CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

PROPOSITION 64. [p. 292]

Problem.

565. *If the resistance of the medium is partially constant and partially proportional to the square of the velocity, to determine the motion of a body oscillating on the cycloid MCB* (Fig.66), *even in the case in which the resistance is very small.*

Solution.

As before let the diameter of the generating circle be $CD = \frac{1}{2}a$, $CP = x$ and the arc $CM = s$. The speed at C is put to correspond to the height b and the speed at M to the height v. The force of the body always acts downwards and is equal to g, the part of the resistance which is constant is equal to h and the part of the resistance proportional to the squares of the speeds is equal to $\frac{v}{k}$ as before; k is an amount very small with respect to v and s, and both a and h are very small with respect to g. Now let the descent be made on the arc MC; then

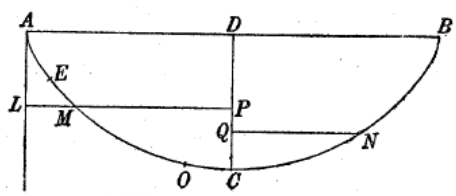

Fig. 66.

$$dv = -gdx + hds + \frac{vds}{k}$$

[In modern terms, we can view this as a work–potential energy transformation, in the same element of time, by the various forces acting over their respective increments] and hence [p. 293]

$$v = e^{\frac{s}{k}}b - e^{\frac{s}{k}}\int e^{-\frac{s}{k}}(gdx - hds)$$

[It may be useful to write down the coordinates of the cycloid, which shares the same coordinates as a point on the uniformly translated and rotated generating circle. We take the origin at C, where the common point lies at the lowest point of the cycloid; the circle with diameter $2a$ rolls along DA at a uniform speed v, and at M its coordinates are $PM = y$, and $CP = x$. Now, the angle turned through by the radius $a/4$ is given by θ and hence

$x=\frac{a}{4}(1-\cos\theta)$, while y is the sum of the distances $\frac{a\theta}{4}+\frac{a}{4}\sin\theta$. Now, it is easily shown that $ds=\frac{a}{2}\cos\theta/2.d\theta$, and the arc CM or $s=a\sin\theta/2$,
while $dx/ds=dx/d\theta.d\theta/ds=\sin\theta/2$. Hence the result $dx=\frac{sds}{a}$ follows.]

But since from the nature of the cycloid curve, $dx=\frac{sds}{a}$, then

$$\int e^{-\frac{s}{k}}gdx=\frac{gk^2-gk^2e^{-\frac{s}{k}}-gke^{-\frac{s}{k}}s}{a}$$

and

$$\int e^{-\frac{s}{k}}hds=hk-hke^{-\frac{s}{k}};$$

hence the equation becomes:

$$v=\frac{e^{\frac{s}{k}}(ab+hak-gk^2)-hak+gk^2+gks}{a}$$

The maximum speed is obtained, if :

$$\frac{gs}{a}=h+\frac{v}{k}$$

or

$$e^{\frac{s}{k}}=\frac{gk^2}{gk^2-ab-hak}$$

A height equal to c is said to correspond to the maximum speed at O; then the arc

$$CO=\frac{ha}{g}+\frac{ac}{gk}$$

and

$$ab=gk^2-hak-gk^2e^{\frac{-ac-hak}{gk^2}}$$

Putting the arc $MO=q$; then

$$s=\frac{ha}{g}+\frac{ac}{gk}+q;$$

hence

$$v=\frac{ac+gk^2+gkq-e^{\frac{q}{k}}gk^2}{a}.$$

Since now v is less than c, I put $c-v=z$; then

$$az+gk^2+gkq=e^{\frac{q}{k}}gk^2.$$

Which equation converted into a series gives, as above :

$$\frac{az}{g}=\frac{q^2}{2}+\frac{q^3}{6k}+\frac{q^4}{24k^2}$$

and

$$q=\frac{\sqrt{2az}}{\sqrt{g}}-\frac{az}{3gk}+\frac{az\sqrt{2az}}{18gk^2\sqrt{g}}.$$

The descent starts from M; where $v=0$ and $z=c$ and thus

$$MO = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

The ascent arc is found from the same formula

$$ON = \frac{\sqrt{2ac}}{\sqrt{g}} + \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}};$$

and since h is not found in these, the time of a semi–oscillation along *MON* is as above equal to

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}}.$$

Now the whole of the descent arc *MC* is equal to

$$\frac{ha}{g} + \frac{\sqrt{2ac}}{\sqrt{g}} + \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}$$

and the ascent arc

$$CN = -\frac{ha}{g} + \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

[p. 294] Whereby if the descent arc *MC* is put equal to *E* and the ascent arc *CN* is equal to *F*, then

$$F = E - \frac{2ha}{g} - \frac{2E^2}{3k} + \frac{4haE}{3gk} + \frac{4E^3}{9k^2}.$$

In the following semi–oscillation the descent arc is F and the ascent arc is

$$G = E - \frac{4ha}{g} - \frac{4E^2}{3k} + \frac{16haE}{3gk} + \frac{16E^3}{9k^2}.$$

And generally in that semi–oscillation, which is indicated by the number n, the ascent arc is equal to

$$E - \frac{2nha}{g} - \frac{2nE^2}{3k} + \frac{4n^2haE}{3gk} + \frac{4n^2E^3}{9k^2} = \frac{3gkE - 6hnak}{3gk + 2gnE}.$$

Whereby if n semi–oscillations are performed, and the first descending arc is called *E* and the last ascending arc is called *L*, then

$$2gnEL = 3gk(E - L) - 6hnak$$

or

$$n = \frac{3gk(E-L)}{2gEL + 6hak}.$$

Now the time, in which some semi–oscillation is completed along *MCN* , is equal to

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi(gE-ha)^2\sqrt{2a}}{24g^2k^2\sqrt{g}}$$

with this value put in place of *c*. Q.E.I.

Corollary 1.

566. If on placing $c = 0$, as the point produced at which the body is at rest, it is found that

$$MC = \frac{ha}{g};$$

hence the body can remain at rest not only at the point *C*, but beyond *C* also at a distance $\frac{ha}{g}$ on either side of *C*. Whereby in a medium of this kind it is understood that a pendulum can be in a state of rest not exactly in a vertical line, but can disagree by an angle of which the sine is $\frac{h}{g}$.

Scholium 1. [p. 295]

567. Resistance of this kind is easily demonstrated from experiments having water in place, obviously in the slowest motions a resistance proportional to the square of the speeds is rarely observed; now in a fluid in addition to resistance proportional to the square of the speeds it is not probable to give another, except a constant resistance. This is confirmed from the experiments set up by La Hirio, in which he showed that it is possible for a pendulum to remain in place in water beyond the vertical position. Which could not occur if the resistance only depended on the speed. From Newton's experiments, which he conducted regarding the retarded motion of pendulums in air, it can be concluded that the constant resistance of a lead ball of diameter 2 inches to be around a millionth part of the weight, or $\frac{h}{g} = \frac{1}{1000000}$. [*Principia*, 2nd ed. Lib. II, sect. VI. This seems to be missing from the 3rd ed. See Cohen's translation.]
Hence here the globe suspended by a thread can depart from the vertical by an angle of around 10''', which error moreover is beyond measurement. But this error can be greater and sensitive to measurement, when a smaller and likewise lighter globe is used.

Corollary 2.

568. It can be deduced on making use of that equation (565) from above, in finding the angle that

$$\frac{h}{g} = \frac{E-L}{2na} - \frac{EL}{3ak}$$

which is equal to the sine of the angle that the pendulum is able to depart from the vertical. But in place of *E* it is convenient to take a small arc in which the higher order terms become negligible

Corollary 3. [p. 296]

569. From the equation

$$n = \frac{3gk(E-L)}{2gEL+6hak}$$

it is apparent, when the arc described by the oscillation is greater, with that the term $6hak$ becomes smaller with respect to $2gEL$. And it is for this reason, that this resistance is perceived only in the smallest oscillations.

Corollary 4.

570. Because h is a very small number according to the hypothesis in a fine medium, to which this proposition has been applied, for the quantity ha almost vanished also before gE and thus the time of a single semi–oscillation equal to :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}.$$

Therefore the constant resistance does not change the times of the oscillations.

Scholium 2.

571. Therefore even if this constant resistance is considered to be joined with the resistance proportional to the squares of the speeds, the calculation for that does not become more extended or difficult. For from the maximum speed $\sqrt{c}$, the whole arc described for one semi–oscillation determined in the same way, if this constant resistance is present or not ; and indeed in either case the same equation is plainly obtained. On this account or from that to follow, it is seen that this law of resistance has a place, now that other resistances besides this and that which is proportional to the squares of the speeds, actually have not been found. [p. 297] Moreover fluids now for a long time have been noted to exercise a twofold resistance, the one proportional to the square of the speeds (Book I, (490)), which are observed in faster speeds only, and the other perceptible in the slowest motions only. The former resistance arises from the force of inertia of the particles of the fluid and through that the body loses its motion, when these particles remove it; since the proportionality to the squares of the speeds cannot be doubted. Now this resistance has arisen from the tenacity of the fluid, by which the particles of the fluid cohere together and in turn and in turn can hardly be separated from one another. Therefore while the body is moving through a given interval, a given number of particles proportional to the interval must be separated from each other; whereby this resistance agrees with the absolute force retarding the motion of the body, clearly which also in equal intervals exercise an equal number of impulses on the body. Therefore this resistance or force is always opposite to the motion of the body and it is a constant retarding tangential force. But in this case the nature demands an exception from the calculation, when the body is at rest. For since this force is constant, it must act equally when the body is at rest and in motion ;
[What Euler has in mind may be the fluid equivalent of static friction as opposed to kinetic friction; a body in static equilibrium can have frictional forces acting on it, though these are not usually of the same size as the constant contact kinetic frictional force acting on a moving body, and in the case of a body at rest on an inclined plane or on a

curve, that can adjust themselves up to a maximum value. At this time, the laws of friction both of the fluid and the contact kind had not been formulated properly.] moreover the resting body, since the particles of fluid are not being separated, this force cannot be considered. To this it is agreed, since this force is opposite the direction of motion of the body, and in the resting body, since it has no direction, it cannot have any effect. But if the motion is investigated upon a curved line, the tangent to the curve always has the direction of the motion, even if is actually at rest, [p. 298], and thus the calculation shows the effect of this force even on the resting body ; hence in this case it is necessary to make an exception to the calculation. Thus it can be said that a pendulum body hence is able to remain at rest in some small interval around *C* , since the striving of this body towards C is not sufficient to separate the particles of fluid from each other. Whereby the body is able to remain at rest at some point of this interval, even if the calculation also shows that the body cannot remain at rest at the point *C*.

Scholium 3.

572. From these, which we have treated generally in part, and in part brought to bear on the cycloid, it is evident how in a medium with resistance in the square ratio of the speeds, the motion of the body on any curve can be determined. Indeed we have considered the medium resisting uniformly and an equal force also to be acting; but from the equation to be resolved that also it can be integrated, as in whatever way the medium is made non–uniform and a variable force acting as well; for always in the equation the height *v* corresponding to the speed has only a single power. Therefore I proceed to other hypotheses of resistances of the medium ; but since then the motion cannot be defined for any curve, first curves are to be found, which allow the motion to be determined. [p. 299] Here we assume these curves, according to our custom, which lead to homogeneous equations, in which the indeterminates maintain a number of the same dimension. If the resistance is proportional to the 2*m* exponent of the speed, this equation is obtained :

$$dv = \pm g dx \pm \frac{v^m ds}{k^m};$$

which since it is homogeneous between *v* and *x*, must be

$$ds = x^{-m}dx, \text{ or } s = a^m x^{1-m}\text{ ; or } x = a^{\frac{m}{m-1}} s^{\frac{1}{1-m}}.$$

Or if *x* and *s* are increased or decreased by given amounts, on the curve of which this is the equation :

$$x == a^{\frac{m}{m-1}}(s+f)^{\frac{1}{1-m}} - a^{\frac{m}{m-1}} f^{\frac{1}{1-m}},$$

the motion can also be determined. Hence in a medium with resistance in the simple ration of the speed the curve is a cycloid, and thus we can determine the motion on that.

