuit commendatum.

1) Vide SERSON, Phil. Trans. 47 (1752), p. 352. C.ß.

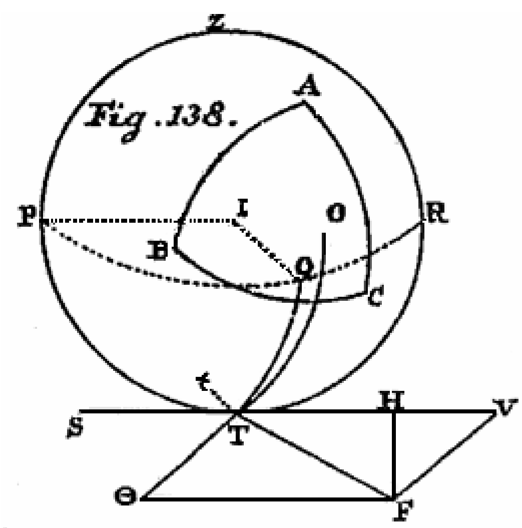
CAPUT V

DE MOTU GLOBORUM CENTRUM INERTIAE IN IPSORUM CENTRO SITUM HABENTIUM SUPER PLANO HORIZONTALI

PROBLEMA 14

1038. Si globus super plano horizontali utcunque tam motu progressivo quam gyratorio moveatur, determinare celeritatem et directionem, qua punctum contactus radii superficiem horizontalem.

SOLUTIO



Sit *I* centrum simulque centrum inertiae globi (Fig. 138), eiusque radius = *f* et contactus fiet in puncto imo *T*. Motus autem globi ita sit comparatus, *I* ut centrum inertiae moveatur secundum directionem *PIR* celeritate = *v*, simul vero gyretur circa axem quemcunque
IO celeritate angulari = γ' in eum sensum, ut punctum *T* circa *O* incedat per arculum *Tt*, ac pro positione puncti *O* statuamus angulum $PTO = \vartheta$ et arcum *TO* = *s*, ubi quidem arcus ita sumo, quasi radius globi esset = 1. Ducatur *TV* ipsi *PIR* parallela, ac si motus gyratorius abesset, punctum contactus *T* rasurum esset planum horizontale celeritate = *v* in directione *TV*. Deinde si globus solo motu gyratorio ferretur, quia punctum *T* per *Tt* moveretur celeritate = $f\gamma' \sin TO = f\gamma' \sin s$,cuius directio cum sit horizontalis, in plano per rectam $T\Theta$ referatur, ita ut sit angulus $ST\Theta = PTt = \vartheta - 90°$ ob *OTt* rectum. Erit ergo $VT\Theta = 270^{\circ} - \vartheta$. Capiantur rectae $TV = v$ et $T\Theta = f\gamma' \sin s$, et quia punctum *T* his duobus motibus coniunctim movetur, eius verus motus fiet secundum rectam *TF* diagonalem parallelogrammi $TVF\Theta$. Ex *F* ad *TV* ducta normali *FH* erit ,

$$VH = f\gamma' \sin s \sin \vartheta \quad \text{et} \quad FH = -f\gamma' \sin s \cos \vartheta$$

unde fit

$$TH = v - f\gamma' \sin s \sin \vartheta$$

atque celeritas radens

$$TF = \sqrt{(vv - 2f\gamma' v \sin s \sin \vartheta + ff\gamma'\gamma' \sin^2 s)}$$

et

$$\text{tang } VTF = -\frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta}$$

Ducatur ex centro *I* ipsi *TF* parallela *IQ*, erit arcus *TQ* quadrans et angulus *RTQ* = *VTF*. Quare si *IQ* sit directioni, secundum quam punctum *T* radit, parallela, erit

$$\text{tang } PTQ = \frac{f\gamma' \sin s \cos\vartheta}{v - f\gamma' \sin s \sin\vartheta},$$

ac posita celeritate radente

$$\sqrt{(vv - 2f\gamma' v \sin s \sin\vartheta + ff\gamma'\gamma' \sin^2 s)} = u$$

erit

$$\sin PTQ = \frac{-f\gamma' \sin s \cos\vartheta}{u} \quad \text{et} \quad \cos PTQ = \frac{f\gamma' \sin s \sin\vartheta - v}{u}$$

COROLLARIUM 1

l039. Fieri ergo potest, ut celeritas radens ideoque et attritus evanescat, quo casu hae duae conditiones locum habere debent, altera $\gamma' \sin s \cos\vartheta = 0$, altera $v = f\gamma' \sin s \sin\vartheta$. Unde statim patet, si nulIus adsit motus progressivus, seu *v* = 0, nullum attritum aflore, si $\sin s = 0$, hoc est si globus circa axem verticalem *Z T* gyretur.

COROLLARIUM 2

1040. Deinde motus globi ab attritu erit liber, si fuerit primo $\cos\vartheta = 0$ seu angulus PTO rectus; deinde celeritas progressiva *v* ad angularem γ' hanc relationem tenere debet, ut sit $v = f\gamma' \sin s$ seu $TV = T\Theta$ et angulus $ST\Theta = 0$.

COROLLARIUM 3

1041. Quando ergo globus huiusmodi motum est consecutus, quia sublato omni attritu etiam nulla adest frictio, globus eundem motum constanter conservabit, siquidem axis gyrationis *IO* habeat proprietatem axis principalis.

SCHOLION 1

1042. Quemadmodum hic frictionem constituimus, ea non obstat, quominus globus super plano horizontali motum suum intemeratum conservare possit, quod tamen minime fieri observamus, cum globus super tabula tali motu latus mox omnem motum amittat, cuius rei causa resistentiae aeris tribui nequit. Verum hic primum animadverto experimenta nunquam Theoriae perfectissime congruere; veluti dum casu hic tractato assumsimus contactum unico fieri puncto, id semper in praxi secus evenit. Interim tamen si arcus *TO* est quadrans et *PTO* = 90°, existente $v = f\gamma'$, tametsi contactus non fiat in uno puncto, tamen attritus evanescit, ideoque hic motus extinctio neutiquam frictioni adscribi potest. Ex quo concludere debemus praeter frictionem, uti hic eam definivimus, aliud adhuc dari motus impedimentum, dum corpora super superficiebus incedunt, a frictione probe distinguendum, cuius ratio utcunque fuerit comparata, eius effectus potius seorsim investigari convenit, quam frictionis indolem hic stabilitam immutari. Et quemadmodum hic a resistentia aeris mentem abstrahimus, ita etiam licebit hoc obstaculum frictionem concomitans a praesenti argumento seiungere.

SCHOLION 2

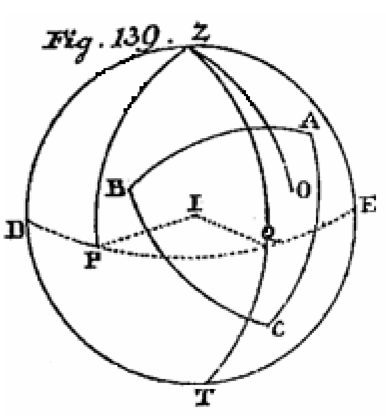
1043. Corpora hic sphaerica considero, in quorum centro situm sit ipsorum centrum inertiae *I*, quod ergo ipsum etiam in plano horizontali moveatur et puncto contactus *T* perpetuo verticaliter immineat. Ex quo patet pressionem in contactu semper fore ponderi corporis *M* aequalem. Si ergo corpus ex materia uniformi constaret, omnes eius diametri proprietate axium principalium gauderent, sed concipiamus materiae distributionem utcunque inaequabilem, ita tamen ut centrum inertiae cadat in centrum figurae. Quare necesse erit in globo ternos axes principales considerari, qui ex centro *I* pertingant in puncta *A, B, C*, quadrantibus a se invicem distantia, quorum respectu sit, ut supra posuimus, momenta inertiae *Maa, Mbb, Mcc.* Quanquam autem deinceps bina vel omnia haec momenta inter se aequalia statuemus, tamen conveniet huiusmodi tria puncta in superficie fixa notasse, quo ex eorum relatione ad spatium absolutum, facilius motus globi definiri possit. Constitutis autem in globo his tribus punctis *A, B, C,* quoniam motus gyrationis circa *O*, quem in plagam *Tt* dirigi assumsimus, sensum habet *CBA* contrarium ei, quem supra statuimus, in applicatione formularum generalium ad hunc casum celeritatem angularem γ' ut negativam spectare debemus.

PROBLEMA 15

1044. Si globus super plano horizontali utcunque moveatur, definire vires, quibus sollicitatur, earumque momenta respectu ternorum axium principalium globi.

SOLUTIO

Inclusus concipiatur globus sphaerae vel fixae vel cum eo parem motum progressivum habenti (Fig. 139), in qua *Z* sit punctum verticale eiusque oppositum *T* punctum contactus, *DE* vero sit diameter horizontalis ad certam mundi plagam tendens, et *DPQE* circulus maximus horizontalis. Nunc autem elapso tempore *t*, moveatur globus motu progressivo secundum directionem *PI* celeritate = *v*, ponaturque arcus *DP* seu angulus $DZP = \varphi$;



axes autem principales nunc sint in *A,B,C* .Tum vero globus iam gyretur circa axem *IO* celeritate angulari = γ' in sensum *ACB;* sitque pro situ puncti *O* angulus *PTO* seu $PZO = \vartheta$ et arcus *ZO* = *s*. Etsi enim ante arcum *TO* posuimus = *s*, quia tantum eius sinus in computum intrat, perinde. est. Erit ergo angulus $DZO = \vartheta + \varphi$ et $EZO = 180^{\circ} - \vartheta - \varphi$.

