CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 402]

PROPOSITION 82.

Problem.

738. *In the rarest medium, which resists in some ratio of the powers of the speeds, and according to the hypothesis of a uniform force acting downwards, to determine the tautochrone curve AM* (Fig. 80)*, upon which either all the descents or all the ascents are completed in equal times.*

Solution.

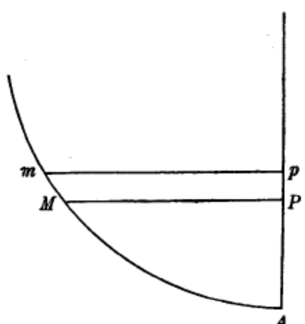

Fig. 80.

With the abscissa $AP = x$ and the arc $AM = s$ let the whole arc of some descent be equal to f. The force acting downwards is equal to g, the height corresponding to the speed at M is equal to v, the exponent of the resistance is equal to k and the force of the resistance itself is $\frac{v^m}{k^m}$, where k is some very large quantity, thus in order that fractions, which in the denominator k have a dimension greater than m can be taken as vanishing. With these in place, from the nature of the descent, it follows that [p. 403] :

$$dv = -gdx + \frac{v^m ds}{k^m}.$$

Now evidently if the resistance is absent then the curve sought is a cycloid, the equation of which is given by :

$$gx = \alpha s^2;$$

moreover since the medium is the rarest, the curve sought does not differ much from a cycloid; therefore this equation is put in place for the curve sought:

$$gx = \alpha s^2 + \frac{\beta s^n}{k^m}.$$

Now since

$$dv = -gdx + \frac{v^m ds}{k^m}.$$

on account of the very small term $\frac{v^m ds}{k^m}$, the solution is approximately :

$$v = C - gx = C - \alpha s^2 - \frac{\beta s^n}{k^m},$$

where the constant C is determined from this condition, that if we make $s = f$, then $v = 0$. On account of which, we have :

$$v = \alpha(f^2 - s^2) - \frac{\beta}{k^m}(f^n - s^n), \text{ as an approximation.}$$

Therefore, on putting

$$v = \alpha(f^2 - s^2) - \frac{\beta}{k^m}(f^n - s^n) + Q;$$

then [as dQ has the form $\frac{v^m ds}{k^m}$.]

$$dv = -2\alpha s ds - \frac{n\beta s^{n-1} ds}{k^m} + dQ = -g dx + \frac{v^m ds}{k^m}$$

$$= -2\alpha s ds - \frac{n\beta s^{n-1} ds}{k^m} + \frac{\alpha^m (f^2 - s^2)^m ds}{k^m}$$

which expression must be put in place for v^m, with the remaining terms ignored, because in these, k has dimensions greater than m in the denominator. From these it now follows that :

$$Q = \frac{\alpha^m}{k^m}\int (f^2 - s^2)^m ds$$

with this integral thus taken, so that it vanishes on putting $s = f$. Therefore,

$$v = \alpha(f^2 - s^2) + \frac{\beta}{k^m}(f^n - s^n) + \frac{\alpha^m}{k^m}\int (f^2 - s^2)^m ds$$

and hence [on inverting and making a binomial expansion of the sq. root; p. 404]

$$\frac{1}{\sqrt{v}} = \frac{1}{\sqrt{\alpha(f^2 - s^2)}} - \frac{\beta(f^n - s^n) + \alpha^m \int (f^2 - s^2)^m ds}{2\alpha^{\frac{3}{2}} k^m (ff - ss)^{\frac{3}{2}}}$$

with terms again to be omitted on account of the reason mentioned. The time is now produced from this equation:

$$\int \frac{ds}{\sqrt{v}} = \int \frac{ds}{\sqrt{\alpha(f^2 - s^2)}} - \frac{\beta}{2\alpha^{\frac{3}{2}} k^m}\int \frac{ds(f^n - s^n)}{(ff - ss)^{\frac{3}{2}}} - \frac{\alpha^m}{2\alpha^{\frac{3}{2}} k^m}\int \frac{ds \int (f^2 - s^2)^m ds}{(ff - ss)^{\frac{3}{2}}},$$

which is integrated, so that it vanishes on putting $s = 0$, and which gives the time in which the descending body completes the arc $MA = s$. Therefore the time of the decent is found for the arc f, if we put $s = f$ after the integration ; since the time must be constant or thus made so that it does not depend on f. Moreover the first term of the time expansion:

$$\int \frac{ds}{\sqrt{\alpha(f^2 - s^2)}}$$

on putting $s = f$ after the integration, now gives an expression of this kind, in which f is no longer present ; indeed $\frac{\pi}{2\sqrt{\alpha}}$ is produced. On account of which, if the two remaining terms are thus to be prepared, so that on putting $s = f$, they cancel each other out, then the question is satisfied ; and indeed this expression $\frac{\pi}{2\sqrt{\alpha}}$ is obtained for the time of the descent, which is constant. Hence it must the case that :

$$\beta \int \frac{ds(f^n - s^n)}{(ff - ss)^{\frac{3}{2}}} = -\alpha^m \int \frac{ds \int (f^2 - s^2)^m ds}{(ff - ss)^{\frac{3}{2}}}$$

on putting $s = f$ after the integration. But [p. 405]

$$\int \frac{ds(f^n - s^n)}{(ff - ss)^{\frac{3}{2}}}$$

on putting $s = f$ on integrating, gives a multiple of the power of f, and the exponent of this is $n - 2$, and the other integral

$$\int \frac{ds \int (f^2 - s^2)^m ds}{(ff - ss)^{\frac{3}{2}}}$$

on putting $s = f$ gives a multiple of the power of f, the exponent of this is $2m - 1$. Therefore on making $n = 2m + 1$ and β thus taken in order that these two powers of f, the exponent of each of which is $2m - 1$, cancel each other out. From which we conclude, moreover, it is apparent that such coefficients of these powers of f are to be had for the simpler cases. Therefore it is found, for the following formulas on integration, in order that they vanish on putting $s = 0$, and then on putting $s = f$:

$$\int \frac{(f - s)ds}{(ff - ss)^{\frac{3}{2}}} = f^{-1}, \quad \int \frac{(f^3 - s^3)ds}{(ff - ss)^{\frac{3}{2}}} = \frac{2}{1} f^1, \text{ and } \int \frac{(f^5 - s^5)ds}{(ff - ss)^{\frac{3}{2}}} = \frac{2 \cdot 4}{1 \cdot 3} f^3;$$

and hence in general this becomes :

$$\int \frac{(f^{2m+1} - s^{2m+1})ds}{(ff - ss)^{\frac{3}{2}}} = \frac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots (2m - 1)} f^{2m-1},$$

from which :

$$\beta \int \frac{ds(f^n - s^n)}{(ff - ss)^{\frac{3}{2}}} = \frac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots (2m - 1)} \beta f^{2m-1}.$$

The coefficient of the other term is known by the same reasoning. For with the integration thus put in place, so that the integral vanishes on putting $s = 0$, [and on setting $s = f$] through the series :

$$\int (f^2 - s^2)^m ds = -f^{2m}(f - s) + \frac{m}{1 \cdot 3} f^{2m-2}(f^3 - s^3) - \frac{m(m-1)}{1 \cdot 2 \cdot 5} f^{2m-4}(f^5 - s^5)$$
$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3 \cdot 7} f^{2m-6}(f^7 - s^7) - \text{etc.}$$

Therefore this series multiplied by

$$-\frac{\alpha^m ds}{(ff - ss)^{\frac{3}{2}}}$$

and integrated, and then on putting $s = f$ gives :

$$\alpha^m f^{2m-1}\left(1 - \frac{2m}{1 \cdot 3} + \frac{4m(m-1)}{1 \cdot 3 \cdot 5} - \frac{8m(m-1)(m-2)}{1 \cdot 3 \cdot 5 \cdot 7} + \text{etc.}\right) = \frac{\alpha^m f^{2m-1}}{2m+1}.$$

Therefore with these two values equal to each other, there is produced : [p. 406]

$$\beta = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2m-1)\alpha^m}{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2m(2m+1)},$$

and with this value substituted in place of β, the following equation is obtained for the tautochrone curve pertaining to the descent, hence on putting $\frac{1}{a}$ in place of the homogeneous term α,

$$s = \sqrt{gax} - \frac{1 \cdot 3 \cdot 5 \cdots (2m-1) g^m a x^m}{2 \cdot 4 \cdot 6 \cdots 2m(2m+1) 2k^m}.$$

And the time of any one descent on this curve is equal to $\frac{\pi\sqrt{\alpha}}{2}$, or the length for the pendulum *in vacuo*, with natural gravity acting equal to 1 to complete the descent in the same time, is equal to $\frac{a}{2}$. Likewise the equation for the tautochrone curve is modified in the equation for the curve, upon which all the ascents in equal times are completed, clearly in the time $\frac{\pi\sqrt{\alpha}}{2}$, if in place of k^m is written $-k^m$. Q.E.I.

Corollary 1.

739. If $2m+1>2$ or $m>\frac{1}{2}$, the radius of osculation of the curve at the point A is the same as that for the equation $gx = \frac{s^2}{a}$, clearly $\frac{ga}{2}$. Therefore in these cases the minimum descent is completed in the same time as *in vacuo*; or the descent on the lowest of the smallest part of the same infinitely small part likewise both *in vacuo* and in a resisting medium, but only if $m > \frac{1}{2}$.

Corollary 2. [p. 407]

740. If $m = \frac{1}{2}$, in each term s has the dimensions 2. Whereby the radius of osculation at A is not greater than $\frac{ga}{2}$, but is less than that value. Therefore in a medium which resists in the ratio of the speeds, the descent time is a minimum along a circular arc greater than *in vacuo* and thus in the ratio given.

Corollary 3.

741. If m should be $< \frac{1}{2}$, yet now > 0, then the radius of osculation is infinitely small at A ; hence upon this curve *in vacuo* the smallest descent is completed in an infinitely small time, as is made in a finite time in a resisting medium.

Scholium 1.

742. Since as concerns the case $m < \frac{1}{2}$, as we have said, this indeed follows from the equation that we put in place, and provides a satisfactory answer for the question in the rarest medium. In the case moreover in which $m < \frac{1}{2}$, the three terms from which the equation is constructed are not satisfied, even if the medium is the rarest. For the two terms which are equal to *gx*, must be considered as the two initial terms of converging series, in which the terms following vanish before the first. Now the whole series of this kind has the form : [p. 408]

$$gx = \frac{s^2}{a} + \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} + \frac{Cs^{6m-1}}{a^{3m-2}k^{3m}} + \text{etc.},$$

in which the exponents of *s* in are progression in an arithmetical progression ; and this form can be gathered partly from analogy, partly from that reason that we have used to find the second term. Now from this form it is apparent that the condition of the curve cannot be known from the two first terms at the lowest point *A*, if $m < \frac{1}{2}$, no matter how great *k* should be. For since the exponents of *s* decrease, yet in the following terms, s changes in the denominator and thus putting $s = 0$ makes $x = \infty$, from which it is apparent that the curve does not terminate in these cases at *A* and nor can the radius of osculation be defined in this place. Which inconvenience does not occur, if the exponents of *s* are increasing.

Scholium 2.

743. Hence it is therefore apparent the manner in which a tautochrone curve can be found, in a medium resisting in some power of the ratio of the speeds, even if the medium is not the most rare. Since indeed the equation for the tautochrone is of this form :

$$gx = \frac{s^2}{a} + \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} + \frac{Cs^{6m-1}}{a^{3m-2}k^{3m}} + \text{etc.},$$

as from the condition of tautochronism we have determined the value of the coefficient *A*, in the same manner also the coefficients of the remaining terms can be defined. But on account of so many formulas being composed, the labour of the integration becomes almost insurmountable, but it is perhaps of help if only a single coefficient from which B, or at the most two, *B* and *C* need to be applied in the determination, since the following can be concluded by analogy. [p. 409] It happens in the case in which $m = 1$, that this series becomes known, and clearly from above (724) we have :

$$gx = \frac{s^2}{a} + \frac{s^3}{6ak} + \frac{s^4}{48ak^2} + \frac{s^5}{480ak^3} + \text{etc.},$$

that bring more than a little help in finding the general series. Now the term *B* must be found from the following equation :

$$B\int\frac{(f^{4m}-s^{4m})ds}{(ff-ss)^{\frac{3}{2}}} = \frac{3}{4}A^2\int\frac{(f^{2m+1}-s^{2m+1})^2 ds}{(ff-ss)^{\frac{5}{2}}} + \frac{3}{2}A\int\frac{ds(f^{2m+1}-s^{2m+1})\int ds(ff-ss)^m}{(ff-ss)^{\frac{5}{2}}}$$
$$+\frac{3}{4}\int\frac{ds(\int ds(f^2-s^2)^m)^2}{(ff-ss)^{\frac{5}{2}}} - mA\int\frac{ds\int ds(ff-ss)^{m-1}(f^{2m+1}-s^{2m+1})}{(ff-ss)^{\frac{3}{2}}}$$
$$-m\int\frac{ds\int ds\,(ff-ss)^{m-1}\int ds(ff-ss)^m}{(ff-ss)^{\frac{3}{2}}},$$

thus the integrals of this equation are to be taken so that they vanish on putting $s = f$, on which being done it is necessary to put $s = 0$; and then the value of *B* can be found. Now the coefficient *A* thus has become known; for we have found that

$$A = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2m(2m+1)}.$$

For the case in which $m = 1$, I have extracted the above equation and found that $B = \frac{1}{48}$ in taking $A = \frac{1}{6}$, which agrees uncommonly well. But if the values of the coefficients are to be known indefinitely, then a certain constant equation should be found for the infinite series for the tautochrone sought ; which moreover, if the rule should be known, by using my method of summing series [L. Euleri Commentatio 25 (E25) : *General method of summing progressions,* Comment. acad. sc. Petrop. 6 (1732/3), 1738; L. Euleri *Opera Omnia*, series 1, vol. 14.] it is possible to transform the equation of the terms into a finite constant number.[p. 410] And this I judge to be about the safest method, and with the help of this, the tautochrones in other hypotheses of the resistance can be found.

