CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 375]

PROPOSITION 78.

Problem.

691. *In a medium with some kind of resistance and with some kind of forces acting to find the brachistochrone AM* (Fig.78)*, upon which a descending body arrives the fastest from A to M.*

Solution. [p. 376]

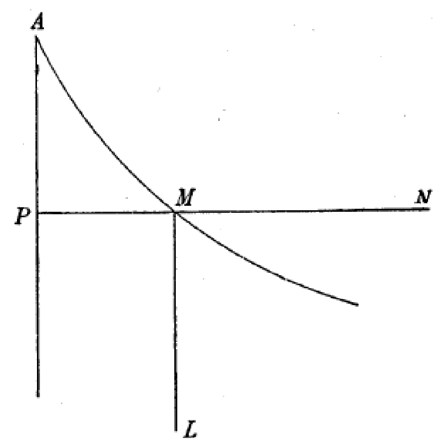

Fig. 78.

Let A be the starting point of the motion, through which some straight line AP is drawn being taken for the axis, on which the abscissa $AP = x$ it taken; to which there corresponds the applied line $PM = y$ and the arc $AM = s$. Again let the speed of the body at M correspond to the height v and the resistance depending somehow on the speed equal to R. Now the body is acted on by some absolute forces act on the body, in place of these two forces can be substituted in the given directions ML and MN, of which the former is parallel to the axis AP, and indeed the latter normal to that axis. Moreover the force acting on the body following ML is equal to P and the force along $MN = Q$. From these forces, there arises the equation :

$$dv = Pdx + Qdy - Rds.$$

And the nature of the brachistochrone curve gives :

$$\frac{2v}{r} = \frac{2vdxddy}{ds^3} = \frac{Pdy - Qdx}{ds}$$

(673) with r denoting the radius of osculation of the cure at M towards the upper direction; hence from which on taking dx as constant we put $\frac{+ds^3}{dxddy}$, since the other direction must give $r = \frac{-ds^3}{dxddy}$. Hence from these two equations :

$$dv = Pdx + Qdy - Rds \text{ and } \frac{2vdxddy}{ds^2} = Pdy - Qdx$$

if v is eliminated, the equation is had for the brachistochrone curve sought ; clearly there is given

$$v = \frac{Pdyds^2 - Qdxds^2}{2dxddy};$$

with the differential of this substituted in place of dv and in place of v itself in the resistance R, the equation is given for the curve sought. Q.E.I.

Corollary 1. [p. 377]

692. The equation for the curve, if v is eliminated in the stated manner, becomes a differential of the third order. Whereby if the threefold integration is to be obtained, then also three constants can be added, in which to be effective, so that on x vanishing, likewise both y and s or v vanish and in addition the curve passes through the given point M .

Corollary 2.

693. Therefore since the brachistochrone curve can always be shown, which has the starting point A and passes through a given point, an infinite number of brachistochrone curves can be drawn from the point A.

Corollary 3.

694. With the equation for the brachistochrone curve AM known, likewise the speed of the descent on that curve can be found at individual points; and as much as

$$v = \frac{Pdyds^2 - Qdxds^2}{2dxddy}.$$

Corollary 4.

695. With the speed given, from that the time can be determined in which the body completes the arc AM ; clearly the time to pass along the arc AM is equal to :

$$\int \frac{ds}{\sqrt{v}} = \int \frac{\sqrt{2dxddy}}{\sqrt{(Pdy - Qdx)}};$$

and that can now be found on account of the equation between x and y, even if it has to be shown by quadrature.

Corollary 5. [p. 378]

696. Therefore if the curve is to be found, which all the brachistochrones drawn from *A* must cross at right angles, then the construction of this line is obtained, if an amount of the same magnitude

$$\int \frac{\sqrt{2dxddy}}{\sqrt{(Pdy - Qdx)}}$$

is cut off from all the brachistochrones. According to this, the arc of the isochrone is cut off from this infinitude of curves which, since all the curves are brachistochrones, terminate at right–angles to the trajectory. [Thus, the beginnings of sets of orthogonal curves is set out.]

Example.

697. Let the resistance be proportional to the square of the resistance and let the exponent of the resistance be some variable q; then $R = \frac{v}{q}$. Hence since

$$dv = Pdx + Qdy - \frac{vds}{q},$$

on integration it becomes :

$$e^{\int \frac{ds}{q}} v = \int e^{\int \frac{ds}{q}} (Pdx + Qdy).$$

But since

$$v = \frac{Pdyds^2 - Qdxds^2}{2dxddy},$$

then the above equation becomes :

$$2dxddy \int e^{\int \frac{ds}{q}} (Pdx + Qdy) = e^{\int \frac{ds}{q}} ds^2 (Pdy - Qdx),$$

in which equation v is no longer present. Yet meanwhile this equation becomes an equation of the third order, if the integration signs are removed by differentiation; besides the indeterminate values of P, Q, and q are reasons by which the equation is less able to be prepared for construction.

Scholium.

698. Here it is clearly evident, concerning the brachistochrone curves that have been deduced from the two forces P and Q, since however many forces are acting on the body, all of these can be resolved in this manner into two, but only if the directions of all the forces lie in the same plane. [p. 379] On this account also in this proposition, the brachistochrones are held for any hypothesis of the centripetal forces acting, which moreover, since neither well–ordered nor constructible equations are forthcoming, we shall not pursued further. Therefore with these disposed of, in which the speed is prescribed by a certain rule, we progress to the following questions, in which curves are required, which sustain a given force on these from the body in motion.

PROPOSITION 79.

Problem.

699. *According to the hypothesis of uniform gravity and a uniform medium, which resists according to some ratio of the speeds, to determine the curve of constant pressure AM* (Fig.79)*, which sustains the same pressing force everywhere by a body descending on it.*

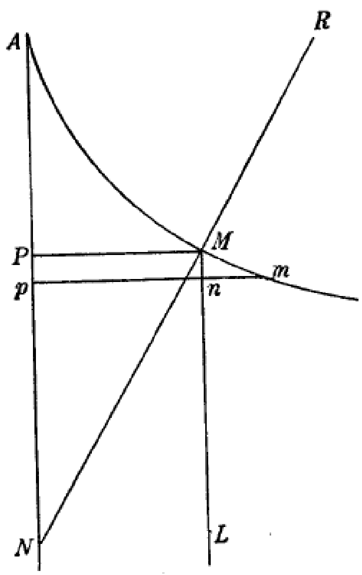

Fig. 79.

Solution.

On putting $AP = x$, $PM = y$, $AM = s$ and for the speed at M to correspond to a height equal to v, let the force of the body pulling downwards following ML be equal to g, and the force of the resistance at M = $\frac{v^m}{k^m}$. Therefore, while the body progresses through the element Mm,

$$dv = gdx - \frac{v^m ds}{k^m}.$$

[p. 380] We put the curve to be convex downwards, thus so that MR is the direction of the radius of curvature and the radius of osculation itself is $MR = \frac{ds^3}{dxddy}$,with dx put constant. Hence the centrifugal force is in the direction of the normal MN and the size of this is equal to $\frac{2vdxddy}{ds^3}$. Now following the same direction, the curve is pressed by a normal force arising from the resolution of the force $ML = g$, which is equal to $\frac{gdy}{ds}$. Therefore the total force, by which the curve is pressed along MN, is equal to $\frac{gdy}{ds} + \frac{2vdxddy}{ds^3}$; which since it has to be constant, that is put equal to αg and there is obtained :

$$\alpha gds^2 = gdyds^2 + 2vdxddy$$

and hence

$$v = \frac{\alpha gds^2 - gdyds^2}{2dxddy}.$$

Let $ds = pdx$ and $dy = dx\sqrt{(p^2 - 1)}$; then we have :

$$ddy = \frac{pdpdx}{\sqrt{(p^2-1)}}.$$

With these substituted, the equation becomes :

$$v = \frac{(\alpha g p^3 dx - g p^2 dx \sqrt{(p^2-1)})\sqrt{(pp-1)}}{2pdp} = \frac{\alpha g p^2 dx \sqrt{(p^2-1)} - gpdx(p^2-1)}{2dp}.$$

Again, let $dx = 2qdp$; then

$$v = gpq(\alpha p \sqrt{(p^2-1)} - p^2 + 1)$$

or

$$v = Pq$$

on putting

$$gp(\alpha p \sqrt{(p^2-1)} - p^2 + 1) = P.$$

Hence this equation becomes

$$dv = Pdq + qdP \quad \text{et} \quad v^m = P^m q^m.$$

With which values substituted into the equation $dv = gdx - \frac{v^m ds}{k^m}$, there is produced :

$$Pdq + qdP = gdx - \frac{P^m q^m ds}{k^m}.$$

Now it is the case that :

$$dx = 2qdp \text{ et } ds = pdx = 2pqdp.$$

On account of which, the above equation becomes :

$$Pdq + qdP = 2gqdp - \frac{2P^m q^{m+1} pdp}{k^m},$$

which contains only the two variables p and q, since P is given in terms of p. In solving this equation, there is put

$$q = \frac{1}{Pu^{\frac{1}{m}}};$$

from which this equation is obtained : [p. 381]

$$du + \frac{2mgudp}{P} = \frac{2mpdp}{k^m P};$$

which multiplied by $e^{2mg\int\frac{dp}{P}}$ and integrated goes into this :

$$u = \frac{2me^{-2mg\int\frac{dp}{P}}}{k^m}\int\frac{e^{2mg\int\frac{dp}{P}}pdp}{P}.$$

Hence therefore there is found u in terms of p, and from u found then $q = \frac{1}{Pu^{\frac{1}{m}}}$ and

$$x=\int\frac{2\,dp}{Pu^{\frac{1}{m}}},\quad s=\int\frac{2p\,dp}{Pu^{\frac{1}{m}}},\text{ and } y=\int\frac{2\,dp\sqrt{(p^2-1)}}{Pu^{\frac{1}{m}}}.$$

Moreover since we have

$$P=gp(\alpha p\sqrt{(p^2-1)}-p^2+1),$$

then

$$\frac{g\,dp}{P}=\frac{dp}{p}+\frac{dp}{\alpha\sqrt{(p^2-1)}}+\frac{(1-\alpha^2)dp}{\alpha^2p-\alpha\sqrt{(p^2-1)}}$$

and

$$\int\frac{g\,dp}{P}=lp-l(\alpha p-\sqrt{(p^2-1)})$$

and

$$e^{2mg\int\frac{dp}{P}}=p^{2m}(\alpha p-\sqrt{(p^2-1)})^{-2m}.^{1})$$

Now,

$$\int\frac{e^{2mg\int\frac{dp}{P}}p\,dp}{P}=\frac{e^{2mg\int\frac{dp}{P}}p}{2mg}-\frac{1}{2mg}\int e^{2mg\int\frac{dp}{P}}dp.$$

On account of which [Paul Stackel has corrected the following formulae in the *O. O.*],

$$u=\frac{p}{gk^m}-\frac{e^{-2mg\int\frac{dp}{P}}}{gk^m}\int e^{2mg\int\frac{dp}{P}}dp=\frac{p}{gk^m}-\frac{\int p^{2m}(\alpha p-\sqrt{(p^2-1)})^{-2m}dp}{gk^m p^{2m}(\alpha p-\sqrt{(p^2-1)})^{-2m}}$$

Therefore since u can be found from p in this manner, the solution of the curve sought can be brought about. Q.E.I.

Corollary 1. [p. 382]

700. The curve found hence has this property, that at any point M it is pressed towards MN by a constant force, which is in the same ratio to the force of gravity $ML = g$, as α is to 1.

Corollary 2.

701. If a negative number is taken for α, the curve is equally pressed everywhere along MR, in the opposite direction. Therefore in this case the curve must be concave downwards, since the centrifugal force is in the opposite direction and is greater than the normal force, and the direction of this force has always been placed along MN.

Corollary 3.

702. If $\alpha = 0$, then the curve is produces that sustains no pressing force from the body. Hence this is the curve that the motion of the body projected freely describes.

Corollary 4.

703. If $\alpha = 1$ or the total pressing force is equal to *g*, then the curve is convex downwards everywhere. For since the normal force alone is everywhere less than *g* except in the case that $ds = dy$, the centrifugal force acts together with that, and thus the radius of osculation falls on the opposite side of MN.