Deinde si a punctis *A*, *B*, *C* tam ad *O* quam ad *Z* arcus circulorum magnorum ducti concipiantur, sint ut hactenus arcus

$$AO=\alpha,\quad BO=\beta\ ,CO=\gamma,$$
$$ZA\ =l,\quad ZB=m,\quad ZC=n$$

et anguli

$$EZA=\lambda,\quad EZB=\mu,\quad EZC=v\,.$$

In praecedente autem problemate ostendimus, punctum contactus *T* planum subiectum radere secundum directionem radio *IQ* parallelam celeritate

$$=\sqrt{(vv-2f\gamma' v\sin s\sin\vartheta+ff\gamma'\gamma'\sin^2 s)}\,.$$

foreque

$$\text{tang } PTQ=\text{tang } PZQ=\frac{f\gamma'\sin s\cos\vartheta}{v-f\gamma'\sin s\sin\vartheta}$$

denotante *f* radium globi. Cum igitur pressio in *T* sit = *M*, frictio erit = δM , quae puncto *T* est applicata secundum directionem ipsi *QI* parallelam. Hac ergo vi resoluta secundum directiones axium principalium *IA*, *IB*, *IC*, prodit

$$\text{vis sec. } IA=-\delta M\cos AQ,$$
$$\text{vis sec. } IB=-\delta M\cos BQ,$$

et

$$\text{vis sec. } IC=-\delta M\cos CQ\,,$$

quae ternae vires in puncto *T* applicatae sunt concipiendae, unde colliguntur momenta

Resp. axis *IA* in sensum $BC=\ -\delta Mf\cos CQ\cos BT+\delta Mf\cos BQ\cos OT=P,$

Resp. axis *IB* in sensum $CA=\ -\delta Mf\cos AQ\cos CT+\delta Mf\cos CQ\cos AT=Q,$

Resp. axis *IC* in sensum $AB=\ -\delta Mf\cos BQ\cos AT+\delta Mf\cos AQ\cos BT=R\ .$

Erunt ergo haec tria momenta:

$$P=\delta Mf\,(\cos m\cos CQ-\cos n\cos BQ),$$
$$Q=\delta Mf\,(\cos n\cos AQ-\cos l\cos CQ),$$
$$R=\delta Mf\,(\cos l\cos BQ-\cos m\cos AQ).$$

Pro puncto autem *Q* ponamus angulum $PZQ=\xi$; ut sit

$$\text{tang }\xi=\frac{f\gamma'\sin s\cos\vartheta}{v-f\gamma'\sin s\sin\vartheta}$$

et posita celeritate radente

$$\sqrt{(vv-2f\gamma' v\sin s\sin\vartheta+ff\gamma'\gamma'\sin^2 s)}=u\,,$$

erit

$$\sin\xi = \frac{-f\gamma'\sin s\cos\vartheta}{u} \quad \text{et} \quad \cos\xi = \frac{f\gamma'\sin s\sin\vartheta - v}{u}.$$

Fit ergo $DZQ = \varphi + \xi$ et $EZQ = 180° - \xi - \varphi$; hincque

$$AZQ = 180^{\circ} - \xi - \varphi - \lambda, \quad BZQ = \mu + \xi + \varphi - 180^{\circ}, \ CZQ = 180^{\circ} - \xi - \varphi - \nu$$

ergo

$$\cos AQ = -\cos(\xi + \varphi + \lambda)\sin l,$$
$$\cos BQ = -\cos(\xi + \varphi + \mu)\sin m,$$
$$\cos CQ = -\cos(\xi + \varphi + \nu)\sin n.$$

Ex relatione igitur, quae inter angulos λ, μ et ν intercedit, concludemus momenta virium:

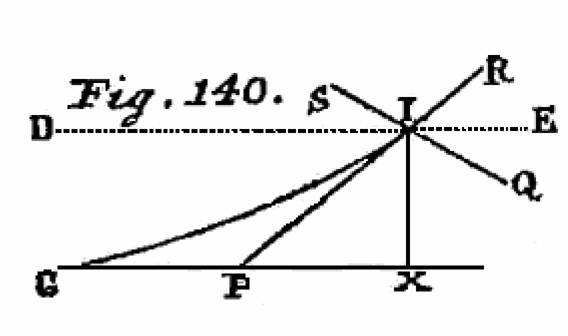
$$P = \delta Mf\ \sin l\sin(\lambda + \varphi + \xi),$$
$$Q = \delta Mf\ \sin m\sin(\mu + \varphi + \xi),$$
$$R = \delta Mf\ \sin n\sin(\nu + \varphi + \xi).$$

PROBLEM 16

1045. Si motum gyratorium ad quodvis tempus ut datum spectemus, definire motum progressivum globi.

SOLUTIO

Quia centrum globi in plano horizontali movetur, descripserit id tempore *t* lineam *GI*, quae referatur ad directricem *GX* superiori directioni fixae *DE* parallelam, ductaque *IX* ad *GX* normali, sint coordinatae *GX* = *X*, *XI* = *Y* (Fig. 140). Per *I* ducatur recta *DE* ipsi *GX* parallela, quae erit ipsa diameter *DE* in figura 139. Ducatur *IP*, ut sit $DIP = EIR = \varphi$, et centrum *I* per hypothesin progreditur in directione *I R* celeritate = *v*, ita ut sit celeritas secundum $GX = v\cos\varphi$ et celeritas secundum $XI = v\sin\varphi$ ideoque



$dX = vdt\cos\varphi$ et $dY = vdt\sin\varphi$. Ducatur recta *QIS*, ut *IQ* sit directioni, qua punctum contactus radit, parallela, erit angulus $EIQ = DIS = 180° - \xi - \varphi$; est enim aequalis angulo *EZQ* in praecedente figura, unde globus sollicitari censendus est vi = δM in directione *IS*. Hinc ergo oritur vis secundum

$ID = -\delta M\cos(\xi + \varphi)$ et vis secundum $XI = -\delta M\sin(\xi + \varphi)$. Ex quibus colligitur

$$\frac{d\cdot v\cos\varphi}{2gdt} = \frac{dv\cos\varphi - vd\varphi v\sin\varphi}{2gdt} = \delta\cos(\xi + \varphi),$$
$$\frac{d\cdot v\sin\varphi}{2gdt} = \frac{dv\sin\varphi + vd\varphi v\cos\varphi}{2gdt} = \delta\sin(\xi + \varphi)$$

hincque porro

$$\frac{dv}{2gdt}=\delta\cos\xi \quad \text{et} \quad \frac{vd\varphi}{2gdt}=\delta\sin\xi$$

ita ut sit

$$\frac{vd\varphi}{dv}=\text{tang }\xi=\frac{f\gamma'\sin s\cos\vartheta}{v-f\gamma'\sin s\sin\vartheta}$$

PROBLEMA 17

1046. Definito motu progressivo globi determinare eius motum gyratorium.

SOLUTIO

Spectetur nunc centrum globi *I* ut quiescens et maneant omnes denominationes in problemate 15 adhibitae sintque *Maa*, *Mbb*, *Mcc* momenta inertiae respectu axium principalium *IA*, *IB*, *IC*, quae primo ut inaequalia consideremus (Fig. 139). Quoniam vero hic celeritatem angularem γ' ut negativam spectare debemus, quia tendit in sensum *ACB*, si ponamus

$$\gamma'\cos\alpha=x, \quad \gamma'\cos\beta=y, \quad \text{et} \quad \gamma'\cos\gamma=z\,,$$

in formulis generalibus has literas *x*, *y*, *z* negative sumi oportet, ex § 810 habebimus has aequationes motum determinantes

$$dx+\frac{bb-cc}{aa}yzdt+\frac{2\delta fg}{aa}dt\sin l\sin(\lambda+\varphi+\xi)=0,$$
$$dy+\frac{cc-aa}{bb}xzdt+\frac{2\delta fg}{bb}dt\sin m\sin(\mu+\varphi+\xi)=0,$$
$$dz+\frac{aa-bb}{cc}xydt+\frac{2\delta fg}{cc}dt\sin n\sin(\nu+\varphi+\xi)=0,$$
$$dl\sin l=dt(z\cos m-y\cos n)\,, \quad d\lambda\sin^2 l=dt(y\cos m+z\cos n)\,,$$
$$dm\sin m=dt(x\cos n-z\cos l)\,, \quad d\mu\sin^2 m=dt(z\cos n+x\cos l),$$
$$dn\sin n=dt(y\cos l-x\cos m)\,, \quad dv\sin^2 n=dt(x\cos l+y\cos m).$$

Tum vero ex motu progressivo habemus:

$$dv=2\delta gdt\cos\xi, \quad vd\varphi=2f\delta gdt\sin\xi$$

et

$$\text{tang }\xi=\frac{f\gamma'\sin s\cos\vartheta}{v-f\gamma'\sin s\sin\vartheta},$$

ubi est $PZO=\vartheta$ et $ZO=s$. Cum ergo sit $EZO=180^{\circ}-\vartheta-\varphi$, erit $AZO=180^{\circ}-\lambda-\vartheta-\varphi$ hincque

$$\begin{aligned}\cos\alpha &= \cos l\cos s - \sin l\sin s\cos(\lambda+\vartheta+\varphi),\\ \cos\beta &= \cos m\cos s - \sin m\sin s\cos(\mu+\vartheta+\varphi),\\ \cos\gamma &= \cos n\cos s - \sin n\sin s\cos(\nu+\vartheta+\varphi),\end{aligned}$$

existente

$$\cos s = \cos l\cos\alpha + \cos m\cos\vartheta + \cos n\cos\gamma,$$

unde sequitur fore

$$\sin l\cos l\cos(\lambda+\vartheta+\varphi)+\sin m\cos m\cos(\mu+\vartheta+\varphi)+\sin n\cos n\cos(\nu+\vartheta+\varphi)=0.$$