Corollary 4.

744. Therefore if the equation :

$$gx = \frac{s^2}{a} + \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} + \text{etc.}$$

expresses the tautochrone of the descent, then the tautochrone of the ascent is produced by this curve :

$$gx = \frac{s^2}{a} - \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} - \frac{Cs^{6m-1}}{a^{3m-2}k^{3m}} + \text{etc.},$$

which arises on putting k^m negative.

Corollary 5.

745. Hence it is evident that whenever *m* is a whole number, so the tautochrone serving for the ascents *ANC* (Fig. 83) is a continuation of the tautochrone of the descent *BMA*. Indeed the same equation arises, and either k^m or *s* is put negative.

Corollary 6.

746. It is understood besides that the curve *ANC*, upon which all the ascents are completed in the same time in which the descents on the curve *BMA*, is less than the curve *BMA* and the cusp *C* to have a higher place, as in the medium that resists in the ratio of the square of the speeds.

Corollary 7. [p. 411]

747. In the medium that resists in the simple ratio of the speeds, all the exponents of *s* are made equal to 2; from which it follows that the tautochrone of the descents as of the ascents are semi-cycloids. Now that [curve] for the ascent, since s^2 has a smaller coefficient than for the descent, has been generated by a greater circle, if indeed the times of the ascent are to be equal to the times of the descent. Now meanwhile the same cycloid produces successive semi-oscillations also; now the times of the ascents are smaller than the times of the descents.

Scholium 3.

748. If *m* is a whole number, the value of *A* can be defined easily from the given form. And if $m = 1$, then $A = \frac{1}{6}$, if $m = 2$, then $A = \frac{3}{40}$ and so on thus. But if m in not a whole number, it is more difficult to show the value of *A* ; indeed the series of values of *A* must be interpolated. Indeed for fractions, the denominator of which is 2, the value of A can be defined through the quadrature of the circle. For on putting π for the perimeter of the circle, of which the diameter is 1, it becomes as follows :

m	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$,	$\frac{7}{2}$	etc.
A	$\frac{1}{\pi}$,	$\frac{2}{3}\cdot\frac{1}{2\pi}$,	$\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{3\pi}$,	$\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{4\pi}$	etc.

From which it appears, if $m=\frac{2n+1}{2}$, that

$$A=\frac{2\cdot 4\cdot 6\cdots\cdot 2n}{3\cdot 5\cdot 7\cdots\cdot(2n+1)}\cdot\frac{1}{(n+1)\pi}$$

and thus [p. 412]

$$gx=\frac{s^2}{a}+\frac{2\cdot 4\cdot 6\cdots\cdot 2n\cdot s^{2n+2}}{3\cdot 5\cdot 7\cdots\cdot(2n+1)(n+1)\pi a^m k^m},$$

clearly on putting $m=\frac{2n+1}{2}$. Moreover, whatever the fraction m becomes, it is always the case that :

$$A=\frac{1}{(2m+1)^2\int dz(1-z)^m}$$

thus on taking this integral, so that it vanishes on putting $z = 0$, and then on putting $z = 1$, as I have deduced in Comment. A acad. sc. Petrop. 5. (1730/1), in the Dissertation *Concerning the progressions of transcendental numbers.* [E19].

Corollary 8.

749. If now for the individual mediums of the rarest resistances the tautochrones of the descents are sought, these themselves are obtained, as follows:

$m = 0$	$gx = \frac{s^2}{a} + s$
$m = \frac{1}{2}$	$gx = \frac{s^2}{a} + \frac{s^2}{\pi\sqrt{ak}}$
$m = 1$	$gx = \frac{s^2}{a} + \frac{s^3}{6ak}$
$m = \frac{3}{2}$	$gx = \frac{s^2}{a} + \frac{s^4}{3\pi ak\sqrt{ak}}$
$m = 2$	$gx = \frac{s^2}{a} + \frac{3s^5}{40a^2k^2}$
$m = \frac{5}{2}$	$gx = \frac{s^2}{a} + \frac{8s^6}{45\pi a^2k^2\sqrt{ak}}$
$m = 3$	$gx = \frac{s^2}{a} + \frac{5s^7}{112a^3k^3}.$

From these the tautochrones of the ascents are formed, if the last terms become negative.

Scholion 4. [p. 413]

750. These are therefore sufficient for the simple tautochrones, upon which either all the descents only or all the ascents are completed in equal times. But as well as these curves, also other curves can be called tautochrones, upon which either all the semi-oscillations or even all the whole oscillations are isochronous; the number of which curves as *in vacuo* is infinite. But since it pertains to whole oscillations, this question of the resistance is appropriate; indeed *in vacuo* all the semi-oscillations are equal to each other. But since for these questions two curves are proposed to be found, of which the first pertains to the ascent, and the second to the descent, before we solve questions of this kind for tautochronism, we present other easer propositions regarding the ascent and descent of curves.

PROPOSITION 83.

Problem.

751. *According to the hypothesis of a uniform force acting downwards and a uniform medium that resists in the square ratio of the speeds, for the given curve MA* (Fig. 84) *to find the other curve AN of this kind joined to that at A, in order that the body descending on some arc MA on the given curve by ascending on the curve sought completes the arc AN which is equal to the arc MA.*

Solution. [p. 414]

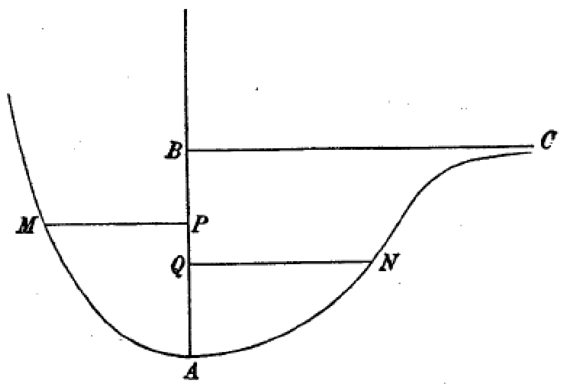

Fig. 84.

On putting the force acting *g*, with the exponent of the resistance *k* and with the speed at *A*, that it has acquired from the descent on the given curve, on each sought curve the ascent *AN* begins, corresponding to the height *b*, for the given curve let the abscissa $AP = x$, the arc $AM = s$; for the sought curve now let the abscissa $AQ = t$ and the arc $AN = r$. With these in place the height corresponding to the speed of the descending body at *M* is equal to :

$$e^{\frac{s}{k}}\left(b-g\int e^{\frac{-s}{k}}dx\right)$$

and the height corresponding to the speed of the ascending body at *N* is equal to :

$$e^{\frac{-r}{k}}\left(b-g\int e^{\frac{r}{k}}dt\right).$$

Therefore the descent of the whole arc comes about from this equation :

$$b=g\int e^{\frac{-s}{k}}dx,$$

now the whole arc of the ascending body comes from this equation :

$$b=g\int e^{\frac{r}{k}}dt.$$

Therefore between the arc of the descent and the arc of the ascent, we have this equation :

$$\int e^{\frac{-s}{k}} dx = \int e^{\frac{r}{k}} dt$$

or the differential of this :

$$e^{\frac{-s}{k}} dx = e^{\frac{r}{k}} dt.$$

But since the curve *MA* is given, the equation is given between *s* and *x*; from which, if in place of *dx* the value of this replaced by *s* and *ds*, the equation is produced between *t* and *s* or between *t* and *r* since $r = s$; which determines the nature of the curve sought *AN*. Q.E.I.

Corollary1. [p. 415]

752. If the lower part of the given curve *MA* is expressed by the equation $x = \alpha s^n$, then for the lower part of the curve AN there is this equation :

$$dt = \alpha n e^{\frac{-2s}{k}} s^{n-1} ds,$$

Now indeed because the smallest arc *s* is given by :

$$e^{\frac{-2s}{k}} = 1 - \frac{2s}{k},$$

thus it becomes :

$$t = \alpha s^n - \frac{2\alpha n s^{n+1}}{(n+1)k}$$

or only $t = \alpha s^n$. Therefore the smallest parts of each curve are similar to each other.

Corollary 2.

753. From the equation $e^{\frac{-2s}{k}} dx = dt$ it is understood that always $dt < dx$ or $t < x$. Therefore the point *N* is always put lower than the corresponding point *M*. From which it follows that the curve *AN* bends less towards *AB* than the curve *AM*.

Corollary 3.

754. On this account the curve *ANC* cannot be similar and equal to the curve *AM*, since in this case the points *M* and *N* are equal to the arcs *AM* and *AN* with the ends to be placed at the same height.

Corollary 4.

755. If the body on the curve *MA* descends from an infinite height, since at *A* it has acquired only a finite speed, then it is able to ascend to a finite altitude only. Therefore in this case, since the curve *AM* is extended to infinity, the curve *ANC* is unable to ascend beyond a certain height, or has a horizontal asymptote *BC*. Is also apparent from this that if one puts $s = \propto$; then indeed there is produced $dt = 0$.

Scholium 1. [p. 416]

756. As it is understood from this proposition, how the curve of ascent *AN* can be found from the given curve of descent *MA*, thus hence in turn it is easy from the given curve *AN* to define the other. If indeed the equation is given between *t* and *s*, then

$$dx = e^{\frac{2s}{k}} dt$$

is the equation for the curve AM.

Corollary 5.

757. Since the curve of the ascents *AN* corresponds to the curve of the descents *MA*, the equation of this is :

$$dt = e^{\frac{-2s}{k}} dx ,$$

thus, if this curve AN is taken for the curve of the descents, the corresponding abscissa of the curve of the ascents is equal to

$$\int e^{\frac{-2s}{k}} dt = \int e^{\frac{-4s}{k}} dx.$$

Corollary 6.

758. If in this manner further corresponding curves are sought, then the following series of equations :

Abscissa of the curve corresponding to the arc s

$$\begin{aligned} \text{I} &= x, \\ \text{II} &= \int e^{\frac{-2s}{k}} dx, \\ \text{III} &= \int e^{\frac{-4s}{k}} dx, \\ \text{IV} &= \int e^{\frac{-6s}{k}} dx, \\ &\vdots \\ n &= \int e^{\frac{-2(n-1)s}{k}} dx. \end{aligned}$$

Corollary 7. [p. 417]

759. Therefore two contiguous curves of this series have this property, that for these connected at the lowest point *A* the body descending on the first curve ascends on the other an arc equal to the descending arc. Moreover, the smallest of this series of curves is the horizontal straight line, on putting $n = \infty$, since $e\frac{-\infty s}{k}$ vanishes, and thus is the abscissa of the curve itself.

Example 1.

