CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 349]

PROPOSITION 73.

Problem.

649. *If the force is uniform and acting downwards and the medium resists according to some power of the ratio of the speed, to determine the curve AM* (Fig.73)*, upon which the body by descending progresses along the horizontal AH at a steady rate.*

Solution.

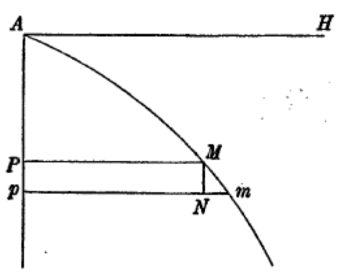


Fig. 73.

Let *A* be the highest point of the curve, through which the vertical axis *AP* is drawn, and the speed by which the body progresses horizontally corresponds to the height *b*. The abscissa $AP = x$ is taken, the applied line $PM = y$ and the arc $AM = s$ and let the speed of the body at *M* correspond to the height *v*, with which speed of the body the element $Mm = ds$ is traversed. Hence then as *ds* is to *dy* thus the speed of the body along *Mm*, which is $\sqrt{v}$, os to the horizontal speed $\sqrt{b}$, hence there arises :

$$v = \frac{bds^2}{dy^2}.$$

Now let the force acting be equal to *g*, the exponent of the resistance equal to *k* and the resistance itself is equal to $\frac{v^m}{k^m}$. With these in place, there arises the equation :

$$dv = gdx - \frac{v^m ds}{k^m},$$

which equation, if the value $\frac{bds^2}{dy^2}$ is put in place of *v*, expresses the nature of the curve sought. Moreover let $ds = pdy$; then [these relations arise]

$$v = bp^2 \text{ and } dx = dy\sqrt{(p^2 - 1)}.$$

On account of which there is obtained :

$$2bpdp = gdy\sqrt{(p^2 - 1)} - \frac{b^m p^{2m+1} dy}{k^m},$$

which separated gives : [p. 350]

$$dy = \frac{2bk^m p dp}{gk^m \sqrt{(p^2-1)} - b^m p^{2m+1}}.$$

Therefore the construction of the curve sought is as follows : on taking

$$y = \int \frac{2bk^m p dp}{gk^m \sqrt{(p^2-1)} - b^m p^{2m+1}}$$

then

$$x = \int \frac{2bk^m p dp \sqrt{(p^2-1)}}{gk^m \sqrt{(p^2-1)} - b^m p^{2m+1}}.$$

Q.E.I.

Corollary 1.

650. If $\frac{v}{b}$ is restored in place of *pp* and *y* and *x* are defined in terms of *v*, then we have

$$y = \int \frac{k^m dv \sqrt{b}}{gk^m \sqrt{(v-b)} - v^m \sqrt{v}}$$

and

$$x = \int \frac{k^m dv \sqrt{(v-b)}}{gk^m \sqrt{(v-b)} - v^m \sqrt{v}}.$$

In a like manner the arc is given by :

$$s = \int \frac{k^m dv \sqrt{v}}{gk^m \sqrt{(v-b)} - v^m \sqrt{v}}.$$

Moreover on taking $\frac{v}{b}$ in place of *pp*, then we have

$$ds = \frac{dy\sqrt{v}}{\sqrt{b}} \text{ and } dx = \frac{dy\sqrt{(v-b)}}{\sqrt{b}} \text{ and } ds = \frac{dx\sqrt{v}}{\sqrt{(v-b)}}.$$

Corollary 2.

651. Because the equation

$$dy = \frac{k^m dv \sqrt{b}}{gk^m \sqrt{(v-b)} - v^m \sqrt{v}}$$

has been separated, from that the solution satisfying a particular condition can be elicited by making the denominator

$$gk^m \sqrt{(v-b)} - v^m \sqrt{v} = 0,$$

thus it is that the speed $\sqrt{v}$ is constant. Hence let $v = c$; then $\sqrt{(v-b)} = \frac{c^m \sqrt{c}}{gk^m}$

and thus

$$ds = \frac{gk^m dx}{c^m}$$

for the inclined straight line, as we have found above(628) [There is a typo' in the *O. O.* here, not present in the original ms.].

Corollary 3. [p. 351]

652. Moreover in order that the body can progress horizontally with a given speed that corresponds to the height *b*, it is possible to define the height *c* from the equation $\sqrt{(c-b)} = \frac{c^m\sqrt{c}}{gk^m}$. With which found, the inclination of the given straight line satisfying the condition is obtained and the speed of the body $\sqrt{c}$ at *A* initially, by which it descends uniformly along the line.

Corollary 4.

653. If the resistance vanishes and the body is moving in a vacuum then $k = \infty$ and thus

$$x = \int \frac{dv}{g} \text{ or } v = g(a+x)$$

and

$$dy = \frac{dx\sqrt{b}}{\sqrt{(ga+gx-b)}}.$$

Therefore on integration the equation becomes

$$y = \frac{2}{g}\sqrt{b(ga+gx-b)} - \frac{2}{g}\sqrt{b(ga-b)},$$

which is the equation for a parabola, as the body projected freely describes.

Example.

654. If the medium should be the most rare and hence *k* very great, then as an approximation :

$$\frac{1}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}} = \frac{1}{gk^m\sqrt{(v-b)}} + \frac{v^m\sqrt{v}}{g^2k^{2m}(v-b)}$$

On this account there is obtained : [p. 352]

$$y = \frac{2\sqrt{b(v-b)}}{g} + \int \frac{v^m dv\sqrt{bv}}{g^2k^m(v-b)} \text{ and } x = \frac{v}{g} + \int \frac{v^m dv\sqrt{v}}{g^2k^m\sqrt{(v-b)}}.$$

From this latter equation there is as an approximation :

$$v = gx - \int \frac{g^m x^m dx \sqrt{gx}}{k^m \sqrt{(gx - b)}},$$

which value substituted in the equation

$$dy = \frac{dx \sqrt{b}}{\sqrt{(v - b)}}$$

gives the equation sought between x and y for the curve.

PROPOSITION 74.

Problem.

655. *To find the curve AM* (Fig.74)*, upon which the body is descending uniformly downwards in a medium with some kind of resistance, with a uniform absolute force directed downwards acting.*

Solution.

With the abscissa $AP = x$ and $AM = s$ let the speed by which the body descends regularly correspond to the height b. Again the uniform force directed downwards is g and the height corresponding to the speed at M is equal to v, and the resistance is equal to $\frac{v^m}{k^m}$; hence the equation arises :

$$dv = gdx - \frac{v^m ds}{k^m}.$$

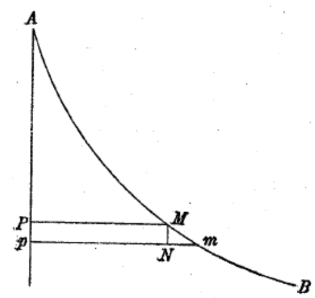


Fig. 74.

Moreover it is the case that $Mm : MN = \sqrt{v} : \sqrt{b}$, thus the equation becomes $v = \frac{bds^2}{dx^2}$. Hence from this equation we have :

$$ds = \frac{dx\sqrt{v}}{\sqrt{b}},$$

with which value substituted in the equation we have :

$$dv = gdx - \frac{v^{m+\frac{1}{2}} dx}{k^m \sqrt{b}} \quad \text{or} \quad dx = \frac{k^m dv \sqrt{b}}{gk^m \sqrt{b} - v^m \sqrt{v}}.$$

On account of which it follows that [p. 353]

$$x = \int \frac{k^m dv \sqrt{b}}{gk^m \sqrt{b} - v^m \sqrt{v}}, \quad s = \int \frac{k^m dv \sqrt{v}}{gk^m \sqrt{b} - v^m \sqrt{v}}$$

and the applied line

$$PM = y = \int \frac{k^m dv \sqrt{(v - b)}}{gk^m \sqrt{b} - v^m \sqrt{v}}.$$

From which equations the construction of the curve sought can be made. Q.E.I.

Corollary 1.

656. From these three equations, if the equation is desired consisting only of *x, y* and *s*, that can be taken, from which the value of *v* can be most conveniently found, and with this subsequently placed in either of the remaining equations.

Corollary 2.

657. Because the equation

$$dx = \frac{k^m dv\sqrt{b}}{gk^m\sqrt{b} - v^m\sqrt{v}}$$

has the indeterminates separated from each other in turn, the particular solution can be obtained depending on the condition :

$$gk^m\sqrt{b} - v^m\sqrt{v} = 0.$$

This is therefore :

$$v = g^{\frac{2}{2m+1}} k^{\frac{2m}{2m+1}} b^{\frac{1}{2m+1}}$$

and thus

$$ds = \frac{g^{\frac{1}{2m+1}} k^{\frac{m}{2m+1}} dx}{b^{\frac{m}{2m+1}}}.$$

Hence the equation is satisfied by the inclined straight line, if the body is moving upon that with the given speed $\sqrt{v}$.

Scholium 1. [p. 354]

658. Because in the preceding and in this problem too an inclined straight line presents a particular solution, from that it can be understood that in a resisting medium an inclined straight line may be found upon which the body moves uniformly, as we have shown above (628). Moreover here each case of the problem is satisfied; if indeed the body advances upon the straight line with a uniform motion, then it is moving horizontally as well as vertically with a uniform motion ; then why not also be carried equally along any direction.

Corollary 3.

659. For the vacuum let $k = \infty$. On account of which then we have $x = \frac{v}{g}$ or $v = gx$ and

$$ds = \frac{dx\sqrt{gx}}{\sqrt{b}} \text{ and } dy = \frac{dx\sqrt{(gx-b)}}{\sqrt{b}},$$

which equation integrated gives :

$$y = \frac{2(gx-b)^{\frac{3}{2}}}{3g\sqrt{b}}$$

and presents the rectifiable cubic parabola, as we have thus found (258).

Example 1.

660. We put the resistance proportional to the speeds, then we have $m = \frac{1}{2}$ and thus :

$$x = \int \frac{dv\sqrt{bk}}{g\sqrt{bk} - v} = \sqrt{bk}\, l \frac{g\sqrt{bk}}{g\sqrt{bk} - v},$$

if the start of the abscissas is taken in that point where the integral vanishes. Moreover from this equation there is produced : [p. 355]

$$e^{\frac{x}{\sqrt{bk}}} = \frac{g\sqrt{bk}}{g\sqrt{bk} - v} \quad \text{or} \quad v = e^{\frac{-x}{\sqrt{bk}}}(e^{\frac{x}{\sqrt{bk}}} - 1)g\sqrt{bk} = g\sqrt{bk}(1 - e^{\frac{-x}{\sqrt{bk}}}).$$

With this value of v substituted, there is obtained :

$$ds = \frac{dx\sqrt{g(1 - e^{\frac{-x}{\sqrt{bk}}})\sqrt{bk}}}{\sqrt{b}}.$$

Or since

$$ds = \frac{dv\sqrt{kv}}{g\sqrt{bk} - v},$$

put $v = u^2$; then

$$ds = \frac{2u^2 du\sqrt{k}}{g\sqrt{bk} - uu} = \frac{2gk\, du\sqrt{b}}{g\sqrt{bk} - uu} - 2du\sqrt{k},$$

which integrated gives:

$$s = \sqrt[4]{g^2bk^3}\, l \frac{\sqrt[4]{g^2bk} + \sqrt{v}}{\sqrt[4]{g^2bk} - \sqrt{v}} - 2\sqrt{kv},$$

in which the value of v found before can be substituted, in order that the equation is produced between x and s.

Example 2.

661. Now the medium resists in the ratio of the square of the speeds; then $m = 1$ and hence

$$dx = \frac{kdv\sqrt{b}}{gk\sqrt{b} - v\sqrt{v}} \text{ and } ds = \frac{kdv\sqrt{v}}{gk\sqrt{b} - v\sqrt{v}}.$$

The integral of this latter equation is :

$$s = \frac{2k}{3} l \frac{gk\sqrt{b}}{gk\sqrt{b} - v\sqrt{v}},$$

from which there arises :

$$e^{\frac{3s}{2k}} = \frac{gk\sqrt{b}}{gk\sqrt{b} - v\sqrt{v}}$$

and

$$v^{\frac{3}{2}} = gk\left(e^{\frac{3s}{2k}} - 1\right) e^{\frac{-3s}{2k}} \sqrt{b} = gk\left(1 - e^{\frac{-3s}{2k}}\right)\sqrt{b}.$$

On account of which it becomes :

$$\sqrt{v} = \sqrt[3]{gk\left(1 - e^{\frac{-3s}{2k}}\right)\sqrt{b}};$$

which value substituted in the equation $dx = \frac{ds\sqrt{b}}{\sqrt{v}}$ gives the equation between s and x sought for the curve.

Scholium 2.