PROPOSITION 65.

Problem.

573. *In a medium that resists in the simple ratio of the speeds, to determine the oscillatory motion of the body on the cycloid ACB* (Fig.66) *with the medium present as with the force acting to be uniform.*

Solution.

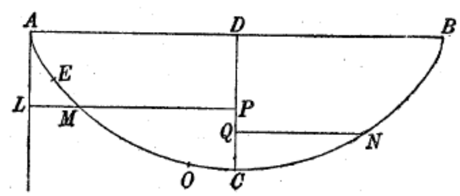

Fig. 66.

Again as before let the diameter of the generating circle be $CD = \frac{1}{2}a$, $CP = x$ and the arc $CM = s$. [p. 300] The speed at *C* is put to correspond to the height *b* and the speed at *M* to the height *v*. The force of the body always draws downwards and is equal to *g*, and the resistance is equal to $\frac{\sqrt{v}}{\sqrt{k}}$. The descent is made on the part *AMC*; the descent produced is then

$$dv = -gdx + \frac{ds\sqrt{v}}{\sqrt{k}}$$

on putting

$$\frac{\sqrt{v}}{\sqrt{k}} = \frac{u}{a};$$

then

$$v = \frac{ku^2}{a^2} \text{ et } dv = \frac{2kudu}{a^2},$$

hence it becomes

$$2kudu = -gasds + auds,$$

which equation must thus be integrated, so that on making s = 0 then $u = \frac{a\sqrt{b}}{\sqrt{k}} = \frac{u}{a}$. Now for the ascent on the arc *CN* this equation is obtained:

$$2kudu = -gasds - auds,$$

Putting $u = ps$, for the descent there is had :

$$2kp^2sds + 2kps^2dp = -gasds + apsds$$

or

$$\frac{2kpdp}{ap - ga - 2kp^2} = \frac{ds}{s}.$$

Hence on integration it becomes :

$$ls = lC - \frac{1}{2}l\left(p^2 - \frac{ap}{2k} + \frac{ga}{2k}\right) + \frac{a}{2\sqrt{(a^2-8gak)}}\,l\frac{4kp - a + \sqrt{(a^2-8gak)}}{4kp - a - \sqrt{(a^2-8gak)}}$$

$$= lC - \frac{1}{2}l\frac{2a^2v\sqrt{k} - a^2s\sqrt{v} + gas^2\sqrt{k}}{2ks^2\sqrt{k}} + \frac{a}{2\sqrt{(a^2-8gak)}}\,l\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}$$

or (with the meaning of the letter *C* changed.)

$$lC = l(2a^2v\sqrt{k} - a^2s\sqrt{v} + gas^2\sqrt{k}) - \frac{a}{\sqrt{(a^2-8gak)}}\,l\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}.$$

Putting $s = 0$ and $v = b$; there becomes $lC = l2a^2b\sqrt{k}$; hence,

$$\frac{2av\sqrt{k} - as\sqrt{v} + gs^2\sqrt{k}}{2ab\sqrt{k}} = \left(\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

Now in the other part of the curve *CN* for the ascent of the body, on putting $CN = s$ this equation is obtained :

$$\frac{2av\sqrt{k} + as\sqrt{v} + gs^2\sqrt{k}}{2ab\sqrt{k}} = \left(\frac{4a\sqrt{kv} + as - s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} + as + s\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

If the height corresponding to the maximum speed, which is at *O*, is called *c*, then

$$CO = \frac{a\sqrt{c}}{g\sqrt{k}}$$

and

$$c = b\left(\frac{4gk - a + \sqrt{(a^2-8gak)}}{4gk - a - \sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

[p. 301] Moreover these equations cannot be considered, unless we have $a^2 > 8gak$ or $k < \frac{a}{8g}$. For if $k > \frac{a}{8g}$, the equations depend on logarithms and likewise on the quadrature of the circle

We can put $k = \frac{a}{8g}$ and then

$$\frac{ds}{s} = \frac{-pdp}{\left(p - \frac{a}{4k}\right)^2} = \frac{-dp}{p - \frac{a}{4k}} - \frac{adp}{4k\left(p - \frac{a}{4k}\right)^2}.$$

Hence on integration there is produced :

$$ls = lC - l\left(p - \frac{a}{4k}\right) + \frac{a}{4kp - a} = lC - l\left(\frac{a\sqrt{v}}{s\sqrt{k}} - \frac{a}{4k}\right) + \frac{as}{4a\sqrt{kv} - as}$$

(or by changing the meaning of C) , this is equal to :

$$lC - l\,(4a\sqrt{kv} - as) + ls + \frac{as}{4a\sqrt{kv} - as}.$$

This therefore becomes $lC = l4a\sqrt{kb}$; hence there is obtained :

$$l\,\frac{4\sqrt{kv} - s}{4\sqrt{kb}} = \frac{s}{4\sqrt{kv} - s}.$$

Hence it is apparent that in the descent the speed can never be equal to zero; for $4\sqrt{kv}$ has to be greater than s always. Therefore in this case, if the speed at the point C is real, the start of the descent shall be imaginary. Whereby, wherever the body starts the descent, the speed at the lowest point C is equal to 0. Therefore the motion has to be investigated, if the descent is from some given point E and it is the case that $CE = f$, then $lC = l - af + 1$ and thus

$$l\,\frac{s - 4\sqrt{kv}}{f} = \frac{4\sqrt{kv}}{4\sqrt{kv} - s}.$$

From which it is understood that it must always be the case that $4\sqrt{kv} < s$, on account of which at the point C, where $s = 0$, also it is the case that $v = 0$. The maximum speed, which is at O, is had by putting

$$\sqrt{v} = \frac{gs\sqrt{k}}{a} \text{ or } s = 8\sqrt{kv}\,,$$

with which in place, there is produced :

$$l\frac{s}{2f} = -1 \text{ or } s = \frac{2f}{e} = CO$$

with e denoting the number, of which the logarithm is equal to 1. Therefore in these cases, in which the arc of the descent is real, the ascent arc is zero. But if the body ascends through the arc CN with a speed at C corresponding to the initial height b, then the motion is expressed by this equation : [p. 302]

$$l\,\frac{4\sqrt{kv} + s}{4\sqrt{kb}} = \frac{-s}{4\sqrt{kv} + s},$$

from which it is apparent to be

$$s + 4\sqrt{kv} < 4\sqrt{kb} \text{ and } \sqrt{v} < \sqrt{b} - \frac{s}{4\sqrt{k}}.$$

The total arc of the ascent CN is found by making $v = 0$ and then it becomes

$$l\frac{s}{4\sqrt{kb}} = -1 \text{ or } CN = \frac{4\sqrt{kb}}{e}.$$

If $k < \frac{a}{8g}$, which case we have treated now, the resistance thus is greater, on account of which a much greater speed at *C* is zero, if indeed the descent if made from a given point, and for the given speed at C, the descent is imaginary. On account of which the equation, that we have deduced for the descent is imaginary, unless the constant is determined from the start of the given descent. Therefore let the arc *CE* be equal to *f*; then we have

$$lC = lgaf^2\sqrt{k} - \frac{a}{\sqrt{(a^2-8gak)}}\, l\,\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}$$

and on making *s* = 0, this becomes :
[the original expression and those that follow have been corrected in the *O.O.*]

$$l\,\frac{gf^2}{2av} = \frac{a}{\sqrt{(a^2-8gak)}}\, l\,\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}$$

if *v* is not equal to 0; for if *v* is equal to 0, then this equation is not valid. Moreover it is apparent that this equation contains a contradiction, since 2*av* must be greater than *gff* or $2v > \frac{gss}{a}$ on putting *s* for *f*. But $\frac{ss}{a} = 2x$ and thus should be that *v* > *gx*, which is absurd; for *in vacuo* only is *v* = *gx* and in a medium with resistance thus it must be smaller. But for the ascent the equation found is of use, and from that the whole arc of the ascent *CN* is found by making *v* = 0, with which in place there is produced :

$$\frac{gs^2}{2ab} = \left(\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}} = \frac{g\cdot CQ}{b}.$$

Therefore from these it is evident, if it should be that either $k < \frac{a}{8g}$ or $k = \frac{a}{8g}$, oscillations cannot be performed, since no ascent is possible after the descent. [The cases of over damped and critically damped systems]. Whereby the most important case remains for us, in which $k > \frac{a}{8g}$, [p. 303] are to be examined ; since indeed in these the resistance is smaller, and however small it is assumed to be, oscillations can always be performed. Therefore with the above substitutions made we have :

$$\frac{-ds}{s} = \frac{pdp}{p^2 - \frac{ap}{2k} + \frac{ga}{2k}},$$

which equation on putting $q = p - \frac{a}{4k}$ turns into this :

$$\frac{-ds}{s} = \frac{qdq + \frac{adq}{4k}}{qq + \frac{ga}{2k} - \frac{a^2}{16k^2}}.$$

On putting

$$\frac{ga}{2k}-\frac{a^2}{16k^2}=B^2,$$

since it is a positive quantity, then we have :

$$lC-ls=l\sqrt{(q^2+B^2)}+\frac{a}{4k}\int\frac{dq}{q^2+B^2}=l\sqrt{(q^2+B^2)}+\frac{a}{4Bk}At.\frac{q}{B},$$

where $At.\frac{q}{B}$ is the arc of a circle, the tangent of which is $\frac{q}{B}$, with the whole sine being equal to 1. Moreover with the value restored for *q* then the equation becomes : (with the constant *C* changed in value)

$$lC=l\sqrt{(2av\sqrt{k}-as\sqrt{v}+gs^2\sqrt{k})}+\frac{a}{4Bk}At.\frac{4a\sqrt{kv}-as}{4Bks}.$$

Put *s* = 0 and *v* = *b* so that *l*C can be defined; then

$$lC=l\sqrt{2ab\sqrt{k}}+\frac{a}{4Bk}At.\infty,$$

whereby we have :

$$l\frac{\sqrt{2ab\sqrt{k}}}{\sqrt{(2av\sqrt{k}-as\sqrt{v}+gs^2\sqrt{k})}}=-\frac{a}{4Bk}At.\frac{4Bks}{4a\sqrt{kv}-as}.$$

Now for the ascent through the arc *CN* it is found that

$$l\frac{\sqrt{2ab\sqrt{k}}}{\sqrt{(2av\sqrt{k}+as\sqrt{v}+gs^2\sqrt{k})}}=\frac{a}{4Bk}At.\frac{4Bks}{4a\sqrt{kv}+as}.$$

Now on putting *v* = 0, the whole arc of the descent *MC* is produced from this equation :

$$l\sqrt{\frac{2ab}{gss}}=\frac{a}{4Bk}At.\frac{4Bk}{a}=l\sqrt{\frac{b}{g\cdot CP}}.$$