760. Let the given line be the right vertical line ; then we have $x = s$. Hence *ANE* is sought for the curve of the ascension (Fig. 85) on assuming $AQ = t$ and $AN = s$ this equation is obtained :

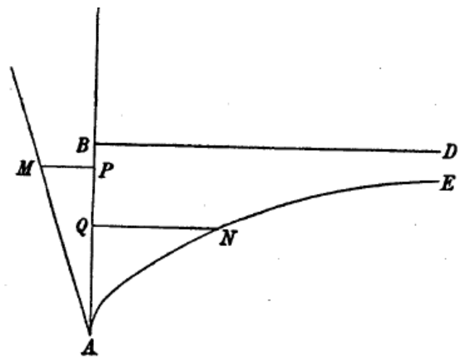

Fig. 85.

$$dt = e^{\frac{-2s}{k}} ds \text{ or } t = \frac{k}{2}\left(1 - e^{\frac{-2s}{k}}\right).$$

And with the exponential quantity eliminated, then the equation is :

$$2tds = kds - kdt \text{ or } \frac{(\frac{1}{2}k - t)ds}{dt} = \frac{1}{2}k.$$

From which equation it is evident that the curve *ANE* is a tractrix arising from a string of length $\frac{1}{2}k$ upon the asymptote to the horizontal *BD*. Whereby the height *AB* of the asymptote is then equal to $\frac{1}{2}k$. Now if this tractrix *ANE* is itself taken for the curve of descent then to that there corresponds the curve sought, the abscissa of this corresponding to the arc s is equal to $\int e^{\frac{-4s}{k}} ds$; [p. 418] which curve therefore is again a tractrix having a horizontal asymptote, of which the asymptote has been raised by the interval $\frac{1}{4}k$ above *A*, to which the length of the string is equal. Now all the curves of the above series are tractrices which are generated by strings of which the lengths constitute this series : $\frac{k}{0}, \frac{k}{2}, \frac{k}{4}, \frac{k}{6}$ etc. Clearly the vertical straight line is to be considered as the tractrix of which the generating string is $\frac{k}{0}$ or infinite. Moreover the last of this series of tractrices becomes the horizontal straight line drawn through *A*.

Example 2.

761. If the line of the descents is a straight line *MA* inclined in some manner to the horizontal (Fig. 85), thus in order that *MA* (s): *AP* (x) = α :1 or $dx = \frac{ds}{\alpha}$, this equation is obtained for the curve sought *AN* :

$$\alpha dt = e^{\frac{-2s}{k}} ds,$$

and the integral of this is :

$$\alpha t = \frac{k}{2}(1 - e^{\frac{-2s}{k}}).$$

From which equations for the adjoining curve arises :

$$2\alpha tds = kds - \alpha kdt \text{ or } \frac{\left(\frac{k}{2\alpha} - t\right)ds}{dt} = \frac{k}{2}.$$

Which is also the equation for generating the above tractrix, with a string of length $\frac{k}{2}$ with the horizontal asymptote BD, on taking $AB=\frac{k}{2\alpha}$; and this tractrix must pass through *A*. The following curves of the series are all tractrices also, as in the preceding example, of which the generating strings are $\frac{k}{0},\frac{k}{2},\frac{k}{4},\frac{k}{6}$ etc., now of these asymptotes the distances from the point *A* are held in this progression [p. 419] $\frac{k}{0\alpha},\frac{k}{2\alpha},\frac{k}{4\alpha},\frac{k}{6\alpha}$ etc. Clearly all these tractrices make an angle with the vertical axis *AB* equal to the angle PAM.

Corollary 8.

762. Of these tractrices that, which first or precedes the line *MA*, hence has this property that the body descending on that afterwards ascending on the line *AM* traverses equal distances.

Corollary 9.

763. Therefore according to the curve of the descents *CA* (Fig. 86) to be found, to that corresponds the right inclined line *AM*, above the asymptote to the horizontal, the tractrix CA is described by a string of length $\frac{k}{2}$ and on that the applied line $Ab=\frac{k}{2\alpha}$ is taken, and from A the right inclined line *AM* is constructed ; and then *CA* is the curve of the descents, to which there corresponds the line *AM* for the ascents.

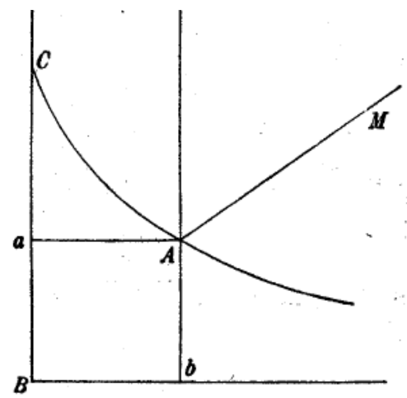

Fig. 86.

Scholium 2.

764. Here the counterpart case can be of service, examples of the inverse problem, in which from a given curve of the ascents the curve of the descents is required.

Example 3. [p. 420]

765. Let the curve of the descents be the given cycloid *MA* (Fig. 84), the nature of which can be expressed by the equation

$$2ax=s^2$$

or the diameter of the generating circle is equal to $\frac{a}{2}$. Hence it follows that

$$dx=\frac{sds}{a},$$

and thus the equation is found for the other curve of the ascents *AN*

$$adt=e^{\frac{-2s}{k}}sds,$$

the integral of this is :

$$at = \frac{k^2}{4}\left(1 - e^{\frac{-2s}{k}}\right) - \frac{k}{2} e^{\frac{-2s}{k}} s,$$

which on account of

$$e^{\frac{-2s}{k}} = \frac{adt}{sds}$$

goes into this equation :

$$atsds = -\frac{ak^2dt}{4} + \frac{k^2sds}{4} - \frac{aksdt}{2}.$$

This curve at *A*, as has now been said, has a horizontal tangent. Now also it has a horizontal asymptote *BC*; the height *BA* of this is found if *s* becomes equal to $\propto$. Moreover in this case making $e^{\frac{-2s}{k}} = 0$; whereby it then follows that $t = AB = \frac{k^2}{4a}$.

From this it is understood that the curve must have a point of inflection somewhere ; which can be found, if on putting *dt* constant there is placed *dds* = 0. Hence now there is produced $1 = \frac{2s}{k}$ or $s = \frac{k}{2}$. Whereby if the arc is taken $AN = \frac{k}{2}$, then *N* is the point of inflection; to which there corresponds the abscissa $AQ = \frac{k^2}{4a} - \frac{k^2}{4ae}$ or $BQ = \frac{k^2}{2ae}$. Concerning which is always the case that *AB* : *BQ* = *e* : 2 = 2,71828 : 2.

Scholium 3.

766. This problem was proposed anonymously in the Act. Lips. A. 1728 [*Problema geometris propositum*, p. 528.] and the solution of this was given in the Comment. Acad. Petrop.A. 1729 by the most distinguished D. Bernoulli who used another method. [Dan. Bernoulli, *A theorem concerning the curvilinear motion of bodies that experience resistance in proportion to the squares of their speed*, Comment. Acad. sc. Petrop.4 (1730/1), 1735, p. 136.] [p. 421] Now beside this condition, that anonymous person required chiefly a single continuous curve, one branch of which was for the descent, and the other serving for the ascent, and innumerable curves of this kind are given, which we uncover in the following proposition.

PROPOSITION 84.

Problem.

767. *With these things put in place as before, to find the continued curve of this kind MAN* (Fig. 84)*, in order that in some semi-oscillation, which always begins on the arc MA, upon that the arc of the descent MA made is equal to the arc of the following ascent AN.*

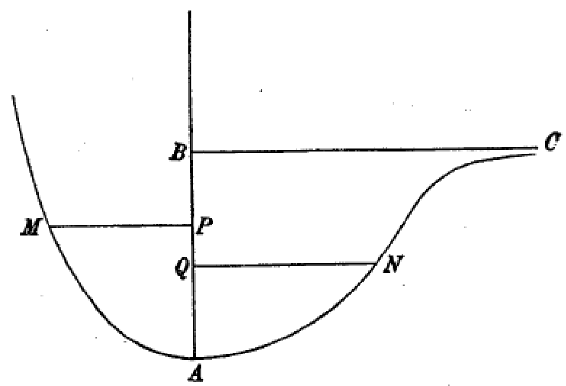

Fig. 84.

Solution.

This proposition differs from the preceding in this, that there the curve *MA* is given; now here that also must be found from this condition, that each curve *MA* and *AN* must make one and the same continuous curve . Therefore with the arcs *AM* and *AN* taken equal to *s* and on putting *AP* = *x* and *AQ* = *t* then

$$dt = e^{\frac{-2s}{k}}dx.$$

Moreover since the curve *MAN* must be continuous, thus it is necessary to compare the equation between *s* and *x*, in order that, if in that equation in place of *s* there is put – *s* , in which case the arc *AM* becomes the arc *AN* , then the value of *x* becomes equal to *t*, or is equal to $\int e^{\frac{-2s}{k}}dx.$ [p. 422] Therefore I put *dx* = *Mds*, where *M* is a certain function of *s*, and that changes into *N*, if in place of *s* there is put – *s*. Hence – *s* is put in place of *s*, in which case *x* becomes *t*, and then *dt* = –*Nds*. Now also we have

$$dt = e^{\frac{-2s}{k}}dx = e^{\frac{-2s}{k}}Mds,$$

on account of which

$$N = -e^{\frac{-2s}{k}}M.$$

Again, let

$$M = -e^{\frac{+s}{k}}P.$$

and *P* is changed into *Q* on putting – *s* in place of *s* and then

$$N = e^{\frac{-s}{k}}Q.$$

With these values substituted in place of *M* and *N* there is produced *Q* = – *P* . From which it is apparent that P must be a function of *s* of this kind, that changes into – *P* on putting – *s* in place of *s,* which functions I have become accustomed to call odd. And thus let *P* be some odd function of *s*, of such a kind as are e. g. $\alpha s, \alpha s^3, \alpha s^5$ etc , and then

$$M = e^{\frac{s}{k}} Pds \text{ or } x = \int e^{\frac{s}{k}} Pds.$$

Which is the equation for the curve sought. Q.E.I.

Corollary 1.

768. Since

$$dx = e^{\frac{s}{k}} Pds,$$

this becomes on taking logarithms :

$$ldx = \frac{s}{k} + lP + lds.$$

This equation is differentiated again on putting ds and there is produced :

$$\frac{ddx}{dx} = \frac{ds}{k} + \frac{dP}{P} \text{ or } kPddx = Pdxds + kdxdP.$$

Which equation is free from exponentials.

Corollary 2.

769. Since *P* must be given in terms of *s*, the equation found does not have mixed variables; on account of which this is sufficient for the curves contained in that equation to be constructed.

Example. [p. 423]

770. We can put $P = \frac{s}{a}$; then the above equation becomes

$$ax = \int e^{\frac{s}{k}} sds = ke^{\frac{s}{k}} s - k^2 e^{\frac{s}{k}} + k^2,$$

which is the equation for perhaps the single most simple satisfactory curve. Now this equation on eliminating the exponential $e^{\frac{s}{k}}$ changes into this :

$$axsds = aksdx - k^2 adx + k^2 sds.$$

Or by setting out $e^{\frac{s}{k}}$ in a series, this equation is produced :

$$ax = \frac{s^2}{2} + \frac{s^3}{1 \cdot 3k} + \frac{s^4}{1 \cdot 2 \cdot 4k^2} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 5k^3} + \text{etc.}$$

Hence this curve has a horizontal tangent at *A* and the radius of oscillation in place is *a*. Because it is necessary that $dx < ds$, it is necessary that $e^{\frac{s}{k}} s < a$. Which therefore with this in place becomes $e^{\frac{s}{k}} s$, which expression with *s* increasing also increases, to equal *a*, where the curve *AM* has a vertical tangent and the point of returning. For the branch *AN* put $-s$ in place of $+s$ and this equation is obtained :

$$ax = \int e^{\frac{-s}{k}} sds \text{ or } dx = \frac{e^{\frac{-s}{k}} sds}{a}.$$

Whereby as long as it is the case that $e^{\frac{-s}{k}} s < a$, the curve does not become imaginary. But if at any point $e^{\frac{-s}{k}} s = a$, there the curve also has a point of reversion and a vertical diameter. [p. 424] But is always possible to happen, if a is taken large enough, that $e^{\frac{-s}{k}} s$ is always less than a, in which case the curve AN goes to infinity and has a horizontal asymptote BC. But $e^{\frac{-s}{k}} s = 0$ in the cases when $s = 0$ and $s = \propto$; hence it has a maximum value, if the differential of this is equal to zero; now in this case it becomes fit $k = s$ and

$$e^{\frac{-s}{k}} s = \frac{k}{e}$$

Whereby if we have $a > \frac{k}{e}$, then the curve has the asymptote BC, the height of which BA is equal to $\frac{k^s}{a}$. But if it is the case that $a < \frac{k}{e}$, then the curve AN using the other branch also has a point of reversion, which is determined from this equation : $a = e^{\frac{-s}{k}} s$. In the first case the curve AN must have a point of inflection, that is found from the equation $1 = \frac{s}{k}$; clearly the point is at N on taking the arc $AN = k$.

PROPOSITION 85.

Problem.

