# CHAPTER FOUR

## CONCERNING THE MOTION OF A POINT ON A GIVEN SURFACE.

[p. 477]

## PROPOSITION 95.

### Problem.

**856.** *If a body moving on a surface is acted on by any forces, to define the normal forces, evidently the normal force pressing* [*on the surface*], *the deflecting force* [*in the tangent plane normal to the curve*], *and the force along the tangent, with all arising from resolution.*

### Solution.

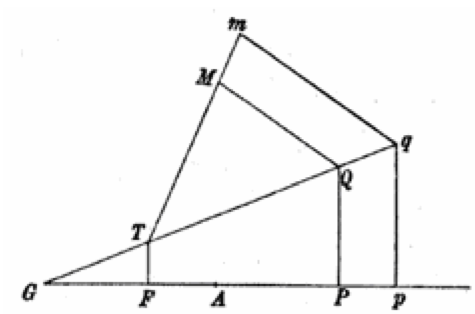

Fig. 92.

Whatever the forces acting should be, these can be reduced to three forces, the directions of which are along the three coordinates *x*, *y*, *z* [at *M*]. Now let the force at *M* (Fig. 92), drawing the body parallel to the abscissa *PA*, be equal to *E*, the force drawing the body parallel to the direction *QP* to be equal to *F*, and the force drawing the body along *MQ* to be equal to *G*. Hence the forces are each to be resolved in turn into three parts, clearly a normal pressing force, a normal force of deflection [in the plane of the tangent], and a force along the tangent [to the curve at *M*]. Moreover since these three directions are normal to each other, from each of these forces *E*, *F* et *G*, the normal and tangential forces themselves can be produced, if these are taken by the cosine of the angle that the directions of these forces make with these [other directions].

We may begin with the force along the tangent, the direction of which is *MT*, for which the equation arises [from Prop. 93] :

$$AF = \frac{zdx}{dz} - x \text{ and } FT = y - \frac{zdy}{dz}$$

and

$$QT = \frac{z\sqrt{(dx^2+dy^2)}}{dz} \text{ and } MT = \frac{z\sqrt{(dx^2+dy^2+dz^2)}}{dz}.$$

Hence the cosine of the angle *QMT*, that the direction the force *G* makes with the force along the tangent, is equal to : [p. 478]

$$\frac{QM}{MT} = \frac{dz}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

If the force G is multiplied by this, then the tangential force due to *G* produced, equal to

$$\frac{Gdz}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Now the cosine of the angle that *MT* makes with the direction of the force *F*, which is parallel to *QP* , is equal to

$$\cos. PQT. \sin. QMT = \frac{PQ-FT}{QT} \cdot \frac{QT}{MT} = \frac{dy}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Hence the tangential force from *F* that has arisen is equal to

$$\frac{Fdy}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Again the cosine of the angle that the direction of the force *E* makes with *MT*, is equal to

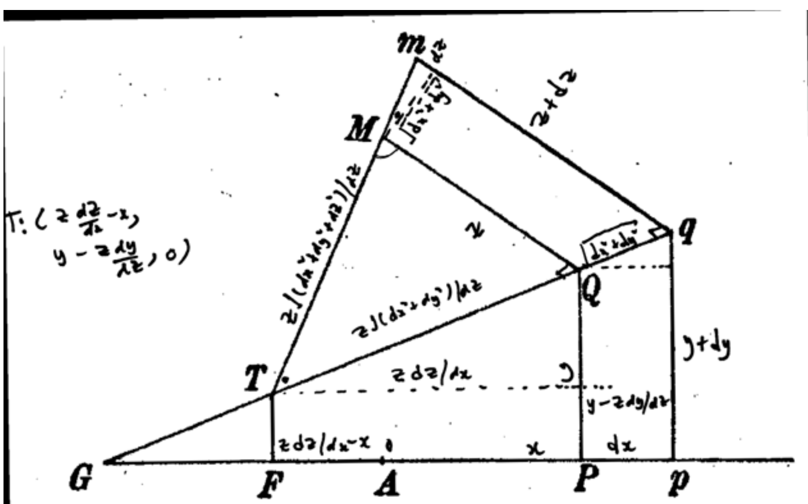


Fig. 92.

$$\frac{dx}{\sqrt{(dx^2+dy^2+dz^2)}}$$

and thus the tangential force arising from *E* is equal to

$$\frac{Edx}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

[Thus, the direction ratios of *Mm* are ( $dx, dy, dz$ ), and the components $E\cos\alpha, F\cos\beta, G\cos\gamma$ lies along this element, where $\cos\alpha = \frac{dx}{\sqrt{dx^2+dy^2+dz^2}}$, etc]

Now the normal pressing force is considered (Fig. 91), and the direction of this is along MN, with the equations

$$AH = x + Pz \text{ and } HN = -y - Qz$$

or

$$PH = Pz\ ,\quad QP + HN = -Qz.$$

From which it follows that

$$QN = z\sqrt{(P^2+Q^2)}\ ,\quad MN = z\sqrt{(1+P^2+Q^2)}.$$

Therefore the cosine of the angle that the direction of the force *G* makes with *MN* is

$$\frac{MQ}{MN} = \frac{1}{\sqrt{(1+P^2+Q^2)}}$$

and thus the normal force arising from *G* is equal to

$$\frac{G}{\sqrt{(1+P^2+Q^2)}}.$$

Again the cosine of the angle that the direction of the force *F*, which is parallel to *QP*, makes with *MN*, is equal to

$$\cos. PQN.\ \sin. QMN = \frac{(PQ+HN)QN}{QN\cdot MN} = -\frac{Q}{\sqrt{(1+P^2+Q^2)}}.$$

Hence the normal pressing force that has come from the force *F* is :

$$-\frac{FQ}{\sqrt{(1+P^2+Q^2)}}.$$

And in a like manner the normal pressing force arising from the force *E* (Fig. 94) is equal to :

$$-\frac{EP}{\sqrt{(1+P^2+Q^2)}}.$$

And finally, since the force of deflection is in the direction MG : as

$$PE = \frac{z(dy + Qdz)}{Qdx - Pdy} \text{ and } QP + EG = \frac{z(dx + Pdz)}{Qdx - Pdy},$$

then the cosine of the angle that *MG* makes with the direction of the force *G* , is equal to [p. 479] [*MQ/MG* in Fig. 94]:

$$\frac{Qdx - Pdy}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}};$$

whereby the deflecting force arising from the force *G* is equal to :

$$\frac{G(Qdx - Pdy)}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}.$$

Again the cosine of the angle that *MG* makes with the direction of the force *F,* is equal to:

$$\frac{PQ + EG}{MG} = \frac{dx + Pdz}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}};$$

[as *PG* + *EG* is the projection of the horizontal component *QG* on the y-axis] on account of which the deflecting force arising is equal to :

$$\frac{F(dx + Pdz)}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}.$$

And then the cosine of the angle that the direction of the force *E* makes with *MG* is equal to :

$$\frac{-PE}{MG} = \frac{-dy - Qdz}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}.$$

[-*PE* is the projection of *QG* on the *x*-axis; note that all the positive forces act along the negative directions of their axis. ]
Hence the deflecting force arising from the force *E* is equal to :

$$\frac{-E(dy + Qdz)}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}.$$

Moreover since previously we have called the tangential force *T*, the normal pressing force *M*, and the deflecting force *N*, we have reduced the three proposed forces *E*, *F*, and *G* to these; namely

$$T = \frac{Edx + Fdy + Gdz}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$

and

$$M = \frac{-EP - FQ + G}{\sqrt{(1 + P^2 + Q^2)}}$$

and

$$N = \frac{-E(dy + Qdz) + F(dx + Pdz) + G(Qdx - Pdy)}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}.$$

Q.E.I.

## Corollary 1.

**857.** Therefore if the body is acted on by three forces *E*, *F* and *G*, on putting *v* for the height corresponding to the speed at *M*, then

$$dv = -Edx - Fdy - Gdz$$

(849), if in place of *T* there is put the tangential force arising from the resolution of the forces *E*, *F* and *G*. [As this is the only force that does work, as we now understand the physics.]

## Corollary 2. [p. 480]

**858.** If in addition, the body is moving in a resistive medium and the resistance at *M* is equal to *R*, then (850)

$$dv = -Edx - Fdy - Gdz - R\sqrt{(dx^2 + dy^2 + dz^2)}.$$

## Corollary 3.

**859.** If, in the equation (851) found, in which the effect of the deflecting force *N* has been determined, in place of *N* the deflecting force arising from the resolution of the forces *E*, *F* and *G* is substituted, then there is produced :

$$\frac{2vddz(Pdy - Qdx) - 2vddy(dx + Pdz)}{dx^2 + dy^2 + dz^2}$$
$$= -E(dy + Qdz) + F(dx + Pdz) - G(Pdy - Qdx).$$

## Corollary 4.

**860.** Therefore if the two equations are solved together with the elimination of *v*, the equation is produced, which joined with the local equation for the surface $dz = Pdx + Qdy$, determines the path described on the surface by the body.

## Corollary 5.

**861.** Moreover the force by which the surface is pressed along its normal, both by the normal pressing force *M* as well as the centrifugal force that has arisen, is equal to:

$$\frac{(G - EP - FQ)(dx^2 + dy^2 + dz^2) + 2v(dPdx + dQdy)}{(dx^2 + dy^2 + dz^2)\sqrt{(1 + P^2 + Q^2)}}$$

(845), on substituting the value found in place of *M*.

## Corollary 6.

**862.** Now from the equation in corollary 3, it has been found that

$$2v = \frac{(dx^2 + dy^2 + dz^2)(E(dy + Qdz) - F(dx + Pdz) + G(Pdy - Qdx))}{-ddz(Pdy - Qdx) + ddy(dx + Pdz)},$$

in which with the value substituted there, the total pressing force is equal to [p. 481] :

$$\frac{(Gdxddy - Fdxddz + Edyddz - Edzddy)\sqrt{(1 + P^2 + Q^2)}}{ddz(Pdy - Qdx) - ddy(dx + Pdz)}.$$

## Scholium.

**863.** Since the three forces *E, F, G* have directions normal to each other in turn, the equivalent force from these is equal to $\sqrt{(E^2 + F^2 + G^2)}$. Now we have found the three forces *M*, *N* and *T* to be equivalent to these three forces, the directions of which also are normal to each other in turn ; whereby from these three, the equivalent force is equal to $\sqrt{(M^2 + N^2 + T^2)}$. On account of which, if in place of *M*, *N* and *T* the values found from *E*, *F* and *G* are substituted, there must be produced $\sqrt{(E^2 + F^2 + G^2)}$; which is the thing to have checked in setting up a calculation. Moreover this serves the part of a criterion, however involved the calculation, by completely resolving the question that the solution has been rightly or wrongly set up. Now from this criterion these formulae are found to be correctly established.