Corollary 5. [p. 383]

704. If we put

$$k^m e^{2mg\int\frac{dp}{P}} u = 2mz,$$

then

$$dz = \frac{e^{2mg\int\frac{dp}{P}} pdp}{P}.$$

Hence a special solution is obtained, since in this equation the indeterminates can be separated from each other, on putting $P = 0$. Moreover this becomes ;

$$gp(\alpha p\sqrt{(p^2-1)} - p^2 + 1) = 0$$

or

$$\alpha p = \sqrt{(p^2-1)} \text{ or } \alpha ds = dy.$$

Thus there is produced :

$$\alpha s = y.$$

Hence making a straight line angle to the vertical AP, the sine of which is α on taking 1 for the whole sine. For in this case the centrifugal force vanishes and the normal force becomes equal to αg.

Example.

705. Let $\alpha = 1$ or the curve is sought, which can be pressed everywhere by a force equal to *g* ; in which case the integration of $\frac{gdp}{P}$ emerges simpler; for then it becomes :

$$e^{2mg\int\frac{dp}{P}} = p^{2m}(p-\sqrt{(p^2-1)})^{-2m} = \left(\frac{p}{p-\sqrt{(p^2-1)}}\right)^{2m} = (p^2 + p\sqrt{(p^2-1)})^{2m}.$$

On this account then,

$$u = \frac{p}{gk^m} - \frac{\int (p^2 + p\sqrt{(p^2-1)})^{2m} dp}{gk^m (p^2 + p\sqrt{(p^2-1)})^{2m}}.$$

Which equation, as long as 2*m* is a whole number, admits to being integrated. For on putting :

$$p^2 + p\sqrt{(p^2-1)} = \frac{r+1}{2}$$

then

$$p = \frac{r+1}{2\sqrt{r}}$$

and

$$u = \frac{r+1}{2gk^m\sqrt{r}} - \frac{1}{4gk^m(r+1)^{2m}} \int \frac{(r-1)(r+1)^{2m} dr}{r\sqrt{r}}.$$

Moreover with this in place :

$$r = 2p^2 - 1 + 2p\sqrt{(p^2-1)} \text{ or } \sqrt{r} = p + \sqrt{(p^2-1)}.$$

As if $m = \frac{1}{2}$ or the resistance becomes proportional to the speeds, then [p. 384]

$$u = \frac{r+1}{2g\sqrt{kr}} - \frac{1}{4g(r+1)\sqrt{k}} \int \frac{(r^2-1)\,dr}{r\sqrt{r}} = \frac{1}{2g\sqrt{k}}\left(\frac{r+1}{\sqrt{r}} - \frac{r^2 + \beta\sqrt{r} + 3}{3(r+1)\sqrt{r}}\right) = \frac{2rr + 6r - \beta\sqrt{r}}{6g(r+1)\sqrt{kr}},$$

or modified by the constant β, it is

$$u = \frac{r\sqrt{r} + 3\sqrt{r} + 2\beta}{3g(r+1)\sqrt{k}}.$$

Moreover with this value put in place of *r* it becomes :

$$u = \frac{p^2 + 1 + p\sqrt{(p^2-1)}}{3gp\sqrt{k}} + \frac{\beta}{3g(p^2 + p\sqrt{(p^2-1)})\sqrt{k}}.$$

Let $\beta = 0$; then

$$u^{\frac{1}{m}} = u^2 = \frac{(p^2 + 1 + p\sqrt{(p^2-1)})^2}{9g^2kp^2}$$

and on this account,

$$P = gp(p - \sqrt{(p^2-1)})\sqrt{(p^2-1)}$$

then

$$Pu^{\frac{1}{m}} = \frac{(p - \sqrt{(p^2-1)})(p^2 + 1 + p\sqrt{(p^2-1)})^2\sqrt{(p^2-1)}}{9gkp}$$

and

$$x = \frac{2gk}{3p^2 - 1 - 3p\sqrt{(p^2-1)}} - 2gk\,l(3p^2 - 1 - 3p\sqrt{(p^2-1)}).$$

Scholium.

706. The integration of a similar formula to which u is equal, also follows if $\alpha = -1$, in which case there is produced a curve concave downwards, in which the centrifugal force is in the opposite direction and is greater than the normal force ; obviously the excess is equal to g. Now the same equation is produced, as for the case $\alpha = 1$, except that the sign of $\sqrt{(p^2 - 1)}$ has to be changed.

As pertains to the remaining questions touched on so far, in which other laws of the pressing force are proposed, these either can be deduced from exceedingly long calculations, or now have been handled. Indeed we have seen that the curves in which the total pressing force is twice as great as either the centrifugal force alone, or the normal force alone, are the brachistochrones, and curves in which another ratio acts, we have now handled these too, while we should investigate curves on which the motion is accelerated the least. [p. 385] Hence it follows that we proceed to finding curves, upon which many diverse descents and ascents are governed among themselves by given laws, which questions are the most difficult. For it is necessary in solving problems of this kind, that the speed of the body at individual points can be expressed by quantities from which the nature of the curve can be determined. Moreover, since they cannot be completed according to any hypothesis of the resistance, as we have noted above, such questions can only be proposed for special hypothesis of the resistance. Therefore this treatment is especially adapted to resistance in proportion to the squares of the speeds, since in that case the canonical equation, by which the speed is determined, leads to separation of the variables, and thus the speed can be determined. Then also to be considered is the resistance which is proportional to the biquadratic of the speeds, since for that hypothesis the speed can become known in a certain way. And finally for whatever law of the resistance, but only if the resistance becomes very small, questions of this kind emerge with a solution more easily. Now in these problems either the ratio of the speeds which have been acquired in different descents on the same curve is investigated, or the ratio of the times in which different descents or ascents have been completed. And in each kind, either from the ratio of the given times or speeds acquired from different descents, the curves are to be found.

PROPOSITION 80. [p. 386]

Problem.

707. *In a uniform medium, which resists in the ratio of the squares of the speeds, and with an absolute force acting downwards, to compare the speeds between themselves at the point A* (Fig.80)*, which are acquired in different descents of the body on the curve MA.*

Solution.

Let the speed at A, acquired in one descent, correspond to the height b, and the speed at M correspond to the height v. Put $AP = x$, $AM = s$, the force acting at M, which is variable in some way, is equal to P and the exponent of the resistance is equal to k. With these in position, we have :

$$dv = -Pdx + \frac{vds}{k},$$

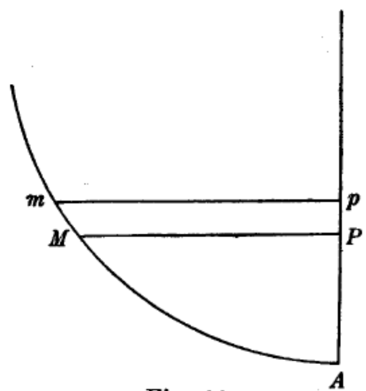

Fig. 80.

which equation integrated gives :

$$v = e^{\frac{s}{k}}(b - \int e^{\frac{-s}{k}} Pdx)$$

with the integral $\int e^{\frac{-s}{k}} Pdx$ thus taken, so that it vanishes on putting $x = 0$. Now let M be the starting point of the descent, where $v = 0$; this point is found from the equation

$$b = \int e^{\frac{-s}{k}} Pdx .$$

Now another descent is put in place to come from a neighbouring point m and the speed acquired at A corresponds to the height $b + db$. Hence we have

$$b + db = \int e^{\frac{-s}{k}} Pdx$$

equal to the sum of all the terms $e^{\frac{-s}{k}} Pdx$ from A as far as m; now in the first equation, $b = \int e^{\frac{-s}{k}} Pdx$ signifies the sum of all the terms $e^{\frac{-s}{k}} Pdx$ only as far as from A to M .

Hence the first sum is greater than this sum by the final element $e^{\frac{-s}{k}} Pdx$, with $AM = s$ and $Pp = dx$. [p. 387] Hence we have

$$db = e^{\frac{-s}{k}} Pdx.$$

From which equation the relation is given between the arc traversed in the descent MA and the speed acquired at the lowest point A. *Q.E.I.*

Corollary 1.

708. Hence from the given arc of the descent $AM = s$, the height corresponding to the speed acquired by the speed at A is given by :

$$b = \int e^{\frac{-s}{k}} Pdx$$

Or if the point M and the speed at A are considered as the only variable quantities, then there is this equation between them :

$$db = e^{\frac{-s}{k}} Pdx.$$

Corollary 2.

709. Hence from this equation, if some ratio was proposed between the arc of the descent and the speeds acquired at the point A, the equation is found for the curve AM satisfying the proposed conditions.

Corollary 3.

710. If the medium is not uniform, but dissimilar in some way with the exponent of this arising set equal to q, in place of the equation found there is produced this equation :

$$db = e^{-\int \frac{s}{q}} Pdx\,,$$

which is of similar use.

Corollary 4. [p. 388]

711. Because the value of e is greater than unity, clearly 2,7182818284, then $e^{-\int \frac{ds}{q}}$ or $e^{\frac{-s}{k}}$ is less than unity and on this account $db < Pdx$. *In vacuo* now it is the case that $db = Pdx$.

Scholium 1.

712. In a similar manner in ascending it occurs that, when the body rises from A along the arc $AM = s$ to a corresponding height b. Then indeed it becomes :

$$db = e^{\frac{s}{k}} Pdx.$$

or in a non-uniform medium :

$$db = e^{\int \frac{ds}{q}} Pdx\,.$$

Which formulas from the descents are found to be made serviceable on putting $-s$ in place of $+s$; with which put in place the descent is always changed into the ascent. Hence it is apparent, as for the descent always it is the case that $db < Pdx$, thus for the ascents it is always the case that $db > Pdx$, since $e^{\frac{s}{k}}$ or $e^{\int \frac{ds}{q}}$ is greater than one.

Corollary 5.

713. Hence in a medium it will not be possible for an ascent of descent if $b = \int Pdx$ or $b = \alpha \int Pdx$; for then we have $e^{\frac{s}{k}} = \alpha$ or s = const., in which equation no line is continued.

Corollary 6.

714. Neither can a curve be found, for which we can write, either in the ascent or descent, $b=\int Qdx$, with Q denoting some function of s and x, unless Q can be prepared thus, so that $\frac{Q}{P}$ becomes equal to 1 on putting s and x = 0. Indeed it happens that $e^{\pm\int\frac{ds}{q}}P=Q$ and $e^{\pm\int\frac{ds}{q}}$ has the value 1 on putting s = 0.

Scholium 2. [p. 389]

715. The reason for this is that we have put s to vanish with x vanishing; and on this account the equation

$$db=e^{\pm\int\frac{ds}{q}}Pdx$$

thus must be integrated, so that b vanishes on putting x = 0. Moreover if b is given thus, so that db is expressed in terms of dx , the equation can be divided by dx. On which account it may not be possible to apply this condition to that equation, unless perhaps the equation freely has this pleasing property. But if such a value of b is given, in order that $db = Rds$ or $b=\int Rds$ with b vanishing on making s = 0, then the equation for the curve sought becomes :

$$Rds=e^{\pm\int\frac{ds}{q}}Pdx\,,$$

which is always for a real curve, provided $\int Rds$ has a positive value, and gives rise to $ds > dx$ or

$$e^{\pm\int\frac{ds}{q}}P>R.$$

Example 1.