771. *According to the preceding hypothesis of gravity and resistance if the curve MA* (Fig. 84) *is given, upon which the descent is completed, to find the curve for the ascents with this property, that the time of each ascent is equal to the time of the preceding descent.*

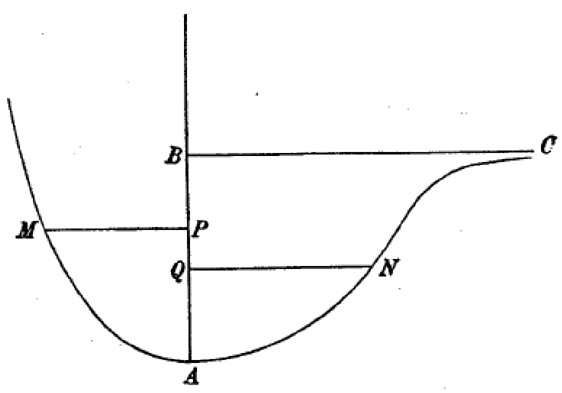

Fig. 84.

Solution.

On putting as before g for the force acting and the exponent of the medium is equal to k, for the curve MA let the abscissa $AP = x$, the arc $AM = s$ and for the curve sought AN, the abscissa $AQ = t$, and the arc $AN = r$. A height equal to b is put corresponding to a certain speed of descent acquired at A, with which speed the body completes the following ascent on the curve AN . [p. 425] With these in place the height corresponding to the speed at M is equal to :

$$e^{\frac{s}{k}}(b - g\int e^{\frac{-s}{k}}dx)$$

and the height corresponding to the speed at N in the ascent is equal to :

$$e^{\frac{-r}{k}}(b - g\int e^{\frac{r}{k}}dt).$$

Therefore the time of descent along the arc *MA* is equal to

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b-g\int e^{\frac{-s}{k}}dx)}}$$

and the time of the ascent along the arc *AN* is equal to :

$$\int \frac{e^{\frac{r}{2k}}dr}{\sqrt{(b-g\int e^{\frac{r}{k}}dt)}},$$

which two times, upon integrating, there is put

$$g\int e^{\frac{-s}{k}}dx = b \text{ and } g\int e^{\frac{r}{k}}dt = b,$$

must be equal. According to this being obtained, put

$$g\int e^{\frac{-s}{k}}dx = X \ , \ g\int e^{\frac{r}{k}}dt = T$$

and

$$\frac{ds}{e^{\frac{s}{2k}}} = dS \text{ and } e^{\frac{r}{2k}}dr = dR,$$

where *X, T, S* and *R* are such functions that vanish on putting *x*, *t*, *s* and *r* = 0. Hence this is brought about in order that the two integrals

$$\int \frac{dS}{\sqrt{(b-X)}} \text{ and } \int \frac{dR}{\sqrt{(b-T)}}$$

become equal to each other, if after integration there is put *X* = *b* and *T* = *b*. But *S* and *R* as both *X* and *T* are clearly quantities not depending on *b* and they must have the same relation between each other, whatever value *b* may have. Therefore what is sought is satisfied, if *R* is such a function of *T*, as *S* of *X*. Or on taking *R* = *S* there also corresponds *T* = *X*. Now [p. 426]

$$S = 2k\left(1 - e^{\frac{-s}{2k}}\right) \text{ and } R = 2k\left(e^{\frac{r}{2k}} - 1\right);$$

hence on making *R* = *S* then

$$2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}} \text{ and } r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Moreover since with this in place it is necessary that $X = T$ or $e^{\frac{-s}{k}}dx = e^{\frac{r}{k}}dt$, then it follows that $t = \int e^{\frac{-s-r}{k}}dx$. While now

$$e^{\frac{r}{k}} = \left(2 - e^{\frac{-s}{2k}}\right)^2 = 4 - 4e^{\frac{-s}{2k}} + e^{\frac{-s}{k}},$$

then

$$t = \int \frac{dx}{e^{\frac{s}{k}}\left(2 - e^{\frac{-s}{2k}}\right)^2} = \int \frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2}.$$

Hence from which the construction of the curve is known, since by taking the arc

$$AN = r = 2kl\left(2 - e^{\frac{-s}{2k}}\right)$$

to this there corresponds the abscissa

$$AQ = t = \int \frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2}.$$

Now the equation for the curve *AN* is more conveniently found from the given equation between *s* and *x*. For since

$$s = -2kl\left(2 - e^{\frac{r}{2k}}\right) \text{ and } x = \int \frac{dt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2},$$

if the values are put in place of *s* and *x*, then the equation is produced between *t* and *r* for the curve *AN* sought. Q.E.I.

Corollary 1.

772. Since it is the case that $r = 2kl(2 - e^{\frac{-s}{2k}})$, then it follows that

$$dr = \frac{e^{\frac{-s}{2k}}ds}{2 - e^{\frac{-s}{2k}}}.$$

Now since it must be the case that $dr > dt$, lest the curve *AN* becomes imaginary, the curve *AN* is real up to that point, and in correspondence

$$ds > \frac{dx}{e^{\frac{s}{2k}}\left(2 - e^{\frac{-s}{2k}}\right)} \text{ or } ds > \frac{dx}{2e^{\frac{s}{2k}} - 1}.$$

Corollary 2. [p. 427]

773. Now $e^{\frac{s}{2k}}$ is always greater than unity; from which it follows, that everywhere $ds > dx$, and from that

$$ds > \frac{dx}{2e^{\frac{s}{2k}} - 1}.$$

Whereby if the given curve is real, then the sought curve is always real also.

Corollary 3.

774. Since it is the case that $s = -2kl(2 - e^{\frac{r}{2k}})$, then

$$ds = \frac{e^{\frac{r}{2k}}dr}{2 - e^{\frac{r}{2k}}} = \frac{dr}{2e^{\frac{-r}{2k}} - 1},$$

thus an easier relation can be taken in the computation between *ds* and *dx*.

Corollary 4.

775. From the solution of the problem it is likewise apparent, how the inverse problem or this shall be solved. For if the ascending curve *AN* is given, or the equation between *t* and *r*, from that the equation between *x* and *s* is formed with the help of the equation

$$t = \int \frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2} \text{ and } r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Corollary 5. [p. 428]

776. Towards investigating the form of the curve around the point A , *s* and *r* are placed very small and then $e^{\frac{-s}{2k}} = 1$, hence *dr* becoming equal to *ds* and $dt = dx$. From which it is evident that the lowest parts of the curves *MA* and *NA* are similar and equal to each other.

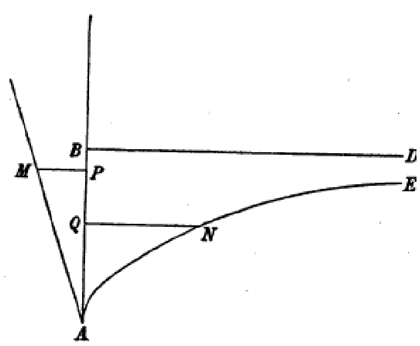

Fig. 85.

Example 1.

777. Let the given line of the descents be the right line MA (Fig. 85) at some inclination, in order that $s = \alpha x$ or $ds = \alpha dx$. Now since we have

$$ds = \frac{dr}{2e^{\frac{-r}{2k}} - 1} \text{ and } dx = \frac{dt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2},$$

this equation is obtained between *t* and *r* for the curve sought :

$$dr = \frac{\alpha dt}{2e^{\frac{-r}{2k}} - 1} \text{ or } \alpha dt = 2e^{\frac{-r}{2k}}dr - dr,$$

and the integral of this is :

$$\alpha t = 4k\left(1 - e^{\frac{-r}{2k}}\right) - r.$$

Which equation converted into a series gives :

$$\alpha t = r - \frac{r^2}{1\cdot 2k} + \frac{r^3}{1\cdot 2\cdot 3\cdot 2k^2} - \frac{r^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^3} + \text{etc.}$$

and thus at the lowest point A it is the case that $dr = \alpha dt$. This curve has a horizontal tangent somewhere, which place is found on putting $dt = 0$; now then $2 = e^{\frac{r}{2k}}$ or $r = 2kl2$, to which there corresponds $\alpha t = 2k - 2kl2$. And if *r* becomes greater than $2kl2$, the value of *dt* is made negative and thus the curve again descends, while $-dt$ becomes equal to *dr*; but this happens, if the equation is

$$1 - \alpha = 2e^{\frac{-r}{2k}} \text{ or } r = 2kl\frac{2}{1-\alpha}.$$

[p. 429] But since α cannot be less than 1, then if $\alpha = 1$, then the vertical tangent from *A* stands apart, and if $\alpha > 1$, then there is no vertical tangent beyond the horizontal tangent. But otherwise [?] beyond the tangent there is obtained a vertical, where it is the case that

$$r = -2kl\frac{1+\alpha}{2}.$$

Hence in this case, in which the given line is vertical or $\alpha = 1$, then $r = 0$ or the tangent at *A* is vertical.

Corollary 6.

778. If *s* denotes the whole length of the arc of descent, *r* expresses the whole arc described on the curve *AN* (Fig. 84) from the following ascent. Whereby if the arc of the descent *s* is given, the arc of the ascent can be found :

$$r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

For since when we put *T* = *X*, the letters *s* and *r* denote the whole arcs of the descent and of the ascent.

Example 2.

779. Let the given curve MA be the tautochrone of the descents, that we have previously found for the same hypothesis of the resistance; the curve AN has this property, that all the ascents are also completed in equal times, clearly the same by which the descents are completed on MA. Whereby the curve AN is a tautochrone of the ascents with the curve MA found before now continued. Moreover so that this can be shown from that calculation, we take the equation for the tautochrone curve of the descents, which is (719) [p. 430]

$$\text{either } adx = k\left(e^{\frac{s}{2k}} - 1\right)ds \text{ or } ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks.$$

Since now

$$ds = \frac{dr}{2e^{\frac{-r}{2k}} - 1} \text{ both } e^{\frac{s}{2k}} = \frac{e^{\frac{-r}{2k}}}{2e^{\frac{-r}{2k}} - 1} \text{ and } dx = \frac{dt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2},$$

with these substituted,

$$\frac{adt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2} = \frac{kdr\left(1 - e^{\frac{-r}{2k}}\right)}{\left(2e^{\frac{-r}{2k}} - 1\right)^2}$$

or

$$adt = k\left(1 - e^{\frac{-r}{2k}}\right)dr.$$

Which equation is formed from the former, if t is put in place of x and $-r$ for s. Whereby this curve *AN* has been continued, and with *MA* the tautochrone of the ascents.

Corollary 7.

780. Hence with the arc of the descent s given on the tautochrone of the descents, then the arc of the following ascent is the tautochrone of the ascents :

$$r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

And if the descent begins from the cusp of the tautochrone, the place of this is given by :

$$e^{\frac{s}{2k}} = \frac{a+k}{k}$$

(729), and the arc of the ascent is :

$$r = 2kl\frac{2a+k}{a+k},$$

as we have found above (732).

Scholium.

781. And thus since the tautochrone according to this hypothesis of the resistance satisfies the question and the curve is continued, hence we can seize the handle and more continuous curves can be investigated, of which the two branches of the curves in turn *MA* and *AN* are able to be sustained – which we present in the following proposition.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 402]

PROPOSITIO 82.

Problema.

738. *In medio rarissimo, quod resistit in quacunque multiplicata ratione celeritatum, et hypothesi potentiae uniformis deorsum tendentis determiare curvam tautochronam AM* (Fig. 80), *super qua vel omnes descensus vel ascensus aequalibus absolvantur temporibus.*

Solutio.

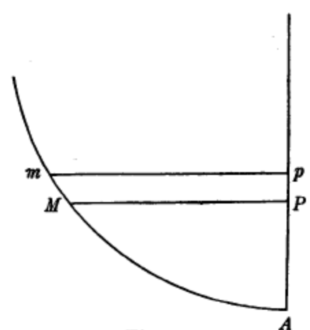

Fig. 80.