## PROPOSITION 96.

## Problem.

**864.** *According to the hypothesis of gravity g acting downwards uniformly, to determine the line that a body projected on some surface in vacuo describes.*

### Solution. [p. 482]

Let *APQ* (Fig. 92) be a horizontal plane and the point *M* is given both on the surface, and on the line described by the body. Hence *MQ* is vertical and therefore in the direction of the force of gravity *g*. On putting $AP = x$, $PQ = y$ and $QM = z$ and with the equation expressing the nature of the surface $dz = Pdx + Qdy$ let the speed at *M*, in which the element *Mm* is traversed, correspond to the height *v*. Therefore as this problem is a case of the preceding, for it becomes $G = g$, $E = 0$ and $F = 0$, these two equations are obtained : $dv = -gdz$ (857) and

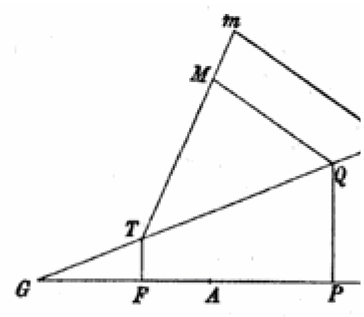

Fig. 92.

$$2vddz(Pdy - Qdx) - 2vddy(dx + Pdz) + g(Pdy - Qdx)(dx^2 + dy^2 + dz^2) = 0$$

(859). Again let a height equal to *b* correspond to the speed that the body is to have, if it arrives in the horizontal plane *APQ*; then $v = b - gz$. Now through the other equation :

$$2v = \frac{g(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}{ddy(dx + Pdz) - ddz(Pdy - Qdx)}.$$

Hence it becomes :

$$\frac{dv}{2v} = \frac{dzddz(Pdy - Qdx) - dzddy(dx + Pdz)}{(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}.$$

Which equation with the help of the equation $dz = Pdx + Qdy$ is changed into this :

$$\frac{dv}{2v} = \frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} - \frac{Pddy}{Pdy - Qdx};$$

which integrated gives :

$$lv = l(dx^2 + dy^2 + dz^2) - 2\int \frac{Pddy}{Pdy - Qdx}.$$

Therefore in which a special case has to be investigated, or

$$\frac{Pddy}{Pdy - Qdx}$$

can be integrated. If that happens, then $v$ is obtained by differentials of the first degree, and since it is $v = b - gz$, an equation of the differential of the first degree arises expressing the nature of the described curve. Now the pressing force on the surface along the normal is equal to :

$$\frac{gdx\,ddy\sqrt{(1+P^2+Q^2)}}{ddz(Pdy - Qdx) - ddy(dx + Pdz)},$$

which with the differentials of the second order removed, (861) changes into this :

$$\frac{-g}{\sqrt{(1+P^2+Q^2)}} - \frac{2(b-gz)(dPdx + dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}.$$

Q.E.I.

## Corollary 1. [p. 483]

**865.** Hence the speed of any motion the body according to the hypothesis of uniform gravity $g$ directed downwards can be known on some surface can become known from the height only, as if everywhere the body is moving in the same plane.

## Corollary 2.

**866.** If the minute time, in which the element $Mm$ is completed, is put as $dt$, then

$$dt = \frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}.$$

Hence by the equation found, we have:

$$ldt = \int \frac{Pddy}{Pdy - Qdx} \text{ and } \frac{ddt}{dt} = \frac{Pddy}{Pdy - Qdx}.$$

## Corollary 3.

**867.** From these equations uncovered, it is found that :

$$ddy = \frac{g(Pdy - Qdx)(dx^2+dy^2+dz^2)}{2dx(b-gz)(1+P^2+Q^2)} + \frac{(dPdx + dQdy)(Pdy - Qdx)}{(1+P^2+Q^2)dx}$$

and

$$ddz = \frac{gQ(Pdy - Qdx)(dx^2+dy^2+dz^2)}{2dx(b-gz)(1+P^2+Q^2)} + \frac{(dPdx + dQdy)(dx + Pdz)}{(1+P^2+Q^2)dx}.$$

Hence

$$dzddy - dyddz = \frac{gP(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}{2(b - gz)(1 + P^2 + Q^2)} - \frac{(dPdx + dQdy)(dy + Qdz)}{1 + P^2 + Q^2}.$$

Now the radius of osculation of this described curve is equal to :

$$\frac{2(dx^2 + dy^2 + dz^2)(b - gz)\sqrt{(1 + P^2 + Q^2)}}{\sqrt{(g^2(Pdy - Qdx)^2(dx^2 + dy^2 + dz^2) + 4(b - gz)^2(dPdx + dQdy)^2)}}.$$

## Scholium.

**868.** We will explain more broadly in the following problems the use of these formulas in particular cases, in which a certain kind of surfaces is considered, to which we add examples of individual surfaces. [p. 484]

## PROPOSITION 97.

## Problem.

**869.** *According to the hypothesis of uniform gravity g acting downwards, to determine the motion of a body on the surface of cylinder of any kind, the axis of which is vertical.*

## Solution.

Because the axis of the cylinder is put as vertical, all the sections are equal to each other; therefore let *ABQC* (Fig. 95) be the base of the cylinder, on the surface of which the body is moving. Putting $AP = x$, $PQ = y$ and let $z$ be the height of the body at the point $Q$ on the surface of the cylinder. Hence the nature of this surface of the cylinder is expressed by this equation: $0 = Pdx + Qdy$ or $Qdy = -Pdx$. Moreover this equation arises from the general equation $dz = Pdx + Qdy$, if $P$ and $Q$ are made infinitely large quantities, or with the coefficient of $dz$ vanishing, if we may assume this. On account of which in the equations found before, $P$ and $Q$ must be considered as infinitely large quantities, even if they are quantities of finite magnitude [i. e. relative to each other; recall that $P = \frac{\partial z}{\partial x}$ and $Q = \frac{\partial z}{\partial y}$, here with a negative sign.] Moreover $P$ and $Q$ are functions of $x$ et $y$ only, and nor is $z$ is present in these. Therefore from these there is obtained from the deductions in the calculation :

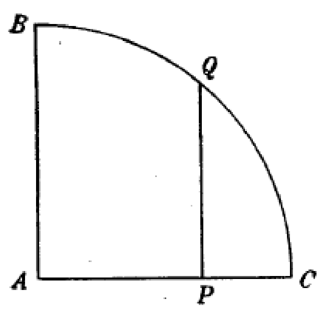

Fig. 95.

$$v = b - gz$$

and

$$2v = \frac{g(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}{Pdzddy - Pdyddz + Qdxddz} = \frac{g(dx^2 + dy^2)(dx^2 + dy^2 + dz^2)}{dydzddy - dy^2ddz - dx^2ddz}$$

on account of the ratio $P : Q = dy : -dx.$ But the logarithm of the second equation is : [p. 485]

$$lv = l(dx^2 + dy^2 + dz^2) - 2\int \frac{dyddy}{dx^2 + dy^2} = l(dx^2 + dy^2 + dz^2) - l(dx^2 + dy^2) + lc.$$

Thus the equation becomes :

$$\frac{v}{c} = \frac{dx^2 + dy^2 + dz^2}{dx^2 + dy^2} = \frac{b - gz}{c} = 1 + \frac{dz^2}{dx^2 + dy^2}.$$

Hence there arises :

$$(b - c - gz)(dx^2 + dy^2) = cdz^2$$

or

$$\sqrt{(dx^2 + dy^2)} = \frac{dz\sqrt{c}}{\sqrt{(b - c - gz)}},$$

and the integral of this is [corrected in the *O. O* edition]:

$$\int \sqrt{(dx^2 + dy^2)} = -\frac{2\sqrt{c(b - c - gz)}}{g} \quad ,$$

where $\int\sqrt{(dx^2 + dy^2)}$ denotes the arc of the base *BQC* traversed in the horizontal motion. If the increment of the time, in which the element *Mm* is completed, is put as *dt*, then

$$\frac{ddt}{dt} = \frac{dyddy}{dx^2 + dy^2} \text{ and } \alpha dt = \sqrt{(dx^2 + dy^2)}$$

[From the equation $\frac{v}{c} = \frac{dx^2+dy^2+dz^2}{dx^2+dy^2}$ it follows that the constant $\alpha = \sqrt{c}$ . Note by Paul St. in the *O. O*.]
and again,

$$\alpha t = \int \sqrt{(dx^2 + dy^2)}.$$

Whereby the times are in proportion to the corresponding arcs in the base
Moreover the equation

$$\int \sqrt{(dx^2 + dy^2)} = -\frac{2\sqrt{c(b - c - gz)}}{g}$$

gives the equation for the curve described on the surface of the cylinder, if this surface is considered to be set out in a plane; for then $\int\sqrt{(dx^2 + dy^2)}$ denotes the abscissa on the horizontal axis and *z* the applied vertical line. Now we have the projection of the curve described in the vertical plane in which we have *AC* cutting the horizontal plane, if with the help of the equation $y$ is eliminated, in order that an equation is produced between $x$ and $z$, which are the coordinates of this projection. Clearly since $dy = -\frac{Pdx}{Q}$ then

$$\sqrt{(dx^2 + dy^2)} = \frac{dx}{Q}\sqrt{(P^2 + Q^2)}$$

and thus

$$\int \frac{dx\sqrt{(P^2 + Q^2)}}{Q} = -\frac{2\sqrt{c(b - c - gz)}}{g},$$

where this value of $x$ must be substituted into $P$ and $Q$ in place of $y$.

Now the pressing force that the surface sustains, arises only from the centrifugal force on account of the force $g$ being placed in the surface itself, and is equal to :

$$\frac{2(b - gz)(dPdx + dQdy)}{(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2)}},$$

by which force the body is trying to recede from the axis of the cylinder, if this expression is positive. Q.E.I. [p. 486]

[There was at the time, and even now, unfortunately, the notion held by some people that bodies try to flee from the centre when involved in circular motion, and that seems to have originated with Huygens in his *Horologium.* The reader may wish to compare Euler's early solutions with more modern solution to these and further problems, such as presented by Whittaker in his *Analytical Dynamics*, in Ch. 4, *The Soluble Problems of Particle Dynamics.* ]

## Corollary 1.

**870.** Therefore the curve, that the body describes on the surface of the cylinder if the surface is set out as a plane, changes into a parabola, clearly that trajectory that a body describes in the vertical plane.

## Corollary 2.

**871.** If the motion of the body on the surface of the cylinder is considered to be composed from a vertical motion, as it progresses either up or down, and from the horizontal, then the horizontal motion is uniform, since the times $t$ are proportional to $\int\sqrt{(dx^2 + dy^2)}$ , *i. e.* to the arcs traversed in the horizontal motion. Now the vertical motion is either uniformly accelerated or decelerated.

## Corollary 3.

**872.** Hence if the horizontal motion vanishes, then the body either ascends or descends along a straight line, and generally if the body can freely ascend or descend. And this case is produced if $c = 0$, when $\int\sqrt{(dx^2 + dy^2)}$ vanishes.

## **Example.** [p. 487]

**873.** Let the base of the cylinder be a circle, the quadrant of which is *BQC* and the radius $AB = a$; then the equation is $x^2 + y^2 = a^2$ and $xdx + ydy = 0$. Hence the equations become $P = x$ and $Q = y = \sqrt{(a^2 - x^2)}$ . Now the projection of the line described by this on the surface of the cylinder in the vertical plane erected from *AC,* is expressed by the equation :

$$\int \frac{adx}{\sqrt{(a^2 - x^2)}} = -\frac{2\sqrt{c(b-c-gz)}}{g}.$$

Therefore if we set $c = 0$, then $dx = 0$ and $x$ is constant; whereby in this case the projection is a straight line. Now the curve, which is the projection for whatever value of $c$, is constructed with the help of the rectification of the circle. Moreover the pressing force, that the surface of the cylinder sustains, is equal to

$$\frac{2(b-gz)(dx^2+dy^2)}{(dx^2+dy^2+dz^2)a} = \frac{2c}{a}$$

on account of the equation

$$\frac{dx^2+dy^2+dz^2}{dx^2+dy^2} = \frac{b-gz}{c}.$$

Whereby the pressing force is constant everywhere and proportional to $c$.

## **Corollary 4.**

**874.** On account of the same equation the pressing force generally that any cylinder sustains,

$$\frac{2c(dPdx+dQdy)}{(dx^2+dy^2)\sqrt{(P^2+Q^2)}} = \frac{2Q^2c(dPdx+dQdy)}{dx^2(P^2+Q^2)^{\frac{3}{2}}} = \frac{2Qc(QdP-PdQ)}{dx(P^2+Q^2)^{\frac{3}{2}}}.$$

## **Scholium.** [p. 488]

**875.** But not only can the motion of a body on the surface of an erect cylinder be easily determined with the help of resolution, but also, if the axis of the cylinder should be horizontal, with the same ease the motion of the body becomes known. And indeed if the body does not have a horizontal motion along the cylinder, then the body perpetually remains on the same section of the cylinder as that is moving on a given line. But if now the motion is agreed to be horizontal, this remains the same always and does not disturb the other motion; and from these motions taken together the motion of the body truly becomes known easily.

