CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 260]

PROPOSITION 58.

Problem.

505. *According to the hypothesis of uniform gravity g, and in a medium with some uniform resistance, to determine the motion of the body with the initial speed of ascent given at A* (Fig.62) *on the straight line AB inclined at some angle to the horizontal.*

Solution.

With the line *AC* drawn to the horizontal and with the perpendicular *MP* drawn to that from *M*, call $PM = x$, and let $AM = nx$. Let the height corresponding to the initial speed at *A* be equal to *b* and the height corresponding to the speed at *M* be equal to *v*; now the resistance at *M* is equal to $\frac{V}{K}$. With these in place, there is the equation :

$$dv = -gdx - \frac{nVdx}{K}$$

(479), hence there is obtained

$$dx = \frac{-Kdv}{gK+nV} \text{ and } x = \int \frac{-Kdv}{gK+nV}$$

with this integral thus taken, so that it vanishes on putting $v = b$. If then we put $v = 0$, then there is produced $x = BC$, where the body has lost all the speed at the point *B*. Now the time, in which the body ascends along *AM*, is equal to

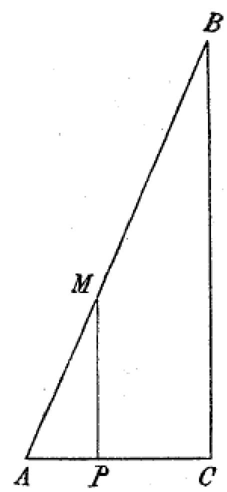

Fig. 62.

$$\int \frac{-Kdv}{(gK+nV)\sqrt{v}}$$ [p. 261]

with this integral also taken so that it vanishes on putting $v = b$; in which if again we put $v = 0$, the time for the whole ascent along *AMB* is found. Moreover the force sustained by the line *AMB* is everywhere constant, and equal to the normal force

$\frac{g\sqrt{(n^2-1)}}{n}$. Q.E.I.

Corollary 1.

506. If the line *AMB* is horizontal, then with the angle *BAC* vanishing, *n* becomes $\propto$. Therefore on putting $AM = z = nz$, there is found

$$z = \int \frac{-Kdv}{V}$$

and the time, in which the body progresses along AM, is equal to

$$\int \frac{-Kdv}{V\sqrt{v}}.$$

Corollary 2.

507. If the resistance is as some power 2*m* of the speeds, then we have $V = v^m$ and $K = k^m$. Therefore in this case, there becomes :

$$x = \int \frac{-k^m dv}{gk^m + nv^m}$$

and the time to pass along *AM* is equal to :

$$\int \frac{-k^m dv}{(gk^m + nv^m)\sqrt{v}}.$$

Corollary 3.

508. This expression and the other converted into series give

$$x = \frac{b-v}{g} - \frac{n(b^{m+1} - v^{m+1})}{(m+1)g^2k^m} + \frac{n^2(b^{2m+1} - v^{2m+1})}{(2m+1)g^3k^{2m}} - \text{etc.}$$

and the time to pass along *AM* is equal to :

$$\frac{2n(\sqrt{b} - \sqrt{v})}{g} - \frac{2n^2(b^{m+\frac{1}{2}} - v^{m+\frac{1}{2}})}{(2m+1)g^2k^m} + \frac{2n^3(b^{2m+\frac{1}{2}} - v^{2m+\frac{1}{2}})}{(4m+1)g^3k^{2m}} - \text{etc.}$$

[p. 262] On account of which, on putting *v* = 0, the series becomes

$$BC = \frac{b}{g} - \frac{nb^{m+1}}{(m+1)g^2k^m} + \frac{n^2b^{2m+1}}{(2m+1)g^3k^{2m}} - \text{etc.}$$

and the time of the whole ascent along *AB* is equal to :

$$\frac{2n\sqrt{b}}{g} - \frac{2n^2b^{m+\frac{1}{2}}}{(2m+1)g^2k^m} + \frac{2n^3b^{2m+\frac{1}{2}}}{(4m+1)g^3k^{2m}} - \text{etc.}$$

Example 1.

509. Let the resistance be proportional to the speed, then $m = \frac{1}{2}$ and

$$x = \int \frac{-dv\sqrt{k}}{g\sqrt{k} + n\sqrt{v}} = \frac{2\sqrt{bk}}{n} - \frac{2\sqrt{kv}}{n} + \frac{2gk}{n^2} l \frac{g\sqrt{k} + n\sqrt{v}}{g\sqrt{k} + n\sqrt{b}}.$$

Hence the total height *BC*, to which the body is able to reach,

$$= \frac{2\sqrt{bk}}{n} - \frac{2gk}{n^2} l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k}}.$$

Now the time in which it rises along *AM* is equal to :

$$\int \frac{-ndv\sqrt{k}}{(g\sqrt{k} + n\sqrt{v})\sqrt{v}} = 2\sqrt{k}\, l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k} + n\sqrt{v}}.$$

Whereby the time of the whole ascent along *AMB* is equal to :

$$2\sqrt{k}\, l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k}}.$$

Therefore if the body descends on the inclined line *AC* (Fig. 63) and with the speed acquired at *C* ascends on *CB* as far *B* and let $AC = N.AD$ and $BC = n.BE$ and the speed at *C* corresponds to the height *b*, then (486)

$$AD = -\frac{2\sqrt{bk}}{N} + \frac{2gk}{N^2} l \frac{g\sqrt{k}}{g\sqrt{k} - N\sqrt{b}},$$

$$AC = -2\sqrt{bk} + \frac{2gk}{N} l \frac{g\sqrt{k}}{g\sqrt{k} - N\sqrt{b}},$$

$$BE = \frac{2\sqrt{bk}}{n} - \frac{2gk}{n^2} l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k}},$$

$$CB = 2\sqrt{bk} - \frac{2gk}{n} l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k}}.$$

And the descent time along *AC* =

$$2\sqrt{k}\, l \frac{g\sqrt{k}}{g\sqrt{k} - N\sqrt{b}}$$

(cit.) and the ascent time along *CB* =

$$2\sqrt{k}\, l \frac{g\sqrt{k} + n\sqrt{b}}{g\sqrt{k}}.$$

[p. 263] Hence the descent and the ascent on the straight lines can be compared with each other.

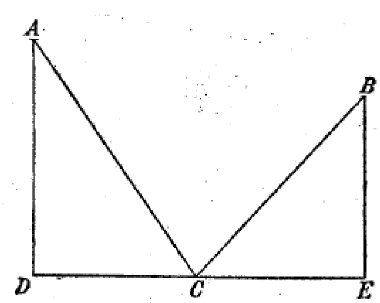

Fig. 63.

Corollary 4.

510. If these logarithms are expressed in series, it is clear that it is not possible that $BE = AD$; for according to any hypothesis of the resistance, as it is understood from the series (508 and 488), that we have $BE < \frac{b}{g}$ and $AD > \frac{b}{g}$. But it can happen that $AC = BC$.

Corollary 5.

511. Moreover it is easily demonstrated that the descent time along AC is equal to the ascent time along CB. Clearly this must become

$$ng\sqrt{k} = Ng\sqrt{k} + Nn\sqrt{b} \text{ or } n = \frac{Ng\sqrt{k}}{g\sqrt{k} - N\sqrt{b}}$$

Hence we have $n > N$ or angle $BCE <$ angle ACD. Moreover, the relation between N and n depends on the speed at the point C.

Corollary 6.

512. But if the angle BCE is equal to the angle ACD or $N = n$, the ascent time along BC is less than the descent time along AC. And this generally is the position for any hypothesis of the resistance; for the time of the descent along $CB < \frac{2n\sqrt{b}}{g}$, as is apparent from the series given above ((488) and (508)). [p. 264]

Example 2.

513. The medium resists in the ratio of the square of the speeds; then $m = 1$. Whereby there is obtained :

$$x = \int \frac{-kdv}{gk + nv} = \frac{k}{n} l \frac{gk+nb}{gk+nv}$$

and (Fig. 62)

$$BC = \frac{k}{n} l \frac{gk+nb}{gk} \text{ and } AB = kl \frac{gk+nb}{gk}.$$

Now the time of the descent along AM is equal to

$$\int \frac{-nkdv}{(gk+nv)\sqrt{v}} = \frac{2\sqrt{kn}}{\sqrt{g}} \left(A.\ \text{tang.} \sqrt{\frac{nb}{gk}} - A.\ \text{tang.} \sqrt{\frac{nv}{gk}} \right)$$

with the radius equal to 1 and with A denoting the arc of the circle. Hence the ascent time along AB is equal to

$$\frac{2\sqrt{kn}}{\sqrt{g}} A.\ \text{tang.} \sqrt{\frac{nb}{gk}}.$$

If now the body descends along the inclined line AC (Fig. 63) and with the speed at C acquired, which corresponds to the height b, it ascends along CB again where $AC = N.AD$ and $BC = n.BE$, then we have :

$$AD = \frac{k}{N} l \frac{gk}{gk-Nb} \text{ and } AC = kl \frac{gk}{gk-Nb}$$

and the time of descent along *AC* is equal to :

$$\frac{\sqrt{Nk}}{\sqrt{g}} l \frac{\sqrt{gk}+\sqrt{Nb}}{\sqrt{gk}-\sqrt{Nb}}$$

(407). Now again we have :

$$BE = \frac{k}{n} l \frac{gk+nb}{gk}, \quad BC = k\, l \frac{gk+nb}{gk}$$

and the time of the ascent along *CB* is equal to :

$$\frac{2\sqrt{nk}}{\sqrt{g}} A.\,\text{tang.} \sqrt{\frac{nb}{gk}}.$$

Corollary 7.

514. In this hypothesis of the resistance it is convenient to put *AC* = *BC*; for it must become

$$ngk = Ngk + Nnb \text{ or } n = \frac{Ngk}{gk-Nb}.$$

Hence we have $n > N$, and hence the angle *BCE* < the angle *ACD.*

Example 3.

515. Let the resistance be taken as very small and proportional to the $2m^{\text{th}}$ power of the speed; then *k* is a very large quantity. [p. 265] If therefore the speed at *C* corresponds to the height *b*, $AC = N.AD$ and $BC = n.BE$, and the body descends on the line *AC* and ascends on the line *CB* , then

$$AD = \frac{b}{g} + \frac{Nb^{m+1}}{(m+1)g^2k^m}$$

and the time of descent along *AC* is equal to :

$$\frac{2N\sqrt{b}}{g} + \frac{2N^2b^m\sqrt{b}}{(2m+1)g^2k^m}$$

(488). Now for the ascent,

$$BE = \frac{b}{g} - \frac{nb^{m+1}}{(m+1)g^2k^m}$$

and the time to pass along *CB* is equal to :

$$\frac{2n\sqrt{b}}{g} - \frac{2n^2b^m\sqrt{b}}{(2m+1)g^2k^m}$$

(508). If therefore it is to be brought about that *AC* = *BC*, then it is required that

$$N+\frac{N^2b^m}{(m+1)gk^m}=n-\frac{n^2b^m}{(m+1)gk^m},$$

thus this becomes :

$$n=N+\frac{2N^2b^m}{(m+1)gk^m}$$

since k is a very large quantity. But where the time of the descent along *AC* is equal to the time of the ascent along *CB*, it must be the case that :

$$N+\frac{N^2b^m}{(2m+1)gk^m}=n-\frac{n^2b^m}{(2m+1)gk^m}$$

or

$$n=N+\frac{2N^2b^m}{(2m+1)gk^m}.$$

Scholium 1. [p. 266]

516. In the case of this example, where the resistance is very small, the curve *AMD* (Fig. 64) can be determined by this property, as the body by ascending from C with a speed corresponding to the height b upon some line CM reaches the curve AMD. For on putting $CM = z$ and $MP = x$ then $n=\frac{z}{x}$ and

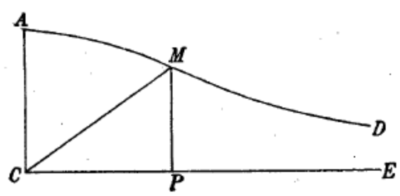

Fig. 64.

$$x=\frac{b}{g}-\frac{nb^{m+1}}{(m+1)g^2k^m}.$$

Whereby this equation is obtained :

$$b^{m+1}z=(m+1)gbk^mx-(m+1)g^2k^mx^2.$$

Let $CP = y$ and let f be written in place of

$$\frac{b^{m+1}}{(m+1)gk^m},$$

and there arises :

$$f\sqrt{(x^2+y^2)}=bx-gx^2$$

or

$$ffyy=(bb-ff)xx-2gbx^3+g^2x^4.$$

If we put $y = 0$, then both $x = 0$ and $x=\frac{b-f}{g}=CA.$ Hence the curve also passes through the point C, which moreover ceases to satisfy the part of this question on account of the following neglected terms, which have been wrongly ignored, if n or $\frac{z}{x}$ is also made very large. Now the equation gives a curve in the form of an ellipse with the maximum length described about the minor axis AC. But the true curve has the form *AMD*, the asymptote of which is the horizontal line *CE*, if indeed $m>1$, the equation of this curve is obtained with all the terms taken, which is :

$$xdz + (m-1)zdx = \frac{bk^m(zdx - xdz)}{gk^m x + b^m z}.$$

If $m < 1$, then the curve does not progress to infinity, but falls on *CE* by taking

$$CE = \frac{bk^m}{(1-m)b^m}.$$

For if $m < 1$, the body is not able to progress to infinity, but all the speed is lost after a finite distance.

Scholium 2. [p. 267]

517. If the resistance of the medium is not uniform, then the line is not straight upon which the motion can be most easily determined; the same too is to be observed, if the force acting is not uniform. As the force is set equal to *P* and the resistance equal to $\frac{V}{Q}$, where Q is such a function of the exponent of the variable resistance *q*, as *V* is of *v*; with these put in place the motion of the body upon any curve is expressed by this equation : i.e.

$$dv = \pm Pdx \pm \frac{Vds}{Q}.$$

Hence this is the equation of this curve, upon which the motion is most easily defined :

$$PQdx = Ads,$$

and from which the following arises :

$$\frac{Adv}{\pm A \pm V} = Pdx$$

determining the motion on this curve, in which the indeterminates can be separated from each other. Whereby if we might wish to pursue hypotheses of this kind, we must assume in place of the straight lines curves expressed by this equation $PQdx = Ads$. But since we have decided to handle further only the hypothesis of uniform forces acting and uniform resistance, with these dismissed we progress to these cases, in which *v* has only a single dimension, that which arises if the resistance is proportional to the square of the speed.

PROPOSITION 59.

Problem.

518. *According to the hypothesis of uniform gravity g and with the resistance proportional to the square of the speed, the body descends on some curve AMB* (Fig.57); *to determine the motion of this body, and the force sustained by the curve at individual points.* [p. 268]

Solution.