662. As we have determined the curves in these two problems, upon which the moving body is carried uniformly either along the horizontal or down along the vertical, thus in a like manner it is possible to solve the problem, [p. 356] if the body should be progressing uniformly along any other direction; but that question, since nothing very pleasing is deduced from the solution, this I have set aside, and for the same reason I do not touch on isochronous problems about a centre for a resisting medium. Now I will apply myself to these problems, in which a certain law of the speeds is proposed, a problem not a little curious for resisting mediums, that has not been treated hitherto by anyone; which is not indeed a problem for the vacuum. Clearly the curve is sought, upon which the body reaches a given point with a maximum speed ; for in the vacuum the body moving on some given curve always obtains the same speed at the same place. [An implicit statement of the law of conservation of mechanical energy when there is no friction.]

PROPOSITION 75.

Problem.

663. *Between all the curves that join the points A and C* (Fig.75), *to determine that curve AMC, upon which the body descending from A to C acquires the maximum speed with the resistance present as some power of the ratio of the speeds, and with the uniform force acting downwards.*

Solution.

In order that the body can arrive at the point *C* with the maximum speed, each two elements of the curve sought *AMC* $Mm, m\mu$ thus have to be put in place, so that the body by running through these can take the maximum increment of the speed [p. 357]. For if the body should acquire a greater increase in the speed by passing through the elements $Mn, n\mu$ in other ways, it will also have a greater speed at *C*. Therefore by the method of the maxima the position of the elements $Mn, n\mu$ can be found, if these elements $Mn, n\mu$ are compared with their neighbouring elements and the increases of the speed, which is generated in each are put equal to each other.

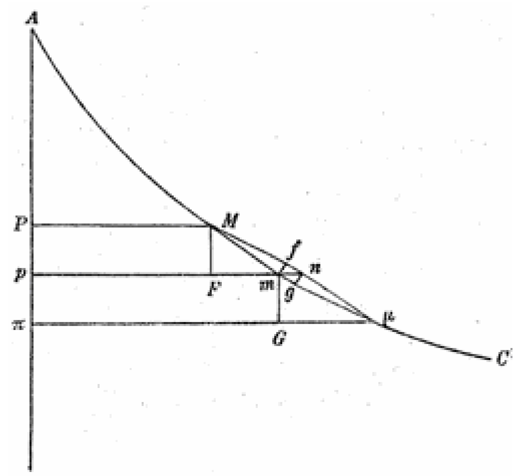


Fig. 75.

The applied lines MP, nmp and $\mu\pi$ are drawn in accordance to the vertical axis, and let the elements $Pp,$ and $p\pi$ be equal. Also the vertical lines *MF* and *mG* are drawn and on the curves the normal elements *mf* and *ng*. Now let the force acting be equal to *g*, the exponent of the resistance be equal to *k*, with the resistance in the ratio of the $2m^{\text{th}}$ power of the speeds, and the height corresponding to the speed at *M* be set equal to *v*. With these in place, the increment of *v* along *Mm* is equal to

$$g.MF - \frac{v^m.Mm}{k^m}$$

and the increment of the height corresponding to the speed, while the body advances along $m\mu$, is equal to

$$g.MG - \frac{(v+g.MF-\frac{v^m.Mm}{k^m})^m m\mu}{k^m}$$

Therefore while the body performs the elements Mm and $m\mu$, the height *v* takes an increase equal to :

$$g(MF+mG)-\frac{v^m\cdot Mm}{k^m}-\frac{\left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m m\mu}{k^m}.$$

But the elements Mn and $n\mu$ on being traversed take an increment in v equal to :

$$g(MF+mG)-\frac{v^m\cdot Mn}{k^m}-\frac{\left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m n\mu}{k^m}.$$

From which equations put in place there is obtained : [p. 358]

$$0=v^m(Mn-Mm)+\left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m n\mu-\left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m m\mu.$$

Now we have $Mn-Mm=nf$ and

$$\left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m=v^m+m\cdot gv^{m-1}\cdot MF-\frac{mv^{2m-1}}{k^m}Mn$$

and

$$\left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m=v^m+m\cdot gv^{m-1}\cdot MF-\frac{mv^{2m-1}}{k^m}Mm.$$

Now on substituting these values, this equation arises :

$$v(nf-mg)-m\cdot g.MF.mg-\frac{mv^m}{k^m}(Mn\cdot n\mu-Mm\cdot m\mu)=0.$$

But [since $Mn = Mm + mg$, and $m\mu = mg + g\mu$,]

$$Mn\cdot n\mu-Mm\cdot m\mu=n\mu\cdot nf-Mm\cdot mg$$

and on account of the similar triangles *nfm*, *mFM* and *mgn*, μGm

$$nf=\frac{mF\cdot mn}{Mm}\ \text{ and }\ mg=\frac{\mu G\cdot mn}{m\mu}.$$

With these substituted and divided by *mn* there is produced :

$$v\left(\frac{mF}{Mm}-\frac{\mu G}{m\mu}\right)-m\cdot g\cdot\frac{MF\cdot\mu G}{m\mu}-\frac{mv^m}{k^m}\left(\frac{n\mu\cdot mF}{Mm}-\frac{Mm\cdot\mu G}{m\mu}\right)=0.$$

The first two members of this equation are differential of the first order, now the third is equivalent to a differential of the second order, that can be rejected, and hence the equation becomes :

$$\frac{m\cdot g\cdot MF\cdot \mu G}{m\mu}+v\left(\frac{\mu G}{m\mu}-\frac{mF}{Mm}\right)=0$$

or

$$\frac{m\cdot g\cdot MF\cdot mF}{Mm}+vd.\frac{mF}{Mm}=0.$$

From which equation the position of the elements *Mm* and $m\mu$ are determined. [p. 359] Moreover in order that we may employ symbols, let *AP* = *x*, *PM* = *y* and *AM* = *s*; then $Pp = p\pi = dx$, $mF = dy$ and $Mm = ds$ and the equation is produced :

$$\frac{m\cdot gdxdy}{ds}+vd.\frac{dy}{ds}=0.$$

Now the canonical equation is :

$$dv=gdx-\frac{v^m ds}{k^m},$$

in which if in place of *gdx* there is substituted from the above equation :

$$\frac{-vds}{mdy}d.\frac{dy}{ds},$$

then there is obtained :

$$dv+\frac{vds}{mdy}d.\frac{dy}{ds}+\frac{v^m ds}{k^m}=0$$

or

$$\frac{mdvdy}{ds}+vd.\frac{dy}{ds}+\frac{mv^m dy}{k^m}=0.$$

Let $dy = pds$ and $v^{1-m}=u;$ and it follows that,

$$pdu+\frac{(1-m)}{m}udp+\frac{(1-m)pds}{k^m}=0,$$

from which by integration there is produced :

$$u=\frac{(m-1)p^{\frac{m-1}{m}}}{k^m}\int p^{\frac{1-m}{m}}ds.$$

From this *u* can be obtained, and in turn $v=u^{\frac{1}{1-m}}$, which value substituted in the above equation gives the equation

$$mgpds\sqrt{(1-pp)}+vdp=0$$

between *p* and *s* and consequently between *y* and *s*.

Moreover it is expedient that the computation be established in this manner towards the construction of the curve. On placing *dy* = *pds* these two equations are obtained :

$$mgpds\sqrt{(1-pp)}+vdp=0$$

and

$$dv=gds\sqrt{(1-pp)}-\frac{v^m ds}{k^m}.$$

From the first equation it follows that

$$ds=\frac{-vdp}{mgp\sqrt{(1-p^2)}},$$

which value substituted in the second equation gives :

$$mpdv+vdp=\frac{v^{m+1}dp}{gk^m\sqrt{(1-pp)}}.$$

This equation divided by $v^{m+1}p^2$ is made integrable and the integral is : [p. 360]

$$\frac{1}{v^m p}=C+\frac{\sqrt{(1-pp)}}{gk^m p}$$

or

$$v^m=\frac{gk^m}{\alpha p+\sqrt{(1-pp)}} \text{ and } v=\frac{k\sqrt[m]{g}}{\sqrt[m]{(\alpha p+\sqrt{(1-pp)})}}.$$

On account of which

$$mgds=\frac{-kdp\sqrt[m]{g}}{p(1-pp)^{\frac{1}{2}}\sqrt[m]{(\alpha p+\sqrt{(1-pp)})}}$$

and

$$mgdx=\frac{-kdp\sqrt[m]{g}}{p\sqrt[m]{(\alpha p+\sqrt{(1-pp)})}}$$

and

$$mgdy=\frac{-kdp\sqrt[m]{g}}{(1-pp)^{\frac{1}{2}}\sqrt[m]{(\alpha p+\sqrt{(1-pp)})}}.$$

From which equations the curve sought can be constructed easily. Q.E.I.

Corollary 1.

664. If the radius of osculation of the curve at M directed towards the axis is called r, then [see diagram here : note that this diagram is slightly inaccurate, as the angle increment should be negative, as the body moves down the slope and not up as assumed. Note also that Euler's geometric derivatives are the geometrical ratios corresponding to the expansion $(1+d)^2 y = 1 + 2dy/dx + d^2y/dx^2$] :

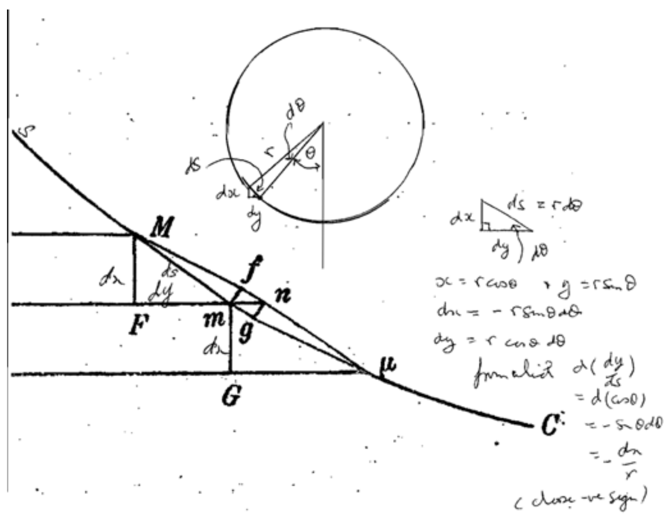
$$d.\frac{dy}{ds} = -\frac{dx}{r}.$$

And with this value substituted into

$$\frac{m \cdot g\,dx\,dy}{ds} + v\,d.\frac{dy}{ds} = 0.$$,

there is obtained:

$$\frac{m.g.dy}{ds} = \frac{v}{r} \text{ or } \frac{2m.g.dy}{ds} = \frac{2v}{r}$$



Now $\frac{2v}{r}$ is the centrifugal force of the body on the curve in this motion, the direction of which is away from the axis, and $\frac{gdy}{ds}$ is the normal force. Whereby in the curve sought the centrifugal force is acting in the opposite direction to the normal force and is in the ratio to the normal force as $2m$ to 1, that is as the exponent of the power of the speed to which the resistance is proportional to unity.

Corollary 2. [p. 361]

665. Therefore all these curves with a concave part are directed downwards. For since the direction of the normal force and of the radius of osculation are considered to be tending towards the same place, the concavity of the curve must also be considered to be downwards.

Corollary 3.

666. In a medium with resistance in the simple ratio of the speeds we have $2m = 1$. Therefore in this case the centrifugal force is equal and opposite to the normal force. On account of which the curve sought satisfying the trajectory is that described by the body projected freely.

Corollary4.

667. Because in the equation

$$mgds = \frac{-kdp\sqrt[m]{g}}{p(1-pp)^{\frac{1}{2}}\sqrt[m]{(\alpha p+\sqrt{(1-pp)})}}$$

the indeterminates have been separated, three particular solutions can be obtained thus. The first gives the equation $\alpha p+\sqrt{(1-pp)}=0$, in which case the speed becomes infinite and the equation is satisfied by some line. In the second case $p = 1$, or $dy = ds$, which is for a horizontal straight line, and in the third case $p = 0$, for a vertical straight line; which has this property, that the body descending along it always accrues the maximum increase in the speed.

Example 1. [p. 362]

668. The medium resists in the simple ratio of the speeds ; then it follows that $m=\frac{1}{2}$. That is taken from the three equations found, which contains dy; it becomes

$$dy = \frac{-2gkdp}{(\alpha p+\sqrt{(1-pp)})^2\sqrt{(1-pp)}},$$

the integral of which is

$$y = C - \frac{2gkp}{\alpha p+\sqrt{(1-pp)}}.$$

But as

$$p=\frac{dy}{ds} \text{ and } \sqrt{(1-pp)}=\frac{dx}{ds},$$

then the equation becomes :

$$y = C - \frac{2gkdy}{\alpha dy+dx},$$

or on neglecting the constant C, which does not change the curve, then we have :

$$\alpha ydy + ydx + 2gkdy = 0.$$

Which equation divided by y and integrated anew gives :

$$\alpha y + x + 2gk\, ly = C.$$

Which is the equation for that logarithmic curve itself, as we found the trajectory under this hypothesis of the resistance in the first book (889).

