CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 233]

PROPOSITION 53.

Problem.

465. *If a body is acted on by a given force g in a medium with whatever resistance, to determine the motion of the body descending on a given curve AM* (Fig.57), *and the compressive force sustained by the curve at particular points.*

Solution.

The abscissa $AP = x$ is placed along the vertical AP, the applied line $PM = y$ and the arc $AM = s$; let the height corresponding to the speed of the body at M be equal to v, and the resistance at M is equal to R. Therefore it is evident from the previous chapter, (93), that if there is no resistance, then [the differential of the height becomes]

$$dv = gdx.$$

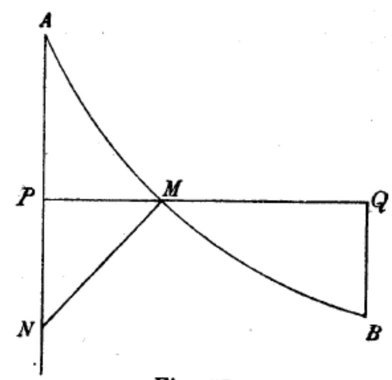

Fig. 57.

Now the resistance has lessened this increment of the speed and is equivalent to a tangential force R; and the effect of this force alone consists of this, that [the increment] becomes

$$dv = -Rds.$$

On account of which, if both the force g and the resistance are both likewise acting on the body, then this equation becomes

$$dv = gdx - Rds,$$

and from which equation the speed of the body at any point M is to be elicited. [p. 234]

[We have remarked on Euler's basic equations occasionally, which resemble but are not identical with certain modern equations describing the same phenomena. We note that no mention is made of mass, so presumably we are to assume that unit mass is used throughout, or that the forces or accelerations are organised so that the masses always cancel. In addition, one cannot perform a dimensional analysis on these equations, without accepting that certain physical quantities have been set equal to 1. Thus, terms such as gdx and $-Rds$ may be viewed in the modern context as the increments of work done on the body by gravity and against the body by friction, thus increasing or decreasing the kinetic energy of the body; however, the concepts of work and energy were in their infancy in Euler's day, although Johan Bernoulli and others had written some papers on what was then termed the *vis viva*.

Euler did not enter into this matter, and adhered to experimental facts when setting down his equations. Thus we should not view these equations from the modern physicist's conventional viewpoint, but rather from the mathematician's viewpoint. The idea that a moving body had something in common with the same body at rest at a greater height could perhaps best be explained at the time by referring to Galileo's experiments concerning bodies descending from rest down an inclined plane : from these observations a height v could be made proportional to the square of the speed at the end of the descent. This gave Euler the idea of replacing squares of speeds with heights : he had no hesitation in ascribing such a function v equal to the height above the earth, for which $\sqrt{v}$ was related to the speed of the body. In the first equation above, the force g is the force of gravity somewhere, while on the earth's surface it has the value $g = 1$. Thus, $1.dv = gdx$ is simply a proportionality between two forces or accelerations per unit mass, while $dv = dx$ for vertical motion near the surface or the earth. Such a proportionality is seen to be true under the circumstances that if V is the speed at the respective heights x and v in the locations where uniform gravity is g and 1, then $V^2 = 2gx = 2.1.v$ gives $V\Delta V = g\Delta x = 1.\Delta v$, here assumed at rest initially. In addition, we have seen that even horizontal speeds are ascribed by such heights, which is thus little more than a mathematical convenience. If we view such equations as energy equations, then a factor of 2 or $\frac{1}{2}$ is missing; hence these are not energy equations, for instead they are equations used to relate accelerations, and to transform squares of speeds into corresponding heights under the earth's gravity, and which has the great advantage of removing all speed and hence time dependence from the equations, which are thus rendered homogeneous for integration, etc. When the need arises, as in centripetal force below, the factor of 2 is inserted to deal with the absolute acceleration. In other places, where actual numbers are required, Euler referred to Huygens pendulum experiments to get the correct length of the pendulum for a swing of one second.
{Recall that Huygens proposed the idea of measuring lengths in terms of seconds via his pendulum, were the second was an arbitrary constant time interval}. In a word then, view Euler's dynamical equations above and elsewhere as mathematically expedient transformations for solving dynamical problems, but realise that these transformations are not motivated by potential/kinetic energy conservation, although these can now be viewed as analogous to Euler's pioneering work. We return to the text :]

