CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 318]

PROPOSITIO 68.

Problem.

602. *In a uniform medium, which resists in the ratio of the quadruple of the speeds, so to determine any ascent or descent of the body on the cycloid ACB* (Fig.66).

Solution .

The body is continually drawn downwards by a uniform force g, and with the abscissa $CP = x$ and the arc $CM = s$, from the nature of the cycloid, it follows that $dx = \frac{sds}{a}$. Let the speed at C correspond to the height b and at M to the height v, and k is the exponent of the resistance ; the resistance at $M = \frac{v^2}{k^2}$. [p. 319] For the descent, this equation is therefore obtained :

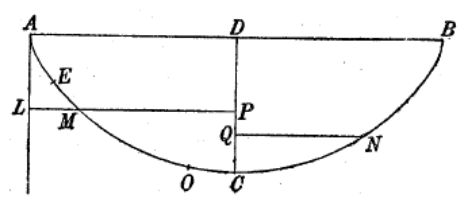

Fig. 66.

$$dv = -gdx + \frac{v^2ds}{k^2} = -\frac{gsds}{a} + \frac{v^2ds}{k^2},$$

and thus for the ascent, this equation :

$$dv = -\frac{gsds}{a} - \frac{v^2ds}{k^2}.$$

For the descent, there is put in place :

$$v = \frac{-k^2dz}{zds};$$

then the equation arises :

$$dv = \frac{-k^2ddz}{zds} + \frac{k^2dz^2}{z^2ds}$$

on putting ds constant. On account of which there is obtained :

$$k^2ddz = \frac{gszds^2}{a};$$

which equation, converted into a series gives :

$$z = f + hs + \frac{fgs^3}{2\cdot 3\,ak^2} + \frac{hgs^4}{3\cdot 4\,ak^2} + \frac{fg^2s^6}{2\cdot 3\cdot 5\cdot 6\,a^2k^4} + \frac{hg^2s^7}{3\cdot 4\cdot 6\cdot 7\,a^2k^4} + \frac{fg^3s^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\,a^3k^6} + \text{etc.}$$

The value of *v* is sought on putting s = 0 in order that the constants *f* and *h* can be found ; hence

$$b = -\frac{k^2h}{f}.$$

And hence, if $s = 0$, there is given

$$dv = \frac{v^2ds}{k^2} = \frac{b^2ds}{k^2};$$

on account of $dv = \frac{-k^2ddz}{zds} + \frac{k^2dz^2}{z^2ds}$ then

$$\frac{b^2}{k^2} = \frac{k^2h^2}{ff},$$

which equation agrees with the former; hence $v = \frac{-k^2dz}{zds}$ becomes with $h = -\frac{bf}{k^2}$ substituted, then :

$$v = \frac{b - \frac{gs^2}{2a} + \frac{bgs^3}{3ak^2} - \frac{g^2s^5}{2\cdot 3\cdot 5a^2k^2} + \frac{bg^2s^6}{3\cdot 4\cdot 6a^2k^4} - \text{etc.}}{1 - \frac{bs}{k^2} + \frac{gs^3}{2\cdot 3ak^2} - \frac{bgs^4}{3\cdot 4ak^4} + \frac{g^2s^6}{2\cdot 3\cdot 5\cdot 6a^2k^4} - \text{etc.}}.$$

For the ascent now on putting –*s* in place of *s* it is found that :

$$v = \frac{b - \frac{gs^2}{2a} - \frac{bgs^3}{3ak^2} + \frac{g^2s^5}{2\cdot 3\cdot 5a^2k^2} + \frac{bg^2s^6}{3\cdot 4\cdot 6a^2k^4} - \text{etc.}}{1 + \frac{bs}{k^2} - \frac{gs^3}{2\cdot 3ak^2} - \frac{bgs^4}{3\cdot 4ak^4} + \frac{g^2s^6}{2\cdot 3\cdot 5\cdot 6a^2k^4} + \text{etc.}}.$$

From these equations the whole arc is found either for the ascent or the descent, if we put $v = 0$ and the value of *s* is found. As if *k* is made a very large quantity, the arc of the descent, which is *E*, is equal to :

$$\frac{\sqrt{2ab}}{\sqrt{g}} + \frac{8ab^2}{15gk^2} + \frac{301a^2b^4\sqrt{g}}{450g^2k^4\sqrt{2ab}} + \text{etc.}$$

But the following arc of the ascent, which is *F*, is equal to :

$$\frac{\sqrt{2ab}}{\sqrt{g}} - \frac{8ab^2}{15gk^2} + \frac{301a^2b^4\sqrt{g}}{450g^2k^4\sqrt{2ab}} - \text{etc.}$$

Q.E.I. [These final two expressions are the corrected values taken from the *O.O.*, Book II, p. 295.]

Corollary 1. [p. 320]

603. Therefore if the resistance is as the smallest value, then the sum of the arcs of the descent and of the ascent, or the arc for one semi-oscillation described, *i. e.*

$$E+F=\frac{2\sqrt{2ab}}{\sqrt{g}}\text{, approx.}$$

Corollary 2.

604. But the difference between the arcs of the ascent and of the descent, clearly is equal to :

$$E-F=\frac{16ab^2}{15gk^2}=\frac{g(E+F)^4}{60ak^2}.$$

Whereby the difference between the arc of the descent and the ascent is as the biquadratic [fourth power] of the sum of the arcs.

Scholium 1.

605. From these it is evident in the rarest medium, which resists as the quadruple of the ratio of the speeds, the difference between the arc of the ascent and of the descent is proportional to the biquadratic of the sum of the arcs, or

$$E-F=\frac{g(E+F)^4}{60ak^2}.$$

Moreover above we saw in the rarest medium, which resisted in the ratio of the square of the speeds, that the arc of the descent

$$E=\frac{\sqrt{2ab}}{\sqrt{g}}+\frac{2ab}{3gk}$$

and the arc of the ascent

$$F=\frac{\sqrt{2ab}}{\sqrt{g}}-\frac{2ab}{3gk};$$

hence this gives

$$E+F=\frac{2\sqrt{2ab}}{\sqrt{g}}\text{ and }E-F=\frac{4ab}{3gk}=\frac{(E+F)^2}{6k}$$

(557). Whereby in this resistance the difference between the arc of the ascent and the arc of the descent is as the square of the sum of the arcs. And in a medium that resists in the simple ratio of the speeds, if it were the rarest, is

$$E=\frac{\sqrt{2ab}}{\sqrt{g}}+\frac{\pi a\sqrt{b}}{4g\sqrt{k}}\text{ and }F=\frac{\sqrt{2ab}}{\sqrt{g}}-\frac{\pi a\sqrt{b}}{4g\sqrt{k}}$$

(582). Whereby :

$$E-F=\frac{\pi a\sqrt{b}}{2g\sqrt{k}}=\frac{\pi(E+F)\sqrt{a}}{4\sqrt{2kg}}.$$

Or, the difference between the arc of the descent and the arc of the ascent is proportional to the sum of the arcs. [p. 321] From which it is seen to be a consequence in any rarest medium, that resists in the ratio of the $2m$ th power of the speeds, the difference between the arc of the ascent and of the descent on a cycloid is proportional to the power of the sum of the arcs of the ascent and of the descent, and the exponent of this is $2m$. And for this hypothesis of the resistance it is allowed to assign the arc of the descent :

$$E=\frac{\sqrt{2ab}}{\sqrt{g}}+\frac{2\cdot 4\cdot 6\cdots 2mab^m}{3\cdot 5\cdot 7\cdots(2m+1)gk^m}$$

and the arc of the ascent :

$$F=\frac{\sqrt{2ab}}{\sqrt{g}}-\frac{2\cdot 4\cdot 6\cdots 2mab^m}{3\cdot 5\cdot 7\cdots(2m+1)gk^m}.$$

Hence this gives :

$$E-F=\frac{1\cdot 2\cdot 3\cdots mg^{m-1}(E+F)^{2m}}{3\cdot 5\cdot 7\cdots(2m+1)\,2^{2m-1}a^{m-1}k^m}.$$

Hence whenever m is an whole number or $2m$ is an even number, it is possible for an equation to be assigned between $E-F$ et $E+F$, but if m is a fractional number, the value of the fraction $\frac{1.2.3......m}{3.5.7....(2m+1)}$ can be investigated by the method of interpolation, as shown in the Comment. Acad. Petrop. A 1730 [L. Euler Comm. 19 (E19) : *Concerning progressions of transcendentals, or of these general terms that are unable to be given algebraically.*] From which indeed it is agreed, that if $2m$ is an odd number, then the value of this fraction involves the quadrature of the circle, as also we have found in the case in which $2m = 1$.

Scholium 2. [p. 322]

606. Because indeed it touches on that proposition that Newton demonstrated in the *Principia* [Book. II, Prop. XXXI] which is, that the difference between the arcs of the descent and of the ascent on a cycloid is as the sum of the arcs raised to the whole power of whatever power the resistance present has in proportion to the ratio of the speeds, if indeed the resistance is the smallest. And it is also possible to derive this from the equation

$$dv=-\frac{gsds}{a}\pm\frac{v^m ds}{k^m}.$$

For on putting

$$v=b-\frac{gs^2}{2a}+Q,$$

where Q is a very small quantity besides b and $\frac{gs^2}{2a}$. Hence on this account there is found:

$$-\frac{gsds}{a} + dQ = \frac{-gsds}{a} + \frac{\left(b - \frac{gs^2}{2a}\right)^m ds}{k^m}$$

for the descent, or

$$Q = \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}$$

thus with this integral taken, so that it vanishes on putting $s = 0$. Hence for the descent it becomes :

$$v = b - \frac{gs^2}{2a} + \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}$$

and for the ascent :

$$v = b - \frac{gs^2}{2a} - \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}.$$

Put $v = 0$, and since $s = \frac{\sqrt{2ab}}{\sqrt{g}}$ approximately, put $s = \frac{\sqrt{2ab}}{\sqrt{g}} + q;$ then

$$0 = \frac{-gq\sqrt{2ab}}{a\sqrt{g}} + \int \frac{(2ab - gss)^m ds}{(2ak)^m}$$

if indeed after the integration there is put in place $s = \frac{\sqrt{2ab}}{\sqrt{g}}$. But since in this equation, the value of q has the dimensions $2m$ of $\sqrt{b}$ and s, then q has a form of this kind Nb^m. On account of which the arc of the descent is given by :

$$E = \frac{\sqrt{2ab}}{\sqrt{g}} + Nb^m$$

and the arc of the ascent by :

$$F = \frac{\sqrt{2ab}}{\sqrt{g}} - Nb^m.$$

Hence there is obtained :

$$E - F = 2Nb^m = \frac{Ng^m(E+F)^{2m}}{2^{3m-1}a^m}.$$

But the number N can be obtained from the formula [p. 323] :

$$\sqrt{\frac{a}{2gb}} \int \left(\frac{2ab - gss}{2abk}\right)^m ds,$$

if after the integration there is put in place : $s = \frac{\sqrt{2ab}}{\sqrt{g}}$. And this is the demonstration of that that we derived by induction in the previous scholium. For N is a rational number, as often as m is a positive integer ; but if $2m$ is an odd integer, finding the number N depends on the quadrature of the circle [by this Euler means of course the evaluation of trigonometric integrals]. Moreover generally the value of q agrees with this expression :

$$\frac{2 \cdot 4 \cdot 6 \cdots 2mab^m}{3 \cdot 5 \cdot 7 \cdots (2m+1)gk^m}.$$

PROPOSITION 69.

Problem.

607. *In a medium that resists in the ratio of the quadruple power of the ratio of the speeds, if the speed of the descent of the body on the curve AMC from a given point A is given* (Fig.67) *at individual points, to find the speed of the same body beginning at some other point E.*

Solution.

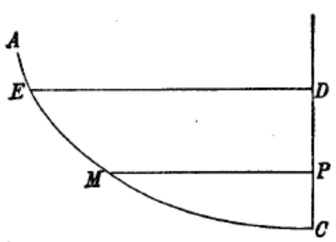

Fig. 67.