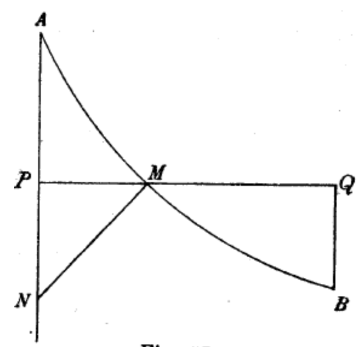

Fig. 57.

On the vertical axis is taken the abscissa $AP = x$ and the arc is put $AM = s$, the speed at M corresponds to the height v, and k is the exponent of the resistance ; the resistance is equal to $\frac{v}{k}$. On account of which this equation is had setting out the motion of the body :

$$dv = gdx - \frac{vds}{k}$$

(465). On multiplying that to be integrated by $e^{\frac{s}{k}}$, there is the integral :

$$e^{\frac{s}{k}}v = \int e^{\frac{s}{k}}gdx$$

Moreover with this integral it must be taken thus, so that on putting $s = 0$ there comes about the height v corresponding to the initial speed at A. Therefore if the descent is put to be made from rest, $e^{\frac{s}{k}}gdx$ thus must be integrated, so that the speed vanishes on putting $s = 0$. And thus with this done, we have :

$$v = ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx.$$

Hence the time to traverse AM is equal to :

$$\int \frac{e^{\frac{s}{2k}}dx}{\sqrt{g\int e^{\frac{s}{k}}dx}}.$$

Now on placing $PM = y$ and on taking dx constant the force sustained by the curve at M along the normal MN is equal to :

$$\frac{gdy}{ds} + \frac{2ge^{\frac{-s}{k}}dxddy\int e^{\frac{s}{k}}dx}{ds^3}.$$

Q.E.I.

Corollary 1. [p. 269]

519. The force that the curve sustains can be changed into this form :

$$\frac{gds}{e^{\frac{s}{k}}dydx}d.\frac{dy^2}{ds^2}\int e^{\frac{s}{k}}dx,$$

which, upon integrating $e^{\frac{s}{k}}dx$ for the given curve, is more convenient to apply to any case.

Corollary 2.

520. The body on descending has the maximum speed where $v=\frac{gkdx}{ds}$. Now this comes about, where

$$e^{\frac{s}{k}}kdx=ds\int e^{\frac{s}{k}}dx$$

or where

$$d.l\int e^{\frac{s}{k}}dx=\frac{ds}{k};$$

at which point it is clear that the tangent is not horizontal.

Corollary 3.

521. If it should be that

$$\int e^{\frac{s}{k}}dx=\frac{e^{\frac{s}{k}}s^n}{a^{n-1}},$$

then

$$dx=\frac{ns^{n-1}ds}{a^{n-1}}+\frac{s^n ds}{a^{n-1}k}$$

and

$$x=\frac{s^n}{a^{n-1}}+\frac{s^{n+1}}{(n+1)a^{n-1}k}.$$

Whereby if this equation expresses the nature of the curve sought, then

$$v=\frac{gs^n}{a^{n-1}}$$

and the time to traverse *AM* is equal to

$$\frac{2a^{\frac{n-1}{2}}s^{\frac{2-n}{2}}}{(2-n)\sqrt{g}},$$

[p. 270] if indeed n is less than two; for if $n=2$ or $n>2$, then the curve has a horizontal tangent at *A* and the body remains there permanently.

Corollary 4.

522. In a similar manner it is also evident, if x is some power of s or a sum of powers of this kind, then it is always possible to integrate $e^{\frac{s}{k}}dx$ and thus the terminal speed can be shown.

Corollary 5.

523. Moreover if the abscissae are taken on the vertical axis BQ and the speed that the body has at B corresponds to the height *b*, and besides calling $BQ = x$ and $BM = s$, then

$$dv = -gdx - \frac{vds}{k}$$

the integral of this equation is :

$$e^{-\frac{s}{k}}v = b - g\int e^{-\frac{s}{k}}dx$$

clearly with the integral $\int e^{-\frac{s}{k}}dx$ thus taken, in order that it vanishes on putting $x = 0$. On this account we have :

$$v = be^{\frac{s}{k}} - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx$$

and the time, in which the arc descended *MB* is completed, is equal to :

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b - g\int e^{-\frac{s}{k}}dx)}}.$$

Corollary 6. [p. 271]

524. Therefore if the speed is given at the point *B*, clearly $\sqrt{b}$, it is possible to find the point *A* on the curve *BMA*, from which the body always begins its descent and has the speed equal to zero. Whereby this is clearly the place where

$$\int e^{-\frac{s}{k}}dx = \frac{b}{g}.$$

And also the expression for the time :

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b - g\int e^{-\frac{s}{k}}dx)}}$$

gives the time of the whole descent along AMB, if we put after the integration,

$$\int e^{-\frac{s}{k}}dx = \frac{b}{g}.$$

Scholium .

525. Thus we have presented two ways in which the motion can be investigated : as it can be adapted to the descent made from a given point, or the descent as far as a given point is considered, as is usually the case in oscillatory motion.

PROPOSITION 60.

Problem.

526. *With the force arising acting uniformly and with a uniform medium with resistance in the square ratio of the speed, to determine the motion of the body ascending on the given curve AMD* (Fig.58) *and the force pressing on the curve sustained at individual points M.*

Solution.

The abscissa $AP = x$ is placed on the vertical line AP, the arc $AM = s$, the speed at A corresponds to the height b and the speed at M to the height v. Let the force acting downwards be equal to g and the resistance is equal to $\frac{v}{k}$. With these in place, then

$$dv = -gdx - \frac{vds}{k}$$

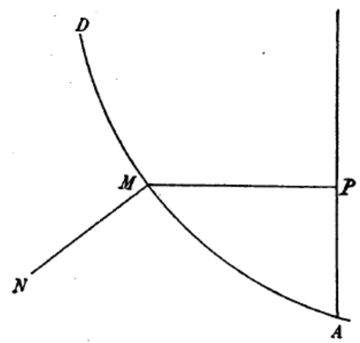

Fig. 58.

(475), which multiplied by $e^{\frac{s}{k}}$ gives on integrating :

$$e^{\frac{s}{k}}v = b - g\int e^{\frac{s}{k}}dx \text{ [p. 272]}$$

Thus with $\int e^{\frac{s}{k}}dx$ taken, so that it vanishes on placing $x = 0$. On account of which the equation becomes :

$$v = be^{-\frac{s}{k}} - ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx$$

From which the time of ascent along the curve AM equals

$$\int \frac{e^{\frac{s}{2k}}ds}{\sqrt{(b - g\int e^{\frac{s}{k}}dx)}}$$

From the speed found the pressing force is obtained [*i. e*. the normal reaction], that is borne by the curve at M along the normal MN, that is equal to :

$$\frac{gdy}{ds} - \frac{2vdxddy}{ds^3}$$

(475) on putting $PM = y$ and on taking dx as constant. Q.E.I.

Corollary 1.

527. Hence on putting $v = 0$ the equation becomes :

$$g\int e^{\frac{s}{k}}dx = b,$$

from which equation the point D is obtained, and to which the body is able to ascend from A. And the time of the whole ascent along AMD can be obtained, if $g\int e^{\frac{s}{k}}dx = b$ is put in the expression for the time.

Corollary 2.

528. If, in the formula showing the force pressing, in place of *v* this value found is substituted, then there is obtained :

$$-\frac{2e^{\frac{-s}{k}}bdxddy}{ds^3}+\frac{gdy}{ds}+\frac{2ge^{\frac{-s}{k}}dxddy\int e^{\frac{s}{k}}dx}{ds^3}.$$

Which it is possible to change into this form :

$$-\frac{2e^{\frac{-s}{k}}bdxddy}{ds^3}+\frac{gds}{e^{\frac{s}{k}}dydx}d.\frac{dy^2}{ds^2}\int e^{\frac{s}{k}}dx.$$

Corollary 3. [p. 273]

529. Now for the descent, if the speed at *A* also corresponds to the height *b*, then the pressing force that the curve sustains at *M* along the normal *MN* is equal to :

$$\frac{2e^{\frac{s}{k}}bdxddy}{ds^3}+\frac{ge^{\frac{s}{k}}ds}{dydx}d.\frac{dy^2}{ds^2}\int e^{\frac{-s}{k}}dx.$$

Corollary 4.

530. If therefore the ascent as well as the descent are defined with respect to the axis AP, the equation determining the ascent can be changed into the equation for the descent by writing $-k$ in place of *k* in turn. Whereby if the descent has been determined on the curve AM, then the ascent can also be determined in turn.

Scholium.

531. Since the formulas determining the ascent and the descent have so much in common, the ascents and the descents can easily be compared with each other, and thus the oscillations on a given curve can be determined. We present this in the following proposition, as generally as it can be done. [p. 274]

PROPOSITION 61.

Problem.

532. *Let whatever curves MA and NA* (Fig.65) *be joined at the lowest point A and the body descends on the curve [MA and ascends on the curve] AN in a medium with uniform resistance following the square of the speeds; to be compares between themselves are the descends on the curve MA and the ascents on the curve AN.*

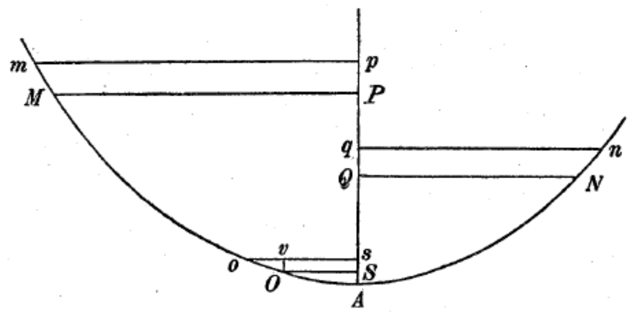

Fig. 65.

Solution.

Let the speed at the point *A* correspond to the height *b* and on the vertical axis *AP* the abscissa *AP* = *x*, and the arc *AM* = *s*. Now for the curve *AN*, the ascent shall be *AQ* = *t* and *AN* = *r*. With these in place, the speed of the body descending at *M* corresponds to the height

$$e^{\frac{s}{k}}b - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx$$

(523). Now the speed of the body ascending on the curve *AN* at *N* corresponds to the height :

$$e^{-\frac{s}{k}}b - ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx$$

(526). Whereby if the speeds at *M* and *N* vanish [as does *v*], thus in order that *MAN* is the arc described by a single semi–oscillation, then

$$\frac{b}{g} = \int e^{-\frac{s}{k}}dx \text{ and } \frac{b}{g} = \int e^{\frac{r}{k}}dt.$$

Now if another semi–oscillation completing the arc *mMn* is taken, in which the speed at the point *A* corresponds to the height *b* + *db*, then

$$\frac{b+db}{g} = \int e^{\frac{s}{k}}dx + e^{-\frac{s}{k}}dx$$

and hence

$$e^{-\frac{s}{k}}dx = \frac{db}{g} \text{ or } e^{-\frac{AM}{k}}.Pp = \frac{db}{g}.$$

Similarly for the ascent, there is

$$e^{\frac{AN}{k}}.Qq = \frac{db}{g}.$$

From which there becomes [p. 275]

$$\frac{Pp}{Qq} = e^{\frac{AM+AN}{k}} \text{ or } lPp - lQq = \frac{AM+AN}{k}.$$

Hence for the given arc *MAN* described by a single oscillation, if the body begins to descend from a point very close above *m*, the point *n* can be found, that pertains to the point above *N* ; clearly it is

$$Qq = \frac{Pp}{e^{\frac{AM+AN}{k}}}.$$

Q.E.I.

Corollary 1.

533. Therefore if *MAN* and *mAn* are two arcs described by neighbouring oscillations, then always $Qq < Pp$ and from this *Qq* is less than *Pp*, in which the sum of the arcs *AM* + *AN* is greater. Therefore also, *AQ* shall always be less than *AP*.

Corollary 2.

534. *In vacuo*, since $k = \infty$, then $e^{\frac{AM+AN}{k}} = 1$; and the above equation becomes $Qq = Pp$ and hence $AQ = AP$. Whereby the body oscillating *in vacuo* ascends to as great a height, as that from which it descends.

Corollary 3.

535. If the resistance is very small and thus *k* extremely large, then

$$e^{\frac{AM+AN}{k}} = 1 + \frac{AM+AN}{k}.$$

Whereby in this case the ratio is :

$$Qq + \frac{MAN.Qq}{k} = Pp \text{ and } Qq = \frac{Pp(k - MAN)}{k}$$

Corollary 4. [p. 276]

536. If the point *O* is the place, at which the descending body has the maximum speed, and there put $AO = s$, $AS = x$, then

$$\frac{gkdx}{ds} = e^{\frac{s}{k}}b - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx$$

or [corrected in *O. O.*]

$$b = g\int e^{-\frac{s}{k}}dx + \frac{gkdx}{ds}.$$

Whereby if the body descends from *m*, the point of the maximum speed arising at *o*

$$db = ge^{-\frac{s}{k}}dx + \frac{gkdy}{p},$$

with $dy = ov$ and *p* radius of osculation at the point *O*, or

$$\frac{db}{g} = e^{-\frac{AO}{k}}Ss + \frac{k.ov}{p}.$$

Corollary 5.

537. The time of a single motion along MAN is found, if in the sum of the integration,

$$\int \frac{e^{\frac{-s}{2k}} ds}{\sqrt{\left(b - g\int e^{\frac{-s}{k}} dx\right)}} + \int \frac{e^{\frac{r}{2k}} ds}{\sqrt{\left(b - g\int e^{\frac{r}{k}} dt\right)}}$$

there is inserted

$$\int e^{\frac{-s}{k}} dx = \frac{b}{g} \text{ and } \int e^{\frac{r}{k}} dt = \frac{b}{g}.$$

And thus the time of a single semi–oscillation is produced.

Corollary 6.

538. From what has been said another elegant property follows, for if the body ascends from *A* with the speed $\sqrt{b}$ to *N* and from *N* it again falls through *NA* and the speed that it then has at *A* corresponds to the height *c*, [p. 277] then from *A* it can reach *n* with a speed corresponding to the height $b + db$, hence again on descending to *A* it acquires a speed corresponding to the height $c + dc$. Hence

$$\frac{db}{g} = e^{\frac{AN}{k}} \cdot Qq \text{ and } \frac{dc}{g} = e^{\frac{-AN}{k}} \cdot Qq$$

and

$$Qq^2 = \frac{db.dc}{g^2}$$

and also

$$\frac{db}{dc} = e^{\frac{2AN}{k}}.$$

Scholion.

539. It remains, that we adapt these generalisations to examples or given curves, where the use of these will there become more apparent. Moreover we take only the cycloid for the given curve, as there the integral $\int e^{\frac{s}{k}} dx$ can be easily demonstrated and also the equation between *s* and *x* is algebraic. Hence we will examine both descents made on cycloids as well as oscillations, from which it is apparent to what extent the oscillations made on the cycloid depart from being isochronous, clearly *in vacuo* all have been shown to be completed in the same time. [(191)].