And the whole arc of the ascent *CN* is found from this equation

$$l\sqrt{\frac{2ab}{gss}}=\frac{a}{4Bk}At.\frac{4Bk}{a}=l\sqrt{\frac{b}{g\cdot CQ}}.$$

From these equations it can be seen that any [consecutive] arcs of ascent and descent are equal to each other, yet [the sequence of ascending and the sequence of descending] arcs are unequal ; for an infinity of arcs is given of which the tangent $\frac{4Bk}{a}$ is the same, and for one the ascent must be taken, and for the other the descent. And since the infinitude of arcs with the tangents $\frac{4Bk}{a}$ is given [p. 304], any of these can be put in place in the proposition. Indeed from these arcs taken in order, all the arcs of the ascent are produced successively as well as of the descent, while the body performs oscillations ; for since the equation found is general, all these places can be shown in which the speed of the oscillating body at some time is equal to zero. Whereby according to this hypothesis of the resistance, this has the convenience that immediately for any oscillation, say the hundredth, the arc of the descent as well as of the ascent can be defined. Let the arc [of the circle] be D, of which the tangent is $\frac{4Bk}{a}$, and on putting the ratio of the diameter to the periphery as $1:\pi$ then there is the same tangent $\frac{4Bk}{a}$ for all the arcs [of the circle] :

$$D,\ \ \pi+D,\ \ 2\pi+D,\ 3\pi+D, \text{etc.}$$

Now for the arc of descent of the first oscillation MC, the arc [i. e. angle] D must be taken and then

$$\frac{2ab}{gss}=e^{\frac{Da}{2Bk}}$$

or the abscissa of the arc MC is equal to $\frac{b}{g}e^{\frac{-Da}{2Bk}}$. Moreover the abscissa of the following ascending arc, or the abscissa of the descending arc of the second semi–oscillation then equal to

$$\frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}.$$

In a similar manner the abscissa of the descending arc in the third semi–oscillation then equal to

$$\frac{b}{g}e^{\frac{-a(2\pi+D)}{2Bk}}.$$

and generally the abscissa of the arc of the descent in the oscillation , which is indicated by n + 1, is equal to

$$\frac{b}{g}e^{\frac{-a(n\pi+D)}{2Bk}},$$

which likewise is the abscissa of the arc of the ascent in the oscillation which is indicated by the number n. As regards the times of the oscillations, that we reserve to the following proposition. Q.E.I.

Corollary 1. [p. 305]

574. Hence unless it is the case that $k>\frac{a}{8g}$, the oscillations are unable to be completed, since the body is reduced to rest at the end of the first descent, or if either $k<\frac{a}{8g}$ or $k=\frac{a}{8g}$. But if $k>\frac{a}{8g}$, the oscillations endure for ever, since the expression $\frac{b}{g}e^{\frac{-a(n\pi+D)}{2Bk}}$ neither vanishes nor is it able to become negative.

Corollary 2.

575. The descending arc to the following ascending is in a given ratio ; for the ratio of the abscissas is

$$\frac{b}{g}e^{\frac{-Da}{2Bk}} \text{ to } \frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}$$

and thus of the arcs themselves 1 to $e^{\frac{-\pi a}{4Bk}}$, which ratio does not depend on the given speed $\sqrt{b}$.

Corollary 3.

576. And in a like manner the first descending arc of the first semi–oscillation to the ascending arc of the semi–oscillation indicated by the number *n* has a given ratio ; for this ratio is as $e^{\frac{\pi n a}{4Bk}}$ to 1. Whereby if the number of semi–oscillations is twice as great, then this ratio is squared.

Corollary 4.

577. Any number of descending arcs of sem–oscillations following each other constitute a geometric progression decreasing in the ratio 1 to $e^{\frac{-\pi a}{4Bk}}$. And thus the arcs of the whole semi–oscillations are described by a geometric progression of the same denominator. [p. 306]

Scholium 1.

578. Moreover since an infinitude of arcs can be taken for *D*, from which it is apparent, from these which can be taken for the descending arc, I take the case in which $k = \frac{a}{8g}$ and the ascending arc is equal to $\frac{4\sqrt{kb}}{e}$, of this the abscissa is equal to $\frac{8kb}{ae^2} = \frac{b}{g}e^{-2}$; moreover in this case B = 0 and the abscissa of the ascending is equal to $\frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}$. Hence it must be the case that $\frac{a(\pi+D)}{2Bk} = 2$ and $\pi + D = \frac{4Bk}{a} = 0.$ Now $\frac{4Bk}{a}$ is the tangent of the arc $\pi + D$, and as $\frac{4Bk}{a}$ is equal to 0, then $\pi + D$ must correspond to the minimum arc of the tangent $\frac{4Bk}{a}$. Hence it is said that the smallest arc of the tangent $\frac{4Bk}{a}$ corresponds to *E*; and then $D = E - \pi$. Concerning which in the first semi oscillation, the abscissa of the descending arc is equal to

$$\frac{b}{g}e^{\frac{a(\pi-E)}{2Bk}} = \frac{MC^2}{2a}$$

and thus the arc *MC* is equal to

$$e^{\frac{a(\pi-E)}{4Bk}}\sqrt{\frac{2ab}{g}}\ .$$

Putting $\frac{4Bk}{a}$ or the tangent to the arc $E = \tau$; then the descending arc of the first oscillation is equal to

$$e^{\frac{\pi-E}{\tau}}\sqrt{\frac{2ab}{g}},$$

the first ascending arc or the second descending arc of the oscillation is equal to

$$e^{\frac{-E}{\tau}}\sqrt{\frac{2ab}{g}},$$

and the descending arc in the semi–oscillation which is indicated by the number $n + 1$, is equal to

$$e^{\frac{-E-(n-1)\pi}{\tau}}\sqrt{\frac{2ab}{g}},$$

which is likewise the ascending arc in the semi–oscillation in the semi–oscillation indicated by the number n. Hence the denominator of the geometric progression established by these arcs is $e^{\frac{-\pi}{\tau}}$.

Corollary 5. [p. 307]

579. From these also, in whatever semi–oscillation, the speed at the lowest point C can be defined. Indeed in the semi–oscillation which is indicated by the number, let the speed at C correspond to the height β; then the arc of the secant is equal to

$$e^{\frac{-E}{\tau}}\sqrt{\frac{2\alpha\beta}{g}},$$

which must be equal to

$$e^{\frac{-E-(n-1)\pi}{\tau}}\sqrt{\frac{2ab}{g}}.$$

Hence it becomes

$$\sqrt{\beta}=e^{\frac{-(n-1)\pi}{\tau}}\sqrt{b}.$$

Therefore the speed at the point C in successive semi–oscillations constitute a geometric progression, the denominator of which is $e^{\frac{-\pi}{\tau}}$.

Corollary 6.

580. If n is put as a negative number, the semi–oscillations are understood to be those which can be made before the first; as in the semi–oscillation preceeding the first, the descending arc must correspond to $e^{\frac{2\pi-E}{\tau}}\sqrt{\frac{2ab}{g}}$.

Corollary 7.

581. If in the first semi–oscillation, the descent is made from the highest point of the cycloid A, then the descent arc is equal to a. Whereby $\sqrt{\frac{ag}{2b}}=e^{\frac{\pi-E}{\tau}}$ and the speed at the lowest point C or $\sqrt{b}$ is $e^{\frac{E-\pi}{\tau}}\sqrt{\frac{ga}{2}}$.

Corollary 8. [p. 308]

582. If the resistance almost disappears or k becomes an exceedingly large quantity, then

$$B=\sqrt{\frac{ga}{2k}} \text{ and } \tau=\frac{4\sqrt{gk}}{\sqrt{2a}}=\frac{2\sqrt{2gk}}{\sqrt{a}}.$$

Then since τ shall be a very large, then $E=\frac{\pi}{2}$ and the descending arc of the first semi–oscillation is equal to

$$e^{\frac{\pi}{2\tau}}\sqrt{\frac{2ab}{g}}=\left(1+\frac{\pi}{2\tau}\right)\sqrt{\frac{2ab}{g}}$$

and the ascending arc is equal to

$$\left(1-\frac{\pi}{2\tau}\right)\sqrt{\frac{2ab}{g}}.$$

Scholium 2.

583. From the solution of this proposition, among others, it is to be understood that often there is a need for much careful thought in drawing conclusions from the equations. For in the case that $k<\frac{a}{8g}$ the equations that we have found for the ascent and the descent, having thus been compared, so that from these it is seen to follow that the ascending arc is equal to the descending arc ; for on making $v = 0$ there is produced from either the equation:

$$\frac{gs^2}{2ab}=\left(\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

And thus they have also this [ratio] in common, unless the descent by necessity makes $b = 0$. Moreover, on putting $b = 0$ no ascent is given, and the equation for the descent evidently has to be changed. Whereby except for the case in which $k=\frac{a}{8g}$, we may turn our attention to b being equal to 0, and it is with difficulty that the truth can be understood from the equation. The same thing also happens, when according to the same hypothesis $k<\frac{a}{8g}$ with the descent made from some given point, and we inquire about the speed at the point C; for on putting $s = 0$ the equation is reduced to the absurd. Thus indeed the equation has been compared with that [ratio], as on making $s = 0$ it is not shown that also $v = 0$, even if actually v is equal to 0; indeed only these terms are to be neglected in which s is found, since the rest containing v by the same rule have to be neglected. [p. 309]
Hence it is found that it is not possible for $v = 0$, if $s = 0$; but since absurdity follows from the equation, unless is should be that $v = 0$, from this it must be concluded that $v = 0$, if s

= 0. Now in other cases, in which absurdity is not easily evident, it is difficult to avoid falling into trouble.

PROPOSITION 66.

Theorem.

573. *In a uniform medium, that offers resistance in the simple ratio of the speeds, all the descends on the cycloid AMC* (Fig.66) *are made in equal times and likewise also all the ascents on the cycloid CNB are completed in equal times, if a certain force is acting and it is uniform and directed downwards.*

Demonstration.

For the descent, if the arc is taken $CM = s$ and the height corresponding to the speed at $M = v$, then this equation is obtained :

$$dv = -gsds + \frac{ds\sqrt{v}}{\sqrt{k}}$$

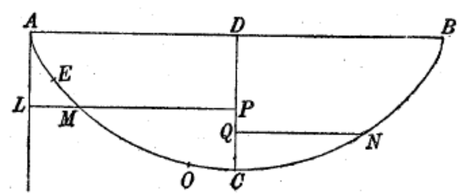

Fig. 66.

Put $\sqrt{v} = u;$ then u is as the speed at M and on this account, $dv = 2udu$ and this equation is obtained :

$$2udu = -\frac{gsds}{a} + \frac{uds}{\sqrt{k}},$$

in which u and s everywhere are raised to the same power. Whereby if the initial descent is put at E and the arc $CE = f$ and the proposed equation is integrated, thus so that $u = 0$ on putting $s = f$, the equation of the integration is produced, in which u, f and s constitute everywhere a number of the same dimension. [p. 310] Therefore from this u is equal to a function of one dimension of f and s. Concerning which the element of the time $\frac{ds}{u}$ is a function of no dimensions of f, s and of the element ds. Hence with the integral of this thus taken, so that it vanishes on putting $s = 0$, there is some function of zero dimensions of f and s and it shows the time to traverse the arc CM. Therefore in this function, if on putting $s = f$, f vanishes everywhere from this function, and the time of the whole descent along EC is equal to a function composed only from the constant quantities g, a and k, in which neither f nor any other quantity respecting the point E is introduced. On account of which the time of the descent along EC is expressed by the same quantity, wherever the point E is taken, and thus all the descents are completed in the equal times. If in the formula for showing the descent time $-\sqrt{k}$ is put in place of $\sqrt{k}$, the ascent time is produced for the arc CNB, which therefore is also constant, however great the arc traversed in ascending. Q.E.D.