# PROPOSITION 98.

## Problem.

**876.** *If a body is moving on the surface of a solid of revolution, of which the axis is the vertical line AL* (Fig. 96)*, in vacuo with gravity g acting uniformly, to determine the motion of the body on a surface of this kind.*

## Solution.

The solid of revolution is generated by the rotation of the curve *AM* about the vertical axis *AL*; all the sections of this are horizontal circles, of which the radii are applied lines of the curve *AM*. Therefore the equation expressing the nature of this surface is $dz = \frac{xdx+ydy}{Z}$, with *Z* some function *z*; indeed it is given by :

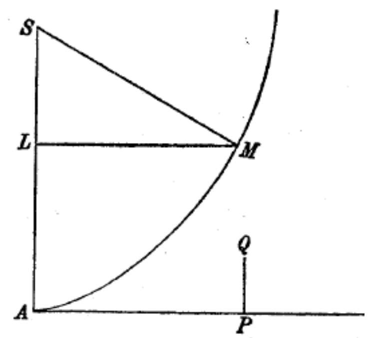

Fig. 96.

$$2\int Zdz = x^2 + y^2 = LM^2 .$$

Therefore if the equation is given between $AL = z$ and $LM = \sqrt{2\int Zdz}$ for the curve *AM*, then *Z* is also given. [p. 489] With these put in place, therefore $P = \frac{x}{Z}$ and $Q = \frac{y}{Z}$; from which values if substituted the two following equations are obtained, from which the described curve as well as the motion on that surface become known :

$$v = b - gz$$

and

$$lv = l(dx^2 + dy^2 + dz^2) - 2\int \frac{xddy}{xdy - ydx} = l(dx^2 + dy^2 + dz^2) - 2l(xdy - ydx) + \text{const.}$$

Whereby this becomes :

$$v = \frac{c^3(dx^2 + dy^2 + dz^2)}{(xdy - ydx)^2} = b - gz.$$

Putting $x^2 + y^2 = u^2$; the function *u* is a certain function of *z*, clearly

$$u^2 = 2\int Zdz,$$

and the above equation changes into this :

$$\sqrt{(dx^2 + dy^2)} = \frac{\sqrt{(c^3dz^2 + u^2du^2(b - gz))}}{\sqrt{(u^2(b - gz) - c^3)}} .$$

Now the projection in the horizontal plane is obtained through the equation between *x* and *y*, if from the equation $x^2 + y^2 = 2\int Zdz$ the value of *z* is substituted in terms of *x* and $y$ ; and the arc of this projection is $\int\sqrt{(dx^2+dy^2)}$ . Moreover in the vertical plane the projection is obtained by eliminating $y$; with which done the equation is produced :

$$\frac{udx-xdu}{c\sqrt{c(u^2-x^2)}}=\sqrt{\frac{du^2+dz^2}{u^2(b-gz)-c^3}},$$

which equation, if it is divided by $u$, allows the construction. Now the pressing force, that the surface sustains towards the axis, is equal to :

$$\frac{-gZ}{\sqrt{(x^2+y^2+Z^2)}}-\frac{2c^3Z(dx^2+dy^2)-2c^3dZ(xdx+ydy)}{Z(xdy-ydx)^2\sqrt{(x^2+y^2+Z^2)}}.$$

Q.E.I.

## Corollary1.

**877.** If the increment of the time in which the element *Mm* is completed, is put as $dt$, then

$$\frac{ddt}{dt}=\frac{xddy}{xdy-ydx},$$

and the integral of this is :

$$\alpha dt=xdy-ydx.$$

[From equation (876) $v=\frac{c3(dx^2+dy^2+dz^2)}{(xdy-ydx)^2}$ it follows that the constant $\alpha=c\sqrt{c}$ . Note in the *O.O.*]
Therefore the time becomes as :

$$xy-2\int ydx$$

with $\int ydx$ denoting the area of the projection in the horizontal plane.

## Corollary 2. [p. 490]

**878.** If the body is considered to be moving in the projection in the horizontal plane, then the speed of this at $Q$ corresponds to the height

$$\frac{c^3(dx^2+dy^2)}{(ydx-xdy)^2};$$

from which motion in the projection the motion on the surface itself can be found.

## Corollary 3.

**879.** Therefore let *BQC* (Fig. 97) be the projection of the curve in the horizontal plane, in which the body is moving, thus so that the motion of this corresponds to the motion of the body on the surface itself; the time in which the arc *BQ* is completed, is as $\frac{xy}{2} - \int ydx$ ,or with the negative as $\int ydx - \frac{xy}{2}$, *i. e.* as the area BAQ drawn by the radius AQ.

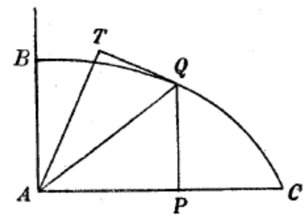

Fig. 97.

## Corollary 4.

**880.** Moreover the element of the area *BAQ* is $\frac{ydx - xdy}{2}$. Therefore with the tangent *QT* drawn, and on sending the perpendicular *AT* to that from *A*, then

$$AT = \frac{ydx - xdy}{\sqrt{(dx^2 + dy^2)}}.$$

Whereby the height corresponding to the speed at $Q$ is equal to $\frac{c^3}{AT^2} = \frac{c^3}{p^2}$ on putting $AT = p$.

## Corollary 5.

**881.** Therefore the body moving in the projection likewise is moving on that surface, and if it is moving freely attracted by some centripetal force to the centre *A* (Book I. (587)).

## Corollary 6. [p. 491]

**882.** The point *B* corresponds to the start of the motion made on the surface, and since the direction for the first motion on the surface is given, the perpendicular to the tangent at B is given. Therefore let $AB = f$ and the perpendicular to the tangent is equal to $h$ ; then the height corresponding to the speed at *B* is equal to $\frac{c^3}{h^2}$.

## Corollary 7.

**883.** Therefore the centripetal force tending towards *A*, which acts, as the body in the projection *BQC* is moving freely, is equal to $\frac{2c^3dp}{p^3du}$ with *u* put in place fore $\sqrt{(x^2 + y^2)}$.
Now the equation between *u* and *z* expresses the nature of the curve, and the rotation of this curve thus has given rise to the proposed surface, and is therefore given.

## Corollary 8.

**884.** Again it is the case that

$$\sqrt{(dx^2 + dy^2)} = \frac{udu}{\sqrt{(u^2 - p^2)}} \text{ and } ydx - xdy = \frac{pudu}{\sqrt{(u^2 - p^2)}}.$$

These values, if they are substituted in the equation found, give

$$c^3u^2du^2 + c^3u^2dz^2 - c^3p^2dz^2 = (b - gz)p^2u^2du^2$$

or

$$p^2 = \frac{c^3u^2(du^2 + dz^2)}{c^3dz^2 + u^2du^2(b - gz)}.$$

## Corollary 9.

**885.** Therefore this quantity, with the differential of this

$$-\frac{c^3dz^2 + u^2du^2(b - gz)}{u^2(du^2 + dz^2)}$$

divided by *du*, gives the centripetal force required at *A*, as the body in the projection *BQC* is moving freely, in a motion corresponding to the motion of the body on the surface.

## Corollary 10.

**886.** If $c = 0$, also making $p = 0$. Whereby in this case the projection is in the horizontal plane with a straight line passing through *A*, upon which the body thus approaching *A* is thus attracted, as it is the centripetal force

$$\frac{1}{du}\, d.\, \frac{-du^2(b - gz)}{du^2 + dz^2}.$$

## Corollary 11. [p. 492]

**887.** If the direction of the body is first horizontal, the tangent at *B* is normal to *AB* and thus $h = f$. Now in this case the speed on the surface is equal to the speed in the projection ; whereby if *i* is the value of *z*, and if $u = f$, then $b - gi = \frac{c^3}{f^2}$.

## Corollary 12.

**888.** If besides the centripetal force at *B* is equal to the centrifugal force, then the curve *BQC* is a circle and therefore so also the curve described on the surface, and both the motion on the surface as well as that in the projection are equal. Let it be the case, when $u = f$ and $z = i$, that $dz = mdu$ ; as $p = f$ and $u = f$ and $z = i$ in the case in which a circle is described, then $2c^3 = gmf^3$. [See (892)]

## Corollary 13.

**889.** If there is put in place $\pi : 1$ as the ratio of the periphery to the diameter, then the perimeter of our circle is equal to $2\pi f$ , which divided by the speed $\sqrt{\frac{c^3}{f^2}}$ , *i. e.* $\sqrt{\frac{gmf}{2}}$ , gives the time of one period in the circle, that hence becomes equal to $\frac{2\pi\sqrt{2f}}{\sqrt{gm}}$. Hence the pendulum completing whole isochronous oscillations with this period is equal to $\frac{f}{m}$ in the same hypothesis of gravity.

## Corollary 14. [p. 493]

**890.** Therefore on the surface which is generated by the rotation of the curve *AM* (Fig. 96) about the vertical axis *AL*, a projected body can describe a circle of radius *LM* in the same time that a pendulum of length equal to the subnormal *LS* can complete a whole oscillation.

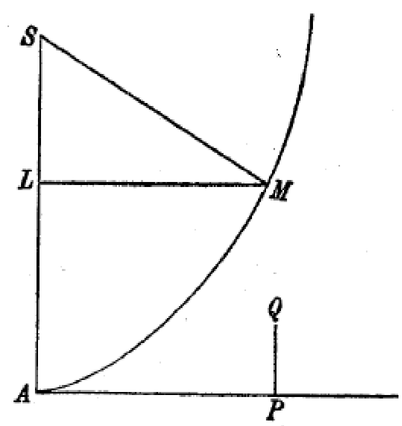

Fig. 96.

## Corollary 15.

**891.** Therefore if the curve *AM* is a parabola, all the periods along horizontal circles in the parabolic conoid are completed in equal times; and pendulums in the same times by performing whole oscillations with a length equal to half the parameter.

## Scholium 1.

**892.** Whatever curve *AM* is assumed, if the centripetal force motion in the projection towards *A* is defined from a given formula and that is put equal to the centrifugal force and joined together with the equation $b - gi = \frac{c^3}{f^2}$, there is produced $2c^3 = gmf^3$; for in this case as the projection is a circle as is the curve described on the surface. Now so that this can be more apparent, there is put $dz = qdu$, and the (884) centripetal force comes about equal to

$$\frac{2q\,dq(bu^2 - gzu^2 - c^3)}{u^2du(1+q^2)^2} + \frac{2c^3q^2 + gqu^3}{u^3(1+q^2)}.$$

Now putting $u = f$, $z = i$ and $bf^2 = gf^2i + c^3$ and $q = m$; the centripetal force is equal to :

$$\frac{2c^3m^2 + gmf^3}{f^3(1+m^2)};$$

which put equal to the centrifugal force $\frac{2c^3}{f^2}$, gives $2c^3 = gmf^3$.

## Example. [p. 494]

**893.** Let the surface be a circular cone or *AM* (Fig. 96 and Fig. 97) a right line inclined at some angle to the axis *AL*; whereby then $z = mu$ and $dz = mdu$. Therefore the centripetal force tending towards *A* which is produced, in order that the body projected in *BQC* is moving freely, is equal to

$$\frac{2c^3m^2 + gmu^3}{u^3(1+m^2)}.$$

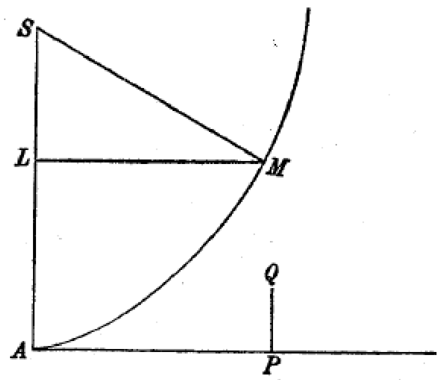

Fig. 96.