On putting $CP = x$ and $CM = s$ let the speed of the body fallen from A at M correspond to the height u, which quantity u by hypothesis is given by x and s. Now if the body starts the descend from some other point E, let the speed at M correspond to the height v. Now the equation determining the motion is :

$$dv = -gdx + \frac{v^2 ds}{k^2},$$

which gives the value of v, wherever the descent should start; hence it becomes also : [p. 324]

$$du = -gdx + \frac{u^2 ds}{k^2}.$$

On putting $v = u - q$, then

$$du - dq = -gdx + \frac{u^2 ds}{k^2} - \frac{2quds}{k^2} + \frac{q^2 ds}{k^2};$$

from which equation because

$$du = -gdx + \frac{u^2 ds}{k^2}$$

there arises :

$$-dq = -\frac{2quds}{k^2} + \frac{q^2 ds}{k^2} \text{ or } -\frac{dq}{q^2} + \frac{2uds}{k^2 q} = \frac{ds}{k^2},$$

which multiplied by $e^{\frac{2\int uds}{k^2}}$ gives this integral :

$$e^{\frac{2\int uds}{k^2}} = cq + q\int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2},$$

from which there is produced [with the meaning of the letter *c* changed]

$$q = \frac{k^2 e^{\frac{2\int uds}{k^2}}}{c + \int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2}}.$$

On account of which :

$$v = u - \frac{k^2 e^{\frac{2\int uds}{k^2}}}{c + \int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2}},$$

in which equation, the integrals

$$\frac{2\int uds}{k^2} \text{ and } \int e^{\frac{2\int uds}{k^2}} ds$$

thus are taken, so that they vanish on putting $s = 0$. Now let the height corresponding to the speed at *C* be equal to *a*, if the descent is made from *A*, but the height corresponding to the speed at *C*, if the descent is made from *E* is equal to *b*; then $b = a - \frac{k^2}{c}$. From which there is obtained

$$v = u - \frac{(a-b)k^2 e^{\frac{2\int uds}{k^2}}}{k^2 + (a-b)\int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2}}$$

Therefore from the given speed at C, clearly $\sqrt{b}$, there is found the point E, at which the descent starts, from this equation :

$$u = \frac{(a-b)k^2 e^{\frac{2\int uds}{k^2}}}{k^2 + (a-b)\int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2}},$$

from which the value of *s* gives the arc *CME*. Therefore because *u* is given through *s*, from this equation the speed of the body fallen from any other point on the curve *AMC* can be found. Q.E.I.

Corollary 1. [p. 325]

608. If the value of v is thus changed, in order that both in the numerator and in the denominator, b appears without a coefficient, there is produced :

$$v=\frac{\left(u\int e^{\frac{2\int uds}{k^2}}ds-k^2e^{\frac{2\int uds}{k^2}}\right)}{\int e^{\frac{2\int uds}{k^2}}ds}\cdot\frac{\left(b-a+\frac{k^2u}{k^2e^{\frac{2\int uds}{k^2}}-u\int e^{\frac{2\int uds}{k^2}}ds}\right)}{\left(b-a-\frac{k^2}{\int e^{\frac{2\int uds}{k^2}}ds}\right)}.$$

And it is the case that $v = 0$, if

$$a+\frac{k^2u}{u\int e^{\frac{2\int uds}{k^2}}ds-k^2e^{\frac{2\int uds}{k^2}}}=b.$$

Corollary 2.

609. Since we have

$$du-\frac{u^2ds}{k^2}=-gdx,$$

the integral of this equation multiplied by $e^{\frac{-\int uds}{k^2}}$ becomes :

$$u=ae^{\frac{-\int uds}{k^2}}-ge^{\frac{\int uds}{k^2}}\int e^{\frac{-\int uds}{k^2}}dx$$

with the integrals thus taken, so that they disappear on putting s or $x = 0$. Or also there is

$$\frac{du}{u}+\frac{gdx}{u}=\frac{uds}{k^2}$$

and hence

$$\int\frac{uds}{k^2}=l\frac{du}{u}+\int\frac{gdx}{u}$$

Whereby this becomes :

$$e^{\int\frac{uds}{k^2}}=\frac{e^{\int\frac{gdx}{u}}u}{a};$$

thus dx can be introduced in place of ds in the above equation.

Corollary 3. [p. 326]

610. If the resistence werw so small that $\int e^{\frac{2\int uds}{k^2}} ds$ vanishes before k^2 and thus it becomes :

$$v = u - (a-b)e^{\frac{2\int uds}{k^2}} = u - (a-b)\left(1 + \frac{2\int uds}{k^2}\right)$$

on account of the large quantity k. On account of which :

$$v = b + \frac{2b\int uds}{k^2} - a - \frac{2a\int uds}{k^2} + u = \left(1 + \frac{2\int uds}{k^2}\right)\left(b - a + \frac{u}{1 + \frac{2\int uds}{k^2}}\right)$$

Corollary 4.

611. Since moreover,

$$\sqrt{\left(1 + \frac{2\int uds}{k^2}\right)} = 1 + \frac{\int uds}{k^2},$$

an element of the time becomes :

$$\frac{ds}{\sqrt{v}} = \frac{k^2 ds}{(k^2 + \int uds)\sqrt{\left(b - a + \frac{k^2 u}{k^2 + 2\int uds}\right)}}.$$

But by the equation :

$$du = -gdx + \frac{u^2 ds}{k^2}$$

which is approximately

$$u = a - gx + \int \frac{(a-gx)^2 ds}{k^2},$$

hence

$$\int uds = \int (a-gx)ds$$

and

$$\frac{ds}{\sqrt{v}} = \frac{k^2 ds}{(k^2 + \int (a-gx)ds)\sqrt{\left(b - a + \frac{k^2 a - gk^2 x + \int (a-gx)^2 ds}{k^2 + 2\int (a-gx)ds}\right)}}$$

$$= \frac{k^2 ds}{(k^2 + \int (a-gx)ds)\sqrt{\left(b - \frac{gk^2 x - \int (a^2 - g^2 x^2)ds}{k^2 + 2\int (a-gx)ds}\right)}}.$$

Scholium. [p. 327]

612. As the hypothesis with the resistance proportional to the squares of the speeds, before other hypothesis except that in which the resistance is constant, has this outstanding property, that the speed of any body moving on any curve can be defined at all places from the equation of the curve, thus this hypothesis of the resistance excels in this respect before the others, that from a single ascent or descent, likewise all the descents and ascents can be determined. For with the other resistances that we have used here, the operation does not succeed, nor can an equation be deduced from which the indeterminates can be separated from each other. On this account the use of a constant resistance is the most simple, and next follows that in which the resistance is proportional to the squares of the speed, and thus after this, the simplest resistance to be had is that in the ratio of the fourth power of the speeds. Indeed it is seen from these that little of convenience is to be obtained for the motion, by defining the resistance according to this hypothesis, since one descent is taken as given, which moreover is found with as much difficulty as any another. But if more descents are considered and compared with each other, the equation

$$dv = -gdx + \frac{v^2 ds}{k^2}$$

itself implies three variables, clearly besides v and s or x, the speed at the point C which changes in different descents. Whereby with the resolution of this equation we are led back to the other equation

$$du = -gdx + \frac{u^2 ds}{k^2},$$

which is observed for one descent, that was inconveniently carried out by the three variables in that manner. Besides with the help of that art, more descents can be compared with each other, which indeed under other hypothesis of the resistance cannot happen. And hence also many inverse problems under this hypothesis of the resistance can be resolved, which are unable to be treated in all the others. [p. 328]

PROPOSITION 70.

Problem.

613. *If the resistance were taken as very small with respect to the force acting and proportional to some power of the speeds, to determine the motion of the body on some curve AM* (Fig.68).

Solution.

The body descends on the curve *AM* with the start of the descent being from *A* ; the abscissa $AP = x$ is put on the vertical axis, the arc $AM = s$, and the force always pulling downwards is equal to *g*. Let the speed at *M* correspond to the height *v*, and the resistance at that place is equal to $\frac{v^m}{k^m}$, thus so that the resistance is proportional to the 2*m* th power of the speed. With these in place, the [governing equation] is :

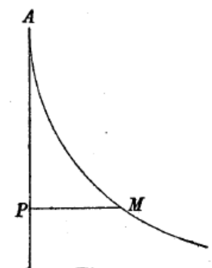

Fig. 68.

$$dv = gdx - \frac{v^m ds}{k^m},$$

but since the resistance is put very small, then the term $\frac{v^m ds}{k^m}$ is exceedingly small and on that account $v = gx$ as an approximation. Substitute *gx* in place of *v* in the term $\frac{v^m ds}{k^m}$; then [p. 329]

$$v = gx - \frac{g^m}{k^m}\int x^m ds$$

and in a like manner a closer [value is]

$$v = gx - \frac{g^m}{k^m}\int x^m ds + \frac{mg^{2m-1}}{k^{2m}}\int x^{m-1} ds \int x^m ds.$$

Which integrals are thus to be taken, so that they vanish on putting $x = 0$. Hence therefore:

$$\frac{1}{\sqrt{v}} = \frac{1}{\sqrt{gx}} + \frac{g^m \int x^m ds}{2k^m gx\sqrt{gx}} - \frac{mg^{2m}\int x^{m-1} ds \int x^m ds}{2k^{2m}g^2 x\sqrt{gx}} + \frac{3g^{2m}(\int x^m ds)^2}{8k^{2m}g^2x^2\sqrt{gx}} + \text{etc.}$$

And the time of the descent along *AM* is equal to :

$$\int \frac{ds}{\sqrt{gx}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int x^{-\frac{3}{2}} ds \int x^m ds$$

as an approximation. But if the descent as far as the fixed point *C* (Fig. 67) is desired, the initial descent is made from the point *E*, the vertical height of the point *E* above *C* is put as $CD = a$, the abscissa $CP = x$ and the arc $CM = s$; with which put in place this case is reduced to the above, if $a - x$ is put in place of x and $-ds$ in place of ds. Whereby if the height corresponding to the speed at M is called v, then

$$v = g(a-x) + \frac{g^m}{k^m}\int (a-x)^m ds + \frac{m g^{2m-1}}{k^{2m}}\int (a-x)^{m-1} ds \int (a-x)^m ds$$

as an approximation. Now these integrals are to be taken, so that they vanish on putting $x = a$. And the time to travel along the arc EM is equal to

$$-\int \frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds$$

as an approximation, where again all the integrals are thus to be taken so that they vanish on putting $x = a$. [p. 330] In a similar manner, if the body from C on the curve CME ascends with such a speed, that it is able to reach as far as the point E, the same equations are in place, only if in place of k^m there is put $-k^m$. Hence on this account it becomes :

$$v = g(a-x) - \frac{g^m}{k^m}\int (a-x)^m ds + \frac{m g^{2m-1}}{k^{2m}}\int (a-x)^{m-1} ds \int (a-x)^m ds$$

and the time of the descent along ME is approximately equal to :

$$-\int \frac{ds}{\sqrt{g(a-x)}} - \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds$$

with all these integrals taken thus, so that they vanish on putting $x = a$. And in this manner both the descents and the oscillations of the body on any suitable curve in the rarest medium can be determined. Q.E.I.

Corollary 1.

614. It is apparent from these, as indeed it is understood by itself, if the body descends in some medium with resistance on the curve AM (Fig. 68), that the speed of the body at M is less than if the body descends along the same curve *in vacuo*. And the time in the resisting medium is more than the time the time of descent along AM *in vacuo*.

Corollary 2.

615. The height corresponding to the speed at the point C (Fig. 67) is produced, if $x = 0$ is put in the expression for v. But with this done it becomes

$$v = ga + \frac{g^m}{k^m}\int (a-x)^m ds$$

as an approximation with the above prescribed method of action $x = 0$ put in place after the integration. [p. 331] But if $\int (a-x)^m ds$ is thus taken, so that it vanishes on putting

$x = 0$, then at the point C there is :

$$v = ga - \frac{g^m}{k^m}\int (a-x)^m ds,$$

if after the integration there is put $x = a$. Since that pertains to the descent.

Corollary 3.

616. But for the ascent the speed of the body at C, by which it prevails to ascend as far as E, must correspond to a height equal to

$$ga + \frac{g^m}{k^m}\int (a-x)^m ds,$$

if this integral is thus taken, so that it vanishes on putting $x = 0$, and after the integration there is put $x = a$. [Euler's quaint way of describing definite integrals.]

Corollary 4.

617. Let the height corresponding to the speed of the body at $C = b$, as now it has acquired by descending along *EMC* and which ascends again on the same curve; the height *DC* from the descent traversed is put as before as a and the height to which by the ascent it reaches, $a - d$; then d is a very small quantity and thus

$$b = ga - \frac{g^m}{k^m}\int (a-x)^m ds = ga - gd + \frac{g^m}{k^m}\int (a-x)^m ds$$

and

$$d = \frac{2g^{m-1}}{k^m}\int (a-x)^m ds$$

or also

$$d = \frac{2ga - 2b}{g}.$$

Corollary 5. [p. 332]

618. Because the time for the descent along *EM* is equal to

$$-\int \frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds$$

with these integrals thus taken, so that they vanish on putting $x = a$, the time to traverse *MC* is equal to :

$$\int \frac{ds}{\sqrt{g(a-x)}} - \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds,$$

if the integrations are thus taken, so that they disappear on putting $x = 0$. And in a like manner with the ascent along *CM* , the time is equal to :

$$\int \frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds.$$

Corollary 6.

619. Hence the whole time or the time for the descent or the time for the ascent along *CME* is obtained, if in these last formulas is put $x = a$.

Scholium.