Positis abscissa $AP = x$ et arcu $AM = s$ sit totus arcus descensu aliquo descriptus = f. Potentia sollicitans deorsum ponatur = g, altitudo celeritati in M debita sit = v, exponens resisteniae = k atque ipsa resistentiae vis $\frac{v^m}{k^m}$, ubi k est quantitas valde magna, ita ut fractiones, quae in denominatore k plurium quam m dimensionum habent, pro evanescentibus haberi queant. His positis erit ex natura descensus
[p. 403]

$$dv = -gdx + \frac{v^m ds}{k^m}.$$

Iam si resistentia prorsus abesset, curva quaesita esset cyclois, cuius aequatio est

$$gx = \alpha s^2;$$

quia autem medium est rarissimum, curva quaesita non multum a cycloide differet; ponatur igitur aequatio pro curva quaesita haec

$$gx = \alpha s^2 + \frac{\beta s^n}{k^m}.$$

Quia vero est

$$dv = -gdx + \frac{v^m ds}{k^m}.$$

propter terminum $\frac{v^m ds}{k^m}$ valde parvum erit proxime

$$v = C - gx = C - \alpha s^2 - \frac{\beta s^n}{k^m},$$

ubi constans C ex hoc determinatur, quod, si fit $s = f$, fiat $v = 0$. Quocirca erit

$$v = \alpha(f^2 - s^2) - \frac{\beta}{k^m}(f^n - s^n) \text{ quam proxime.}$$

Ponatur ergo

$$v = \alpha(f^2 - s^2) - \frac{\beta}{k^m}(f^n - s^n) + Q;$$

erit

$$dv = -2\alpha s ds - \frac{n\beta s^{n-1} ds}{k^m} + dQ = -g dx + \frac{v^m ds}{k^m}$$

$$= -2\alpha s ds - \frac{n\beta s^{n-1} ds}{k^m} + \frac{\alpha^m (f^2 - s^2)^m ds}{k^m}$$

neglectis reliquis terminis, qui pro v^m poni deberent, quia in iis k plures quam m in denominatore habet dimensiones. Ex his iam sequitur fore

$$Q = \frac{\alpha^m}{k^m}\int (f^2 - s^2)^m ds$$

integrali hoc ita accepto, ut evanescat posito $s = f$. Erit ergo

$$v = \alpha(f^2 - s^2) + \frac{\beta}{k^m}(f^n - s^n) + \frac{\alpha^m}{k^m}\int (f^2 - s^2)^m ds$$

atque hinc [p. 404]

$$\frac{1}{\sqrt{v}} = \frac{1}{\sqrt{\alpha(f^2 - s^2)}} - \frac{\beta(f^n - s^n) + \alpha^m \int (f^2 - s^2)^m ds}{2\alpha^{\frac{3}{2}} k^m (ff - ss)^{\frac{3}{2}}}$$

omittendis iterum terminis sequentibus ob memoratam rationem. Hinc nunc prodit tempus

$$\int \frac{ds}{\sqrt{v}} = \int \frac{ds}{\sqrt{\alpha(f^2 - s^2)}} - \frac{\beta}{2\alpha^{\frac{3}{2}} k^m}\int \frac{ds(f^n - s^n)}{(ff - ss)^{\frac{3}{2}}} - \frac{\alpha^m}{2\alpha^{\frac{3}{2}} k^m}\int \frac{ds \int (f^2 - s^2)^m ds}{(ff - ss)^{\frac{3}{2}}},$$

quod ita integratum, ut evanescat posito $s = 0$, dabit tempus, quo corpus descendens arcum $MA = s$ absolvit. Totum ergo descensus tempus per arcum f obtinebitur, si post integrationem ponatur $s = f$; quod tempus debet esse constans seu ita comparatum, ut non pendeat ab f. Primus autem temporis terminus

$$\int \frac{ds}{\sqrt{\alpha(f^2 - s^2)}}$$

posito post integrationem $s = f$ iam dat huiusmodi expressionem, in qua non amplius inest f; prodit enim $\frac{\pi}{2\sqrt{\alpha}}$. Quamobrem si duo reliqui termini ita essent comparati, ut, postquam est $s = f$, sese destruant, quaesito foret satisfactum; haberetur enim pro integro descensus tempore haec expressio $\frac{\pi}{2\sqrt{\alpha}}$, quae est constans. Debebit ergo esse

$$\beta\int\frac{ds(f^n-s^n)}{(ff-ss)^{\frac{3}{2}}}=-\alpha^m\int\frac{ds\int(f^2-s^2)^m ds}{(ff-ss)^{\frac{3}{2}}}$$

posito post integrationem $s = f$. Dat autem [p. 405]

$$\int\frac{ds(f^n-s^n)}{(ff-ss)^{\frac{3}{2}}}$$

posito $s = f$ multiplum potestatis ipsius f, cuius exponens est $n - 2$, atque alterum integrale

$$\int\frac{ds\int(f^2-s^2)^m ds}{(ff-ss)^{\frac{3}{2}}}$$

posito $s = f$ dat multiplum potestatis ipsius f, cuius exponens est $2m - 1$. Fiat igitur $n = 2m + 1$ atque β ita sumatur, ut hae duae potestates ipsius f, quarum utriusque expons erit $2m - 1$, sese destruant. Quo autem appareat, quales coefficientes habiturae sint istae ipsius f potestates, ex casibus simpliciorbus concludemus. Integratis igitur sequentibus formulis, ita ut evanescant posito $s = 0$, tumque facto $s = f$ reperietur

$$\int\frac{(f-s)ds}{(ff-ss)^{\frac{3}{2}}}=f^{-1}\quad\text{atque}\quad\int\frac{(f^3-s^3)ds}{(ff-ss)^{\frac{3}{2}}}=\frac{2}{1}f^1\quad\text{et}\quad\int\frac{(f^5-s^5)ds}{(ff-ss)^{\frac{3}{2}}}=\frac{2\cdot 4}{1\cdot 3}f^3;$$

hincque erit generaliter

$$\int\frac{(f^{2m+1}-s^{2m+1})ds}{(ff-ss)^{\frac{3}{2}}}=\frac{2\cdot 4\cdot 6\cdots\cdot 2m}{1\cdot 3\cdot 5\cdots\cdot(2m-1)}f^{2m-1},$$

unde erit

$$\beta\int\frac{ds(f^n-s^n)}{(ff-ss)^{\frac{3}{2}}}=\frac{2\cdot 4\cdot 6\cdots\cdot 2m}{1\cdot 3\cdot 5\cdots\cdot(2m-1)}\beta f^{2m-1}.$$

Alterius termini coefficiens ex eadem analogia innotescat. Est enim integratione ita instituta, ut integrale evanescat posito $s = 0$, per seriem

$$\int(f^2-s^2)^m ds=-f^{2m}(f-s)+\frac{m}{1\cdot 3}f^{2m-2}(f^3-s^3)-\frac{m(m-1)}{1\cdot 2\cdot 5}f^{2m-4}(f^5-s^5)$$
$$+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3\cdot 7}f^{2m-6}(f^7-s^7)-\text{etc.}$$

Haec igitur series in

$$-\frac{\alpha^m ds}{(ff-ss)^{\frac{3}{2}}}$$

ducta et integrata atque tum posito $s = f$ dabit

$$\alpha^m f^{2m-1}\left(1-\frac{2m}{1\cdot 3}+\frac{4m(m-1)}{1\cdot 3\cdot 5}-\frac{8m(m-1)(m-2)}{1\cdot 3\cdot 5\cdot 7}+\text{etc.}\right)=\frac{\alpha^m f^{2m-1}}{2m+1}.$$

His igitur duobus valoribus inter se aequatis prodit [p. 406]

$$\beta=\frac{1\cdot 3\cdot 5\cdot 7\cdots(2m-1)\alpha^m}{2\cdot 4\cdot 6\cdot 8\cdots 2m(2m+1)},$$

quo valore loco β substituto habebitur pro curva tautochrona ad descensus pertinente, posita $\frac{1}{a}$ loco α homogeneitatis ergo, sequens aequatio

$$s=\sqrt{gax}-\frac{1\cdot 3\cdot 5\cdots(2m-1)g^m ax^m}{2\cdot 4\cdot 6\cdots 2m(2m+1)2k^m}.$$

Atque tempus uniuscuiusque descensus super hac curva erit = $\frac{\pi\sqrt{\alpha}}{2}$ seu longitudo pendulo in vacuo a gravitate naturali = 1 sollicitati eodem tempore descensus absolventis erit = $\frac{a}{2}$. Eadem aequatio pro curva tautochrona mutatur in aequationem pro curva, super qua omnes ascensus aequalibus temporibus, tempore scilicet $\frac{\pi\sqrt{\alpha}}{2}$, absolvuntur, si loco k^m scribatur $-k^m$. Q.E.I.

Corollarium 1.

739. Si $2m+1>2$ seu $m>\frac{1}{2}$, curvae radius osculi in puncto A idem erit qui pro aequatione $gx=\frac{s^2}{a}$, scilicet $\frac{ga}{2}$. In his ergo casibus corpus minimum descensum absolvit eodem tempore quo in vacuo; seu descensus super infima curvae portiuncula infinite parva idem erit in vacuo et in medio resistente, si modo $m>\frac{1}{2}$.

Corollarium 2. [p. 407]

740. Si $m=\frac{1}{2}$, in utroque termino s habebit duas dimensiones. Quare radius osculi in A non amplius erit $\frac{ga}{2}$, sed eo erit minor. In hoc ergo medio, quod in ratione celeritatum resistit, tempus descensus minimi per arculum circularem maius erit quam in vacuo hocque in data ratione.

Corollarium 3.

741. Si m fueret $< \frac{1}{2}$, verum tamen > 0, radius osculi in A erit infinite parvus; super hac ergo curva in vacuo minimus descensus absolveretur tempore infinite parvo, cum in medio resistente finito tempore perficiatur.

Scholion 1.

742. Quod ad hunc casum $m < \frac{1}{2}$ attinet, hoc, quod diximus, quidem ex aequatione sequitur, quam in medio rarissimo quaesito plene satisfacere ponimus. In casu autem, quo $m < \frac{1}{2}$, tres termini, ex quibus aequatio consistit, etiamsi medium sit rarissimum, non satisfaciunt. Duo enim termini quibus gx aequatur, considerari debent tanquam dup termini initiales seriei convergentis, in qua sequentes prae primis evanescant. Tota vero series huiusmodi habebit formam [p. 408]

$$gx = \frac{s^2}{a} + \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} + \frac{Cs^{6m-1}}{a^{3m-2}k^{3m}} + \text{ etc.},$$

in qua exponentes ipsius s in progressione arithmetica progrediuntur; haecque forma partim ex analogia, partim ea ipsa ratione, qua ad secundum terminum inveniendum usi sumus, colligi potest. Ex hac iam forma apparet curvae in puncto infimo A conditionem, si $m < \frac{1}{2}$, ex duobus terminis primis cognosci non posse, quantumvis k sit magnum. Quia enim exponentes ipsius s descrescunt, in sequentibus terminis s tandem in denominatorem migrabit ideoque posito $s = 0$ fiet $x = \infty$, ex quo apparet curvam his casibus in A non terminari neque radium osculi in hoc loco posse definiri. Quod incommodum non habet locum, si exponentes ipsius s crescunt.

Scholion 2.

743. Hinc igitur patet modus curvam tautochronam in medio in quacunque celeritatum ratione multiplicata resistente inveniendi, etiamsi medium non fuerit rarissimum. Cum enim aequatio pro tautochrona sit hius formae

$$gx = \frac{s^2}{a} + \frac{As^{2m+1}}{a^m k^m} + \frac{Bs^{4m}}{a^{2m-1}k^{2m}} + \frac{Cs^{6m-1}}{a^{3m-2}k^{3m}} + \text{ etc.},$$

quaemadmodum ex conditione tautochronism valorem coefficientis A determinavimus, eodem modo etiam coefficientes reliquorum terminorum poterunt definiri. At propter tantopere compositas formulas integrales labor fere fit insuperabilis, qui autem forte sublevabitur, si quis tantum unicum coefficientem B vel ad summum duos B et C determinandi operam adhibuerit, quoniam sequentes ex analogia concludi possent. [p. 409] Ad haec accedit, quod haec series in casu, quo m = 1, cognita sit, quippe ex superioribus (724) habemus

$$gx = \frac{s^2}{a} + \frac{s^3}{6ak} + \frac{s^4}{48ak^2} + \frac{s^5}{480ak^3} + \text{ etc.},$$

quae series ad generalem inveniendam non parum subsidii afferet. Terminus vero *B* inveniri debet ex sequente aequatione

$$B\int\frac{(f^{4m}-s^{4m})ds}{(ff-ss)^{\frac{3}{2}}}=\frac{3}{4}A^2\int\frac{(f^{2m+1}-s^{2m+1})^2ds}{(ff-ss)^{\frac{5}{2}}}+\frac{3}{2}A\int\frac{ds(f^{2m+1}-s^{2m+1})\int ds(ff-ss)^m}{(ff-ss)^{\frac{5}{2}}}$$
$$+\frac{3}{4}\int\frac{ds(\int ds(f^2-s^2)^m)^2}{(ff-ss)^{\frac{5}{2}}}-mA\int\frac{ds\int ds(ff-ss)^{m-1}(f^{2m+1}-s^{2m+1})}{(ff-ss)^{\frac{3}{2}}}$$
$$-m\int\frac{ds\int ds\,(ff-ss)^{m-1}\int ds(ff-ss)^m}{(ff-ss)^{\frac{3}{2}}},$$

cuius aequationis integralia ita sunt sumenda, ut evanescant posito $s = f$, quo facto poni debet $s = 0$; atque tum valor ipsius *B* invenietur. Coefficiens vero *A* iam cognitus est; invenimus enim

$$A=\frac{1\cdot 3\cdot 5\cdots\cdots(2m-1)}{2\cdot 4\cdot 6\cdots\cdots 2m(2m+1)}.$$

Pro casu, quo $m = 1$, superiorem aequationem evolvi invenique $B=\frac{1}{48}$ existente $A=\frac{1}{6}$, id quod egregie congruit. Si autem valores coefficientium in infinitum innotescerent, tum haberetur quidem aequatio serie infinita constans pro tautochrona quaesita; quae autem, si eius lex cognita fuerit, per methodum meam seriem summandi [L. Euleri Commentatio 25 (E25) : *Methodus generalis summandi progressiones*, Comment. acad. sc. Petrop. 6 (1732/3), 1738; L. Euleri *Opera Omnia*, series 1, vol. 14.] in aequationem terminorum numero finito constantem transformari poterit. [p. 410] Atque hanc propemodum unicam ut tutissimam iuciso methodum, cuius ope tautochronae in aliis resistentiae hypothesibus invenire queant.