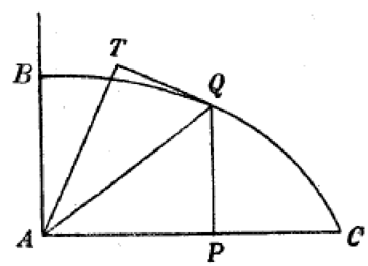

Fig. 97.

Therefore this force is composed from a constant force and from a force that varies inversely as the cube of the distance from the centre *A*. If $c = 0$, then the projection is a right line passing through *A* and the centripetal force is equal to $\frac{gm}{1+m^2}$, or is constant. Therefore the body approaches *A* with a uniformly accelerated motion; now the motion on the surface of the cone agrees with the descent or the ascent on a right line inclined equally and the acceleration is the same. But if the projection is curvilinear and the tangent at *B* is normal to *AB*, then

$i = mf$ and $b - gmf = \frac{c^3}{f^2}$ on putting $AB = f$. Therefore in this case we have

$b = \frac{c^3 + gmf^3}{f^2}$, and if the body is revolving in a horizontal circle, it becomes above:

$2c^3 = gmf^3$. Hence it comes about that the speed of the body corresponds to the height $\frac{c^3}{f^2} = \frac{gmf}{2}$. And the periods completed in the same times in this circle, in which

pendulums with lengths $\frac{f}{m}$ complete whole oscillations. But if it is not the case that

$2c^3 = gmf^3$, now still $b = \frac{c^3 + gmf^3}{f^2}$, the curve BQC has a certain tangent at B normal to AB, but this projection is not a circle. But if $2c^3$ is almost equal to $gmf^3$, then the curve does not depart much from a circle ; but it has various apses, at which the tangent is perpendicular to the radius. Now the positions of these apses has been determined by proposition 91 of the preceding book. For since the centripetal force is

$$\frac{2c^3m^2 + gmu^3}{u^3(1+m^2)},$$

if this is compared with that centripetal force $\frac{P}{y^3}$, [p. 495] on account of $y = u$ then

$$P = \frac{2c^3m^2 + gmu^3}{1+m^2}, \quad \frac{dP}{P} = \frac{3gmu^2du}{2c^3m^2 + gmu^3}, \text{ and } \frac{Pdy}{ydP} = \frac{2mc^3 + gu^3}{3gu^3};$$

and thus on putting $u = f$, the line of the apses also is distant from the preceding apse by the angle

$$180\sqrt{\frac{2mc^3 + gf^3}{3gf^3}} \text{ degrees.}$$

But since $2c^3$ is approximately equal to $gmf^3$, the angle intercepted between the two apses is equal to $180\sqrt{\frac{m^2+1}{3}}$ degrees. Hence as $\frac{m^2+1}{3}$ is always greater than $\frac{1}{3}$, then the angle intercepted is greater than $103^0\ 55'$.

## Scholium 2.

**894.** I do not include here the example of the surface of the sphere, but I resolve the motion on that boundary in the following proposition, since this matter is worthy of being handled most carefully. For if a pendulum is not set in motion following a vertical plane, then the body moves on the surface of a sphere and the body describes a circle or another not less elegant curves, as becomes known from any experiment set up. Indeed the case, in which a pendulum completes a circle, has now been published in the Acta Erud. Lips. A. 1715, [p.242] by the most celebrated Johann Bernoulli under the title *De centro turbinationis inventa nova*. [*Opera omnia*, Tom. 2, p. 187]. But if the curve is not circular, no one as far as I know, has consider either the motion of this pendulum nor determined it. [Isaac Newton had considered an analogous problem in the *Principia*, London, 1687, Book. I sect. X Prop. LV and LVI, in connection with the apses of the moon, where gravity follows the inverse square law and therefore is not constant; see Cohen's translation and commentary.]

## Definition 6. [p. 496]

**895.** *The rotational* [or whirling] *motion of a pendulum is the name given to the motion imparted to a pendulum which is not in a vertical plane.* Therefore in this case the pendulum is not moving in the same vertical plane, but describes some curve on the surface of the sphere of which the radius is the length of the pendulum in place. [This is now called the spherical pendulum.]

## PROPOSITION 99.

## Problem.

**896.** *Pendulums are set in motion in rotational motion ; to determine the motion and the curved line described on the surface of a sphere.*

## Solution.

Since the moving body is bound by the pendulum, it moves on the surface of a sphere of which the radius is the length of the pendulum. Let this length or the radius of the sphere be equal to $a$ ; from the nature of the circle $AM$ (Fig. 96 and Fig. 97) it follows that $z = a - \sqrt{(a^2 - u^2)}$ . Hence we have

$$q = \frac{u}{\sqrt{(a^2 - u^2)}} \text{ and } dq = \frac{a^2 du}{(a^2 - u^2)^{\frac{3}{2}}}$$

and thus

$$\frac{2q\,dq}{u^2 du} = \frac{2a^2}{u(a^2 - u^2)^2}.$$

From these is found the centripetal force attracting towards $A$, which put in place, in order that the body can move freely in the projection $BQC$ , is equal to

$$\frac{2bu - 2gau + 3gu\sqrt{(a^2 - u^2)}}{a^2}.$$

And this equation is obtained for the curve $BQC$ :

$$p^2 = \frac{a^2 c^3}{c^3 + (b - ga)(a^2 - u^2) + g(a^2 - u^2)^{\frac{3}{2}}},$$

which is sufficient for the construction of the projection $BQC$. Let the tangent at $B$ be perpendicular to the radius $AB$, which must always happen somewhere, unless the projection is a straight line, since the centripetal force decreases with decreasing $u$. Putting $AB = f$; then $i = a - \sqrt{(a^2 - f^2)}$ and [p. 497]

$$b - ga + g\sqrt{(a^2 - f^2)} = \frac{c^3}{f^2}.$$

If besides it should be that

$$2c^3 = \frac{gf^4}{\sqrt{(a^2 - f^2)}},$$

then the body is rotating in a circle, the radius of which is $f$, with a speed corresponding to the height

$$\frac{c^3}{f^2} = \frac{gf^2}{2\sqrt{(a^2 - f^2)}}.$$

Now the length of the pendulum completing whole oscillations in the same time is equal to $\sqrt{(a^2 - f^2)}$. But if it is not the case that $2c^3 = \frac{gf^4}{\sqrt{(a^2 - f^2)}}$, and now the difference is very small, then the curve BQC does not depart much from the circle. In order that the positions of cusps of this curve can be found, just as in Proposition 91 of the preceding book :

$$y = u \text{ and } P = \frac{u^4}{a^2}\left(2b - 2ga + 3g\sqrt{(a^2 - u^2)}\right).$$

Hence this becomes :

$$\frac{dP}{P} = \frac{4du}{u} - \frac{3gudu}{(2b - 2ga)\sqrt{(a^2 - u^2)} + 3g(a^2 - u^2)}$$

and

$$\frac{ydP}{Pdy} = 4 - \frac{3gu^2}{(2b - 2ga)\sqrt{(a^2 - u^2)} + 3g(a^2 - u^2)}.$$

Since the curve is nearly a circle, put $u = f$ and

$$2b - 2ga = \frac{gf^2}{\sqrt{(a^2 - f^2)}} - 2g\sqrt{(a^2 - f^2)} = \frac{3gf^2 - 2ga^2}{\sqrt{(a^2 - f^2)}},$$

with which done the is produced :

$$\frac{ydP}{Pdy} = 4 - \frac{3f^2}{a^2}.$$

Hence it follows that the interval between the apses is the angle :

$$\frac{180a}{\sqrt{(4a^2 - 3f^2)}} \text{ degrees.}$$

Clearly the position for an angle of such a size, at which the pendulum is at a maximum distance from the axis, is placed at intervals with the position at which the pendulum is closest to the axis. Q.E.I.

## Corollary 1.

**897.** Therefore in order that the pendulum $AB = a$ (Fig. 98) describes the circle *BCDE* in a rotational motion, it is necessary that the speed corresponds to the height $\frac{g.BO^2}{2.AO}$.

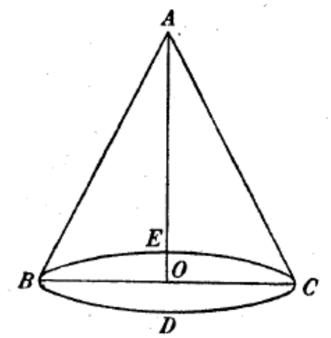

Fig. 98.

## Corollary 2.

**898.** Now the length of the pendulum, that completes the smallest whole oscillations in the same time, in which the period is completed in the circle *BDCE*, is equal to *AO*.

## Corollary 3. [p. 498]

**899.** Therefore the times, in which different circles are traversed by the rotating pendulum *AB*, are in the square root ratio of the heights *AO*.

## Corollary 4.

**900.** Therefore since the pendulum of length *a* makes a maximum horizontal circle, the radius of which is *a*, an infinitely great speed is required and each period is completed in an infinitely short time.

## Corollary 5.

**901.** If the radius of the circle *BO* should be very small with respect to the pendulum $AB = a$, the periods of the rotational motion are in agreement with the whole oscillations of the same pendulum.

## Corollary 6.

**902.** If the curve described by the pendulum should not be a circle, but a close figure and *BO* is very small, then the angle between two apses is $90^0$ or a right angle.

## Corollary 7.

**903.** Now in this case the curve described by the body is an ellipse having the centre at *A*. Which can be gathered from the centripetal force, which then becomes proportional to the distances.

## Corollary 8. [p. 499]

**904.** Moreover since the larger the radius *BO* becomes, from that also the greater becomes the angle intercepted between the two apses. And on making *BO* = *BA*, the angle here is $180^0$.

## Corollary 9.

**905.** If the angle *BAO* is 30 degrees, then $BO = \frac{1}{2}BA$ or $f = \frac{1}{2}a$. Therefore the angle intercepted between the two apses is $\frac{360}{\sqrt{13}}$ degrees or $99^0$ 50'. Hence this figure *abcdefghik*, etc is of the projection in the horizontal plane (Fig. 99), in which the highest apses are at *a, c, e, g, i* and the deepest at *b, d, f, h, k.*

## Corollary 10.

**906.** Therefore in this figure the line of the apses is moving in succession; for each period progresses by almost $39^0$ in succession.

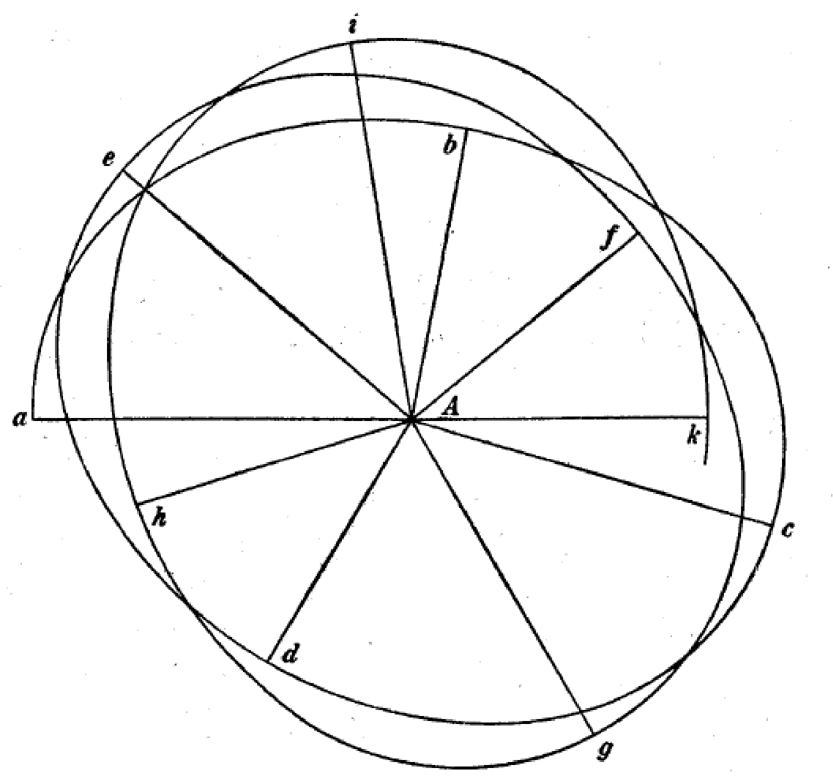

Fig. 99.