716. Let the uniform force acting or $P = g$ and the medium resists uniformly and the curve MA has this required property, that the body in individual descents, descends from A until it has acquires speeds which are in the square root ratio of the descended arcs traversed. . Hence $\sqrt{b}$ is as $\sqrt{s}$ or $b=\alpha s$; thus the equation becomes : [p. 390]

$$\alpha ds=ge^{\frac{-s}{k}}dx \text{ or } \alpha e^{\frac{s}{k}}ds=gdx\,,$$

the integral of which is :

$$\alpha k(e^{\frac{s}{k}}-1)=gx$$

with a constant added, in order that s vanishes on making x = 0. There is hence had

$$e^{\frac{s}{k}}=\frac{\alpha k+gx}{\alpha k} \text{ and } \frac{\mathrm{s}}{\mathrm{k}}=l(\alpha k+gx)-l\alpha k.$$

Which differentiated gives :

$$\frac{ds}{k}=\frac{gdx}{\alpha k+gx};$$

from which it is understood that the curve is tractrix for a thread of length k on a horizontal base at a distance down from the point A by the interval $\frac{\alpha k}{g}$. Hence the curve can be constructed in this manner : upon the horizontal base CE (Fig. 81) and with the string $BC = k$ the tractrix BA is described; then the horizontal line DA is drawn at a distance $DC=\frac{\alpha k}{g}$ from CE; with which done the part of the curve BA sought is satisfied. Moreover we can place the vertical line BC and B the highest point of the tractrix; from which it is understood that by necessity α must be less than g. If indeed it should be greater, then it follows that $CD > CB$ and thus the point A becomes imaginary. But if it is the case that $\alpha = g$, then the point A falls on B and thus is the only satisfactory point. If it is the case that $\alpha = 0$, then the point A is at an infinite distance and the descending body loses all its speed. Therefore since it must be the case that $\alpha < g$, then $b < gs$.

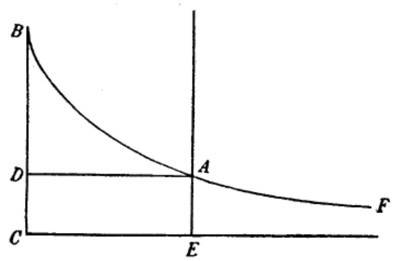

Fig. 81.

Example 2.

717. As according to the above hypothesis of resistance and force acting, the curve AMF (Fig. 82) is sought, upon which all the ascents made from the point A thus have in common, that the whole arcs for each of the individual complete ascents are proportional to the squares of the speeds of the motion starting from A. [p. 391] Hence as before we have $b=\alpha s$ and $db=\alpha ds$. But since for the ascents, $db=ge^{\frac{s}{k}}dx,$ then $\alpha e^{\frac{-s}{k}}ds=gdx$ and on integrating,

$$\alpha k(1-e^{\frac{-s}{k}})=gx.$$

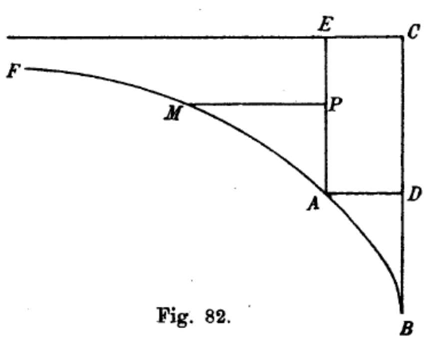

Fig. 82.

Hence therefore it is found that

$$e^{\frac{-s}{k}}=\frac{\alpha k-gx}{\alpha k}$$

and

$$\frac{ds}{k}=\frac{gdx}{\alpha k-gx} \text{ or } (\frac{\alpha k}{g}-x)\frac{ds}{dx}=k.$$

From which it is apparent that the satisfying curve is again a tractrix upon the horizontal base CE made by a line of length k, but considered downwards, the cusp of which is at B with BC becoming equal to k. Moreover taking $CD=\frac{\alpha k}{g}$; and with DA drawn to the horizontal then A is the point, in which all the ascents must begin. Hence therefore it is also understood that α cannot be greater than g, as previously the point *A* becomes imaginary. But if $\alpha = g$ or $b = gs$, then *A* falls on *B* and the arc traversed for any ascent is equal to $\frac{b}{g}$.

Scholium 3.

718. Many examples of this kind I omit here, since they can be resolved so easily from the general formula ; nor do I report also on questions of this kind for other hypothesis of the resistance, for which the solution of these are able to be found, since such questions are now neither subject to discussion nor are considered curious enough that the solutions of these are required. Therefore we proceed to more worthy problems, in which the curves sought are tautochrones, upon which either all the ascents or descents are completed in equal times.

PROPOSITION 81. [p. 392]

Problem.

719. *According to the hypothesis of a uniform force acting in the downwards direction and with a uniform medium, which resists in the ratio of the square of the speeds, to find the tautochrone curve AM* (Fig.80)*, upon which all the descents as far as the point A are completed in equal times.*

Solution.

Some descent is considered, in which the speed that the body has acquired at the lowest point *A* corresponds to the height *b*. Putting $AP = x$, $AM = s$, the height corresponding to the speed at *M* is equal to *v* and the force acting is equal to *g* and the exponent of the medium is equal to *k*, thus so that the resistance at *M* is to the force of gravity as $\frac{v}{k}$ to 1. With these in place, we have :

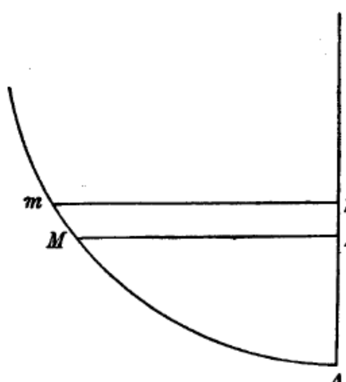

Fig. 80.

$$dv = -gdx + \frac{vds}{k},$$

which equation on integration gives :

$$v = e^{\frac{s}{k}}\left(b - \int e^{\frac{-s}{k}} gdx\right)$$

thus with the integral $\int e^{\frac{-s}{k}} gdx$ taken thus, so that it vanishes on putting *x* or $s = 0$. Hence from this equation the start of the descent is found by putting $v = 0$ or $\int e^{\frac{-s}{k}} gdx = b$.

Now the time, in which the arc *MA* is completed, hence is equal to

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b - \int e^{\frac{-s}{k}} gdx\right)}},$$

from which the time of the whole descent is produced, if after the integration putting

$$\int e^{\frac{-s}{k}} gdx = b.$$

For the sake of brevity we put [p. 393]

$$\int e^{\frac{-s}{k}} gdx = t \text{ and } \frac{ds}{e^{\frac{s}{2k}}} = du,$$

in order that the time of the whole descent thus becomes equal to $\int \frac{du}{\sqrt{(b-t)}}$, on putting $t = b$ after the integration. Now since this expression always gives the same value, it is necessary that $\int \frac{du}{\sqrt{(b-t)}}$ is a function of zero dimensions of b and t, so that on putting $t = b$, b vanishes from the formula. On this account du must be a function of dimensions one half only of t, since u cannot depend on b. Therefore it is necessary that $du = \frac{\alpha dt}{\sqrt{t}}$ with the quantity α arising not containing b. With this in place the time of one descent is equal to $\alpha \int \frac{dt}{\sqrt{(bt-tt)}}$, putting $t = b$ after the integration. Or with the ratio of the diameter to the periphery put as $1 : \pi$, then the time of one descent is equal to $\alpha\pi$; which value always remains, in whatever manner b or the start of the descent is changed. Hence the tautochrone curve sought is determined from this equation $: du = \frac{\alpha dt}{\sqrt{t}} = \frac{ds}{e^{\frac{s}{2k}}},$ the integral of which is :

$$2\alpha\sqrt{t} = 2k\left(1 - e^{\frac{-s}{2k}}\right) \text{ or } t = \frac{k^2}{\alpha^2}\left(1 - e^{\frac{-s}{2k}}\right)^2$$

clearly with a constant added, which makes t disappear on putting $s = 0$. But since

$$t = \int e^{\frac{-s}{k}} gdx,$$

then

$$dt = e^{\frac{-s}{k}} gdx = \frac{k}{\alpha^2}\left(1 - e^{\frac{-s}{2k}}\right) e^{\frac{-s}{2k}} ds.$$

We put $\alpha^2 g = a$ or $\alpha = \sqrt{\frac{a}{g}}$; then we have

$$adx = k\left(e^{\frac{s}{2k}} - 1\right) ds,$$

the integral of which is :

$$ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks;$$

[p. 394] which equations indeed, since the variables s and x have been separated from each other, are sufficient to construct the curve. But if the equation is desired to be free from the exponentials, since from the other equation we have :

$$k\left(e^{\frac{s}{2k}} - 1\right) = \frac{ax + ks}{2k},$$

and with this value substituted in the other equation, it becomes $axds + ksds = 2akdx.$
Q.E.I.

Corollary 1.

720. Since $\alpha = \sqrt{\frac{a}{g}}$, then the time of one descent is equal to $\pi\sqrt{\frac{a}{g}}$. Moreover *in vacuo* and with gravity equal to 1, the time of descent of a pendulum of length f is equal to : $\frac{\pi\sqrt{2f}}{2}$ (166). Whereby the length of an isochronous pendulum *in vacuo* is equal to $\frac{2a}{g}$.

Corollary 2.

721. Therefore if $\frac{2a}{g}$ = 3166 thousandth parts [or scruples] of a Rhenish foot (170), the descent is completed in half a second; this therefore arises, if we put $a = 1583g$ scruples of a Rhenish foot.
[If we use the elementary formula for the period of a simple pendulum of length f, $T = 2\pi\sqrt{\frac{f}{g}}$, then the time for a quarter period, or one descent, is given by $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{f}{g}}$.
If this time T/4 is put as half a second, then $1 = \pi\sqrt{\frac{f}{g}}$; if we approximate the Rhenish foot with the English or American foot, then g = 32000 scruples/sec^2 approx. In this case the corresponding length f is given by 3240 scruples, while the Euler value for 2a = 3170, which is in approximate agreement. Note that the diameter of the generating circle of the cycloid is the radius of circle for the simple pendulum, and f is approx. 39 inches, or 1 metre. Is it a 'happy accident' that $\pi^2 = 9.87 \approx g$?]

Corollary 3.

722. The height corresponding to the speed at M is v and this is equal to

$$e^{\frac{s}{k}}\left(b - \int e^{\frac{-s}{k}} g\,dx\right) = e^{\frac{s}{k}}(b - t)$$

and because of this,

$$t = \frac{gk^2\left(1 - e^{\frac{-s}{2k}}\right)^2}{a}$$

then

$$v = e^{\frac{s}{k}}\left(b - \frac{gk^2}{a}\left(1 - e^{\frac{-s}{2k}}\right)^2\right) = \frac{abe^{\frac{s}{k}} - gk^2\left(e^{\frac{s}{2k}} - 1\right)^2}{a}.$$

Corollary 4. [p. 395]

723. On putting $v = 0$ the whole of the descent arc is produced from this equation :

$$ab = gk^2\left(1 - e^{\frac{-s}{2k}}\right)^2.$$

Hence if the descent arc is put equal to f, then

$$ab = gk^2\left(1 - e^{\frac{-f}{2k}}\right)^2.$$

Whereby from the given arc of the descent f, then

$$v = \frac{gk^2 e^{\frac{s}{k}}}{a}\left(\left(1 - e^{\frac{-f}{2k}}\right)^2 - \left(1 - e^{\frac{-s}{2k}}\right)^2\right)$$

Corollary 5.

724. The equation for this curve

$$ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks$$

can be changed into a series of the exponential $e^{\frac{s}{2k}}$, which is

$$1 + \frac{s}{2k} + \frac{s^2}{1\cdot 2\cdot 4k^2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 8k^3} + \text{etc.},$$

and hence it becomes :

$$ax = \frac{s^2}{1\cdot 2\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 4k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 8k^2} + \text{etc.}$$

or

$$2ax = \frac{s^2}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 2k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^2} + \text{etc.}$$

Scholium 1.

725. Here it is appropriate to observe that the curve is expressed by an equation similar to that serving to express the ascent for the above brachistochrone; indeed there it was :

$$at = \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3k} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4k^2} + \text{etc.}$$

(687), which equation only differs from this one of ours found for the tautochrone, as here it is $2a$, and there it was a, and the exponent of the resistance of the brachistochrone is twice as larges as the exponent for the resistance of the tautochrone. Hence the brachistochrone curve can be adapted to produce tautochronism also, with the arc of the ascent attributed to the descent in the resisting medium, the exponent of which is twice as small. [p. 396]

Corollary 6.

726. To find the continuation of the curve *MA* beyond *A, s* must be made negative, with which done there is obtained :

$$ax = 2k^2\left(e^{\frac{-s}{2k}} - 1\right) + ks$$

or

$$2ax = \frac{s^2}{1\cdot 2} - \frac{s^3}{1\cdot 2\cdot 3\cdot 2k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^2} - \text{etc.}$$

Which same equation emerges, if we make *k* negative. But with *k* made negative the descent is changed into ascent ; on account of which the curve *MA* continued beyond *A* serves to ascend, and above that point all the ascents are completed in the same times, clearly $\pi\sqrt{\frac{a}{g}}$.