And if the body descends from rest at A, the integration is thus to be put in place so that on placing $x = 0$ the height $v = 0$ also. Now if the body starts the descent at A with a given speed, on putting $x = 0$ in the integration, v must be made equal to the height of that initial speed. Moreover since the speed of the body is found, likewise the time is obtained, in which any arc AM you please is completed, by taking $\int \frac{ds}{\sqrt{v}}$. As for the force sustained by the curve at M, it is observed that the curve at M is pressed on by two forces, clearly the centrifugal force and the normal force. We put the curve to be convex downwards and the element dx constant; the length of the radius of osculation directed along the opposite normal MN is equal to

$$\frac{ds^3}{dxddy},$$

and thus the centrifugal force is equal to

$$\frac{2vdxddy}{ds^3},$$

which is pressing on the curve along the direction MN . Now the curve is pressed upon by a normal force along the same direction which is given by

$$\frac{gdy}{ds};$$

for the normal force arises only from the absolute force acting g, since the direction of the force of resistance is placed along the tangent and thus does not generate a normal force. Consequently the total force, which presses on the curve at M along the direction of the normal MN , is equal to

$$\frac{gdy}{ds}+\frac{2vdxddy}{ds^3}.$$

Q.E.I.

Corollary 1.

466. Therefore the expression of the force pressing the curve agrees with that which we found in the case of the vacuum (83). Yet the curve is pressed in a medium with resistance by the same force as in a vacuum on account of the speed, upon which the centrifugal force depends, which is not varies by the resistance of the medium. [p. 235]

Corollary 2.

467. In this descent the body does not have the maximum speed as in a vacuum in which the tangent is horizontal at the point B, but at the place where $dv = 0$ at which it has the maximum speed, is found from this equation :

$$gdx = Rds \text{ or } \frac{dx}{ds}=\frac{R}{g}$$

in that point, where the sine of the angle, that the tangent to the curve makes with the line to the horizontal, is to the whole sine as the absolute force g to the resistance R at that place.

Corollary 3.

468. Therefore the speed of the body is increased as far as this point, in which the speed is a maximum; now beyond this point the speed decreases again since then Rds exceeds gdx and on this account makes dv negative.

Corollary 4.

469. If the resistance is as some power of the speed of which the exponent is $2m$, and if the resisting medium is uniform, the exponent of this is k, where k is the height corresponding to the speed with which the body is moving, the resistance is allowed be made equal to the force of gravity; in this case

$$R = \frac{v^m}{k^m}$$

and this equation is obtained defining the motion [note : the speed corresponds to the square root of the height always, hence v^m is proportional to the speed] : p. 236]

$$dv = gdx - \frac{v^m ds}{k^m}.$$

Corollary 5.

470. But if abscissae in taken on the axis *BQ* and we put *BQ* = *x*, *QM* = *y* and *BM* = *s*, on account of these a differential quantity negative with respect to the former is obtained :
$dv = -gdx + Rds$.
Which equation is thus to be integrated, so that on putting *x* = 0 makes *v* = *b*, if indeed the body has acquired a certain speed at *B* to which the height *b* corresponds. But the force pressing along *MN*, that the curve sustains, is equal to $\frac{gdy}{ds} - \frac{2vdxddy}{ds^3}$.

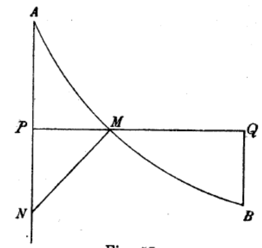

Fig. 57.

Corollary 6.

471. If the medium is uniform, and the exponent of this is *k*, the resistance is now proportional to some function of *v*, which is *V*, and such a function *K* of *k* is taken, as *V* is of *v*; the resistance is given by $R = \frac{V}{K}$ and thus this equation is obtained :

$$dv = -gdx + \frac{Vds}{K}$$

with the axis *BQ* taken.

Scholium 1.