## Corollary 11. [p. 500]

**907.** But if that angle *BAO* should be less than 30 degrees, then the progression of the apses is less also. For whatever the angle *BAO* which is known at some point, I resolve the fraction

$$\frac{a}{\sqrt{(4a^2-3f^2)}}$$

in a series, which is the following :

$$\frac{1}{2}+\frac{3f^2}{16a^2}+\frac{27f^4}{256a^4}+\text{etc.}$$

Hence in one period the line of the apses is moved forwards by the angle $\frac{135f^2}{a^2}$ degrees as an approximation, if $f$ is very small.

## Corollary 12.

**908.** From these it is apparent that the movement of the line of the apses in individual periods is almost in the square ratio of the sines of the angle *BAO*.

# CAPUT QUARTUM

## DE MOTU PUNCTI SUPER DATA SUPERFICIE

[p. 477]

## PROPOSITIO 95.

### Problema.

**856.** *Si corpus super superficie motum a quotcunque potentiis sollicitetur, definire vires normales, prementem scilicet et deflectentem, atque vim tangentialem ex resolutione omnium ortam.*

### Solutio.

Quaecunque fuerint potentiae sollicitantes, eae ad tres reduci possunt, quarum directiones sint secundum tres coordinatas $x$, $y$, $z$. Sit nunc vis corpus in $M$ (Fig. 92) secundum parallelam abscissae $PA$ trahens = $E$, vis secundum parallelam ipsi $QP$ trahens = $F$ et vis secundum $MQ$ trahens = $G$. Singulae ergo hae vires resolvendae sunt in ternas, normalem prementem scilicet, normalem deflectentem et tangentialem. Quia autem hae tres directiones sunt inter se normales, ex quaque ipsarum $E$, $F$ et $G$ vires normales et tangentiales prodibunt, si illae ducantur in cosinum anguli, quem illarum virium directiones cum istis constituunt.

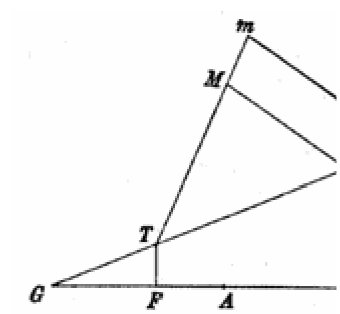

Fig. 92.

Incipiamus a vi tangentiali, cuius directio est $MT$, existente

$$AF = \frac{z\,dx}{dz} - x \quad \text{et} \quad FT = y - \frac{z\,dy}{dz}$$

atque

$$QT = \frac{z\sqrt{(dx^2 + dy^2)}}{dz} \quad \text{et} \quad MT = \frac{z\sqrt{(dx^2 + dy^2 + dz^2)}}{dz}.$$

Unde erit cosinus anguli $QMT$, quem directio vis $G$ cum vi tangentiali constituit, [p. 478] =

$$\frac{QM}{MT} = \frac{dz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

In quem si ducatur vis *G*, prodibit vis tangentialis ex ea orta =

$$\frac{Gdz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Cosinus vero anguli, quem *MT* constituit cum directione vis *F*, quae est ipsi *QP* parallela, est =

$$\cos. PQT. \sin. QMT = \frac{PQ - FT}{QT} \cdot \frac{QT}{MT} = \frac{dy}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Vis ergo tantentialis ex F orta est =

$$\frac{Fdy}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Cosinus porro anguli, quem directio vis *E* constituit cum *MIT*, est =

$$\frac{dx}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$

ideoque vis tangentialis ex vi *E* orta =

$$\frac{Edx}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Iam consideretur (Fig. 91) vis premens, cuius directio est MN, existente

$$AH = x + Pz \text{ et } HN = -y - Qz$$

seu

$$PH = Pz \text{ et } QP + HN = -Qz.$$

Ex quo erit

$$QN = z\sqrt{(P^2 + Q^2)} \text{ et } MN = z\sqrt{(1 + P^2 + Q^2)}.$$

Anguli ergo, quem directio potentiae *G* cum *MN* constituit, cosinus est

$$\frac{MQ}{MN} = \frac{1}{\sqrt{(1+P^2+Q^2)}}$$

ideoque vis premens ex *G* orta =

$$\frac{G}{\sqrt{(1+P^2+Q^2)}}.$$

Porro anguli, quem directio vis *F*, quae est parallela ipsi *QP*, cum *MN* constituit, cosinus est =

$$\cos. PQN. \ \sin. QMN = \frac{(PQ+HN)QN}{QN \cdot MN} = -\frac{Q}{\sqrt{(1+P^2+Q^2)}}.$$

Ergo vis premens ex vi *F* orta est

$$-\frac{FQ}{\sqrt{(1+P^2+Q^2)}}.$$

Atque simili modo (Fig. 94) vis premens ex vi *E* orta est =

$$-\frac{EP}{\sqrt{(1+P^2+Q^2)}}.$$

Denique cum vis deflectentis directio sit MG atque

$$PE = \frac{z(dy+Qdz)}{Qdx-Pdy} \quad \text{et} \quad QP+EG = \frac{z(dx+Pdz)}{Qdx-Pdy},$$

erit anguli, quem *MG* cum directione vis *G* constituit, [p. 479] cosinus =

$$\frac{Qdx-Pdy}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}};$$

quare vis deflectens ex vi *G* orta est =

$$\frac{G(Qdx-Pdy)}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}.$$

Porro anguli, quem *MG* cum directione vis *F* constituit, cosinus est =

$$\frac{PQ+EG}{MG} = \frac{dx+Pdz}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}};$$

quamobrem vis deflectens ex vi *F* orta est =

$$\frac{F(dx+Pdz)}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}.$$

Denique anguli, quem directio vis *E* cum *MG* constituit, cosinus est =

$$\frac{-PE}{MG}=\frac{-dy-Qdz}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}.$$

Vis ergo deflectens ex vi *E* orta est =

$$\frac{-E(dy+Qdz)}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}.$$

Cum autem ante vim tangentialem vocaverimus *T*, vim prementem = *M* et vim deflectentem = *N*, ad has vires tres propositas *E*, *F*, et *G* reduximus ; erit namque

$$T=\frac{Edx+Fdy+Gdz}{\sqrt{(dx^2+dy^2+dz^2)}}$$

et

$$M=\frac{-EP-FQ+G}{\sqrt{(1+P^2+Q^2)}}$$

atque

$$N=\frac{-E(dy+Qdz)+F(dx+Pdz)+G(Qdx-Pdy)}{(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}.$$

Q.E.I.

## Corollarium 1.

**857.** Si igitur corpus a tribus potentiis *E*, *F* et *G* sollicetur, erit posito *v* pro altitudine debita celeritati in *M*

$$dv=-Edx-Fdy-Gdz$$

(849), si loco *T* ponatur vis tangentialis ex resolutione potentiarum *E*, *F* et *G* orta.

## Corollarium 2. [p. 480]

**858.**Si praeterea corpus in medio resistente moveatur atque resistentia in *M* fuerit = *R*, erit (850)

$$dv=-Edx-Fdy-Gdz-R\sqrt{(dx^2+dy^2+dz^2)}.$$

## Corollarium 3.

**859.** Si in aequatione (851) inventa, in qua effectus vis deflectentis N est determinatus, loco N substituatur vis deflectens ex resolutione virium E, F et G orta, prodit

$$\frac{2vddz(Pdy-Qdx)-2vddy(dx+Pdz)}{dx^2+dy^2+dz^2}$$
$$=-E(dy+Qdz)+F(dx+Pdz)-G(Pdy-Qdx).$$

## Corollarium 4.

**860.** Si ergo ex istis duabus aequationibus confletur una eliminanda $v$, prodibit aequatio, quae cum locali ad superficiem $dz=Pdx+Qdy$ coniuncta determinabit viam corpore in superficie descriptam.

## Corollarium 5.

**861.** Vis autem, qua superficies secundum normalem in eam premitur, tam ex vi normali premente $M$ quam ex vis centrifuga orta est =

$$\frac{(G-EP-FQ)(dx^2+dy^2+dz^2)+2v(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}$$

(845) substituto loco $M$ valore invento.

## Corollarium 6.

**862.** Est vero ex aequatione corollorio 3 inventa

$$2v=\frac{(dx^2+dy^2+dz^2)(E(dy+Qdz)-F(dx+Pdz)+G(Pdy-Qdx))}{-ddz(Pdy-Qdx)+ddy(dx+Pdz)},$$

quo valore ibi substituto prodibit tota pressio [p. 481] =

$$\frac{(Gdxddy-Fdxddz+Edyddz-Edzddy)\sqrt{(1+P^2+Q^2)}}{ddz(Pdy-Qdx)-ddy(dx+Pdz)}.$$

## Scholion.

**863.** Quia tres potentiae $E, F, G$ directiones habent invicem normales, erit potentia iis aequivalens = $\sqrt{(E^2+F^2+G^2)}$. His vero tribus viribus aequivalere invenimus tres $M$, $N$ et $T$, quarum directiones sunt quoque invicem normales; quare istis tribus aequivalens vis erit = $\sqrt{(M^2+N^2+T^2)}$. Quamobrem, si loco $M$, $N$ et $T$ substituantur valores inventi ex $E$, $F$ et $G$, prodire debet quoque $\sqrt{(E^2+F^2+G^2)}$; id quod calculo instituto re ipsa se habere deprehendetur. Inservit autem hoc instar probationis, utrum calculus prolixus, quo haec resolutio est absoluta, recte fuerit institutus, an vero secus. Hac vero probatione instituta reperientur se hae formulae recte habere

## PROPOSITIO 96.

### Problema.

**864.** *In hypothesi gravitatis uniformis et deorsum directae g determinare lineam, quam corpus super quacunque superficie proiectum in vacuo describit.*

### Solutio. [p. 482]

Sit *APQ* (Fig. 92) planum horizontale et *M* punctum tam in superficie data quam in linea a corpore descripta. Erit ergo *MQ* verticalis et propterea directio vis gravitatis *g*. Positis $AP = x$, $PQ = y$ et $QM = z$ atque aequatione superficiei naturam exprimente $dz = Pdx + Qdy$ sit celeritas in *M*, qua elementum Mm percurritur, debita altitudine *v*. Cum igitur problema hoc sit casus praecedentis, fit enim $G = g$, $E = 0$ et $F = 0$, habebuntur hae duae aequationes $dv = -gdz$ (857) atque

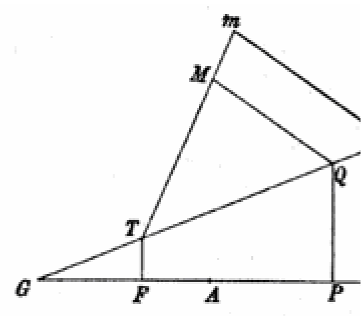

Fig. 92

$$2vddz(Pdy - Qdx) - 2vddy(dx + Pdz) + g(Pdy - Qdx)(dx^2 + dy^2 + dz^2) = 0$$

(859). Sit porro altitudo debita celeritati, quam corpus habiturum esset, si in planum horizontale *APQ* perveniret, = *b*; erit $v = b - gz$. Per alteram vero aequationem est

$$2v = \frac{g(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}{ddy(dx + Pdz) - ddz(Pdy - Qdx)}.$$

Unde erit

$$\frac{dv}{2v} = \frac{dzddz(Pdy - Qdx) - dzddy(dx + Pdz)}{(Pdy - Qdx)(dx^2 + dy^2 + dz^2)}.$$

Quae aequatio ope aequationis $dz = Pdx + Qdy$ transmutatur in hanc

$$\frac{dv}{2v} = \frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} - \frac{Pddy}{Pdy - Qdx};$$

quae integrata dat

$$lv = l(dx^2 + dy^2 + dz^2) - 2\int \frac{Pddy}{Pdy - Qdx}.$$

In quovis ergo case speciali investigari debet, an

$$\frac{Pddy}{Pdy - Qdx}$$

integrationem admittat. Quod si contigerit, habebitur $v$ in differentialibus primi gradus; atque cum sit $v = b - gz$, orietur aequatio differentialis primi curvae descriptaqe naturam exprimens. Pressio vero in superficiem secundum normalem erit =

$$\frac{g\,dx\,ddy\sqrt{(1+P^2+Q^2)}}{ddz(Pdy-Qdx)-ddy(dx+Pdz)},$$

quae sublatis differentialibus secundi gradus (861) abit in hanc

$$\frac{-g}{\sqrt{(1+P^2+Q^2)}}-\frac{2(b-gz)(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}.$$

Q.E.I.