Ponamus $\gamma'\cos s = p$ et $\gamma'\sin s = q$, ut sit

$$\text{tang }\xi = \frac{fq\cos\vartheta}{v-fq\sin\vartheta} = \frac{vd\varphi}{dv};$$

eritque

$$\begin{aligned}x &= p\cos l - q\sin l\cos(\lambda+\vartheta+\varphi),\\ y &= p\cos m - q\sin m\cos(\mu+\vartheta+\varphi),\\ z &= p\cos n - q\sin n\cos(\nu+\vartheta+\varphi),\end{aligned}$$

ex quibus valoribus fit

$$\begin{aligned}dl &= qdt\sin(\lambda+\vartheta+\varphi), & d\lambda &= pdt + qdt\cot l\cos(\lambda+\vartheta+\varphi),\\ dm &= qdt\sin(\mu+\vartheta+\varphi), & d\mu &= pdt + qdt\cot m\cos(\mu+\vartheta+\varphi),\\ dn &= qdt\sin(\nu+\vartheta+\varphi), & d\nu &= pdt + qdt\cot n\cos(\nu+\vartheta+\varphi)\end{aligned}$$

indeque porro

$$\begin{aligned}dx &= dp\cos l - dq\sin l\cos(\lambda+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin l\sin(\lambda+\vartheta+\varphi),\\ dy &= dp\cos m - dq\sin m\cos(\mu+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin m\sin(\mu+\vartheta+\varphi),\\ dz &= dp\cos n - dq\sin n\cos(\nu+\vartheta+\varphi) + q(d\vartheta+d\varphi)\sin n\sin(\nu+\vartheta+\varphi).\end{aligned}$$

At sine subsidio harum substitutionum ex aequationibus ternis primis, cum in genere sit

$$\sin l\cos l\sin(\lambda+A)+\sin m\cos m\sin(\mu+A)+\sin n\cos n\sin(\nu+A)=0,$$

elicimus hanc aequationem

$$aadx\cos l + bbdy\cos m + ccdz\cos n - aaxdl\sin l - bbydm\sin m - cczdn\sin n = 0$$

cuius integrale est

$$aax\cos l + bby\cos m + ccz\cos n = C,$$

quae adhibitis substitutionibus abit in hanc formam:

$$p(aa\cos^2 l+bb\cos^2 m+cc\cos^2 n)-q(aa\sin l\cos l\cos(\lambda+\vartheta+\varphi)$$
$$+bb\sin m\cos m\cos(\mu+\vartheta+\varphi)+cc\sin n\cos n\cos(\nu+\vartheta+\varphi))=\text{ Const.}$$

Deinde etiam per reductiones § 934 traditas pro vi viva colligitur haec aequatio differentialis

$$aaxdx+bbydy+cczdz=2\delta fgqdt\sin(\xi-\vartheta).$$

SCHOLION

1047. Ad reductiones hic factas intelligendas ex formulis supra traditis, ubi angulos Π et ν per λ, l, m, n expressimus, notari convenit fieri:

$$\cos(\mu+\vartheta+\varphi) = \frac{-\cos l\cos m\cos(\lambda+\vartheta+\varphi)+\cos n\sin(\lambda+\vartheta+\varphi)}{\sin l\sin m},$$
$$\cos(\nu+\vartheta+\varphi) = \frac{-\cos l\cos n\cos(\lambda+\vartheta+\varphi)-\cos m\sin(\lambda+\vartheta+\varphi)}{\sin l\sin n},$$
$$\sin(\mu+\vartheta+\varphi) = \frac{-\cos l\cos m\sin(\lambda+\vartheta+\varphi)-\cos n\cos(\lambda+\vartheta+\varphi)}{\sin l\sin m},$$
$$\sin(\nu+\vartheta+\varphi) = \frac{-\cos l\cos n\sin(\lambda+\vartheta+\varphi)+\cos m\cos(\lambda+\vartheta+\varphi)}{\sin l\sin n}.$$

Ac simili modo anguli $\mu+\varphi+\xi$ et $\nu+\varphi+\xi$ ad angulum $\lambda+\varphi+\xi$ revocari possunt. Deinde etiam pro sequentibus reductionibus haec forma imprimis est notanda

$$\sin(\mu+B)\cos(\nu+C)-\sin(\nu+B)\cos(\mu+C)$$

quae ob

$$\sin M\cos N=\tfrac{1}{2}\sin(M+N)+\tfrac{1}{2}\sin(M-N)$$

reducitur ad

$$\sin(\mu-\nu)\cos(B-C);$$

hocque modo reductionem pro allis formulis instituendo reperimus:

$$\sin(\mu+B)\cos(\nu+C) - \sin(\nu+B)\cos(\mu+C) = \sin(\mu-\nu)\cos(B-C),$$
$$\sin(\mu+B)\sin(\nu+C) - \sin(\nu+B)\sin(\mu+C) = -\sin(\mu-\nu)\sin(B-C),$$
$$\cos(\mu+B)\cos(\nu+C) - \cos(\nu+B)\cos(\mu+C) = -\sin(\mu-\nu)\sin(B-C),$$

ubi $sin(\mu - v)$ per formulas usurpatas datur, est enim

$$sin(\mu - v) = \frac{cos\, l}{sin\, m\, sin\, n}$$

PROBLEMA 18

1048. Si globus ex materia uniformi constet vel saltem ita fuerit comparatus, ut omnia momenta inertiae sint inter se aequalia eique initio impressus fuerit motus quicunque, determinare eius continuationem.

SOLUTIO

Cum hic sit *aa* = *bb* = *cc* seu momentum inertiae respectu omnium diametrorum = *Maa*, prima aequatio integrata praebet *aap* = Const., unde *p* erit quantitas constans. Statuatur ergo *p* = *h*, et ternae aequationes differentiales priores induent has formas:

$$\text{I. } -dq\, cos(\lambda + \vartheta + \varphi) + q(d\vartheta + d\varphi)\, sin(\lambda + \vartheta + \varphi) + \frac{2\delta fg}{aa} dt\, sin(\lambda + \varphi + \xi) = 0,$$

$$\text{II. } -dq\, cos(\lambda + \vartheta + \varphi) + q(d\vartheta + d\varphi)\, sin(\mu + \vartheta + \varphi) + \frac{2\delta fg}{aa} dt\, sin(\mu + \varphi + \xi) = 0,$$

$$\text{III.} -dq\, cos(\lambda + \vartheta + \varphi) + q(d\vartheta + d\varphi)\, sin(\nu + \vartheta + \varphi) + \frac{2\delta fg}{aa} dt\, sin(\nu + \varphi + \xi) = 0,$$

quarum autem sufficit binas considerasse, quia inde iam nota est conclusio *p* = *h*. Iam per superiores reductiones binae posteriores ita combinentur:

$$\text{II. } cos(v + \vartheta + \varphi) - \text{ III. } cos(\mu + \vartheta + \varphi)$$

praebet

$$q(d\vartheta + d\varphi)\, sin(\mu - \nu) + \frac{2\delta fg}{aa} dt\, sin(\mu - \nu)\, cos(\xi - \vartheta) = 0$$

seu

$$q(d\vartheta + d\varphi) + \frac{2\delta fg}{aa} dt\, cos(\xi - \vartheta) = 0.$$

Deinde

$$\text{II. } sin(\nu + \vartheta + \varphi) - \text{ III. } sin(\mu + \vartheta + \varphi)$$

dat

$$q\, sin(\mu - \nu) - \frac{2\delta fg}{aa} dt\, sin(\mu - \nu)\, sin(\xi - \vartheta) = 0$$

seu

$$q = \frac{2\delta fg}{aa} dt\, sin(\xi - \vartheta)$$

qui valor in ultima aequatione pro viribus vivis substitutus praebet

$$xdx + ydy + zdz = qdq$$

hincque

$$xx + yy + zz = \gamma'\gamma' = \text{ Const.} + qq = \text{ Const.} + \gamma'\gamma' \sin^2 s$$

ita ut sit $\gamma'\gamma' \cos^2 s = \text{const}$., ut iam invenimus ob $\gamma' \cos s = p = h$. Hinc istas habemus aequationes a litteris $l,\ m,\ n,\ \lambda,\ \mu,\ \nu$ immunes:

$$\text{1. } q(d\vartheta + d\varphi) + \frac{2\delta fg}{aa}\, dt \cos(\xi - \vartheta) = 0\ ,$$

$$\text{II. } dq = \frac{2\delta fg}{aa} dt \sin(\xi - \vartheta)\ ,$$

$$\text{III. } dv = 2\delta gdt\cos\xi,$$

$$\text{IV.} vd\varphi = 2\delta gdt \sin\xi,$$

quibus adiungatur haec finita

$$\operatorname{tang}\xi = \frac{fq\cos\vartheta}{v - fq\sin\vartheta}\,,$$

quae transformata in hanc

$$v\sin\xi - fq\cos(\xi - \vartheta) = 0\,,$$

differentietur

$$dv\sin\xi + vdv\cos\xi - fdq\cos(\xi - \vartheta) + fqd\xi\sin(\xi - \vartheta) - fqd\vartheta\sin(\xi - \vartheta) = 0.$$

Iam

$$\text{I. } \sin(\xi - \vartheta) + \text{ II. } \cos(\xi - \vartheta)$$

dat

$$q(d\vartheta + d\varphi)\sin(\xi - \vartheta) + dq\cos(\xi - \vartheta) = 0,$$

quae per f multiplicata illi addatur

$$dv\sin\xi + vd\xi\cos\xi + fq(d\xi + d\varphi)\sin\left(\xi - \vartheta\right) = 0\,.$$