PROPOSITION 62.

Problem.

540. *Let the given curve be the cycloid ACB* (Fig.66) *described by the circle of diameter CD rolling along upon the given horizontal base AB, and the body descends on that cycloid from A in a medium with the resistance proportional to the square of the speed; to determine the motion of the descending body.* [p. 278]

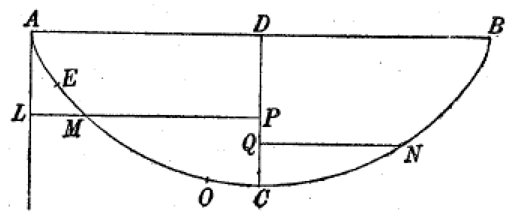

Fig. 66.

Solution.

On placing $2CD = a$, $AL = x$ and $AM = s$ then from the nature of the cycloid :

$$s = a - \sqrt{(a^2 - 2ax)} \text{ or } 2ax = 2as - ss.$$

Now the speed at M corresponds to the height v; then

$$v = ge^{-\frac{s}{k}} \int e^{\frac{s}{k}} dx \,.$$

Moreover, since

$$dx = ds - \frac{sds}{a},$$

then

$$\int e^{\frac{s}{k}} dx = ke^{\frac{s}{k}} + \frac{k^2 e^{\frac{s}{k}}}{a} - \frac{ke^{\frac{s}{k}} s}{a} - k - \frac{k^2}{a} = e^{\frac{s}{k}} \frac{ak + k^2 - ks}{a} - \frac{ak + k^2}{a}.$$

In which with the value substituted,

$$v = \frac{gak + gk^2 - gks}{a} - \frac{ge^{\frac{-s}{k}}(ak + k^2)}{a}.$$

The speed of the body is a maximum where

$$v = \frac{gkdx}{ds} = \frac{gak - gks}{a};$$

therefore this happens, where

$$e^{\frac{s}{k}} k = a + k \text{ or } s = kl \frac{a+k}{k}.$$

Also it is possible to find the point N, at which the body has lost all its speed, by making $v = 0$ or

$$e^{\frac{s}{k}} = \frac{a+k}{a+k-s}, \text{or } s = kl \frac{a+k}{a+k-s},$$

from which equation the value of s gives the arc ACN. The time, in which the arc AM has been completed, is given by :

$$\int \frac{ds\sqrt{a}}{\sqrt{gk}\left(a+k-s-e^{\frac{-s}{k}}(a+k)\right)}.$$

Then in order that the pressing force can be found :

$$\frac{dy}{ds}=\frac{\sqrt{(2as-ss)}}{a}=\frac{\sqrt{2ax}}{a},$$

from which the force itself produced at the point M is equal to:

$$\frac{g\sqrt{2ax}}{a}+\frac{2gak+2gkk-2gks}{a\sqrt{2ax}}-\frac{2gak+2gkk}{e^{\frac{s}{k}}a\sqrt{2ax}}.$$

[p. 279] Hence we have found the speed, the time, and the pressing force from the known motion of the body. Q.E.I.

Corollary 1.

541. Since

$$e^{-\frac{s}{k}}=1-\frac{s}{1.k}+\frac{s^2}{1.2.k^2}-\frac{s^3}{1.2.3.k^3}+\text{etc.},$$

if we put $a+k=c$ or $a=c-k$, then

$$\frac{(c-k)v}{gk}=-s+\frac{cs}{1.k}-\frac{cs^2}{1.2.k^2}+\frac{cs^3}{1.2.3.k^3}-\text{etc.}=\frac{as}{1.k}-\frac{cs^2}{1.2.k^2}+\frac{cs^3}{1.2.3.k^3}-\text{etc.}$$

Whereby

$$v=\frac{gs}{a}\left(\frac{a}{1}-\frac{s}{1.2}-\frac{as}{1.2.k}+\frac{(a+k)s^2}{1.2.3.k^2}-\frac{(a+k)s^3}{1.2.3.4k^3}+\text{etc.}\right)$$

Corollary 2.

542. Therefore *in vacuo*, where k is infinite, then

$$v=gs-\frac{gs^2}{2a}=gx,$$

is taken as a constant. But if the resistance is only very small, and therefore k becomes very large, then

$$v=gs-\frac{gs^2}{2a}-\frac{gs^2}{2k}+\frac{gs^3}{6ak}.$$

Corollary 3.

543. *In vacuo* it is apparent that the speed of the body is zero at two points, where $s = 0$ and $s = 2a$, *i.e.* at the two cusps A et B. Now in a resisting medium, the one place is where $s = 0$; the other must be elicited from the equation :

$$a = \frac{cs}{1 \cdot 2k} - \frac{cs^2}{1 \cdot 2 \cdot 3k^2} + \frac{cs^3}{1 \cdot 2 \cdot 3 \cdot 4k^3} - \text{etc.},$$

hence it is found that :

$$s = \frac{2ak}{c} + \frac{4a^2k}{3c^2} + \frac{10a^3k}{9c^3} + \frac{136\,a^4k}{135c^4} + \text{etc.}$$

If indeed k is very large, then

$$s = \frac{2ak}{a+k} + \frac{4a^2k}{3(a+k)^2} = 2a - \frac{2a^2}{3k}$$

as an approximation.

Corollary 4. [p. 280]

544. This same series found on substituting $a + k$ in place of c is transformed into this :

$$s = 2a - \frac{2a^2}{3k} + \frac{4a^3}{9k^2} - \frac{44a^4}{135k^3} + \text{etc.}$$

from which the value of s gives the arc length ACN, by which the body is able to travel so far by its own motion.

Corollary 5.

545. The arc AO from A as far as to O, where the body has its maximum speed, is equal to :

$$k\,l\frac{a+k}{k} = a - \frac{a^2}{2k} + \frac{a^3}{3k^2} - \frac{a^4}{4k^3} + \frac{a^5}{5k^4} - \text{etc.}$$

Whereby the arc is given by [the following expressions have been corrected as in *O. O.*]:

$$ON = a - \frac{a^2}{6k} + \frac{a^3}{9k^2} - \text{etc.}$$

$$OC = \frac{a^2}{2k} - \frac{a^3}{3k^2} + \frac{a^4}{4k^3} - \text{etc.}$$

and

$$-AO + ON = \frac{a^2}{3k} - \frac{2a^3}{9k^2} + \text{etc.}$$

Corollary 6.

546. Now the speed at the point C is found corresponding to the height, is equal to

$$\frac{gk^2 - ge^{\frac{-a}{k}}(ak+kk)}{a} = g\left(\frac{a}{2} - \frac{a^2}{1\cdot 3k} + \frac{a^3}{1\cdot 2\cdot 4k^2} - \frac{a^4}{1\cdot 2\cdot 3\cdot 5k^3} + \frac{a^5}{1\cdot 2\cdot 3\cdot 4\cdot 6k^4} - \text{etc.}\right)$$

From which it is evident that the speed at C cannot vanish ; for the height corresponding to this is equal to :

$$\frac{gk^2}{e^{\frac{a}{k}}a}\left(e^{\frac{a}{k}}-1-\frac{a}{k}\right)$$

and $e^{\frac{a}{k}}$ is always greater than $1+\frac{a}{k}$; moreover the excess is greater than $\frac{a^2}{2k^2}$. Whereby the height corresponding to the speed at *C* is greater than $\frac{ga}{2e^{\frac{a}{k}}}$ and thus *ACN* is greater than *AC*.

Corollary 7. [p. 281]

547. The height corresponding to the maximum speed at *O* is equal to :

$$gk-\frac{gk^2}{a}l\frac{a+k}{k}=g(\frac{a}{2}-\frac{a^2}{3k}+\frac{a^3}{4k^2}-\frac{a^4}{5k^3}+\text{etc})$$

Whereby the excess of this height over the height corresponding to the speed at *C* is equal to

$$g(\frac{a^3}{1.2.4k^2}-\frac{5a^4}{1.2.3.5k^3}+\frac{23a^5}{1.2.3.4,6k^4}\text{ - etc})=ge^{\frac{-a}{k}}(\frac{a^3}{8k^2}-\frac{a^4}{24k^3}+\text{etc.})$$

This expression is certainly obtained if $kl\frac{a+k}{k}$ is substituted in place of *s* in the general value of *v*, and clearly the arc *AO* is equal to this quantity.

Scholium.

548. In the solution of this proposition it is required to be examined, since by determining only the formula for the descents we have also derived the ascent of the body on the curve *CN* ; from which it is possible that doubt arises, or that the true ascent is defined. But this is evidently easy to be derived from the formula for the ascent. Indeed we have been using this formula $dv=gdx-Rds,$ in which with the point *M* falling beyond the point *C* as *dx* has been made negative is changed into : $dv=-gdx-Rds,$ which actually contains the nature of the ascent. From these the continuity between the ascent and the descent is understood, which touch each other with no jump in between. [p. 282] Where indeed the curve itself begins to change direction and rise, where the like formula serving the descent is changed freely into the formula for the ascent. And this junction has a place in a medium with some kind of resistance, as is apparent from the general formulas, which only disagree with the sign of *dx* . On account of which it is not necessary from the given equation for some curve, that the body ascending or descending on one part of the curve or the other is sought, as either formula can be adapted to the equation and gives the true motion on the proposed curve. Only this has to be understood, that the abscissae are taken on the vertical axis, and that formula either for the ascent or of the descent is adhered to, which agrees with the initial motion.

PROPOSITION 63.

Problem.

549. *Let the given curve ACB* (Fig.66) *be a cycloid described on the horizontal base AB and considering the body to complete downwards oscillations on that curve in a medium with resistance in the square ratio of the speeds; to determine the motion of the oscillations.*

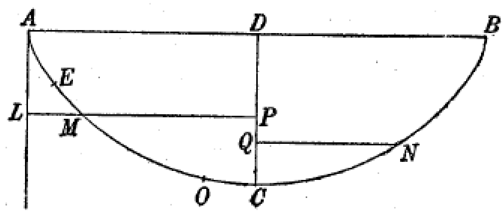

Fig. 66.

Solution.

The diameter of the circle is put as $CD = \frac{1}{2}a$ and on that is taken the abscissa $CP = x$ and the arc C*M* is called s ; then from the nature of the cycloid, it follows that

$$s = \sqrt{2ax},\ x = \frac{ss}{2a},\ \text{and } dx = \frac{sds}{a}.\ \text{[p. 283]}$$

Now the body descends on the arc *MC* and let the speed of this at *C* correspond to the height *b*; then the height that corresponds to the speed at *M* is equal to :

$$e^{\frac{s}{k}}b - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx.$$

(523). Now, it is the case that :

$$\int e^{\frac{-s}{k}}dx = \int \frac{e^{\frac{-s}{k}}sds}{a} = \frac{k^2 - k^2e^{\frac{-s}{k}} - ke^{\frac{-s}{k}}s}{a};$$

whereby the height corresponding to the speed at *M* is equal to :

$$\frac{e^{\frac{s}{k}}(ab - gk^2) + gk^2 + gks}{a} = e^{\frac{s}{k}}b - \frac{g}{a}\left(\frac{s^2}{2} + \frac{s^3}{2\cdot 3k} + \frac{s^4}{2\cdot 3\cdot 4k^2} + \frac{s^5}{2\cdot 3\cdot 4\cdot 5k^3} + \text{etc.}\right).$$

Therefore the arc is obtained, in which the whole descent is contained, if the value of *s* is sought from this equation :

$$e^{\frac{s}{k}} = \frac{gk^2 + gks}{gk^2 - ab}.$$

Moreover, this can be made into the series [corrected in the *O. O.* from the original] :

$$s = A + \frac{A^2}{3k} + \frac{11A^3}{72k^2} + \frac{43A^4}{540k^3} + \text{etc.}$$

hence on putting A in place of $\frac{\sqrt{2ab}}{\sqrt{g}}$ for brevity. The arc CM is the equal of this series , if indeed the body starts to descend from the point M. The body has the maximum speed at O on taking $CO = s$ from this equation

$$e^{\frac{s}{k}} = \frac{gk^2}{gk^2 - ab} \text{ or } CO = kl\frac{gk^2}{gk^2 - ab}$$

and the height corresponding to this maximum speed is equal to :

$$\frac{gk \cdot CO}{a} = \frac{gk^2}{a} l \frac{gk^2}{gk^2 - ab} = b + \frac{ab^2}{2gk^2} + \frac{a^2b^3}{3g^2k^4} + \text{etc.}$$

Towards determining the time, it is agreed to consider the maximum speed at the point O and to define the time along MO. On this account I put the height corresponding to the speed at O equal to c and the arc $MO = q$; then we have : [p. 284]

$$CO = \frac{ac}{gk} \text{ and } ab = gk^2\left(1 - e^{\frac{-ac}{gk^2}}\right),\ s = \frac{ac}{gk} + q.$$

With these put in place, the height corresponding to the speed at M, or v, is equal to:

$$\frac{gk^2 + ac + gkq - e^{\frac{q}{k}} gk^2}{a}.$$

Now because v in less than c, put $c - v = z$ and the equation becomes :

$$az + gk^2 + gkq = e^{\frac{q}{k}} gk^2$$

and in the series :

$$\frac{az}{g} = \frac{q^2}{2} + \frac{q^3}{6k} + \frac{q^4}{24k^2} + \frac{q^5}{120k^3} + \text{etc.}$$

From which on being converted, it becomes

$$q = \frac{\sqrt{2az}}{\sqrt{g}} - \frac{az}{3gk} + \frac{az\sqrt{2az}}{18gk^2\sqrt{g}} - \frac{2a^2z^2}{135g^2k^3} + \frac{a^2z^2\sqrt{2az}}{1080g^2k^4\sqrt{g}} - \text{etc.}$$

The descent starts from the point M; there $v = 0$ and $z = c$ and thus :

$$OM = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} - \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} - \text{etc.}$$

From the same formula, if q is made negative, then the motion along OCN is obtained; but since in the same way, if q or k is made negative, if N is the point of maximum ascent of the body, the arc ON is equal to :

$$\frac{\sqrt{2ac}}{\sqrt{g}}+\frac{ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}+\frac{2a^2c^2}{135g^2k^3}+\frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}}+\text{etc.}$$

Now the time for the motion along MO is found in this manner: since

$$ds=dq=\frac{adz}{\sqrt{2gaz}}-\frac{adz}{3gk}+\frac{adz\sqrt{2az}}{12gk^2\sqrt{g}}-\frac{4a^2zdz}{135g^2k^3}+\frac{a^2zdz\sqrt{2az}}{432g^2k^4\sqrt{g}}-\text{etc.},$$

this divided by $\sqrt{v}=\sqrt{(c-z)}$ gives an element of the time equal to

$$\frac{dz\sqrt{a}}{\sqrt{2g(cz-z^2)}}-\frac{adz}{3gk\sqrt{(c-z)}}+\frac{azdz\sqrt{2a}}{12gk^2\sqrt{g(cz-zz)}}-\frac{4a^2zdz}{135g^2k^3\sqrt{(c-z)}}$$
$$+\frac{a^2z^2dz\sqrt{2a}}{432g^2k^4\sqrt{g(cz-z^2)}}-\text{etc.}$$

Which thus must be integrated, so that it vanishes on putting $v = c$ or $z = 0$; then if we put $z = c$, the time is obtained in which the body descends along the arc *MO*. Therefore this time, with the ratio of the periphery of the circle to the diameter put as π to 1, is equal to [p. 285]

$$\frac{\pi\sqrt{a}}{\sqrt{2g}}-\frac{2a\sqrt{c}}{3gk}+\frac{\pi ac\sqrt{a}}{12gk^2\sqrt{2g}}-\frac{16a^2c\sqrt{c}}{405g^2k^3}+\text{etc.}$$

Therefore on putting *k* negative, the time in which the body ascends from *O* as far as *N*, is equal to :

$$\frac{\pi\sqrt{a}}{\sqrt{2g}}+\frac{2a\sqrt{c}}{3gk}+\frac{\pi ac\sqrt{a}}{12gk^2\sqrt{2g}}+\frac{16a^2c\sqrt{c}}{405g^2k^3}+\text{etc.}$$

Therefore the time to travel along *MCN* or the time of one half oscillation, is equal to :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}}+\text{etc.}$$

Q.E.I.