472. I have given this formula twice for the increment of the speed for the two axes *AP* and *BQ*, since that is soon to be used in what follows. [p. 237] Clearly when the descent is made from the fixed point *A*, we use the first formula by taking *AP* for the axis. But if on the same curve several descents as far as the fixed point *B* are to be considered, as usually comes about in oscillatory motion, we use the second formula, in which *BQ* is had for the axis.

Scholium 2.

473. Since the formula, from which the motion of the body on the given line must be determined, has thus been prepared, in order that a few indeterminate cases can be separated from each other, often nothing can be concluded from that about the motion considered. On account of which it is agreed to pursue only these cases, for which the equation

$$dv = \pm gdx \mp \frac{Vds}{K}$$

can be either separated or integrated. Moreover all these cases can be reduced to three general cases. The first is, when the line on which the body is moving is straight ; for then on account of $ds = ndx$ the equation is changed into this :

$$\frac{\pm Kdv}{gK - nV} = dx,$$

in which the indeterminates are separated from each other. The second case is, when *V* is present as a first power of *v*; for then the equation admits to integration. The third case is when *v* has thus been prepared in the equation, that in the equation *v* and *x* everywhere constitute a number of the same power ; for then by the rule noted by Bernoulli the indeterminates can be separated from each other. [*Concerning the integration of differential equations, where an example of a method of integration is presented without the previous separation of the indeterminates.* Comment. acad. sc. Petrop. 1 (1726), 1728, p. 168; *Opera Omnia* Tom. 3, Lausannae et Geverae 1742, p. 108.] Moreover this comes about, if *Vds* is a single power of *v* and *x*. Besides these there are two other cases that permit integration, but these are not relevant here. [p. 238] The first is, if the resistance vanishes, which case has indeed been set out sufficiently in the previous chapter. The other case is, if the force g vanishes; concerning which there is no need that we treat this, since the motion on any given line agrees with the motion on a straight line, about which enough has been said in the previous book. Besides also the equation can be separated in many more cases if it is the case that $V = v^2$, clearly whenever the equation for the curve has thus been prepared, so that the equation can be reduced to the case that formerly Com. Riccati proposed. Now generally it is also possible to show the speed by a series in this case and for a finite expression to be defined, as I have given a general solution of the Riccati equation. [E31 in this series of translations] Therefore as the solution of a problem may be required, in addition to these three cases set out, immediately afterwards we explain this case also, in which the resistance is proportional to the fourth power of the speed.

Scholium 3.

474. Since this tract on the motion in a resisting medium is by all accounts difficult and complex, we will not apply ourselves to many hypotheses of the forces acting, as we did in the previous chapter, but the force acting shall be uniform and directed downwards and we do not elicit many centripetal forces to be acting. And when the force acting is put uniform, [p. 239] the resistance of the medium also can be put to agree with that ; for a fluid, that generates resistance, diminishes the force of gravity itself, and if that was not uniform, the absolute force cannot be put to be uniform in a straight line. Then also for that reason we can assume that the curve in which the body is moving lies in the same plane, by which we are able to remove many useless difficulties.

PROPOSITION 54.
Problem.

475. *If the body is always acted on by a uniform force g downwards in a medium with some resistance, to determine the motion of the ascending body on the given curve AM* (Fig.58) *and the force pressing on the curve sustained at individual points M.*

Solution.

In the vertical *AP* place the abscissa $AP = x$, $PM = y$ and $AM = s$, and let the height corresponding to the speed of the body at *A* be equal to *b* and at *M* the corresponding height is *v*, and the resistance at *M* is equal to *R*. Therefore it is the case, while the body rises, so the force acting *g* as well as the resistance *R* to be contrary to the motion. On this account likewise as in the previous proposition,

$$dv = -gdx - Rds.$$

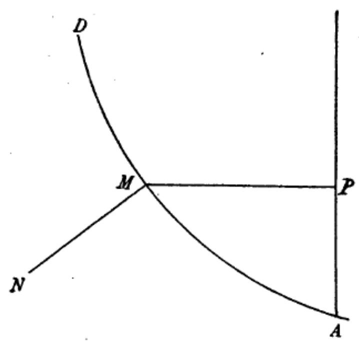

Fig. 58.