620. Now with these sufficiently well explained, which pertain to finding the motion of a body on a given curve, I progress to the inverse questions, in which from these quantities given, these which are unknown are investigated. And indeed at first problems of this kind occur, in which the law of the acceleration or the scale of the speeds is given and the curve is sought, which the motion of that scale agreed upon is produced in a medium with some kind of resistance; now we assume the absolute force as constant and acting downwards, as up to the present.

PROPOSITION 71. [p. 333]

Problem.

621. *In a medium which resists in the ratio of some power of the speeds, to find the curve AM* (Fig.69), *upon which the body thus descends, so that at the individual points M it has a speed corresponding to the height, which is equal to the corresponding applied line PL of the given curve BL.*

Solution.

On putting $AP = x$ and $PL = v$ the equation is given between x and v on account of the given curve *BL*. Now let the arc AM = s and the exponent of the ratio of the multiple of the speeds be equal to2*m*, to which the resistance is proportional ; with which put in place, there arises :

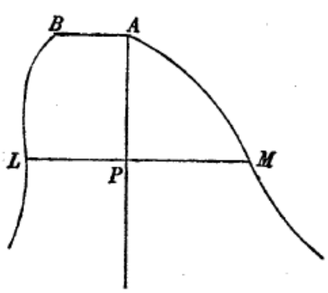

Fig. 69.

$$dv = gdx - \frac{v^m ds}{k^m},$$

with g denoting the uniform force pulling downwards and k the exponent of the resistance. Therefore from this equation it is found that

$$ds = \frac{gk^m dx - k^m dv}{v^m},$$

which, since v can be given in terms of x, the equation has the variables separated from each other and thus it is sufficient to construct the curve *AM* sought. But since ds must always be greater than dx, lest the curve *AM* becomes imaginary, it is required that [p. 334] $gk^m dx - k^m dv > v^m dx$ or

$$dx > \frac{k^m dv}{gk^m - v^m}.$$

And if where

$$dx = \frac{k^m dv}{gk^m - v^m},$$

there the tangent to the curve *AM* is vertical; and where

$$dx < \frac{k^m dv}{gk^m - v^m},$$

there the curve AM is unable to have any part. Q.E.I.

Corollary 1.

622. Since, where the tangent to the curve *BL* is vertical, is the case that *dv* = 0, then in the corresponding place to the curve *AM*

$$ds = \frac{gk^m dx}{v^m},$$

in which point the body descending has either a maximum or minimum speed. Therefore lest this point of the curve *AM* is imaginary, it is necessary that $ak^m > v^m$ or $v < k\sqrt[m]{g}$.

Corollary 2.

623. If the curve *BL* is incident at some point on the axis *AP*, so that *v* = 0 there, then in the corresponding place of the curve *AM*, it is the case that $ds = \propto$, if indeed *m* is a positive number. Therefore in this place the curve *AM* has a horizontal tangent, in which the curve comes to an end.

Corollary 3.

624. The curve *AM* can have a horizontal tangent somewhere ; in this place *dx* can vanish before *ds*. On this account it becomes :

$$ds = \frac{-k^m dv}{v^m}. \text{ [p. 335]}$$

From which it appears that in place of the curve *BL* with the corresponding applied line must decrease and there the tangent to the curve *BL* is horizontal, since *dv* is also infinitely greater than *dx*.

Corollary 4.

625. The tangent *AM* of the curve is vertical, as we see, where

$$dx = \frac{k^m dv}{gk^m - v^m}.$$

Therefore in order that the curve at the beginning A, where the speed is zero, has a vertical tangent, it is necessary there that *dv* = *gdx* or *v* = *gx*. Therefore in this case the tangent of the angle, that the curve *BL* at *A* makes with *AP*, is equal to *g*.

Corollary 5.

626. Since

$$ds = \frac{gk^m dx - k^m dv}{v^m},$$

then the element of time is given by

$$\frac{ds}{\sqrt{v}} = \frac{gk^m dx - k^m dv}{v^{m+\frac{1}{2}}}.$$

On account of which the descent time along *AM* is equal to :

$$\frac{2k^m}{(2m-1)v^{m-\frac{1}{2}}} + gk^m \int \frac{dx}{v^{m+\frac{1}{2}}}.$$

Scholion 1. [p. 336]

627. If the curve *BL* rises above *AB*, then the curve *AM* also inclines upwards and in place of the descent, for the problem can satisfy the ascent upon that curve. Thus indeed in this case let the abscissa *x* be negative and let the element of this is be *dx*; on which account we have this equation :

$$ds = \frac{-gk^m dx - k^m dv}{v^m},$$

which arises from the equation :

$$dv = -gdx - \frac{v^m ds}{k^m}$$

from containing the nature of the ascent. In a similar manner, if the curve *BL* thus has been compared, so that it ascends again, then the curve *AM* also is directed up and satisfies the prescribed condition by partially descending and partially rising.

Example 1.

628. If the curve *AM* is sought, upon which the body is moving uniformly, with a speed clearly corresponding to the height *b*, then *BL* is a straight line parallel to the axis AP and $v = b$. On this account,

$$ds = \frac{gk^m dx}{b^m}.$$

Thus it follows that the line AM is a straight inclined line and the cosine of the angle, that it makes with the vertical AP, is equal to $\frac{b^m}{gk^m}$ with 1 put for the whole sine. Therefore so that the greater *b* or the speed shall be, by which the body is to be carries, for that less will be the angle with the vertical *AP*, and if it should be that $b^m = gk^m$, then the line sought is the vertical *AP* itself. But if b^m is put in place greater than gk^m, then the solution is lead to being imaginary, clearly to an angle of which the cosine is greater than the whole sine. Again the time, in which the portion of the line *AM* is completed on descending, is equal to $\frac{gk^m x}{b^m \sqrt{b}} = \frac{s}{\sqrt{b}}$.

Example 2. [p. 337]

629. The curve AM is sought, upon which the body thus descends, so that its speed at individual points is in the ratio of the square root from the height AP, which is the property agreeing for all motion *in vacuo*. Therefore we have $v = \alpha x$ and $dv = \alpha dx$ and with these substituted there is obtained :

$$ds = \frac{gk^m dx - \alpha k^m dx}{\alpha^m x^m} \text{ and } s = \frac{(g-\alpha)k^m x^{1-m}}{(1-m)\alpha^m},$$

where there is no need to add a constant, if $m < 1$. But if $m = 1$, the curve is a tractrix on the line described to the horizontal passing through *A* ; upon which the descent begins from an infinite distance clearly by drawing together with the asymptote. In a similar manner, if $m > 1$, the curve has a form similar to the tractrix.
[The tractrix is the curve described by a body dragged along a rough horizontal plane, attached to the end of a massless string of constant length, the other end of which moves along a straight line at a constant speed. It is the evolute of the catenary. One parametric form of the curve is : See e.g. Lockwood, *A Book of Curves,* p. 124, for some of the history of this curve; or one of the websites that delves into such things.]
Moreover it is always necessary that $\alpha < g$; from which it is evident that the body in the resisting medium is not able to acquire as much speed as in a vacuum. Then since *ds* must be greater than *dx*, it follows that $(g-\alpha)k^m > \alpha^m x^m$; therefore the body is unable to descent further than through the distance equal to $\frac{k}{\alpha}\sqrt[m]{(g-\alpha)}$; in which place the tangent to the curve is vertical and the curve has the point of reversion. Moreover the time, in which the body descends along the arc *AM* , is equal to :

$$\int \frac{ds}{\sqrt{\alpha x}} = \frac{2(g-\alpha)k^m x^{\frac{1}{2}-m}}{(1-2m)\alpha^{m+\frac{1}{2}}}.$$

[p. 338] Whereby unless $m < \frac{1}{2}$, the time cannot be finite, but is of infinite size; for if $m = \frac{1}{2}$, the curve is a cycloid pointing downwards, upon the vertex *A* of which the body remains for ever.

Corollary 6.

630. Therefore it is evident in a resisting medium that the motion cannot be continued beyond a given point, thus in order that the speeds are always in the square root ratio of the heights.

Corollary 7.

631. From these it is apparent that all the curves found in this way have a horizontal tangent at *A*. Hence as long as the radius of osculation at *A* is of finite magnitude, the body never descends. But if the radius of osculation becomes infinitely small, which eventuates if $m < \frac{1}{2}$, then the body can descend; from which it is understood that the time is finite.

Scholium 2.

632. If the body starts to descend from *A* and the tangent to the curve *AM* at *A* is not horizontal, then the initial resistance of the motion is zero; hence there $dv = gdx$. On which account at the place where the curve *AM* is produced does not have a horizontal tangent at *A*, p. 339] the curve *BL*, which at *A* agrees with *AP*, thus has to be prepared, so that it is given initially by $v = gx$; or the tangent of the curve *BL* at *A* must make an angle with the axis *AP*, the tangent of which is equal to *g*; indeed otherwise the curve *AM* cannot make an acute angle with *AP*. But on descending more, it is always the case that $v < gx$; for in a resisting medium the speed acquires from any height is less than the speed that it acquires from the same height *in vacuo*. Again in a resisting medium the body cannot acquire a greater speed than if it were dropped vertically; since indeed a vertical line is the shortest and fastest by which the descent is completed, as the body is exposed to the least action of the resistance. On this account in a resisting medium the curve *BL* thus must be prepared, so that everywhere *v* is less than the height corresponding to the speed that is acquired by a body falling through the same resistive medium along *AP*. For where *v* surpasses this height, there the curve *AM* is imaginary.

Scholium 3.

633. In like manner the situation arises, if the curve *BLD* (Fig. 70) is given for the ascent, the applied lines of which *PL* are the heights corresponding to the speeds upon the curve to be found *AME* at the points *M*. For on calling $AP = x$, $PL = v$ and $AM = s$ then there arises

$$ds = \frac{-gk^m dx - k^m dv}{v^m} .$$

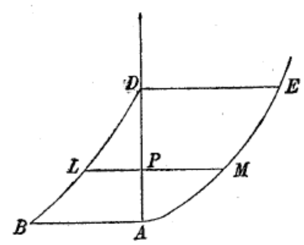

Fig. 70.

Therefore lest *ds* is negative, it is necessary that *dv* has a negative value [p. 340], i. e. in order that the curve *BL* continuously converges to the axis *AD*. For then $-dv$ must be greater than *gdx* or the tangent of the curve *BL* everywhere must make a greater angle with the axis *AP* than is that angle, the tangent of which is equal to g. Nor truly does this suffice, but also $-dv - gdx$ must be greater than $\frac{v^m dx}{k^m}$ or

$$-dv > \frac{(gk^m + v^m)dx}{k^m} ,$$

or the difference between $-dv$ and *gdx* must be greater than $\frac{v^m dx}{k^m}$. Here this last condition is returned in the form, that *PL* is less than the height corresponding to the speed that the body has at P, if from A with a speed corresponding to the height AB it could ascend to AP. For in the ascent along the vertical line by reason of the height traversed, the body suffers the smallest decrease of speed from the resistance. Moreover at the point D the angle ADB has to be of such a size, that the tangent of this is equal to *g*, since near the point E, at which the speed is zero, the effect of the resistance vanishes; but now if this angle should be greater, then the curve AME has a horizontal tangent at E, as we cautioned in the preceding scholium regarding the descent.

PROPOSITION 72.

Problem. [p. 341]

634. *If the curve AM* (Fig.71) *is given, upon which the body is moving in vacuo, to find the curve am, upon which the body moves in a medium with resistance thus descends, so that the speed at a is equal to the speed at A and with equal arcs AM and am taken, so that the speeds at individual points M and m are also equal.*

Solution.

With the vertical axes *AP* and *ap* drawn, and with the horizontal lines *MP, mp* let *AM* = *am* = *s*, *AP* = *t* and *ap* = *x*; the given curve *AM* is given by means of the equation between *s* et *t*. Now the speeds at the points *A* and *a* correspond to the height *b* and the speeds at *M* and *m* correspond to the height *v*. Let the absolute force acting downwards be *g* and the resistance as the 2*m*th power of the speeds. With these put in place, then for the motion *in vacuo* upon the curve *AM*:

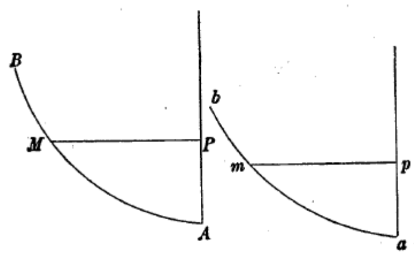

Fig. 71.

$$dv = -gdt \text{ or } v = b - gt$$

and with the descent in the resisting medium upon the curve *ma* :

$$dv = -gdx + \frac{v^m ds}{k^m};$$

in which equation if in place of *dv* and *v* there are substituted values found from the previous equation, there is produced :

$$-gdt = -gdx + \frac{(b-gt)^m ds}{k^m} \text{ or } dx = dt + \frac{(b-gt)^m ds}{gk^m}.$$

Since moreover the equation is given between *t* and *s*, if in place of *t* the value of this in terms of *s* is substituted, the equation is obtained between *x* and *s* for the curve sought *am*. Q.E.I.