Corollary 1.

550. Therefore if the maximum speed of the descending body corresponds to the height *c*, since

$$CO=\frac{ac}{gk}$$

then

$$MC=\frac{\sqrt{2ac}}{\sqrt{g}}+\frac{2ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}-\frac{2a^2c^2}{135g^2k^3}+\frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}}-\text{etc.}$$

Now the total ascent *CN* is equal to :

$$ON - CO = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} + \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} + \text{etc.}$$

Hence

$$CM - CN = \frac{4ac}{3gk} - \frac{4a^2c^2}{135g^2k^3} - \text{etc.}$$

and

$$MCN = \frac{2\sqrt{2ac}}{\sqrt{g}} + \frac{ac\sqrt{2ac}}{9gk^2\sqrt{g}} + \frac{a^2c^2\sqrt{2ac}}{540g^2k^4\sqrt{g}} + \text{etc.}$$

Corollary 2.

551. If the whole arc of the descent *MC* is put equal to *E* and the following arc of the ascent *CN* is equal to *F* and the height corresponding to the height at *C* = *b*, then

$$\frac{ab}{g} = k^2 - e^{\frac{-E}{k}}(k^2 + kE) = \frac{E^2}{2} - \frac{E^3}{3k} + \frac{E^4}{8k^2} - \frac{E^5}{30k^3} + \frac{E^6}{144k^4} - \text{etc.}$$

And with *k* made negative it is found in the same way:

$$\frac{ab}{g} = \frac{F^2}{2} + \frac{F^3}{3k} + \frac{F^4}{8k^2} + \frac{F^5}{30k^3} + \frac{F^6}{144k^4} + \text{etc.}$$

From which there becomes :

$$F = E - \frac{2E^2}{3k} + \frac{4E^3}{9k^2} - \frac{44E^4}{135k^3} + \frac{104E^5}{405k^4} - \text{etc.}$$

and the height corresponding to the maximum speed is given by : [p. 286]

$$c = \frac{gE^2}{2a} - \frac{gE^3}{3ak} + \frac{gE^4}{4ak^2} - \text{etc.}$$

Corollary 3.

552. Since *F* is the arc of the ascent in the first half oscillation, the arc *F* is likewise the arc of the descent in the following oscillation ; therefore as it can be joined with the ascent arc,

$$G = E - \frac{4E^2}{3k} + \frac{16E^3}{9k^2} - \frac{328E^4}{135k^3} + \frac{1376E^5}{405k^4} - \text{etc.}$$

And in a like manner the following oscillations can be defined, however many it might please to consider.

Corollary 4.

553. It is apparent from the equation setting out the time, that the time in which the body arrives at *O* from *M*, is always less than the time in which the body reaches as far as *N* from *O*. In a similar manner also the arc *ON* is greater than the arc *OM*, and now the arc *CN* is less than the arc *MC*.

Corollary 5.

554. If the oscillations should become infinitely small or *c* becomes a vanishing quantity, then the oscillations agree with the oscillations made *in vacuo* ; for in the individual expressions the same terms vanish, which vanish on putting $k = \propto$. Therefore isochronous with the smallest oscillations of a pendulum of length *a in vacuo* acted on by a force *g* , or of a pendulum according to the hypothesis of gravity equal to 1, the length of which is equal to $\frac{a}{g}$.

[This agrees with modern analysis, and reinforces the belief considered earlier, that Euler is using a set of units in which the second and the acceleration of gravity are taken as one.]

Corollary 6. [p. 287]

554. But if the oscillations become greater, then the times of the oscillations also become greater; whereby according to the hypothesis of resistance the cycloid is not favoured with the property of tautochronism. For when the maximum speed is made greater in some oscillation, the greater also is the excess of the time of the oscillation of this kind over the time of the smallest oscillation.

Scholium 1.

556. Since we have said that the smallest oscillations agree with oscillations *in vacuo*, then the situation arises that *a* and *k* are quantities of finite size. For if *a* should be infinitely large or *k* infinitely small, the following terms express the time :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}} + \text{etc.}$$

that do not vanish, even if *c* should be indefinitely small. Moreover, therefore, only the smallest oscillations upon some curve agree with the oscillations *in vacuo* and in a resisting medium, when neither the radius of osculation of the curve at the lowest point is infinitely large nor the resistance infinitely great.

Example.

557. We explain the case by means of an example, in which the resistance is so small and thus the quantity *k* so large that the fractions in the denominators of which *k* has more than two dimensions can be put equal to zero without harm. Therefore with the aforesaid height *c* corresponding to the maximum speed at *O*, thus so that $CO = \frac{ac}{gk}$, then the arc of the descent is given by : [p. 288]

$$MC = E = \frac{\sqrt{2ac}}{\sqrt{g}} + \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}$$

and the following arc of the ascent is given by :

$$CN = F = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

From these it is found

$$\sqrt{c} = \frac{E\sqrt{g}}{\sqrt{2a}} - \frac{E^2\sqrt{g}}{3k\sqrt{2a}} + \frac{7E^3\sqrt{g}}{36k^2\sqrt{2a}}$$

or

$$c = \frac{gE^2}{2a} - \frac{gE^3}{3ak} + \frac{gE^4}{4ak^2}$$

and

$$F = E - \frac{2E^2}{3k} + \frac{4E^3}{9k^2}.$$

Therefore the time of half an oscillation along *MCN* is equal to :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}.$$

In the following half oscillation the arc of descent is equal to :

$$F = E - \frac{2E^2}{3k} + \frac{4E^3}{9k^2},$$

and the arc of ascent that follows is equal to :

$$G = E - \frac{4E^2}{3k} + \frac{16E^3}{9k^2};$$

and the time of this half oscillation is equal to :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}} - \frac{\pi E^3\sqrt{2a}}{18k^3\sqrt{g}},$$

where the final term can be neglected on account of k^3 in the denominator. In the third half oscillation, the descent arc is equal to :

$$G = E - \frac{4E^2}{3k} + \frac{16E^3}{9k^2}$$

and the ascending arc is equal to :

$$H = E - \frac{6E^2}{3k} + \frac{36E^3}{9k^2}.$$

And generally in that half oscillation, which in indicated by the number *n*, the arc of descent is equal to : =

$$E - \frac{2(n-1)E^2}{3k} + \frac{4(n-1)^2E^3}{9k^2}$$

and the arc of ascent is equal to :

$$E - \frac{2nE^2}{3k} + \frac{4n^2E^3}{9k^2}.$$

On account of which after *n* half oscillations the body is at a distance from the lowest point *C* by the arc :

$$E - \frac{2nE^2}{3k} + \frac{4n^2E^3}{9k^2},$$

which is less than the first arc of descent by the amount :

$$\frac{2nE^2}{3k} - \frac{4n^2E^3}{9k^2}.$$

Moreover the time of the half oscillation indicated by the given number *n* is :

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}} - \frac{\pi(n-1)E^3\sqrt{2a}}{18k^3\sqrt{g}}.$$

But if the whole arc of the first half oscillation *MCN* is called *A*, then

$$A = 2E - \frac{2E^2}{3k} + \frac{4E^3}{9k^2} \text{ and } E = \frac{A}{2} + \frac{A^2}{12k}$$

with the following term vanishing spontaneously. Hence the whole arc described by the oscillation indicated by the number *n* is equal to

$$2E - \frac{2(2n-1)E^2}{3k} + \frac{4(2n^2-2n+1)E^3}{9k^2} = A - \frac{(n-1)A^2}{3k} + \frac{(n-1)^2A^3}{9k^2}.$$

Corollary 7. [p. 289]

558. If n half oscillations are made, the descending arc of the first oscillation is E, and the final ascending arc is equal to L, then

$$L = E - \frac{2nE^2}{3k} + \frac{4n^2E^3}{9k^2},$$

which expression, if taken with the nearby series, almost agrees with the geometric progression of the same starting value, [and ratio $\frac{2nE}{3k}$]and on this account it follows that

$$L = \frac{3Ek}{3k + 2nE} \text{ or } 3k(E - L) = 2nEL.$$

Corollary 8.

559. Hence, for some number of oscillations performed, if the first descending arc of the first oscillation E is given together with the final ascending arc L, it is possible to find the number of oscillations; for it is in fact

$$n = \frac{3k(E-L)}{2EL}.$$

Corollary 9.

560. Hence it is clear that the diminution of the arcs does not depend on the length of the pendulum, but from n and E given the same arc L is found, whatever the length a of the pendulum should be. and n is always proportional to $\frac{1}{L} - \frac{1}{E}$.

Scholium 2. [p. 290]

561. Newton examines experiments of this kind concerning oscillations in a resisting medium in *Phil*. Book. II, where he observes the first descending arc, the final descending arc, and the number of oscillations for [a pendulum] in air, as in water and mercury. Whereby if the mediums resisted perfectly in the ratio of the square of the speed, there should be agreement with these formulas, thus in order that the decrease of the arc should be proportional to the number of oscillations, and both the first and the final arcs taken together. Since also the situation arises for larger oscillations to be observed, in which the speed is not extremely small. But in the smallest oscillations the greatest aberration from this rule is observed. From which it is gathered, when the speed of the body in the fluid is made greater, the resistance to that falls closer to the ratio of the square of the speeds, but the slowest motion is liable to be affected by other resistances, which vanishes in motions for speeds with the previous resistance, which is proportional to the square of the speeds. Also in these experiments, Newton assumed that the resistance was in part simply proportional to the ratio of the speed, partly as the speed to the three on two power, and partly as the square of the speeds, yet this was not satisfactory for the slowest motions. Now in the final *Phil*. edition Newton recognises the insufficiency of his own initial theory and by many other reasons he shows that another resistance of the fluid to be constant, or proportional to the lengths of time, as before he had considered for the proportionality with the speeds. Hence on account of this, in the following proposition we will consider joining that resistance with this, which is proportional to the square of the speed, since the resolution of the equation and the determination of the speed with this addition does not present any more special difficulty. [The interested reader should consult Cohen's translation of the third ed. of the *Principia*, where these matters of Newton are discussed at some length. Cohen's work is remarkable for the complete lack of comment on the follow-up to Newton that we are providing here. One wonders why Newton has been 'done to death' as it were by numerous commentaries, while this wonderful work of Euler has been hidden in the closet, as it were, for centuries.........]

Corollary 10. [p. 291]

562. Since according to the times of the oscillations and the semi oscillations [the pendulum] attains, it is evident that this decreases when the arc described becomes smaller, and if the arcs completely vanish, then the time of half an oscillation becomes equal to $\frac{\pi\sqrt{2a}}{\sqrt{g}}$.

Corollary 11.

563. Moreover the excess of the time of this half oscillation over the smallest half oscillation in the case of the smallest resistance is $\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}$ with E denoting the descent of this half oscillation. Whereby the excess itself is in proportion to the square of the arc descended or also to the square of the whole arc of the half oscillation described.

Scholium 3.

564. Therefore the cycloid, that has been suitably described by Huygens according to the production of isochronous pendulums, loses this property on account of the resistance in proportional to the square of the speeds, and hence is not serviceable in air, unless the oscillations are very small or are almost equal to each other. [p. 292] Now from this, since larger oscillations endure a long time, it is possible to gather that the true tautochrone curve according to this hypothesis of the resistance to be more curved than the cycloid. As clearly the cycloid is continued in a circle of the same radius, of which the cycloid is the lowest point, thus also the true tautochrone will be continued in a cycloid and with the curvature decreased more from the lowest point than with the curvature of the cycloid.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 260]

PROPOSITIO 58.

Problema.

505. *In hypothesi gravitatis uniformis g et medio quocunque resistente uniformi determinare motum corporis data cum celeritate initiali ex A* (Fig.62) *ascendis super linea recta AB utcunque inclinata ad horizontem.*

Solutio.

Ducta horizontali *AC* et ex *M* ad eam perpendiculari *MP* vocetur $PM = x$, sitque $AM = nx$. Sit altitudo debita celeritati initiali in $A = b$ et altitudino debita celeritati in $M = v$; resistentia vero in *M* sit = $\frac{V}{K}$. His positus erit

$$dv = -gdx - \frac{nVdx}{K}$$

(479), unde habetur

$$dx = \frac{-Kdv}{gK+nV} \text{ atque } x = \int \frac{-Kdv}{gK+nV}$$

hoc integrali ita accepto, ut evanescat posito $v = b$. Si deinde ponatur $v = 0$, prodibit $x = BC$, ubi in puncto *B* corpus omnem celeritatem amittit. Tempus vero, quo corpus per *AM* ascendit, est =

$$\int \frac{-Kdv}{(gK+nV)\sqrt{v}}$$ [p. 261]

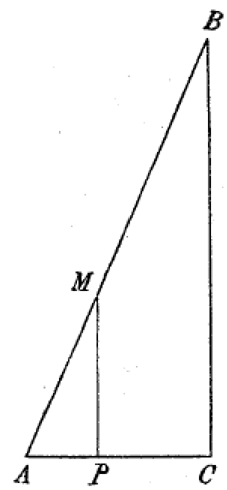

Fig. 62.

hoc integrali quoque ita accepto, ut evanescat posito $v = b$; in quo si porro ponatur $v = 0$, prodibit tempus totius ascensus per *AMB*. Pressio autem, quam linea *AMB* sustinet, ubique est constans et aequalis vi normali =

$$\frac{g\sqrt{(n^2-1)}}{n}$$

Q.E.I.