From which equation *v* is thus to be determined, so that on putting $x = 0$ makes $v = b$. Then with the resistance not present in the pressing force experienced by the curve, as above the total pressing force [which we would call the normal reaction now], that the curve sustains at *M* along the direction of the normal *MN*, [p. 240],

$$= \frac{gdy}{ds} - \frac{2vdxddy}{ds^3}$$

with *dx* put constant; where $\frac{gdy}{ds}$ the normal force and $-\frac{2vdxddy}{ds^3}$ the centrifugal force, both placed along the direction *MN*. Q.E.I.

Corollary 1.

476. Hence in the ascent of the body on any curve, the speed of the body is continually made less and the point *D* is reached, at which the speed of the ascent of the body vanishes, if in the equation :

$$dv = -gdx - Rds$$

after integration, putting $v = 0$.

Corollary 2.

477. If the body descending on the curve *DMA* should have this [hypothetical] equation

$$dv = -gdx + Rds \ \ (470),$$

from which it is understood that the ascent is not similar to the descent, as in a vacuum. But if [also] the resistance were to become negative or accelerating, then the ascent would be similar to the descent. Whereby the descent in the medium with resistance agrees just as much in the resisting medium with the ascent, and in turn with the acceleration.

Corollary 3.

478. Since the equation for this ascent yet differs from the equation for the descent, since the value of the resistance R is put negative, it is understood from the same cases, [p. 241] in which the equation for the descent are to be separated or integrable, from which the equation for the ascent too can be treated in the same way.

Corollary 4.

479. If we set $R = \frac{V}{K}$, then this equation is found for the ascent on the curve AM :

$$dv = -gdx - \frac{Vds}{K}.$$

But for the descent there is had : $dv = -gdx + \frac{Vds}{K}$.

Whereby if the other equation is to be integrated, likewise also the integral of this equation is had only on putting $-K$ in place of K. [On rising, both dx and ds are considered as positive, and thus the height corresponding to the speed diminishes; however, on descending, dx is negative and thus $-dx$ gives a positive contribution to the height and speed; however, energy dissipation means that the resistance term is negative. Thus, Euler's musings are here more connected with solving an equation than correctly handling the physical situation.]

Scholium.

480. Therefore following the three cases mentioned above, in which the equation found can be either separated or integrated, so that we can handle the descent as well as the ascent, clearly if the curve is given upon which the motion can be performed. Moreover then, from the given force acting, we can investigate the curve for resistance and the force acting on it. Thirdly if the motion should have a certain proposed property, we determine the curve which it satisfies according to the hypothesis of resistance it satisfies. Besides other problems follow, in which of these four quantities – resistance, motion, force pressing, and the curve – two are given, and the remaining two are required. Then we also have indeterminate problems, for which all the curves are required, upon which the descending body either acquires the same speed or completes these in the same time. [p. 242] Then the doctrine of brachistochrone lines follows, and finally the chapter ends with a treatment of oscillatory motion.

PROPOSITION 55.

Problem.

481. *In a medium with some kind of uniform resistance and under the hypothesis of uniform gravity g acting, to determine the motion of the body descending on the given straight line AMB* (Fig.59) *inclined in some manner to the horizontal.*

Solution.

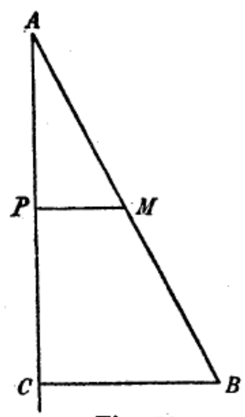
Fig. 59.

With $AP = x$ then $AM = s = nx$; $PM = y$ and since the resistance of the medium is uniform, then let the resistance be $R = \frac{V}{K}$. Therefore with the height corresponding to the speed at M equal to v, then

$$dv = -gdx - \frac{nVdx}{K} \text{ (465).}$$

Hence this equation becomes :

$$\frac{Kdv}{gK - nV} = dx,$$

in which equation the indeterminates are separable from each other in turn; hence the equation is :

$$x = \int \frac{Kdv}{gK - nV},$$

in which the integration is effected so that putting $x = 0$ makes $v = 0$, if indeed with the descent starts from A at rest. But if now it has an initial speed, this has to be introduced through the integration. The time to traverse the interval AM is equal to

$$\int \frac{ndx}{\sqrt{v}}.$$

Therefore with the value in terms of v put in place of dx, the time to traverse AM is equal to

$$\int \frac{nKdv}{(gK - nV)\sqrt{v}},$$

which integral is thus to be taken, so that it vanishes on putting $\sqrt{v}$ equal to the initial speed at A. Now the pressing force, that the line sustains at any point M is constant, surely equal to the normal force :

$$\frac{gdy}{ds} = \frac{g\sqrt{(n^2-1)}}{n},$$

since the centrifugal force vanishes on account of $ddy = 0$. Q.E.I. [p. 243]

Corollary 1.