Corollary 7.

727. Hence the same curve continued BMANC (Fig. 83) is a tautochrone for the ascent as well as for the ascent. In fact on the curve BMA all the descents are completed in the same time and on the arc ANC all the ascents. Whereby all the half oscillations which begin on the arc BMA are isochronous with each other, and the time of one semi-oscillation is equal to $2\pi\sqrt{\frac{a}{g}}$.

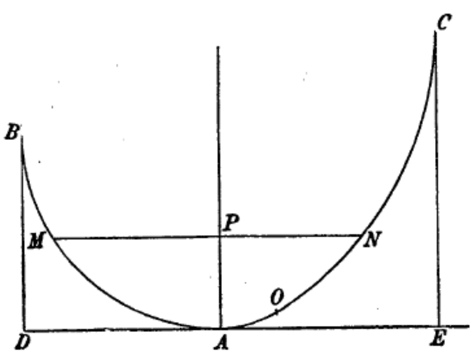

Fig. 83.

Corollary 8. [p. 397]

728. If the resistance vanishes, in which case *k* becomes $\propto$, this curve is changed into a cycloid, which is the tautochrone curve *in vacuo*. This itself indicates equation expressed through the series; for it becomes $2ax = \frac{s^2}{1.2}$ or $4ax = s^2$, the equation for the cycloid.

Corollary 9.

729. Hence the curve *BMANC* exactly as the cycloid has vertical cusps at *B* and *C*, and so that these can be found, put $dx = ds$, and hence for the arc *BMA*

$$a = k\left(e^{\frac{s}{2k}} - 1\right) \text{ or } s = 2kl\frac{a+k}{k} = AMB;$$

and the height of this *BD* becomes equal to

$$2k - \frac{2k^2}{a}l\frac{a+k}{k}.$$

Now for the arc of the ascent *ANC* we have :

$$ANC = 2kl\frac{k}{k-a} \text{ and } CE = \frac{2k^2}{a}l\frac{k}{k-a} - 2k.$$

Or through series we have :

$$BD = a - \frac{2a^2}{3k} + \frac{2a^3}{4k^2} - \frac{2a^4}{5k^3} + \text{etc.}$$

and

$$CE = a + \frac{2a^2}{3k} + \frac{2a^3}{4k^2} + \frac{2a^4}{5k^3} + \text{etc.}$$

Corollary 10.

730. From these it is evident that the cusp *C* of the ascending arc is higher than the cusp *A* of the descending arc. And the arc *ANC* of the cusps becomes infinite, if $k = a$; and if $a > k$, the cusp *C* becomes imaginary. Likewise it is apparent from the equation that *BD* as well as *CE* are diameters of the curve sought.

Corollary 11. [p. 398]

731. If the body in the half–oscillation has a speed corresponding to the height *b*, then the arc of the descent is equal to :

$$2kl\frac{k\sqrt{g}}{k\sqrt{g} - \sqrt{ab}}$$

(723) or through the series is equal to :

$$\frac{2\sqrt{ab}}{\sqrt{g}} + \frac{2ab}{2gk} + \frac{2ab\sqrt{ab}}{3gk^2\sqrt{g}} + \text{etc.}$$

and the following arc of the ascent is equal to :

$$2kl\frac{k\sqrt{g} + \sqrt{ab}}{k\sqrt{g}} = \frac{2\sqrt{ab}}{2g} - \frac{2ab}{2gk} + \frac{2ab\sqrt{ab}}{3gk^2\sqrt{g}} - \text{etc.}$$

Corollary 12.

732. Hence if the descent is made from the point B, thus in order that the arc of the descent is equal to AMB = $2kl\frac{a+k}{k}$,

then

$$b = \frac{gak^2}{(a+k)^2};$$

now the arc of the following ascent is equal to $2kl\frac{a+k}{k}$.

Corollary 13.

733. From the equation for this curve it appears that the curve has a horizontal tangent at the point *A*. Moreover since on putting *ds* constant, the radius of osculation at *M* is equal to :

$$\frac{dsdy}{ddx} = \frac{ds\sqrt{(ds^2 - dx^2)}}{ddx},$$

since (719) :

$$dx = \frac{k}{a}\left(e^{\frac{s}{2k}} - 1\right)ds,$$

then the equation becomes ;

$$ddx = \frac{1}{2a}e^{\frac{s}{2k}}ds^2$$

and

$$\sqrt{(ds^2 - dx^2)} = ds\sqrt{\left(1 - \frac{k^2}{a^2}\left(e^{\frac{s}{2k}} - 1\right)^2\right)};$$

and the radius of osculation at *M* is equal to :

$$\frac{\sqrt{\left(4a^2 - 4k^2\left(e^{\frac{s}{2k}} - 1\right)^2\right)}}{e^{\frac{s}{2k}}}.$$

Hence on putting $s = 0$ the radius of osculation at the lowest point $A = 2a$. Now at *B* and *C* the radius of osculation vanishes.

Corollary 14. [p. 399]

734. The radius of osculation is not a maximum at the lowest point *A*; but through the method of maxima the maximum is found in the arc of the ascension, and that is present at the point *O* :

$$AO = 2kl\frac{k^2}{k^2 - a^2}.$$

For at this point the radius of osculation is equal to $\frac{2ak}{\sqrt{(k^2-a^2)}}$. From which is inferred that, unless $k > a$, the curvature of the curve *ANC* is to be continually diminished and the point *O* does not exist anywhere.

Scholium 2.

735. Therefore in this problem we have found two curves, upon the first of which all the descents are to be completed, and indeed on the second all the ascents are completed in equal times. And while on the whole curve *BAC* the whole journeys or semi–oscillations are performed in equal times, if they start from a certain portion of the curve *BA*, this curve is suitable for isochronous oscillations to be made in fluids, but only if the returns also should be isochronous, concerning which there is no agreement. [As the initial descent-ascent and the following descent-ascent are made in opposite directions; if one is a tautochrone, then the other is not.]Moreover because in fluids besides resistance proportional to the square of the speeds another is observed above, that jointly with that resistance which for proportional or constant moments of time it is probably worth the effort to determine the tautochrone, since ; because that can be easily done from the preceding. For let the constant resistance be equal to h; then for the descent we have :

$$dv = -gdx + hds + \frac{vds}{k}.$$

Whereby if in the former operation in place of gdx only, everywhere $gdx - hds$ is substituted, a satisfying tautochrone is obtained in a like manner ; clearly for the descent this curve is obtained : [p. 400]

$$ax = \left(\frac{ah}{g} - k\right)s + 2k^2\left(e^{\frac{s}{2k}} - 1\right)$$

and now for the ascent this equation :

$$ax = \left(k - \frac{ah}{g}\right)s + 2k^2\left(e^{\frac{-s}{2k}} - 1\right);$$

which curve constitutes a continuation with the first curve; indeed the one goes into the other on putting s negative. It is to be observed that if $k = \frac{ah}{g}$, then the tautochrone curve is the tractrix *BAF* (Fig. 82) having the horizontal asymptote *CE*, which is distant from the point *A* by the interval $AE = \frac{2k^2}{a}$. Moreover the length of the thread , by which the tractrix is described, is equal to $2k$.

But because above with the curve of this kind never having a horizontal tangent the semi–oscillation can be completed and somewhere a point of equilibrium *A* exists, it is no wonder as now we have observed above, that in such a hypothesis of the resistance the body is able to stop at some place on the slope. Moreover in these cases, in which the curve is allowed to descend beyond *A*, there is no return and thus no oscillations can be performed, since the body is able to come to rest somewhere on the inclined plane, yet is unable to ascend above that ; for in some part of the curve *AF* the body is able to remain at rest.

Scholium 3.

736. The solution of the problem is not made much more difficult, if the force tending downwards is not constant, but P is made variable somehow, and the exponent of the resistance also is put as the variable q . Indeed there is had for the element of time in the descent :

$$\frac{ds}{e^{\frac{1}{2}\int\frac{ds}{q}}\sqrt{\left(b-\int e^{-\int\frac{ds}{q}}Pdx\right)}}.$$

If now there is put as before (719) [p. 401]

$$\int e^{-\int\frac{ds}{q}}Pdx = t \text{ and } \frac{ds}{e^{\frac{1}{2}\int\frac{ds}{q}}} = du,$$

there also corresponds :

$$du = \frac{\alpha dt}{\sqrt{t}} \text{ or } t = \frac{\alpha^2 dt^2}{du^2} = \frac{\alpha^2 P^2 dx^2}{e^{\int\frac{ds}{q}}ds^2}.$$

Which equation differentiated again on putting *ds* constant and with this value substituted in place of *dt* there is given :

$$\frac{qds^2}{2\alpha^2} = Pqddx + qdPdx - \frac{1}{2}Pdxds$$

for the curve having the isochronous descent. And the continuation of this curve beyond *A* serves as the ascent curve.

Scholium 4.

737. I first gave this tautochrone according to the hypothesis of resistance proportional to the square of the speeds in Book IV of the Commentaries [of the St. Petersburg Ac. of Sc.;E013 in these translations], in which I have used the same method here. Since then also the Cel. Joh. Bernoulli has signified to me by letter that he too has found the same tautochrone for the same hypothesis of the resistance [See the letter from Joh. Bernoulli 27 Dec. 1729 given to Leonhard Euler, that G. Enestrom published in the Biblioth. Mathem. 4, 1903, p.378. Note by P. St. in the *O. O.*]; the method of which can be seen in the Comm. Acad.Paris. A 1730 [Joh. Bernoulli, *Méthode pour trouver les tautochrones, dans les milieaux résistans comme le quarré des vitesses*, Mém. de l'acad. d. sc. de Paris 1730, p. 78; Opera Omnia, Book. 3, Lausannae et Genevae 1742, p. 173.]. Now for other hypotheses of the resistance with that exception, in which it is proportional to the speeds, nobody to date as far as I know has determined the tautochrones. For that which pertained to these curves , to which I gave name of tautochrones in the Act. Lips. A. 1726 [E001], did not give a satisfactory answer to this question, as the Cel. Hermann, who first fell upon the same, and I afterwards, have shown. [Jac. Hermann, General theory of the motion, which arises from any forces acting constantly on bodies, Comment. Acad. Sc. Petrop. 2 (1727), p. 139; see in particular p. 158.] But the difficulty of this method of finding tautochrones rests of this, that [p. 402] from other hypotheses of the resistance the speed cannot generally be determined from the canonical equations. Now how nevertheless the tautochrones are able to be investigated for other resistances of the tautochrones, can be gathered from the following proposition, in which it is proposed how for the rarest mediums with the resistance in some ratio of the speeds the tautochrones are to be found.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 375]

PROPOSITIO 78.

Problema.

691. *In medio resistente quocunque et potentiis sollicitantibus quibuscunque invenire curvam brachystochronam AM* (Fig.78)*, super qua corpus descendus ex A ad M citissime perveniat.*

Solutio. [p. 376]

Sit A motus initium, per quod ducatur recta quaecunque AP pro axe habenda, in qua sumatur abscissa $AP = x$; cui respondeat applicata $PM = y$ et arcus $AM = s$. Sit porro corporis in M celeritas debita altitudini v et resistentia utcunque a celeritate pendens = R. Quaecunque nunc corpus sollicitent potentiae absolutae, earum loco duae potentiae substituti possunt in datis directionibus ML et MN, quarum illa axi AP sit parallela, haec vero ad illum normalis. Vis autem corpus secundum ML sollicitans sit = P et vis secundum $MN = Q$. Ex his viribus oritur

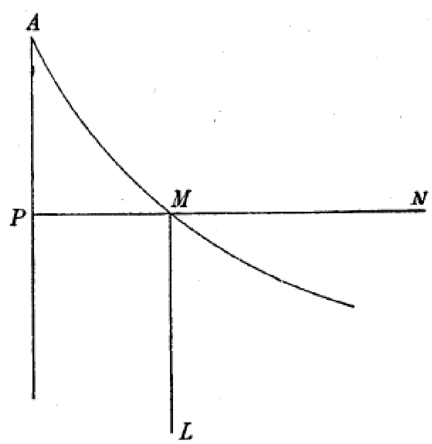

Fig. 78.

$$dv = Pdx + Qdy - Rds.$$

Atque natura brachystochronismi dat

$$\frac{2v}{r} = \frac{2vdxddy}{ds^3} = \frac{Pdy - Qdx}{ds}$$

(673) denotante r radium osculi curvae in M versus superiora directum; pro quo ergo sumto dx constante ponimus $\frac{+ds^3}{dxddy}$, cum alias deberet esse $r = \frac{-ds^3}{dxddy}$. Ex his ergo duabus aequationibus

$$dv = Pdx + Qdy - Rds \text{ et } \frac{2vdxddy}{ds^2} = Pdy - Qdx$$

si eliminetur *v,* habebitur aequatio pro curva brachystochrona quaesita; est nempe

$$v = \frac{Pdyds^2 - Qdxds^2}{2dxddy};$$

cuius differentiale loco *dv* atque ipsum *v* in resistentia *R* substitutum dabit aequationem pro curva quaesita. Q.E.I.