Corollarium 1.

506. Si linea *AMB* fit horizontalis, evanescente angulo *BAC* fiet $n = \infty$. Posito igitur *AM* = *z* = *nz* erit

$$z = \int \frac{-Kdv}{V}$$

et tempus, quo per *AM* progreditur, erit =

$$\int \frac{-Kdv}{V\sqrt{v}}.$$

Corollarium 2.

507. Si resistentia fuerit ut potestas exponentis 2*m* celeritatum, erit $V = v^m$ et $K = k^m$. Hoc ergo casu erit

$$x = \int \frac{-k^m dv}{gk^m + nv^m}$$

atque tempus per *AM* =

$$\int \frac{-k^m dv}{(gk^m + nv^m)\sqrt{v}}.$$

Corollarium 3.

508. Utraque haec expressio in seriem conversa dat

$$x = \frac{b-v}{g} - \frac{n(b^{m+1} - v^{m+1})}{(m+1)g^2k^m} + \frac{n^2(b^{2m+1} - v^{2m+1})}{(2m+1)g^3k^{2m}} - \text{etc.}$$

atque tempus per *AM* =

$$\frac{2n(\sqrt{b} - \sqrt{v})}{g} - \frac{2n^2(b^{m+\frac{1}{2}} - v^{m+\frac{1}{2}})}{(2m+1)g^2k^m} + \frac{2n^3(b^{2m+\frac{1}{2}} - v^{2m+\frac{1}{2}})}{(4m+1)g^3k^{2m}} - \text{etc.}$$

[p. 262] Quamobrem posito $v = 0$ erit

$$BC = \frac{b}{g} - \frac{nb^{m+1}}{(m+1)g^2k^m} + \frac{n^2b^{2m+1}}{(2m+1)g^3k^{2m}} - \text{etc.}$$

atque tempus totius ascensus per *AB* =

$$\frac{2n\sqrt{b}}{g} - \frac{2n^2b^{m+\frac{1}{2}}}{(2m+1)g^2k^m} + \frac{2n^3b^{2m+\frac{1}{2}}}{(4m+1)g^3k^{2m}} - \text{etc.}$$

Exemplum 1.

509. Sit resistentia celeritatibus proportionalis; erit $m=\frac{1}{2}$ atque

$$x=\int\frac{-dv\sqrt{k}}{g\sqrt{k}+n\sqrt{v}}=\frac{2\sqrt{bk}}{n}-\frac{2\sqrt{kv}}{n}+\frac{2gk}{n^2}l\frac{g\sqrt{k}+n\sqrt{v}}{g\sqrt{k}+n\sqrt{b}}.$$

Hinc erit tota altitudo *BC*, ad quam corpus pertingere valet,

$$=\frac{2\sqrt{bk}}{n}-\frac{2gk}{n^2}l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}}.$$

Tempus vero, quo per *AM* ascendit, est =

$$\int\frac{-ndv\sqrt{k}}{(g\sqrt{k}+n\sqrt{v})\sqrt{v}}=2\sqrt{k}\,l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}+n\sqrt{v}}.$$

Quare tempus totius ascensus per *AMB* erit =

$$2\sqrt{k}\,l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}}.$$

Si igitur corpus super linea inclinata *AC* (Fig. 63) descenderit et celeritate in *C* acquisita ascendat in *CB* usque ad *B* sitque $AC=N.AD$ et $BC=n.BE$ atque celeritas in *C* debita altitudini *b*, erit (486)

$$AD=-\frac{2\sqrt{bk}}{N}+\frac{2gk}{N^2}l\frac{g\sqrt{k}}{g\sqrt{k}-N\sqrt{b}},$$

$$AC=-2\sqrt{bk}+\frac{2gk}{N}l\frac{g\sqrt{k}}{g\sqrt{k}-N\sqrt{b}},$$

$$BE=\frac{2\sqrt{bk}}{n}-\frac{2gk}{n^2}l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}},$$

$$CB=2\sqrt{bk}-\frac{2gk}{n}l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}}.$$

Atque tempus descensus per *AC* =

$$2\sqrt{k}\,l\frac{g\sqrt{k}}{g\sqrt{k}-N\sqrt{b}}$$

(cit.) et tempus ascensus per *CB* =

$$2\sqrt{k}\,l\frac{g\sqrt{k}+n\sqrt{b}}{g\sqrt{k}}.$$

[p. 263] Unde descensus et ascensus super lineis rectis inter se comparari possunt.

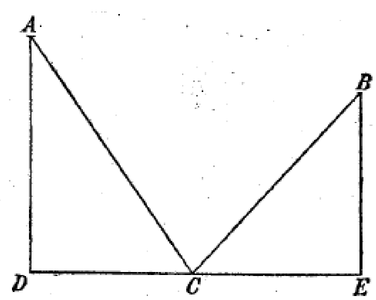

Fig. 63.

Corollarium 4.

510. Si hi logarithmi per series exprimantur, patet fieri non posse, ut sit $BE=AD$; est enim in quacunque resistentiae hypothesi, ut ex seriebus (508 et 488) intelligitur, $BE<\frac{b}{g}$ et $AD>\frac{b}{g}$. Fieri autem potest, ut sit $AC=BC$.

Corollarium 5.

511. Effici autem facile potest, ut tempus descensus per *AC* aequale sit tempori ascensus per *CB*. Fieri solicet debet

$$ng\sqrt{k} = Ng\sqrt{k} + Nn\sqrt{b} \quad \text{seu} \quad n = \frac{Ng\sqrt{k}}{g\sqrt{k} - N\sqrt{b}}.$$

Est igitur $n > N$ seu ang. *BCE* < ang. *ACD*. Relatio autem inter *N* et *n* pendet a celeritate in puncto *C*.

Corollarium 6.

512. Si autem angulus *BCE* aequalis fuerit angulo *ACD* seu $N = n$, tempus ascensus per *BC* minus erit tempore descensus per *AC*. Atque hoc generaliter locum habet in quacunque resistentiae hypothesi; est enim tempus descensus per $CB < \frac{2n\sqrt{b}}{g}$, ut ex seriebus supra datis ((488) et (508)) apparet. [p. 264]

Exemplum 2.

513. Resistat medium in duplicata ratione celeritatum; erit $m = 1$. Quare habebitur

$$x = \int \frac{-kdv}{gk + nv} = \frac{k}{n} l \frac{gk+nb}{gk+nv}$$

atque (Fig. 62)

$$BC = \frac{k}{n} l \frac{gk+nb}{gk} \text{ et } AB = kl \frac{gk+nb}{gk}.$$

Tempus vero ascensus per *AM* erit =

$$\int \frac{-nkdv}{(gk+nv)\sqrt{v}} = \frac{2\sqrt{kn}}{\sqrt{g}} \left(A.\ \text{tang.} \sqrt{\frac{nb}{gk}} - A.\ \text{tang.} \sqrt{\frac{nv}{gk}} \right)$$

existente radio = 1 et *A* denotante arcum circuli. Erit ergo tempus ascensus per *AB* =

$$\frac{2\sqrt{kn}}{\sqrt{g}} A.\ \text{tang.} \sqrt{\frac{nb}{gk}}.$$

Si nunc corpus super linea inclinata *AC* (Fig. 63) descenderit et celeritate in C acquisita, quae debita sit altitudini *b*, rursus ascendat per *CB* fueritque $AC = N.AD$ et $BC = n.BE$, erit

$$AD = \frac{k}{N} l \frac{gk}{gk - Nb} \quad \text{et} \quad AC = kl \frac{gk}{gk - Nb}$$

atque tempus descensus per *AC* =

$$\frac{\sqrt{Nk}}{\sqrt{g}} l \frac{\sqrt{gk} + \sqrt{Nb}}{\sqrt{gk} - \sqrt{Nb}}$$

(407). Porro vero erit

$$BE = \frac{k}{n} l \frac{gk+nb}{gk}, \quad BC = kl \frac{gk+nb}{gk}$$

et tempus ascensus per *CB* =

$$\frac{2\sqrt{nk}}{\sqrt{g}} A. \text{tang.} \sqrt{\frac{nb}{gk}}.$$

Corollarium 7.

514. In hac resistentiae hypothesi commode effici potest, ut sit $AC = BC$; debebit enim esse

$$ngk = Ngk + Nnb \text{ seu } n = \frac{Ngk}{gk - Nb}.$$

Est igitur $n > N$ hincque ang. $BCE <$ ang. ACD.

Exemplum 3.

515. Sit resistentia quam minima et proportionalis potestati $2m$ celeritatum; erit k quantitas vehementer magna. [p. 265] Si ergo celeritas in C fuerit debita altitudini b et $AC = N.AD$ atque $BC = n.BE$ corpusque super AC descendat et super CB ascendat, erit

$$AD = \frac{b}{g} + \frac{Nb^{m+1}}{(m+1)g^2k^m}$$

et tempus descensus per AC =

$$\frac{2N\sqrt{b}}{g} + \frac{2N^2b^m\sqrt{b}}{(2m+1)g^2k^m}$$

(488). Pro ascensu vero erit

$$BE = \frac{b}{g} - \frac{nb^{m+1}}{(m+1)g^2k^m}$$

et tempus per CB =

$$\frac{2n\sqrt{b}}{g} - \frac{2n^2b^m\sqrt{b}}{(2m+1)g^2k^m}$$

(508). Si igitur effici debeat, ut sit $AC = BC$, oportet esse

$$N + \frac{N^2b^m}{(m+1)gk^m} = n - \frac{n^2b^m}{(m+1)gk^m},$$

unde fit

$$n = N + \frac{2N^2b^m}{(m+1)gk^m}$$

propter quantitatem k valde magnam. At quo tempus descensus per AC aequale sit tempori ascensus per CB, debet esse

$$N + \frac{N^2b^m}{(2m+1)gk^m} = n - \frac{n^2b^m}{(2m+1)gk^m}$$

seu

$$n = N + \frac{2N^2 b^m}{(2m+1)gk^m}.$$

Scholion 1. [p. 266]

516. In casu huius exempli, quo resistentia est valde parva, curva *AMD* (Fig. 64) potest determinari huius proprietatis, ut corpus ex *C* celeritate altitudini b debita ascendendo super quavis recta CM ad curvam AMD pertingat. Posita enim $CM = z$ et $MP = x$ erit $n = \frac{z}{x}$ atque

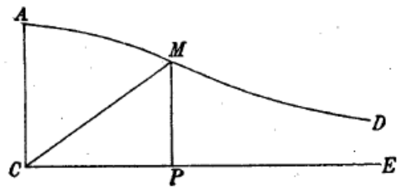
Fig. 64.

$$x = \frac{b}{g} - \frac{nb^{m+1}}{(m+1)g^2k^m}.$$

Quare habebitur ista aequatio

$$b^{m+1}z = (m+1)gbk^m x - (m+1)g^2k^m x^2.$$

Sit $CP = y$ et loco

$$\frac{b^{m+1}}{(m+1)gk^m}$$

scribatur *f*; orietur

$$f\sqrt{(x^2+y^2)} = bx - gx^2$$

seu

$$ffyy = (bb-ff)xx - 2gbx^3 + g^2x^4.$$

Si ponatur $y = 0$, erit et $x = 0$ et $x = \frac{b-f}{g} = CA.$ Curva ergo etiam per punctum C transit, quae autem eius pars quaestioni satisfacere cessat propter sequentes terminos neglectos, qui perperam negliguntur, si n seu $\frac{z}{x}$ fit quoque valde magnum. Aequatio vero dat curvam ellipsoformem maxime oblongam circa axem minorem AC descriptam. Vera autem curva habet formam *AMD*, cuius asymtotos est horizontalis *CE*, si quidem est $m > 1$, eiusque aequatio habetur omnibus sumendis terminis, quae erit

$$xdz + (m-1)zdx = \frac{bk^m(zdx - xdz)}{gk^m x + b^m z}.$$

Si $m < 1$, curva non in infinitum progredietur, sed incidet in *CE* sumendo

$$CE = \frac{bk^m}{(1-m)b^m}.$$

Nam si $m < 1$, corpus horizontaliter non in infinitum progredi potest, sed in distantia finita omnem amittit celeritatem.

Scholion 2. [p. 267]

517. Si medium resistens non sit uniforme, tum linea non esset recta, super qua motus facillime determinari posset; idem quoque est notandum, si potentia sollicitans non fuerit uniformis. Ut sit potentia = *P* et resistentia = $\frac{V}{Q}$, ubi Q talis est functio exponentis resistentiae variabilis *q*, qualis *V* est ipsius *v*; his positis motus corporis super quacunque curva exprimitur hac aequatione

$$dv = \pm Pdx \pm \frac{Vds}{Q}.$$

Eius ergo curvae, super qua motus facillime definitur, haec erit aequatio

$$PQdx = Ads,$$

ex qua oritur sequens

$$\frac{Adv}{\pm A \pm V} = Pdx$$

motum super hac curva determinans, in qua indeterminatae sunt a se invicem separatae. Quare si etiam huiusmodi hypotheses persequi vellemus, loco linearum rectarum huiusmodi curvas hac aequatione expressas $PQdx = Ads$ assumere deberemus. Sed quia statuimus hypothesin potentiae sollicitantis et resistentiae uniformis tantum fusius pertractare, missis his ad eos casus progredimur, in quibus *v* unicam tantum habet dimensionem, id quod evenit, si resistentia proportionalis fuerit quadratis celeritatum

PROPOSITIO 59.

Problema.

518. *In hypothesi gravitatis uniformis g resistentiae quadratis celeritatum proportionalis descendat corpus super curva quacunque AMB* (Fig.57); *determinare eius motum et pressionem, quam curvam in singulis punctis sustinet.* [p. 268]

Solutio.

In axe verticali sumatur abscissa $AP = x$ et ponatur arcus $AM = s$, celeritas in M debita altutudini v et exponens resistentiae k ; erit resistentia = $\frac{v}{k}$.

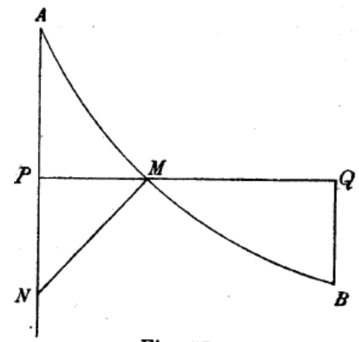

Fig. 57.