Porro ob

$$\frac{dv}{vd\varphi} = \frac{\cos\xi}{\sin\xi}$$

erit

$$v(d\varphi + d\xi)\cos\xi + fq(d\xi + d\varphi)\sin(\xi - \vartheta) = 0$$

seu

$$(d\varphi + d\xi)\left(v\cos\xi + fq\sin(\xi - \vartheta)\right) = 0\,,$$

quorum factorum finitus

$$v\cos\xi + fq\sin(\xi - \vartheta)$$

evanescere nequit ob

$$v\sin\xi - fq\cos(\xi - \vartheta) = 0\,;$$

sequeretur enim inde

$$v \cos \vartheta = 0 \text{ et } fq \cos \vartheta = 0,$$

quod nonnisi casu $\vartheta = 90^\circ$ locum habet. Relinquitur ergo ut sit $d\varphi + d\xi = 0$ ideoque

$$\varphi + \xi = \text{Const.}$$

Hoc impetrato reliqua non difficulter expedientur; ad integrationes autem determinandas pro statu initiali $t = 0$ ponamus fuisse celeritatem progressivam $v = e$, ang. $\varphi = 0$, ang.$PZO = \vartheta = \mathfrak{h}$, arcum $ZO = s = \mathfrak{f}$ et celeritatem angularem $\gamma' = \varepsilon$ in sensum *ACB;* hincque

$$p = h = \gamma' \cos s = \varepsilon \cos \mathfrak{f} \text{ et } q = \varepsilon \sin \mathfrak{f};$$

porro

$$\text{tang } \xi = \frac{\varepsilon f \sin \mathfrak{f} \cos \mathfrak{h}}{\varepsilon - \varepsilon f \sin \mathfrak{f} \sin \mathfrak{h}}$$

Statuatur

$$\frac{\varepsilon f \sin \mathfrak{f} \cos \mathfrak{h}}{\varepsilon - \varepsilon f \sin \mathfrak{f} \sin \mathfrak{h}} = \text{tang } \zeta,$$

ut fuerit initio $\xi = \zeta$, ac perpetuo erit $\xi + \varphi = \zeta$ ita ut angulus $DZQ = \zeta$ maneat constans. Quare cum sit $\xi = \zeta - \varphi$, erit

$$v \sin(\zeta - \varphi) = fq \cos(\zeta - \vartheta - \varphi).$$

Supra autem invenimus:

$$\frac{d \cdot v \cos \varphi}{2gdt} = \delta \cos(\xi + \varphi) = \delta \cos \zeta \quad \text{et} \quad \frac{d \cdot v \sin \varphi}{2gdt} = \delta \sin(\xi + \varphi) = \delta \sin \zeta,$$

unde integrando colligimus

$$v \cos \varphi = e + 2\delta gt \cos \zeta \quad \text{and} \quad v \sin \varphi = e + 2\delta gt \sin \zeta$$

hincque

$$v = \sqrt{(ee + 4\delta egt \cos \zeta + 4\delta\delta ggtt)} \text{ et tang } \varphi = \frac{2\delta gt \sin \zeta}{e + 2\delta gt \cos \zeta}$$

atque

$$\text{tang}(\zeta - \varphi) = \frac{e \sin \zeta}{e \cos \zeta + 2\delta gt} = \frac{fq \cos \vartheta}{v - fq \sin \vartheta} = \text{tang} \xi.$$

Deinde ob $d\varphi = -d\xi$ binae priores aequationes abeunt in

$$\text{I. } q(d\xi - d\vartheta) = \frac{2\delta fg}{aa}\, dt \cos(\xi - \vartheta) \ ,$$
$$\text{II. } dq = \frac{2\delta fg}{aa} dt \sin(\xi - \vartheta) \ ,$$

quarum haec per illam divisa dat:

$$\frac{dq}{q(d\xi - d\vartheta)} = \frac{\sin(\xi-\vartheta)}{\cos(\xi-\vartheta)}$$

quae integrata dat $q\cos(\xi - \vartheta)$ = Const. : ideoque

$$q\cos(\xi - \vartheta) = \varepsilon \sin\mathfrak{f} \cos(\zeta - \mathfrak{h}) \,,$$

unde valor ipsius q in prima substitutus praebet:

$$\frac{\varepsilon(d\xi - d\vartheta)\sin\mathfrak{f}\cos(\zeta-\mathfrak{h})}{\cos^2(\xi-\vartheta)} = \frac{2\delta fg}{aa} dt$$

et integrando

$$\varepsilon \sin\mathfrak{f} \cos(\zeta - \mathfrak{h}) \operatorname{tang}(\xi - \vartheta) = C + \frac{2\delta fg}{aa} t \ ,$$

ubi $C = \varepsilon \sin\mathfrak{f} \sin(\zeta - \mathfrak{h})$. At

$$\operatorname{tang}(\xi - \vartheta) = \operatorname{tang}(\zeta - \varphi - \vartheta) = \frac{\operatorname{tang}(\zeta-\varphi) - \operatorname{tang}\vartheta}{1 + \operatorname{tang}\vartheta \operatorname{tang}(\zeta-\varphi)}$$

et

$$\operatorname{tang}\vartheta = \frac{\operatorname{tang}\xi - \operatorname{tang}(\xi-\vartheta)}{1+\operatorname{tang}\xi \operatorname{tang}(\xi-\vartheta)}$$

Sed per hypothesin est

$$\varepsilon \sin\mathfrak{f} = \frac{e\sin\zeta}{f\cos(\zeta-\mathfrak{h})}$$

unde fit

$$\operatorname{tang}(\xi - \vartheta) = \operatorname{tang}(\zeta - \mathfrak{h}) + \frac{2\delta ffgt}{eaa\sin\zeta} \ ,$$

at

$$\operatorname{tang}\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt}$$

hincque angulus ϑ facile determinatur, indeque

$$q = \frac{e\sin\zeta}{f\cos(\xi-\vartheta)}$$

Verum hic notari oportet, cum sit

$$\text{tang}\,\zeta = \frac{\varepsilon f \sin\mathfrak{f} \cos\mathfrak{h}}{e - \varepsilon f \sin\mathfrak{f} \sin\mathfrak{h}},$$

esse ut supra de angulo ξ ostendimus,

$$\sin\zeta = -\frac{\varepsilon f \sin\mathfrak{f} \cos\mathfrak{h}}{\sqrt{\left(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + eeff \sin^2\mathfrak{f}\right)}}$$

et

$$\cos(\zeta - \mathfrak{h}) = -\frac{e\cos\mathfrak{h}}{\sqrt{\left(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + eeff \sin^2\mathfrak{f}\right)}}$$

unde

$$\cos(\zeta - \mathfrak{h}) = -\frac{e\cos\mathfrak{h}}{\sqrt{\left(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + eeff \sin^2\mathfrak{f}\right)}}.$$

His inventis cum sit $\gamma' \cos s = \varepsilon \cos\mathfrak{f}$ et $\gamma' \sin s = q$, erit

$$\gamma' = \sqrt{\left(qq + \varepsilon\varepsilon \cos^2\mathfrak{f}\right)} \text{ et tang } s = \frac{q}{\varepsilon \cos\mathfrak{f}}$$

Sicque tam motus progressivus, quam ad quodvis tempus axis gyrationis *O*cum celeritate angulari γ' poterit assignari, id quod ad motus cognitionem sufficit. Determinatio autem situs punctorum *A*, *B*, *C* ad quodvis tempus nimis est ardua, quam ut eam perficere liceat.

COROLLARIUM I

1049. Cum sit ce1eritas angu1aris $\gamma' = \frac{\varepsilon \cos\mathfrak{f}}{\cos s}$ seu cosinui arcus *SO* reciproce proportionalis, sequitur, si polus gyrationis *O* initio fuerit in superiori hemisphaerio *DZE*, eum nunquam in inferius pervenire posse; in transitu enim per circulum horizontalem *DE* prodiret celeritas angularis γ' infinita.

COROLLARIUM 2

1050. Ob eandem rationem, si polus gyrationis *O* initio fuerit in hemisphaerio inferiori *DTE*, is nunquam in superius ascendet. Sin autem initio fuerit in ipso circulo horizontali *DE*, perpetuo in eodem manebit. Scilicet si initio axis gyrationis fuerit horizontalis, perpetuo horizontalis manebit.

COROLLARIUM 3

1051. Si fuerit initio angulus $DZO = \mathfrak{h}$ rectus, fiet $\sin\zeta = 0$ et ob

$$\text{tang}(\zeta - \mathfrak{h}) = \frac{\varepsilon f \sin\mathfrak{f} - e\sin\mathfrak{h}}{e\cos\mathfrak{h}}$$

erit etiam $\xi-\vartheta$ rectus. Sed ob

$$\text{tang}\,\xi=\frac{e\sin\zeta}{e\cos\zeta+2\delta gt}$$

angulus ξ evanescit, unde angulus $\vartheta=PZO$ prodit rectus. Simulatque igitur angulus *PZO* factus fuerit rectus, perpetuo rectus manebit.

COROLLARIUM 4

1052. Memorabilis est etiam proprietas, quod angulus $\xi+\varphi$ seu *DZQ* et angulus *DIQ* in fig. 140 sit constans. Recta enim *QIS* sibi perpetuo manebit parallela, et quia globus in motu progressivo sollicitatur vi constante δM secundum eandem directionem *IS,* curva ab eo descripta *GI* parabola sit necesse est.

SCHOLION 1

1053. Hic autem motus globi, uti nostris formulis est definitus, diutius non durat, quam revera frictio adest, seu planum horizontale in puncto contactus T raditur. Si enim eveniat, ut rasio cesset seu celeritas radens in *T* evanescat, subito frictio evanescit formulaeque inventae non amplius locum habent. Tum igitur globus motu tam progressivo quam gyratorio uniformiter in directum progredietur, neque axis gyrationis ullam amplius mutationem patietur. Ac si statim initio motus globo impressus ita fuerit comparatus, ut frictio fuerit nulla, quod evenit, si tam $\varepsilon f\ \sin\mathfrak{f}\cos\mathfrak{h}=0$ quam $e=\varepsilon f\ \sin\mathfrak{f}\sin\mathfrak{h}$, deinceps etiam globus nullam frictionem sentiet; et statim ab initio motum progressivum uniformiter in directum prosequetur simulque uniformiter circa eundem axem gyrabitur. Verum si corpori ab initio alius motus quicunque fuerit impressus, semper aliquo tempore elapso eo reducetur, ut frictio evanescat indeque motum suum uniformiter prosequetur; quod memorabile temporis punctum in sequenti problemate investigabimus.