Corollary 1. [p. 342]

635. If on the curve *AM* the point *B* is the start of the descent and thus the height of this above $A = \frac{b}{g}$, there is also obtained the start of the descent *b* on the curve *am*, on taking the arc *amb* = *AMB.*

Corollary 2.

636. From the solution it is apparent that *dx* > *dt* always; whereby the height *ap* is greater than the height *AP*; for in the medium with the resistance there is a need for more height to generate the same speed as *in vacuo*.

Corollary 3.

637. Because on the curves *AM* and *am* taken with equal arcs the speeds at these places are the same, the times too are equal, in which the equal arcs *AM* et *am* are described. And thus the time of the descent in the medium with resistance along *bma* is equal to the time of the descent *in vacuo* along *BMA*.

Corollary 4.

638. Lest the curve *bma* becomes imaginary, it is necessary that everywhere $dx < ds$. On this account it is necessary that :

$$dt + \frac{(b-gt)^m ds}{gk^m} < ds \text{ or } gk^m dt < gk^m ds - (b-gt)^m ds .$$

Moreover this is thus obtained, if it is the case that [p. 343]

$$gk^m dt < (gk^m - b^m)ds ,$$

which is so if t vanishes and [the inequality] pertains to the point *a*, unless t has a negative value somewhere. Concerning this it is only required to be considered that the point *a* is made real, which comes about if $gk^m dt$ is not greater than $(gk^m - b^m)ds$.

Corollary 5.

639. Therefore lest the curve *am* becomes imaginary, before everything it is necessary that $b < k\sqrt[m]{g}$. At the point *A*, let $ds = \alpha dt$; α is a number greater than one, and thus

$$gk^m < \alpha(gk^m - b^m)ds \text{ or } b < k\sqrt[m]{\frac{g(\alpha-1)}{\alpha}} .$$

Therefore if the curve *MA* in *A* has a horizontal tangent, it must be the case that $b < k\sqrt[m]{g}$ on account of $\alpha = \propto$.

Corollary 6.

640. Moreover, if $ds = \alpha dt$ at the point A, let $b^m = \frac{g(\alpha-1)}{\alpha}k^m$; then at this point *a*

$$dx = \frac{ds}{\alpha} + \frac{(\alpha-1)ds}{\alpha} = ds.$$

Therefore in this case the curve *am* has a vertical tangent at *a*.

Corollary 7.

641. At the start of the motion at *B* let $b = gt$ or $t = \frac{b}{g}$. Therefore for the point *b*, $dx = dt$ with the elements of the curves taken equal. Whereby the tangents at the points *B* and *b* are equally inclined.

Scholion 1. [p. 344]

642. Since it is not possible to completely construct the curve *am* from the curve *AM*, for besides it is necessary to know the speed at the point *A* or at the start of the descent *B*, if another starting point of the descent is taken on the curve *AM* then another curve *am* is found. Therefore for this reason the curves *BMA* and *bma* are thus only in agreement for a single descent, in order that the speeds in which the intervals are traversed are equal to each other, and if the starts of the descents are placed at other points, then this agreement is no longer in place. Therefore two curves are not given, upon which all the descents as far as to a given point are in agreement with each other, the one *in vacuo*, and the other placed in a resisting medium.

Example 1.

643. Let *AMB* be a straight line inclined at some inclination, thus so that $s = \alpha t$, and the curve *amb* is sought, upon which the body in a like manner is progressing in a resisting medium upon as the above *AMB in vacuo*. Moreover in putting $\frac{s}{\alpha}$ in place of t, the following equation is produced between *x* and *s* for the curve sought *amb* :

$$dx = \frac{ds}{\alpha} + \frac{(\alpha b - gs)^m}{g\alpha^m k^m} ds,$$

and the integral of this is :

$$x = \frac{s}{\alpha} + \frac{\alpha b^{m+1}}{(m+1)g^2 k^m} - \frac{(\alpha b - gs)^{m+1}}{(m+1)g^2\alpha^m k^m}.$$

[p. 345] Lest the point *a* becomes imaginary, it is necessary that :

$$\frac{1}{\alpha} + \frac{b^m}{gk^m} < 1.$$

For if it happens that

$$b^m = \frac{(\alpha - 1)gk^m}{\alpha},$$

then the curve has a vertical tangent at *a* nor therefore can *b* have a greater value. Therefore the body can put on the inclined line *BMA* to descend from such a height, as it becomes :

$$b = k\sqrt[m]{\frac{g(\alpha - 1)}{\alpha}};$$

then the equation becomes :

$$dx = \frac{ds}{\alpha} + \frac{(k\sqrt[m]{g\alpha^{m-1}(\alpha - 1)} - gs)^m ds}{g\alpha^m k^m},$$

which is the equation for the curve *amb*, at which the start of the descent is taken from *b*, where $ds = \alpha dx$, or the arc *amb* is equal to:

$$k\sqrt[m]{\frac{\alpha^{m-1}(\alpha - 1)}{g^{m-1}}}.$$

If the resistance were proportional to the squares of the speeds, then $m = 1$ and thus

$$dx = \frac{ds}{\alpha} + \frac{(gk(\alpha-1)-gs)ds}{g\alpha k} = ds - \frac{sds}{\alpha k}$$

and on integrating :

$$x = s - \frac{ss}{2\alpha k}.$$

Which is the equation for a cycloid described on a horizontal base, the diameter of the generating circle of which is $\frac{\alpha k}{2}$.

Corollary 8.

644. Therefore let the cycloid *AMB* be described on the horizontal base *CB* (Fig. 72) by the generating circle *ANC*, and let the medium be resisting in the ratio of the square of the speeds, the exponent of which is equal to *k*. If now in the circle *ANC* the chord $AN = \frac{k}{2}$ is taken, and the horizontal *PNM* is drawn and from *M* the tangent *MT* and the two bodies are placed to descend, the one on *MT in vacuo* and the other on the curve *MB* in the resisting medium, both these bodies complete equal distances in equal times. [p. 346]

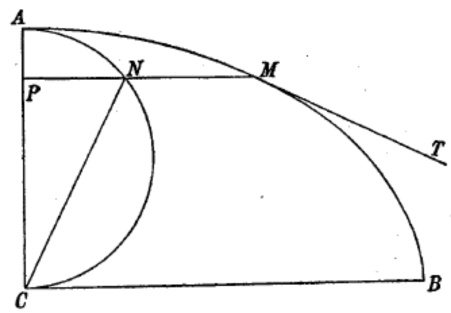

Fig. 72.

Example 2.

645. Let *AMB* (Fig. 71) be a cycloid considered downwards, the diameter of the generating circle of which is $\frac{\alpha}{2}$; then we have $ss = 2at$, $t = \frac{ss}{2a}$ and $dt = \frac{sds}{a}$. Hence with these substituted there is produced the following equation for the curve *amb* :

$$dx = \frac{sds}{a} + \frac{(2ab - gss)^m ds}{2^m g a^m k^m};$$

or if the whole arc *AMB*, which *in vacuo* is put for the completed descent, is called *c*, then $b = \frac{gcc}{2a}$ and thus for the curve *amb* this equation arises :

$$dx = \frac{sds}{a} + \frac{g^{m-1}(cc - ss)^m ds}{2^m a^m k^m}.$$

Therefore lest this curve becomes imaginary at the point *a*, it is necessary that $g^{m-1} < 2^m a^m k^m$ or the height of the arc AB must be less than $\frac{k}{g}\sqrt[m]{g}$; for if the height of the arc $AB = \frac{k}{g}\sqrt[m]{g}$, then the tangent of the curve *amb* is vertical at *b*. Therefore if B is the cusp of the cycloid, then $\frac{cc}{2a} = \frac{a}{2}$ or $c = a$, and if in addition $g^{m-1}a^m = 2^m k^m$, in order that the tangent of the curve *amb* becomes vertical at *a*, this equation is obtained :

$$dx = \frac{sds}{a} + \frac{(aa-ss)^m ds}{a^{2m}};$$

which curve has vertical tangents at *a* and *b*.

Corollary 9. [p. 347]

646. Therefore when the body thus descends on the cycloid, in order that the accelerations of this body are proportional to the distances traversed, then descents on the curve *amb* in the resisting medium have the same the same property in place, if the start of the descent is taken at the point *b*, that is determined through the equation to the curve *amb*; clearly it is *amb* = *c*.

Corollary 10.

647. If another starting point is taken at *B* on the curve *AMB*, another whole curve *amb* is found, since the length of the arc *AMB* = *c* is contained in the equation of this. Whereby, even if the cycloid is a tautochronous curve *in vacuo*, yet such a curve *amb* is not [a tautochrone] in a medium with resistance, since for several descents on the same curve *AMB* as many different curves correspond in the resisting medium.

Scholium 2.

648. In this example the curves have been elicited that the Cel. Hermann in Comm. Book II found for tautochrones in resisting mediums [Jacob Hermann, *General theory of the motion that arises with forces acting constantly on bodies*. Commen. acad. sc. Petrop. 2 (1727), 1729, p. 139]; but likewise he has shown that it is not possible to give a satisfactory answer to the question. Furthermore from these is understood, in a like manner, that it is possible to find in a curve in a resisting medium, upon which a body on ascending is moving in the same way as that on a given curve *in vacuo*. [p. 348] For *A* and *a* are the initial points of the ascent, that one *in vacuo*, and this in a resisting medium, and let the initial speed correspond to the height *b*; this equation is obtained for the curve *amb* :

$$dx = dt - \frac{(b-gt)^m ds}{gk^m};$$

from which equation it is understood that the curve *amb* does not become imaginary, unless the curve itself *AMB* were such. For because, lest the curve *amb* is imaginary, it must be that $dx < ds$, here dx is less than dt, that *per se* must be less than ds. As if the line

AMB is a vertical line, the other *amb* is possible to be assigned; for on putting $AB = c$ then $b = gc$ and $s = t$; wherefore for the curve *amb* this equation is found :

$$dx = ds - \frac{g^{m-1}(c-s)^m ds}{k^m},$$

the integral of which is :

$$x = s + \frac{g^{m-1}(c-s)^{m+1} - g^{m-1}c^{m+1}}{(m+1)k^m}.$$

This equation can be adapted to resistance proportional to the square of the speed; let $m = 1$ and thus

$$x = s + \frac{(c-s)^2 - c^2}{2k} = \frac{2(k-c)s + ss}{2k},$$

which is related to the cycloid in this way: the cycloid *AMB* (Fig.72) is described with the generating circle of diameter $AC = \frac{k}{2}$ upon the horizontal base *BC*; then the arc $AM = k - c$ is taken; then *M* is the start of the ascent, from which point, if the body ascends on the curve *MA* with a speed corresponding to the height *gc*, in the medium resisting as the square ratio of the speeds, the body is moving in the same way as *in vacuo* with the same initial speed, rising vertically up.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 318]

PROPOSITIO 68.

Problema.

602. *In medio uniformi, quod resistit in quadruplicata ratione celeritatum, determinare tam descensum quam ascensum corporis quemcunque super cycloide ACB* (Fig.66).

Solutio.

Posita potentia uniformi corpus perpetuo deorsum trahente g et abscissa $CP = x$ et arcu $CM = s$ erit ex natura cycloidis $dx = \frac{sds}{a}$. Sit celeritas in C debita altitudini b et in M altitudini v atque k exponens resistentiae ; erit resistentia in $M = \frac{v^2}{k^2}$. [p. 319]

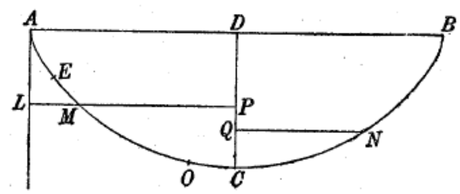

Fig. 66.