Quamobrem ista habebitur aequatio motum corporis exponens

$$dv = gdx - \frac{vds}{k}$$

(465). Ad hanc integrandam multiplico per $e^{\frac{s}{k}}$ eritque integralis

$$e^{\frac{s}{k}}v = \int e^{\frac{s}{k}}gdx$$

Ita autem sumi debet hoc integrale, ut posito $s = 0$ abeat *v* in altitudinem celeritati initiali in *A* debitam. Si igitur descensus ex quiete fieri ponatur $e^{\frac{s}{k}}gdx$ ita debet integrari, ut evanescat posito $s = 0$. Hoc itaque facto erit

$$v = ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx.$$

Unde tempus per *AM* erit =

$$\int \frac{e^{\frac{s}{2k}} dx}{\sqrt{g \int e^{\frac{s}{k}} dx}}.$$

Posito nunc *PM* = *y* sumtoque *dx* constante erit pressio, quam curva in *M* secundum normalem *MN* sustinet, =

$$\frac{g\,dy}{ds} + \frac{2ge^{\frac{-s}{k}} dx ddy \int e^{\frac{s}{k}} dx}{ds^3}.$$

Q.E.I.

Corollarium 1. [p. 269]

519. Pressio, quam curva sustinet, in hanc formam potest transmutari

$$\frac{g\,ds}{e^{\frac{s}{k}} dy dx} d. \frac{dy^2}{ds^2} \int e^{\frac{s}{k}} dx,$$

quae, postquam pro data curva integratum est $e^{\frac{s}{k}} dx$, commodius ad quosvis casus accommodatur.

Corollarium 2.

520. Corpus in descensu maximam habet celeritatem, ubi est $v = \frac{gkdx}{ds}$. Hoc vero evenit, ubi est

$$e^{\frac{s}{k}} kdx = ds \int e^{\frac{s}{k}} dx$$

seu ubi

$$d.l \int e^{\frac{s}{k}} dx = \frac{ds}{k};$$

in quo puncto patet tangentem non esse horizontalem.

Corollarium 3.

521. Si fuerit

$$\int e^{\frac{s}{k}} dx = \frac{e^{\frac{s}{k}} s^n}{a^{n-1}},$$

erit

$$dx = \frac{ns^{n-1} ds}{a^{n-1}} + \frac{s^n ds}{a^{n-1} k}$$

atque

$$x = \frac{s^n}{a^{n-1}} + \frac{s^{n+1}}{(n+1) a^{n-1} k}.$$

Quare si haec aequatio exprimat curvae quaesitae naturam, erit

$$v = \frac{gs^n}{a^{n-1}}$$

et tempus per *AM* =

$$\frac{2a^{\frac{n-1}{2}}s^{\frac{2-n}{2}}}{(2-n)\sqrt{g}},$$

[p. 270] si quidem n fuerit minor binario; nam si $n = 2$ vel $n > 2$, curva in A tangentem horizontalem habebit atque corpus ibi perpetuo permanebit.

Corollarium 4.

522. Simili modo etiam perspicitur, si x fuerit potestas quaecunque ipsius s vel huiusmodi potestatum aggregatum, semper integrari posse $e^{\frac{s}{k}}dx$ atque ideo celeritatem terminus finitis exhiberi.

Corollarium 5.

523. Si autem abscissae in axe verticali BQ sumantur et celeritas, quam corpus in B habebit, debita sit altitudini b praetereaque vocetur $BQ = x$ et $BM = s$, erit

$$dv = -gdx - \frac{vds}{k}$$

cuius integralis est

$$e^{-\frac{s}{k}}v = b - g\int e^{-\frac{s}{k}}dx$$

integrali scilicet $\int e^{-\frac{s}{k}}dx$ ita accepto, ut evanescat posito $x = 0$. Hanc ob rem erit

$$v = be^{\frac{s}{k}} - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx$$

et tempus, quo in descensu arcus MB absolvitur, =

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b-g\int e^{-\frac{s}{k}}dx)}}.$$

Corollarium 6. [p. 271]

524. Si igitur detur celeritas in puncto B, nempe $\sqrt{b}$, inveniri potest in curva BMA punctum A, ex quo corpus descendere incepit ubique celeritatem habuit = 0. Quari debet scilicet locus, ubi est

$$\int e^{-\frac{s}{k}}dx = \frac{b}{g}.$$

Atque etiam expressio temporis

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{(b-g\int e^{-\frac{s}{k}}dx)}}$$

dabit tempus totius descensus per AMB, si post integrationem ponatur

$$\int e^{-\frac{s}{k}}dx = \frac{b}{g}.$$

Scholion .

525. Duplicem hic motum investigandi modum ideo attulimus, ut tam ad descensus ex dato puncto facto quam ad descensus usque ad datum punctum, ut in motu oscillatorio fieri solet, accommodari possit.

PROPOSITIO 60.

Problema.

526. *Existente potentia sollicitante uniformi et medio uniformi resistente in duplicata ratione celeritatum, determinare motum corporis ascendentis super data curva AMD* (Fig.58) *et pressionem, quam curvam sustinet in singulis punctis M.*

Solutio.

In linea verticali *AP* ponatur abscissa *AP* = *x*, the arc *AM* = *s*, the speed at *A* corresponds to the height *b* and the speed at *M* to the height *v*. Let the force acting downwards be equal to *g* and let the resistance be equal to $\frac{v}{k}$. With these in place, then

$$dv = -gdx - \frac{vds}{k}$$

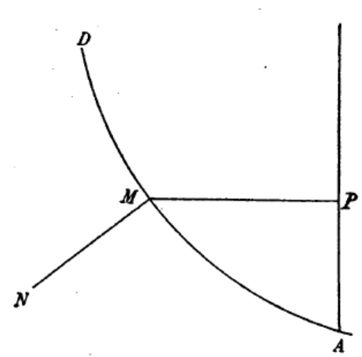

Fig. 58.

(475), which on being multiplies by $e^{\frac{s}{k}}$, the integral gives

$$e^{\frac{s}{k}}v = b - g\int e^{\frac{s}{k}}dx \text{ [p. 272]}$$

Thus on taking $\int e^{\frac{s}{k}}dx$ so that it vanishes on putting $x = 0$. On this account, this becomes:

$$v = be^{-\frac{s}{k}} - ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx$$

Hence the time of the ascent along the arc *AM* is equal to :

$$\int \frac{e^{\frac{s}{2k}}ds}{\sqrt{(b - g\int e^{\frac{s}{k}}dx)}}$$

From the speed found there is obtained the pressing force, that is allowed on the curve at M along the normal, equal to

$$\frac{gdy}{ds} - \frac{2vdxddy}{ds^3}$$

(475) on putting $PM = y$ and on taking dx as constant. Q.E.I.

Corollarium 1.

527. Posito ergo $v = 0$ erit

$$g\int e^{\frac{s}{k}}dx = b,$$

ex qua aequatione obtinebitur punctum *D*, quousque corpus ex *A* ascendere poterit. Atque tempus totius ascensus per *AMD* habebitur, si in expressione temporis ponatur

$$g\int e^{\frac{s}{k}}dx = b.$$

Corollarium 2.

528. Si in formula pressionem exhibente loco *v* eius valor inventus substituatur, habebitur

$$-\frac{2e^{\frac{-s}{k}}bdxddy}{ds^3}+\frac{gdy}{ds}+\frac{2ge^{\frac{-s}{k}}dxddy\int e^{\frac{s}{k}}dx}{ds^3}.$$

Quae transmutari potest in hanc formam

$$-\frac{2e^{\frac{-s}{k}}bdxddy}{ds^3}+\frac{gds}{e^{\frac{s}{k}}dydx}\,d.\,\frac{dy^2}{ds^2}\int e^{\frac{s}{k}}dx.$$

Corollarium 3. [p. 273]

529. Pro descensu vero, si celeritas in *A* debita quoque est altitudini *b*, pressio, quam curva in *M* secundum normalem *MN* sustinet, est =

$$\frac{2e^{\frac{s}{k}}bdxddy}{ds^3}+\frac{ge^{\frac{s}{k}}ds}{dydx}\,d.\,\frac{dy^2}{ds^2}\int e^{\frac{-s}{k}}dx.$$

Corollarium 4.

530. Si igitur tam ascensus quam descensus respectu axis *AP* definiatur, aequatio ascensum determinans transmutari potest in aequationem descensus scribendo – *k* loco *k* atque vicissim. Quare si super curva *AM* descensus fuerit determinatus, habebitur quoque ascensus ascensus et vicissim.

Scholion.

531. Quia formulae ascensum et descensum determinantes tantam inter se habent affinitatem, ascensus et descensus facile poterunt inter se comparari atque ideo oscillationes super data curva determinari. Id quod in sequente propositione, quantum generaliter fieri potest, praestabimus. [p. 274]

PROPOSITIO 61.

Problema.

532. *Sint curvae quaecunque MA et NA* (Fig.65) *in infimo puncto A coniunctae atque corpus descendat super curva [MA et ascendat super curva]AN in medio resistente uniformi secundum quadrata celeritum; inter se comparare descensum super curva MA et ascensum super cura AN.*

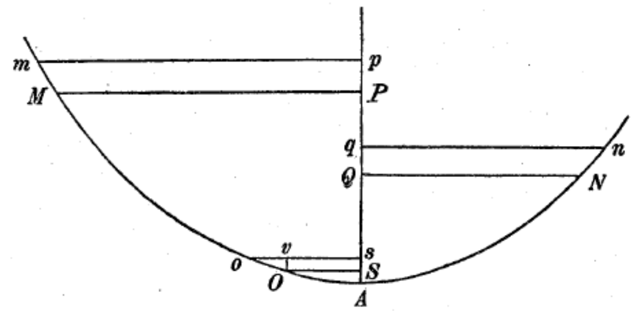

Fig. 65.

Solutio.

Sit celeritas in puncto A debita altitudini b atque in axe verticali AP abscissa $AP = x$, arcus $AM = s$. Pro curva ascensus AN vero sit $AQ = t$ et $AN = r$. His positis erit corporis descendentis celeritas in M debita altutudini

$$e^{\frac{s}{k}}b - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx$$

(523). Corporis vero ascendentis super curva AN celeritas in N debita est altitudini

$$e^{-\frac{s}{k}}b - ge^{-\frac{s}{k}}\int e^{\frac{s}{k}}dx$$

(526). Quare si celeritates in M et N evanescant, ita ut MAN sit arcus una semioscillatione descriptus, erit

$$\frac{b}{g} = \int e^{-\frac{s}{k}}dx \text{ et } \frac{b}{g} = \int e^{\frac{r}{k}}dt.$$

Si nunc concipiatur alia semioscillatio arcum mMn absolvens, in qua celeritas in puncto A debita sit altitudini $b + db$, erit

$$\frac{b+db}{g} = \int e^{\frac{s}{k}}dx + e^{-\frac{s}{k}}dx$$

hincque

$$e^{-\frac{s}{k}}dx = \frac{db}{g} \text{ seu } e^{-\frac{AM}{k}}.Pp = \frac{db}{g}.$$

Similiter pro ascensu erit

$$e^{\frac{AN}{k}}.Qq = \frac{db}{g}.$$

Ex quibus fiet [p. 275]

$$\frac{Pp}{Qq} = e^{\frac{AM+AN}{k}} \text{ seu } lPp - lQq = \frac{AM+AN}{k}.$$

Dato ergo arcu *MAN* una semioscillatione descripto, si corpus in puncto proximo superiore *m* descendere incipiat, invenietur punctum *n*, ad quod supra *N* pertinget; erit nempe

$$Qq = \frac{Pp}{e^{\frac{AM+AN}{k}}}.$$

Q.E.I.

Corollarium 1.

533. Si igitur *MAN* et *mAn* fuerint duo arcus oscillationibus proximis descripti, erit semper $Qq < Pp$ eoque minus erit *Qq* quam *Pp*, quo maior fuerit summa arcuum $AM + AN$. Semper igitur quoque erit *AQ* minor quam *AP*.

Corollarium 2.

534. In vacuo, quia est $k = \infty$, erit $e^{\frac{AM+AN}{k}} = 1$; fietque $Qq = Pp$ atque hinc $AQ = AP$. Quare corpus oscillans in vacuo ad tantam ascendit altitudinem, quantas erat illa, ex qua descendit.

Corollarium 3.

535. Si resistentia fuerit valde parva ideoque *k* vehementer magnum, erit

$$e^{\frac{AM+AN}{k}} = 1 + \frac{AM+AN}{k}.$$

Quare hoc casu erit

$$Qq + \frac{MAN.Qq}{k} = Pp \text{ atque } Qq = \frac{Pp(k - MAN)}{k}$$

Corollarium 4. [p. 276]

536. Si punctum *O* fuerit locus, in quo corpus descendens maximam habet celeritatem, ibique ponatur $AO = s$, $AS = x$, erit

$$\frac{gkdx}{ds} = e^{\frac{s}{k}} b - ge^{\frac{s}{k}} \int e^{-\frac{s}{k}} dx$$

seu [corrigi]

$$b = g\int e^{-\frac{s}{k}} dx + \frac{gkdx}{ds}.$$

Quare si corpus ex *m* descendat, erit punctum maximae celeritas in *o*

$$db = ge^{-\frac{s}{k}} dx + \frac{gkdy}{p},$$

existente $dy = ov$ et *p* radio osculi in puncto *O*, seu

$$\frac{db}{g} = e^{-\frac{AO}{k}} Ss + \frac{k.ov}{p}.$$

Corollarium 5.

537. Tempus unius itus per MAN habetur, si in integralium summa

$$\int \frac{e^{\frac{-s}{2k}}ds}{\sqrt{\left(b - g\int e^{\frac{-s}{k}}dx\right)}} + \int \frac{e^{\frac{r}{2k}}ds}{\sqrt{\left(b - g\int e^{\frac{r}{k}}dt\right)}}$$

ponatur

$$\int e^{\frac{-s}{k}}dx = \frac{b}{g} \quad \text{atque} \quad \int e^{\frac{r}{k}}dt = \frac{b}{g}.$$

Sicque prodibit tempus unius semioscillationis.

Corollarium 6.

538. Ex dictis alia elegans sequitur proprietas, ut si corpus ex *A* celeritate $\sqrt{b}$ ascendat ad *N* atque ex *N* iterum decidat per *NA* sitque celeritas, quam tum in *A* habebit, debita altitudini *c*, [p. 277] deinde ex *A* celeritate altitudini $b + db$ debita ascendat pertingatque ad *n*, unde rursus descendendo in *A* acquirat celeritatem altitudini $c + dc$ debitam. Erit ergo

$$\frac{db}{g} = e^{\frac{AN}{k}} \cdot Qq \quad \text{et} \quad \frac{dc}{g} = e^{\frac{-AN}{k}} \cdot Qq$$

atque

$$Qq^2 = \frac{db.dc}{g^2}$$

vel etiam

$$\frac{db}{dc} = e^{\frac{2AN}{k}}.$$

Scholion.