## Corollarium 1. [p. 483]

**865.** Celeritas ergo corporis in hypothesi gravitatis uniformis $g$ et deorsum directae in superficie quacunque moti ex sola altitudine cognosci potest, omnino ut si corpus in eodem plano moveretur.

## Corollarium 2.

**866.** Si tempusculum, quo elementum $Mm$ absolvitur, ponatur $dt$, erit

$$dt=\frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}.$$

Per aequationem ergo inventam erit

$$ldt=\int\frac{Pddy}{Pdy-Qdx}\quad\text{atque}\quad\frac{ddt}{dt}=\frac{Pddy}{Pdy-Qdx}.$$

## Corollarium 3.

**867.** Ex inventis aequationibus reperietur

$$ddy=\frac{g(Pdy-Qdx)(dx^2+dy^2+dz^2)}{2dx(b-gz)(1+P^2+Q^2)}+\frac{(dPdx+dQdy)(Pdy-Qdx)}{(1+P^2+Q^2)dx}$$

atque

$$ddz=\frac{gQ(Pdy-Qdx)(dx^2+dy^2+dz^2)}{2dx(b-gz)(1+P^2+Q^2)}+\frac{(dPdx+dQdy)(dx+Pdz)}{(1+P^2+Q^2)dx}.$$

Hinc erit

$$dzddy-dyddz=\frac{gP(Pdy-Qdx)(dx^2+dy^2+dz^2)}{2(b-gz)(1+P^2+Q^2)}-\frac{(dPdx+dQdy)(dy+Qdz)}{1+P^2+Q^2}.$$

Curvae vero descriptae radius osculi erit =

$$\frac{2(dx^2+dy^2+dz^2)(b-gz)\sqrt{(1+P^2+Q^2)}}{\sqrt{(g^2(Pdy-Qdx)^2(dx^2+dy^2+dz^2)+4(b-gz)^2(dPdx+dQdy)^2)}}.$$

## Scholion.

**868.** Harum formularum usum in casibus particularibus, quibus certa quaedam species superficierum consideratur, in sequentibus problematis fusius exponemus, quibus singularium superficierum exempla adiungemus. [p. 484]

## PROPOSITIO 97.

## Problema.

**869.** *In hypothesi gravitatis uniformis deorsum tendentis g determinare motum corporis in superficie cuiuscunque cylindri, cuius axis sit verticalis.*

## Solutio.

Quia axis cylindri ponatur verticalis, erunt omnes sectiones inter se aequales; sit igitur *ABQC* (Fig. 95) basis cylindri, in cuius superficie movetur corpus. Ponatur $AP = x$, $PQ = y$ sitque $z$ corporis altitudo super puncto $Q$ in superficie cylindri. Natura ergo huius superficiei cylindricae hac exprimetur aequatione $0 = Pdx + Qdy$ seu $Qdy = -Pdx$. Haec autem aequatio orietur ex generali $dz = Pdx + Qdy$, si $P$ et $Q$ fiant quantitates infinitae magnae, seu evanescente coefficiente ipsius $dz$, si quem assumsissemus. Quamobrem in aequationibus ante inventis $P$ et $Q$ quasi quantitates infinite magnae considerari debent, etiamsi fint finitae magnitudinis. Erunt autem $P$ et $Q$ functiones ipsarum $x$ et $y$ tantum neque in iis inerit $z$. His ergo in calculum deductis habebitur

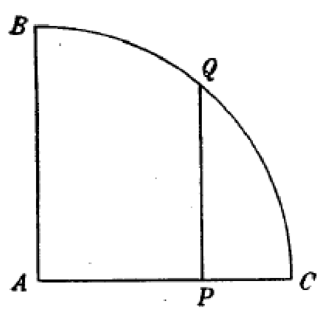

Fig. 95.

$v = b - gz$

atque

$$2v = \frac{g(Pdy-Qdx)(dx^2+dy^2+dz^2)}{Pdzddy-Pdyddz+Qdxddz} = \frac{g(dx^2+dy^2)(dx^2+dy^2+dz^2)}{dydzddy-dy^2ddz-dx^2ddz}$$

propter analogiam $P : Q = dy : -dx$. At posterior aequatio logarithmica erit [p. 485]

$$lv = l(dx^2+dy^2+dz^2) - 2\int\frac{dyddy}{dx^2+dy^2} = l(dx^2+dy^2+dz^2) - l(dx^2+dy^2) + lc.$$

Unde fit

$$\frac{v}{c} = \frac{dx^2 + dy^2 + dz^2}{dx^2 + dy^2} = \frac{b - gz}{c} = 1 + \frac{dz^2}{dx^2 + dy^2}.$$

Hinc oritur

$$(b - c - gz)(dx^2 + dy^2) = c\,dz^2$$

seu

$$\sqrt{(dx^2 + dy^2)} = \frac{dz\sqrt{c}}{\sqrt{(b - c - gz)}},$$

cuius integralis est

$$\int\sqrt{(dx^2 + dy^2)} = -\frac{2\sqrt{c(b - c - gz)}}{g},$$

ubi $\int\sqrt{(dx^2 + dy^2)}$ denotat arcum basis *BQC* motu horizontali percursum. Si tempusculum, quo elementum *Mm* absolvitur, ponatur *dt*, erit

$$\frac{ddt}{dt} = \frac{dy\,ddy}{dx^2 + dy^2} \quad \text{atque} \quad \alpha dt = \sqrt{(dx^2 + dy^2)}$$

porroque

$$\alpha t = \int\sqrt{(dx^2 + dy^2)}.$$

Quare tempora erunt proportionalia arcubus in basi respondentibus.

Aequatio autem

$$\int\sqrt{(dx^2 + dy^2)} = -\frac{2\sqrt{c(b - c - gz)}}{g}$$

dabit aequationem pro curva in superficie cylindrica descripta, si haec superficies in planum concipiatur explicata; denotabit enim tum $\int\sqrt{(dx^2 + dy^2)}$ abscissam in axe horizontali et *z* applicatam verticalem. Proiectionem vero curvae descriptae in plano verticali planum horizontale iuxta *AC* secante habebimus, si ope aequationis $Pdx + Qdy = 0$ eliminetur *y*, ut prodeat aequatio inter *x* et *z*, quae erunt coordinatae huius proiectionis. Scilicet ob $dy = -\frac{Pdx}{Q}$ erit

$$\sqrt{(dx^2 + dy^2)} = \frac{dx}{Q}\sqrt{(P^2 + Q^2)}$$

ideoque

$$\int\frac{dx\sqrt{(P^2 + Q^2)}}{Q} = -\frac{2\sqrt{c(b - c - gz)}}{g},$$

ubi in *P* et *Q* loco *y* eius valor in *x* debet substitui.

Pressio vero, quam superficies sustinebit, a sola vi centrifuga orietur propter potentiam *g* in ipsa superficie sitam eritque =

$$\frac{2(b-gz)(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2)}},$$

qua vi corpus ab axe cylindri recedere conabitur, si haec expressio fuerit affirmativa. Q.E.I. [p. 486]

## Corollarium 1.

**870.** Curva ergo, quam corpus in superficie cylindri describit, si superficies in planum explicetur, abibit in parabolam, ipsam scilicet proiectoriam, quam corpus proiectum in plano verticali describeret.

## Corollarium 2.

**871.** Si motus corporis super superficie cylindrica compositus consideretur ex motu verticali, quod vel sursum vel deorsum progreditur, et ex horizontali, erit motus horizontalis aequabilis, quia tempora *t* proportionalia sunt $\int\sqrt{(dx^2+dy^2)}$, i. e. arcubus horizontali motu percursis. Motus vero verticalis erit vel aequaliter acceleratus vel retardatus.

## Corollarium 3.

**872.** Si ergo motus horizontalis evanescit, corpus recta vel ascendet vel descendet, omnino ac si libere ascenderet vel descenderet. Hicque casus prodit, si fuerit $c = 0$, quo $\int\sqrt{(dx^2+dy^2)}$ evanescit.

## Exemplum. [p. 487]

**873.** Sit basis cylindri circulus, cuius quadrans sit *BQC* et radius $AB = a$; erit $x^2+y^2=a^2$ et $xdx + ydy = 0$. Fiet ergo $P = x$ et $Q = y = \sqrt{(a^2-x^2)}$. Proiectio vero lineae in superficie cylindrica hac descriptae in plano verticali ex *AC* erecto exprimetur aequatione

$$\int\frac{adx}{\sqrt{(a^2-x^2)}} = -\frac{2\sqrt{c(b-c-gz)}}{g}.$$

Si ergo fuerit $c = 0$, fiet $dx = 0$ atque $x$ = constanti; quare hoc casu proiectio erit linea recta. Curva vero, quae est proiectio pro quocunque ipsius *c* valore, ope rectificationis circuli construetur. Pressio autem, quam superficies cylindri sustinet, est =

$$\frac{2(b-gz)(dx^2+dy^2)}{(dx^2+dy^2+dz^2)a} = \frac{2c}{a}$$

propter aequationem

$$\frac{dx^2+dy^2+dz^2}{dx^2+dy^2} = \frac{b-gz}{c}.$$

Quare pressio ubique erit constans et ipsi *c* proportionalis.

## Corollarium 4.

**874.** Propter eandem aequatonem erit generaliter pressio, quam cylinder quicunque sustinet,

$$\frac{2c(dPdx + dQdy)}{(dx^2 + dy^2)\sqrt{(P^2 + Q^2)}} = \frac{2Q^2c(dPdx + dQdy)}{dx^2(P^2 + Q^2)^{\frac{3}{2}}} = \frac{2Qc(QdP - PdQ)}{dx(P^2 + Q^2)^{\frac{3}{2}}}.$$

## Scholion. [p. 488]

**875.** Non solum autem ope resolutionis motus super superficie cylindrica erecta corporis motus facile potest determinari, sed etiam, si cylindri axis fuerit horizontalis, eadem facilitate motus corporis cognoscetur. Namque si corpus non habeat motum horizontalem iuxta cylindri, corpus perpetuo in eadem cylindri sectione permanebit in eaque movebitur tanquam super linea data. Sin vero accedat motus horizontalis, is perpetuo idem manebit neque alterum motum perturbabit; atque his motibus coniungendis verus corporis motus facile cognoscetur.