Corollarium 1. [p. 377]

692. Aequatio pro curva, si dicto modo *v* eliminatur, fit differentialis tertii gradiis. Quare si triplex integratio adhibeatur, tres quoque constantes adiici poterunt, quibus effici potest, ut evanescente *x* simul quoque *y* et *s* et *v* evanescant atque praeteria curva per datum punctum *M* transeat.

Corollarium 2.

693. Quia igitur semper curva brachystochrona potest exhiberi, quae initium habeat in A et per datum punctum transeat, infinitae curvae brachystochronae ex puncto A educi possunt.

Corollarium 3.

694. Inventa aequatione pro curva brachystochrona AM innotescat simul corporis super ea descendentis celeritas in singulis punctis; erit namque

$$v = \frac{Pdyds^2 - Qdxds^2}{2dxddy}.$$

Corollarium 4.

695. Data celeritate determinari ex ea poterit tempus, quo corpus arcum *AM* absolvit; erit scilicet tempus per *AM* =

$$\int \frac{ds}{\sqrt{v}} = \int \frac{\sqrt{2dxddy}}{\sqrt{(Pdy - Qdx)}};$$

quod propter aequationem inter *x* et *y* iam inventam poterit saltem per quadraturas exhiberi.

Corollarium 5. [p. 378]

696. Si igitur curva esset invenienda, quae omnes brachystochronas ex *A* eductas ad angulos rectos traiicere deberet, tum eius lineae constructio haberetur, si ab omnibus abscinderetur

$$\int \frac{\sqrt{2dxddy}}{\sqrt{(Pdy - Qdx)}}$$

eiusdem magnitudinis. Hac enim ratione ab istis curvis infinitis arcus isochroni abscinduntur, qui, quoniam omnes curvae sunt brachystochronae, terminabuntur ad traiectoriam orthogonalem.

Exemplum.

697. Sit resistentia quadratis celeritatum proportionalis atque exponens resistentiae utcunque variabilis q; erit $R=\frac{v}{q}$. Cum ergo sit

$$dv=Pdx+Qdy-\frac{vds}{q},$$

erit integrando

$$e^{\int\frac{ds}{q}}v=\int e^{\int\frac{ds}{q}}(Pdx+Qdy).$$

Cum autem sit

$$v=\frac{Pdyds^2-Qdxds^2}{2dxddy},$$

erit

$$2dxddy\int e^{\int\frac{ds}{q}}(Pdx+Qdy)=e^{\int\frac{ds}{q}}ds^2(Pdy-Qdx),$$

in qua aequatione non amplius inest v. Interm tamen haec aequatio fit differentialis tertii gradus, si differentiatione signa integralia tollantur; indeterminati praeterea ipsarum *P, Q*, et q valores in causa sunt, quo minus aequatio ad constructionem praeparari queat.

Scholion.

698. Quae hic ex duabus potentiis P et Q circa curvas brachystochronas sunt deducta, latissime patent, quia, quotcunque potentiae corpus sollicitaverint, eae omnes in huiusmodi duas possunt resolvi, si modo omnium directiones in eodem plano fuerint positae. [p. 379] Quamobrem in hac quoque propositione continentur brachystochronae pro quacunque virium centripetarum hypothesi, quas autem, quia neque concinnae neque construibiles aequationes proveniunt, ulterius non persequemur. Missis igitur his, in quibus celeritatum quaedam lex praescribitur, progredimur ad sequentes quaestiones, in quibus curvae requiruntur, quae a corpore super iis moto datam sustineant pressionem.

PROPOSITIO 79.

Problema.

699. *In hypothesi gravitatis uniformis et medio uniformi, quod in ratione quacunque celeritatum resistat, determinare curvam aequalibis pressionis AM* (Fig.79)*, quae a corpore super ea descendente ubique eandem sustineat pressionem..*

Solutio.

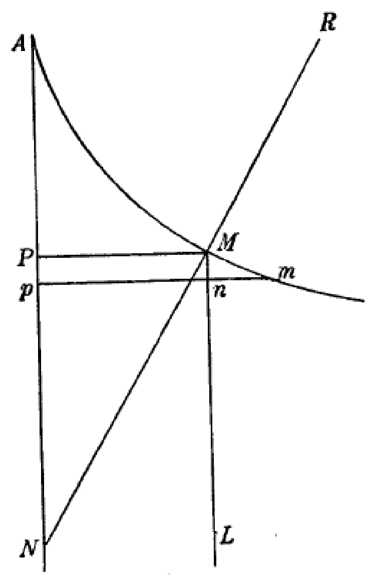

Fig. 79.

Positis $AP = x$, $PM = y$, $AM = s$ et celeritati in M altitudine debita $= v$ et sit potentia corpus deorsum secundum ML trahens $= g$ et vis resistentiae in $M = \frac{v^m}{k^m}$. Erit ergo, dum corpus per elementum Mm progreditur,

$$dv = gdx - \frac{v^m ds}{k^m}.$$

[p. 380] Ponamus curvam deorsum esse convexam, ita ut MR sit radii osculi directio atque ipse radius osculi $MR = \frac{ds^3}{dxddy}$ posito dx constante. Vis ergo centrifugae directio erit in normali MN eiusque quantitas est $= \frac{2vdxddy}{ds^3}$. Secundum eandem vero directionem curva premitur a vi normali ex resolutione potentiae $ML = g$ orta, quae est $= \frac{gdy}{ds}$.

Tota ergo vis, qua curva secundum MN premitur, est $= \frac{gdy}{ds} + \frac{2vdxddy}{ds^3}$; quae cum debeat esse constans, ponatur ea aequalis αg habebiturque

$$\alpha gds^2 = gdyds^2 + 2vdxddy$$

atque hinc

$$v = \frac{\alpha gds^2 - gdyds^2}{2dxddy}.$$

Sit $ds = pdx$ atque $dy = dx\sqrt{(p^2 - 1)}$; erit

$$ddy = \frac{pdpdx}{\sqrt{(p^2-1)}}.$$

His substitutis erit

$$v = \frac{(\alpha gp^3dx - gp^2dx\sqrt{(p^2-1)})\sqrt{(pp-1)}}{2pdp} = \frac{\alpha gp^2dx\sqrt{(p^2-1)} - gpdx(p^2-1)}{2\,dp}.$$

Sit porro $dx = 2qdp$; erit

$$v = gpq(\alpha p\sqrt{(p^2-1)} - p^2 + 1)$$

vel

$$v = Pq$$

posito

$$gp(\alpha p\sqrt{(p^2-1)} - p^2 + 1) = P.$$

Erit ergo

$$dv = Pdq + qdP \quad \text{et} \quad v^m = P^m q^m.$$

Quibus valoribus in aequatione $dv = gdx - \frac{v^m ds}{k^m}$ substitutis prodibit

$$Pdq + qdP = gdx - \frac{P^m q^m ds}{k^m}.$$

Est vero

$$dx = 2qdp \text{ et } ds = pdx = 2pqdp.$$

Quamobrem proveniet ista aequatio

$$Pdq + qdP = 2gqdp - \frac{2P^m q^{m+1} pdp}{k^m},$$

quae duas tantum continet variables p et q, quia P per p datur. Ad hanc aequationem construendam ponatur

$$q = \frac{1}{Pu^{\frac{1}{m}}};$$

quo facto obtinebitur ista aequatio [p. 381]

$$du + \frac{2mgudp}{P} = \frac{2mpdp}{k^m P};$$

quae ducta in $e^{2mg\int \frac{dp}{P}}$ et integrata abit in hanc

$$u = \frac{2me^{-2mg\int \frac{dp}{P}}}{k^m}\int \frac{e^{2mg\int \frac{dp}{P}} pdp}{P}.$$

Hinc ergo invenitur u per p, atque u invento erit $q = \frac{1}{Pu^{\frac{1}{m}}}$ atque

$$x = \int \frac{2dp}{Pu^{\frac{1}{m}}} \quad \text{et} \quad s = \int \frac{2pdp}{Pu^{\frac{1}{m}}} \quad \text{et} \quad y = \int \frac{2dp\sqrt{(p^2-1)}}{Pu^{\frac{1}{m}}}.$$

Cum autem sit

$$P = gp(\alpha p\sqrt{(p^2-1)} - p^2 + 1),$$

erit

$$\frac{gdp}{P} = \frac{dp}{p} + \frac{dp}{\alpha\sqrt{(p^2-1)}} + \frac{(1-\alpha^2)dp}{\alpha^2 p - \alpha\sqrt{(p^2-1)}}$$

atque

$$\int \frac{gdp}{P} = lp - l(\alpha p - \sqrt{(p^2-1)})$$

et

$$e^{2mg\int\frac{dp}{P}} = p^{2m}(\alpha p - \sqrt{(p^2-1)})^{-2m.1})$$

Est vero

$$\int \frac{e^{2mg\int\frac{dp}{P}} p dp}{P} = \frac{e^{2mg\int\frac{dp}{P}} p}{2mg} - \frac{1}{2mg}\int e^{2mg\int\frac{dp}{P}} dp.$$

Quodcirca erit

$$u = \frac{p}{gk^m} - \frac{e^{-2mg\int\frac{dp}{P}}}{gk^m}\int e^{2mg\int\frac{dp}{P}} dp = \frac{p}{gk^m} - \frac{\int p^{2m}(\alpha p - \sqrt{(p^2-1)})^{-2m} dp}{gk^m p^{2m}(\alpha p - \sqrt{(p^2-1)})^{-2m}}$$

Cum igitur hoc modo ex p inveniri possit u, curvae quaesitae constructio hinc perficietur. Q.E.I.

Corollarium 1. [p. 382]

700. Curva inventa ergo hanc habebit proprietatem, ut in quovis puncto M prematur versus MN vi constanti, quae est ad vim gravitatis $ML = g$ ut α ad 1.

Corollarium 2.

701. Si pro α sumatur numerus negativus, curva ubique secundum MR, directionem priori oppositam, aequabiliter premetur. Hoc ergo casu curva debet esse concava deorsum, quia vis centrifuga contraria atque maior esse debet quam vis normalis, cuius directio semper in MN est sita.

Corollarium 3.

702. Si $\alpha = 0$, tum curva prodibit, quae nullam omnino pressionem a corpore sustinet. Quae ergo curva est ea ipsa, quam corpus proiectum libere motum describit.

Corollarium 4.

703. Si $\alpha = 1$ seu tota pressio = g, tum curva erit convexa deorsum ubique. Nam quia vis normalis sola ubique est minor quam g nisi casu, quo $ds = dy$, vis centrifuga cum ea conspirare debet ideoque radius osculi in plagam ipsi MN oppositam cadere.

Corollarium 5. [p. 383]

704. Si ponatur

$$k^m e^{2mg\int\frac{dp}{P}} u = 2mz,$$

erit

$$dz = \frac{e^{2mg\int\frac{dp}{P}} p\,dp}{P}.$$

Particularis ergo solutio, quia in hac aequatione indeterminatae sunt a se invicem separatae, habebitur, si ponatur $P = 0$. Hinc autem fit

$$gp(\alpha p\sqrt{(p^2-1)} - p^2 + 1) = 0$$

seu

$$\alpha p = \sqrt{(p^2-1)} \quad \text{sive} \quad \alpha ds = dy.$$

Unde prodit

$$\alpha s = y.$$

Satisfacit ergo recta angulum cum verticali *AP* constituens, cuius sinus est α sumto 1 pro sinu toto. Hoc enim casu vis centrifuga evanescit et vis normalis fit = αg.