Example 2.

668. Now let the resistance be in proportion to the square of the speeds; then it is the case that $m = 1$. This equation is taken :

$$ds = \frac{-kdp}{p(1-pp)^{\frac{1}{2}}(\alpha p + \sqrt{(1-pp)})}.$$

Moreover, the integral of this is

$$s = k\, l\frac{\alpha p + \sqrt{(1-pp)}}{\beta p} \text{ or } e^{\frac{s}{k}} = \frac{\alpha p + \sqrt{(1-pp)}}{\beta p} = \frac{\alpha dy + dx}{\beta dy}.$$

Hence it becomes :

$$\beta e^{\frac{s}{k}} dy - \alpha dy = dx \text{ and } ds = dy\sqrt{\left(1 + \left(\beta e^{\frac{s}{k}} - \alpha\right)^2\right)}.$$

Which is the equation for the curve sought, which has this property, that the centrifugal force of the body is twice as great as the normal force. Therefore the curve is constantly pressed upwards by a force equal either to the normal force, or to half the centrifugal force.

[There is the continual problem with Euler's mechanics that he does not restrict himself to the forces acting on the body alone, but also includes these reaction forces acting on the curve. Thus, what Euler calls the centrifugal force exerted on the curve is the reaction of the centripetal force, which in turn is the contact force due to the curve acting on the body; again, the normal force is the reaction force of the curve corresponding to the normal component of the weight.]

Now the body thus moves on this curve so that the height corresponding to the speed at *M* is equal to :

$$\frac{gk}{e^{\frac{s}{k}}\beta p} = \frac{gkds}{\beta e^{\frac{s}{k}} dy} = \frac{gkds}{dx + \alpha dy}.$$

Scholium 1. [p. 363]

670. Although according to any hypothesis of the resistance, there is a place for the particular ratio between the centrifugal force and the normal force, the vacuum moreover should be considered as well as the case for each resistance, and it follows *in vacuo* that any curve is satisfactory [*i.e.* as conservative forces only apply.] Also all the curves *in vacuo* have this property, that upon these curves from the equality of the heights equal speeds are generated, and thus nothing extra can be defined that provide satisfaction to the question rather than the rest of the equations.

Scholium 2.

671. It is worth noting in all these curves found that the speed of the body is nowhere equal to zero. And therefore the problem cannot be resolved by this method, that among all the descents made from A to C from rest, this curve in which the body reaches the maximum speed can be determined ; to which question only the vertical straight line passing through *C* and joined to the horizontal drawn through *A* joined together is satisfactory. Moreover our solution has been prepared, so that the positions of any two neighbouring elements can thus be defined, which produce either the maximum or minimum increase in the speed. On account of which by this method that curve is found, upon which the motion of the body acquires either a greater or smaller increase in the speed than upon another curve connecting *A* and *C*, [p. 364] if the body begins the descent from *A* with the same speed. Moreover, by this reason it is possible to pick out from the curves found that curve produced, upon which the smallest increase in the speed is generated, or upon which the body is carried with the maximum uniform speed. And in this sense it is easily observed that the motion cannot begin from rest. Though it is certainly the case that if the points *A* and *C* have been placed on a vertical straight line, then upon this line from the vertical motion made from rest at *A*, the maximum speed is produced at *C*; yet the calculation does not give this solution, even if the vertical line is present, if the initial speed at *A* is made to correspond to the height $k\sqrt[m]{g}$, which speed is of such a size, that no further increase can be taken. Therefore with this speed the body descends uniformly from *A* to *C*; and for this reason zero is the minimum increase taken in the speed. Therefore the problem, in order that it has an agreeable solution, must be proposed thus : among all the lines joining the points A and C to determine that line, upon which the motion of the body takes the smallest increase in the speed, and likewise to define a fitting initial speed of the body at *A*.
[It is evident that Euler finds that the method fails under certain boundary conditions. Thus, if the initial speed is zero, that value cannot be treated as a variable; again, if the body has reached its terminal velocity at A, then the method fails, as no further increase in speed is possible.]

Scholium 3.

672. Now problems of this kind ought to follow in a prescribed order, in which suitable curves are to be investigated by a certain given relation of the times ; but since many relations of the times can be reduced to relations of the speeds, I do not bring forwards questions of this kind [p. 365]. However, in this work I will examine the question of the brachistochrone curve, since that, even if the condition of the time has to be prescribed, cannot be reduced to ratios of the speeds, and to which we will now attend. Whereby I will use the same premises as in the above treatment of the brachistochrone in vacuo (361 – 366).

PROPOSITION 76.

Theorem.

673. *In a medium with any kind of resistance and under the hypothesis of absolute forces of any kind, that curve AMC is a brachistochrone or that produces the shortest time of descent between A and C; in which the centrifugal force is equal to the normal force, and directed in the same plane.*

Demonstration.

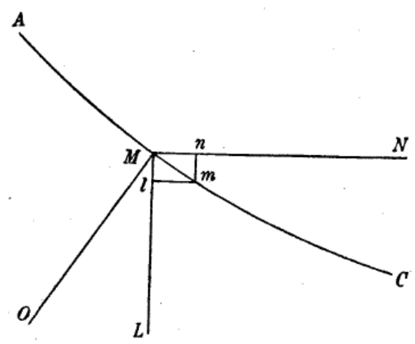


Fig. 76.

Whatever the absolute forces should be acting on the body at M, these can be resolved into two forces normal to each other, of which the one is $ML = P$, and the other is $MN = Q$. With the element of the curve taken $Mm = ds$ and with the perpendiculars ml and mn drawn, let $Ml = mn = dx$ and $ml = Mn = dy$. The height corresponding to the speed at M is equal to v, the resistive force is equal to R, and the radius of osculation at $M = r$, which I put directed upwards, thus in order that on putting dx constant, then $r = \frac{ds^3}{dxddy}$. With these in place, [the governing equation becomes :]

$$dv = Pdx + Qdy - Rds,$$

since $\frac{Pdx+Qdy}{ds}$ is the tangential force arising from the forces P and Q. But always from the nature of brachistochronism, if [p. 366]

$$dv = Pdx + Qdy + Rds,$$

we find the equation to be

$$\frac{2v}{r} = \frac{Pdy - Qdx}{ds}$$

(364), which formulas differ only in the sign of the letter R from that of ours, and thus does not come into the computation. Moreover $\frac{2v}{r}$ denotes the centrifugal force acting along the normal MO and $\frac{Pdy-Qdx}{ds}$ is the normal force acting along MO and arises from each force P and Q. Whereby if the centrifugal force is equal to the normal force and along the same direction in the same plane, then the curve is a brachistochrone. Q.E.D.

Corollary 1.

674. If the normal force, which arises from the resolution of the absolute forces acting on the body, is called *N* and the tangential force arising from the same resolution is put as *T*, then [the basic equations become] :

$$dv = (T - R)ds \text{ and } \frac{2v}{r} = N,$$

which two equations joined together must give the brachistochrone curve.

Corollary 2.

675. Therefore, whatever the resistance should be, always it is the case that $v = \frac{Nr}{2}$, hence the speed of the body on the brachistochrone is easily found. For the ratio : force of gravity 1 is to the normal force *N* thus as half the radius of osculation is to the height corresponding to the speed at *M*.

Scholium. [p. 367]

675. The same proportion is also in place in the free motion of a projected body ; indeed it is also the case for free motion that the centrifugal force is equal to the normal force. But there is a distinction : in free motion the centrifugal and normal forces are opposite to each other in direction, but for brachistochrones they act together; or in free motion the directions of the radius of osculation *r* and the normal force *N* coincide, while in brachistochrones they are in contrary directions to each other. On this account here we have accepted $r = \frac{ds^3}{dxddy}$, while for free motion, it is the case that $r = \frac{-ds^3}{dxddy}$.

Corollary 3.

677. Since $v = \frac{Nr}{2}$ is produced naturally from the formula of the brachistochrone, if this value is substituted in place of *v* everywhere in the other equation $dv = (T\text{-}R)ds$, then equation showing the nature of the brachistochrone curve is obtained.

Corollary 4.

678. Therefore in whatever the resisting medium and for whatever the forces acting on the body, these curves are all brachistochrones, in which the total force sustained by the curve exerts a force which is twice as great as either the centrifugal force alone or the forces acting alone that arise from the resolution of the normal force. [p. 368]

PROPOSITION 77.

Problem.

679. *In a uniform medium, which resists in some ratio of the powers of the speeds, and with the absolute force present uniform and directed downwards, to determine the brachistochrone AM* (Fig.74) *upon which the descending body arrives at A from M in the shortest possible time.*

Solution.

With the abscissa $AP = x$ put in place on the vertical axis and with the corresponding applied line to that $PM = y$ and with the arc of the curve sought $AM = s$ let g be the force acting downwards and $\frac{v^m}{k^m}$ be the resistance at M, if indeed the speed at M corresponds to the height v. With these in place the normal force is equal to $\frac{gdy}{ds}$; to which the centrifugal force must be equal , which is :

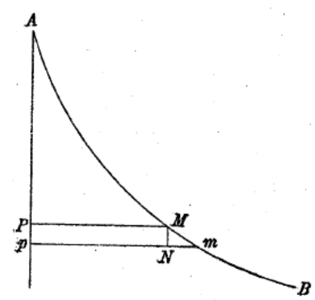


Fig. 74.

$$\frac{2v}{r} = \frac{2vdxddy}{ds^3}$$

(676) on taking dx constant. With this equation in place, hence :

$$v = \frac{gds^2dy}{2dxddy}.$$

Now the canonical equation for the descent in a resisting medium gives :

$$dv = gdx - \frac{v^m ds}{k^m}.$$

But in the former equation on putting $dsdds$ in place of $dyddy$, on account of constant dx, then that equation becomes :

$$v = \frac{gdsdy^2}{2dxdds},$$

from which there arises :

$$dv = \frac{gdy^2}{2dx} + \frac{gdsdyddy}{dxdds} - \frac{gdsdy^2d^3s}{2dxdds^2} = \frac{gdy^2}{2dx} + \frac{gds^2}{dx} - \frac{gdsdy^2d^3s}{2dxdds^2} = gdx - \frac{v^m ds}{k^m},$$

which equation reduced gives :

$$\frac{gdsdy^2d^3s}{2dxdds^2} - \frac{3gdy^2}{2dx} = \frac{g^m ds^{m+1}dy^{2m}}{2^m k^m dx^m dds^m}$$

or [p. 369]

$$dsd^3s - 3dds^2 = \frac{g^{m-1}ds^{m+1}dy^{2m-2}}{2^{m-1}k^m dx^{m-1}dds^{m-2}};$$

and this equation sets out the nature of the curve sought. In order that this equation can be reduced, and then prepared for the construction of the curve, I put $ds = pdx$, so that

$$dy = dx\sqrt{(p^2-1)},$$

and

$$dds = dpdx \text{ and } [ddds =]\ d^3s = dxddp.$$

With these substituted we have this equation :

$$pddp - 3dp^2 = \frac{g^{m-1}p^{m+1}dx^m(pp-1)^{m-1}}{2^{m-1}k^m dp^{m-2}}.$$

Now again, let $dx = qdp$, then

$$ddx = 0 = dqdp + qddp \text{ or } ddp = \frac{-dpdq}{q}$$

and this equation arises :

$$-\frac{pdq}{q} - 3dp = \frac{g^{m-1}p^{m+1}q^m dp(p^2-1)^{m-1}}{2^{m-1}k^m}$$

or

$$\frac{-pdq - 3qdp}{q^{m+1}} = \frac{g^{m-1}p^{m+1}dp(p^2-1)^{m-1}}{2^{m-1}k^m}.$$

This equation is multiplied by mp^{-3m-1} so that it becomes integrable, and there is obtained:

$$-mp^{-3m}q^{-m-1}dq - 3mp^{-3m-1}q^{-m}dp = \frac{mg^{m-1}p^{-2m}dp(p^2-1)^{m-1}}{2^{m-1}k^m},$$

and the integral of this is :

$$p^{-3m}q^{-m} = \frac{mg^{m-1}}{2^{m-1}k^m}\int\frac{(p^2-1)^{m-1}dp}{p^{2m}}.$$

There is now put in place :

$$\frac{mg^{m-1}}{2^{m-1}k^m}\int\frac{(p^2-1)^{m-1}dp}{p^{2m}} = P^{-m};$$

P is a certain function of p and hence is given, even with quadrature permitted. Therefore with these in place, we have

$$p^3q = P \text{ and } q = \frac{P}{p^3}.$$

Now since $dx = qdp$, then we have : [p. 370]

$$x = \int\frac{Pdp}{p^3},\quad s = \int\frac{Pdp}{p^2},\ \text{and}\quad y = \int\frac{Pdp\sqrt{(p^2-1)}}{p^3}.$$

Hence the construction of the brachistochrone curve follows. Q.E.I.