Corollary 1.

585. Since *u* is equal to a function of one dimension of *f* and *s*, $\frac{u}{f}$ is equal to a function of zero dimensions of *f* and *s*. Whereby if $s = nf$ is put in place, then $\frac{u}{f}$ is equal to a constant quantity, in which *f* is not present. Therefore in different descents the speeds at homologous points of the whole arcs are proportional to the arcs *f*.

Corollary 2. [p. 311]

586. Since in the descent the maximum speed shall be where $u = \frac{gs\sqrt{k}}{a}$, the point *O* is found, or the arc *CO* from the equation, in which *f* and *s* everywhere constitute a number of the same dimensions ; from which hence *s* or *CO* is proportional to *f* . Therefore in a number of descents the maximum speeds are in proportion to the whole arcs *CO* of the descents.

Corollary 3.

587. Since the time to pass along *MC* is equal to a function of zero dimensions of *f* and *s*, the time also to traverse *EM* is equal to a function of zero dimensions of *f* and s or also of *f* and the arc *EM*.

Corollary 4.

588. Hence consequently not only the times of the whole descents, but also the times of the descents along similar parts of the whole arcs are equal to each other. And in a like manner this is in place for the ascents.

Corollary 5.

589. Therefore since as all the descents are isochronous as are all the ascents, also all the semi–oscillations are completed in equal times. And in the case $k > \frac{a}{8g}$, in which the body continues oscillations perpetually, all are completed in equal times.

Corollary 6. [p. 312]

590. Therefore the cycloid, which is the tautochronous curve *in vacuo*, retains the same property in a medium that resists in the simple proportion of the speeds. In addition the cycloid is also a tautochrone in a medium in which the resistance is constant or in the words of Newton, proportional to the moments of time (570).

Scholium 1.

591. This triple tautochronism of the cycloid Newton demonstrated too in *Prin. Phil.* (Book. II, Prop. XXV & XXVI, 1687; 2nd ed.), and regarding the resistance in proportion to the speeds, from this he formulated the demonstration, that in different descents, if the parts of the arcs in proportion to the whole arcs are taken, in these places the speeds are also in proportion to the whole arcs. For if the speeds should be proportional to the whole arcs, if the elements also are taken in proportion to the whole arcs, then the times for these are equal to each other.

Scholium 2.

592. But although from these it is apparent that the times of the ascents as for the descents are equal to each other, yet it is not possible to determine from these, how much the time of either the ascents or the descents shall be, and neither also can the times of the ascents or descents be compared between each other. [p. 313] For the equation defining the relation between s and u is thus complicated, for that element of time $\frac{ds}{u}$ cannot be expressed by one variable. Besides the infinitely small oscillations, which before the calculation in determining the times returned most easily, under this hypothesis of the resistance are of no help. For also if the whole arc described is made infinitely small, in the equation

$$l\frac{\sqrt{2ac^2\sqrt{k}}}{\sqrt{(2au^2\sqrt{k}-aus+gss\sqrt{k})}}=\frac{a}{4Bk}\,At.\frac{4Bks}{as-4au\sqrt{k}},$$

which arises from the above integration, indeed not a single term vanishes from those remaining. Moreover the time of the ascent depends on the quantities a, k and g; but how it is composed from these is not evident. Yet meanwhile this is certain, in which the greater g shall be with the other parts, there the time is less, but with a increasing, the time also increases, now with k increased to be diminished, since the resistance is made smaller. Therefore in this hypothesis of the resistance, the resistance in the slowest motion does not disappear, as in the resistances proportional to the square of the speeds. From which it is seen to be a consequence, if the resistance increases in a ratio greater than the square of the speeds, then in the slowest motions can be neglected, but if the resistance is in a smaller ratio, in the slowest motions the resistance has to be considered.

PROPOSITION 67. [p. 314]

Problem.

593. *In a uniform medium, in which the resistance is in a ratio of the multiple of the speeds of which the exponent is 2m, to determine the motion of the body on the curve CMA* (Fig.66) *, in which any arc CM is proportional to a power of the abscissa CP, of which the exponent is 1 – m.*

Solution.

With the abscissa put as $CP = x$ and with the arc $CM = s$, then $ds = \frac{a^m dx}{x^m}$. The speed at M corresponds to the height v ; the resistance at M is equal to $\frac{v^m}{k^m}$ and thus, if the body is placed to descend on the curve CM, this equation is obtained:

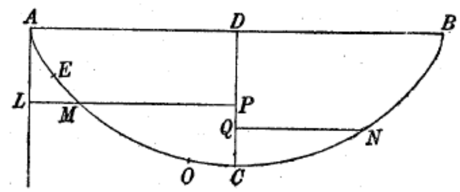

Fig. 66.

$$dv = -gx + \frac{a^m v^m dx}{k^m x^m} .$$

Moreover for the ascent on the same curve, this equation is used :

$$dv = -gx - \frac{a^m v^m dx}{k^m x^m}$$

Now each equation admits to being separated, if $v = TX$ is put in place ; indeed for the descent there is produced :

$$xdt = -tdx - gx + \frac{a^m t^m dx}{k^m}$$

or

$$\frac{-k^m dt}{k^m (g+t) - a^m t^m} = \frac{dx}{x}$$

and for the ascent this equation :

$$\frac{-k^m dt}{k^m (g+t) + a^m t^m} = \frac{dx}{x} .$$

[p. 315] In which equations the variables t and x are separated from each other in turn, thus so that t can be determined in terms of x with the help of quadrature. The constant in the integration to be added has to be defined either from the given speed at the point C or from the location of the curve, at which the descent begins or the ascent ends. If the abscissa of the whole arc of the descent or of the ascent corresponds to f, then v is equal to a function of one dimension of f and x so in the descent as in the ascent on account of the homogeneous differential equations. On this account, $\sqrt{v}$ is equal to a function of half the dimensions of f and x. Therefore the time of the descent through MC, which is equal to :

$$\int \frac{ds}{\sqrt{v}} = a^m \int \frac{ds}{x^m \sqrt{v}} ,$$

is equal to a function of dimensions $\frac{1}{2}-m$ of f and x. Moreover, since on putting $x = f$ the time of the whole descent is produced as $f^{\frac{1}{2}-m}$, also proportional to that is the time of the whole ascent, if indeed f designates the abscissa of the whole ascent arc. If the whole arc of the ascent or of the descent is put equal to A, since A is as f^{1-m}, then the time of either the whole ascent or descent becomes as $\frac{A}{f^{\frac{1}{2}}}$ or as $A^{\frac{1-2m}{2-2m}}$. Therefore the times of more of the ascents are in the $\frac{1-2m}{2-2m}$ th multiple ratio of the whole arcs described. And in the same ratio also are the times of the ascents between themselves, but the times of the ascents and of the descents cannot be compared with each other. Q.E.I.

Corollary 1. [p. 316]

594. Since the speed or $\sqrt{v}$ is equal to a function of half the dimensions of f and x, in more descents the speeds acquired at the point C are in the square root ratio of the heights, from which the body descended. And the heights, to which the ascending body can reach, are in the square ratio of the initial speeds at C.

Corollary 2.

595. Since the times of the descents as well as of the ascents are as $f^{\frac{1}{2}-m}$, then all the descents are completed in equal times, if it should be that $m=\frac{1}{2}$ or the resistance in proportion to the speeds. And equally with this for the hypothesis, the times of the ascents are equal to each other. Moreover the curve is a cycloid, as we have shown before.

Corollary 3.

596. Since $ds=\frac{a^m dx}{x^m}$, then the arc becomes $s=\frac{a^m x^{1-m}}{1-m}$. From which it is evident, unless $m < 1$, that the curve AMC becomes negative or imaginary, which is the same thing. Indeed the curve must always be greater than the abscissa.

Corollary 4.

597. Besides always it must be the case that $ds > dx$; whereby, when this happens, if $x = 0$, m must be a positive number. Hence it is required in our proposition, that m is contained between the limits 0 and 1.

Corollary 5. [p. 317]

598. In these cases the maximum values of x is a and there $ds = dx$ or the vertical tangent. And at this place the curve has a cusp ; indeed it cannot ascend higher, since if $x > a$, it becomes $ds < dx$, which is not able to happen.

Corollary 6.

599. If m is contained between the limits 0 and 1, the curve has a horizontal tangent at *C* and the radius of osculation at *C* becomes equal to $\frac{sds}{dx} = \frac{a^{2m}x^{1-2m}}{1-m}$ on putting $x = 0$. Whereby the radius of osculation at *C* becomes infinitely small, if $m < \frac{1}{2}$, finite if $m = \frac{1}{2}$, and infinitely great if $m > \frac{1}{2}$.

Corollary 7.

600. The times of the minimum descents and ascents are infinitely small, if $m < \frac{1}{2}$, but finite if $m = \frac{1}{2}$. And finally they are infinitely great, if $m > \frac{1}{2}$. Therefore they hold the ratio of the radii of osculation at the lowest point *C*.

Scholion.

601. Hence we have here examples of curves for resistance with a ratio held less than the squares of the speeds, upon which the motion of bodies can be determined. [p. 318] But if the medium resists in a ratio greater than the square, the curve never has a horizontal tangent and thus descents and ascents are never able to be finished. Moreover since it becomes apparent in an example, such is the kind of motion in a medium which resists in a ratio greater than the square of the speeds; it may please to investigate the motion of a body on a cycloid even in a medium, which resists as the fourth power of the speeds. Now I select this hypothesis of resistance before others, since in this the speed can be conveniently defined by a series.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 292]

PROPOSITIO 64.

Problema.

565. *Si medii resistentia partim fuerit constans, partim quadratis celeritatum proportionalis, determinare motum oscillatorium corporis super cycloide MCB* (Fig.66), *saltem in casu, quo resistentia est valde parva*

Solutio.