Exemplum.

705. Sit $\alpha = 1$ seu quaeratur curva, quae ubique vi = g prematur; quo casu integratio ipsius $\frac{gdp}{P}$ simplicior evadit; erit enim

$$e^{2mg\int\frac{dp}{P}} = p^{2m}(p-\sqrt{(p^2-1)})^{-2m} = \left(\frac{p}{p-\sqrt{(p^2-1)}}\right)^{2m} = (p^2+p\sqrt{(p^2-1)})^{2m}.$$

Hanc ob rem erit

$$u = \frac{p}{gk^m} - \frac{\int(p^2+p\sqrt{(p^2-1)})^{2m}dp}{gk^m(p^2+p\sqrt{(p^2-1)})^{2m}}.$$

Quae aequatio, quoties $2m$ est numerus integer, integrationem admittit. Posito enim

$$p^2+p\sqrt{(p^2-1)} = \frac{r+1}{2}$$

erit

$$p = \frac{r+1}{2\sqrt{r}}$$

atque

$$u = \frac{r+1}{2gk^m\sqrt{r}} - \frac{1}{4gk^m(r+1)^{2m}}\int\frac{(r-1)(r+1)^{2m}dr}{r\sqrt{r}}.$$

Est autem hac positione

$$r = 2p^2 - 1 + 2p\sqrt{(p^2 - 1)} \quad \text{seu} \quad \sqrt{r} = p + \sqrt{(p^2 - 1)}.$$

Ut si fuerit $m = \frac{1}{2}$ seu resistentia celeritatibus proportionalis, erit [p. 384]

$$u = \frac{r+1}{2g\sqrt{kr}} - \frac{1}{4g(r+1)\sqrt{k}}\int\frac{(r^2-1)dr}{r\sqrt{r}} = \frac{1}{2g\sqrt{k}}\left(\frac{r+1}{\sqrt{r}} - \frac{r^2+\beta\sqrt{r}+3}{3(r+1)\sqrt{r}}\right) = \frac{2rr+6r-\beta\sqrt{r}}{6g(r+1)\sqrt{kr}},$$

seu mutata constante β est

$$u = \frac{r\sqrt{r} + 3\sqrt{r} + 2\beta}{3g(r+1)\sqrt{k}}.$$

Posito autem loco r eius valore erit

$$u = \frac{p^2 + 1 + p\sqrt{(p^2-1)}}{3gp\sqrt{k}} + \frac{\beta}{3g(p^2 + p\sqrt{(p^2-1)})\sqrt{k}}.$$

Sit $\beta = 0$; erit

$$u^{\frac{1}{m}} = u^2 = \frac{(p^2 + 1 + p\sqrt{(p^2-1)})^2}{9g^2kp^2}$$

et propter

$$P = gp(p - \sqrt{(p^2-1)})\sqrt{(p^2-1)}$$

erit

$$Pu^{\frac{1}{m}} = \frac{(p - \sqrt{(p^2-1)})(p^2 + 1 + p\sqrt{(p^2-1)})^2\sqrt{(p^2-1)}}{9gkp}$$

atque

$$x = \frac{2gk}{3p^2 - 1 - 3p\sqrt{(p^2-1)}} - 2gk\,l(3p^2 - 1 - 3p\sqrt{(p^2-1)}).$$

Scholion.

706. Similis integratio formulae, cui u est aequalis, etiam succedit, si $\alpha = -1$, quo casu prodit curva concava deorsum, in qua vis centrifuga contraria est et maior quam vis normalis; quippe excessus est = g. Eadem vero ipsa prodit aequatio, quae pro casu $\alpha = 1$, nisi quod signum ipsius $\sqrt{(p^2 - 1)}$ debet immutari.

Quod ad reliquas quaestiones huc pertinentes attinet, in quibus aliae pressionum leges proponuntur, eae vel ad nimis prolixos calculos deducunt vel iam sunt pertractatae. Vidimus enim curvas, in quibus pressio totalis sit duplo maior quam vel sola vis centrifuga vel sola normalis, esse brachystochronas, atque curvas, in quibus alia obtinet ratio, supra quoque iam pertractavimus, cum curvas investigaremus, super quibus motus, super quibus motus quam minime acceleratur. [p. 385] Sequitur ergo ut ad curvas inveniendas progrediamur, super quibus plures diversi descensus vel ascensus datas inter se teneant leges, quae quaestiones plurimum difficultatis in se habent. Necesse enim est ad huiusmodi problemata solvenda, ut celeritas corporis in singulis locis possit exprimi per quantitates, quibus curvae natura determinatur. Quod autem cum non in quavis resistentiae hypothesi possit perfici, uti supra notavimus, tales quaestiones tantum pro specialibus resistentiae hypothesibus poterunt proponi. Praecipue ergo ista tractatio ad resistantiam quadratis celeritatum proportionalem est accommodanda, quia hoc casu aequatio canonica, qua celeritas determinatur, separationem variabilium admittit atque ipsa celeritas potest exhiberi. Tum etiam considerari potest resistentia, quae biquadratis celeritatum est proportionalis, cum pro hac hypothesi celeritas quodammodo cognoscatur. Denique quaecunque fuerit resistentiae lex, si modo resistentia est vale parva, huiusmodi quaestiones solutu faciliores evadent. In his vero problematibus vel ratio celeritatum, quae in diversis descensibus super eadem curva acquiruntur, investigatur vel temporum, quibus diversi descensus aut ascensus absolvuntur, ratio. Atque in utroque genere ex data vel temporum vel celeritatum variis descensibus acquisitarum ratione curvae sunt invenienda.

PROPOSITIO 80. [p. 386]

Problema.

707. *In medio uniformi, quod resistit in duplicata ratione celeritatum, atque potentia absoluta deorsum tendente comparare inter se celeritates in puncto A* (Fig.80)*, quae in diversis descensibus corporis super curva MA acquiruntur.*

Solutio.

Sit celeritas in A, quam uno descensu acquisivit, debita altitudini b et celeritas in M debita altitudini v. Ponatur $AP = x$, $AM = s$, potentia sollicitans in M , quae sit utcunque variabilis, $= P$ atque exponens resistantiae $= k$. His positis erit

$$dv = -Pdx + \frac{vds}{k},$$

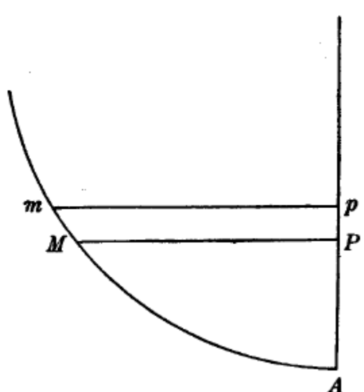

Fig. 80.

quae aequatio integrata dat

$$v = e^{\frac{s}{k}}\left(b - \int e^{\frac{-s}{k}} Pdx\right)$$

integrali $\int e^{\frac{-s}{k}} Pdx$ ita accepto, ut evanescat posito $x = 0$. Sit nunc M initium descensus, ubi est $v = 0$; invenietur hoc punctum ex aequatione

$$b = \int e^{\frac{-s}{k}} Pdx .$$

Iam ponatur alius descensus fieri ex puncto proximo

m atque celeritas in *A* acquisita sit debita altitudini *b* + *db*. Erit ergo
$$b+db=\int e^{\frac{-s}{k}}Pdx$$
= summae omnium $e^{\frac{-s}{k}}Pdx$ ab *A* usque ad *m*; in aequatione vero priore $b=\int e^{\frac{-s}{k}}Pdx$ significat summam omnium $e^{\frac{-s}{k}}Pdx$ ab *A* usque ad *M* tantum. Illa ergo summa superat hanc summam ultimo elemento $e^{\frac{-s}{k}}Pdx$ existente *AM* = *s* et *Pp* = *dx*. [p. 387] Erit ergo
$$db=e^{\frac{-s}{k}}Pdx.$$
Ex qua aequatione datur relatio inter arcum *MA* descensu percursum et inter celeritatem in puncto infimo *A* acquisitam. *Q.E.I.*

Corollarium 1.

708. Dato ergo arcu descensus *AM* = *s* erit altitudo celeritati in *A* acquisitae debita
$$b=\int e^{\frac{-s}{k}}Pdx$$
Seu si punctum *M* et celeritas in *A* tanquam variabilis quantitates considerantur, erit aequatio inter eas
$$db=e^{\frac{-s}{k}}Pdx.$$

Corollarium 2.

709. Ex hac ergo aequatione, si proposita fuerit quaecunque ratio inter arcus descensu et celeritates in puncto *A* acquisitas, invenietur aequatio pro curva *AM* propositae conditioni satisfaciens.

Corollarium 3.

710. Si medium non fuerit uniforme, sed difforme utcunque existente eius exponente = *q*, loco aequationis inventae prodibit ista aequatio
$$db=e^{-\int\frac{s}{q}}Pdx,$$
cuius similis est usus.

Corollarium 4. [p. 388]

711. Quia valor ipsius *e* est unitate maior, quippe 2,7182818284, erit $e^{-\int\frac{ds}{q}}$ seu $e^{\frac{-s}{k}}$ unitate minor et hanc ob rem *db* < *Pdx*. In vacuo vero esset *db* = *Pdx*.

Scholion 1.

712. Simili modo res se habet in ascensu, quando corpus celeritate altitudini *b* debita ex *A* per arcum *AM* = *s* ascendit. Tum enim erit
$$db=e^{\frac{s}{k}}Pdx.$$
vel in medio difformi
$$db=e^{\int\frac{ds}{q}}Pdx.$$

Quae formulae illis descensui inservientibus invenientur ponendo – s loco + s; qua substitutione semper descensus in ascensum transmutatur. Hinc apparet, quemadmodum pro descensu semper erat $db < Pdx$, ita fore pro ascensu semper $db > Pdx$, quia $e^{\frac{s}{k}}$ seu $e^{\int \frac{ds}{q}}$ est unitate maior.

Corollarium 5.

713. In media ergo resistente neque pro ascensu neque pro descensu esse potest $b = \int Pdx$ vel $b = \alpha \int Pdx$; tum enim foret $e^{\frac{s}{k}} = \alpha$ seu s = const., in qua aequatione nulla linea continetur.

Corollarium 6.

714. Neque etiam curva poterit inveniri, pro qua vel in descensu vel in ascensu foret $b = \int Qdx$ denotante Q functionem quamcunque ipsarum s et x, nisi Q ita sit comparata, ut $\frac{Q}{P}$ fiat = 1 positis s et x = 0. Fit enim $e^{\pm \int \frac{ds}{q}} P = Q$ et $e^{\pm \int \frac{ds}{q}}$ abit in 1 posito s = 0.

Scholion 2. [p. 389]

715. Ratio huius est, quod posuimus s evanescere evanescente x; atque hanc ob rem aequatio

$$db = e^{\pm \int \frac{ds}{q}} Pdx$$

ita debet integrari, ut evanescat b posito x = 0. Si autem b ita detur, ut db per dx exprimatur, aequatio per dx dividi poterit. Quocirca ea ad hanc legem non potest accommodari, nisi forte sponte aequatio hac proprietate iam gaudeat. Sin autem datus ipsius b valor talis fuerit, ut esset $db = Rds$ seu $b = \int Rds$ evanescente b facto s = 0, tum aequatio pro curva quaesita erit

$$Rds = e^{\pm \int \frac{ds}{q}} Pdx,$$

quae semper est pro curva reali, dummodo $\int Rds$ habeat valorem affirmativum prodeatque $ds > dx$ seu

$$e^{\pm \int \frac{ds}{q}} P > R.$$

Exemplum 1.