Corollary 1.

680. Let the point *A*, at which the motion begins and the speed is zero; then there we have $v = 0$ or

$$\frac{gdsdy^2}{2dxdds} = 0,$$

hence $dy = 0$, since it is not possible for ds to vanish. The curve therefore has a vertical tangent at the point *A*.

Corollary 2.

681. Because in the initial motion, the motion in a resisting medium does not disagree with the motion in a vacuum, then the start *A* of the curve *AM* does not disagree with the cusp of the cycloid, which is the brachistochrone *in vacuo*. And thus at *A* not only is the tangent vertical, but also the radius of osculation at that location is infinitely small.

Corollary 3.

682. Because $dy = 0$ at *A* and also $dy = dx\sqrt{(p^2 - 1)}$, then at the point *A*, $p = 1$. Therefore from the given construction of the curve the point *A* is obtained on making $p = 1$. Therefore that integral must be taken, so that *x*, *s* and *y* vanish on putting $p = 1$.

Corollary 4.

683. Since we have :

$$v = \frac{gdsdy^2}{2dxdds},$$

this becomes, as $ds = pdx$ and $dds = dpdx$

$$v = \frac{gpdx(p^2 - 1)}{2dp}$$

and since $dx = qdp$ then [p. 371]

$$v = \frac{gpq(pp - 1)}{2} = \frac{gP(pp - 1)}{2p^2}.$$

Thus it is apparent that *v* vanishes if we put $p = 1$.

Corollary 5.

684. The radius of osculation at some point *M* is equal to

$$\frac{ds^3}{dxddy} = \frac{ds^2dy}{dxdds}.$$

Whereby on account of $ds = pdx$ the radius of osculation becomes :

$$r = \frac{p^2dx\sqrt{(p^2 - 1)}}{dp} = p^2q\sqrt{(p^2 - 1)} = \frac{P\sqrt{(p^2 - 1)}}{p}.$$

Therefore at the point *A*, where $p = 1$, the radius of osculation $r = 0$.

Corollary 6.

685. Let B be the point of the brachistochrone, at which the tangent is horizontal; then there $dy = \infty$ and thus $p = \infty$. Therefore the point B is found on putting $p = \infty$. Hence at this point $v = \frac{gP}{2}$ and the radius of osculation $r = P$.

Example 1.

686. We can put the resistance to be vanishing, thus so that the motion becomes that in a vacuum; then $k = \infty$ and thus the equation is obtained :

$$dsds^3 - 3dds^2 = 0 .$$

Which equation divided by *dsdds* and integrated gives :

$$ldds - 3lds = lC$$

or

$$\frac{dds}{ds^3} = \frac{1}{adx} = \frac{dx}{adx^2} .$$

This equation integrated again gives :

$$-\frac{1}{2ds^2} = \frac{x}{adx^2} + C .$$

Or with the constants changed, and with $ds = pdx$ put in place, then $-a = ppx + Cpp;$ since on putting $p = 1$ x must vanish, and the equation becomes

$$x = \frac{a(pp-1)}{pp} \text{ or } p = \frac{\sqrt{a}}{\sqrt{(a-x)}} \text{ and thus } ds = \frac{dx\sqrt{a}}{\sqrt{(a-x)}} ,$$

which is the equation for the cycloid as agreed.

Example 2. [p. 372]

687. The medium resists in the ratio of the square of the speeds ; then $m = 1$ and

$$\frac{1}{P} = \frac{1}{k}\int\frac{dp}{p^2} = C - \frac{1}{kp} .$$

Thus this becomes :

$$P = \frac{kp}{Ckp-1} = \frac{akp}{kp-a} .$$

On this account we have :

$$x = \int\frac{akdp}{p^2(kp-a)} \text{ and } s = \int\frac{akdp}{p(kp-a)} .$$

Hence we have

$$s = kl\frac{kp-a}{(k-a)p} \text{ and } e^{\frac{s}{k}} = \frac{kp-a}{(k-a)p} = \frac{kds-adx}{(k-a)ds}$$

on account of $ds = pdx$. Hence again we have :

$$(k-a)e^{\frac{s}{k}}ds = kds - adx,$$

which integrated gives :

$$k(k-a)e^{\frac{s}{k}} = ks - ax + k(k-a).$$

Or by eliminating the exponential quantity $e^{\frac{s}{k}}$ it becomes :

$$ksds - axds - akds + akdx = 0.$$

But if we want to express the exponential $e^{\frac{s}{k}}$ by a series, then

$$k(k-a)e^{\frac{s}{k}} - k(k-a) = k(k-a)\left(\frac{s}{k} + \frac{ss}{1\cdot 2k^2} + \frac{s^3}{1\cdot 2\cdot 3k^3} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4k^4} + \text{ etc.}\right).$$

Which series substituted gives :

$$\frac{a(s-x)}{k-a} = \frac{s^2}{1\cdot 2k} + \frac{s^3}{1\cdot 2\cdot 3k^2} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4k^3} + \text{ etc.}$$

At any point M

$$v = \frac{gak(pp-1)}{2p(kp-a)}.$$

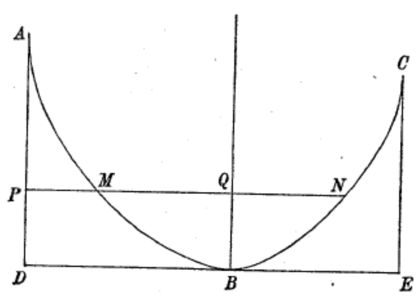


Fig. 77.

Now for the point B, at which the tangent is horizontal, the arc length is given by
$s = kl\frac{k}{k-a}$ and $e^{\frac{s}{k}} = \frac{k}{k-a}$
and thus
$x = -k + \frac{kk}{a}l\frac{k}{k-a}$.

Now the curve BNC is continued from B (Fig. 77) ; the nature of this can be found as, on the axis BQ on putting the abscissa $BQ = t$ and the arc $BN = z$. With these in place it follows that :

$$AP = x = -k - t + \frac{k^2}{a}l\frac{k}{k-a} \text{ and } AMN = s = z + kl\frac{k}{k-a}.$$

Hence it follows that

$$e^{\frac{s}{k}} = \frac{k}{k-a}e^{\frac{z}{k}};$$

with which values substituted in the above equation there is produced :

$$k^2 e^{\frac{z}{k}} = kz + k^2 + at \text{ or } at = k^2(e^{\frac{z}{k}} - 1) - kz.$$

And through the series[p. 373]

$$at = \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3k} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4k^2} + \text{etc.}$$

for the curve *BNC*; but for the branch *BMA*, in which the arc *BM* = *z* is negative, the equation becomes :

$$at = k^2(e^{\frac{-z}{k}} - 1) + kz = \frac{z^2}{1 \cdot 2} - \frac{z^3}{1 \cdot 2 \cdot 3k} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4k^2} - \text{etc.}$$

Now the curve BNC also has a vertical tangent at C, since the point is found on putting $dz = dt$. Now in this position this equation becomes :

$$a = ke^{\frac{z}{k}} - k \text{ or } z = kl\frac{a+k}{k} = BNC$$

and

$$t = CE = k - \frac{kk}{a} l\frac{a+k}{k} = \frac{a}{2} - \frac{a^2}{3k} + \frac{a^3}{4k^2} - \text{etc.},$$

while for the other branch, it becomes

$$AD = \frac{a}{2} + \frac{a^2}{3k} + \frac{a^3}{4k^2} + \text{etc.}$$

From which it is apparent that the point *A* is to be placed higher than the point *C* and the curve has cusps at *A* and *C* or the points of return, thus so that both *AD* and CE are diameters of the curve; that which is understood from this, is that

$$y = \int \frac{Pdp\sqrt{(pp-1)}}{p^3},$$

where $\sqrt{(p^2-1)}$ takes positive values as well as negative ones.

Scholium 1.

688. Below it is observed that this brachistochrone curve is congruent with the tautochrone curve for the same hypothesis of the resistance. Now there is a difference between tautochronous and brachistochronous motions, as according to tautochronous motion being obtained, the body must descend on the branch *CNB* and ascend on the other; while the opposite is true for the brachistochrone motion, as it must descend along *AMB.* Yet meanwhile each of these curves certainly is worthy of attention, and since *in vacuo* the same agreement is observed. [p. 374]

Example 3.

689. The medium resists in the ratio of the square of the speeds, thus so that $m = 2$. This equation is obtained for the curve sought :

$$dsd^3s - 3dds^2 = \frac{gds^3dy^2}{2k^2dx},$$

now in constructing the curve :

$$\frac{1}{P^2} = \frac{g}{k^2}\int\left(\frac{dp}{p^2} - \frac{dp}{p^4}\right);$$

thus P becomes :

$$P = \frac{kp\sqrt{3np}}{\sqrt{g(p^3 - 3np^2 + n)}}$$

and

$$s = \int\frac{kdp\sqrt{3n}}{\sqrt{g(p^4 - 3np^3 + np)}}$$

and

$$x = \int\frac{kdp\sqrt{3n}}{p\sqrt{g(p^4 - 3np^3 + np)}} \text{ and } y = \int\frac{kdp\sqrt{3n}(pp-1)}{p\sqrt{g(p^4 - 3np^3 + np)}}.$$

Therefore the general construction of the curve is obtained. But since n can denote any number, let $n = \frac{1}{2}$;then

$$y = \int\frac{kdp\sqrt{3}}{p\sqrt{g(2p^2 - p)}} = \frac{2k\sqrt{3(2p-1)}}{\sqrt{gp}} - \frac{2k\sqrt{3}}{\sqrt{g}};$$

as thus we add a constant so that $y = 0$ on putting $p = 1$.
[On putting $n = \frac{1}{2}$ then

$$y = \int\frac{kdp\sqrt{3}(pp-1)}{p\sqrt{g(2p^4 - 3p^3 + p)}},$$

which formula, since $2p^4 - 3p^3 + p$ does not contain the factor $pp - 1$, does not admit to the reduction made by Euler. Thus also neither can the following formulas be put in place. Note by P. St. in the *Opera Omnia*]
Now $p = \infty$ gives the applied line

$$DB = \frac{2k(\sqrt{6} - \sqrt{3})}{\sqrt{g}}.$$

Moreover,

$$p = \frac{12k^2}{12k^2 - 4ky\sqrt{3g} - gy^2}$$

and

$$\sqrt{(p^2-1)} = \frac{dy}{dx} = \frac{\sqrt{(96k^3y\sqrt{3g} - 24gk^2y^2 - 8gky^3\sqrt{3g} - g^2y^4)}}{12k^2 - 4ky\sqrt{3g} - gy^2}.$$

From which it follows

$$x = \int \frac{12k^2dy - 4kydy\sqrt{3g} - gy^2dy}{\sqrt{(96k^3y\sqrt{3g} - 24gk^2y^2 - 8gky^3\sqrt{3g} - g^2y^4)}}$$

$$= \int \frac{12k^2dy - 4kydy\sqrt{3g} - gy^2dy}{\sqrt{(gy^2 + 4ky\sqrt{3g})(24k^2 - 4ky\sqrt{3g} - gy^2)}},$$

which is the equation between the coordinates x and y for the curve sought.

Scholium 2. [p. 375]

690. In a medium, which resists in the simple ratio of the speeds, it is not possible to determine the simpler brachistochrone, as follows at once from the general construction. On account of which we do not describe this case in detail by an example. Moreover that may be checked concerning the remaining propositions here, in which the curve is sought, upon which the body descends the quickest to arrive at a given line, which is either straight or a curve, and that can be solved in a similar way for a resisting medium as for the vacuum. While clearly from the same point *A*, innumerable brachistochrone curves can be sent off, from these one has to be selected, which for the given line – either straight or curved, it meets at right angles; indeed upon this line, as shown in the previous chapter, the body arrives at that line in the shortest possible time. By similar reasoning the curve, which all the brachistochrones cross at right angles, cuts from all these curves the isochronous arcs or [the corresponding] arcs which the descending body completes in equal intervals of time. And all these are obtained in the same way, whatever should be the resistance and whatever the absolute forces. But now we will set out most generally an understanding of the brachistochrone problem.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 349]

PROPOSITIO 73.

Problema.

649. *Si potentia fuerit uniformis et deorsum directa mediumque in ratione quacunque multiplicata celeritatum resistat, determinare curvam AM* (Fig.73)*, super qua corpus descendendo secundum horizontalem AH aequalibiter progrediatur.*

Solutio.