Sit ut ante diameter circuli generatoris $CD = \frac{1}{2}a$, $CP = x$ et arcus $CM = s$. Ponatur celeritas in C debita altitudini b et celeritas in M altitudini v. Potentia corpus perpetuo deorsum sollicitans sit = g, pars resistentiae, quae est constans, sit = h et pars resistantiae quadratis celeritatum proportionalis sit = $\frac{v}{k}$ ut ante; erit k quantitas valde magna respecta v et s et a atque h valde parvum respectu g. Iam fiat descensus super arcu MC; erit

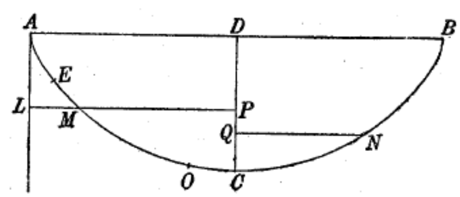

Fig. 66.

$$dv = -gdx + hds + \frac{vds}{k}$$

atque hinc [p. 293]

$$v = e^{\frac{s}{k}}b - e^{\frac{s}{k}}\int e^{-\frac{s}{k}}(gdx - hds)$$

At quia est ex natura cycloidis $dx = \frac{sds}{a}$, erit

$$\int e^{-\frac{s}{k}}gdx = \frac{gk^2 - gk^2e^{-\frac{s}{k}} - gke^{-\frac{s}{k}}s}{a}$$

et

$$\int e^{-\frac{s}{k}}hds = hk - hke^{-\frac{s}{k}};$$

unde fit

$$v = \frac{e^{\frac{s}{k}}(ab+hak-gk^2)-hak+gk^2+gks}{a}$$

Cereritas maxima habetur, si fuerit

$$\frac{gs}{a} = h + \frac{v}{k}$$

seu

$$e^{\frac{s}{k}} = \frac{gk^2}{gk^2-ab-hak}$$

Dicatur altitudo debita celeritati maximae in $O = c$; erit $CO = \frac{ha}{g} + \frac{ac}{gk}$

et

$$ab = gk^2 - hak - gk^2 e^{\frac{-ac-hak}{gk^2}}$$

Ponatur arcus $MO = q$; erit

$$s = \frac{ha}{g} + \frac{ac}{gk} + q;$$

unde fit

$$v = \frac{ac+gk^2+gkq-e^{\frac{q}{k}}gk^2}{a}.$$

Quia nunc v minus est quam c, pono $c - v = z$; erit

$$az + gk^2 + gkq = e^{\frac{q}{k}} gk^2.$$

Quae aequatio in seriem conversa dat ut supra

$$\frac{az}{g} = \frac{q^2}{2} + \frac{q^3}{6k} + \frac{q^4}{24k^2}$$

atque

$$q = \frac{\sqrt{2az}}{\sqrt{g}} - \frac{az}{3gk} + \frac{az\sqrt{2az}}{18gk^2\sqrt{g}}.$$

Incipiat descensus ex M; erit ibi $v = 0$ et $z = c$ ideoque

$$MO = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

Ex eadem formula reperitur arcus ascensus

$$ON = \frac{\sqrt{2ac}}{\sqrt{g}} + \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}};$$

atque cum in his h non reperiatur, erit ut supra tempus semioscillationis per MON =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}}.$$

Totus vero arcus descensus MC erit =

$$\frac{ha}{g}+\frac{\sqrt{2ac}}{\sqrt{g}}+\frac{2ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}$$

atque arcus ascensus

$$CN=-\frac{ha}{g}+\frac{\sqrt{2ac}}{\sqrt{g}}-\frac{2ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

[p. 294] Quare si arcus descensus *MC* ponatur *E* et arcus ascensus *CN* = *F*, erit

$$F=E-\frac{2ha}{g}-\frac{2E^2}{3k}+\frac{4haE}{3gk}+\frac{4E^3}{9k^2}.$$

In sequente semioscillatione est arcus descensus F et arcus ascensus

$$G=E-\frac{4ha}{g}-\frac{4E^2}{3k}+\frac{16haE}{3gk}+\frac{16E^3}{9k^2}.$$

Atque generaliter in ea semioscillatione, quae indicatur numero *n*, arcus ascensus est =

$$E-\frac{2nha}{g}-\frac{2nE^2}{3k}+\frac{4n^2haE}{3gk}+\frac{4n^2E^3}{9k^2}=\frac{3gkE-6hnak}{3gk+2gnE}.$$

Quare si peractis *n* semioscillationis dicatur arcus descensus primae *E* et arcus ascensus ultimae *L*, erit

$$2gnEL=3gk(E-L)-6hnak$$

seu

$$n=\frac{3gk(E-L)}{2gEL+6hak}.$$

Tempus vero, quo quaelibet semioscillatio per *MCN* absolvitur, est =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi(gE-ha)^2\sqrt{2a}}{24g^2k^2\sqrt{g}}$$

loco *c* eius valore substituto. Q.E.I.

Corollarium 1.

566. Si ponatur $c = 0$, ut locus prodeat, in quo corpus sit quieturum, invenitur

$$MC=\frac{ha}{g};$$

corpus ergo in quiete permanere potest non solum in puncto *C*, sed extra *C* quoque in distantia $\frac{ha}{g}$ cis et ultra *C*. Quare in huiusmodi medio resistente ex statu quietis penduli non exacte linea verticalis potest cognosci, sed angulo, cuius sinus est $\frac{h}{g}$, aberri potest.

Scholion 1. [p. 295]

567. Huiusmodi resistentiam in aqua locum habere experimentis facile evincitur, quippe in motibus tardissimis resistentia quadratis celeritatum minime proportionalis observatur; in fluido vero praeter resistentiam quadratis celeritatum proportionalem aliam non dari probabile est, nisi resistentiam constantem. Confirmatur hoc etiam experimentis a La Hirio instituitis, quibus monstravit pendulum in aqua extra situm verticalem in quiete permanere posse. Quod fieri non posset, si resistentia a sola celeritate penderet. Ex experimentis Neutoni, quae circa retardationem motus pendulorum in aere instituit, concludi potest globi plumbei diameteri 2 dig. resistentiam constantem esse circiter partem millionesimam gravitatis seu $\frac{h}{g}=\frac{1}{1000000}$. [*Principia*, 2nd ed. Lib. II, sect. VI.] Hic ergo globus filo suspensus a linea verticali aberrare potest angulo circuliter 10''', qui autem error est insensibilis. Maior autem et sensibilis esse hic poterit error, quo minor simulque levior globus adhibeatur.

Corollarium 2.

568. Ad hunc angulum inveniendum inservit ista aequatio ex superori (565) deducta

$$\frac{h}{g}=\frac{E-L}{2na}-\frac{EL}{3ak}$$

= sinui anguli, quo pendulum a linea verticali declinare potest. At loco E arcum parvum accipi convenit, quo termini neglecti eo magis fiant insensibiles.

Corollarium 3. [p. 296]

569. Ex aequatione

$$n=\frac{3gk(E-L)}{2gEL+6hak}$$

apparet, quo maior sit arcus oscillatione descriptus, eo minorem fieri terminum $6hak$ respectu $2gEL$. Atque hoc in casu est, quod haec resistentia tantum in minimis oscillationis sentiatur.

Corollarium 4.

570. Qia h est numerus tum per hypothesin valde parvus tum in subtilibus fluidis, ad quae haec propositio est accommodata, reipsa fere evanescet quoque ha prae gE ideoque tempus unius semioscillationis erit =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}.$$

Resistantia igitur constans non immutabit tempora oscillationum.

Scholion2.

571. Etiamsi ergo haec resistentia constans cum resistantia quadratis celeritatum proportionali coniuncta consideretur, calculus eo neque fit prolixior neque difficilior. Nam ex celeritate maxima $\sqrt{c}$ totus arcus una semioscillatione descriptus eodem modo determinatur, sive haec resistentia constans adsit sive non; utroque enim casu plane eadem obtineatur aequatio. Quamobrem vel ex hoc ipso sequi videtur hanc resistentiae legem locum habere, alias vero resistentias praeter hanc et eam, quae quadratis celeritatum est proportionalis,actu non inveniri. [p. 297] Fluida autem iam pridem (Tom I, (490)) duplicem resistentiam exercere observata sunt, alteram quadratis celeritatum proportionalem, quae in motibus celerioribus sola observetur, alteram in motibus tardissimis tantum sensibilem. Illa resistentia oritur a vi inertiae particularum fluidi et per eam corpus de moto suo amittit, quando particulas eas removet; quam quadratis celeritatum proportionalem esse dubitari nequit. Haec vero resistentia a tenacitate fluidi ortum habet, qua particulae fluide inter se cohaerent et difficulter a se invicem separantur. Dum igitur corpus per datum spatium movetur, datus particularum numerus ipsi spatio proportionalis a se invicem divelli debet; quare haec resistentia congruit cum potentia absoluta motum corporis retardante, quippe quae etiam per aequalia spatia aequalem ictuum numerum in corpus exerit. Haec igitur resistentia seu vis motui corporis semper est vis tangentialis perpetuo constans et retardans. At hoc casu natura a calculo exceptionem postulat, quando corpus quiescit. Quoniam enim haec vis est constans, aeque agere deberet in corpus quiescent ac in motum; quiescens autem corpus, quia fluidi particulas non divellit, hanc vim sentire nequit. Accedit ad hoc, quod, cum haec vis directioni motus sit contraria, ea in corpus quiescens, quod nullam habeat directionem, nullum effectum habere queat. At si motus super linea curva investigatur, tangens curvae semper pro directione motus habetur, [p. 298] etiamsi corpus actu quiescat, atque ideo calculus effectum huius vis etiam in corpore quiescente ostendit; hoc igitur casu a calculo exceptionem fieri oportet. Corpus pendulum ergo per aliquod parvum spatium circa *C* in quiete permanere posse ideo est dicendum, quia eius nisus versus C non sufficit ad particulas fluidi a se invicem separandas. Quare corpus quiescere potest in quolibet puncto illius spatioli, etiamsi calculus ostendat etiam in ipso puncto *C* corpus non in quiete perseverare posse.

Scholion 3.

572. Ex his, quae partim generaliter tradidimus, partim de cycloide attulimus, perspicitur, quomodo in medio resistente in duplicata ratione celeritatum motus corporis super quacunque curva possit determinari. Consideravimus quidem medium resistens uniforme et potentiam sollicitantem quoque aequabilem; sed ex aequatione resolvenda apparet eam quoque integrari posse, quomodcunque tum medium sit difforme tum potentia sollicitans variabilis; semper enim in aequatione altitudo celeritati debita v unicam tantum habet dimensionem. Progredior igitur ad alias medii resistentis hypotheses; sed quia tum non pro quavis curva motus potest definire, curva primo sunt inveniendae, quae determinationem motus admittunt. [p. 299] Assumimus hic autem iuxta institutum nostrum eas curvas, quae ad aequationem homogeneam deducunt, in qua indeterminatae ubique eundem obtinent dimensionum numerum. Si resistentia fuerit potestati celeritatum exponentis $2m$ proportionalis, habetur ista aequatio

$$dv = \pm gdx \pm \frac{v^m ds}{k^m};$$

quae quo sit homogenea inter v et x, debet esse

$$ds = x^{-m}dx \text{ seu } s = a^m x^{1-m} \text{ seu } x = a^{\frac{m}{m-1}} s^{\frac{1}{1-m}}.$$

Vel si x et s datis quantitatibus augeantur vel diminuantur, in curva, cuius haec est aequatio

$$x == a^{\frac{m}{m-1}}(s+f)^{\frac{1}{1-m}} - a^{\frac{m}{m-1}} f^{\frac{1}{1-m}},$$

motus quoque determinari potest. In medio ergo resistente in simplici ratione celertitatum curva fit cyclois ideoque motum super ea determinemus.

PROPOSITIO 65.

Problema.

573. *In medio, quod resistit in simplici ratione celeritatum, determinare motum oscillatorium corporis super cycloide ACB* (Fig.66) *existente tam medio quam potentia sollicitante uniformi.*

Solutio.