Pro descensu ergo habebitur haec aequatio

$$dv = -gdx + \frac{v^2ds}{k^2} = -\frac{gsds}{a} + \frac{v^2ds}{k^2},$$

at pro ascensu ista

$$dv = -\frac{gsds}{a} - \frac{v^2ds}{k^2}.$$

Pro descensu ponatur

$$v = \frac{-k^2dz}{zds};$$

erit

$$dv = \frac{-k^2ddz}{zds} + \frac{k^2dz^2}{z^2ds}$$

posito ds constante. Quamobrem habebitur

$$k^2ddz = \frac{gszds^2}{a};$$

quae aequatio in seriem conversa dat

$$z = f + hs + \frac{fgs^3}{2\cdot 3ak^2} + \frac{hgs^4}{3\cdot 4ak^2} + \frac{fg^2s^6}{2\cdot 3\cdot 5\cdot 6a^2k^4} + \frac{hg^2s^7}{3\cdot 4\cdot 6\cdot 7a^2k^4} + \frac{fg^3s^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9a^3k^6} + \text{etc.}$$

Ad constantes *f* et *h* determinandas quaeratur valor ipsius *v* posito s = 0; erit ergo

$$b = -\frac{k^2h}{f}.$$

Et quia, si *s* = 0, est

$$dv = \frac{v^2ds}{k^2} = \frac{b^2ds}{k^2},$$

propter $dv = \frac{-k^2ddz}{zds} + \frac{k^2dz^2}{z^2ds}$ erit

$$\frac{b^2}{k^2} = \frac{k^2h^2}{ff},$$

quae aequatio cum illa congruit; erit ergo $h = -\frac{bf}{k^2}$. Hoc substituto erit

$$v = \frac{b - \frac{gs^2}{2a} + \frac{bgs^3}{3ak^2} - \frac{g^2s^5}{2\cdot 3\cdot 5a^2k^2} + \frac{bg^2s^6}{3\cdot 4\cdot 6a^2k^4} - \text{etc.}}{1 - \frac{bs}{k^2} + \frac{gs^3}{2\cdot 3ak^2} - \frac{bgs^4}{3\cdot 4ak^4} + \frac{g^2s^6}{2\cdot 3\cdot 5\cdot 6a^2k^4} - \text{etc.}}.$$

Pro ascensu vero posito –*s* loco *s* habebitur

$$v = \frac{b - \frac{gs^2}{2a} - \frac{bgs^3}{3ak^2} + \frac{g^2s^5}{2\cdot 3\cdot 5a^2k^2} + \frac{bg^2s^6}{3\cdot 4\cdot 6a^2k^4} - \text{etc.}}{1 + \frac{bs}{k^2} - \frac{gs^3}{2\cdot 3ak^2} - \frac{bgs^4}{3\cdot 4ak^4} + \frac{g^2s^6}{2\cdot 3\cdot 5\cdot 6a^2k^4} + \text{etc.}}.$$

Ex his aequationibus totus arcus vel descensus vel ascensus invenitur, si ponatur *v* = 0 atque valor ipsius s investigetur. Ut si *k* fuerit quantitas valde magna, erit arcus descensus, qui sit *E*, =

$$\frac{\sqrt{2ab}}{\sqrt{g}} + \frac{8ab^2}{15gk^2} + \frac{301a^2b^4\sqrt{g}}{450g^2k^4\sqrt{2ab}} + \text{etc.}$$

At sequens arcus ascensus, qui sit *F*,erit =

$$\frac{\sqrt{2ab}}{\sqrt{g}} - \frac{8ab^2}{15gk^2} + \frac{301a^2b^4\sqrt{g}}{450g^2k^4\sqrt{2ab}} - \text{etc.}$$

Q.E.I.

Corollarium 1. [p. 320]

603. Si ergo resistentia est quam minima, erit summa arcuum descensus et ascensus seu arcus una semioscillatione descriptus, i. e.

$$E+F=\frac{2\sqrt{2ab}}{\sqrt{g}}\text{ quam proxime.}$$

Corollarium 2.

604. Differentia autem inter arcus ascensu et descensus, scilicet

$$E-F=\frac{16ab^2}{15gk^2}=\frac{g(E+F)^4}{60ak^2}.$$

Quare differentia inter arcus descensu et ascensus est ut biquadratum summae arcuum.

Scholion 1.

605. Ex his perspicitur in medio rarissimo, quod resistit in quadruplicata ratione celeratum, differentiam inter arcus ascensus et descensus proportionalem esse biquadrato summae arcuum seu

$$E-F=\frac{g(E+F)^4}{60ak^2}.$$

Supra autem vidimus in medio rarissimo, quod in duplicata ratione celeritatum resistit, esse arcum descensus

$$E=\frac{\sqrt{2ab}}{\sqrt{g}}+\frac{2ab}{3gk}$$

et arcum ascensus

$$F=\frac{\sqrt{2ab}}{\sqrt{g}}-\frac{2ab}{3gk};$$

hinc erit

$$E+F=\frac{2\sqrt{2ab}}{\sqrt{g}}\quad\text{et}\quad E-F=\frac{4ab}{3gk}=\frac{(E+F)^2}{6k}$$

(557). Quare in hac resistentia est differentia inter arcus ascensus et descensus ut quadratum summae arcuum. Atque in medio, quod in simplici ratione celeritatum resistit, si fuerit rarissimum, est

$$E=\frac{\sqrt{2ab}}{\sqrt{g}}+\frac{\pi a\sqrt{b}}{4g\sqrt{k}}\quad\text{et}\quad F=\frac{\sqrt{2ab}}{\sqrt{g}}-\frac{\pi a\sqrt{b}}{4g\sqrt{k}}$$

(582). Quare erit

$$E-F=\frac{\pi a\sqrt{b}}{2g\sqrt{k}}=\frac{\pi(E+F)\sqrt{a}}{4\sqrt{2kg}}.$$

Seu differentia inter arcus descensus et ascensus est ipsi summae arcuum proportionalis. [p. 321] Ex quo consequi videtur in medio quocunque rarissimo, quod resistit in $2m$ – multiplicata ratione celeritatum, differentiam inter arcus ascensus et descensus super cycloide proportionalem esse potestati summae arcuum ascensus et descensus, cuius exponens sit $2m$. Atque in hac resistentiae hypothesi coniectare licet fore arcum descensus

$$E = \frac{\sqrt{2ab}}{\sqrt{g}} + \frac{2 \cdot 4 \cdot 6 \cdots 2mab^m}{3 \cdot 5 \cdot 7 \cdots (2m+1)gk^m}$$

et arcum ascensus

$$F = \frac{\sqrt{2ab}}{\sqrt{g}} - \frac{2 \cdot 4 \cdot 6 \cdots 2mab^m}{3 \cdot 5 \cdot 7 \cdots (2m+1)gk^m}.$$

Unde fit

$$E - F = \frac{1 \cdot 2 \cdot 3 \cdots mg^{m-1}(E+F)^{2m}}{3 \cdot 5 \cdot 7 \cdots (2m+1)\, 2^{2m-1}a^{m-1}k^m}.$$

Quoties ergo m est numerus integer seu $2m$ numerus par, assignari potest aequatio inter $E - F$ et $E + F$, at si m fuerit numerus fractus, valor fraction $\frac{1.2.3......m}{3.5.7....(2m+1)}$ per methodum interpolationum, quam exhibui in Comment. Acad. Petrop. A 1730 [L. Eulero Commentatio 19 (E19) : *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequent*.] investigari potest. Ex qua quidem constat, si $2m$ fuerit numerus impar, valorem huius fractionsi involvere quadraturam circuli, quemadmodem etiam in casu, quo $2m = 1$, repperimus.

Scholion 2.

606. Quod quidem ad ipsam propositionem attinet, esse differentiam inter arcus descensus et ascensus super cycloide in totuplicata ratione summae arcuum, [p. 322] in quotuplicata ratione celeritatum sit resistentia, si quidem fuerit minimia, Neutonus in Princip. demonstravit [Lib. II, Prop. XXXI]. Atque demonstrationem etiam ex ipsaa aequatione

$$dv = -\frac{gsds}{a} \pm \frac{v^m ds}{k^m}$$

derivare licet. Ponatur enim

$$v = b - \frac{gs^2}{2a} + Q,$$

ubi Q erit quantitas valde parva prae b et $\frac{gs^2}{2a}$. Hanc ob rem habebitur

$$-\frac{gsds}{a} + dQ = \frac{-gsds}{a} + \frac{\left(b - \frac{gs^2}{2a}\right)^m ds}{k^m}$$

pro descensu seu

$$Q = \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}$$

hoc integrali ita accepto, ut evanescat posito $s = 0$. Pro descensu ergo erit

$$v = b - \frac{gs^2}{2a} + \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}$$

et pro ascensu

$$v = b - \frac{gs^2}{2a} - \int \frac{(2ab - gs^2)^m ds}{(2ak)^m}.$$

Ponatur $v = 0$, et quia tunc proxime est $s = \frac{\sqrt{2ab}}{\sqrt{g}}$, ponatur $s = \frac{\sqrt{2ab}}{\sqrt{g}} + q;$ erit

$$0 = \frac{-gq\sqrt{2ab}}{a\sqrt{g}} + \int \frac{(2ab - gss)^m ds}{(2ak)^m}$$

si quidem post integrationem ponatur $s = \frac{\sqrt{2ab}}{\sqrt{g}}$. At quia in hoc ipsius q valore ipsarum $\sqrt{b}$ et s sunt $2m$ dimensiones, habebit q huiusmodi formam Nb^m. Quocirca erit arcus descensus

$$E = \frac{\sqrt{2ab}}{\sqrt{g}} + Nb^m$$

et arcus ascensus

$$F = \frac{\sqrt{2ab}}{\sqrt{g}} - Nb^m.$$

Hinc ergo habebitur

$$E - F = 2Nb^m = \frac{Ng^m(E+F)^{2m}}{2^{3m-1}a^m}.$$

At numerus N obtinebitur ex formula [p. 323]

$$\sqrt{\frac{a}{2gb}} \int \left(\frac{2ab - gss}{2abk}\right)^m ds,$$

si post integrationem ponatur $s = \frac{\sqrt{2ab}}{\sqrt{g}}$. Atque haec est demonstatio illius ipsius, quod in praecedente scholio ex inductione derivabamus. Erit enim N numerus rationalis, quoties m fuerit integer affirmativus; at si $2m$ fuerit numerus integer impar, inventio numeri N a quadratura circuli pendebit. Generaliter autem valor ipsius q cum hac expressione

$$\frac{2 \cdot 4 \cdot 6 \cdots 2mab^m}{3 \cdot 5 \cdot 7 \cdots (2m+1)gk^m}$$

congruit.

PROPOSITIO 69.

Problema.

607. *In medio, quod resistit in quadruplicata ratione celeritatum, si detur corporis super curva AMC* (Fig.67) *ex dato puncto A descendentis in singulis locis celeritas, invenire celeritatem eiusdem corporis descensum in quocunque alio puncto E incepientis.*

Solutio.

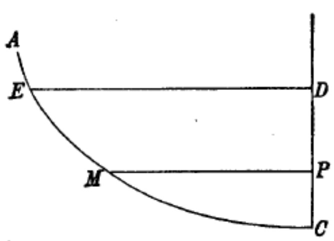

Fig. 67.

Posita $CP = x$ et $CM = s$ sit corporis ex A delapsi celeritas in M debita altitudini u, quae quantitas u ergo per hypothesin datur per x et s. Iam si corpus descensum ex quocunque puncto E incipiat, sit celeritas in M debita altitudini v. Aequatio vero motum determinans erit

$$dv = -gdx + \frac{v^2 ds}{k^2},$$

quae dat valorem ipsius v, ubicunque descensus inceperit; erit ergo etiam [p. 324]

$$du = -gdx + \frac{u^2 ds}{k^2}.$$

Ponatur $v = u - q$; erit

$$du - dq = -gdx + \frac{u^2 ds}{k^2} - \frac{2qu ds}{k^2} + \frac{q^2 ds}{k^2};$$

ex qua aequatione propter

$$du = -gdx + \frac{u^2 ds}{k^2}$$

oritur

$$-dq = -\frac{2quds}{k^2} + \frac{q^2 ds}{k^2} \quad \text{seu} \quad -\frac{dq}{q^2} + \frac{2uds}{k^2 q} = \frac{ds}{k^2},$$

quae multiplicata per $e^{\frac{2\int uds}{k^2}}$ dat hanc integalem

$$e^{\frac{2\int uds}{k^2}} = cq + q\int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2},$$

ex qua prodit [mutata significatione litterae c]

$$q = \frac{k^2 e^{\frac{2\int uds}{k^2}}}{c + \int \frac{e^{\frac{2\int uds}{k^2}} ds}{k^2}}.$$

Quocirca erit

$$v = u - \frac{k^2 e^{\frac{2\int u ds}{k^2}}}{c + \int \frac{e^{\frac{2\int u ds}{k^2}} ds}{k^2}},$$

in qua aequatione integralia

$$\frac{2\int u ds}{k^2} \text{ et } \int e^{\frac{2\int u ds}{k^2}} ds$$

ita sint accepta, ut evanescant posito $s = 0$. Sit nunc altitudo celeritati in *C* debita = a, si descensus ex *A* fiat, at altitudo celeritati in *C* debita, si descensus ex *E* fit, = b; erit $b = a - \frac{k^2}{c}$. Ex quo habebitur

$$v = u - \frac{(a-b)k^2 e^{\frac{2\int u ds}{k^2}}}{k^2 + (a-b)\int \frac{e^{\frac{2\int u ds}{k^2}} ds}{k^2}}$$

Data ergo celeritate in C, nempe $\sqrt{b}$, invenietur punctum E, in quo descensus incepit, ex hac aequaitione

$$u = \frac{(a-b)k^2 e^{\frac{2\int u ds}{k^2}}}{k^2 + (a-b)\int \frac{e^{\frac{2\int u ds}{k^2}} ds}{k^2}},$$

ex qu valor ipsius s dabit arscum *CME*. Quia igitur datur u per s, ex hac aequatione celeritas corporis ex quocunque alio puncto delapsi super curva *AMC* invenietur. Q.E.I.