539. Restat, ut haec generalia ad exempla seu datas curvas accommodemus, quo usus eorum eo magis pateat. Accipiemus autem pro curva data cycloidem tantum, eo quod $\int e^{\frac{s}{k}}dx$ in ea facile possit exhiberi et aequatio quoque inter *s* et *x* sit algebraica. Investigabimus ergo tam descensus super cycloides factos quam oscillationes, quo appareat, quantum oscillationes, quae in cycloide fiunt, ab isochronismo discrepent, quippe in vacuo omnes eodem tempore absolvi sunt demonstratae [(191)].

PROPOSITIO 62.

Problema.

540. *Sit curva data cyclois ACB* (Fig.66) *super basi horizontali AB provolutione circuli diametri CD descripta corpusque super ea ex A descendat in medio resistente in duplicata ratione celeritatum; determinare motum corporis descendentis.* [p. 278]

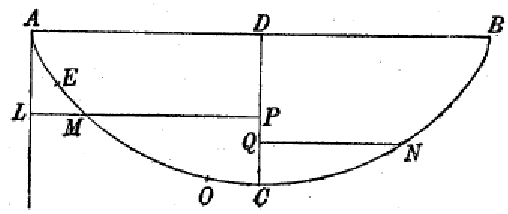

Fig. 66.

Solutio.

Posita $2CD = a$, $AL = x$ et $AM = s$ erit ex natura cycloidis

$$s = a - \sqrt{(a^2 - 2ax)} \text{ seu } 2ax = 2as - ss.$$

Celeritas vero in M debita sit altitudini v; erit

$$v = ge^{-\frac{s}{k}} \int e^{\frac{s}{k}} dx .$$

Quia autem est

$$dx = ds - \frac{sds}{a},$$

erit

$$\int e^{\frac{s}{k}} dx = ke^{\frac{s}{k}} + \frac{k^2 e^{\frac{s}{k}}}{a} - \frac{ke^{\frac{s}{k}} s}{a} - k - \frac{k^2}{a} = e^{\frac{s}{k}} \frac{ak + k^2 - ks}{a} - \frac{ak + k^2}{a}.$$

Quo valore substituto erit

$$v = \frac{gak + gk^2 - gks}{a} - \frac{ge^{\frac{-s}{k}}(ak + k^2)}{a}.$$

Maxima corporis erit celeritas, ubi est

$$v = \frac{gkdx}{ds} = \frac{gak - gks}{a};$$

hoc igitur accidit, ubi est

$$e^{\frac{s}{k}} k = a + k \text{ seu } s = kl\frac{a+k}{k}.$$

Inveniri etiam potest punctum N, in quo corpus omnem celeritatem perdit, faciendo v = 0 seu

$$e^{\frac{s}{k}} = \frac{a+k}{a+k-s} \text{ seu } s = kl\frac{a+k}{a+k-s},$$

ex qua aequatione valor ipsius s dat arcum ACN. Tempus, quo arcus AM descendu absolvitur, est =

$$\int \frac{ds\sqrt{a}}{\sqrt{gk\left(a+k-s-e^{\frac{-s}{k}}(a+k)\right)}}.$$

Deinde ad pressionem inveniendam est

$$\frac{dy}{ds} = \frac{\sqrt{(2as-ss)}}{a} = \frac{\sqrt{2ax}}{a},$$

unde pressio ipsa in puncto *M* prodit =

$$\frac{g\sqrt{2ax}}{a} + \frac{2gak+2gkk-2gks}{a\sqrt{2ax}} - \frac{2gak+2gkk}{e^{\frac{s}{k}}a\sqrt{2ax}}.$$

[p. 279] Determinavimus ergo celeritatem et tempus et pressionem, unde motus corporis innotescit. Q.E.I.

Corollarium 1.

541. Quia est

$$e^{-\frac{s}{k}} = 1 - \frac{s}{1.k} + \frac{s^2}{1.2.k^2} - \frac{s^3}{1.2.3.k^3} + \text{etc.},$$

si ponatur $a + k = c$ seu $a = c - k$, erit

$$\frac{(c-k)v}{gk} = -s + \frac{cs}{1.k} - \frac{cs^2}{1.2.k^2} + \frac{cs^3}{1.2.3.k^3} - \text{etc.} = \frac{as}{1.k} - \frac{cs^2}{1.2.k^2} + \frac{cs^3}{1.2.3.k^3} - \text{etc.}$$

Quare erit

$$v = \frac{gs}{a}\left(\frac{a}{1} - \frac{s}{1.2} - \frac{as}{1.2.k} + \frac{(a+k)s^2}{1.2.3.k^2} - \frac{(a+k)s^3}{1.2.3.4k^3} + \text{etc.}\right)$$

Corollarium 2.

542. In vacuo igitur, ubi *k* est infinitum, erit

$$v = gs - \frac{gs^2}{2a} = gx,$$

ut constat. At si resistentia tantum sit valde parva et propterea *k* valde magnum, erit

$$v = gs - \frac{gs^2}{2a} - \frac{gs^2}{2k} + \frac{gs^3}{6ak}.$$

Corollarium 3.

543. In vacuo apparet celeritatem corporis esse nullam in duobus punctis, ubi est $s = 0$ et $s = 2a$, i.e. in duobus cuspidibus *A* et *B*. In medio vero resistente alter locus est $s = 0$; alter vero ex hac aequatione erui debet

$$a = \frac{cs}{1\cdot 2k} - \frac{cs^2}{1\cdot 2\cdot 3k^2} + \frac{cs^3}{1\cdot 2\cdot 3\cdot 4k^3} - \text{etc.},$$

unde invenitur

$$s = \frac{2ak}{c} + \frac{4a^2k}{3c^2} + \frac{10a^3k}{9c^3} + \frac{136a^4k}{135c^4} + \text{etc.}$$

Si igitur k fuerit valde magnum, erit

$$s = \frac{2ak}{a+k} + \frac{4a^2k}{3(a+k)^2} = 2a - \frac{2a^2}{3k}$$

quam proxime.

Corollarium 4. [p. 280]

544. Eadem haec series inventa substituto $a + k$ loco c transformatur in hanc

$$s = 2a - \frac{2a^2}{3k} + \frac{4a^3}{9k^2} - \frac{44a^4}{135k^3} + \text{etc.}$$

ex qua valor ipsius s dat arcum *ACN*, quo usque corpus motu suo pervenire potest.

Corollarium 5.

545. Arcus *AO* ab *A* usque ad *O*, ubi corpus maximum habet celeritatem, est =

$$kl\frac{a+k}{k} = a - \frac{a^2}{2k} + \frac{a^3}{3k^2} - \frac{a^4}{4k^3} + \frac{a^5}{5k^4} - \text{etc.}$$

Quare erit arcus

$$ON = a - \frac{a^2}{6k} + \frac{a^3}{9k^2} - \text{etc.}$$

$$OC = \frac{a^2}{2k} - \frac{a^3}{3k^2} + \frac{a^4}{4k^3} - \text{etc.}$$

et

$$-AO + ON = \frac{a^2}{3k} - \frac{2a^3}{9k^2} + \text{etc.}$$

Corollarium 6.

546. Celeritas vero in puncto *C* reperitur debita altitudini =

$$\frac{gk^2 - ge^{\frac{-a}{k}}(ak+kk)}{a} = g\left(\frac{a}{2} - \frac{a^2}{1\cdot 3k} + \frac{a^3}{1\cdot 2\cdot 4k^2} - \frac{a^4}{1\cdot 2\cdot 3\cdot 5k^3} + \frac{a^5}{1\cdot 2\cdot 3\cdot 4\cdot 6k^4} - \text{etc.}\right)$$

Unde perspicitur celeritatem in C nunquam posse esse evanescentem; nam altitudo huic celeritati debita est =

$$\frac{gk^2}{e^{\frac{a}{k}}a}\left(e^{\frac{a}{k}} - 1 - \frac{a}{k}\right)$$

atque $e^{\frac{a}{k}}$ semper maius est quam $1+\frac{a}{k}$; excessus autem maior est quam $\frac{a^2}{2k^2}$. Quare altitudo debita celeritati in *C* maior est quam $\frac{ga}{2e^{\frac{a}{k}}}$ ideoque *ACN* maior est quam *AC*.

Corollarium 7. [p. 281]

547. Altitudo debita celeritati maximae in O est =

$$gk - \frac{gk^2}{a} l \frac{a+k}{k} = g(\frac{a}{2} - \frac{a^2}{3k} + \frac{a^3}{4k^2} - \frac{a^4}{5k^3} + \text{etc})$$

Quare excessus huius altitudinis supra altitudinem celeritati in C debitam est =

$$g(\frac{a^3}{1.2.4k^2} - \frac{5a^4}{1.2.3.5k^3} + \frac{23a^5}{1.2.3.4,6k^4} - \text{etc}) = ge^{\frac{-a}{k}}(\frac{a^3}{8k^2} - \frac{a^4}{24k^3} + \text{etc.})$$

Haec nempe expressio obtinetur, si in generali valore ipsius v substituatur $kl\frac{a+k}{k}$ loco s, quippe cui quantitati arcus AO est aequalis.

Scholion.

548. In solutione huius propositionis considerandum venit, quod ex formula descensum tantum determinante etiam ascensum corporis super arcu CN derivavimus; ex quo dubium oriri potest, an iste ascensus legitime sit definitus. Hoc autem ex ipsa formula ascensum determinante facile perspicitur. Usi enim fuimus formula hac $dv = gdx - Rds$, quae puncto M ultra punctum C cadente propter dx factum negativum abit in hanc $dv = -gdx - Rds$, quae revera naturam ascensus continet. Ex his intelligitur continuitas inter ascensum et descensum, qua nullo interiecto saltu inter se cohaerent. [p. 282] Ubi enim curva se sursum flectere incipit, ibi simul formula descensui inserviens transmutatur sponte in formulam ascensus. Atque haec connexio locum habet in medio quocunque resistente, uti ex generalibus formulis apparet, quae tantum signo ipsius dx descrepant. Quamobrem data aequatione pro curva quacunque non est necesse, ut inquiratur, super quanam parte corpus ascendat descendatve, sed alterutra formula ad aequationem accommodata verum dabit motum super curva proposita. Hoc tantum est tenendum, ut abscissae in axe verticali capiantur atque ea formula sive ascensus sive descensus adhibeatur, quae cum motus initio congruat.

PROPOSITIO 63.

Problema.

549. *Sit curva data ACB* (Fig.66) *cyclois super basi horizontali AB descripta et deorsum spectans corpusque super ea oscillationes peragat in medio resistente in duplicata ratione celeritatum; determinare motum oscillatorium.*

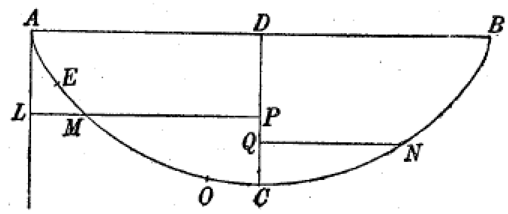

Fig. 66.

Solutio.

Ponatur diameter circuli $CD = \frac{1}{2}a$ in eaque sumatur abscissa $CP = x$ et arcus CM vocetur s ; erit ex natura cycloidis

$$s = \sqrt{2ax} \text{ et } x = \frac{ss}{2a} \text{ atque } dx = \frac{sds}{a}. \text{ [p. 283]}$$

Descendat nunc corpus super arcu MC sitque eius celeritas in C debita altitudini b; erit altitudo debita celeritati in M =

$$e^{\frac{s}{k}}b - ge^{\frac{s}{k}}\int e^{-\frac{s}{k}}dx .$$

(523). Est vero

$$\int e^{\frac{-s}{k}}dx = \int \frac{e^{\frac{-s}{k}}sds}{a} = \frac{k^2 - k^2e^{\frac{-s}{k}} - ke^{\frac{-s}{k}}s}{a};$$

quare altitudo debita celeratiati in M est =

$$\frac{e^{\frac{s}{k}}(ab - gk^2) + gk^2 + gks}{a} = e^{\frac{s}{k}}b - \frac{g}{a}\left(\frac{s^2}{2} + \frac{s^3}{2\cdot 3k} + \frac{s^4}{2\cdot 3\cdot 4k^2} + \frac{s^5}{2\cdot 3\cdot 4\cdot 5k^3} + \text{etc.}\right).$$

Arcus ergo, in quo integer fit descensus, habebitur, si ipsius s valor ex hac aequatione quaeratur

$$e^{\frac{s}{k}} = \frac{gk^2 + gks}{gk^2 - ab}.$$

Fiet autem hinc in serie

$$s = A + \frac{A^2}{3k} + \frac{11A^3}{72k^2} + \frac{43A^4}{540k^3} + \text{etc.}$$

posito brevitatis ergo *A* loco $\frac{\sqrt{2ab}}{\sqrt{g}}$. Huic ergo seriei aequatur arcus *CM*, si quidem corpus ex puncto *M* descendere inceperit. Celeritatem maximam corpus habebit in *O* sumto *CO* = *s* ex hac aequatione

$$e^{\frac{s}{k}} = \frac{gk^2}{gk^2 - ab} \text{ seu } CO = kl\frac{gk^2}{gk^2 - ab}$$

atque altitudo huic maximae celeritati debita est =

$$\frac{gk \cdot CO}{a} = \frac{gk^2}{a} l \frac{gk^2}{gk^2 - ab} = b + \frac{ab^2}{2gk^2} + \frac{a^2b^3}{3g^2k^4} + \text{etc.}$$

Ad tempus determinandum convenit ad punctum *O* celeritatemque maximam respicere et tempus per *MO* definire. Hanc ob rem pono altitudinem celeritati in *O* debitam = *c* et arcum *MO* = *q*; erit [p. 284]

$$CO = \frac{ac}{gk} \quad \text{et} \quad ab = gk^2\left(1 - e^{\frac{-ac}{gk^2}}\right), \quad s = \frac{ac}{gk} + q.$$

His substitutis erit altitudo debita celeritati in *M*, seu *v*, =

$$\frac{gk^2 + ac + gkq - e^{\frac{q}{k}} gk^2}{a}.$$

Quia nunc *v* minor est quam *c*, ponatur *c* – *v* = *z* eritque

$$az + gk^2 + gkq = e^{\frac{q}{k}} gk^2$$

atque in serie

$$\frac{az}{g} = \frac{q^2}{2} + \frac{q^3}{6k} + \frac{q^4}{24k^2} + \frac{q^5}{120k^3} + \text{etc.}$$

Ex qua convertendo fit

$$q = \frac{\sqrt{2az}}{\sqrt{g}} - \frac{az}{3gk} + \frac{az\sqrt{2az}}{18gk^2\sqrt{g}} - \frac{2a^2z^2}{135g^2k^3} + \frac{a^2z^2\sqrt{2az}}{1080g^2k^4\sqrt{g}} - \text{etc.}$$

Incipiat descensus in puncto *M*; erit ibi *v* = 0 et *z* = *c* ideoque

$$OM = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} - \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} - \text{etc.}$$