Sit iterum ut ante diameter circuli generatoris $CD = \frac{1}{2}a$, $CP = x$ et arcus $CM = s$. [p. 300] Ponatur celeritas in C debita altitudini b et celeritas in M altitudini v. Potentia corpus perpetuo deorsum trahens sit = g, et resistentiae $= \frac{\sqrt{v}}{\sqrt{k}}$. Fiat super parte AMC descensus; erit ex natura descensus

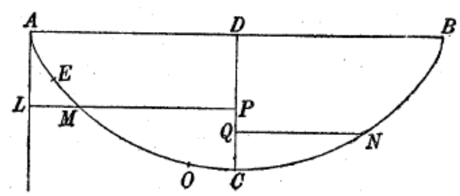

Fig. 66.

$$dv = -gdx + \frac{ds\sqrt{v}}{\sqrt{k}}$$

ponatur

$$\frac{\sqrt{v}}{\sqrt{k}} = \frac{u}{a};$$

erit

$$v = \frac{ku^2}{a^2} \text{ et } dv = \frac{2kudu}{a^2},$$

unde fit

$$2kudu = -gasds + auds,$$

quae aequatio ita debet integrari, ut facto s = 0, erit $u = \frac{a\sqrt{b}}{\sqrt{k}} = \frac{u}{a}$. Pro ascensu vero super arcu CN haec habetur aequatio

$$2kudu = -gasds - auds,$$

Ponatur $u = ps$, habebitur pro descensu

$$2kp^2sds + 2kps^2dp = -gasds + apsds$$

seu

$$\frac{2kpdp}{ap - ga - 2kp^2} = \frac{ds}{s}.$$

Hinc fit integrando

$$ls = lC - \frac{1}{2}l\left(p^2 - \frac{ap}{2k} + \frac{ga}{2k}\right) + \frac{a}{2\sqrt{(a^2-8gak)}}\,l\frac{4kp - a + \sqrt{(a^2-8gak)}}{4kp - a - \sqrt{(a^2-8gak)}}$$
$$= lC - \frac{1}{2}l\frac{2a^2v\sqrt{k} - a^2s\sqrt{v} + gas^2\sqrt{k}}{2ks^2\sqrt{k}} + \frac{a}{2\sqrt{(a^2-8gak)}}\,l\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}$$

seu (mutata significatione litterae *C*)

$$lC = l(2a^2v\sqrt{k} - a^2s\sqrt{v} + gas^2\sqrt{k}) - \frac{a}{\sqrt{(a^2-8gak)}}\,l\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}.$$

Ponatur $s = 0$ et $v = b$; fiet $lC = l2a^2b\sqrt{k}$; hinc fiet

$$\frac{2av\sqrt{k} - as\sqrt{v} + gs^2\sqrt{k}}{2ab\sqrt{k}} = \left(\frac{4a\sqrt{kv} - as + s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} - as - s\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

In altera vero curvae parte *CN* pro ascensu corporis posito $CN = s$ habetur haec aequatio

$$\frac{2av\sqrt{k} + as\sqrt{v} + gs^2\sqrt{k}}{2ab\sqrt{k}} = \left(\frac{4a\sqrt{kv} + as - s\sqrt{(a^2-8gak)}}{4a\sqrt{kv} + as + s\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

Si altitudino celeritati maximae, quae sit in *O*, debita dicatur *c*, erit $CO = \frac{a\sqrt{c}}{g\sqrt{k}}$
atque

$$c = b\left(\frac{4gk - a + \sqrt{(a^2-8gak)}}{4gk - a - \sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}.$$

[p. 301] Hae autem aequationes locum non habent, nisi sit $a^2 > 8gak$ seu $k < \frac{a}{8g}$. Nam si $k > \frac{a}{8g}$, aequationes a logarithmis et quadratura circuli simul pendebunt.

Ponemus esse $k = \frac{a}{8g}$ eritque

$$\frac{ds}{s} = \frac{-pdp}{\left(p - \frac{a}{4k}\right)^2} = \frac{-dp}{p - \frac{a}{4k}} - \frac{adp}{4k\left(p - \frac{a}{4k}\right)^2}.$$

Integrando ergo prodibit

$$ls = lC - l\left(p - \frac{a}{4k}\right) + \frac{a}{4kp - a} = lC - l\left(\frac{a\sqrt{v}}{s\sqrt{k}} - \frac{a}{4k}\right) + \frac{as}{4a\sqrt{kv} - as}$$

(seu mutata significatione litterae *C*) =

$$lC - l(4a\sqrt{kv} - as) + ls + \frac{as}{4a\sqrt{kv} - as}.$$

Fit igitur $lC = l4a\sqrt{kb}$; unde habebitur

$$l\frac{4\sqrt{kv} - s}{4\sqrt{kb}} = \frac{s}{4\sqrt{kv} - s}.$$

Apparet hinc in descensu celeritatem nusquam esse posse = 0; semper enim $4\sqrt{kv}$ maius esse debet quam *s*. Hoc ergo casu, si celeritas in puncto *C* sit realis, descensus initium fit imaginarium. Quare ubicunque corpus descensum inceperit, celeritas in puncto infimo *C* erit = 0. Ad motum igitur investigandum, si descensus fit ex puncto dato *E* fueritque *CE* = *f*, erit $lC = l - af + 1$ ideoque

$$l\frac{s - 4\sqrt{kv}}{f} = \frac{4\sqrt{kv}}{4\sqrt{kv} - s}.$$

Ex quo intelligitur semper esse debere $4\sqrt{kv} < s$, quamobrem in puncto *C*, ubi *s* = 0, debet quoque esse *v* = 0. Maxima celeritas, quae sit in *O*, habetur ponendo

$$\sqrt{v} = \frac{gs\sqrt{k}}{a} \text{ seu } s = 8\sqrt{kv},$$

quo posito prodit

$$l\frac{s}{2f} = -1 \text{ seu } s = \frac{2f}{e} = CO$$

denotante *e* numerum, cuius logarithmus est = 1. His igitur casibus, quibus arcus descensus est realis, nullus est arcus ascensus. At si corpus per arcum *CN* celeritate initiali in *C* altitudini *b* debita ascendat, motus hac aequationne exprimetur [p. 302]

$$l\frac{4\sqrt{kv} + s}{4\sqrt{kb}} = \frac{-s}{4\sqrt{kv} + s},$$

ex qua patet esse

$$s + 4\sqrt{kv} < 4\sqrt{kb} \quad \text{et} \quad \sqrt{v} < \sqrt{b} - \frac{s}{4\sqrt{k}}.$$

Totus arcus ascensus *CN* reperitur faciendo *v* = 0 tumque erit

$$l\frac{s}{4\sqrt{kb}} = -1 \text{ seu } CN = \frac{4\sqrt{kb}}{e}.$$

Si est $k < \frac{a}{8g}$, quem casum iam tractavimus, resistentia adhuc fit maior, quamobrem multo magis celeritas in *C* erit = 0, si quiedem descensus ex puncto dato fiat, atque pro data celeritate in C initium descensus erit imaginarium. Quamobrem aequatio, quam pro descensu dedimus, est imaginaria, nisi constans determinatur ex dato descensus initio. Sit igitur arcus *CE* = *f*; erit

$$lC = lgaf^2\sqrt{k} - \frac{a}{\sqrt{(a^2 - 8gak)}}\, l\frac{a - \sqrt{(a^2 - 8gak)}}{a + \sqrt{(a^2 - 8gak)}}$$

factoque *s* = 0 erit

$$l\,\frac{gf^2}{2av}=\frac{a}{\sqrt{(a^2-8gak)}}\;l\,\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}$$

si *v* non esset = 0; nam si v est = 0, haec aequatio non valet. Apparet autem hanc aequationem contradictionem continere, quia 2*av* maius esse deberet quam *gff* seu $2v>\frac{gss}{a}$ posito *s* pro *f*. At est $\frac{ss}{a}=2x$ atque ita esset $v>gx$, quod est absurdum; nam in vacuo est tantum $v=gx$ atque in medio resistente adhuc minor esse debet. Pro ascensu autem inservit aequatio inventa atque ex ea totius arcus ascensus *CN* reperitur faciendo $v=0$, quo posito prodit

$$\frac{gs^2}{2ab}=\left(\frac{a-\sqrt{(a^2-8gak)}}{a+\sqrt{(a^2-8gak)}}\right)^{\frac{a}{\sqrt{(a^2-8gak)}}}=\frac{g\cdot CQ}{b}.$$

Ex his igitur perspicitur, si fuerit vel $k<\frac{a}{8g}$ vel $k=\frac{a}{8g}$, oscillationes peragi non posse, quia post nullum descensum ascensus sequi potest. Quare nobis potissimum reliqui casus, quibus est $k>\frac{a}{8g}$, [p. 303] sunt investigandi; quia enim in his resistentia est minor atque quantumvis parva assumi potest, oscillationes utique perfici poterunt. Factis ergo superioribus substitutionibus habemus

$$\frac{-ds}{s}=\frac{pdp}{p^2-\frac{ap}{2k}+\frac{ga}{2k}},$$

quae aequatitio posito $q=p-\frac{a}{4k}$ abit in hanc

$$\frac{-ds}{s}=\frac{qdq+\frac{adq}{4k}}{qq+\frac{ga}{2k}-\frac{a^2}{16k^2}}.$$

Ponatur

$$\frac{ga}{2k}-\frac{a^2}{16k^2}=B^2,$$

quia est quantitas affirmativa, eritque

$$lC-ls=l\sqrt{(q^2+B^2)}+\frac{a}{4k}\int\frac{dq}{q^2+B^2}=l\sqrt{(q^2+B^2)}+\frac{a}{4Bk}At.\frac{q}{B},$$

ubi $At.\frac{q}{B}$ est arcus circuli, cuius tangens est $\frac{q}{B}$ existente sinu toto = 1. Restituto autem pro *q* valore debito erit (mutata significatione litterae *C*)

$$lC=l\sqrt{(2av\sqrt{k}-as\sqrt{v}+gs^2\sqrt{k})}+\frac{a}{4Bk}\,At.\frac{4a\sqrt{kv}-as}{4Bks}.$$

Ponatur ad *l*C definiendum $s=0$ et $v=b$; erit

$$lC = l\sqrt{2ab\sqrt{k}} + \frac{a}{4Bk}At.\infty,$$

quare erit

$$l\frac{\sqrt{2ab\sqrt{k}}}{\sqrt{(2av\sqrt{k} - as\sqrt{v} + gs^2\sqrt{k})}} = -\frac{a}{4Bk}At.\frac{4Bks}{4a\sqrt{kv} - as}.$$

Pro ascensu vero per arcum *CN* invenitur

$$l\frac{\sqrt{2ab\sqrt{k}}}{\sqrt{(2av\sqrt{k} + as\sqrt{v} + gs^2\sqrt{k})}} = \frac{a}{4Bk}At.\frac{4Bks}{4a\sqrt{kv} + as}.$$

Posito nunc $v = 0$ prodit integer arcus descensus *MC* ex hac aequatione

$$l\sqrt{\frac{2ab}{gss}} = \frac{a}{4Bk}At.\frac{4Bk}{a} = l\sqrt{\frac{b}{g\cdot CP}}.$$

Atque totus arcus ascensus *CN* invenietur ex hac aequatione

$$l\sqrt{\frac{2ab}{gss}} = \frac{a}{4Bk}At.\frac{4Bk}{a} = l\sqrt{\frac{b}{g\cdot CQ}}.$$

Ex his aequationibus quanquam videantur arcus ascensus et descensus inter se aequales, tamen sunt inaequales; dantur enim infiniti arcus, quorum tangens est eadem $\frac{4Bk}{a}$, atque pro ascensu alius accipi debet, alius pro descensu. Atque cum infiniti dentur [p. 304] arcus tangentis $\frac{4Bk}{a}$, quilibet eorum ad propositum accomodari potest. His enim sumtis arcubus in ordine prodeunt successive omnes arcus tam ascensus quam descensus, quamdiu corpus oscillationes peragit; nam quia aequatio inventa est generalis, ea omnia loca ostendere debet, in quibus corporis oscillationis celeritas unquam est = 0. Quare in hac resistentiae hypothesi hoc habetur commodum, quod statim pro qualibet oscillatione, centesima v. gr., arcus tam descensus quam ascensus possit definiri. Sit arcus *D*, cuius tangens est $\frac{4Bk}{a}$, et posita ratione diametri ad peripheriam $1:\pi$ erit eadem tangens $\frac{4Bk}{a}$ omnium horum arcuum

$$D,\ \pi + D,\ 2\pi + D,\ 3\pi + D, \text{etc.}$$

Pro arcu descensu nunc primae oscillationis *MC* sumi debet arcus *D* eritque

$$\frac{2ab}{gss} = e^{\frac{Da}{2Bk}}$$

seu abscissa arcus *MC*

$$= \frac{b}{g}e^{\frac{-Da}{2Bk}}.$$

Abscissa autem arcus ascensus sequentis seu abscissa arcus descensus secundae semioscillationis erit

$$=\frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}.$$

Simili modo abscissa arcus descensus in tertia semioscillatione erit

$$=\frac{b}{g}e^{\frac{-a(2\pi+D)}{2Bk}}.$$

Atque generaliter abscissa arcus descensus in oscillatione, quae indicatur per $n + 1$, est

$$=\frac{b}{g}e^{\frac{-a(n\pi+D)}{2Bk}},$$

quae simul est abscissa arcus ascensus in oscillatione, quae indicatur numero *n*. Quod ad tempora oscillationum attinet, ea sequenti propositioni reservamus. Q.E.I.