Corollarium 1. [p. 325]

608. Si valor ipsius v ita immutetur, ut tam in numeratore quam in denominatore b sine coefficiente appareat, prodibit

$$v = \frac{\left(u\int e^{\frac{2\int u ds}{k^2}} ds - k^2 e^{\frac{2\int u ds}{k^2}}\right)}{\int e^{\frac{2\int u ds}{k^2}} ds} \cdot \frac{\left(b - a + \frac{k^2 u}{k^2 e^{\frac{2\int u ds}{k^2}} - u\int e^{\frac{2\int u ds}{k^2}} ds}\right)}{\left(b - a - \frac{k^2}{\int e^{\frac{2\int u ds}{k^2}} ds}\right)}.$$

Atque erit $v = 0$, si est

$$a + \frac{k^2 u}{u\int e^{\frac{2\int u ds}{k^2}} ds - k^2 e^{\frac{2\int u ds}{k^2}}} = b.$$

Corollarium 2.

609. Quia est

$$du - \frac{u^2 ds}{k^2} = -gdx,$$

erit huius aequationis per $e^{\frac{-\int uds}{k^2}}$ multiplicatae integralis haec

$$u = ae^{\frac{-\int uds}{k^2}} - ge^{\frac{\int uds}{k^2}} \int e^{\frac{-\int uds}{k^2}} dx$$

integralibus ita sumtis, ut evanescant posito s vel $x = 0$. Vel etiam est

$$\frac{du}{u} + \frac{gdx}{u} = \frac{uds}{k^2}$$

atque hinc

$$\int \frac{uds}{k^2} = l\frac{du}{u} + \int \frac{gdx}{u}$$

Quare erit

$$e^{\int \frac{uds}{k^2}} = \frac{e^{\int \frac{gdx}{u}} u}{a};$$

unde dx loco ds in aequatione superiore potest introduci.

Corollarium 3. [p. 326]

610. Si resistentia fuerit quam minima, evanescet $\int e^{\frac{2\int uds}{k^2}} ds$ prae k^2 et ideo erit

$$v = u - (a-b)e^{\frac{2\int uds}{k^2}} = u - (a-b)\left(1 + \frac{2\int uds}{k^2}\right)$$

propter k quantitatem maximam. Quamobrem erit

$$v = b + \frac{2b\int uds}{k^2} - a - \frac{2a\int uds}{k^2} + u = \left(1 + \frac{2\int uds}{k^2}\right)\left(b - a + \frac{u}{1 + \frac{2\int uds}{k^2}}\right)$$

Corollarium 4.

611. Cum autem sit

$$\sqrt{\left(1 + \frac{2\int uds}{k^2}\right)} = 1 + \frac{\int uds}{k^2},$$

erit elementum temporis

$$\frac{ds}{\sqrt{v}} = \frac{k^2 ds}{(k^2 + \int u ds)\sqrt{\left(b - a + \frac{k^2 u}{k^2 + 2\int u ds}\right)}}.$$

Per aequationem autem

$$du = -gdx + \frac{u^2 ds}{k^2}$$

est quam proxime

$$u = a - gx + \int \frac{(a - gx)^2 ds}{k^2},$$

unde erit

$$\int u ds = \int (a - gx) ds$$

atque

$$\frac{ds}{\sqrt{v}} = \frac{k^2 ds}{(k^2 + \int (a - gx) ds)\sqrt{\left(b - a + \frac{k^2 a - gk^2 x + \int (a - gx)^2 ds}{k^2 + 2\int (a - gx) ds}\right)}}$$

$$= \frac{k^2 ds}{(k^2 + \int (a - gx) ds)\sqrt{\left(b - \frac{gk^2 x - \int (a^2 - g^2 x^2) ds}{k^2 + 2\int (a - gx) ds}\right)}}.$$

Scholion. [p. 327]

612. Quemadmodum hypothesis resistentia quadratis celeritatum proportionalis prae aliis hypothesibus excepta ea, quae est constans, hanc habet praerogativam, ut corporis super quacunque curva moti celeritas in omnibus locis ex aequatione curvae possit definiri, ita haec resistentiae hypothesis in hoc prae reliquis excellit, quod ex dato unico descensu vel ascensu simul omnes descensus et ascensus possint determinari. In aliis enim resistentiis operatio, qua hic usi sumus, non succedit neque ad aequationem deducit, in qua indeterminatae a se invicem separari possunt. Hanc ob rem uti resistentia constans est simplicissima eamque sequitur ea, quae quadratis celeritatum est proportionalis, ita post has pro simplicissima resistentiae hypothesi est habenda ea, quae fit in quadruplicata ratione celeritatum. Videtur quidem ex his parum commodi ad motum in hac resistentiae hypothesi definiendum obtineri, quia unus descensus tanquam datus accipitur, qui autem inventa aeque est difficilis ac quisque alius. At si plures descensus considerantur et inter se comparantur, aequatio

$$dv = -gdx + \frac{v^2 ds}{k^2}$$

reipsa tres variabiles implicat, nempe praeter v et s seu x celeritatem in puncto C, quae in variis descensibus variatur. Quare cum resolutionem huius aequationis ad hanc aequationem

$$du = -gdx + \frac{u^2 ds}{k^2}$$

reduxerimus, quae ad unicum descensum spectat, illud incommodum trium variabilium hoc modo tollitur. Praeterea ope isitius artificii plures descensus inter se comparari possunt, quod in aliis resistentiae hypothesibus ne quidem fieri potest. Atque hinc etiam multa problemata inversa pro hac resistentiae hypothesi resolvi possunt, quae in aliis omnino tractari nequeant. [p. 328]

PROPOSITIO 70.

Problema.

613. *Si resistantia fuerit quam minima respectu potentia sollicitantis absolutae et proportionalis potestati cuicunque celeritatum, determinare motum corporis super quacunque curva AM* (Fig.68).

Solutio.

Descendat corpus super curva *AM* descensus initio in *A* existente; ponatur super axe verticali abscissa $AP = x$, arcus $AM = s$ et potentia perpetuo deorsum trahens = *g*. Sit celeritas in *M* debita altitudini *v*, et resistentia ibidem = $\frac{v^m}{k^m}$, ita ut resistentia sit proportionalis potestati exponentis 2*m* celeritatum. His positis erit

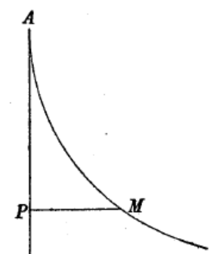
Fig. 68.

$$dv = gdx - \frac{v^m ds}{k^m},$$

at quia resistentia ponitur valde parva, erit terminus $\frac{v^m ds}{k^m}$ vehementer exiguus atque propterea $v = gx$ quam proxime. Substituatur *gx* loco *v* in termino $\frac{v^m ds}{k^m}$; erit
[p. 329]

$$v = gx - \frac{g^m}{k^m}\int x^m ds$$

atque simili modo adhuc propius

$$v = gx - \frac{g^m}{k^m}\int x^m ds + \frac{mg^{2m-1}}{k^{2m}}\int x^{m-1}ds\int x^m ds.$$

Quae integralia ita sunt accipienda, ut evanescant posito $x = 0$. Hinc ergo erit

$$\frac{1}{\sqrt{v}} = \frac{1}{\sqrt{gx}} + \frac{g^m\int x^m ds}{2k^m gx\sqrt{gx}} - \frac{mg^{2m}\int x^{m-1}ds\int x^m ds}{2k^{2m}g^2x\sqrt{gx}} + \frac{3g^{2m}(\int x^m ds)^2}{8k^{2m}g^2x^2\sqrt{gx}} + \text{etc.}$$

Atque tempus descensus per *AM* erit =

$$\int\frac{ds}{\sqrt{gx}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int x^{-\frac{3}{2}}ds\int x^m ds$$

quam proxime. At si descensus ad fixum punctum *C* (Fig. 67) usque desideretur initio descensus ex puncto *E* facto, ponatur puncti *E* supra *C* altitudo verticalis $CD = a$, abscissa $CP = x$ et arcus $CM = s$; quibus positis hic casus ad superiorem reducitur, si ibi loco *x* ponatur $a - x$ et $-ds$ loco *ds*. Quare si altitudo celeritati in *M* debita vocetur *v*, erit

$$v = g(a-x) + \frac{g^m}{k^m}\int(a-x)^m ds + \frac{mg^{2m-1}}{k^{2m}}\int(a-x)^{m-1}ds\int(a-x)^m ds$$

quam proxime. Haec vero integralia ita sunt accipienda, ut evanescant posito $x = a$. Atque tempus per arcum *EM* est =

$$-\int\frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int(a-x)^{-\frac{3}{2}}ds\int(a-x)^m ds$$

quam proxime, ubi iterum omnia integralia ita sunt capienda, ut evanescant posito $x = a$. [p. 330] Simili modo, si corpus ex *C* super curva *CME* ascendat tanta celeritate, qua ad punctum *E* usque pertingere possit,eaedem aequationes locum habebunt, si modo loco k^m ponatur $-k^m$. Hanc ob rem erit

$$v = g(a-x) - \frac{g^m}{k^m}\int(a-x)^m ds + \frac{mg^{2m-1}}{k^{2m}}\int(a-x)^{m-1}ds\int(a-x)^m ds$$

quam proxime atque tempus ascensus per *ME* =

$$-\int\frac{ds}{\sqrt{g(a-x)}} - \frac{g^{m-\frac{3}{2}}}{2k^m}\int(a-x)^{-\frac{3}{2}}ds\int(a-x)^m ds$$

quam proxime, omnibus his integralibus quoque ita acceptis, ut evanescant posito $x = a$. Atque hoc modo tam descensus corporis super curva quacunque quam oscillationes super curva idonea in medio rarissimo poterunt determinari. Q.E.I.

Corollarium 1.

614. Apparet ex his, quod quidem per se intelligitur, si corpus in medio resistente super curva *AM* (Fig. 68) descendat, fore celeritatem in *M* minorem, quam si corpus in vacuo super eadem curva descendisset. Atque tempus in medo resistente maius est quam tempus descensus per *AM* in vacuo.

Corollarium 2.

615. Altitudo celeratati in puncto infimo *C* (Fig. 67) debita prodibit, si in expressione ipsius *v* ponatur $x = 0$. Hoc autem facto fit

$$v = ga + \frac{g^m}{k^m}\int(a-x)^m ds$$

quam proxime posito post integrationem supra praescipto modo peractam $x = 0$. [p. 331] At si $\int(a-x)^m ds$ ita capiatur, ut evanescat posito $x = 0$, tum erit in puncto *C*

$$v = ga - \frac{g^m}{k^m}\int(a-x)^m ds,$$

si post integrationem ponatur $x = a$. Id quod ad descensum pertinet.

Corollarium 3.

616. At pro ascensum celeritas corporis in C, qua ad E usque ascendere valet, debita erit altitudini =

$$ga + \frac{g^m}{k^m}\int (a-x)^m ds,$$

si hoc integrale ita accipiatur, ut evanescat posito $x = 0$, atque post integrationem ponatur $x = a$.

Corollarium 4.

617. Sit altitudo debita celeritati corporis in *C* = *b*, quam iam descendendo per *EMC* acquisivit et qua iterum super eadem curva ascendit; ponatur altitudo *DC* descensu percursa ut ante *a* et altitudo, ad quam ascensu pertinget, *a* – *d* ; erit d quantitas valde parva atque ideo

$$b = ga - \frac{g^m}{k^m}\int (a-x)^m ds = ga - gd + \frac{g^m}{k^m}\int (a-x)^m ds$$

et

$$d = \frac{2g^{m-1}}{k^m}\int (a-x)^m ds$$

vel etiam

$$d = \frac{2ga - 2b}{g}.$$

Corollarium 5. [p. 332]

618. Quia tempus descensu per *EM* est =

$$-\int \frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds$$

his integralibus ita acceptis, ut evanescant posito $x = a$, erit tempus per *MC* =

$$\int \frac{ds}{\sqrt{g(a-x)}} - \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds,$$

si integralia ita accipiantur, ut evanescant posito $x = 0$. Atque simili modo in ascensu erit tempus per *CM* =

$$\int \frac{ds}{\sqrt{g(a-x)}} + \frac{g^{m-\frac{3}{2}}}{2k^m}\int (a-x)^{-\frac{3}{2}} ds \int (a-x)^m ds.$$

Corollarium 6.