Ex eadem formula, si ponatur *q* negativum, habebitur motus per *OCN*; at quia perinde est, sive *q* ponatur negativum sive *k*, erit arcus *ON*, si *N* fuerit punctum, quousque corpus ascendit, =

$$\frac{\sqrt{2ac}}{\sqrt{g}} + \frac{ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} + \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} + \text{etc.}$$

Tempus vero per *MO* hoc modo invenitur: quia est

$$ds = dq = \frac{adz}{\sqrt{2gaz}} - \frac{adz}{3gk} + \frac{adz\sqrt{2az}}{12gk^2\sqrt{g}} - \frac{4a^2zdz}{135g^2k^3} + \frac{a^2zdz\sqrt{2az}}{432g^2k^4\sqrt{g}} - \text{etc.},$$

hoc divisum per $\sqrt{v} = \sqrt{(c-z)}$ dat elementum temporis =

$$\frac{dz\sqrt{a}}{\sqrt{2g(cz-z^2)}} - \frac{adz}{3gk\sqrt{(c-z)}} + \frac{azdz\sqrt{2a}}{12gk^2\sqrt{g(cz-zz)}} - \frac{4a^2zdz}{135g^2k^3\sqrt{(c-z)}} + \frac{a^2z^2dz\sqrt{2a}}{432g^2k^4\sqrt{g(cz-z^2)}} - \text{etc.}$$

Quod ita integrari debet, ut posito $v = c$ vel $z = 0$ evanescat; deinde si ponatur $z = c$, habebitur tempus, quo corpus per arcum *MO* descendit. Hoc igitur tempus, posita peripheriae ad diametrum ratione π ad 1, erit [p. 285] =

$$\frac{\pi\sqrt{a}}{\sqrt{2g}} - \frac{2a\sqrt{c}}{3gk} + \frac{\pi ac\sqrt{a}}{12gk^2\sqrt{2g}} - \frac{16a^2c\sqrt{c}}{405g^2k^3} + \text{etc.}$$

Posito igitur *k* negativo erit tempus, quo corpus ex *O* ad *N* usque ascendit, =

$$\frac{\pi\sqrt{a}}{\sqrt{2g}} + \frac{2a\sqrt{c}}{3gk} + \frac{\pi ac\sqrt{a}}{12gk^2\sqrt{2g}} + \frac{16a^2c\sqrt{c}}{405g^2k^3} + \text{etc.}$$

Tempus ergo per *MCN* seu tempus unius dimidiae oscillationis est =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}} + \frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}} + \text{etc.}$$

Q.E.I.

Corollarium 1.

550. Si ergo celeritas maxima corporis descendentis fuerit debita altitudini *c*, propter

$$CO = \frac{ac}{gk}$$

erit

$$MC = \frac{\sqrt{2ac}}{\sqrt{g}} + \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} - \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} - \text{etc.}$$

Totus vero ascensus *CN* erit =

$$ON - CO = \frac{\sqrt{2ac}}{\sqrt{g}} - \frac{2ac}{3gk} + \frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}} + \frac{2a^2c^2}{135g^2k^3} + \frac{a^2c^2\sqrt{2ac}}{1080g^2k^4\sqrt{g}} + \text{etc.}$$

Hinc erit

$$CM - CN = \frac{4ac}{3gk} - \frac{4a^2c^2}{135g^2k^3} - \text{etc.}$$

atque

$$MCN = \frac{2\sqrt{2ac}}{\sqrt{g}} + \frac{ac\sqrt{2ac}}{9gk^2\sqrt{g}} + \frac{a^2c^2\sqrt{2ac}}{540g^2k^4\sqrt{g}} + \text{etc.}$$

Corollarium 2.

551. Si totus arcus descensus *MC* ponatur = *E* et sequens arcus ascensus *CN* = *F* atque altitudo debita celeritati in *C* = *b*, erit

$$\frac{ab}{g} = k^2 - e^{\frac{-E}{k}}(k^2 + kE) = \frac{E^2}{2} - \frac{E^3}{3k} + \frac{E^4}{8k^2} - \frac{E^5}{30k^3} + \frac{E^6}{144k^4} - \text{etc.}$$

Atque posito *k* negativo eodem modo invenitur

$$\frac{ab}{g} = \frac{F^2}{2} + \frac{F^3}{3k} + \frac{F^4}{8k^2} + \frac{F^5}{30k^3} + \frac{F^6}{144k^4} + \text{etc.}$$

Ex quibus fit

$$F = E - \frac{2E^2}{3k} + \frac{4E^3}{9k^2} - \frac{44E^4}{135k^3} + \frac{104E^5}{405k^4} - \text{etc.}$$

atque altitudo debita celeritati maximae [p. 286]

$$c = \frac{gE^2}{2a} - \frac{gE^3}{3ak} + \frac{gE^4}{4ak^2} - \text{etc.}$$

Corollarium 3.

552. Quia *F* est arcus ascensus in prima dimidia oscillatione, erit idem arcus *F* arcus descensus in sequente oscillatione; cum quo ergo coniungetur arcus ascensus

$$G = E - \frac{4E^2}{3k} + \frac{16E^3}{9k^2} - \frac{328E^4}{135k^3} + \frac{1376E^5}{405k^4} - \text{etc.}$$

Atque simili modo sequentes oscillationes, quotquot libuerit, definire possunt.

Corollarium 4.

553. Ex aequatione tempus exponente apparet tempus, quo corpus ex *M* ad *O* pervenit, semper minus esse tempore, quo corpus ex *O* ad *N* usque pertingit. Simili modo etiam arcus *ON* maior est quam arcus *OM*, arcus vero *CN* minor est arcu *MC*.

Corollarium 5.

554. Si oscillationes fuerint infinite parvae seu *c* quantitas evanescens, congruent oscillationes cum oscillationibus in vacuo factis; in singulis enim expressionibus iidem termini evanesent, qui evanescerent posito $k = \infty$. Minimis ergo oscillationibus isochronae erunt oscillationes penduli longitudinis a in vacuo sollicitati a potentia *g* seu penduli in hypothesi gravitatis = 1, cuius longitudo est = $\frac{a}{g}$.

Corollarium 6. [p. 287]

554. At si oscillationes fiant maiores, tempora oscillationum quoque fient maiora; quare in hac resistentiae hypothesi cyclois tautochronismi proprietate non gaudet. Quo enim in quaque oscillatione maior fuerit celeritas maxima, maior quoque erit excessus temporis oscillationis huiusmodi supra tempus oscillationis minimae.

Scholion 1.

556. Quod diximus oscillationes minimas cum oscillationibus in vacuo congruere, locum habet, si *a* et *k* fuerint quantitates finitae magnitudinis. Si enim *a* esset infinite magnum seu *k* infinite parvum, sequentes termini tempus exprimentes

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi ac\sqrt{2a}}{12gk^2\sqrt{g}}+\text{ etc.}$$

non evanescerent, etiamsi c esset infinite parvum. Tum igitur tantum oscillationes minimae super curva quacunque in vacuo et medio resistente inter se congruent, quando neque radius osculi curvae in infimo puncto fuerit infinite magnus neque resistentia infinita magna.

Exemplum.

557. Exempli loco evolvamus casum, quo resistentia tam fit exigua ideoque *k* quantitas tam magna, ut fractiones, in quarum denominatoribus *k* plures duabus habet dimensiones, tuto pro nihilo haberi possint. Dicta igitur altitudine celeritati maximae in *O* debita *c*, ita ut sit $CO=\frac{ac}{gk}$, erit arcus descensus [p. 288]

$$MC=E=\frac{\sqrt{2ac}}{\sqrt{g}}+\frac{2ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}$$

et sequens arcus ascensus

$$CN=F=\frac{\sqrt{2ac}}{\sqrt{g}}-\frac{2ac}{3gk}+\frac{ac\sqrt{2ac}}{18gk^2\sqrt{g}}.$$

Unde invenitur

$$\sqrt{c}=\frac{E\sqrt{g}}{\sqrt{2a}}-\frac{E^2\sqrt{g}}{3k\sqrt{2a}}+\frac{7E^3\sqrt{g}}{36k^2\sqrt{2a}}$$

seu

$$c=\frac{gE^2}{2a}-\frac{gE^3}{3ak}+\frac{gE^4}{4ak^2}$$

atque

$$F=E-\frac{2E^2}{3k}+\frac{4E^3}{9k^2}.$$

Tempus ergo dimidiae oscillationis per *MCN* est =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}.$$

In sequente dimidia oscillatione est arcus descensu =

$$F=E-\frac{2E^2}{3k}+\frac{4E^3}{9k^2},$$

quem sequetur arcus ascensus =

$$G=E-\frac{4E^2}{3k}+\frac{16E^3}{9k^2};$$

atque tempus huius dimidiae oscillationis erit =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}-\frac{\pi E^3\sqrt{2a}}{18k^3\sqrt{g}},$$

ubi ultimus terminus negligi potest ob k^3 in denominatore. In tertia semioscillatione est arcus descensus =

$$G=E-\frac{4E^2}{3k}+\frac{16E^3}{9k^2}$$

et arcus descensus =

$$H=E-\frac{6E^2}{3k}+\frac{36E^3}{9k^2}.$$

Atque generaliter in ea semioscillatione, quae indicatur numero *n*, est arcus descensus =

$$E-\frac{2(n-1)E^2}{3k}+\frac{4(n-1)^2E^3}{9k^2}$$

et arcus ascensus =

$$E-\frac{2nE^2}{3k}+\frac{4n^2E^3}{9k^2}.$$

Quamobrem post *n* semioscillationes corpus ab infimo puncto *C* distabit arcu

$$E-\frac{2nE^2}{3k}+\frac{4n^2E^3}{9k^2},$$

qui minor est quam arcus descensus primae oscillationis quantitate

$$\frac{2nE^2}{3k}-\frac{4n^2E^3}{9k^2}.$$

Tempus autem semioscillationis numero *n* indicatae erit =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}+\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}-\frac{\pi(n-1)E^3\sqrt{2a}}{18k^3\sqrt{g}}.$$

At si totus arcus primae semioscillationis *MCN* dicatur *A*, erit

$$A=2E-\frac{2E^2}{3k}+\frac{4E^3}{9k^2}\quad\text{et}\quad E=\frac{A}{2}+\frac{A^2}{12k}$$

evanescente sponte termino sequente. Hinc totus arcus oscillatione per numerum *n* indicata descriptus erit =

$$2E-\frac{2(2n-1)E^2}{3k}+\frac{4(2n^2-2n+1)E^3}{9k^2}=A-\frac{(n-1)A^2}{3k}+\frac{(n-1)^2A^3}{9k^2}.$$

Corollarium 7. [p. 289]

558. Si *n* fiant oscillationes dimidiae et arcus descensus primae oscillationis fuerit *E* et arcus ascensus ultimae = *L*, erit

$$L=E-\frac{2nE^2}{3k}+\frac{4n^2E^3}{9k^2},$$

quae expressio, si per seriem propius capiatur, fere congruet cum progressione geometrica eiusdem initiali hancque ob rem erit

$$L=\frac{3Ek}{3k+2nE}\quad\text{seu}\quad 3k(E-L)=2nEL.$$

Corollarium 8.

559. Hinc, si peractis aliquot semioscillationibus detur arcus descensus primae oscillationis E una cum arcu ascensus ultimae *L*, inveniri potest numerus semioscillationum; namque est

$$n=\frac{3k(E-L)}{2EL}.$$

Corollarium 9.

560. Patet ergo diminutionem arcuum non a longitudine penduli pendere, sed ex *n* et *E* datis idem reperitur arcus *L*, quacunque fuerit longitudo penduli *a*. Atque est semper *n* proportionalis ipsi $\frac{1}{L}-\frac{1}{E}$.

Scholion 2. [p. 290]

561. Huiusmodi experimenta circa oscillationes in medio resistente multa recenset Neutonus in *Phil*. Lib. II, ubi notat arcum descensus primae, arcum descensus ultimae oscillationis atque numerum oscillationum tam in aere quam in aqua et mercurio. Quare si haec media perfecte resisterent in duplicata celeritatum ratione, congrua esse deberent cum hisce formulis, ita ut decrementum arcus proportionale esset numero oscillationum et arcui primo et ultimo coniunctum. Quod etiam locum habere observavi in maioribus oscillationibus, in quibus celeritas non est nimis exigua. At in oscillationibus minimis maxima aberratio ab hac regula conspicitur. Ex quo colligitur, quo maior fuerit corporis celeritas in fluido, eo propius resistentiam accedere ad rationem duplicatum celeritatum, motum autem tardissimum alii resistentiae insuper esse obnoxium, quae in motibus celerioribus prae resistentia, quae quadratis celeritatum est proportionalis, evanescat. In hisce quoque expermentis Neutonus resistentiam partim simplici celeritatum rationi, partim sesquiplicatae, partim duplicatae proportionalem assumsit neque tamen pro motibus tardissimis satisfecit. In ultima vero *Phil*. editione ipse Neutonus quoque insufficentem priorem theoriam suam agnoscit atque pluribus rationibus ostendit alteram illam fluidorum resistentiam esse constantem seu temporis momentis proportionalem, quam antea ipsis celeritatibus proportionalem erat arbitratus. Hanc ob rem istam resistentiam cum ea, quae quadratis celeritatum est proportionalis, in sequents propositione coniunctum considerabimus, cum praesertim aequationum resolutio et celeritatum determinatio hac adiectione non difficilior evadat.

Corollarium 10. [p. 291]

562. Quod ad tempora oscillationum et semioscillationum attinet, perspicuum est ea decrescere, quo minores fiant arcus descripti, atque si arcus plane evanescant, tempus dimidiae oscillationis fore = $\frac{\pi\sqrt{2a}}{\sqrt{g}}$.

Corollarium 11.

563. Excessus autem cuiusque semioscillationis temporis supra tempus minimae semioscillationis in casu resistentaie minimae est $\frac{\pi E^2\sqrt{2a}}{24k^2\sqrt{g}}$ denotate E arcum descensus illius semioscillationis. Quare iste excessus proportionalis est quadrato arcus descensus vel etiam quadrato totius arcus semioscillatione descripti.

Scholion 3.

564. Cylois igitur, quae ab Hugenio apta est demonstrata ad isochronismum pendulorum producendum, hanc proprietatem in medio resistente in duplicata celeritatum ratione amittit et hanc ob rem in aere non inservit, nisi vel oscillationes sint valde parvae vel inter se proxime aequales. [p. 292] Ex hoc vero, quod maiores oscillationes diutius durent, colligi licet veram curvam tautochronam in hac resistentiae hypothesi magis esse curvam quam cycloidem. Quemadmodum scilicet cyclois in circulo eiusdem radii, cuius cyclois est in infimo puncto, continetur, ita quoque vera tautochrona in cycloide continebitur atque eius curvedo a puncto infimo magis decrescet quam curvedo cycloidis.