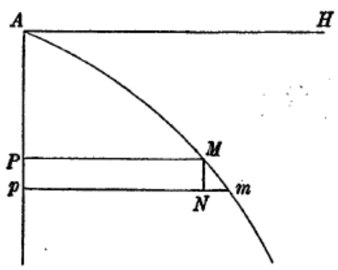


Fig. 73.

Sit A curvae punctum supremum, per quod ducatur axis verticalis AP, celeritasque, qua corpus horizontaliter progreditur, sit debita altitudini b. Sumatur abscissa $AP = x$, applicata $PM = y$ et arcus $AM = s$ sitque corporis in M celeritas debitia altitudini v, qua celeritate corporis elementum $Mm = ds$ percurret. Erit ergo ut ds ad dy ita corporis celeritas per Mm, quae est $\sqrt{v}$, ad celeritatem horizontalem $\sqrt{b}$, unde oritur

$$v = \frac{bds^2}{dy^2}.$$

Iam sit potentia sollicitans = g, exponens resistentiae = k et ipsa resistentia = $\frac{v^m}{k^m}$. His positis erit

$$dv = gdx - \frac{v^m ds}{k^m},$$

quae aequatio, si loco v valor $\frac{bds^2}{dy^2}$ substituatur, exprimet naturam curvae quaesitae. Sit autem $ds = pdy$; erit

$$v = bp^2 \text{ et } dx = dy\sqrt{(p^2 - 1)}.$$

Quocirca habebitur

$$2bpdp = gdy\sqrt{(p^2 - 1)} - \frac{b^m p^{2m+1} dy}{k^m},$$

quae separata dat [p. 350]

$$dy = \frac{2bk^m p\,dp}{gk^m\sqrt{(p^2-1)} - b^m p^{2m+1}}.$$

Curvae igitur quaesitae sequens erit constructio: sumto

$$y = \int \frac{2bk^m p\,dp}{gk^m\sqrt{(p^2-1)} - b^m p^{2m+1}}$$

erit

$$x = \int \frac{2bk^m p\,dp\sqrt{(p^2-1)}}{gk^m\sqrt{(p^2-1)} - b^m p^{2m+1}}.$$

Q.E.I.

Corollarium 1.

650. Si loco *pp* restituatur $\frac{v}{b}$ atque per *v* definiantur *y* et *x*, erit

$$y = \int \frac{k^m dv\sqrt{b}}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}}$$

atque

$$x = \int \frac{k^m dv\sqrt{(v-b)}}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}}.$$

Similique modo hinc erit arcus

$$s = \int \frac{k^m dv\sqrt{v}}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}}.$$

Sumto autem $\frac{v}{b}$ loco *pp* erit

$$ds = \frac{dy\sqrt{v}}{\sqrt{b}} \quad \text{et} \quad dx = \frac{dy\sqrt{(v-b)}}{\sqrt{b}} \quad \text{atque} \quad ds = \frac{dx\sqrt{v}}{\sqrt{(v-b)}}.$$

Corollarium 2.

651. Quia aequatio

$$dy = \frac{k^m dv\sqrt{b}}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}}$$

est separata, ex ea solutio particularis quaesito satisfaciens erui potest faciendo denominatorem

$$gk^m\sqrt{(v-b)} - v^m\sqrt{v} = 0,$$

unde erit ipsa celeritas $\sqrt{v}$ constans. Sit ergo $v = c$; erit $\sqrt{(v-b)} = \frac{c^m\sqrt{c}}{gk^m}$

atque ideo

$$ds = \frac{gk_m dx}{gk^m}$$

pro linea recta inclinata, ut supra iam invenimus (628).

Corollarium 3. [p. 351]

652. Quo autem corpus data celeritate, quae debita est altitudini b, horizontaliter progrediatur, ex aequatione $\sqrt{(c-b)} = \frac{c^m\sqrt{c}}{gk^m}$ definiri debet altitudo c. Qua inventa habebitur inclinatio rectae satisfacientis et celeritas corporis initialis $\sqrt{c}$ in A, qua aequalibiter per rectam descendet.

Corollarium 4.

653. Si resistentia evanescat corpusque in vacuo moveatur, fiet $k = \propto$ ideoque

$$x = \int \frac{dv}{g} \quad \text{seu} \quad v = g(a+x)$$

atque

$$dy = \frac{dx\sqrt{b}}{\sqrt{(ga+gx-b)}}.$$

Integrando ergo fiet

$$y = \frac{2}{g}\sqrt{b(ga+gx-b)} - \frac{2}{g}\sqrt{b(ga-b)},$$

quae est aequatio pro parabola, quam corpus proiectum libere describit.

Exemplum.

654. Si medium fuerit rarissimum atque ideo k valde magnum, erit

$$\frac{1}{gk^m\sqrt{(v-b)} - v^m\sqrt{v}} = \frac{1}{gk^m\sqrt{(v-b)}} + \frac{v^m\sqrt{v}}{g^2k^{2m}(v-b)}$$

quam proxime.
Hanc ob rem habebitur [p. 352]

$$y = \frac{2\sqrt{b(v-b)}}{g} + \int \frac{v^m dv\sqrt{bv}}{g^2k^m(v-b)} \quad \text{et} \quad x = \frac{v}{g} + \int \frac{v^m dv\sqrt{v}}{g^2k^m\sqrt{(v-b)}}.$$

Ex hac posteriori aequatione est quam proxime

$$v = gx - \int \frac{g^m x^m dx\sqrt{gx}}{k^m\sqrt{(gx-b)}},$$

qui valor in aequatione

$$dy = \frac{dx\sqrt{b}}{\sqrt{(v-b)}}$$

substitutus dat aequationem inter x et y pro curva quaesita.

PROPOSITIO 74.

Problema.

655. *Invenire curvam AM* (Fig.74)*, super qua corpus descendens in medio quocunque resistente aequalibiter deorsum progrediatur existente potentia absoluta uniformi et deorsum directa.*

Solutio.

Positaq abscissa $AP = x$, $AM = s$ sit celeritas, qua corpus uniformiter descendere debet, debita altitudini b. Potentia porro uniformis deorsum directa sit g et altitudo debita celeritati in $M = v$ atque resistentia = $\frac{v^m}{k^m}$; erit ergo

$$dv = gdx - \frac{v^m ds}{k^m}.$$

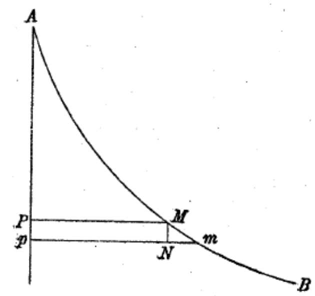


Fig. 74.

Debebit autem esse ut $Mm : MN = \sqrt{v} : \sqrt{b}$, unde erit $v = \frac{bds^2}{dx^2}$. Ex hac ergo aequatio erit

$$ds = \frac{dx\sqrt{v}}{\sqrt{b}},$$

quo valore in aequatione substituto habebitur

$$dv = gdx - \frac{v^{m+\frac{1}{2}}dx}{k^m\sqrt{b}} \quad \text{seu} \quad dx = \frac{k^m dv\sqrt{b}}{gk^m\sqrt{b} - v^m\sqrt{v}}.$$

Quamobrem erit [p. 353]

$$x = \int \frac{k^m dv\sqrt{b}}{gk^m\sqrt{b} - v^m\sqrt{v}}, \quad s = \int \frac{k^m dv\sqrt{v}}{gk^m\sqrt{b} - v^m\sqrt{v}}$$

atque applicata

$$PM = y = \int \frac{k^m dv\sqrt{(v-b)}}{gk^m\sqrt{b} - v^m\sqrt{v}}.$$

Ex quibus aequationibus constructio curvae quaesitae conficitur. Q.E.I.

Corollarium 1.

656. Ex his tribus aequationibus, si desideretur aequatio ex x, y et s tantum consistens, accipi potest ea, ex qua valor ipsius v commodissime poterit inveniri, isque deinceps in alterutra reliquarum substitui.

Corollarium 2.

657. Quia aequatio

$$dx = \frac{k^m dv \sqrt{b}}{gk^m \sqrt{b} - v^m \sqrt{v}}$$

indeterminatas a se invicem habet separatas, poterit solutio particularis obtineri ponendo

$$gk^m \sqrt{b} - v^m \sqrt{v} = 0.$$

Hinc igitur erit

$$v = g^{\frac{2}{2m+1}} k^{\frac{2m}{2m+1}} b^{\frac{1}{2m+1}}$$

ideoque

$$ds = \frac{g^{\frac{1}{2m+1}} k^{\frac{m}{2m+1}} dx}{b^{\frac{m}{2m+1}}}.$$

Satisfacit ergo linea recta inclinata, si corpus data celeritate $\sqrt{v}$ super ea moveatur.

Scholion 1. [p. 354]

658. Quod in praecedente et hoc problemate linea recta inclinata solutionem praebeat particularem, ex eo intelligi potest, quod in medio resistente recta inclinata inveniri possit, super qua corpus aequabiliter moveatur, ut supra (628) ostendimus. Hic autem ipse casus utrique problemati satisfacit; si enim corpus super recta aequabili motu incedit, tam horizontaliter quam verticaliter quoque promovetur; quin etiam secundum quamcunque plagam aequabiliter fertur.

Corollarium 3.

659. Pro vacuo fit $k = \propto$. Quamobrem erit $x = \frac{v}{g}$ seu $v = gx$ atque

$$ds = \frac{dx \sqrt{gx}}{\sqrt{b}} \quad \text{et} \quad dy = \frac{dx \sqrt{(gx - b)}}{\sqrt{b}},$$

quae aequatio integrata dat

$$y = \frac{2(gx - b)^{\frac{3}{2}}}{3g\sqrt{b}}$$

et praebet parabolam cubicalem rectificabilem, ut (258) iam invenimus.

Exemplum 1.

660. Ponamus resistentiam ipsis celeritatibus proportionalem; erit $m = \frac{1}{2}$ ideoque

$$x = \int \frac{dv \sqrt{bk}}{g\sqrt{bk} - v} = \sqrt{bk}\, l \frac{g\sqrt{bk}}{g\sqrt{bk} - v},$$

si initium abscissarum in eo puncto accipiatur, ubi evanescit. Ex hac autem aequatione prodit [p. 355]

$$e^{\frac{x}{\sqrt{bk}}} = \frac{g\sqrt{bk}}{g\sqrt{bk}-v} \quad \text{seu} \quad v = e^{\frac{-x}{\sqrt{bk}}}\left(e^{\frac{x}{\sqrt{bk}}}-1\right)g\sqrt{bk} = g\sqrt{bk}\left(1-e^{\frac{-x}{\sqrt{bk}}}\right).$$

Valore hoc ipsius *v* substituto habebitur

$$ds = \frac{dx\sqrt{g\left(1-e^{\frac{-x}{\sqrt{bk}}}\right)\sqrt{bk}}}{\sqrt{b}}.$$

Vel cum sit

$$ds = \frac{dv\sqrt{kv}}{g\sqrt{bk}-v},$$

ponatur $v = u^2$; erit

$$ds = \frac{2u^2du\sqrt{k}}{g\sqrt{bk}-uu} = \frac{2gk\,du\sqrt{b}}{g\sqrt{bk}-uu} - 2du\sqrt{k},$$

quae integrata dat

$$s = \sqrt[4]{g^2bk^3}\,l\,\frac{\sqrt[4]{g^2bk}+\sqrt{v}}{\sqrt[4]{g^2bk}-\sqrt{v}} - 2\sqrt{kv},$$

in qua valor ipsius *v* ante inventus substutui potest, quo prodeat aequatio inter *x* et *s*.

Exemplum 2.

661. Resistat nunc medium in duplicata celeritatum ratione; erit *m* = 1 ideoque

$$dx = \frac{kdv\sqrt{b}}{gk\sqrt{b}-v\sqrt{v}} \quad \text{et} \quad ds = \frac{kdv\sqrt{v}}{gk\sqrt{b}-v\sqrt{v}}.$$

Huius posterioris aequationis integralis est

$$s = \frac{2k}{3}\,l\,\frac{gk\sqrt{b}}{gk\sqrt{b}-v\sqrt{v}},$$

ex qua oritur

$$e^{\frac{3s}{2k}} = \frac{gk\sqrt{b}}{gk\sqrt{b}-v\sqrt{v}}$$

atque

$$v^{\frac{3}{2}} = gk\left(e^{\frac{3s}{2k}}-1\right)e^{\frac{-3s}{2k}}\sqrt{b} = gk\left(1-e^{\frac{-3s}{2k}}\right)\sqrt{b}.$$

Quocirca erit

$$\sqrt{v} = \sqrt[3]{gk\left(1 - e^{\frac{-3s}{2k}}\right)}\sqrt{b};$$

qui valor in aequatione $dx = \frac{ds\sqrt{b}}{\sqrt{v}}$ substitutus dat aequatinem inter s et x pro curva quaesita.