619. Integrum ergo tempus vel descensus vel ascensus per *CME* habebitur, si in his posterioribus formulis ponatur $x = a$.

Scholion.

620. Satis iam expositis iis, quae ad motum corporis super data curva inveniendum pertinent, progredior ad quaestiones inversas, in quibus ex aliis datis, quae incognita sunt, investigantur. Et primum quidem occurrunt huiusmodi problemata, in quibus lex accelerationis seu scala celeritatum datur et curva quaeritur, quae motum illi scalae convenientem producat in medio quocunque resistente; potentiam vero absolutam ut hactenus constantem et deorsum directam assumemus.

PROPOSITIO 71. [p. 333]

Problema.

621. *In medio, quod resistit in ratione quacunque multiplicata celeritatum, invenire curvam AM* (Fig.69), *super qua corpus ita descendat, ut in singulis punctis M celeritatem habeat debitam altitudini, quae aequalis sit applicatae respondenti PL datae curvae BL.*

Solutio.

Posita $AP = x$ et $PL = v$ dabitur aequatio inter x et v propter curvam *BL* datam. Iam sit arcus $AM = s$ et exponens rationis multiplicatae celeritatum $2m$, cui resistentia est proportionalis; quibus positis erit

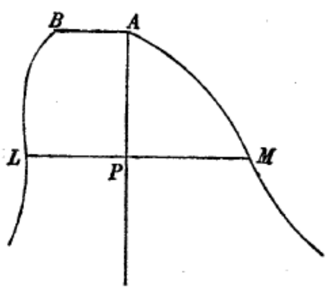

Fig. 69.

$$dv = gdx - \frac{v^m ds}{k^m},$$

denotante g potentiam uniformem deorsum trahentem et k exponentem resistentiae. Ex hac igitur aequatione est

$$ds = \frac{gk^m dx - k^m dv}{v^m},$$

quae, quia v per x dari ponitur, variabliles habet a se invicem separatas atque idcirco sufficit ad curvam quaesitam *AM* construendam. At quia ds semper maius esse debet quam dx, ne curva AM fiat imaginaria, oportet, ut sit $gk^m dx - k^m dv > v^m dx$ seu

$$dx > \frac{k^m dv}{gk^m - v^m}. \text{ [p. 334]}$$

Namque ubi est

$$dx = \frac{k^m dv}{gk^m - v^m},$$

ibi curvae *AM* tangens fit verticalis; et ubi

$$dx < \frac{k^m dv}{gk^m - v^m},$$

ibi curva AM omnino partem habere nequit. Q.E.I.

Corollarium 1.

622. Cum, ubi curvae *BL* tangens est verticalis, sit $dv = 0$, erit in loco respondente curvae *AM*

$$ds = \frac{gk^m dx}{v^m},$$

in quo puncto corpus descendens maximam vel minimam habebit celeritatem. Ne igitur hoc curvae *AM* punctum sit imaginarium, oportet sit $ak^m > v^m$ seu $v < k\sqrt[m]{g}$.

Corollarium 2.

623. Si curva *BL* alicubi incidat in axem *AP*, ut ibi sit $v = 0$, erit in loco curvae *AM* respondente $ds = \infty$, si quidem *m* fuerit numerus affirmativus. Hoc ergo loco curva *AM* habebit tangentem horizontalem, in quam curva desinet.

Corollarium 3.

624. Habeat curva *AM* alicubi tangentem horizontalem; evanescet illo loco *dx* prae *ds*. Quamobrem erit

$$ds = \frac{-k^m dv}{v^m}. \text{ [p. 335]}$$

Ex quo apparet in loco curvae *BL* respondente applicatas decrescere debere atque ibi curvae *BL* tangentem fore horizontalem, quia *dv* infinities quoque maius erit quam *dx*.

Corollarium 4.

625. Curvae *AM* tangens, ut vidimus, est verticalis, ubi est

$$dx = \frac{k^m dv}{gk^m - v^m}.$$

Quo igitur curva in initio A, ubi celeritas sit nulla, habeat tangentem verticalem, oportet, ut ibi sit $dv = gdx$ seu $v = gx$. Hoc ergo casu tangens anguli, quem curva *BL* in *A* cum *AP* constituet, erit = *g*.

Corollarium 5.

626. Cum sit

$$ds = \frac{gk^m dx - k^m dv}{v^m},$$

erit elementum temporis

$$\frac{ds}{\sqrt{v}} = \frac{gk^m dx - k^m dv}{v^{m+\frac{1}{2}}}.$$

Quocirca tempus descensus per *AM* erit =

$$\frac{2k^m}{(2m-1)v^{m-\frac{1}{2}}} + gk^m\int\frac{dx}{v^{m+\frac{1}{2}}}.$$

Scholion 1. [p. 336]

627. Si curva *BL* super *AB* ascenderet, tum curva *AM* quoque sursum vergeret atque loco descensus problemati satisfaceret ascensus super illa curva. Fit enim hoc casu abscissa *x* negativa ideoque et eius elementum *dx*; quamobrem habebitur ista aequatio

$$ds = \frac{-gk^m dx - k^m dv}{v^m},$$

quae oritur ex aequatione

$$dv = -gdx - \frac{v^m ds}{k^m}$$

naturam ascensus continente. Simili modo, si curva *BL* ita est comparata, ut rursus ascendat, tum curva *AM* quoque sursum dirigetur et partim descensu partim ascensu conditioni praescriptae satisfaciet.

Exemplum 1.

628. Si quaeratur curva *AM*, super qua corpus aequabiliter moveatur, celeritate scilicet altitudini *b* debita, erit *BL* linea recta parallela axi AP atque $v = b$. Hanc ob rem erit

$$ds = \frac{gk^m dx}{b^m}.$$

Unde sequitur lineam AM fore rectam inclinatam et cosinum anguli, quem cum verticali AP constituet, fore = $\frac{b^m}{gk^m}$ posito 1 pro sinu toto. Quo igitur maior fuerit b seu celeritas, qua corpus ferri debet, eo minor erit angulus cum verticali *AP*, atque si fuerit $b^m = gk^m$, tum linea quaesita ipsa erit verticalis *AP*. At si b^m maior proponeretur quam gk^m, tum solutio perduceret ad imaginarium, ad angulum scilicet, cuius cosinus esset maior sinu toto. Tempus porro, quo lineae portio *AM* descensu absolvitur, erit = $\frac{gk^m x}{b^m\sqrt{b}} = \frac{s}{\sqrt{b}}$.

Exemplum 2. [p. 337]

629. Quaeratur curva AM, super qua corpus ita descendat, ut eius celeritas in singulis punctis sit ut radix quadrata ex altitudine AP, quae est proprietas omni motui in vacuo competens. Erit igitur $v = \alpha x$ et $dv = \alpha dx$ hisque substitutis habebitur

$$ds = \frac{gk^m dx - \alpha k^m dx}{\alpha^m x^m} \text{ et } s = \frac{(g-\alpha)k^m x^{1-m}}{(1-m)\alpha^m},$$

ubi constantis additione non est opus, si $m < 1$. At si $m = 1$, curva est tractoria super linea horizontali per *A* transeunte descripta; super qua corpus ab infinita distantia, concursu scilicet tractoriae cum asymtoto, descensum incipit. Simili modo, si $m > 1$, curva formam habebit tractoriae similem. Semper autem esse debet $\alpha < g$; ex quo perspicitur corpus in medio resistente non tantam acquirere posse celeritatem quantam in vacuo. Deinde quia *ds* maius esse debet quam *dx*, erit $(g-\alpha)k^m > \alpha^m x^m$; corpus igitur profundius descendere nequit quam per altitudinem = $\frac{k}{\alpha}\sqrt[m]{(g-\alpha)}$; quo loco curvae tangens erit verticalis curvaque punctum reversionis habebit. Tempus autem, quo corpus per arcum *AM* descendit, est =

$$\int \frac{ds}{\sqrt{\alpha x}} = \frac{2(g-\alpha)k^m x^{\frac{1}{2}-m}}{(1-2m)\alpha^{m+\frac{1}{2}}}.$$

[p. 338] Quare nisi sit $m < \frac{1}{2}$, tempus non potest esse finitum, sed est infinite magnum; nam si $m = \frac{1}{2}$, curva erit cyclois deorsum versa, super cuius vertice A corpus perpetuo permanebit.

Corollarium 6.

630. Perspicitur ergo in medio resistente motum non ultra datum punctum posse continuari, ita ut celeritates semper sint in subduplicata ratione altitudinum.

Corollarium 7.

631. Ex his apparet omnes curvas hoc modo inventas habere in *A* tangentem horizontalem. Quamdiu ergo radius osculi in *A* est finitae magnitudinis, corpus nunquam descendet. At si radius osculi fit infinite parvus, quod evenit, si $m < \frac{1}{2}$, tum corpus descendere poterit; id quod ex eo, quod tempus fit finitum, intelligitur.

Scholion 2.

632. Si corpus ex *A* descensum incipiat atque curvae *AM* in *A* tangens non fuerit horizontalis, tum ipso motus initio resistentia est nulla. erit ergo ibi $dv = gdx$. Quamobrem quo loco *AM* curva prodeat non habens in *A* tangentem horizontalem, [p. 339] curva *BL*, quae in *A* conveniet cum *AP*, ita debet esse comparata, ut in ipso initio sit $v = gx$; seu tangens curvae *BL* in *A* cum axe *AP* angulum constiture debet, cuius tangens sit = *g*; alioquin enim curva *AM* non angulum acutum cum *AP* conficeret. Magis autem descendendo semper esse debet $v < gx$; in medio enim resistente celeritas ex quacunque altitudine acquisita minor est celeritate, quae in vacuo ex eadem altitudine acquiritur. Porro in medio resistente corpus maiorem celeritatem acquirere non potest, quam si per lineam verticalem delaberetur; quia enim linea verticalis est brevissima et citissime descensu absolvitur, corpus quam minime resistentiae actioni est expositum. Quamobrem in medio resistente curva *BL* ita debet esse comparata, ut *v* ubique sit minor quam altitudo debita celeritati, quae a corpore in eodem medio resistente per *AP* cadendo acquiritur. Ubi enim *v* hanc altitudinem superat, ibi curva *AM* fit imaginaria.

Scholion 3.

633. Simili modo res se habet, si pro ascensu detur curva *BLD* (Fig. 70), cuius applicatae *PL* sint altitudines debitae celeritatibus corporis ascendentis super curva invenienda *AME* in punctis *M*. Dictis enim *AP = x, PL = v* et *AM = s* erit

$$ds = \frac{-gk^m dx - k^m dv}{v^m} .$$

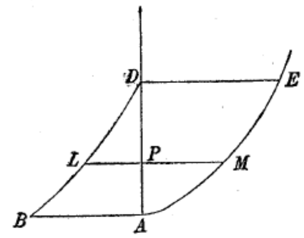

Fig. 70.

Ne igitur *ds* sit negativum, oportet, ut *dv* habeat valorem [p. 340] negativum, i. e. ut curva *BL* continuo ad axem *AD* convergat. Deinde etiam $-dv$ maius esse debet quam *gdx* seu tangens curvae *BL* ubique cum axe *AP* maiorem angulum constiture debet, quam est is, cuius tangens est = *g*. Neque vero hoc sufficit, sed praeterea $-dv - gdx$ maius esse debet quam $\frac{v^m dx}{k^m}$ seu

$$-dv > \frac{(gk^m + v^m)dx}{k^m} ,$$

sive differentia inter $-dv$ et *gdx* maior esse debet quam $\frac{v^m dx}{k^m}$. Haec postrema conditio huc redit, ut *PL* sit minor quam altitudo debita celeritati, quam corpus in *P* haberet, si ex *A* celeritate altitudini *AB* debita per *AP* ascendisset. In ascensu enim per lineam verticalem corpus pro ratione altitudinis percursae minimum celeritatis detrimentum a resistentia patitur. In ipso autem puncto *D* angulus *ADB* tantus esse debet, ut eius tangens sit = *g*, quia prope punctum *E*, in quo celeritas est nulla, resistentiae effectus evanescit; sin vero iste angulus esset maior, curva *AME* in *E* tangentem haberet horizontalem, uti in praecedente scholio quoque de descensu monuimus.

PROPOSITIO 72.

Problema. [p. 341]

634. *Si detur curva AM* (Fig.71), *super qua corpus in vacuo moveatur, invenire curvam am, super qua corpus in medio resistente ita descendat, ut celeritas in a aequalis sit celeritati in A et sumtis arcubus AM et am aequalibus ut celeritates in singulis punctis M et m sint quoque aequales.*

Solutio.