## PROPOSITIO 98.

## Problema.

**876.** *Si corpus moveatur in superficie solidi rotundi, cuius axis est verticalis AL* (Fig. 96)*, in vacuo a gravitate uniformi g sollicitatum, determinare motum corporis super huiusmodi surfacie.*

## Solutio.

Generatur solidum rotundum conversione curvae *AM* circa axem verticalem *AL*; erunt omnes eius sectiones horizontales circuli, quorum radii sunt applicatae curvae *AM*. Aequatio ergo naturalam huius superficiei exprimens erit $dz = \frac{xdx+ydy}{Z}$ denotante *Z* functionem quamcunque ipsius *z*; erit enim

$$2\int Zdz = x^2 + y^2 = LM^2 .$$

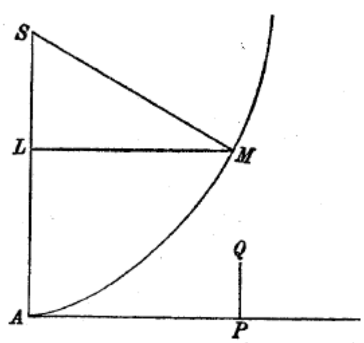

Fig. 96.

Si ergo detur aequatio pro curva *AM* inter $AL = z$ et $LM = \sqrt{2}\int Zdz$ , dabitur quoque *Z*. [p. 489] His praemissis erit itaque $P = \frac{x}{Z}$ et $Q = \frac{y}{Z}$; qui valores si substituantur, habebuntur duae sequentes aequationes, ex quibus tam curva descripta quam motus super ea cognescetur $v = b - gz$ atque

$$lv = l(dx^2 + dy^2 + dz^2) - 2\int \frac{xddy}{xdy - ydx} = l(dx^2 + dy^2 + dz^2) - 2l(xdy - ydx) + \text{const.}$$

Quare erit

$$v = \frac{c^3(dx^2 + dy^2 + dz^2)}{(xdy - ydx)^2} = b - gz.$$

Ponatur $x^2 + y^2 = u^2$; erit *u* functio quaedam ipsius *z*, nempe

$$u^2 = 2\int Zdz,$$

atque superior aequatio abibit in hanc

$$\sqrt{(dx^2 + dy^2)} = \frac{\sqrt{(c^3dz^2 + u^2du^2(b - gz))}}{\sqrt{(u^2(b - gz) - c^3)}}.$$

Proiectio vero in plano horizontali per aequationem inter *x* et *y* habebitur, si ex aequatione $x^2 + y^2 = 2\int Zdz$ valor ipsius *z* in *x* et *y* substituatur; huiusque proiectionis arcus est $\int\sqrt{(dx^2 + dy^2)}$. In plano autem verticali habebitur proiectio eleminanda *y*; quo facto prodit aequatio

$$\frac{udx - xdu}{c\sqrt{c(u^2 - x^2)}} = \sqrt{\frac{du^2 + dz^2}{u^2(b - gz) - c^3}},$$

quae aequatio, si per *u* dividatur, constructionem admittit. Pressio vero, quam superficies sustinet axem versus, erit =

$$\frac{-gZ}{\sqrt{(x^2 + y^2 + Z^2)}} - \frac{2c^3Z(dx^2 + dy^2) - 2c^3dZ(xdx + ydy)}{Z(xdy - ydx)^2\sqrt{(x^2 + y^2 + Z^2)}}.$$

Q.E.I.

## Corollarium 1.

**877.** Si tempusculum, quo elementum *Mm* absolvitur, ponatur *dt*, erit

$$\frac{ddt}{dt} = \frac{xddy}{xdy - ydx},$$

cuius integralis est

$$\alpha dt = xdy - ydx.$$

Ipsum ergo tempus erit ut

$$xy - 2\int ydx$$

denotante $\int ydx$ aream proiectionis in plano horizontali.

## Corollarium 2. [p. 490]

**878.** Sit corpus in proiectione in plano horizontali moveri concipiatur, erit celeritas eius in $Q$ debita altitudini

$$\frac{c^3(dx^2+dy^2)}{(ydx-xdy)^2};$$

ex quo motu in proiectione ipse motus in superficie invenietur.

## Corollarium 3.

**879.** Si igitur $BQC$ (Fig. 97) proiectio curvae in plano horizontali, in qua moveatur corpus, ita ut motus eius respondeat motui corporis in superficie ipsa; erit tempus, quo arcus $BQ$ absolvitur, ut $\frac{xy}{2}-\int ydx$ seu negative ut $\int ydx-\frac{xy}{2}$, i. e. ut area BAQ ducto radio AQ.

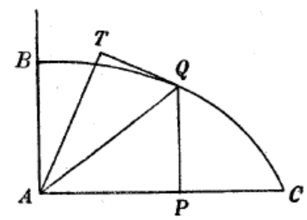

Fig. 97.

## Corollarium 4.

**880.** Areae autem $BAQ$ elementum est $\frac{ydx-xdy}{2}$. Ducto ergo tangente $QT$ et demisso in eam ex $A$ perpendiculo $AT$ erit

$$AT=\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}.$$

Quare altitudo celeritati in $Q$ debita est = $\frac{c^3}{AT^2}=\frac{c^3}{p^2}$ posito $AT=p$.

## Corollarium 5.

**881.** Corpus ergo in proiectione motum perinde in ea movebitur, ac si libere moveretur attractum vi quadam centripeta ad centrum $A$ (T.I. (587)).

## Corollarium 6. [p. 491]

**882.** Respondeat punctum $B$ motus in superficie facti initio, et cum detur directio prima motus in superficie, dabitur perpendiculum in tangentem in $B$. Sit ergo $AB=f$ et perpendiculum in tangentem = $h$ ;erit altitudo celeritati in $B$ debita $\frac{c^3}{h^2}$.

## Corollarium 7.

**883.** Vis ergo centripeta versus $A$ tendens, quae faciet, ut corpus in proiectione $BQC$ libere moveatur, erit = $\frac{2c^3dp}{p^3du}$ posito $u$ pro $\sqrt{(x^2+y^2)}$. Aequatio vero inter $u$ et $z$ exprimet naturam curvae, cuius conversione genita est superficies proposita, ideoque datur.

## Corollarium 8.

**884.** Est porro

$$\sqrt{(dx^2+dy^2)}=\frac{udu}{\sqrt{(u^2-p^2)}} \quad \text{et} \quad ydx-xdy=\frac{pudu}{\sqrt{(u^2-p^2)}}.$$

Hi valores, si in aequatione inventa substituantur, dabunt

$$c^3u^2du^2+c^3u^2dz^2-c^3p^2dz^2=(b-gz)p^2u^2du^2$$

seu

$$p^2=\frac{c^3u^2(du^2+dz^2)}{c^3dz^2+u^2du^2(b-gz)}.$$

## Corollarium 9.

**885.** Huius ergo quantitas

$$-\frac{c^3dz^2+u^2du^2(b-gz)}{u^2(du^2+dz^2)}$$

differentiale per du divisum dabit vim centripetam in *A* requisitam, ut corpus in proiectione *BQC* moveatur libere, motu respondente motui corporis in superficie.

## Corollarium 10.

**886.** Si $c = 0$, fiet quoque $p = 0$. Quare hoc casu proiectio in plano horizontali erit recta per *A* transiens, super qua corpus ad *A* accedens ita attrahetur, ut sit vis centripeta

$$\frac{1}{du}\, d.\frac{-du^2(b-gz)}{du^2+dz^2}.$$

## Corollarium 11. [p. 492]

**887.** Si corporis directio prima fuerit horizontalis, tangens in *B* ad *AB* erit normalis ideoque $h = f$. Hoc vero casu celeritas in superficie aequalis erit celeritati in proiectione; quare si fuerit *i* valor ipsius *z*, si est $u = f$, erit $b-gi=\frac{c^3}{f^2}$.

## Corollarium 12.

**888.** Si praeterea vis centripeta in *B* aequalis fuerit vi centrifugae, curva *BQC* erit circulus et propterea ipsa quoque curva in superficie descripta atque motus tam in superficie quam in proiectione aequabilis. Sit, ubi est $u = f$ et $z = i$, $dz = mdu$; erit ob $p = f$ et $u = f$ et $z = i$ in casu, quo circulus describitur, $2c^3=gmf^3$.

## Corollarium 13.

**889.** Si ponatur $\pi:1$ ut peripheria ad diametrum, erit circuli nostri peripheria $=2\pi f$, quae divisa per celeritatem $\sqrt{\frac{c^3}{f^2}}$, i. e. $\sqrt{\frac{gmf}{2}}$, dat tempus unius periodi in circulo, quod ergo erit = $\frac{2\pi\sqrt{2f}}{\sqrt{gm}}$. Pendulum ergo integrae oscillationes his periodis isochronas absolvens erit = $\frac{f}{m}$ in eadem gravitatis hypothesi.

## Corollarium 14. [p. 493]

**890.** In superficie ergo, quae generatur conversione curvae *AM* (Fig. 96) circa axem verticalem *AL*, corpus proiectum circulum radii *LM* describere potest eodem tempore, quo pendulum longitudinis = subnormali *LS* integram oscillationem absolvit.

## Corollarium 15.

**891.** Si ergo curva *AM* fuerit parabola, omnes periodi per circulos horizontales in conoide parabolica aequalibus absolventur temporibus; atque penduli iisdem temporibus oscillationes integras peragentis longitudo aequabitur dimidiae parti parametri.

## Scholion 1.

**892.** Quaecunque assumatur curva *AM*, si vis centripeta motus in proiectione versus *A* ex data formula definiatur eaque aequalis ponatur vi centrifugae atque coniungatur cum aequatione $b - gi = \frac{c^3}{f^2}$, prodibit $2c^3 = gmf^3$; hoc enim casu tam proiectio erit circulus quam ipsa curva in superficie descripta. Quo vero hoc melius appareat, ponatur $dz = qdu$ provenietque (884) vis centripeta =

$$\frac{2qdq(bu^2 - gzu^2 - c^3)}{u^2du(1+q^2)^2} + \frac{2c^3q^2 + gqu^3}{u^3(1+q^2)}.$$

Iam ponatur $u = f$, $z = i$ et $bf^2 = gf^2i + c^3$ atque $q = m$; erit vis centripeta =

$$\frac{2c^3m^2 + gmf^3}{f^3(1+m^2)};$$

aequalis posita vi centrifugae $\frac{2c^3}{f^2}$, dat $2c^3 = gmf^3$.

## Exemplum. [p. 494]

**893.** Sit superficies conica circulis seu *AM* (Fig. 96 et Fig. 97) linea recta utcunque inclinata ad axem *AL*; quare erit $z = mu$ et $dz = mdu$. Vis ergo centripeta ad *A* tendens, quae faciet, ut corpus libere in proiectione *BQC* moveatur, erit =

$$\frac{2c^3m^2 + gmu^3}{u^3(1+m^2)}.$$

Haec ergo vis erit composita ex vi constante et vi reciproce cubis distantiarum a centro *A* proportionali. Si $c = 0$, tum proiectio erit linea recta per *A* transiens atque vis centripeta erit = $\frac{gm}{1+m^2}$ sive constans. Corpus ergo motu aequabiliter accelerato ad *A* accedet; motus vero in superficie conica congruit cum descensu vel ascensu super recta inclinata eritque pariter aequabiliter acceleratus. Sin autem proiectio fuerit curvilinea et tangens in *B* normalis ad *AB*, erit $i = mf$ atque $b - gmf = \frac{c^3}{f^2}$ posito $AB = f$. Erit ergo hoc casu $b = \frac{c^3+gmf^3}{f^2}$, atque si corpus in circulo horizontali revolvitur, erit insuper $2c^3 = gmf^3$. Quo ergo hoc accidat, debet esse celeritas corporis debita altitudini $\frac{c^3}{f^2} = \frac{gmf}{2}$. Atque periodi in hoc circulo iisdem absolventur temporibus, quibus penduli longitudinis $\frac{f}{m}$ integrae oscillationes. At si non fuerit $2c^3 = gmf^3$, verum tamen $b = \frac{c^3+gmf^3}{f^2}$, curva BQC habebit quidem in B tangentem ad AB normalem, sed proiectio haec non erit circulus. At si $2c^3$ proxime aequale fuerit ipsi $gmf^3$, curva quoque a circulo non multum discrepabit; habebit autem absidum varias, in quibus tangens ad radium est perpendicularis. Harum absidum vero positio per propositionem 91 libri praecedentis determinatur. Quoniam enim vis centripeta est

$$\frac{2c^3m^2 + gmu^3}{u^3(1+m^2)},$$

si haec comparetur cum illa vi centripeta $\frac{P}{y^3}$, [p. 495] ob $y = u$ erit

$$P = \frac{2c^3m^2 + gmu^3}{1+m^2} \quad \text{atque} \quad \frac{dP}{P} = \frac{3gmu^2du}{2c^3m^2 + gmu^3} \quad \text{atque} \quad \frac{Pdy}{ydP} = \frac{2mc^3 + gu^3}{3gu^3};$$

ideoque posito $u = f$ distabit quoque linea absidum a praecedente abside angulo

$$180\sqrt{\frac{2mc^3+gf^3}{3gf^3}} \text{ grad.}$$

Quoniam autem proxime est $2c^3 = gmf^3$, erit angulus inter duas absides interceptus = $180\sqrt{\frac{m^2+1}{3}}$ graduum. Cum ergo $\frac{m^2+1}{3}$ semper sit maius quam $\frac{1}{3}$, erit angulus inter duas absides interceptus maior quam $103^0$ 55'.