Scholion 2.

662. Quemadmodum in his duobus problematibus curvas determinavimus, super quibus corpus motum vel secundum horizontalem vel deorsum aequabiliter feratur, ita simili modo problema resolvi potest, [p. 356] si corpus secundum quamvis aliam plagam aequabiliter progredi debeat ; ipsam autem quaestionem, quia nihil concinni ex solutione deduci potest, hic omisi; atque ob eandem causam problema isochronae paracentricae in medio resistente non attingo. Adiungam vero his, in quibus celeritatum quaedam lex proponitur, non parum curiosum problema, quod a nemine adhuc est tractatum, pro mediis resistentibus; quod pro vacuo propositum ne problem quidem est. Quaeritur scilicet curva, super qua corpus ad datum punctum maxima celeritate pertingat; in vacuo enim corpus super quacunque curva motum in eodem loco semper eandem obtenit celeritatem.

PROPOSITIO 75.

Problema.

663. *Inter omnes curvas puncta A et C* (Fig.75) *iungentes determinare eam AMC, super qua corpus ex A ad C descendens maximam acquirat celeritatem existente resistentia in quacunque multiplicata ratione celeritatum et potentia uniformi deorsum tendente.*

Solutio.

Quo corpus ad punctum C maxima cum celeritate perveniat, curvae quaesitae AMC duo quaeque elementa $Mm, m\mu$ ita posita esse debent, ut corpus ea percurrendo [p. 357] maximum accipiat celeritatis incrementum. Nam si corpus per alia elementa $Mn, n\mu$ maius acquireret celeritatis augmentum, maiorem quoque in C habiturum esset celeritatem. Per methodum igitur maximorum positio elementorum $Mn, n\mu$ invenietur, si elementa haec cum proximis $Mn, n\mu$ comparentur et celeritatis augmenta, quae per utraque generantur, inter si aequalia ponantur. Ducantur ad hoc ad axem verticalem applicatae MP, nmp et $\mu\pi$ sintque elementa $Pp, p\pi$ aequalia. Ducantur quoque verticales MF, mG et in curvae elementa normales mf, ng. Iam sit potentia

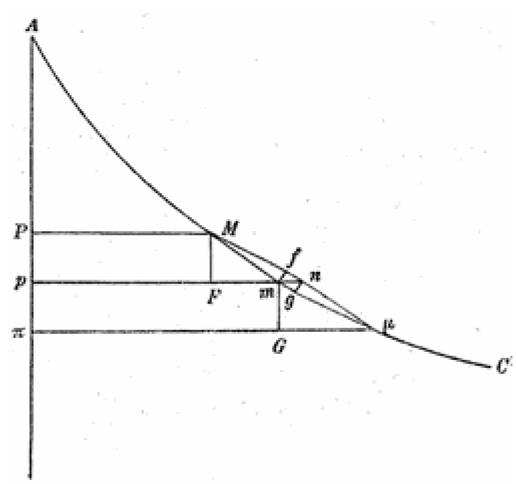


Fig. 75.

sollicitans = g, exponens resistentiae = k, ipsa resistentia in $2m$-multiplicata ratione celeritatum et altitudo celeritati in M debita = v. His positis erit incrementum ipsius v per Mm =

$$g.MF - \frac{v^m.Mm}{k^m}$$

et incrementum altitudinis celeritati debitae, dum corpus per $m\mu$ progreditur,

$$= g.MG - \frac{(v+g.MF-\frac{v^m.Mm}{k^m})^m m\mu}{k^m}$$

Dum ergo corpus elementa Mm et $m\mu$ conficit, altitudo v accipit augmentum =

$$g(MF+mG) - \frac{v^m\cdot Mm}{k^m} - \frac{\left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m m\mu}{k^m}.$$

At elementa Mn, $n\mu$ percurrendo accipiet v augmentum =

$$g(MF+mG) - \frac{v^m\cdot Mn}{k^m} - \frac{\left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m n\mu}{k^m}.$$

Quibus sibi aequalibus positis habebitur [p. 358]

$$0 = v^m(Mn-Mm) + \left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m n\mu - \left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m m\mu.$$

Est vero $Mn - Mm = nf$ et

$$\left(v+g.MF-\frac{v^m}{k^m}Mn\right)^m = v^m + m\cdot gv^{m-1}\cdot MF - \frac{mv^{2m-1}}{k^m}Mn$$

atque

$$\left(v+g.MF-\frac{v^m}{k^m}Mm\right)^m = v^m + m\cdot gv^{m-1}\cdot MF - \frac{mv^{2m-1}}{k^m}Mm.$$

Nunc vero his valoribus substituendis proveniet haec aequatio

$$v(nf-mg) - m\cdot g.MF.mg - \frac{mv^m}{k^m}(Mn\cdot n\mu - Mm\cdot m\mu) = 0.$$

At est

$$Mn\cdot n\mu - Mm\cdot m\mu = n\mu\cdot nf - Mm\cdot mg$$

atque ob triangula nfm, mFM et mgn, μGm similia est

$$nf = \frac{mF\cdot mn}{Mm} \quad \text{atque} \quad mg = \frac{\mu G\cdot mn}{m\mu}.$$

His substitutis et per mn diviso prodit

$$v\left(\frac{mF}{Mm}-\frac{\mu G}{m\mu}\right)-m\cdot g\cdot\frac{MF\cdot\mu G}{m\mu}-\frac{mv^m}{k^m}\left(\frac{n\mu\cdot mF}{Mm}-\frac{Mm\cdot\mu G}{m\mu}\right)=0.$$

Huius aequationis duo priora membra sunt differentialia primi gradus, tertium vero, quia differentiali secundi gradus aequipollet, reiici potest; fiet ergo

$$\frac{m\cdot g\cdot MF\cdot\mu G}{m\mu}+v\left(\frac{\mu G}{m\mu}-\frac{mF}{Mm}\right)=0$$

sive

$$\frac{m\cdot g\cdot MF\cdot mF}{Mm}+v\,d.\frac{mF}{Mm}=0.$$

Ex qua aequatione determinatur positio elementorum Mm et $m\mu$. [p. 359] Quo autem symbolis utamur, sit $AP = x$, $PM = y$ et $AM = s$; erit $Pp = p\pi = dx$, $mF = dy$ et $Mm = ds$ prodibitque

$$\frac{m\cdot g\,dx\,dy}{ds}+v\,d.\frac{dy}{ds}=0.$$

Aequatio vero canonica est

$$dv = gdx-\frac{v^m ds}{k^m},$$

in qua si loco gdx ex superiore aequatione substituatur

$$\frac{-vds}{mdy}\,d.\frac{dy}{ds},$$

habebitur

$$dv+\frac{vds}{mdy}\,d.\frac{dy}{ds}+\frac{v^m ds}{k^m}=0$$

seu

$$\frac{mdvdy}{ds}+v\,d.\frac{dy}{ds}+\frac{mv^m dy}{k^m}=0.$$

Sit $dy = pds$ et $v^{1-m}=u;$ erit, ut sequitur,

$$pdu+\frac{(1-m)}{m}udp+\frac{(1-m)p\,ds}{k^m}=0,$$

ex qua integrata prodit

$$u=\frac{(m-1)p^{\frac{m-1}{m}}}{k^m}\int p^{\frac{1-m}{m}}ds.$$

Ex hoc u obtinebitur ergo vicissim $v=u^{\frac{1}{1-m}}$, qui valor in superiore aequatione

$$mgpds\sqrt{(1-pp)}+vdp=0$$

substitutus dabit aequationem inter p et s et consequenter inter y et s.

Ad curvam autem construendam hoc modo computum instituti expedit. Posito *dy = pds* habentur hae duae aequationes

$$mgpds\sqrt{(1-pp)} + vdp = 0$$

et

$$dv = gds\sqrt{(1-pp)} - \frac{v^m ds}{k^m}.$$

Ex illa est

$$ds = \frac{-vdp}{mgp\sqrt{(1-p^2)}},$$

qui valor in hac substitutus dat

$$mpdv + vdp = \frac{v^{m+1}dp}{gk^m\sqrt{(1-pp)}}.$$

Haec divisa per $v^{m+1}p^2$ fit integrabilis eritque integrale [p. 360]

$$\frac{1}{v^m p} = C + \frac{\sqrt{(1-pp)}}{gk^m p}$$

seu

$$v^m = \frac{gk^m}{\alpha p + \sqrt{(1-pp)}} \quad \text{et} \quad v = \frac{k\sqrt[m]{g}}{\sqrt[m]{(\alpha p + \sqrt{(1-pp)})}}.$$

Quocirca erit

$$mgds = \frac{-kdp\sqrt[m]{g}}{p(1-pp)^{\frac{1}{2}}\sqrt[m]{(\alpha p + \sqrt{(1-pp)})}}$$

et

$$mgdx = \frac{-kdp\sqrt[m]{g}}{p\sqrt[m]{(\alpha p + \sqrt{(1-pp)})}}$$

atque

$$mgdy = \frac{-kdp\sqrt[m]{g}}{(1-pp)^{\frac{1}{2}}\sqrt[m]{(\alpha p + \sqrt{(1-pp)})}}.$$

Ex quibus aequationibus facile est curvam quaesitam construere. Q.E.I.

Corollarium 1.

664. Si radius osculi curvae in *M* versus axem directus vocetur *r*, erit

$$d.\frac{dy}{ds} = -\frac{dx}{r}.$$

Hocque valore substituto habetur

$$\frac{mgdy}{ds} = \frac{v}{r} \quad \text{seu} \quad \frac{2mgdy}{ds} = \frac{2v}{r}.$$

Est vero $\frac{2v}{r}$ vis centrifuga corporis in hac curva moti, cuius directio est ab axe directa, et $\frac{gdy}{ds}$ est vis normalis. Quare in curva quaesita vis centrifuga est contraria vi normali et se habet ad vim normalem ut 2*m* ad 1, id est ut exponens potestatis celeritatis, cui resistentia est proportionalis, ad unitatem.

Corollarium 2. [p. 361]

665. Hae igitur omnes curvae parte concava sunt deorsum directae. Quia enim vis normalis directio deorsum respecit et radius osculi in eandem plagam tendit, concavitas curvae quoque deorsum respicere debet.

Corollarium 3.

666. In medio resistente in simplici ratione celeritatum erit $2m = 1$. Hoc ergo casu vis centrifuga aequalis est et contraria vi normali. Quamobrem curva quaesita satisfaciens erit ipsa proiectoria, quam corpus proiectum libere describit.

Corollarium 4.

667. Quia in aequatione

$$mgds = \frac{-kdp\sqrt[2m]{g}}{p(1-pp)^{\frac{1}{2}}\sqrt[2m]{(\alpha p+\sqrt{(1-pp)})}}$$

indeterminatae sunt separatae, tres solutiones particulares inde obtinentur. Primam dat aequatio $\alpha p + \sqrt{(1-pp)} = 0$, quo casu celeritas fit infinita et quaevis recta satisfacit. Secunda est $p = 1$, seu $dy = ds$, quae est pro recta horizontali, et tertia est $p = 0$, pro recta verticali; quae semper hanc habet proprietatem, ut corpus in ea descendens maxima celeritatis augmenta accipiat

Exemplum 1. [p. 362]

668. Resistat medium in simplici ratione celeritatum; erit $m = \frac{1}{2}$. Sumatur ex tribus inventis aequationibus ea, quae *dy* continet; erit

$$dy = \frac{-2gkdp}{(\alpha p+\sqrt{(1-pp)})^2\sqrt{(1-pp)}},$$

cuius integralis est

$$y = C - \frac{2gkp}{\alpha p+\sqrt{(1-pp)}}.$$

Cum autem sit

$$p = \frac{dy}{ds} \quad \text{et} \quad \sqrt{(1-pp)} = \frac{dx}{ds},$$

erit

$$y = C - \frac{2gkdy}{\alpha dy + dx},$$

seu neglecta constante *C*, quia curvam non immutat, erit

$$\alpha y dy + y dx + 2gk dy = 0.$$

Quae aequatio per *y* divisa et denuo integrata dat

$$\alpha y + x + 2gk\, ly = C.$$

Quae est aequatio pro curva logarithmicali ea ipsa, quam libro primo (889) proiectoriam in hac resistentiae hypothesi invenimus.

Exemplum 2.