Corollarium 4.

744. Si igitur aequatio

$$gx=\frac{s^2}{a}+\frac{As^{2m+1}}{a^mk^m}+\frac{Bs^{4m}}{a^{2m-1}k^{2m}}+\text{etc.}$$

exprimat tautochronam descensuum, ascensus tautochronos producet ista curva

$$gx=\frac{s^2}{a}-\frac{As^{2m+1}}{a^mk^m}+\frac{Bs^{4m}}{a^{2m-1}k^{2m}}-\frac{Cs^{6m-1}}{a^{3m-2}k^{3m}}+\text{etc.},$$

quae oritur ponendo k^m negativum.

Corollarium 5.

745. Perspicitur hinc, quoties m fuerit numerus integer, toties tautochronam ascensibus inservientem *ANC* (Fig. 83) esse continuam tautochronae descensuum *BMA*. Eadem enim aequatio oritur, sive k^m sive *s* ponatur negativum.

Corollarium 6.

746. Praeterea intelligitur curvam *ANC*, super qua omnes ascensa eodem tempore absolvuntur quo descensus super curva *BMA*, minus esse curvam quam *BMA* et cuspidem *C* altius habere positam, quemadmodem in medio, quod in duplicata celeritatun ratione resistit.

Corollarium 7. [p. 411]

747. In medio, quod in simplici ratione celeritatum resistit, omnes ipsius s exponentes fiunt = 2; ex quo sequitur tam tautochronam descensuum quam ascensum esse semicycloides. Ea vero pro ascensu, quia s^2 minorem habet coefficientem quam pro descensu, maore circulo erit genita, si quidem tempora ascensum aequalia esse debeant temporibus descensuum. Interim vero eadem cyclois continua semioscillationes quoque producit isochronas; ascensuum vero tempora minora quam tempora descensuum.

Scholion 3.

748. Si *m* est numerus integer, facile ex data forma potest ipsius *A* valor definiri. Namque si *m* = 1, erit $A = \frac{1}{6}$, si *m* = 2, erit $A = \frac{3}{40}$ et ita porro. At si *m* non est numerus integer, difficilius est valorem *A* exhibere; series enim valorum ipsius *A* debet interpolari. Pro fractionibus quidem, quarum denominator est 2, valor ipsius *A* per quadraturam circuli potest definire. Posito enim π pro perimetro circuli, cuius diametri est 1, erit , ut sequitur :

m	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$,	$\frac{7}{2}$	etc.
A	$\frac{1}{\pi}$,	$\frac{2}{3}\cdot\frac{1}{2\pi}$,	$\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{3\pi}$,	$\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{4\pi}$	etc.

Ex quo patet, si fuerit $m = \frac{2n+1}{2}$, fore

$$A = \frac{2\cdot 4\cdot 6\cdots\cdot 2n}{3\cdot 5\cdot 7\cdots\cdot(2n+1)}\cdot\frac{1}{(n+1)\pi}$$

atque adeo [p. 412]

$$gx = \frac{s^2}{a} + \frac{2\cdot 4\cdot 6\cdots\cdot 2n\cdot s^{2n+2}}{3\cdot 5\cdot 7\cdots\cdot(2n+1)(n+1)\pi a^m k^m},$$

existente scilicet $m = \frac{2n+1}{2}$. Quaecunque autem *m* fuerit fractio, erit perpetuo

$$A = \frac{1}{(2m+1)^2\int dz(1-z)^m}$$

hoc integrali ita accepto, ut evanescat posito *z* = 0, atque tum posito *z* = 1, quemadmodum ducui Comment. A acad. sc. Petrop. 5. (1730/1), in Dissertatione *De progressionibus transcendentibus*. [E19].

Corollarium 8.

749. Si nunc pro singulis mediis resistentibus rarissimis quaerantur tautochronae descensuum, eae se habebunt, ut sequitur :

$m = 0$	$gx = \frac{s^2}{a} + s$
$m = \frac{1}{2}$	$gx = \frac{s^2}{a} + \frac{s^2}{\pi\sqrt{ak}}$
$m = 1$	$gx = \frac{s^2}{a} + \frac{s^3}{6ak}$
$m = \frac{3}{2}$	$gx = \frac{s^2}{a} + \frac{s^4}{3\pi ak\sqrt{ak}}$
$m = 2$	$gx = \frac{s^2}{a} + \frac{3s^5}{40a^2k^2}$
$m = \frac{5}{2}$	$gx = \frac{s^2}{a} + \frac{8s^6}{45\pi a^2k^2\sqrt{ak}}$
$m = 3$	$gx = \frac{s^2}{a} + \frac{5s^7}{112a^3k^3}$.

Ex his formabuntur tautochronae ascensuum, si ultimi termini fiant negativi.

Scholium 4. [p. 413]

750. Haec igitur sufficiant de tautochronis simplicibus, super quibus vel omnes descensus tantum vel omnes ascensus aequalibus absolvuntur temporibus. Praeter has autem curvas etiam aliae tautochronarum nomine appellari possunt, super quibus vel omnes semioscillationes vel etiam solum omnes integrae oscillationes sint isochronae; quarum curvarum numerus uti etiam in vacuo est infinitus. Quod autem ad integras oscillationes attinet, haec questio resistentiae propria est; in vacuo enim omnes semioscillationes inter se sunt aequales. Quia autem in his quaestionibus duae curvae inveniendae proponuntur, quarum altera ad ascensus, altera ad descensus pertineat, antequam huiusmodi quaestiones pro tautochronismo solvamus, alias facioles propositiones circa binas curvas ascensum et descensum spectants praemittemus.

PROPOSITIO 83.

Problema.

751. *In hypothesi potentiae uniformis deorsum tendentis et medio uniformi, quod resistit in duplicata ratione celeritatum, data curva MA* (Fig. 84) *invenire alteram AN illi in A iungendam huius indolis, ut corpus per arcum quemcunque MA super curva data descendens super curva quaesita ascensu conficiat arcum AN, qui aequalis sit arcui MA.*

Solutio. [p. 414]

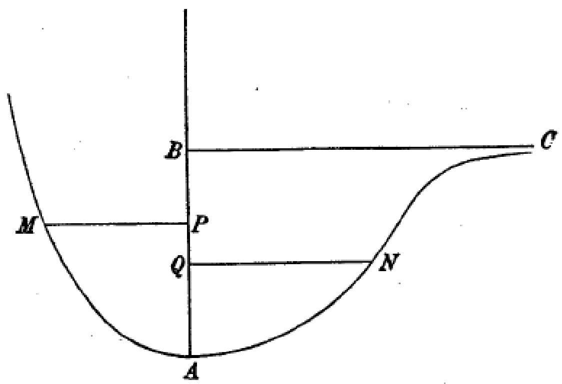

Fig. 84.

Positis potentia sollicitante *g*, exponente resistentiae *k* et celeritate in *A*, quam ex descensu acquisivit quaque super curva quaesita *AN* ascensum inchoat, debita altitudini *b* sit pro curva data abscissa $AP = x$, arcus $AM = s$; pro quaesita vero sit abscissa $AQ = t$ atque arcus $AN = r$. His positis altitudo celeritati corporis descendentis in *M* debeta =

$$e^{\frac{s}{k}}\left(b - g\int e^{\frac{-s}{k}}dx\right)$$

et altitudo celeritati corporis ascendentis in *N* debita =

$$e^{\frac{-r}{k}}\left(b - g\int e^{\frac{r}{k}}dt\right).$$

Integer ergo arcus descensus provenit ex hac aequatione

$$b = g\int e^{\frac{-s}{k}}dx,$$

integer vero arcus ascensus ex hac aequatione

$$b = g\int e^{\frac{r}{k}}dt.$$

Inter arcus igitur descensus et ascensus haec havebitur aequatio

$$\int e^{\frac{-s}{k}} dx = \int e^{\frac{r}{k}} dt$$

seu huius differentialis

$$e^{\frac{-s}{k}} dx = e^{\frac{r}{k}} dt.$$

Cum autem curva *MA* data sit, dabitur aequatio inter *s* et *x*; ex qua, si loco *dx* eius valor per *s* et *ds* substituatur, prodibit aequatio inter *t* et *s* seu inter *t* et *r* propter *r* = *s*; quae determinabit naturam curvae quaesitae *AN*. Q.E.I.

Corollarium 1. [p. 415]

752. Si curvae datae *MA* portio infima exprimatur aequatione $x = \alpha s^n$, erit pro portione infima curvae AN haec aequatio

$$dt = \alpha n e^{\frac{-2s}{k}} s^{n-1} ds,$$

Est vero propter *s* arcum minimum

$$e^{\frac{-2s}{k}} = 1 - \frac{2s}{k},$$

unde erit

$$t = \alpha s^n - \frac{2\alpha n s^{n+1}}{(n+1)k}$$

seu tantum $t = \alpha s^n$. Portiones ergo infimae utriusque curvae erunt inter se similes.

Corollarium 2.

753. Ex aequatione $e^{\frac{-2s}{k}} dx = dt$ intelligitur esse semper *dt* < *dx* seu *t* < *x*. Punctum ergo *N* semper humilius erit positum quam punctum respondens *M*. Ex quo sequitur curvam *AN* minus esse curvam versus *AB* quam curvam *AM*.

Corollarium 3.

754. Hanc ob rem curva *ANC* non poterit esse similis et aequalis curvae *AM*, quia hoc casu puncta *M* et *N* arcus aequales *AM* et *AN* terminatia forent in eadem altitudine sita.

Corollarium 4.

755. Si corpus super curva *MA* ex altitudine infinita descenderet, quia in *A* celeritatem tantum finitam acquirit, ad altitudinem tantum finitam ascendere poterit. Hoc ergo casu, quo curva *AM* in infiniturm porrigitur, curva *ANC* non ultra datam altitudinem ascendere poterit, sed asymptoton habebit horizontalem *BC*. Id quod etiam ex hoc apparet, si fit $s = \propto$; tum enim prodit *dt* = 0.

Scholion 1. [p. 416]

756. Quemadmodum ex hac propositione intelligitur, quomodo ex data curva descensuum *MA* invenire debeat curva ascensuum *AN*, ita vicissim hinc facile erit ex data curva *AN* alteram definire. Si enim detur aequatio inter *t* et *s*, erit

$$dx = e^{\frac{2s}{k}} dt$$

aequatio pro curva AM.

Corollarium 5.

757. Quia curva descensuum MA respondet curva ascensuum AN, cuius haec est aequatio

$$dt = e^{\frac{-2s}{k}} dx,$$

ita, si haec curva AN pro curva descensuum accipiatur, erit respondentis curvae ascensuum abscissa =

$$\int e^{\frac{-2s}{k}} dt = \int e^{\frac{-4s}{k}} dx.$$

Corollarium 6.

758. Si hoc modo ulteriores curvaes respondentes quaerentur, obinebitur sequens aequationum series :

Abscissa curvae respondens arcui s

$$\begin{aligned}
\text{I} &= x,\\
\text{II} &= \int e^{\frac{-2s}{k}} dx,\\
\text{III} &= \int e^{\frac{-4s}{k}} dx,\\
\text{IV} &= \int e^{\frac{-6s}{k}} dx,\\
\vdots &\quad \vdots\\
n &= \int e^{\frac{-2(n-1)s}{k}} dx.
\end{aligned}$$

Corollarium 7. [p. 417]

759. Huius igitur seriei duae curvae contiguae hanc habebunt proprietatem, ut iis in infino puncto A coniunctis corpus super priori descendens super altera per arcum ascendat aequalia arcui descensus. Infinitesima autem huius seriei curva facto $n = \infty$ fit recta horizontalis, quia evanescit $e\frac{-\infty s}{k}$ ideoque ipsa curvae abscissa.

Exemplum 1.