482. Hence the speed of the body for some time is accelerating, as during that time $gK > nV$. But if once $gK = nV$, then the body is neither accelerated nor retarded. Now the speed of the body is diminished if at the start at A it should be the case that $nV > gK$.

Corollary 2.

483. Therefore if the body starts to descend from A at rest, then the motion is always increasing thus yet as $gK > nV$ always, clearly the final speed is that acquired at last from an infinite descent distance

Corollary 3.

484. Since the greater the angle *BAC*, the less is the terminal speed that the body is able to acquire. Now the maximum terminal speed by which the body can progress uniformly is acquired by descending along a vertical straight line *AC*.

Corollary 4.

485. If the resistance were as the 2*m* power of the index of the speed, then $V = v^m$ and $K = k^m$, hence this equation is obtained : [p. 244]

$$x = \int \frac{k^m dv}{gk^m - nv^m}$$

and the time to pass along AM = $\int \frac{Nk^m dv}{(gk^m - nv^m)\sqrt{v}}$.

Example 1.

486. The medium resists in the simple ratio of the speeds ; hence 2*m* = 1 and

$$dv = gdx - \frac{ndx\sqrt{v}}{\sqrt{k}}.$$

Hence this gives :

$$x = \int \frac{dv\sqrt{k}}{g\sqrt{k} - n\sqrt{v}} = -\frac{2\sqrt{kv}}{n} + \frac{2gk}{n^2} l \frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{v}}$$

or by the series :

$$x = \frac{2v}{2g} + \frac{2nv\sqrt{v}}{3g^2\sqrt{k}} + \frac{2n^2v^2}{4g^3k} + \frac{2n^3v^2\sqrt{v}}{5g^4k\sqrt{k}} + \text{etc.},$$

if indeed the descent starts from *A* at rest. Moreover the time to complete the interval *AM* is equal to :

$$\int \frac{ndv\sqrt{k}}{g\sqrt{kv} - nv} = 2\sqrt{k}\, l \frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{v}}.$$

Whereby if the time for *AM* is put equal to *t*, then this gives :

$$nx + 2\sqrt{kv} = \frac{gt\sqrt{k}}{n}$$

and in the series expansion:

$$t = \frac{2n\sqrt{v}}{g} + \frac{2n^2v}{2g^2\sqrt{k}} + \frac{2n^3v\sqrt{v}}{3g^3k} + \frac{2n^4v^2}{4g^4k\sqrt{k}} + \text{etc.}$$

Hence if the body in the descent along *AB* acquires the speed corresponding to the height *b*, from this there is found the height :

$$AC = \frac{-2\sqrt{bk}}{n} + \frac{2gk}{n^2} l \frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{b}}.$$

Example 2.

486. The medium resists in the square ratio of the speeds; hence $m = 1$ and

$$x = \int \frac{kdv}{gk-nv} = \frac{k}{n} l \frac{gk}{gk-nv},$$

if indeed the body starts its descent from *A* at rest. Whereby if *e* is the number of which the logarithm is one, then [p. 245]

$$e^{\frac{nx}{k}} = \frac{gk}{gk-nv} \text{ and } v = \frac{gk\left(e^{\frac{nx}{k}}-1\right)}{ne^{\frac{nx}{k}}} = \frac{gk}{n}\left(1-e^{\frac{-nx}{k}}\right).$$

On this account, if the body has a speed at *B* corresponding to the height *b*, then

$$AC = \frac{k}{n} l \frac{gk}{gk-nb}.$$

And if the body descends through an infinite distance, it has a speed corresponding to the height $\frac{gk}{n}$. Now the time to traverse the distance *AM* is equal to :

$$\int \frac{nkdv}{(gk-nv)\sqrt{v}} = \frac{\sqrt{nk}}{\sqrt{g}} l \frac{\sqrt{gk}+\sqrt{nv}}{\sqrt{gk}-\sqrt{nv}}.$$

So the distance *x* as the time can conveniently be expressed by a series, that we show generally for any value of the letter *m* in the following example.