668. Sit nunc resistentia quadratis celeritatum proportionalis; erit *m* = 1. Sumatur aequatio ista

$$ds = \frac{-kdp}{p(1-pp)^{\frac{1}{2}}(\alpha p + \sqrt{(1-pp)})}.$$

Huius autem integralis est

$$s = k\, l\frac{\alpha p + \sqrt{(1-pp)}}{\beta p} \quad \text{sive} \quad e^{\frac{s}{k}} = \frac{\alpha p + \sqrt{(1-pp)}}{\beta p} = \frac{\alpha dy + dx}{\beta dy}.$$

Hinc fit

$$\beta e^{\frac{s}{k}} dy - \alpha dy = dx \quad \text{atque} \quad ds = dy\sqrt{\left(1 + \left(\beta e^{\frac{s}{k}} - \alpha\right)^2\right)}.$$

Quae est aequatio pro curva quaesita, quae hanc habebit proprietatem, ut vis centrifuga corporis sit duplo maior quam his normalis. Curva igitur perpetuo sursum premetur vi aequali vel ipsi vi normali vel dimidio vis centrifugae. In hac vero curva corpus ita movebitur, ut altitudo celeritati in *M* debita sit =

$$\frac{gk}{e^{\frac{s}{k}}\beta p} = \frac{gkds}{\beta e^{\frac{s}{k}} dy} = \frac{gkds}{dx + \alpha dy}.$$

Scholion 1. [p. 363]

670. Cum in quavis resistentiae hypothesi peculiaris ratio inter vim centrifugam et vim normalem locum habeat, vacuum autem tanquam casus cuiusque resistentiae considerari queat, sequitur in vacuo quamvis curvam satisfacere debere. Omnes etiam curvae in vacuo hanc habent proprietatem, ut super iis ex aequalibus altitudinibus aequales generentur celeritates ideoque nulla potest definiri, quae potius quam reliquae quaesito satisfaciant.

Scholion 2.

671. Notatu dignum est, quod in omnibus his curvis inventis nusquam corporis celeritas sit aequalis nihilio. Atque idcirco problema hac methodo non ita resolvi potest, ut determinetur inter omnes descensus ex *A* ad *C* ex quiete factos is, in quo corpus maximam acquirit celeritatem; cui quaestioni sola recta verticalis per *C* transiens et cum horizontali per *A* ducta coniuncta satisfacit. Nostra autem solutio ita est comparata, ut duorum elementorum quorumque contiguorum positionem eam difiniat, quae maximum vel minimum celeritas augmentum producat. Quamobrem hac methodo ea curva invenitur, super qua corpus motum vel maius vel minus celeritatis augmentum acquirit quam super alia quacunque curva *A* et *C* iungente, [p. 364] si corpus ex *A* eadem celeitate descensum inchoet. Ex inventis autem colligi potest hac ratione eam prodire curvam, super qua minimum celeritatis incrementum generetur, vel super qua corpus motu maxime uniformi feratur. Atque hoc sensu facile perspicitur motum ex quiete incipere non posse. Quanquam enim certum est, si puncta *A* et *C* in linea verticali sunt posita super hac verticali motu in *A* ex quiete facto maximam in *C* generari celeritatem, tamen calculus non hanc dat solutionem, etiamsi praebeat lineam verticalem, sed celeritatem initialem in *A* facit debitam altitudini $k\sqrt[m]{g}$, quae celeritas tanta est, ut non amplius augmentum accipere queat. Hac igitur celeritate corpus aequalibitur ex *A* ad *C* descendet; hacque ratione nullum, hoc est minimum, capit celeritatis incrementum. Problema ergo, ut solutioni consentaneum fuesset, ita proponi debuisset : inter omnes lineas puncta *A* et *C* iungentes eam determinare, super qua corpus motum minima accipiat celeritatis augmenta, atque simul celeritatem initialem in A huic quaesito accommodatam definire.

Scholion 3.

672. Secundum ordinem praescriptum sequi deberent nunc huiusmodi problemata, in quibus temporum quadam lege data curvae essent investigandae idoneae; sed cum temporum leges pleraeque [p. 365] ad celeritatum leges possint reduci, huiusmodi quaestiones non profero. Sed unicam in hoc negotio quaestionem de curvis brachystochronis tractabo, quia ea, etsi temporis praescripta est conditio, ad celeritatum rationes, quas iam pervolvimus, reduci non potest. Qua in re iisdem praemissis utar, quae supra (361 – 366) circa brachystochronas in vacuo sunt tradita.

PROPOSITIO 76.

Theorema.

673. *In medio quocunque resistente et potentiarum absolutorum hypothesi quacunque ea curva AMC est brachystochrona seu brevissimum ab A ad C producit descensum; in qua vis centrifuga est aequalis vi normali, et in eadem plagam directa.*

Demonstratio.

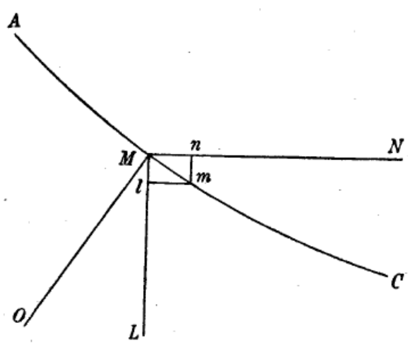


Fig. 76.

Quaecunque fuerint potentiae absolutae in corpus in *M* agentes, eae in duas inter se normales possunt resolvi, quarum altera sit *ML* = *P*, altera *MN* = *Q*. Sumto curvae elemento *Mm* = *ds* ductisque perpendicularis *ml*, *mn* sit *Ml* = *mn* = *dx* et *ml* = *Mn* = *dy*. Ponantur altitudo celeritati in *M* debita = *v* et vis resistentia = *R* atque radius osculi in *M* = *r*, quem pono sursum directum, ita ut posito *dx* constante sit $r = \frac{ds^3}{dxddy}$. His positis erit

$$dv = Pdx + Qdy - Rds,$$

quia $\frac{Pdx+Qdy}{ds}$ est vis tangentialis ex potentiis P et Q orta. At semper ex natura brachystochronismi, si fuerit [p. 366]

$$dv = Pdx + Qdy + Rds,$$

invenimus fore

$$\frac{2v}{r} = \frac{Pdy-Qdx}{ds}$$

(364), quae formulae ab hac nostra tantum in signo litterae *R* differunt, haecque in computum non venit. Denotat autem $\frac{2v}{r}$ vim centrifugam secundum normalem *MO* agentem atque $\frac{Pdy-Qdx}{ds}$ est vis normalis iuxta *MO* agens ex utraque vi *P* et *Q* orta. Quare si fuerit vis centrifuga vi normali aequalis et in eandem plagam directa, curva erit brachystochrona. Q.E.D.

Corollarium 1.

674. Si vis normalis, quae oritur ex resolutione potentiarum absolutarum corpus sollicitantium, vocetur *N* et vis tangentialis ex eadem resolutione orta ponatur *T*, erit

$$dv = (T - R)ds \quad \text{et} \quad \frac{2v}{r} = N,$$

quae duae aequationes coniunctae dabunt curvam brachystochronam.

Corollarium 2.

675. Quaecunque igitur fuerit resistentia, erit semper $v = \frac{Nr}{2}$, unde celeritas corporis super brachystochrona facile invenietur. Erit enim ut vis gravitatis 1 ad vim normalem *N*, ita dimidium radii osculi ad altitudinem celeritati in *M* debitam.

Scholion. [p. 367]

675. Haec eadem proporito quoque locum habet in motu corporum proiectorum libero;; est enim pariter pro motu libero vis centrifuga aequalis vi normali. Discrimen autem in hoc consistit, ut in motu libero vires centrifuga et normalis sint inter se oppositae, pro curvis brachystochronis autem conspirantes, sive in motu libero directiones radii osculi *r* et vis normalis *N* coincidunt, in brachystochronis vero inter se sunt contrariae. Hanc ob rem hic sumsimus

$$r = \frac{ds^3}{dxddy},$$

cum in motu libero sit

$$r = \frac{-ds^3}{dxddy}.$$

Corollarium 3.

677. Cum ex formula brachystochronismi indolem continente prodeat $v = \frac{Nr}{2}$, si hic valor ubique loco *v* in altera aequatione $dv = (T\text{-}R)ds$ substituatur, habebitur aequatio naturam curvae brachystochronae exhibens.

Corollarium 4.

678. In quocunque ergo medio resistente et quibuscunque sollicitantibus corpus potentiis eae curvae omnes erunt brachystochronae, in quibus tota, quam sustinent, pressio duplo maior est quam vel sola vis centrifuga vel sola ex potentiarum sollicitantium resolutione orta vis normalis. [p. 368]

PROPOSITIO 77.

Problema.

679. *In media uniformi, quod resistit in ratione quacunque multiplicata celeritatum, et potentia absoluta existente uniformi et deorsum directa determinare curvam brachystochronam AM* (Fig.74) *super qua corpus descendens tempore brevissimo ex ex A ad M perveniat.*

Solutio.

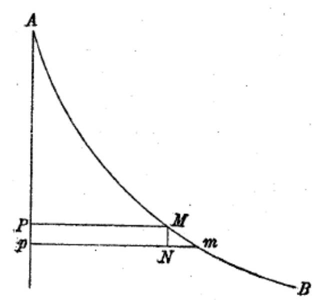


Fig. 74.

Positis in axe verticali abscissa $AP = x$ eique respondente applicata $PM = y$ arcuque curvae quaesitae $AM = s$ sit g potentia deorsum sollicitans et $\frac{v^m}{k^m}$ resistentia in M, si quidem celeritas in M fuerit debita altitudini v. His positis erit vis normalis = $\frac{gdy}{ds}$; cui aequalis esse debet vis centrifuga, quae est

$$\frac{2v}{r} = \frac{2vdxddy}{ds^3}$$

(676) sumto dx pro constante. Facta ergo aequatione est

$$v = \frac{gds^2dy}{2dxddy}.$$

Aequatio vero canonica pro descensu in hoc medio resistente dat

$$dv = gdx - \frac{v^m ds}{k^m}.$$

Prior autem aequatio posito $dsdds$ loco $dyddy$ propter dx constans abit in hanc

$$v = \frac{gdsdy^2}{2dxdds},$$

ex qua fit

$$dv = \frac{gdy^2}{2dx} + \frac{gdsdyddy}{dxdds} - \frac{gdsdy^2d^3s}{2dxdds^2} = \frac{gdy^2}{2dx} + \frac{gds^2}{dx} - \frac{gdsdy^2d^3s}{2dxdds^2} = gdx - \frac{v^m ds}{k^m},$$

quae aequatio reducta dat

$$\frac{gdsdy^2d^3s}{2dxdds^2} - \frac{3gdy^2}{2dx} = \frac{g^m ds^{m+1} dy^{2m}}{2^m k^m dx^m dds^m}$$

seu [p. 369]

$$dsd^3s - 3dds^2 = \frac{g^{m-1} ds^{m+1} dy^{2m-2}}{2^{m-1} k^m dx^{m-1} dds^{m-2}};$$

haecque aequatio exponit naturam curvae quaesitae. Quae aequatio quo reducatur et ad constructionem praeparetur, pono $ds = pdx$, ut sit

$$dy = dx\sqrt{(p^2 - 1)},$$

eritque

$$dds = dpdx \text{ et } d^3s = dxddp.$$

His substitutis habebitur ista aequatio

$$pddp - 3dp^2 = \frac{g^{m-1}p^{m+1}dx^m(pp-1)^{m-1}}{2^{m-1}k^m dp^{m-2}}.$$

Nunc sit porro $dx = qdp$, erit

$$ddx = 0 = dqdp + qddp \text{ seu } ddp = \frac{-dpdq}{q}$$

orietur haec aequatio

$$-\frac{pdq}{q} - 3dp = \frac{g^{m-1}p^{m+1}q^m dp(p^2-1)^{m-1}}{2^{m-1}k^m}$$

seu

$$\frac{-pdq - 3qdp}{q^{m+1}} = \frac{g^{m-1}p^{m+1}dp(p^2-1)^{m-1}}{2^{m-1}k^m}.$$

Multiplicetur haec aequatio, quo integrabilis fiat, per mp^{-3m-1} et habebitur

$$-mp^{-3m}q^{-m-1}dq - 3mp^{-3m-1}q^{-m}dp = \frac{mg^{m-1}p^{-2m}dp(p^2-1)^{m-1}}{2^{m-1}k^m},$$

cuius integralis est

$$p^{-3m}q^{-m} = \frac{mg^{m-1}}{2^{m-1}k^m}\int\frac{(p^2-1)^{m-1}dp}{p^{2m}}.$$

Ponatur

$$\frac{mg^{m-1}}{2^{m-1}k^m}\int\frac{(p^2-1)^{m-1}dp}{p^{2m}} = P^{-m};$$

erit P functio quaedam ipsius p et proinde dabitur, concessis saltem quadraturis. His igitur positis erit

$$p^3q = P \text{ atque } q = \frac{P}{p^3}.$$

Quia vero est $dx = qdp$, erit [p. 370]

$$x = \int\frac{Pdp}{p^3} \quad \text{et} \quad s = \int\frac{Pdp}{p^2} \quad \text{atque} \quad y = \int\frac{Pdp\sqrt{(p^2-1)}}{p^3}.$$

Unde constructio curvae brachystochronae sequitur. Q.E.I.