SCHOLION 2

1054. Quae in solutione problematis elicuimus, hue redeunt: Ex motu primum impresso habemus celeritatem motus progressivi = *e* secundum directionem *DI;* ac si gyretur circa axem *IO* celeritate angulari ε in sensum *ACB* seu *ZETD* (Fig. 139), qui sensus *antrorsum tendens* dici solet, fueritque arcus $ZO=\mathfrak{f}$ et angulus $DZO=\mathfrak{h}$; tum vero radius globi sit = *f* eiusque momentum inertiae = *Maa* respectu omnium diametrorum, existente *M* eius massa; ex his datis colligitur celeritas radens in puncto contactus

$$\sqrt{(ee-2\varepsilon ef\ \sin\mathfrak{f}\sin\mathfrak{h}\ +\varepsilon\varepsilon ff\ \sin^2\mathfrak{f})}\,,$$

quae si ponatur = *k*, quaeratur angulus ζ ut sit

et

$$\sin\zeta=-\frac{\varepsilon f\ \sin\mathfrak{f}\cos\mathfrak{h}}{k}\quad\text{et}\quad\cos\zeta=\frac{\varepsilon f\ \sin\mathfrak{f}\sin\mathfrak{h}-e}{k}\,,$$

qui sit $DZQ = \zeta$, eritque *IQ* directio motus radentis. Tum si elapso tempore *t* globi centrum proferatur celeritate *v* secundum directionem *PI* et gyretur celeritate angulari $= \gamma'$ in sensum *ZETD* circa polum *O* ponaturque $DZP = \varphi$, $PZO = \vartheta$ et $ZO = s$, invenimus primo:

$$\operatorname{tang}\varphi = \frac{2\delta gt \sin\zeta}{e+2\delta gt\cos\zeta}$$

et celeritatem centri

$$D = \sqrt{(ee + 4\delta egt\cos\zeta \ + 4\delta\delta ggtt)}\,,$$

at celeritas radens etiamnum fiet in directione *IQ*, existente $DZQ = \zeta$: unde posito $PZQ = \xi$ erit

$$\operatorname{tang}\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt}$$

Porro est

$$\operatorname{tang}(\xi - \vartheta) = \operatorname{tang}(\zeta - \mathfrak{h}) + \frac{2\delta ffgt}{eaa\sin\zeta}$$

existente

$$\operatorname{tang}(\zeta - \mathfrak{h}) = \frac{ef\sin\mathfrak{f} - e\sin\mathfrak{h}}{e\cos\mathfrak{h}}$$

unde angulus ϑ innotescit, hincque ob $DZ0 = \varphi + \vartheta = \zeta - \xi + \vartheta$ concluditur

$$\operatorname{tang} DZO = \operatorname{tang}(\varphi + \vartheta) = \frac{\varepsilon aak\sin\mathfrak{f}\sin\mathfrak{h} + 2\delta fgt(e - \varepsilon f\sin\mathfrak{f}\sin\mathfrak{h})}{\varepsilon aak\sin\mathfrak{f}\cos\mathfrak{h} - 2\delta\varepsilon ffgt\sin\mathfrak{f}\cos\mathfrak{h}}$$

Atque ex his tandem nacti sumus:

$$\gamma'\cos s = \varepsilon\cos\mathfrak{f} \text{ et } \gamma'\sin s = \frac{e\sin\zeta}{f\cos(\xi - \vartheta)}$$

.

Denique pro celeritate radente secundum *IQ* ea est =

$$\sqrt{(vv - 2\gamma' fv\sin s\sin\vartheta + \gamma'\gamma' ff\sin^2 s)}\,;$$

quae si vocetur = *w*, supra ostendimus esse

$$\sin\xi = \frac{-\gamma' f\sin s\cos\vartheta}{w} \quad \text{et} \quad \cos\xi = \frac{\gamma' f\sin s\sin\vartheta - v}{w}\,,$$

unde γ' et *s* definiuntur. Sed pro situ punctorum *A, B, C* in globe fixorum ad quodvis tempus determinando formulae adeo fiunt intricatae, ut nihil inde concludi queat. Interim si pro puncto *A* vocetur $ZA = l$ et $EZA = \lambda$, ad has binas aequationes totum negotium reducitur:

$$\text{I. } dl = dt\left(\varepsilon \sin \mathfrak{f} \sin(\mathfrak{h} + \lambda) - \frac{2\delta fgt}{aa} \cos(\zeta + \lambda)\right),$$

$$\text{II. } d\lambda \sin l = \varepsilon dt \cos \mathfrak{f} \sin l + dt \cos l\left(e \sin \mathfrak{f} \cos(\mathfrak{h} + \lambda) + \frac{2\delta fgt}{aa} \sin(\zeta + \lambda)\right),$$

quarum resolutio vereor ne frustra suscipiatur. Cum autem ad quodvis tempus axem gyrationis cum celeritate angulari assignare valeamus, quod ad motus cognitionem, qualis vulgo desideratur, sufficere potest, eo magis mirum videtur, quod motus singulorum globi punctorum quasi vires analyseos superet. Multo minus igitur de motu globorum, in quibus momenta inertiae non sunt aequalia, quicquam definire licebit.

PROBLEMA 19

1055. Si globo, cuius omnia momenta inertiae sunt inter se aequalia, motus quicunque fuerit impressus, assignare temporis punctum, ubi celeritas radens ideoque frictio evanescit indeque globus motu uniformi progredi pergat.

SOLUTIO

Supra § 1039 vidimus, ut attritus evanescat, has duas requiri conditiones, alteram $\gamma' \sin s \cos \vartheta = 0$ alteram $v = f\gamma' \sin s \sin \vartheta$, seu in expressione

$$\text{tang } \zeta = \frac{f\gamma' \sin s \cos \vartheta}{v - f\gamma' \sin s \sin \vartheta}$$

tam numeratorem quam denominatorem simul evanescere debere. Cum autem invenerimus

$$\text{tang } \zeta = \frac{e \sin \zeta}{e \cos \zeta + 2\delta gt},$$

ubi numerator $e \sin \zeta$ est constans, si in illa formula numerator evanescit, necesse est, denominator simul evanescat, quia alioquin aequalitas inter has duas fractiones subsistere nequit. Unde positio $\cos \vartheta = 0$ tempus quaesitum declarabit. Verum idem luculentius determinabimus, si ad quodvis tempus elapsum *t* celeritatem radentem *w* investigemus. Cum igitur ex formula

$$\sin \xi = -\frac{\gamma' f \sin s \cos \vartheta}{w}$$

sit

$$w = -\frac{\gamma' f \sin s \cos \vartheta}{\sin \xi}$$

quae expressio ob

$$\gamma' \sin s = \frac{e \sin \zeta}{f \cos(\xi - \vartheta)}$$

abit in

$$w = \frac{-e \sin \zeta \cos \vartheta}{\sin \xi \cos(\xi - \vartheta)}$$

atque ob $\vartheta = \xi - (\xi - \vartheta)$

in hanc

$$w = -e \sin \zeta \left(\cot \xi + \text{tang}\left(\xi - \vartheta\right)\right)$$

si hic pro $\text{tang}\xi$ et $\text{tang}(\xi - \vartheta)$ valores supra inventos substituamus, reperiemus:

$$w = -\left(e \cos \zeta + 2\delta gt + e \sin \zeta \,\text{tang}\left(\zeta - \mathfrak{h}\right) + \frac{2\delta ffgt}{aa}\right)$$

At

$$\cos \zeta + \sin \zeta \,\text{tang}\,(\zeta - \mathfrak{h}) = \frac{\cos \mathfrak{h}}{\sin(\zeta - \mathfrak{h})} \quad \text{et} \quad \cos(\zeta - \mathfrak{h}) = -\frac{e \cos \mathfrak{h}}{k};$$

unde

$$e \cos \zeta + e \sin \zeta \,\text{tang}\,(\zeta - \mathfrak{h}) = -k,$$

ubi *k* denotat celeritatem radentem initialem. Quamobrem elapso tempore *t* habebimus celeritatem radentem

$$w = k - 2\delta g \left(1 + \frac{ff}{aa}\right) t$$

ita ut ea labente tempore uniformiter decrescat. Tandem ergo certo evanescet, idque fiet elapso tempore

$$t = \frac{aak}{2\delta g(aa + ff)}$$

eritque tum $\cos \vartheta = 0$ et $\vartheta = 90^{\circ} = PZO$. Quod ergo cum evenerit, videamus, quomodo reliquae motus determinationes se sint habiturae; et quoniam $2\delta gt = \frac{aak}{aa + ff}$, erit

$$\text{tang}\ \varphi = \frac{aak \sin \zeta}{e(aa + ff) + aak \cos \zeta} \quad \text{et} \quad \text{tang}\ \xi = \frac{e(aa + ff) \sin \zeta}{e(aa + ff) \cos \zeta + aak}.$$

Hinc fit

$$\gamma' \sin s = \frac{e \sin\zeta}{f \sin\xi}.$$

Cum autem sit

$$v = \sqrt{\left(ee + \frac{2aaek\cos\zeta}{aa+ff} + \frac{a^4 kk}{(aa+ff)^2}\right)},$$

erit

$$\sin\varphi = \frac{aak\sin\zeta}{(aa+ff)v} \quad \text{et} \quad \cos\varphi = \frac{e(aa+ff)+aak\cos\zeta}{(aa+ff)v}$$

atque

$$\sin\xi = \frac{e\sin\zeta}{v} \quad \text{ideoque} \quad \gamma'\sin s = \frac{v}{f}.$$

Porro quia $\gamma'\cos s = \varepsilon\cos f$, erit

$$\text{tang } s = \frac{v}{\varepsilon f\cos\mathfrak{f}} \quad \text{et} \quad \gamma' = \sqrt{\left(\frac{vv}{ff} + \varepsilon\varepsilon\cos^2\mathfrak{f}\right)}$$

seu

$$\gamma' = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf\sin\mathfrak{f}\sin\mathfrak{h} + \varepsilon\varepsilon a^4\sin^2\mathfrak{f} + \varepsilon\varepsilon(aa+ff)^2\cos^2\mathfrak{f}\right)}}{aa+ff}$$

ob $kk = ee - 2\varepsilon ef\sin\mathfrak{f}\sin\mathfrak{h} + eeff\sin^2\mathfrak{f}$.