Corollarium 1. [p. 305]

574. Nisi ergo fuerit $k>\frac{a}{8g}$, oscillationes absolvi non possunt, quia finito primo descensu corpus ad quietem redigitur, si vel $k<\frac{a}{8g}$ vel $k=\frac{a}{8g}$. At si $k>\frac{a}{8g}$, oscillationes perpetuo durabunt, quia expressio $\frac{b}{g}e^{\frac{-a(n\pi+D)}{2Bk}}$ neque evanescere neque negativa fieri potest.

Corollarium 2.

575. Arcus descensus se habet ad arcum sequentem ascensus in data ratione; est enim abscissarsum ratio

$$\frac{b}{g}e^{\frac{-Da}{2Bk}} \text{ ad } \frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}$$

ideoque ipsorum arcuum 1 ad $e^{\frac{-\pi a}{4Bk}}$, quae ratio non pendet a celeritate data $\sqrt{b}$.

Corollarium 3.

576. Atque simili modo arcus descensus primae semioscillationis ad arcum ascensus semioscillationis numero n indicatae datam havet rationem; est enim haec ratio ut $e^{\frac{\pi na}{4Bk}}$ ad 1. Quare si numerus semioscillationum duplo fit maior, haec ratio fit duplicata.

Corollarium 4.

577. Arcus descensus quotcunque semioscillationum se insequentium constituunt progressionem geometricam decrescentem in ratione 1 ad $e^{\frac{-\pi a}{4Bk}}$. Atque ideo integri etiam arcus semioscillationibus descripti erunt in progressione geometrica eiusdem denominatoris. [p. 306]

Scholion 1.

578. Quia autem pro *D* infiniti arcus accipi possunt, quo appareat, quinam ex iis pro arcu descensus accipi debeat, sumo casum, quo $k=\frac{a}{8g}$ atque arcus ascensus $=\frac{4\sqrt{kb}}{e}$ seu eius abscissa = $\frac{8kb}{ae^2}=\frac{b}{g}e^{-2}$; hoc autem casu est B = 0 atque abscissa arcus ascensus $=\frac{b}{g}e^{\frac{-a(\pi+D)}{2Bk}}$. Debet ergo esse $\frac{a(\pi+D)}{2Bk}=2$ et $\pi+D=\frac{4Bk}{a}=0.$ Est vero $\frac{4Bk}{a}$ tangens arcus $\pi+D$, et cum $\frac{4Bk}{a}$ sit = 0, debet $\pi+D$ esse minimum arcum tangenti $\frac{4Bk}{a}$ respondentem. Dicatur ergo minimus arcus tangenti $\frac{4Bk}{a}$ respondens *E*; erit $D=E-\pi$. Quocirca in prima semioscillatione erit abscissa arcus descensus = $\frac{b}{g}e^{\frac{a(\pi-E)}{2Bk}}=\frac{MC^2}{2a}$ ideoque ipse arcus $MC=e^{\frac{a(\pi-E)}{4Bk}}\sqrt{\frac{2ab}{g}}$. Ponatur $\frac{4Bk}{a}$ seu tangerns arcus $E=\tau$; erit arcus descensus primae semioscillationis = $e^{\frac{\pi-E}{\tau}}\sqrt{\frac{2ab}{g}}$, arcus ascensus primae seu arcus descensus secundae semioscillationis $=e^{\frac{-E}{\tau}}\sqrt{\frac{2ab}{g}}$ atque arcus descensus in semioscillatione, quae indicatur numero *n* + 1, erit $=e^{\frac{-E-(n-1)\pi}{\tau}}\sqrt{\frac{2ab}{g}}$, qui est simul arcus ascensus in semioscillatione numero *n* indicata. Progressionis geometricae ergo, quam hi arcus ascensus constituunt, denominator est $e^{\frac{-\pi}{\tau}}$.

Corollarium 5. [p. 307]

579. Ex his etiam in qualibet semioscillatione celeritas in puncto infimo *C* potest definiri. Sit enim in semioscillatione, quae numero *n* indicatur, celeritas in *C* debita altitudini β; erit arcus ascensus = $e^{\frac{-E}{\tau}}\sqrt{\frac{2\alpha\beta}{g}}$, qui aequalis esse debet ipsi $e^{\frac{-E-(n-1)\pi}{\tau}}\sqrt{\frac{2ab}{g}}$. Hinc fit $\sqrt{\beta}=e^{\frac{-(n-1)\pi}{\tau}}\sqrt{b}$. Celeritas ergo in puncto *C* in semioscillationibus successivis progressionem geometricam quoque constituunt, cuius denominator est $e^{\frac{-\pi}{\tau}}$.

Corollarium 6.

580. Si *n* ponatur numerus negativus, semioscillationes, quae ante primam factae esse possent, cognoscuntur; ut in semioscillatione primam praecedente arcus descensus esse debuisset $e^{\frac{2\pi-E}{\tau}}\sqrt{\frac{2ab}{g}}$.

Corollarium 7.

581. Si in prima semioscillatione descensus fiat ex puncto cycloidis supremo *A*, erit arcus descensus = *a*. Quare est $\sqrt{\frac{ag}{2b}} = e^{\frac{\pi - E}{\tau}}$ et celeritas in puncto infimo C seu $\sqrt{b}$ erit $e^{\frac{E-\pi}{\tau}}\sqrt{\frac{ga}{2}}$.

Corollarium 8. [p. 308]

582. Si resistentia fere evanescat seu *k* fuerit quantitas vehementer magna, erit

$$B = \sqrt{\frac{ga}{2k}} \quad \text{et} \quad \tau = \frac{4\sqrt{gk}}{\sqrt{2a}} = \frac{2\sqrt{2gk}}{\sqrt{a}}.$$

Cum igitur sit τ valde magnum, erit $E = \frac{\pi}{2}$ atque arcus descensus primae semioscillationis =

$$e^{\frac{\pi}{2\tau}}\sqrt{\frac{2ab}{g}} = \left(1 + \frac{\pi}{2\tau}\right)\sqrt{\frac{2ab}{g}}$$

et arc ascensus =

$$\left(1 - \frac{\pi}{2\tau}\right)\sqrt{\frac{2ab}{g}}.$$

Scholion 2.

583. Ex solutione huius propositionis inter cetera intelligi potest, quanta circumspectione saepe opus sit ad conclusiones ex aequationibus deducendas. Nam in casu $k < \frac{a}{8g}$ aequationes, quas pro descensu et ascensu invenimus, ita sunt comparatae, ut ex iis sequi videatur arcum ascensum aequalem esse arcui descensus; nam facto $v = 0$ ex utraque aequatione prodit

$$\frac{gs^2}{2ab} = \left(\frac{a - \sqrt{(a^2 - 8gak)}}{a + \sqrt{(a^2 - 8gak)}}\right)^{\frac{a}{\sqrt{(a^2 - 8gak)}}}.$$

Atque hoc ita se quoque haberet, nisi descensus necessario faceret $b = 0$. Posito autem $b = 0$ nullus datur ascensus et aequatio pro descensu prorsus est immutanda. Quare nisi ex casu quo $k = \frac{a}{8g}$, advertissemus *b* esse = 0, difficulter ex aequatione veritas cognosci potuisset. Idem quoque accidit, ubi in eadem hypothesi $k < \frac{a}{8g}$ descensu ex dato puncto facto celeritatem in puncto *C* investigavimus; posito enim $s = 0$ aequatio ad absurdum deduxit. Ita enim est comparata illa aequatio, ut facto $s = 0$ non ostendat esse quoque $v = 0$, etiamsi revera sit $v = 0$; ii enim tantum termini sunt negligi, in quibus reperiebatur *s*, cum reliqui v contintentes eodem iure negligi debuissent. [p. 309]

Inveniri ergo non potest esse $v = 0$, si $s = 0$; sed quia ex aequatione absurdum sequitur, nisi esset $v = 0$, ex hoc concludi potest esse $v = 0$, si $s = 0$. In aliis vero casibus, in quibus absurdum non tam facile perspicatur, difficulter lapsus evitari poterit.

PROPOSITIO 66.

Theorema.

584. *In medio uniformi, quod resistit in simplici ratione celeritatum, omnes descensus super cycloide AMC* (Fig.66) *fiunt aequalibus temporis atque similiter etiam omnes ascensus super cycloide CNB aequalibus temporibus absolvuntur, si quidem potentia sollicitans fuerit uniformis et deorsum directa.*

Demonstratio.

Pro descensu, si dicatur arcus $CM = s$ et altitudo debita celeritati in $M = v$, habetur ista aequatio

$$dv = -gsds + \frac{ds\sqrt{v}}{\sqrt{k}}$$

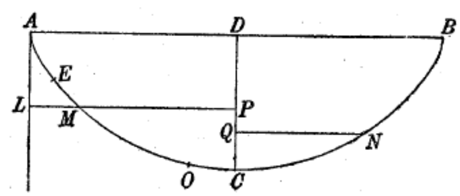

Fig. 66.

Ponatur $\sqrt{v} = u;$ erit u ut ipsa celeritas in M atque ob $dv = 2udu$ habetur ista aequatio

$$2udu = -\frac{gsds}{a} + \frac{uds}{\sqrt{k}},$$

in qua u et s ubique eundem tenent dimensionem numerum. Quare si ponatur initium descensus in E et arcus $CE = f$ et integratur aequatio proposita, ita ut fiat $u = 0$ posito $s = f$, prodibit aequatio integralis, in qua u, f et s ubique eundem dimensionum numerum constituunt. [p. 310] Ex hac igitur aequabitur u functioni unius dimensionis ipsarum f et s. Quocirca elementum temporis $\frac{ds}{u}$ erit functio nullius dimensionis ipsarum f, s atque elementi ds. Eius ergo integrale ita acceptum, ut evanescat posito $s = 0$, erit functio quoque nullius dimensionis ipsarum f et s et exhibebit tempus per arcum CM. In hac igitur functione si ponatur $s = f$, evanescat f ubique ex ea functione aequabiturque tempus totius descensus per EC functioni ex quantitatibus constantibus g, a et k tantum compositae, in quam neque f neque alia quantitas punctum E respiciens ingredietur. Quamobrem tempus descensus per EC eadem exprimetur quantitate, ubicunque punctum E accipiatur, atque ideo omnes descensus aequalibus absolventur temporibus. Si in formula tempus descensus exhibente ponatur $-\sqrt{k}$ loco $\sqrt{k}$, prodibit tempus ascensus in arcu CNB, quod propteria quoque erit constans, quantuscunque fuerit arcus ascensu percursus. Q.E.D.