## Scholion 2.

**894.** Exemplum superficiei sphaericae hic non adiungo, sed motus super ea determinationi sequentem propositionem destino, quia haec materia particulari pertractatione est digna. Si enim pendulum non secundum planum verticale impellitur, tum corpus in superficie sphaericae movebitur et vel circulos describet vel alias curvas non parum elegantes, quemadmodum cuivis experimentum instituenti innotescet. Casus quidem, quo pendulum circulos absolvit, a Cel. Ioh. Bernoulli in Act. Lips. A. 1715 iam est expositus sub titulo *motus turbinatorii*. [*Opera omnia*, Tom. 2, p. 187]. At si curva non fuerit circulus, nemo, quantum scio, hunc penduli motum vel consideravit vel determinavit. [I. Neutonum hoc problema considerasse nec non solutionis compotem fuisse verisimile est; vide Philpsophae naturalis principia mathematica, London, 1687, Lib. I sect. X prop. LV et LVI.]

## Definitio 6. [p. 496]

**895.** *Motus turbinatorius vocatur penduli non in plano verticali impulsi motus.* Hoc ergo casu pendulum non in eodem plano verticali movetur, sed curvam quandam describet in superficie sphaerica, cuius radius est ipsa penduli longitudo, sitam.

## PROPOSITIO 99.

## Problema.

**896.** *Penduli ad motum turvinatorium incitati determinare motum et lineam curvam, quam in superficie sphaerica describi.*

## Solutio.

Quia corpus motum pendulo est alligatum, in superficie sphaerica movebitur, cuius radius est longitudo penduli. Sit haec longitudo seu radius sphaerae = $a$ ; erit $z = a - \sqrt{(a^2 - u^2)}$ ex natura circuli *AM* (Fig. 96 et Fig. 97), Erit ergo

$$q = \frac{u}{\sqrt{(a^2 - u^2)}} \quad \text{et} \quad dq = \frac{a^2 du}{(a^2 - u^2)^{\frac{3}{2}}}$$

ideoque

$$\frac{2q\,dq}{u^2 du} = \frac{2a^2}{u(a^2 - u^2)^2}.$$

Ex his invenitur vis centripeta ad *A* tendens, quae facit, ut corpus in proiectione *BQC* libere moveatur, =

$$\frac{2bu - 2gau + 3gu\sqrt{(a^2 - u^2)}}{a^2}.$$

Atque pro curva *BQC* haec habebitur aequatio

$$p^2 = \frac{a^2 c^3}{c^3 + (b - ga)(a^2 - u^2) + g(a^2 - u^2)^{\frac{3}{2}}},$$

quae ad proiectionem *BQC* construendam sufficit. Sit tangens in *B* perpendicularis ad radium *AB*, id quod semper alicubi contingere debet, nisi proiectio sit linea recta, quia vis centripeta decrescit decrescente $u$. Ponatur $AB = f$; erit $i = a - \sqrt{(a^2 - f^2)}$ atque [p. 497]

$$b - ga + g\sqrt{(a^2 - f^2)} = \frac{c^3}{f^2}.$$

Si praeterea fuerit

$$2c^3 = \frac{gf^4}{\sqrt{(a^2 - f^2)}},$$

corpus in circulo revolvetur, cuius radius est *f*, celeritate debita altitudini

$$\frac{c^3}{f^2} = \frac{gf^2}{2\sqrt{(a^2 - f^2)}}.$$

Penduli vero integras oscillationes iisdem temporibus absolventis longitudo est = $\sqrt{(a^2 - f^2)}$. Sin autem non fuerit $2c^3 = \frac{gf^4}{\sqrt{(a^2 - f^2)}}$, differentia vero sit valde exigua, curva BQC a circulo non multum discrepabit. Cuius curvae quo positiones absidium inveniantur, erit iuxta propositionem 91 libri praecedentis

$$y = u \quad \text{et} \quad P = \frac{u^4}{a^2}\left(2b - 2ga + 3g\sqrt{(a^2 - u^2)}\right).$$

Hinc est

$$\frac{dP}{P} = \frac{4du}{u} - \frac{3gudu}{(2b - 2ga)\sqrt{(a^2 - u^2)} + 3g(a^2 - u^2)}$$

atque

$$\frac{ydP}{Pdy} = 4 - \frac{3gu^2}{(2b - 2ga)\sqrt{(a^2 - u^2)} + 3g(a^2 - u^2)}.$$

Quia curva fere est circulus, ponatur $u = f$ atque

$$2b - 2ga = \frac{gf^2}{\sqrt{(a^2 - f^2)}} - 2g\sqrt{(a^2 - f^2)} = \frac{3gf^2 - 2ga^2}{\sqrt{(a^2 - f^2)}},$$

quo facto prodibit

$$\frac{ydP}{Pdy} = 4 - \frac{3f^2}{a^2}.$$

Hinc sequitur intervallum inter duas absides esse angulum

$$\frac{180a}{\sqrt{(4a^2 - 3f^2)}} \text{ grad.}$$

Tanto scilicet angulo locus, in quo pendulum ab axe maxime distat, dissitus est a locus, in quo pendulum axi est proximum. Q.E.I.

## Corollarium 1.

**897.** Quo igitur pendulum $AB = a$ (Fig. 98) motu turbinatorio circulum *BCDE* describat, oportet, ut eius celeritas debita sit altitudini $\frac{g.BO^2}{2.AO}$.

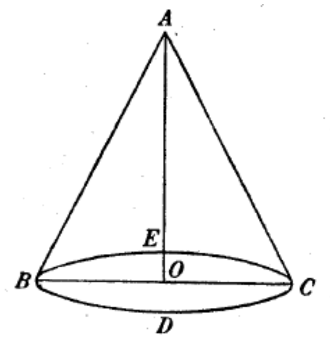

Fig. 98.

## Corollarium 2.

**898.** Longitudo vero penduli, quod oscillationes minimas integras eodem tempore absolvit, quo periodus in circulo *BDCE* conficitur, est = *AO*.

## Corollarium 3. [p. 498]

**899.** Tempora ergo, quibus diversi circuli motu turbinatorio a pendulo *AB* percurruntur, sunt in subduplicata ratione altitudinum *AO*.

## Corollarium 4.

**900.** Quo igitur pendulum longitudinis *a* circulum horizontalem maximum conficiat, cuius radius est *a*, celeritas infinite magna requiritur atque quaeque periodus tempore infinite parva absolvitur.

## Corollarium 5.

**901.** Si radius circuli *BO* fuerit valde parvus respectu penduli $AB = a$, congruent periodi motus turbinatorii cum oscillationibus integris eiusdem penduli.

## Corollarium 6.

**902.** Si curva descripta non fuerit circulus, sed figura proxima atque *BO* valde parvum, erit angulus inter duas absides $90^0$ seu rectus.

## Corollarium 7.

**903.** Hoc vero casu curva a corpore descripta erit ellipsis centrum habens in *A*. Quod ex vi centripeta colligitur, quae tum ipsis distantiis fit proportionalis.

### Corollarium 8. [p. 499]

**904.** Quo maior autem est radius *BO*, eo maior quoque erit angulus inter duas absides interceptus. Atque si fiat *BO* = *BA*, erit hic angulus $180^0$.

### Corollarium 9.

**905.** Si angulus *BAO* fuerit 30 graduum, erit $BO = \frac{1}{2}BA$ seu $f = \frac{1}{2}a$. Angulus ergo inter duas absides interceptus erit $\frac{360}{\sqrt{13}}$ graduum seu $99^0$ 50'. Proiectionis ergo in plano horizontali haec erit figura *abcdefghik* etc. (Fig. 99) in qua absides summae sunt in *a, c, e, g, i* et imae in *b, d, f, h, k.*

### Corollarium 10.

**906.** In hac igitur figura linea absidum movetur in consequentia; singulis enim periodis circiter $39^0$ progredietur in consequentia.

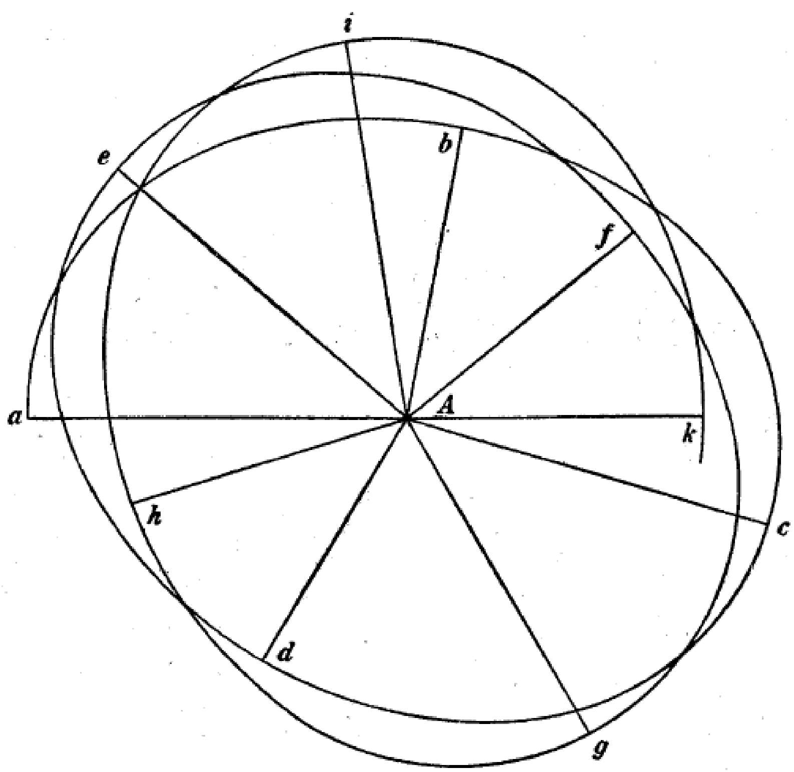


Fig. 99.

### Corollarium 11. [p. 500]

**907.** Sin autem ille angulus *BAO* minor fuerit quam 30 graduum, tum minor etiam erit absidum progressio. Quae quo pro quovis angulo *BAO* statim cognoscatur, fractionem

$$\frac{a}{\sqrt{(4a^2 - 3f^2)}}$$

in seriem resolvo, quae erit sequens

$$\frac{1}{2}+\frac{3f^2}{16a^2}+\frac{27f^4}{256a^4}+\text{ etc.}$$

Una ergo periodo linea absidum provovetur angulo $\frac{135f^2}{a^2}$ graduum quam proxime, si $f$ valde est parvum.

## Corollarium 12.

**908.** Ex his apparet promotionem lineae absidum in singulis periodis proxime esse in duplicata ratione sinus anguli BAO.