COROLLARIUM 1

1056. Quo maior ergo initio fuerit celeritas radens *k*, eo diutius motus durat, antequam cessante frictione ad uniformitatem redigatur. Ac si globus constet ex materia homogenea, fit $aa = \frac{2}{5}ff$, ideoque motus uniformitas incipit elapso tempore $t = \frac{k}{7\delta g}$ min. sec. hinc in hypothesi $\delta = \frac{1}{3}$ fit $t = \frac{3k}{7g}$, existente $g = 15\frac{5}{8}$ ped. Rhen.

COROLLARIUM 2

1057. Ut centrum globi eodem tempore ad quietem redigatur, status initialis ita comparatus esse debet, ut sit $\cos\zeta = -1$ et $e = \frac{aak}{aa+ff}$. Fit ergo
$k = e - \varepsilon f\sin\mathfrak{f}\sin\mathfrak{h}$ et $\sin\mathfrak{h} = 1$ seu $\mathfrak{h} = 90°$ et $k = e - \varepsilon f\sin\mathfrak{f}$;

hincque

$$\varepsilon \sin \mathfrak{f} = -\frac{ef}{aa}.$$

Porro ob $v = 0$ fit $s = 0$ et $\gamma' = \varepsilon \cos \mathfrak{f}$, qua celeritate angulari iam globus circa axem verticalem quiescentem gyrabitur elapso ab initio tempore $t = \frac{e}{2\delta g}$ min. sec.

COROLLARIUM 3

1058. Hoc autem casu, quo initio est $\mathfrak{h} = 90°$ et $\varepsilon = -\frac{ef}{aa \sin \mathfrak{f}}$, fit $\zeta = 180°$, $\varphi = 0$, $\xi = 180°$, $\vartheta = 90°$, $v = e - 2\delta gt$; tum vero

$$\gamma' \cos s = \frac{-ef \cos \mathfrak{f}}{aa \sin \mathfrak{f}},$$

$$\gamma' \sin s = \frac{-ef}{aa}\left(1 - \frac{2\delta gt}{e}\right),$$

hincque

$$\operatorname{tang} s = \left(1 - \frac{2\delta gt}{e}\right) \operatorname{tang} \mathfrak{f}$$

et

$$\gamma' = \frac{-ef}{aa \sin \mathfrak{f}} \sqrt{\left(1 - \frac{4\delta gt}{e} \sin^2 \mathfrak{f} + \frac{4\delta\delta ggtt}{ee} \sin^2 \mathfrak{f}\right)}.$$

At initio erat celeritas radens

$$k = e\left(1 + \frac{ff}{aa}\right),$$

elapso autem tempore t ea est

$$w = e\left(1 + \frac{ff}{aa}\right)(e - 2\delta gt),$$

sicque posito $t = -\frac{e}{2\delta g}$ simul fit $w = 0$, $v = 0$ et $s = 0$, ut ante.

COROLLARIUM 4

1059. Ne valor

$$\gamma' \sin s = \frac{e \sin \zeta}{f \cos(\xi - \vartheta)}$$

indefinitus videatur, quod fit si numerator ac denominator evanescant seu $\zeta = 0$, conveniet loco $\sin \zeta$ et $\cos(\xi - \vartheta)$ valores ex superioribus substitui, atque hinc reperietur:

$$\gamma' \sin s = \sqrt{\left(\varepsilon\varepsilon \sin^2 \mathfrak{f} - \frac{4\delta\varepsilon fgt(\varepsilon f \sin \mathfrak{f} - e \sin \mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)},$$

unde ob $\gamma' \cos s = e \cos \mathfrak{f}$ prodit:

$$\gamma'\gamma' = \varepsilon\varepsilon - \frac{4\delta\varepsilon\, fgt \sin\mathfrak{f}\left(\varepsilon f \sin\mathfrak{f} - e\sin\mathfrak{h}\right)}{aak} + \frac{4\delta\delta\, ffggtt}{a^4}.$$

COROLLARIUM 5

1060. Cum sit vis viva globi = $M\left(vv + aa\gamma'\gamma'\right)$, erat ea initio = $M(ee + \varepsilon\varepsilon aa)$, elapso autem tempore t ea erit =

$$M\left(ee + \varepsilon\varepsilon aa - 4\delta gkt + 4(1 + \frac{ff}{aa})\delta\delta ggtt\right).$$

At elapso tempore

$$t = \frac{aak}{2\delta g(aa + ff)}$$

vis viva fiet

$$\frac{M\left(eeff + 2\varepsilon eaaf \sin\mathfrak{f}\sin\mathfrak{h}\right) + \varepsilon\varepsilon aa\left(aa + ff\cos^2\mathfrak{f}\right)}{aa + ff},$$

cuius defectus ab initiali est

$$\frac{\left(Maaee - 2\varepsilon ef \sin\mathfrak{f}\sin\mathfrak{h} + \varepsilon\varepsilon\, ff\cos^2\mathfrak{f}\right)}{aa + ff} = \frac{Maakk}{aa + ff},$$

ita ut ista vis viva sit =

$$M\left(ee + \varepsilon\varepsilon aa - \frac{aakk}{aa + ff}\right)$$

SCHOLlON

1061. Ex his ergo formulis totus motus globi assignari potest, quicunque motus ei initio fuerit impressus; interim tamen hae formulae non parum sunt complexae, unde ad clariore explicationem haud abs re erit, casus quosdam magis notabiles evolvi. Cuiusmodi sunt, uti iam supra innuimus, duo potissimum, alter quo arcus *ZO* initio erat quadrans, alter vero quo angulus $DZO = \mathfrak{h}$ erat rectus; utrumque igitur seorsim explicemus.

PROBLEMA 20

1062. Si globo, in quo omnia momenta inertiae sunt aequalia, initio motus gyratorius circa axem horizontalem fuerit impressus praeter motum progressivum, definire continuationem motus.

SOLUTIO

Cum initio axis gyrationis fuerit horizontalis, erit $\mathfrak{f} = ZO = 90°$. Denotante ergo e celeritatem progressivam secundum directionem *DIE* (Fig. 139) et ε celeritatem angularem circa axem *IO* in sensum *ZETD,* sit pro puncto *O* angulus $DZO = \mathfrak{h}$, manente f = radio globi et Maa = momento inertiae. Ex his erat initio celeritas radens

$$k=\sqrt{\left(ee-2e\varepsilon f\ sin\mathfrak{h}+\varepsilon\varepsilon ff\right)}$$

et pro eius directione *1Q* angulus $DZQ=\zeta$, ut sit (Fig. 140).

$$sin\,\zeta=-\frac{\varepsilon f\ cos\mathfrak{h}}{k}\quad \text{et}\quad cos\,\zeta=\frac{\varepsilon f\ sin\mathfrak{h}-e}{k}.$$

His pro statu initiali constitutis elapso tempore *t* centrum globi descripserit viam *GI*, ut iam sit in *I*, ubi eius celeritas secundum *1R* erit =

$$v=\sqrt{\left(ee+\frac{4\delta egt\left(\varepsilon f\ sin\mathfrak{h}-e\right)}{k}+4\delta\delta ggtt\right)};$$

unde positis coordinatis *GX* = *X* et *XI* = *Y* ob

$$\text{tang}\ EIR=\ \text{tang}\varphi=-\frac{2\delta\varepsilon\ fgt\ cos\mathfrak{h}}{ek+2\delta\text{gt}\left(\varepsilon f\ sin\mathfrak{h}\ -e\right)}$$

erit

$$dX=edt+\frac{2\delta gtdt}{k}\left(\varepsilon f\ sin\mathfrak{h}-e\right)$$

et

$$dY=\frac{-2\delta\varepsilon\ fgtdt\ cos\mathfrak{h}}{k}$$

ideoque

$$GX=X=et+\frac{\delta gtt}{k}\left(\varepsilon f\ sin\mathfrak{h}-e\right)\quad \text{et}\quad XI=Y=-\frac{\delta gtt}{k}cos\mathfrak{h}.$$

Tum vero pro motu gyratorio, qui iam fiat in sensum *ZETD* celeritate angulari $=\gamma'$ circa polum *O* existente $ZO=s$, $PZO=\vartheta$ et $DZQ=\varphi+\xi$, ubi *IQ* refert directionem celeritatis radentis, quia constanter est $\varphi+\xi=\zeta$, seu directio *IQ* constans, erit

$$\text{tang}\ \xi=\ -\frac{\varepsilon ef\ cos\mathfrak{h}}{\varepsilon ef\ sin\mathfrak{h}-ee+2\delta gkt}$$

et

$$\text{tang}\left(\xi-\vartheta\right)=\ \frac{\varepsilon f-e\,sin\mathfrak{h}}{a\,cos\mathfrak{h}}\ -\frac{2\delta gkt}{\varepsilon eaa\,cos\mathfrak{h}},$$

unde ambo anguli ξ et ϑ definiuntur. Vel erit

$$\text{tang}\left(\varphi+\vartheta\right)=\ \frac{\varepsilon aak\ sin\mathfrak{h}+2\delta gkt\left(e-\varepsilon f\ sin\mathfrak{h}\right)}{\varepsilon aak\ cos\mathfrak{h}-2\delta\varepsilon\ ffgt\ cos\mathfrak{h}}.$$

Celeritas autem radens secundum directionem *I Q* est

$$w = k - 2\delta g\left(1 + \frac{ff}{aa}\right)t \ .$$

Tum vero ob $\gamma' \cos s = 0$ erit arcus $ZO = s$ quadrans et

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt(\varepsilon f - e\sin\mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)}$$