760. Sit linea data recta verticalis; erit $x = s$. Pro curva ergo ascensuum quaesita *ANE* (Fig. 85) existente $AQ = t$ et $AN = s$ habebitur ista aequatio

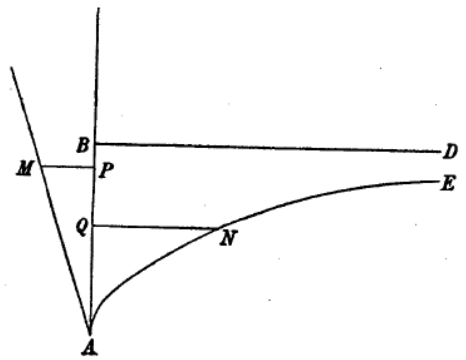

Fig. 85.

$$dt = e^{\frac{-2s}{k}} ds \quad \text{seu} \quad t = \frac{k}{2}\left(1 - e^{\frac{-2s}{k}}\right).$$

Atque eliminata quantitate exponentiali erit

$$2tds = kds - kdt \quad \text{sive} \quad \frac{(\frac{1}{2}k - t)ds}{dt} = \frac{1}{2}k.$$

Ex qua aequatione perspicitur curvam *ANE* esse tractoriam super asymtoto horizontali *BD* filo longitudinis $\frac{1}{2}k$ genitam. Quare altitudo asymtoti *AB* erit $= \frac{1}{2}k$. Si nunc haec ipsa tractoria *ANE* pro curva descensuum accipiatur ei respondebit curva quaesita, cuius abscissa arcui *s* respondens erit $= \int e^{\frac{-4s}{k}} ds$; [p. 418] quae ergo curva iterum erit tractoria asymtoton horizontalem habens, cuius asymtotos supra *A* elevata est intervallo $\frac{1}{4}k$, cui longitudo fili aequatur. Seriei vero superioris omnes curvae erunt tractoriae, quae filis generantur, quorum longitudines constituunt hanc seriem $\frac{k}{0}, \frac{k}{2}, \frac{k}{4}, \frac{k}{6}$ etc. Recta scilicet verticalis tanquam tractoria considerari potest, cuius filum generans est $\frac{k}{0}$ seu infinitum. Ultima autem huius seriei tractoria in rectam abit horizontalem per *A* ductam.

Exemplum 2.

761. Si linea descensuum data fuerit recta utcunque ad horizontalem inclinata *MA* (Fig. 85), ita ut sit $MA(s) : AP(x) = \alpha : 1$ seu $dx = \frac{ds}{\alpha}$, habebitur pro curva quaesita *AN* ista aequatio

$$\alpha dt = e^{\frac{-2s}{k}} ds,$$

cuius integralis est

$$\alpha t = \frac{k}{2}(1 - e^{\frac{-2s}{k}}).$$

Ex quibus aequationibus coniunctis oritur

$$2\alpha tds = kds - \alpha kdt \quad \text{seu} \quad \frac{\left(\frac{k}{2\alpha} - t\right)ds}{dt} = \frac{k}{2}.$$

Quae aequatio quoque est pro tractoria filo longitudinis $\frac{k}{2}$ super asymtoto horizontali *BD* genita existente $AB = \frac{k}{2\alpha}$; haecque tractoria per *A* transire debet. Seriei sequentes curvae omnes quoque tractoriae ut in praecedentes exemplo, quarum fila generantia sunt $\frac{k}{0}, \frac{k}{2}, \frac{k}{4}, \frac{k}{6}$ etc., earum vero asymtotorum a puncto *A* distantiae tenent hanc [p. 419] progressionem $\frac{k}{0\alpha}, \frac{k}{2\alpha}, \frac{k}{4\alpha}, \frac{k}{6\alpha}$ etc. Hae scilicet omnes tractoriae cum axe verticali *AB* angulum constituent aequalem angulo PAM.

Corollarium 8.

762. Tractoriarum harum ea, quae primam seu rectam *MA* praecedit, hanc ergo habebit proprietatam, ut corpus super ea descendens posteaque super recta *AM* ascendens aequalia spatia percurrat.

Corollarium 9.

763. Ad curvam igitur descensuum *CA* (Fig. 86) inveniendam, cui respondeat recta inclinata *AM*, super asymtoto horizontali filo longitudinis $\frac{k}{2}$ describatur tractoria CA in eaque sumatur applicata $Ab = \frac{k}{2\alpha}$ et ex A constituatur recta inclinata AM; eritque CA curva descensuum, cui respondet recta AM pro ascensibus.

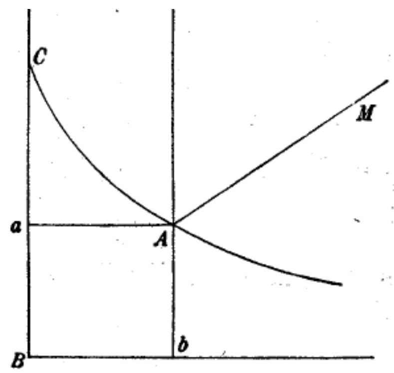

Fig. 86.

Scholion 2.

764. Inservire potest hic casus instar exempli problematis inversi, quo ex data curva ascensuum curva descensuum requiritur.

Exemplum 3. [p. 420]

765. Sit curva descensuum data cyclois *MA* (Fig. 84), cuius natura hac aequatione sit expressa

$$2ax = s^2$$

seu circuli genitoris diameter = $\frac{a}{2}$. Erit ergo

$$dx = \frac{sds}{a},$$

unde pro curva altera ascensuum *AN* haec invenitur aequatio

$$adt = e^{\frac{-2s}{k}} sds,$$

cuius integralis est

$$at = \frac{k^2}{4}\left(1 - e^{\frac{-2s}{k}}\right) - \frac{k}{2} e^{\frac{-2s}{k}} s,$$

quae propter

$$e^{\frac{-2s}{k}} = \frac{adt}{sds}$$

abit in hanc

$$atsds = -\frac{ak^2dt}{4} + \frac{k^2sds}{4} - \frac{aksdt}{2}.$$

Haec curva in *A*, ut iam est dictum, tangentem habebit horizontalem. Habebit vero etiam asymtoton *BC* horizontalem; cuius altitudo *BA* reperietur si *s* fiat = $\propto$. Fiet autem hoc casu $e^{\frac{-2s}{k}} = 0$; quare erit $t = AB = \frac{k^2}{4a}$. Ex hoc intelligitur curvam alicubi punctum flexus contrarii habere debere; quod invenietur, si posito *dt* constante ponatur *dds* = 0. Hinc vero prodibit $1 = \frac{2s}{k}$ seu $s = \frac{k}{2}$. Quare si sumatur arcus $AN = \frac{k}{2}$, erit *N* punctum flexus contrarii; cui respondet abscissa $AQ = \frac{k^2}{4a} - \frac{k^2}{4ae}$ seu $BQ = \frac{k^2}{2ae}$. Quo circa erit semper $AB : BQ = e : 2 = 2{,}71828 : 2$.

Scholion 3.

766. Problema hoc propositum extat ab anonymo in Act. Lips. A. 1728 [*Problema geometris propositum*, p. 528.] eiusque solutionem dedit in Comment. Acad. Petrop.A. 1729 Cl. D. Bernoulli alia usus methodo. [Dan. Bernoulli, *Theorema de motu curvilineo corporum, quae resistentiam patiuntur velocitatis suae quadrato proportionalem*, Comment. Acad. sc. Petrop.4 (1730/1), 1735, p. 136.] [p. 421] Praeter hanc vero conditionem anonymus ille potissimum requirit unam curvam continuam, cuius alter ramus descensibus, alter ascensibus inserviat, cuiusmodi curvae dantur innumerabiles, quas in sequente propositione detegemus.

PROPOSITIO 84.

Problema.

767. *Iisdem positis ut ante invenire curvam continuam MAN* (Fig. 84) *huiusmodi, ut in quavis semioscillatione, quae semper in arcu MA incipiat, super ea facta arcus descensus MA aequalis sit arcus ascensus sequentis AN.*

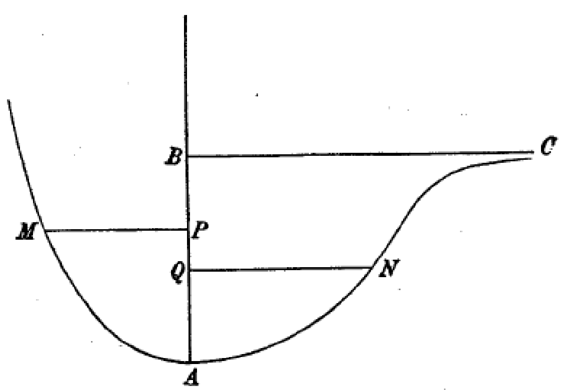

Fig. 84.

Solutio.

Propositio haec a praecedente in hoc tantum differt, quod ibi data fuerit curva *MA*; hic vero ea quoque quari debeat ex hac conditione, quod utraque curva *MA* et *AN* unam eandemque curvam continuam constituere debeant. Sumtis igitur arcubus *AM* et *AN* aequalibus = *s* et posita *AP* = *x* atque *AQ* = *t* erit

$$dt = e^{\frac{-2s}{k}} dx.$$

Quia autem curva *MAN* debet esse continua, aequationem inter s et x ita oportet esse comparatam, ut, si in ea loco s ponatur $-s$, quo casu arcus *AM* in arcum *AN* abit, valor ipsius x fiat $= t$ seu $= \int e^{\frac{-2s}{k}} dx$. [p. 422] Pono igitur $dx = Mds$, ubi M sit functio quaedam ipsius s, eaque abeat in N, si loco s ponatur $-s$. Ponatur ergo $-s$ loco s, quo casu x abit in t, eritque $dt = -Nds$. Est vero quoque

$$dt = e^{\frac{-2s}{k}} dx = e^{\frac{-2s}{k}} Mds,$$

quocirca erit

$$N = -e^{\frac{-2s}{k}} M.$$

Sit porro

$$M = -e^{\frac{+s}{k}} P.$$

abeatque P in Q posito $-s$ loco s eritque

$$N = e^{\frac{-s}{k}} Q.$$

Quibus valoribus loco M et N substitutis prodibit $Q = -P$. Ex quo apparet P huiusmodi esse debere functionem ipsius s, quae abeat in $-P$ posito $-s$ loco s, quas functiones impares appellare consuevi. Sit itaque P functio quaecunque impar ipsius s, cuiusmodi sunt e. gr. $\alpha s, \alpha s^3, \alpha s^5$ etc, eritque

$$M = e^{\frac{s}{k}} Pds \quad \text{seu} \quad x = \int e^{\frac{s}{k}} Pds.$$

Quae est aequatio pro curva quaesita. Q.E.I.

Corollarium 1.

768. Quia est

$$dx = e^{\frac{s}{k}} Pds,$$

erit sumendis logarithmis

$$ldx = \frac{s}{k} + lP + lds.$$

Differentietur haec aequatio denuo positis ds constante prodibitque

$$\frac{ddx}{dx} = \frac{ds}{k} + \frac{dP}{P} \quad \text{seu} \quad kPddx = Pdxds + kdxdP.$$

Quae aequatio ab exponentialibus est libera.

Corollarium 2.

769. Quoniam P per s dari debet, aequatio inventa non habet variabiles inter se permixtas; quamobrem ea sufficit ad curvas in ea contentas construendas.

Exemplum. [p. 423]

770. Ponamus esse $P = \frac{s}{a}$; erit

$$ax = \int e^{\frac{s}{k}} sds = ke^{\frac{s}{k}} s - k^2 e^{\frac{s}{k}} + k^2,$$

quae est aequatio pro una et fortasse simplicissima curva satisfaciente. Haec vero aequatio eliminato exponentiali $e^{\frac{s}{k}}$ abit in hanc

$$axsds = aksdx - k^2 adx + k^2 sds.$$

Vel exposito $e^{\frac{s}{k}}$ per seriem prodibit ista aequatio

$$ax = \frac{s^2}{2} + \frac{s^3}{1 \cdot 3k} + \frac{s^4}{1 \cdot 2 \cdot 4k^2} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 5k^3} + \text{etc.}$$

Haec ergo curva in *A* habet tangentem horizontalem eiusque radius osculi in loco est *a*. Quia, ne curva fiat imaginaria, esse debet $dx < ds$, debebit esse $e^{\frac{s}{k}} s < a$. Quo ergo loco fit $e^{\frac{s}{k}} s$, quae expressio crescente s quoque crescet, aequalis a, ibi curva *AM* habebit tangentem verticalem atque punctum reversionis. Pro ramo *AN* posit – *s* loco + *s* haec habetur aequatio

$$ax = \int e^{\frac{-s}{k}} sds \quad \text{seu} \quad dx = \frac{e^{\frac{-s}{k}} sds}{a}.$$

Quare quamdiu fuerit $e^{\frac{-s}{k}} s < a$, curva non fit imaginaria. At si usquam $e^{\frac{-s}{k}} s = a$, ibi curva quoque habebit punctum reversionis et diametrum verticalem. [p. 424] Fieri autem potest, si *a* satis magnum accipiatur, ut $e^{\frac{-s}{k}} s$ semper minus sit quam *a*, quo casu curva *AN* in infinitum abibit asymtotonque habebit horizontalem *BC*. Fit autem $e^{\frac{-s}{k}} s = 0$ casibus $s = 0$ et $s = \infty$; habebit ergo valorem maximum, si eius differentiale = 0; hoc vero casu fit $k = s$ et

$$e^{\frac{-s}{k}} s = \frac{k}{e}$$

Quare si fuerit $a > \frac{k}{e}$, curva habebit asymtoton *BC*, cuius altitudo *BA* erit = $\frac{k^s}{a}$. At si fuerit $a < \frac{k}{e}$, curva *AN* uti alter ramus habebit quoque punctum reversionis, quod ex hac aequatione determinatur $a = e^{\frac{-s}{k}} s$. In priori casu curva *AN* habere debet punctum flexus contrarii, quod reperietur ex hac aequatione $1 = \frac{s}{k}$; erit scilicet in *N* sumto arcu *AN* = *k*.