716. Sit potentia sollicitans uniformis seu $P = g$ et medium resistents uniforme requiraturque curva MA hanc habens proprietatem, ut corpus in singulis descensibus ad A usque acquirat celeritates, quae sint in subduplicata ratione arcuum descensu percursorum. Erit ergo $\sqrt{b}$ ut $\sqrt{s}$ seu $b = \alpha s$; unde fit [p. 390]

$$\alpha ds = g e^{\frac{-s}{k}} dx \text{ seu } \alpha e^{\frac{s}{k}} ds = gdx,$$

cuius integralis est

$$\alpha k(e^{\frac{s}{k}} - 1) = gx$$

addita constante, quo fiat $x = 0$ evanescente s. Habebitur ergo

$$e^{\frac{s}{k}} = \frac{\alpha k + gx}{\alpha k} \text{ atque } \frac{s}{k} = l(\alpha k + gx) - l\alpha k.$$

Quae differentia dat

$$\frac{ds}{k} = \frac{gdx}{\alpha k + gx};$$

ex quo intelligitur curvam esse tractoriam filo longitudinis k super basi horizontali a puncto A deorsum distante intervallo $\frac{\alpha k}{g}$.

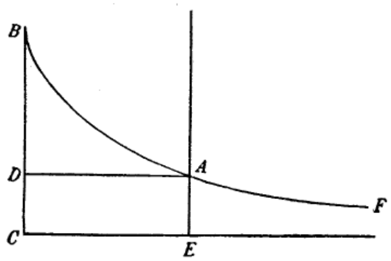

Fig. 81.

Construetur ergo curva hoc modo : super basi horizontali CE (Fig. 81) et filo $BC = k$ describatur tractoria BA; tum ducatur horizontalis DA a CE ad distantium $DC = \frac{\alpha k}{g}$; quo facto satisfaciet curvae portio BA quaesito. Ponimus autem BC verticalem et B punctum tractoriae summum; ex quo intelligitur α necessario minus esse debere quam g. Si enim esset maior, foret $CD > CB$ idoeque punctum A imaginarium. Sin autem esset $\alpha = g$, punctum A in B caderet adeoque nonnisi punctum satisfaceret. Si fuerit $\alpha = 0$, punctum A infinite distaret et corpus descendens omnem amitteret celeritatem. Cum igitur debeat esse $\alpha < g$, erit $b < gs$.

Exemplum 2.

717. In superiori tam resistentiae quam potentiae sollicitatis hypothesi quaeretur curva AMF (Fig. 82), super qua omnes ascensus ex puncto A facti ita se habeant, ut toti arcus ascensibus singulis absoluti sint quadratis celeritatum initialium in A proportionales. [p. 391] Erit ergo ut ante $b = \alpha s$ atque $db = \alpha ds$. Cum autem pro ascensibus sit $db = ge^{\frac{s}{k}}dx$, erit

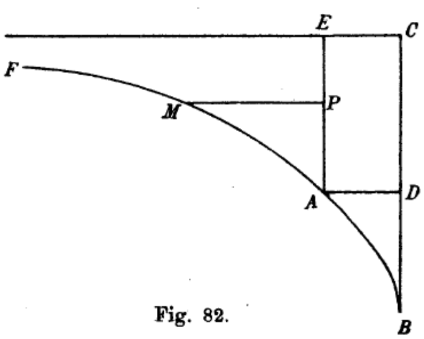

Fig. 82.

$$\alpha e^{\frac{-s}{k}} ds = gdx$$

atque integrando

$$\alpha k(1 - e^{\frac{-s}{k}}) = gx.$$

Hinc igitur habetur

$$e^{\frac{-s}{k}} = \frac{\alpha k - gx}{\alpha k}$$

atque

$$\frac{ds}{k} = \frac{gdx}{\alpha k - gx} \text{ seu } (\frac{\alpha k}{g} - x)\frac{ds}{dx} = k.$$

Ex quo apparet curvam satisfacientem esse iterum tractoriam super basi horizontali CE filo longitudinis k constructam, sed deorsum spectantem, cuius cuspis sit in B existente

$BC = k$. Sumatur autem $CD = \frac{\alpha k}{g}$; ductaque horizontali DA erit A punctum, in quo ascensus omnes incipere debent. Hinc ergo quoque intelligetur α non posse esse maius quam g, quia alias punctum A foret imaginarium. At si fuerit $\alpha = g$ seu $b = gs$, incidet A in B eritque arcus quolibet ascensu percursus = $\frac{b}{g}$.

Scholion 3.

718. Plura huiusmodi exempla, quia tam facile ex universali formula inventa resolvi possunt, hic praetermitto; neque etiam huiusmodi quaestiones pro aliis resistentiae hypothesibus, quibus solutio earum inveniri queat, affero, quoniam tales quaestiones neque iam sunt agitatae neque satis sunt curiosae, ut earum solutiones requirantur. Ad digniora igitur progredior problema, in quibus curvae quaeruntur tautochronae, super quibus omnes vel ascensus vel descensus aequalibus absolvantur temporibus.

PROPOSITIO 81. [p. 392]

Problema.

719. *In hypothesi potentiae uniformis deorsum directae et medio uniformi, quod resistit in duplicata ratione celeritatum, invenire curvam tautochronam AM* (Fig.80)*, super qua omnes descensus ad punctum A usque absolvantur aequalibus temporibus.*

Solutio.

Consideretur quicunque descensus, in quo celeritas, quam corpus in puncto infimo A acquirit, debita sit altitudini b. Ponatur $AP = x$, $AM = s$, altitudo celeritati in M debita = v atque potentia sollicitans = g et medii exponens = k, ita ut resistentia in M sit ad vim gravitatis ut $\frac{v}{k}$ ad 1. His positis erit

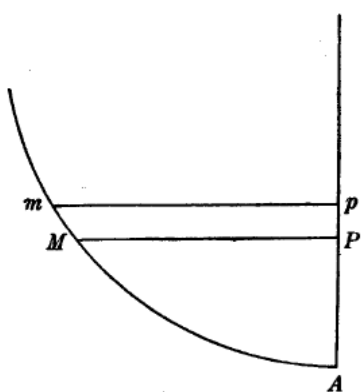

Fig. 80.

$$dv = -gdx + \frac{vds}{k},$$

quae aequatio integrata dat

$$v = e^{\frac{s}{k}}\left(b - \int e^{\frac{-s}{k}} gdx\right)$$

integrali $\int e^{\frac{-s}{k}} gdx$ ita sumto, ut evanescat posito x vel $s = 0$. Ex hac ergo aequatione initium descensus invenitur ponendo v = 0 seu $\int e^{\frac{-s}{k}} gdx = b.$

Tempus vero, quo arcus MA absolvitur, hinc erit =

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b - \int e^{\frac{-s}{k}} gdx\right)}},$$

ex quo prodibit totius descensus tempus, si post integrationem fiat $\int e^{\frac{-s}{k}} gdx = b.$

Ponamus brevitas gratia [p. 393]

$$\int e^{\frac{-s}{k}} g dx = t \text{ et } \frac{ds}{e^{\frac{s}{2k}}} = du,$$

ita ut sit tempus totius descensus $= \int \frac{du}{\sqrt{(b\text{-}t)}}$ posito post integrationem $t = b$. Quo nunc haec expressio perpetuo eundem obtineat valorem, debebit $\int \frac{du}{\sqrt{(b\text{-}t)}}$ esse functio nullius dimensionis ipsarum b et t, ut posito $t = b$ ex formula b evanescat. Hanc ob rem du debet esse function dimediae dimensionis ipsius t tantum, quia u a b pendere non potest. Fieri ergo necesse est $du = \frac{\alpha dt}{\sqrt{t}}$ existente α quanititate constante b non continente. Hoc posito erit tempus unius descensus $= \alpha \int \frac{dt}{\sqrt{(bt\text{-}tt)}}$ post integrationem $t = b$. Vel posita ratione diametri ad peripheriam $1 : \pi$ erit tempus unius descensus $= \alpha\pi$; qui valor perpetuo manet, quodmodocunque b seu descensus initium mutetur. Curva ergo tautochrona quaesita determinabitur ex hac aequatione $du = \frac{\alpha dt}{\sqrt{t}} = \frac{ds}{e^{\frac{s}{2k}}}$,

cuius integralis est

$$2\alpha \sqrt{t} = 2k\left(1 - e^{\frac{-s}{2k}}\right) \quad \text{seu} \quad t = \frac{k^2}{\alpha^2}\left(1 - e^{\frac{-s}{2k}}\right)^2$$

addita scilicet constante, quae faciat t evanescere posito $s = 0$. Cum autem sit

$$t = \int e^{\frac{-s}{k}} g dx,$$

erit

$$dt = e^{\frac{-s}{k}} g dx = \frac{k}{\alpha^2}\left(1 - e^{\frac{-s}{2k}}\right) e^{\frac{-s}{2k}} ds.$$

Ponamus $\alpha^2 g = a$ seu $\alpha = \sqrt{\frac{a}{g}}$; erit

$$a dx = k\left(e^{\frac{s}{2k}} - 1\right) ds,$$

cuius integralis est

$$ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks;$$

[p. 394] quae quidem aequationes, quia variabiles s et x a se invicem sunt separatae, ad curvam construendam sufficiunt. Sin autem aequatio ab exponentialibus libera desideratur, quia ex altera aequatione est

$$k\left(e^{\frac{s}{2k}} - 1\right) = \frac{ax + ks}{2k},$$

erit hoc valore in altera substituto $axds + ksds = 2akdx$. Q.E.I.

Corollarium 1.

720. Quia est $\alpha = \sqrt{\frac{a}{g}}$, erit tempus unius descensus = $\pi\sqrt{\frac{a}{g}}$. In vacuo autem et gravitate = 1 est tempus descensus penduli $f = \frac{\pi\sqrt{2f}}{2}$ (166). Quare longitudo penduli isochroni in vacuo est = $\frac{2a}{g}$.

Corollarium 2.

721. Si igitur fuerit $\frac{2a}{g}$ = 3166 part. millesimarum pedis Rhenani (170), descensus absolvetur dimedio minuto secundo; hoc ergo evenit, si sit a = 1583g scrup. pedis Rhen.

Corollarium 3.

722. Altitudo celeritati in M debita seu v est =

$$e^{\frac{s}{k}}\left(b - \int e^{\frac{-s}{k}} g\,dx\right) = e^{\frac{s}{k}}(b - t)$$

atque ob

$$t = \frac{gk^2\left(1 - e^{\frac{-s}{2k}}\right)^2}{a}$$

erit

$$v = e^{\frac{s}{k}}\left(b - \frac{gk^2}{a}\left(1 - e^{\frac{-s}{2k}}\right)^2\right) = \frac{abe^{\frac{s}{k}} - gk^2\left(e^{\frac{s}{2k}} - 1\right)^2}{a}.$$

Corollarium 4. [p. 395]

723. Posito $v = 0$ prodibit totus arcus descensus ex hac aequatione

$$ab = gk^2\left(1 - e^{\frac{-s}{2k}}\right)^2.$$

Si ergo arcus descensus ponatur = f, erit

$$ab = gk^2\left(1 - e^{\frac{-f}{2k}}\right)^2.$$

Quare dato arcu descensus f erit

$$v = \frac{gk^2 e^{\frac{s}{k}}}{a}\left(\left(1 - e^{\frac{-f}{2k}}\right)^2 - \left(1 - e^{\frac{-s}{2k}}\right)^2\right)$$

Corollarium 5.

724. Aequatio pro curva haec

$$ax = 2k^2\left(e^{\frac{s}{2k}} - 1\right) - ks$$

in seriem exponentiali $e^{\frac{s}{2k}}$ convertendo, quae est

$$1 + \frac{s}{2k} + \frac{s^2}{1\cdot 2\cdot 4k^2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 8k^3} + \text{etc.},$$

abit in hanc

$$ax = \frac{s^2}{1\cdot 2\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 4k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 8k^2} + \text{etc.}$$

seu

$$2ax = \frac{s^2}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3\cdot 2k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^2} + \text{etc.}$$

Scholion 1.

725. Notari hic convenit hanc curvam simili aequatione exprimi, qua supra brachystochrona ascensui inserviens exprimebatur; ibi enim erat

$$at = \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3k} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4k^2} + \text{etc.}$$

(687), quae aequatio ab hac nostra pro tautochrona inventa in hoc tantum differt, quod hic sit $2a$, quod ibi erat a, atque exponens resistentiae brachystochronae duplo maior est exponente resistentiae pro tautochrona. Curva ergo brachystochrona quoque ad tautochronismum producendum accomodari potest arcu ascensus descensui tributo in medio resistente, cuius exponens est duplo minor. [p. 396]

Corollarium 6.