Hic motus inaequabilis autem tantum durabit per tempus

$$t = \frac{aak}{2\delta g(aa+ff)} \ ,$$

quo elapso est

$$\text{tang}\varphi = -\frac{\varepsilon aaf\cos\mathfrak{h}}{e(aa+ff)+aa(\varepsilon f\sin\mathfrak{h} - e)} = -\frac{\varepsilon aa\cos\mathfrak{h}}{ef + \varepsilon aa\sin\mathfrak{h}}$$

$$v = \sqrt{\left(ee + \frac{2aae(\varepsilon f\sin\mathfrak{h} - e)}{aa+ff} + \frac{a^4kk}{(aa+ff)^2}\right)}$$

$$s = 90^{\circ} \text{ et } \gamma' = \frac{v}{f} = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf\sin\mathfrak{h} + \varepsilon\varepsilon a^4\right)}}{aa+ff},$$

substituto pro *kk* valore. Tum autem fit angulus $\vartheta = 90^{\circ}$ et

$$\sin\xi = \frac{e\sin\zeta}{v} .$$

COROLLARIUM 1

1063. Si initio fuerit angulus $DZO = \mathfrak{h} = 0$, erit

$$k = \sqrt{(ee + \varepsilon\varepsilon ff)} ;$$

pro angulo $DZQ = \zeta$ fit

$$\sin\zeta = -\frac{\varepsilon f}{k}, \quad \cos\zeta = -\frac{e}{k} ;$$

tum post tempus *t* prodit

$$v = \sqrt{\left(ee - \frac{4\delta eegt}{k} + 4\delta\delta ggtt\right)}, \quad \text{tang } \varphi = -\frac{2\delta\varepsilon fgt}{e(k - 2\delta gt)},$$

$$X = et\left(1-\frac{\delta gt}{k}\right) \quad , \quad Y = -\frac{\delta\varepsilon fgtt}{k}.$$

Porro

$$\text{tang } \xi = -\frac{eef}{eef-2\delta gkt} \ , \quad \text{tang}\left(\xi-\vartheta\right) = \frac{\varepsilon f}{a} - \frac{2\delta gkt}{\varepsilon eaa};$$

$$\text{tang}\left(\varphi+\vartheta\right) = \frac{2\delta efgt}{\varepsilon aak-2\delta\varepsilon ffgt},$$

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon\varepsilon ffgt}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)} \quad \text{et} \quad w = k-2\delta g\left(1+\frac{ff}{aa}\right)t$$

lapso autem tempore $t = \frac{aak}{2\delta g(aa+ff)}$ erit

$$\text{tang}\varphi = -\frac{\varepsilon aa}{ef}, \quad v = \frac{f\sqrt{\left(eeff+\varepsilon\varepsilon a^4\right)}}{aa+ff} = f\gamma',$$

$$\vartheta = 90^{\circ} \text{ et } \tan\xi = \frac{\varepsilon ef(aa+ff)}{ee(aa+ff)-aakk} = \frac{\varepsilon e(aa+ff)}{f(ee-\varepsilon\varepsilon aa)}$$

COROLLARIUM 2

1064. Si angulus $DZO = \mathfrak{h}$ esset = 180°, eaedem formulae motum indicabunt, sumta celeritate angulari ε negativa seu motu gyratorio in contrarium verso. At si sit ε = 0 seu globo solus motus progressivus fuerit impressus, fit $k=e$, $\zeta = 180°$,

$v = e-2\delta gt$, $\varphi = 0$, $X = t\left(e-\delta gt\right)$, $Y = 0$, $\xi = 180°$, $\vartheta = 90^{\circ}$, $\gamma' = \frac{2\delta fgt}{aa}$; et elapso tempore $t = \frac{aae}{2\delta g\left(gaa+ff\right)}$

fit

$$v = \frac{eff}{aa+ff}, \quad \gamma' = \frac{ef}{aa+ff} \quad \text{et} \quad X = \frac{et\left(aa+2ff\right)}{2(aa+ff)} = \frac{aaee(aa+2ft)}{4\delta g(aa+ff)^2}.$$

SCHOLION

1065. Casus hic, quo globus initio nullum motum gyratorium est adeptus, in genere valet, neque ad ullam hypothesin angulorum $\mathfrak{f}$ et $\mathfrak{h}$ est adstrictus. Tum igitur globus in directum progreditur motu progressivo retardato, motumque paulatim gyratorium accipiet, donec elapso tempore $t = \frac{aae}{2\delta g(aa+ff)}$ motum uniformem acquirat, quo deinceps continuo progrediatur. Hinc deducimur ad casum, quo globus initio motum tantum gyratorium acceperit sine ullo motu progressivo, cuius evolutio est facilis. Posito enim e = 0 erit

$k = \varepsilon f \; sin\mathfrak{f}$ hincque fit

$$sin\zeta = -cos\mathfrak{h} \;\text{ et }\; cos\zeta = sin\mathfrak{h},$$

ergo $\zeta = \mathfrak{h} - 90°;$ ubi pro axe gyrationis initio impressae *IO* est $ZO = \mathfrak{f}$ et $DZO = \mathfrak{h}$, existente celeritate angulari in sensum $ZETD = \varepsilon$. Elapso ergo tempore *t* fit $\varphi = \zeta$, scilicet sublato ab angulo $DZO = \mathfrak{h}$ angulo recto *PZO* erit *PI* directio motus progressivi, quem globus acquiret, cuius celeritas erit $v = 2\delta gt$ ideoque tempori proportionalis. Tum vero erit $\text{tang}\xi = 0$ et $\text{tang}(\xi - \vartheta) = \infty$, ergo ob $\varphi + \xi = \zeta = \mathfrak{h} - 90°$ erit $\xi = 0$ et $\vartheta = 90°$, hinc $DZO = \zeta + 90° = \mathfrak{h}$, ita ut polus gyrationis *O* in eodem perpetuo circulo verticali reperiatur. Denique ex §1059 est

$$\gamma' \, sin\, s = \sqrt{\left(\varepsilon\varepsilon \, sin^2 \mathfrak{f} - \frac{4\delta\varepsilon \, fgt \, sin\mathfrak{f}}{aa} + \frac{4\delta\delta \, ffggtt}{a^4}\right)} = \varepsilon \, sin\mathfrak{f} - \frac{2\delta \, fgt}{aa}$$

et

$$\gamma' \, cos\, s = \varepsilon \, cos\mathfrak{f},$$

unde fit

$$\text{tang } s = \text{tang } \mathfrak{f} - \frac{2\delta \, fgt}{\varepsilon aa \, cos\mathfrak{f}}$$

ita ut arcus *ZO* diminuatur, nisi fuerit quadrans vel eo maior, et

$$\gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon \, fgt \, sin\mathfrak{f}}{aa} + \frac{4\delta\delta \, ffggtt}{a^4}\right)}.$$

Motus autem ad uniformitatem reducetur elapso tempore

$$t = \frac{\varepsilon aaf \; sin\mathfrak{f}}{aa + ff};$$

fitque tum

$$\gamma' = \frac{\varepsilon\sqrt{\left(a^4 \, sin^2 \mathfrak{f} + (aa + ff)^2 \, cos^2 \mathfrak{f}\right)}}{aa + ff}, \quad v = \frac{\varepsilon aaf \; sin\mathfrak{f}}{aa + ff} \quad \text{et} \quad \text{tang } s = \frac{aa \, \text{tang } \mathfrak{f}}{aa + ff}.$$

Si ergo fuisset $\mathfrak{f} = 0$ seu globo motus gyratorius circa axem verticalem impressus esset sine ullo motu progressivo, eundem motum sine ulla mutatione esset conservaturus.

PROBLEMA 21

1066. Si globo, in quo omnia momenta inertiae sunt aequalia, motus gyratorius fuerit impressus circa axem ad motus progressivi directionem normalem, definire continuationem motus.

SOLUTIO

Cum motus progressivi initio impressi directio sit recta *DIE* (Fig. 139) et celeritas = *e*, angulus $DZO = \mathfrak{h}$ est rectus, et sumto $ZO = \mathfrak{f}$ erat *O* polus, circa quem initio globus accepit

celeritatem angularem = ε in sensum *ZETD*. Habemus ergo $k = \pm(e - \varepsilon f\ sin\,\mathfrak{f})$, ubi valorem positivum pro *k* sumi oportet, ita ut hic prodeant duo casus seorsim tractandi.
Casus I. Sit $e > \varepsilon f\ sin\,\mathfrak{f}$, erit $k = e - \varepsilon f\ sin\,\mathfrak{f}$, quae est celeritas radens initio, eiusque directio *IQ*, ut sit sin *DQ* = 0 et cos *DQ* = –1 ideoque $DQ = \zeta = 180^{\circ}$ et *Q* cadat in *E;* globusque a frictione δM secundum *ID* constanter retrahatur, unde statim colligitur globi centrum *I* in eadem recta *DE* esse incessurum. Elapso tempore *t* ergo ob $cos\,\zeta = -1$ fit celeritas centri $v = e - 2\delta gt$ et celeritas radens

$$w = e - \varepsilon\ fsin\mathfrak{f} - 2\delta g\left(1 + \frac{f}{aa}\right)t\ ;$$

tum vero $\varphi = 0$ et $\xi = 180^{\circ}$ atque $\vartheta = 90^{\circ}$. Quare pro axe gyrationis praesente *IO* est *DIO* = 90° et posito arcu *ZO* = *s* et celeritate angulari = γ' habemus

$$\gamma' cos\,s = \varepsilon\,cos\,\mathfrak{f}$$

et ex § 1059

$$\gamma' sin\,s = \varepsilon\,sin\,\mathfrak{f} + \frac{2\delta\,fgt}{aa},$$

unde colligitur

$$\text{tang } s = \text{tang } \mathfrak{f} + \frac{2\delta\,fgt}{\varepsilon aa\,cos\,\mathfrak{f}} \quad \text{et} \quad \gamma' = \sqrt{\left(\varepsilon\varepsilon + \frac{4\delta\varepsilon\,fgt\,sin\,\mathfrak{f}}{aa} + \frac{4\delta\delta\,ffggtt}{a^4}\right)}.$$