Corollarium 1.

680. Sit A punctum, in quo motus incipit atque celeritas est nulla; erit ibi $v = 0$ seu

$$\frac{gdsdy^2}{2dxdds} = 0,$$

unde fit $dy = 0$, quia ds evanescere non potest. In puncto A ergo curva habebit tangentem verticalem.

Corollarium 2.

681. Quia in ipso motus initio motus in medio resistente a motu in vacuo non discrepat, curvae AM initium A a cycloidis cuspide, quae est brachystochrona in vacuo, non discrepabit. Ideoque in A non solum tangens erit verticalis, sed etiam radius osculi in eo loco infinite parvus.

Corollarium 3.

682. Quia in A est $dy = 0$ atque est $dy = dx\sqrt{(p^2-1)}$, erit pro puncto A $p = 1$. Ex data ergo curvae constructione punctum A obtinebitur, si fiat $p = 1$. Integralia ergo illa ita debebunt accipi, ut x, s et y evanescant positio $p = 1$.

Corollarium 4.

683. Quoniam est

$$v = \frac{gdsdy^2}{2dxdds},$$

erit propter $ds = pdx$ et $dds = dpdx$

$$v = \frac{gpdx(p^2-1)}{2dp}$$

atque ob $dx = qdp$ erit [p. 371]

$$v = \frac{gpq(pp-1)}{2} = \frac{gP(pp-1)}{2p^2}.$$

Unde patet v evanescere, si sit $p = 1$.

Corollarium 5.

684. Radius osculi in puncto quocunque M est =

$$\frac{ds^3}{dxddy} = \frac{ds^2dy}{dxdds}.$$

Quare ob $ds = pdx$ erit radius osculi

$$r = \frac{p^2dx\sqrt{(p^2-1)}}{dp} = p^2q\sqrt{(p^2-1)} = \frac{P\sqrt{(p^2-1)}}{p}.$$

In puncto ergo A, ubi est $p = 1$, erit radius osculi $r = 0$.

Corollarium 6.

685. Sit B punctum brachystochronae, in quo tangens est horizontalis; erit ibi $dy = \infty$ ideoque $p = \infty$. Punctum igitur B invenietur ponendo $p = \infty$. Erit ergo in hoc puncto $v = \frac{gP}{2}$ et radius osculi $r = P$.

Exemplum 1.

686. Ponamus resistentiam evanescentem, ita ut motus fiat in vacuo; erit $k=\infty$ ideoque habebitur

$$dsds^3-3dds^2=0.$$

Quae aequatio divisa per *dsdds* et integrata dat

$$ldds-3lds=lC$$

seu

$$\frac{dds}{ds^3}=\frac{1}{adx}=\frac{dx}{adx^2}.$$

Haec aequatio denuo integrata dat

$$-\frac{1}{2ds^2}=\frac{x}{adx^2}+C.$$

Vel mutatis constantibus positoque $ds=pdx$ erit $-a=ppx+Cpp;$ quia posito $p=1$ x debet evanescere, abit in hanc

$$x=\frac{a(pp-1)}{pp} \text{ seu } p=\frac{\sqrt{a}}{\sqrt{(a-x)}} \text{ ideoque } ds=\frac{dx\sqrt{a}}{\sqrt{(a-x)}},$$

quae est aequatio pro cycloide, ut constat.

Exemplum 2. [p. 372]

687. Resistat medium in duplicata ratione celeritatum; erit $m=1$ atque

$$\frac{1}{P}=\frac{1}{k}\int\frac{dp}{p^2}=C-\frac{1}{kp}.$$

Unde fit

$$P=\frac{kp}{Ckp-1}=\frac{akp}{kp-a}.$$

Hanc ob rem erit

$$x=\int\frac{akdp}{p^2(kp-a)} \quad \text{et} \quad s=\int\frac{akdp}{p(kp-a)}.$$

Fit ergo

$$s=kl\frac{kp-a}{(k-a)p} \quad \text{atque} \quad e^{\frac{s}{k}}=\frac{kp-a}{(k-a)p}=\frac{kds-adx}{(k-a)ds}$$

ob $ds=pdx$. Porro ergo habebitur

$$(k-a)e^{\frac{s}{k}}ds=kds-adx,$$

quae integrata dat

$$k(k-a)e^{\frac{s}{k}} = ks - ax + k(k-a).$$

Vel eliminata quantitate exponentiali $e^{\frac{s}{k}}$ erit

$$ksds - axds - akds + akdx = 0.$$

At si exponentialem $e^{\frac{s}{k}}$ per seriem exprimere velimus, erit

$$k(k-a)e^{\frac{s}{k}} - k(k-a) = k(k-a)\left(\frac{s}{k} + \frac{ss}{1\cdot 2k^2} + \frac{s^3}{1\cdot 2\cdot 3k^3} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4k^4} + \text{ etc.}\right).$$

Quae series substituta dat

$$\frac{a(s-x)}{k-a} = \frac{s^2}{1\cdot 2k} + \frac{s^3}{1\cdot 2\cdot 3k^2} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4k^3} + \text{ etc.}$$

In quovis puncto *M* est

$$v = \frac{gak(pp-1)}{2p(kp-a)}.$$

Pro puncto *B* vero, in quo tangens est horizontalis, erit

$s = kl\frac{k}{k-a}$ atque $e^{\frac{s}{k}} = \frac{k}{k-a}$

et idcirco

$x = -k + \frac{kk}{a}l\frac{k}{k-a}$.

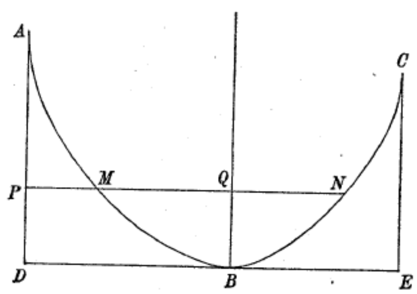


Fig. 77.

Continuetur nunc curva *B* in *BNC* (Fig. 77) ; cuius natura ut inveniatur, in axe *BQ* ponatur abscissa *BQ* = *t* et arcus *BN* = *z*. His positis erit

$$AP = x = -k - t + \frac{k^2}{a}l\frac{k}{k-a} \quad \text{et} \quad AMN = s = z + kl\frac{k}{k-a}.$$

Erit ergo

$$e^{\frac{s}{k}} = \frac{k}{k-a}e^{\frac{z}{k}};$$

quibus valoribus in superiore aequatione substitutis prodibit

$$k^2e^{\frac{z}{k}} = kz + k^2 + at \quad \text{seu} \quad at = k^2(e^{\frac{z}{k}} - 1) - kz.$$

Atque per seriem [p. 373]

$$at = \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3k} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4k^2} + \text{etc.}$$

pro curva *BNC*; at pro ramo *BMA*, in quo erit arcus $BM = z$ negativus, erit

$$at = k^2(e^{\frac{-z}{k}} - 1) + kz = \frac{z^2}{1\cdot 2} - \frac{z^3}{1\cdot 2\cdot 3k} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4k^2} - \text{etc.}$$

Curva vero BNC in C habebit quoque tangentem verticalem, quod punctum invenitur ponendo $dz = dt$. Fiet vero hoc posito

$$a = ke^{\frac{z}{k}} - k \quad \text{seu} \quad z = kl\frac{a+k}{k} = BNC$$

atque

$$t = CE = k - \frac{kk}{a}l\frac{a+k}{k} = \frac{a}{2} - \frac{a^2}{3k} + \frac{a^3}{4k^2} - \text{etc.},$$

cum contra sit

$$AD = \frac{a}{2} + \frac{a^2}{3k} + \frac{a^3}{4k^2} + \text{etc.}$$

Ex quo apparet punctum *A* esse altius positum quam punctum *C* atque in *A* in *C* curvam habere cuspides seu puncta reversionis, ita ut tam *AD* quam CE sint curvae diametri; id quod ex hoc intelligitur, quod sit

$$y = \int \frac{Pdp\sqrt{(pp-1)}}{p^3},$$

ubi $\sqrt{(p^2-1)}$ valorem habet tam affirmativum quam negativum.

Scholion 1.

688. Infra perspicietur hanc curvam brachystochronam congruere cum curva tautochrona in eadem resistentiae hypothesi. Haec vero inter motus tautochronos et brachychronos interest differentia, ut ad tautochronismum obtinendum corpus in ramo *CNB* descendere, in altero ascendere debeat, cum e contrario pro brachystochronismo per *AMB* fieri debeat. Interim tamen haec utriusque curvae convenientia attentione digna videtur, cum et in vacuo eadem congruentia observetur. [p. 374]

Exemplum 3.

689. Resistat medium in quadruplicata ratione celeritatum, ita ut sit $m = 2$. Habebitur ergo pro curva quaesita ista aequatio

$$dsd^3s - 3dds^2 = \frac{gds^3dy^2}{2k^2dx},$$

ad construendam vero curvam

$$\frac{1}{P^2} = \frac{g}{k^2}\int\left(\frac{dp}{p^2} - \frac{dp}{p^4}\right);$$

unde fit

$$P = \frac{kp\sqrt{3np}}{\sqrt{g}(p^3 - 3np^2 + n)}$$

atque

$$s = \int \frac{kdp\sqrt{3n}}{\sqrt{g(p^4 - 3np^3 + np)}}$$

et

$$x = \int \frac{kdp\sqrt{3n}}{p\sqrt{g(p^4 - 3np^3 + np)}} \quad \text{ac} \quad y = \int \frac{kdp\sqrt{3n(pp-1)}}{p\sqrt{g(p^4 - 3np^3 + np)}}.$$

Huius ergo curvae constructio uti generalis habetur. Quia autem n numerum quemcunque denotat, sit $n = \frac{1}{2}$;erit

$$y = \int \frac{kdp\sqrt{3}}{p\sqrt{g(2p^2 - p)}} = \frac{2k\sqrt{3(2p-1)}}{\sqrt{gp}} - \frac{2k\sqrt{3}}{\sqrt{g}};$$

quam constantem ideo adiecimus, quo fiat y = 0 posito p = 1.
[Posito $n = \frac{1}{2}$ erit

$$y = \int \frac{kdp\sqrt{3(pp-1)}}{p\sqrt{g(2p^4 - 3p^3 + p)}},$$

quae formula, quia $2p^4 - 3p^3 + p$ factorem $pp - 1$ non continet, reductionem ab Eulero factam non admittit. Itaque etiam formuale sequentes locum non habent. P. St.]
$p = \infty$ erit applicata

$$DB = \frac{2k(\sqrt{6} - \sqrt{3})}{\sqrt{g}}.$$

Fit autem

$$p = \frac{12k^2}{12k^2 - 4ky\sqrt{3g} - gy^2}$$

atque

$$\sqrt{(p^2 - 1)} = \frac{dy}{dx} = \frac{\sqrt{(96k^3y\sqrt{3g} - 24gk^2y^2 - 8gky^3\sqrt{3g} - g^2y^4)}}{12k^2 - 4ky\sqrt{3g} - gy^2}.$$

Ex quo oritur

$$x = \int \frac{12k^2dy - 4kydy\sqrt{3g} - gy^2dy}{\sqrt{(96k^3y\sqrt{3g} - 24gk^2y^2 - 8gky^3\sqrt{3g} - g^2y^4)}}$$

$$= \int \frac{12k^2dy - 4kydy\sqrt{3g} - gy^2dy}{\sqrt{(gy^2 + 4ky\sqrt{3g})(24k^2 - 4ky\sqrt{3g} - gy^2)}},$$

quae est aequatio inter coordinatas x et y pro curva quaesita.

Scholion 2. [p. 375]

690. In medio, quod in simplici celeritatum ratione resistit, brachystochronam simplicius determinare non licet, quam statim ex universali constructione consequitur. Quamobrem hunc resistentiae casum exemplo non sumus prosecuti. Quod autem ad reliquas huc perintentes propositiones attinet, in quibus curva quaeritur, super qua corpus descendens citissime ad datam lineam, sive rectam sive curvam, perveniat, ea simili modo pro resistente medio solvuntue quo pro vacuo. Cum scilicet ex eodem puncto A innumerabiles egrediantur curvae brachystochronae, ex iis ea es eligenda, quae datae lineae, sive rectae sive curvae, ad angulos rectos occurrat; super hac enim corpus ad istam lineam brevissimo tempore pervenire capite praecedente est demonstratum. Simili ratione curva, quae omnes brachystochronas ad angulos rectos traiicit, ab omnibus arcus abscindet isochronos seu quos corpus descendens aequalibus temporibus absolvit. Haecque omnia eodem se havent modo, quaecunque fuerit resistentia et quaecunque potentiae absolutae. Problema autem brachystochronarum generalissime conceptum evolvemus.