726. Ad inveniendam continuationem curvae *MA* ultra *A* poni debet *s* negativum, quo facto habitur

$$ax = 2k^2\left(e^{\frac{-s}{2k}} - 1\right) + ks$$

vel

$$2ax = \frac{s^2}{1\cdot 2} - \frac{s^3}{1\cdot 2\cdot 3\cdot 2k} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 4k^2} - \text{etc.}$$

Quae eadem aequatio prodisset, si *k* negativum fecissemus. Facto autem *k* negativo descensus mutatur in ascensum; quocirca curva *MA* ultra *A* continuata ascensui inserviet atque super ea omnes ascensus eodem absolventur tempore, scilicet $\pi\sqrt{\frac{a}{g}}$.

Corollarium 7.

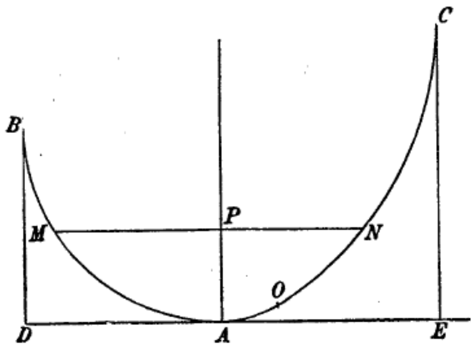

Fig. 83.

727. Eadem ergo curva continua BMANC (Fig. 83) erit tautochrona tam pro descensu quam pro ascensu. Namque super arcu BMA omnes descensus eodem tempore absolvuntur atque super arcu ANC omnes ascensus. Quare omnes dimidiae oscillationes. quae in arcu BMA incipiunt, erunt inter se isochronae atque tempus unius semioscillatonis erit = $2\pi\sqrt{\frac{a}{g}}$.

Corollarium 8. [p. 397]

728. Si resistentia evanescit, quo casu k fit $\propto$,curva haec in cycloidem abire debet, quae est curva tautochrona in vacuo. Hoc ipsum aequatio per seriem expressa indicat; fit enim $2ax = \frac{s^2}{1.2}$ seu $4ax = s^2$, aequation pro cycloide.

Corollarium 9.

729. Curva ergo *BMANC* prout cyclois habebit cuspides verticales in *B* et *C*, ad quas inveniendas ponatur $dx = ds$, eritque pro arcu *BMA*

$$a = k\left(e^{\frac{s}{2k}} - 1\right) \quad \text{seu} \quad s = 2kl\frac{a+k}{k} = AMB;$$

atque eius altitudo *BD* erit =

$$2k - \frac{2k^2}{a}l\frac{a+k}{k}.$$

Pro arcu ascensus vero *ANC* erit

$$ANC = 2kl\frac{k}{k-a} \quad \text{et} \quad CE = \frac{2k^2}{a}l\frac{k}{k-a} - 2k.$$

Sive per series erit

$$BD = a - \frac{2a^2}{3k} + \frac{2a^3}{4k^2} - \frac{2a^4}{5k^3} + \text{etc.}$$

atque

$$CE = a + \frac{2a^2}{3k} + \frac{2a^3}{4k^2} + \frac{2a^4}{5k^3} + \text{etc.}$$

Corollarium 10.

730. Ex his perspicitur cuspidem *C* arcus ascensus elevatiorem esse cuspide *A* arcus descensus. Atque arcus *ANC* cuspis in infinitum abit, si $k = a$; et si $a > k$, cuspis *C* erit imaginaria. Ceterum ex aequatione patet tam *BD* quam *CE* esse diametros curvae inventae.

Corollarium 11. [p. 398]

731. Si corpus in dimidia oscillatione habuerit celeritatem altitudini *b* debitam, erit arcus descensus =

$$2kl\frac{k\sqrt{g}}{k\sqrt{g}-\sqrt{ab}}$$

(723) seu per seriem =

$$\frac{2\sqrt{ab}}{\sqrt{g}}+\frac{2ab}{2gk}+\frac{2ab\sqrt{ab}}{3gk^2\sqrt{g}}+\text{etc.}$$

atque sequens arcus ascensus =

$$2kl\frac{k\sqrt{g}+\sqrt{ab}}{k\sqrt{g}}=\frac{2\sqrt{ab}}{2g}-\frac{2ab}{2gk}+\frac{2ab\sqrt{ab}}{3gk^2\sqrt{g}}-\text{etc.}$$

Corollarium 12.

732. Si ergo descensus fiat ex puncto B, ita ut arcus descensus sit =

$$\text{AMB} = 2kl\frac{a+k}{k},$$

erit

$$b=\frac{gak^2}{(a+k)^2};$$

sequentis vero ascensus arcus erit = $2kl\frac{a+k}{k}$.

Corollarium 13.

733. Ex aequatione pro hac curva apparet curvam in puncto *A* habituram esse tangentem horizontalem. Cum porro posito ds constante radius osculi in *M* sit =

$$\frac{dsdy}{ddx}=\frac{ds\sqrt{(ds^2-dx^2)}}{ddx},$$

quia est (719)

$$dx=\frac{k}{a}\left(e^{\frac{s}{2k}}-1\right)ds,$$

erit

$$ddx=\frac{1}{2a}e^{\frac{s}{2k}}ds^2$$

et

$$\sqrt{(ds^2 - dx^2)} = ds\sqrt{\left(1 - \frac{k^2}{a^2}\left(e^{\frac{s}{2k}} - 1\right)^2\right)};$$

erit radius osculi in *M* =

$$\frac{\sqrt{\left(4a^2 - 4k^2\left(e^{\frac{s}{2k}} - 1\right)^2\right)}}{e^{\frac{s}{2k}}}.$$

Posito ergo *s* = 0 erit radius osculi in puncto infimo *A* = 2*a*. In *B* vero et *C* radius osculi evanescit.

Corollarium 14. [p. 399]

734. Radius osculi non est maximus in puncto infimo *A*; sed per methodum maximorum maximus invenitur in arcu ascensus idque in puncto *O* existente

$$AO = 2kl\frac{k^2}{k^2 - a^2}.$$

In hoc enim puncto est radius osculi = $\frac{2ak}{\sqrt{(k^2-a^2)}}$. Ex quo concluditur, nisi sit $k > a$, curvaturam curvae *ANC* perpetuo diminui neque punctum *O* usquam existere.

Scholion 2.

735. Hoc igitur problemate duas invenimus curvas, super quarum altera omnes descensus, super altera vero omnes ascensus aequalibus absolvuntur temporibus. Atque cum super tota curva *BAC* omnes itus seu semioscillationes aequalibus peragantur temporibus, si quidem in curvae portione *BA* incipiant, haec curva ad oscillationes in fluido isochronas faciendas esset idonea, si modo reditus quoque inter se essent isochroni, de quo vero non constat. Quia autem in fluidis praeter resistentiam quadratis celeritatum proportionalem seu constantem alia insuper observatur, quam momentis temporum proportionalem seu constantem esse probabile est, etiam coniunctim cum ista resistentia tautochronam determinare operae pretium est; quod vero facile ex praecedente effici potest. Sit enim resistentiae constans = *h*; erit pro descensu

$$dv = -gdx + hds + \frac{vds}{k}.$$

Quare si in priore operatione tantum loco *gdx* ubique *gdx – hds* substituatur, tautochrona satisfaciens simil modo obtinebitur; pro descensu scilicet prodibit ista curva [p. 400]

$$ax = \left(\frac{ah}{g} - k\right)s + 2k^2\left(e^{\frac{s}{2k}} - 1\right)$$

atque pro ascensu vero haec

$$ax = \left(k - \frac{ah}{g}\right)s + 2k^2\left(e^{\frac{-s}{2k}} - 1\right);$$

quae curva quoque cum priore eandem curvam continuam constituat; abit enim altera in alteram ponendo s negativum. Notandum hic est si fuerit $k = \frac{ah}{g}$, fore curvam tautochronam tractoriam BAF (Fig. 82) asymtotam habentem horizontalem CE, quae a puncto A distet intervallo $AE = \frac{2k^2}{a}$. Fili autem longitudo, quo haec tractoria describatur, est = $2k$.

Quod autem super huiusmodi curva tangentem horizontalem nusquam habente semioscillatione absolvi possit atque alicubi punctum aequilibrii A existere, mirum non est, cum, ut supra iam observavimus, in tali resistentiae hypothesi corpus in loco subsistere queat declivi. His autem casibus, quibus curva ultra A descendere pergit, nulli reditus atque ideo nullae oscillationes peragi possunt, quia corpus quanquam superl plano declivi ad quietem pervenire potest, tamen super eo ascendere nequit; in quovis enim curvae portionis AF puncto corpus in quiete perseverare potest.

Scholion 3.

736. Non multo difficilior fit problematis solutio, si potentia deorsum tendens non constans, sed variabilis utcunque P atque exponens resistentiae etiam variabilis q ponatur. Habebitur enim pro elemento temporis in descensu

$$\frac{ds}{e^{\frac{1}{2}\int\frac{ds}{q}}\sqrt{\left(b - \int e^{-\int\frac{ds}{q}}Pdx\right)}}.$$

Si nunc ut ante (719) ponatur [p. 401]

$$\int e^{-\int\frac{ds}{q}}Pdx = t \quad \text{et} \quad \frac{ds}{e^{\frac{1}{2}\int\frac{ds}{q}}} = du,$$

debebit quoque esse

$$du = \frac{\alpha dt}{\sqrt{t}} \quad \text{seu} \quad t = \frac{\alpha^2 dt^2}{du^2} = \frac{\alpha^2 P^2 dx^2}{e^{\int\frac{ds}{q}}ds^2}.$$

Quae aequatio denuo differentiata posito ds constante et loco dt eius valore substuto dabit

$$\frac{qds^2}{2\alpha^2} = Pqddx + qdPdx - \frac{1}{2}Pdxds$$

pro curva descensus isochronos habente. Huiusque curvae continua ultra A ascensibus inserviet.

Scholion 4.

737. Tautochronam hanc in hypothesi resistentiae quadratis celeritatum proportioalis primus ego dedi in Comment. Tomo IV [E013], ubi eadem qua hic sum usus methodo. Deinceps vero etiam Cel. Ioh. Bernoulli mihi per letteras significavit se quoque in eadem resistentiae hypothesii istam tautochronam repperisse [Vide letteras ab Ioh. Bernoulli 27 Dec. 1729 ad Leonhardum Eulerum datas, quas G. Enestroem edidit Biblioth. Mathem. 4, 1903, p.378. P. St.]; cuius methodus extat in Comm. Acad.Paris. A 1730 [Ioh. Bernoulli, *Méthode pour trouver les tautochrones, dans les milieaux résistans comme le quarré des vitesses*, Mém. de l'acad. d. sc. de Paris 1730, p. 78; Opera Omnia, Tom. 3, Lausannae et Genevae 1742, p. 173.]. In aliis vero resistentiae hypothesibus exceptaq ea, quae ipsis celeritatibus est proportionalis, nemo adhuc, quantum mihi constat, tautochronas determinavit. Quod enim ad eas curvas attinet, quas in Act. Lips. A. 1726 tautochronarum nomine dedi [E001], eae quaesito non satisfaciunt, uti Cel. Hermannus, qui primum in easdem inciderat, atque ego postea monstravimus. [Iac. Hermann, Theoria generalis motuum, qui nascuntur a potentiis quibusvis in corpora indesinenter agentibus, Comment. acad. sc. Petrop. 2 (1727), p. 139; vide praecipue p. 158.] Difficultas autem methodi huius tautochronas inveniendi in hoc consistit, [p. 402] quod in aliis resistentiae hypothesibus celeritas non possit universaliter ex aequatione canonica determinari. Quomodo vero nihilominus pro aliis resistentiis tautochronae investigare queant, ex sequente propositone colligi poterit; in qua pro mediis rarissimis in quacunque celeritatum rationem multiplicata resistentibus tautochronae inveniendae proponuntur.