Hocque tempore *t* percurrit centrum *I* lineam rectam $GX = \mathrm{X} = t\left(e - \delta gt\right)$.
Hic autem motus inaequabilis durabit per tempus

$$t = \frac{aa\left(e - \varepsilon f\ sin\,\mathfrak{f}\right)}{2\delta g\left(aa + ff\right)},$$

quo elapso erit spatium

$$X = \frac{aa(e - \varepsilon f\ sin\,\mathfrak{f})(e(aa + 2ff) + \varepsilon aaf\ sin\,\mathfrak{f})}{2\delta g\,(aa + ff)^2}$$

et celeritas

$$v = \frac{f\,(ef + \varepsilon aa\,sin\,\mathfrak{f})}{aa + ff}.$$

At pro motu gyratorio

$$\text{tang } s = \text{tang } ZO = \text{tang } \mathfrak{f} = \frac{f\,(e - \varepsilon f\ sin\,\mathfrak{f})}{\varepsilon(aa + ff)\,cos\,\mathfrak{f}} = \frac{ef + \varepsilon aa\,sin\,\mathfrak{f}}{\varepsilon(aa + ff)\,cos\,\mathfrak{f}}.$$

existente perpetuo $DIO = 90^{\circ}$ et celeritas angularis

$$\gamma' = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf \sin\mathfrak{f} + \varepsilon\varepsilon a^4 \sin^2\mathfrak{f} + \varepsilon\varepsilon\left(aa+ff\right)^2 \cos^2\mathfrak{f}\right)}}{aa+ff}$$

Casus II. Sit $e < \varepsilon f \sin\mathfrak{f}$ seu $k = \varepsilon f \sin\mathfrak{f} - e$, quae est celeritas radens initio, eiusque directio *IQ* talis, ut sit $\sin DQ = 0$ et $\cos DQ = 1$, ergo $DQ = \zeta = 0$ et *Q* in *D* cadat. Globus ergo a frictione δM secundum directionem *IE* constanter acceleratur eiusque centrum *I* in eadem recta *IE* progreditur; atque elapso tempore *t* erit eius celeritas $v = e + 2\delta gt$ et celeritas radens

$$w = \varepsilon f\sin\mathfrak{f} - e - 2\delta \mathrm{g}\left(1 + \frac{f}{aa}\right)t$$

Tum vero fit $\varphi = 0$ et $\xi = 0$ atque $\vartheta = 90°$. Quare pro axe gyrationis praesente *IO* est $DIO = 90°$ et posito arcu *ZO* = *s* et celeritate angulari $= \gamma'$ habemus

$$\gamma' \cos s = \varepsilon \sin\mathfrak{f} \text{ et } \gamma' \sin s = \varepsilon \sin\mathfrak{f} - \frac{2\delta fgt}{aa}$$

unde fit

$$\text{tang } s = \text{tang } \mathfrak{f} - \frac{2\delta fgt}{\varepsilon aa \cos\mathfrak{f}} \quad \text{et} \quad \gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt \sin\mathfrak{f}}{aa} + \frac{4\delta\delta ffggtt}{a^4}\right)}$$

hocque tempore *t* centrum globi percurrit lineam rectam

$$GX = X = t\left(e + \delta gt\right)$$

Hic autem motus inaequabilis durabit tantum tempore

$$t = \frac{aa\left(\varepsilon f \sin\mathfrak{f} - e\right)}{2\delta g\left(aa+ff\right)},$$

quo elapso erit celeritas

$$v = \frac{f\,(ef - \varepsilon aa \sin\mathfrak{f})}{aa+ff}$$

et spatium

$$X = \frac{aa(\varepsilon f \sin\mathfrak{f} - e)(e(aa+2ff) + \varepsilon aaf \sin\mathfrak{f})}{2\delta g\,(aa+ff)^2}$$

At pro motu gyratorio reperitur

$$\text{tang } s = \text{tang } ZO = \frac{ef + \varepsilon aa \sin\mathfrak{f}}{\varepsilon(aa+ff)\cos\mathfrak{f}}$$

existente perpetuo *DIO* = 90° et celeritas angularis

$$\gamma' = \frac{\sqrt{\left(eeff + 2\varepsilon eaaf \sin\mathfrak{f} + \varepsilon\varepsilon a^4 \sin^2\mathfrak{f} + \varepsilon\varepsilon\left(aa+ff\right)^2 \cos^2\mathfrak{f}\right)}}{\left(aa+ff\right)}$$

COROLLARIUM 1

1067. Si fuerit $e = \varepsilon f \ sin ſ$, globus statim ab initio motum prosequetur uniformem, tam progressivum quam gyratorium, qui casus limitem constituit inter binos tractatos.

COROLLARIUM 2

1068. Ad priorem' casum, quo $e > \varepsilon f \ sin ſ$, referendi sunt ii, quibus ε habet valorem negativum seu globo impressus fuerit initio motus gyratorius in sensum *ZDTE.* Posito autem $-\varepsilon$ *;* loco ε fieri potest, ut globus revertatur, antequam ad uniformitatem pervenerit, scilicet si fuerit $\varepsilon > \frac{ef}{aa \, sin ſ}$.

COROLLARIUM 3

1069. Casu hoc, quo ε negative capitur, habebimus ad tempus *t:*

$$\varphi = 0, \quad \vartheta = 90^{\circ} \quad \xi = 180^{\circ}, \quad v = e - 2g\delta gt, \quad w = e + \varepsilon f \ sin ſ - 2\delta g\left(1 + \frac{ff}{aa}\right)t,$$

$$\text{tang } s = \text{tang } ſ - \frac{2\delta \, fgt}{\varepsilon aa \cos ſ} \quad \text{et} \quad \gamma' = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon \, fgt \sin ſ}{aa} + \frac{4\delta\delta \, ffggtt}{a^4}\right)}$$

At post tempus

$$t = \frac{aa\left(e + \varepsilon f \ sin ſ\right)}{2\delta g\left(aa + ff\right)}$$

percurso spatio

$$X = \frac{aa(e + \varepsilon f \ sin ſ)(e(aa + 2ff) - \varepsilon aaf \ sin ſ)}{2\delta g(aa + ff)^2}$$

uniformitatem attinget, eritque tum

$$v = \frac{f(ef - \varepsilon aa \sin ſ)}{aa + ff}, \quad \text{tang } s = \frac{\varepsilon aa \sin ſ - ef}{\varepsilon(aa + ff)\cos ſ}$$

et

$$\gamma' = \frac{\sqrt{\left(eeff - 2\varepsilon eaaf \ sin ſ + \varepsilon\varepsilon a^4 \sin^2 ſ + \varepsilon\varepsilon\left(aa + ff\right)^2 \cos^2 ſ\right)}}{aa + ff}$$

SCHOLION

1070. Casus hic praecipue est memorabilis, quo globo eiusmodi motus imprimi potest, ut primo recedat, mox autem iterum revertatur, quod experimento ostendi solet, dum digito ad globum circa *D* applicato et deorsum presso duplex motus globo imprimitur, alter progressivus in directione *DIE*, alter gyratorius in sensum *ZDTE.* Sed ut phaenomenon succedat, necesse est, ut celeritas angularis prae progressiva certum quendam limitem excedat; quem quo facilius agnoscamus, calculum ad istum casum accommodemus, quo motus gyratorius globo circa axem horizontalem et ad directionem motus progressivi normalem imprimitur. Quodsi ergo e denotet celeritatem progressivam secundum directionem *DIE* et γ' celeritatem angularem retro gyrantem in sensum *ZNTE*, existente f radio globi et *Maa* eius momento inertiae, frictioneque = δM ; primo globus in directione *DIE* procedet, et elapso tempore t eius celeritas secundum eandem directionem erit $v = e - 2\delta gt$, confecto spatio $X = t(e - \delta gt)$; tum vero etiamnum circa eundem axem retro volvetur celeritate angulari

$$\gamma' = \varepsilon - \frac{2\delta fgt}{aa}$$

Motus autem aequabilis evadit elapso tempore

$$t = \frac{aa(e+\varepsilon f)}{2\delta g(aa+ff)}$$

eritque tum

celeritas progressiva $v = \frac{f(ef-\varepsilon aa)}{aa+ff}$ et angularis $\gamma' = \frac{\varepsilon aaf - ef}{aa+ff}$.

Quare si fuerit $\varepsilon > \frac{ef}{aa}$, globus nunc retro movetur, gyratorio adhuc retro vergente; sin autem fuerit $\varepsilon < \frac{ef}{aa}$, globus adhuc procedit et gyratio in sensum contrarium est versa. Illo casu globus regredi coepit elapso tempore $t = \frac{e}{2\delta g}$ et percurso spatio $X = \frac{ee}{4\delta g}$.

Si globus sit homogeneus, erit $aa = \frac{2}{5}ff$ et εf exprimit celeritatem gyrationis in puncto contactus, quae si vocetur = h, erit post tempus t celeritas progressiva $v = e - 2\delta gt$ et gyratoria in puncto contactus, quae sit $u = h - 5\delta gt$, et spatium percursum = $t(e - \delta gt)$; motus vero aequabilis evadet elapso tempore $t = \frac{e+h}{7\delta g}$ et confecto spatio = $\frac{(6e-h)(e+h)}{49\delta g}$; ubi erit

$$v = \frac{5e-2h}{7} \quad \text{et} \quad u = \frac{2h-5e}{7} .$$

Ut ergo phaenomenum memoratum succedat, debet esse initio $h > \frac{5}{2}e$. Sin autem esset $h = \frac{5}{2}e$ uterque motus simul extingueretur elapso tempore $= \frac{e}{2\delta g}$ min. sec. et confecto spatio = $\frac{ee}{4\delta g}$.