Ductis axibus verticalibus *AP* et *ap* et horizontalibus *MP, mp* sit *AM* = *am* = *s*, *AP* = *t* et *ap* = *x*; propter curvam *AM* datam dabitur aequatio inter *s* et *t*. Iam sint celeritates in punctis *A* et *a* debitae altitudini *b* et celeritates in *M* et *m* debitae altitudini *v*. Sit potentia absoluta deorsum sollicitans *g* et resistantia ut potestas exponentis 2*m* celeritatum. His positis erit pro motu in vacuo super curva *AM*

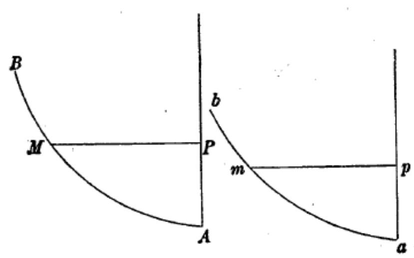

Fig. 71.

$$dv = -gdt \text{ seu } v = b - gt$$

et pro descensu in medio resistente super curva *ma* erit

$$dv = -gdx + \frac{v^m ds}{k^m};$$

in qua aequatione si loco *dv* et *v* substituantur valores ex priori aequatione inventi, prodibit

$$-gdt = -gdx + \frac{(b-gt)^m ds}{k^m} \text{ seu } dx = dt + \frac{(b-gt)^m ds}{gk^m}.$$

Quia autem datur aequatio inter *t* et *s*, si loco *t* eius valor in *s* substituatur, habebitur aequatio inter *x* et *s* pro curva quaesita *am*. Q.E.I.

Corollarium 1. [p. 342]

635. Si in curva *AM* punctum *B* fuerit initium descensus ideoque eius altitudo supra $A = \frac{b}{g}$, habebitur quoque in curva *am* initium descensus *b* sumendo arcum *amb* = *AMB.*

Corollarium 2.

636. Ex solutione apparet esse semper *dx* > *dt*; quare altitudo *ap* maior erit quam altitudo *AP*; in medio enim resistente maiore opus est altutudine ad eandem celeritatem generandam quam in vacuo.

Corollarium 3.

637. Quia in curvis *AM* et *am* sumtis aequalibus arcubus celeritates in illis locis sunt aequales, tempora quoque , quibus aequales arcus *AM* et *am* describuntur, erunt aequalia. Atque ideo tempus descensus in medio resistente per *bma* aequale est tempori descensus in vacuo per *BMA*.

Corollarium 4.

638. Ne curva bma fiat imaginaria, oportet, ut sit ubique $dx < ds$. Hanc ob rem debet esse

$$dt + \frac{(b-gt)^m ds}{gk^m} < ds \text{ seu } gk^m dt < gk^m ds - (b-gt)^m ds .$$

Hoc autem ita se habet, si fuerit [p. 343]

$$gk^m dt < (gk^m - b^m)ds ,$$

qui est casus, si t evanescit et pertinet ad punctum a, nisi t alicubi habeat valorem negativum. Quodcirca ad hoc tantum est respiciendum, ut punctum a fiat reale, quod evenit, si $gk^m dt$ non maius fuerit quam $(gk^m - b^m)ds$.

Corollarium 5.

639. Ne igitur curva *am* fiat imaginaria, ante omnia necesse est, ut sit $b < k\sqrt[m]{g}$. Sit in puncto A $ds = \alpha dt$; erit α numerus unitate maior et ideo erit

$$gk^m < \alpha(gk^m - b^m)ds \text{ seu } b < k\sqrt[m]{\frac{g(\alpha-1)}{\alpha}} .$$

Si ergo curva *MA* in *A* habeat tangentem horizontalem, debet esse $b < k\sqrt[m]{g}$ propter $\alpha = \propto$.

Corollarium 6.

640. Sit autem, si fuerit $ds = \alpha dt$ in puncto A, $b^m = \frac{g(\alpha-1)}{\alpha} k^m$; erit in ipso puncta a

$$dx = \frac{ds}{\alpha} + \frac{(\alpha-1)ds}{\alpha} = ds.$$

Hoc ergo casu curva *am* in *a* tangentem habebit verticalem.

Corollarium 7.

641. In initio motus in *B* fit $b = gt$ seu $t = \frac{b}{g}$. Pro puncto *b* igitur erit $dx = dt$ curvarum elementis sumtis aequalibus. Quare tangentes in punctis *B* et *b* aequaliter erunt inclinatae.

Scholion 1. [p. 344]

642. Quia curva am non absolute ex curva *AM* constui potest, sed praeterea nosse oportet celeritatem in puncto *A* seu descensus initium *B*, si in curva *AM* aliud descensus initium accipiatur, alia invenietur curva am. Curvae ergo *BMA* et *bma* ratione unici descensus tantum ita congruunt, ut celeritates aequalibus percursis spatiis sint inter se aequales, et si in aliis punctis initia descensus ponantur, haec convenentia non amplius locum habebit. Non igitur dantur duae curvae, super quibus descensus omnes ad datum punctum inter se congruant, altera in vacuo, altera in medio resistente constituta.

Exemplum 1.

643. Sit *AMB* linea recta utcunque inclinata, ita ut sit $s = \alpha t$, et quaeratur curva *amb*, super qua corpus simili modo progrediatur in medio resistente quo super *AMB* in vacuo. Posito autem $\frac{s}{\alpha}$ loco *t* prodibit sequens aequatio inter *x* et *s* pro curva quaesita *amb*

$$dx = \frac{ds}{\alpha} + \frac{(\alpha b - gs)^m}{g\alpha^m k^m} ds,$$

cuius integralis est

$$x = \frac{s}{\alpha} + \frac{\alpha b^{m+1}}{(m+1)g^2 k^m} - \frac{(\alpha b - gs)^{m+1}}{(m+1)g^2 \alpha^m k^m}.$$

[p. 345] Ne punctum *a* fiat imaginarium, oportet, ut

$$\frac{1}{\alpha} + \frac{b^m}{gk^m} < 1.$$

Nam si fuerit

$$b^m = \frac{(\alpha - 1)gk^m}{\alpha},$$

curva in *a* habebit tangentem verticalem neque propterea *b* maiorem habere poterit valorem. Ponatur igitur corpus super linea inclinata *BMA* ex tanta altitudine descendisse, ut fiat

$$b = k\sqrt[m]{\frac{g(\alpha - 1)}{\alpha}};$$

erit

$$dx = \frac{ds}{\alpha} + \frac{(k\sqrt[m]{g\alpha^{m-1}(\alpha - 1)} - gs)^m ds}{g\alpha^m k^m},$$

quae est aequatio pro curva *amb*, in qua initium descensus in *b* est capiendum, ubi est $ds = \alpha dx$, seu arcus *amb* erit =

$$k\sqrt[m]{\frac{\alpha^{m-1}(\alpha - 1)}{g^{m-1}}}.$$

Si resistentia fuerit quadratis celeritatum proportionalis, erit *m* = 1 ideoque

$$dx = \frac{ds}{\alpha} + \frac{(gk(\alpha - 1) - gs)ds}{g\alpha k} = ds - \frac{sds}{\alpha k}$$

et integrando

$$x = s - \frac{ss}{2\alpha k}.$$

Quae est aequatio pro cycloide super base horizontali descripta, cuius circuli generatoris diameter est $\frac{\alpha k}{2}$.

Corollarium 8.

644. Sit ergo super basi horizontali *CB* (Fig. 72) descripta cyclois *AMB* circulo generatore *ANC* et sit medium resistens in duplicata ratione celeritatum, cuius exponens sit = *k*. Si nunc in circulo *ANC* sumatur chorda $AN = \frac{k}{2}$ ducatur horizontalis *PNM* et ex *M* tangens *MT* atque duo corpora ponantur descendere, alterum super *MT* in vacuo et alterum super curva *MB* in medio resistente, ambo haec corpora aequalibus temporibus aequalia spatia absolvent. [p. 346]

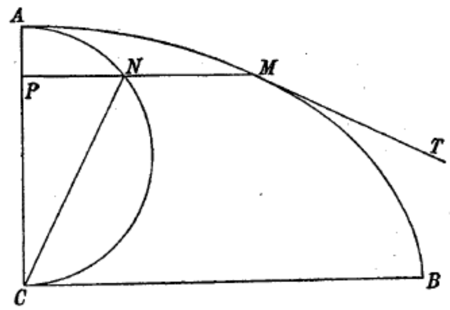

Fig. 72.

Exemplum 2.

645. Sit curva *AMB* (Fig. 71) cyclois deorsum spectans, cuius circuli generatoris diameter $\frac{a}{2}$; erit *ss* =*2at* et $t = \frac{ss}{2a}$ atque $dt = \frac{sds}{a}$. His ergo substitutis prodibit pro curva *amb* sequens aequatio

$$dx = \frac{sds}{a} + \frac{(2ab - gss)^m ds}{2^m g a^m k^m};$$

vel si totus arcus *AMB*, qui in vacuo descensu absolvi ponitur, vocetur *c*, erit $b = \frac{gcc}{2a}$ ideoque pro curva *amb* orietur ista aequatio

$$dx = \frac{sds}{a} + \frac{g^{m-1}(cc - ss)^m ds}{2^m a^m k^m}.$$

Ne igitur haec curva in puncto *a* fiat imaginaria, oportet sit $g^{m-1} < 2^m a^m k^m$, vel altitudo arcus AB minor esse debet quam $\frac{k}{g}\sqrt[m]{g}$; nam si fuerit altitudo arcus $AB = \frac{k}{g}\sqrt[m]{g}$, tum curvae *amb* tangens in *b* erit verticalis. Si ergo *B* sit cuspis cyclois, erit $\frac{cc}{2a} = \frac{a}{2}$ seu *c* = *a*, et si preaterea $g^{m-1} a^m = 2^m k^m$, quo curvae *amb* in *a* tangens fiat verticalis, habebitur ista aequatio

$$dx = \frac{sds}{a} + \frac{(aa - ss)^m ds}{a^{2m}};$$

quae curva in duobus punctis *a* et *b* habebit tangentes verticales.

Corollarium 9. [p. 347]

646. Cum igitur corpus in vacuo super cycloide ita descendat, ut eisus accelerationes sint spatiis percurrendis proportionales, eandem proprietatem habebit descensus in medio resistente super curva *bma*, si initium descensus sumatur in puncto *b*, quod per aequationem ad curvam amb determinatur; erit scilicet *amb* = *c*.

Corollarium 10.

647. Si in curva *AMB* aliud descensus initium *B* capiatur, tota curva amb alia reperietur, quia in eius aequatione longitudo arcus *AMB* = *c* continetur. Quare, etiamsi cyclois sit curva tautochrona in vacuo, curva amb talis tamen non erit in medio resistente, quia pluribus descentibus super *AMB* totidem curvae diversae in medio resistente respondent.

Scholion 2.

648. Curvae hoc exemplo erutae sunt eae ipsae, quas Clar. Hermannus in Comm. Tom. II [Iac. Hermann, *Theoria generalis motuum, qui nascuntur a potentiis quibus in corpora indesinenter agentibus*. Commen. acad. sc. Petrop. 2 (1727), 1729, p. 139.] pro tautochronis in mediis resistentibus invenit; sed simul ipse demostravit eas quaesito satisfacere non posse. Ceterum ex his intelligitur simili modo in medio resistente curvam posse inveniri, super qua corpus ascendendo eodem modo moveatur quo super data curva in vacuo. [p. 348] Sint enim puncta *A* et a initia ascensus, illud in vacuo, hoc in medio resistente, sitque celeritas initialis debita altitudini *b*; habebitur pro curva amb ista aequatio

$$dx = dt - \frac{(b-gt)^m ds}{gk^m};$$

ex qua aequatione intelligitu curvam amb non fieri posse imaginariam, nisi ipsa curva *AMB* talis fuerit. Nam quia, ne curva amb sit imaginaria, esse debet $dx < ds$, hic dx minus est quam dt, quod per se minus est quam ds. Ut si linea *AMB* fit linea verticalis, altera *amb* poterit assignari; nam posita $AB = c$ erit $b = gc$ et $s = t$; quare pro curva *amb* invenitur ista aequatio

$$dx = ds - \frac{g^{m-1}(c-s)^m ds}{k^m},$$

cuius integralis est

$$x = s + \frac{g^{m-1}(c-s)^{m+1} - g^{m-1}c^{m+1}}{(m+1)k^m}.$$

Accommodetur haec aequatio ad resistentiam quadratis celeritatum proportionalem; fiet $m = 1$ ideoque erit

$$x = s + \frac{(c-s)^2 - c^2}{2k} = \frac{2(k-c)s + ss}{2k},$$

quae est ad cycloidem hoc modo : describatur cyclois *AMB* (Fig.72) circulo generatore diametri $AC = \frac{k}{2}$ super basi horizontali *BC*; tum capiantur arcus $AM = k - c$; erit *M* initium ascensus, ex quo puncto, si corpus super *MA* ascendat celeritate altitudini *gc* debita, in medio resisten in duplicata ratione celeritatum eodem modo movebitur quo in vacuo eadem celeritate initiali sursum ascendens verticaliter.