Corollarium 1.

585. Quia u aequalis est functioni unius dimensionis ipsarum f et s, aequabitur $\frac{u}{f}$ functioni nullius dimensionis ipsarum f et s. Quare si ponatur $s = nf$, aequabitur $\frac{u}{f}$ quantitati constanti, in qua non inerit f. In variis ergo descensibus celeritates in punctis homologis totorum arcuum erunt ipsis arcubus f proportionales.

Corollarium 2. [p. 311]

586. Cum in descensu maxima celeritas sit, ubi est $u = \frac{gs\sqrt{k}}{a}$, invenietur punctum O seu arcus CO ex aequatione, in qua f et s ubique eundem dimensionum numerum constituunt; ex qua ergo erit s seu *CO* ipsi f proportionalis. In pluribus ergo descensibus tam maximae celeritates ipsae quam arcus *CO* arcubus descensuum totis erunt proportionales.

Corollarium 3.

587. Quia tempus per *MC* aequale est functioni nullius dimensionis ipsarum f et s, tempus quoque per *EM* aequabitur functioni nullius dimensionis ipsarum f et s seu etiam ipsius f et arcus *EM*.

Corollarium 4.

588. Hinc consequitur non solum tempora integrorum descensuum, sed etiam tempora descensuum per partes similes arcuum totorum esse inter se aequalia. Similique modo hoc locum habet in ascensibus.

Corollarium 5.

589. Cum igitur tam omnes descensus sint isochroni quam omnes ascensus, etiam omnes semioscillationes aequalibus absolventur temporibus. Atque in casu $k > \frac{a}{8g}$, quo corpus perpetuo oscillationes continuat, omnes absolventur temporibus aequalibus.

Corollarium 6. [p. 312]

590. Cyclois ergo, quae est curva tautochrona in vacuo, eandem proprietatem retinet in medio, quod resistit in simplici ratione celeritatum. Praeterea cyclois quoque tautochronismum obtinet in medio, cuius resistentia est constans seu momentis temporum, ut Neutonus loquitur, proportionalis (570).

Scholion 1.

591. Hunc triplicem cycloidis tautochronismum Neutonus quoque demonstravit in *Prin. Phil.* (Lib. II, Prop. XXV et XXVI, 1687.), atque quod ad resistentiam ipsis celeritatibus proportionalem attinet, ex hoc demonstrationem formavit, quod in diversis descensibus, si arcuum partes totis arcubus proportionales accipiantur, in iis locis celeritates sint toties arcubus quoque proportionales. Nam si celeritates totis arcubus fuerint proportionales, si elementa quoque capiantur totis arcubus proportionalia, tempora per ea erunt inter se aequalia.

Scholion 2.

592. Etsi autem ex his appareat tempora tam ascensuum quam descensuum inter se esse aequalia, tamen determinari not potest, quantum sit tempus sive descensuum sive ascensuum, neque etiam tempora descensuum et ascensuum inter se possunt comparari. [p. 313] Aequatio enim relationem inter *s* et *u* definiens ita est complicata, ut ex ea elementum temporis $\frac{ds}{u}$ per unicam variabilem non possit exprimi. Praeterea oscillationes infinite parvae, quae ante in determinandis temporibus calculum valde facilem reddiderunt, in hac resistentiae hypothesi nihil adiuvant. Nam etiam si arcus totus descriptus ponatur infinite parvus, in aequatione

$$l\frac{\sqrt{2ac^2\sqrt{k}}}{\sqrt{(2au^2\sqrt{k}-aus+gss\sqrt{k})}}=\frac{a}{4Bk}At.\frac{4Bks}{as-4au\sqrt{k}},$$

quae ex superiori integrata oritur, ne unicus quidem terminus evanescit praeceteris. Pendebit autem tempus ascensus a quantitatibus *a*, *k* et *g*; at quomodo ex his sit compositum, non liquet. Interim tamen hoc certum est, quo maior sit *g* ceteris paribus, eo minus esse tempus, at crescente *a* tempus quoque crescere, *k* vero crescente diminuti, quia resistentia fit minor. In hac igitur resistentiae hypothesi resistentia in motibus tardissimis non evanescit, quemadmodum in resistentia quadratis celeritatum proportionali. Ex quo consequi videtur, si resistentia in maiore quam duplicata celeritatum ratione crescat, in motibus tardissimis resistentiam neglici posse, at si resistentia fuerit in minore ratione, etiam in motibus tardissimis resistentiam considerari debere.

PROPOSITIO 67. [p. 314]

Problema.

593. *In medio uniformi, quod resistit in ratione multiplicata celeritatum, cuius exponens est 2m, determinare motum corporis super curva CMA* (Fig.66) *, in qua arcus quisque CM proportionalis est potestati abscissae CP, cuius exponens est 1 – m.*

Solutio.

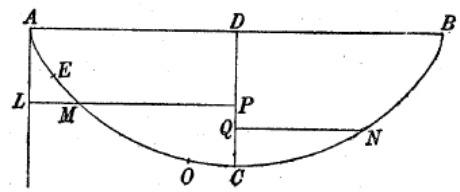

Fig. 66.

Positis abscissa $CP = x$ et arcu CM = s et altitudo erit $ds = \frac{a^m dx}{x^m}$. Sit celeritas in M debita altitudini v ; erit resistentia in M = $\frac{v^m}{k^m}$ atque ideo, si corpus descendere ponatur super arcu CM, habetur ista aequatio

$$dv = -gx + \frac{a^m v^m dx}{k^m x^m}.$$

Pro ascensu autem super eadem curva inservit ista aequatio

$$dv = -gx - \frac{a^m v^m dx}{k^m x^m}$$

Utraque vero aequatio separationem admittet, si ponatur $v = tx$; prodibit enim pro descensu

$$xdt = -tdx - gx + \frac{a^m t^m dx}{k^m}$$

seu

$$\frac{-k^m dt}{k^m(g+t)-a^m t^m} = \frac{dx}{x}$$

atque pro ascensu haec aequatio

$$\frac{-k^m dt}{k^m(g+t)+a^m t^m} = \frac{dx}{x}.$$

[p. 315] In quibus aequationibus variabiles t et x a se invicem sunt separatae, itu ut t per x ope quadraturarum possit determinari. Constans in integratione addenda definire debet vel ex data celeritate in puncto C vel ex loco curvae, in quo vel descensus incipit vel ascensus finitur. Si ponatur abscissa toti arcui descensus vel ascensu respondens f, aequalis erit v functioni unius dimensionis ipsarum f et x tam in descensu quam ascensu propter aequationes differentiales homogeneas. Hanc ob rem $\sqrt{v}$ aequabitur functioni dimidiae dimensionis ipsarum f et x. Tempus igitur descensus per MC , quod est =

$$\int \frac{ds}{\sqrt{v}} = a^m \int \frac{ds}{x^m \sqrt{v}},$$

aequale erit functioni $\frac{1}{2} - m$ dimensionum ipsarum f et x. Quia autem proposito $x = f$ prodit tempus totius descensus ut $f^{\frac{1}{2}-m}$, ei etiam proportionale est tempus totius ascensus, si quidem f abscissam arcus totius ascensus designat. Si ponatur totus arcus vel ascensus vel descensus = *A*, quia est A ut f^{1-m}, erit tempus totum vel ascensus vel descensus ut $\frac{A}{f^{\frac{1}{2}}}$ vel ut $A^{\frac{1-2m}{2-2m}}$. Plurium ergo descensuum tempora sunt in ratione $\frac{1-2m}{2-2m}$ - multiplicata totorum arcuum descriptorum. Atque in eadem ratione sunt quoque tempora ascensuum inter se; sed tempora ascensuum et descensuum inter se non comparantur. Q.E.I.

Corollarium 1. [p. 316]

594. Quia celeritas seu $\sqrt{v}$ aequalis est functioni dimidiae dimensionis ipsarum *f* et *x*, in pluribus descensibus celeritates in puncto *C* acquisitae sunt in subduplicata ratione altitudinum, ex quibus corpus descendit. Atque altitudines, ad quas corpus ascendens pertingit, sunt in duplicata ratione celeritatum initialium in C.

Corollarium 2.

595. Cum tam tempora descensus quam ascensus sint ut $f^{\frac{1}{2}-m}$, omnes descensus aequalibus absolventur temporibus, si fuerit $m = \frac{1}{2}$ seu resistentia ipsis celeritatibus proportinalis. Atque hac hypothesi pariter tempora ascensuum inter se erunt aequalia. Curva autem erit cyclois, ut ante ostendimus.

Corollarium 3.

596. Quia est $ds = \frac{a^m dx}{x^m}$, erit $s = \frac{a^m x^{1-m}}{1-m}$. Ex quo perspicitur, nisi $m < 1$, curvam *AMC* fore negativam seu, quod perinde est, imaginariam. Semper enim curva maior esse debet quam abscissa.

Corollarium 4.

597. Praeter semper debet esse $ds > dx$; quare, quo hoc accidat, si $x = 0$, debet m esse numerus positivus. Hinc nostra propositio requirit, ut *m* inter limites 0 et 1 contineatur.

Corollarium 5. [p. 317]

598. In his casibus maximus ipsius *x* valor erit *a* ibique erit $ds = dx$ seu tangens verticalis. Hocque loco curva habebit cuspidem; altius enim ascendere nequit, quia, si $x > a$, foret $ds < dx$, quod fieri nequit.

Corollarium 6.

599. Si *m* continetur intra limites 0 et 1, curva in *C* habebit tangentem horizontalem atque radius osculi in *C* erit = $\frac{sds}{dx} = \frac{a^{2m} x^{1-2m}}{1-m}$ posito $x = 0$. Quare radius osculi in *C* erit infinite parvus, si $m < \frac{1}{2}$, finitus si $m = \frac{1}{2}$, et infinite magnus, si $m > \frac{1}{2}$.

Corollarium 7.

600. Tempora minimorum descensuum et ascensuum sunt infinite parva, si $m < \frac{1}{2}$, at finita, si $m = \frac{1}{2}$. Infinite magna denique erunt, si $m > \frac{1}{2}$. Tenent ergo radiorum osculi in infimo puncto *C* rationem.

Scholion.

601. Habemus hic ergo exempla curvarum pro resistentia minorem quam duplicatam rationem celeritatum tenente, super quibus motus corporis potest determinari. [p. 318] At si medium in maiore quam duplicata ratione resistit, curva nusquam habebit tangentem horizontalem atque ideo descensus et ascensus nunquam finiri possunt. Quo autem in exemplo appareat, qualis sit motus corporis in medio, quod in maiore quam duplicata ratione celeritatum resistit, investigare lubet motum corporis super cycloide saltem in medio, quod resistit in quadruplicata rationne celeritatum. Hanc vero resistentiae hypothesin prae aliis eligo, quia in ea celeritas commode per seriem potest definiri.