PROPOSITIO 85.

Problema.

771. *In hypothesi gravitatis et resistentiae praecedente si data fuerit curva MA* (Fig. 84)*, super qua descensus absolvantur, invenire pro ascensibus curvam AN huius proprietatis, ut ascensus cuiusque tempus aequale sit tempori descensus praecedentis.*

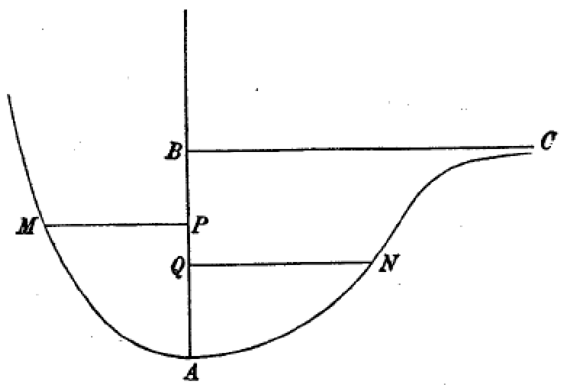

Fig. 84.

Solutio.

Positis ut ante potentia sollicitante = g et medii exponente = k sit pro curva MA abscissa $AP = x$, arcus $AM = s$ atque pro curva quaesita AN abscissa $AQ = t$, arcus $AN = r$. Ponatur altitudo celeritati descensu quodam in A acquisitae debita = b, qua celeritate corpus sequentem ascensum in curva AN absolvet. [p. 425] His positis erit altitudo celeritati in M debita =

$$e^{\frac{s}{k}}(b - g\int e^{\frac{-s}{k}}dx)$$

et altitudo in ascensu celeritati in N debita =

$$e^{\frac{-r}{k}}(b - g\int e^{\frac{r}{k}}dt).$$

Tempus ergo descensus per arcum MA erit =

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b - g\int e^{\frac{-s}{k}}dx)}}$$

et tempus ascensus per arcum AN =

$$\int \frac{e^{\frac{r}{2k}}dr}{\sqrt{(b - g\int e^{\frac{r}{k}}dt)}}$$

quae duo tempora, si post integrationem ponatur

$$g\int e^{\frac{-s}{k}}dx = b \text{ atque } g\int e^{\frac{r}{k}}dt = b,$$

debent esse aequalia. Ponatur ad hoc obtinendum

$$g\int e^{\frac{-s}{k}}dx = X \text{ , } g\int e^{\frac{r}{k}}dt = T$$

atque

$$\frac{ds}{e^{\frac{s}{2k}}} = dS \text{ et } e^{\frac{r}{2k}}dr = dR,$$

ubi *X, T, S* et *R* sint tales functiones, quae evanescant posito *x, t, s* et *r* = 0. Efficiendum ergo est, ut haec duo integralia

$$\int\frac{dS}{\sqrt{(b-X)}} \quad \text{et} \quad \int\frac{dR}{\sqrt{(b-T)}}$$

fiant inter se aequalia, si post integrationem ponatur *X* = *b* et *T* = *b*. At *S* et *R* uti ex *X* et *T* sunt quantitates a *b* prorsus non pendentes atque eandem inter se relationem tenere debent, quemcunque valorem *b* habuerit. Quaesito ergo satisfiet, si *R* fuerit talis functio ipsius *T*, qualis *S* est ipsius *X*. Vel sumto *R* = *S* esse quoque debet *T* = *X*. Est vero [p. 426]

$$S = 2k\left(1 - e^{\frac{-s}{2k}}\right) \quad \text{et} \quad R = 2k\left(e^{\frac{r}{2k}} - 1\right);$$

facto igitur *R* = *S* erit

$$2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}} \quad \text{atque} \quad r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Quia autem hoc posito esse debet *X* = *T* seu $e^{\frac{-s}{k}}dx = e^{\frac{r}{k}}dt$, fiet $t = \int e^{\frac{-s-r}{k}}dx$. Cum vero sit

$$e^{\frac{r}{k}} = \left(2 - e^{\frac{-s}{2k}}\right)^2 = 4 - 4e^{\frac{-s}{2k}} + e^{\frac{-s}{k}},$$

erit

$$t = \int\frac{dx}{e^{\frac{s}{k}}\left(2 - e^{\frac{-s}{2k}}\right)^2} = \int\frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2}.$$

Ex quibus ergo constructio curvae innotescit, quia sumto arcu

$$AN = r = 2kl\left(2 - e^{\frac{-s}{2k}}\right)$$

huic respondet abscissa

$$AQ = t = \int\frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2}.$$

Aequatio vero pro curva *AN* commodius invenietur ex data aequatione inter *s* et *x*. Nam quia est

$$s = -2kl\left(2 - e^{\frac{r}{2k}}\right) \quad \text{et} \quad x = \int\frac{dt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2},$$

si loco *s* et *x* valores substituantur, prodibit aequatio inter *t* et *r* pro curva quaesita *AN*. Q.E.I.

Corollarium 1.

772. Quia est $r = 2kl(2 - e^{\frac{-s}{2k}})$, erit

$$dr = \frac{e^{\frac{-s}{2k}}ds}{2 - e^{\frac{-s}{2k}}}.$$

Cum vero, ne curva *AN* fiat imaginaria, esse debeat $dr > dt$, curva *AN* eousque realis, quousque

$$ds > \frac{dx}{e^{\frac{s}{2k}}\left(2 - e^{\frac{-s}{2k}}\right)} \quad \text{seu} \quad ds > \frac{dx}{2e^{\frac{s}{2k}} - 1}.$$

Corollarium 2. [p. 427]

773. Est vero $e^{\frac{s}{2k}}$ semper maius unitate; ex quo sequitur, ubi fuerit $ds > dx$, eo magis fore

$$ds > \frac{dx}{2e^{\frac{s}{2k}} - 1}.$$

Quare si curva data fuerit realis, quaesita quoque semper erit realis.

Corollarium 3.

774. Cum sit $s = -2kl(2 - e^{\frac{r}{2k}})$, erit

$$ds = \frac{e^{\frac{r}{2k}}dr}{2 - e^{\frac{r}{2k}}} = \frac{dr}{2e^{\frac{-r}{2k}} - 1},$$

unde facilius relatio inter *ds* et *dx* in computum duci potest.

Corollarium 4.

775. Ex solutione problematis simul apparet, quomodo eius inversum sit solvendum. Si enim curva ascensum *AN* datur seu aequatio inter *t* et *r*, ex ea aequatio inter *x* et *s* formabitur ope aequationum

$$t = \int \frac{dx}{\left(2e^{\frac{s}{2k}} - 1\right)^2} \quad \text{et} \quad r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Corollarium 5. [p. 428]

776. Ad curvae formam circa punctum *A* indagandam ponantur *s* et *r* valde exigua eritque $e^{\frac{-s}{2k}}=1$, unde fiet *dr* = *ds* atque *dt* = *dx*. Ex quo perspicitur curvarum *MA* et *NA* infimas portiones esse inter se similes et aequales.

Exemplum 1.

777. Sit linea descensuum data recta *MA* (Fig. 85) utcunque inclinata, ut sit $s=\alpha x$ seu $ds=\alpha dx$. Cum nunc sit

$$ds=\frac{dr}{2e^{\frac{-r}{2k}}-1} \quad \text{et} \quad dx=\frac{dt}{\left(2e^{\frac{-r}{2k}}-1\right)^2},$$

habebitur inter *t* et *r* pro curva quaesita ista aequatio

$$dr=\frac{\alpha dt}{2e^{\frac{-r}{2k}}-1} \quad \text{seu} \quad \alpha dt=2e^{\frac{-r}{2k}}dr-dr,$$

cuius integralis est

$$\alpha t=4k\left(1-e^{\frac{-r}{2k}}\right)-r.$$

Quae aequatio in seriem conversa dat

$$\alpha t=r-\frac{r^2}{1\cdot 2k}+\frac{r^3}{1\cdot 2\cdot 3\cdot 2k^2}-\frac{r^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^3}+\text{etc.}$$

ideoque in puncto infimo A est $dr=\alpha dt$. Curva haec alicubi habebit tangentem horizontalem, qui locus invenietur ponendo dt = 0; tum vero erit $2=e^{\frac{r}{2k}}$ seu $r=2kl2$, cui respondet $\alpha t=2k-2kl2$. Atque si *r* fiat maius quam $2kl2$, valor ipsius *dt* fiet negativus ideoque curva iterum descendet, donec – *dt* fiat = *dr*; hoc autem accidit, si est

$$1-\alpha=2e^{\frac{-r}{2k}} \quad \text{seu} \quad r=2kl\frac{2}{1-\alpha}.$$

[p. 429] At quia α non potest minus quam 1, si est $\alpha=1$, tangens verticalis in infinitum ab *A* destabit, atque si $\alpha>1$, ultra tangentem horizontalem nusquam habebit tangentem verticalem. Sed antrorsum ultra tangentem habebit verticalem, ubi est

$$r=-2kl\frac{1+\alpha}{2}.$$

Casu ergo, quo linea data est verticalis seu $\alpha=1$, fit *r* = 0 seu tangens in *A* erit verticalis.

Corollarium 6.

778. Si *s* denotet totum arcum descensus, *r* exprimet totum arcum sequenti ascensu super curva *AN* (Fig. 84) descriptum. Quare si detur arcus descensus *s*, reperietur arcus ascensus

$$r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Quoniam enim posuimus *T* = *X*, integros arcus descensus et ascensus litterae *s* et *r* denotant.

Exemplum 2.

779. Sit curva data MA ipsa tautochrona descensuum, quam ante pro eadem resistentiae hypothesi invenimus; habebit curva AN hanc proprietatem, ut omnes ascensus aequalibus quoque absolvantur temporibus, iisdem nempe, quibus descensus super MA. Quare curva AN erit ipsa tautochrona ascensuum cum curva MA contina iam ante inventa. Quo hoc autem ex isto calculo ostendatur, sumamus aequationem pro curva tautochrona descensuum, quae est (719) [p. 430]

$$\text{vel} \quad adx = k\left(e^{\frac{s}{2k}} - 1\right)ds \quad \text{vel} \quad ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks.$$

Cum nunc sit

$$ds = \frac{dr}{2e^{\frac{-r}{2k}} - 1} \quad \text{et} \quad e^{\frac{s}{2k}} = \frac{e^{\frac{-r}{2k}}}{2e^{\frac{-r}{2k}} - 1} \quad \text{atque} \quad dx = \frac{dt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2},$$

his substitutis erit

$$\frac{adt}{\left(2e^{\frac{-r}{2k}} - 1\right)^2} = \frac{kdr\left(1 - e^{\frac{-r}{2k}}\right)}{\left(2e^{\frac{-r}{2k}} - 1\right)^2}$$

seu

$$adt = k\left(1 - e^{\frac{-r}{2k}}\right)dr.$$

Quae aequatio ex illa formatur, si pro *x* ponatur *t* atque – *r* pro *s*. Quare haec curva *AN* est continua cum *MA* atque tautochrona ascensuum.

Corollarium 7.

780. Dato ergo arcu descensus *s* super tautochrona descensuum erit arcus ascensus sequentis tautochrona ascensuum

$$r = 2kl\left(2 - e^{\frac{-s}{2k}}\right).$$

Atque si descensus a cuspide tautochronae descensuum incipiat, cuius locum dat

$$e^{\frac{s}{2k}} = \frac{a+k}{k}$$

(729), erit arcus ascensus

$$r = 2k\, l\, \frac{2a + k}{a + k},$$

ut supra invenimus (732).

Scholion.

781. Cum itaque tautochrona in hac resistentiae hypothesi quaesito satisfaciat atque sit curva continua, hinc ansam arripimus investigandi plures curvas continuas, quarum duo rami vices curvarum *MA* et *AN* sustinere quaeant; id quod in sequente propositione praestabimus.