Example 3.

488. Let the resistance be expressed by the $2m$ th power of the speed; then

$$dx = \frac{k^m dv}{gk^m - nv^m};$$

[recall that always Euler takes the speed as proportional to the square root of a height *v*.] which gives on conversion to a series :

$$dx = \frac{dv}{g} + \frac{nv^m dv}{g^2k^m} + \frac{n^2v^{2m}dv}{g^3k^{2m}} + \text{etc.}$$

From which there is found :

$$x = \frac{v}{g} + \frac{nv^{m+1}}{(m+1)g^2k^m} + \frac{n^2v^{2m+1}}{(2m+1)g^3k^{2m}} + \frac{n^3v^{3m+1}}{(3m+1)g^4k^{3m}} + \text{etc.}$$

But if the time to traverse $AM = t$, which is

$$dt = \frac{ndx}{\sqrt{v}},$$

then we have :

$$t = \frac{2n\sqrt{v}}{g} + \frac{2n^2v^m\sqrt{v}}{(2m+1)g^2k^m} + \frac{2n^3v^{2m}\sqrt{v}}{(4m+1)g^3k^{2m}} + \frac{2n^4v^{3m}\sqrt{v}}{(6m+1)g^4k^{3m}} + \text{etc.}$$

[p. 246]

PROPOSITION 56.

Problem.

489. *A medium resists in some multiple ratio of the speeds, and the point A is given* (Fig.60), *from which an infinite number of straight lines AM can be drawn; to determine the curve CMD of this kind, so that a body descending along any line AM has the same speed at M.*

Solution.

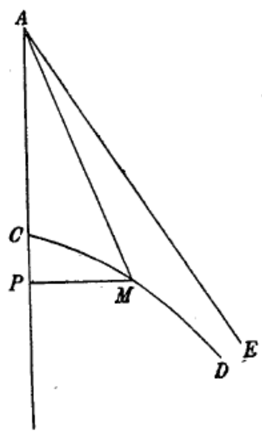

Fig. 60.

Let $2m$ be the exponent of the power of the speed, to which the resistance is proportional, and calling $AP = x$ and $AM = z$ and putting $z = nx$. Let the height corresponding to the speed at M be equal to v, which must be constant, namely equal to b. Hence we have:

$$dx = \frac{k^m dv}{gk^m - nv^m}$$

(485) with k detonating as above the exponent of the resistance and g the force acting downwards. Hence in finding the nature of the curve CM it is necessary to integrate the equation: $dx = \frac{k^m dv}{gk^m - nv^m}$ thus so that on putting v = 0 also makes $x = 0$, moreover then b is to be put in place of v and $\frac{z}{x}$ in place of n, and in this way we obtain the equation between x and z expressing the nature of he curve. Moreover we have integrated the above equation proposed by a series (488), thus on putting b in place of v we have :

$$x = \frac{b}{g} + \frac{nb^{m+1}}{(m+1)g^2k^m} + \frac{n^2b^{2m+1}}{(2m+1)g^3k^{2m}} + \text{etc.}$$

Put q^m in place of n and it is multiplied everywhere by q; with which accomplished we have :

$$qx = \frac{bq}{g} + \frac{b^{m+1}q^{m+1}}{(m+1)g^2k^m} + \frac{b^{2m+1}q^{2m+1}}{(2m+1)g^3k^{2m}} + \text{etc.}$$

The differential is taken and it is divided by bdg; there is obtained : [p. 247]

$$\frac{qdx + xdq}{bdq} = \frac{1}{g} + \frac{b^m q^m}{g^2k^m} + \frac{b^{2m}q^{2m}}{g^3k^{2m}} + \text{etc.} = \frac{k^m}{gk^m - b^m q^m}$$

or

$$qdx + xdq = \frac{bk^m dq}{gk^m - b^m q^m}.$$

But since $q^m = n$ and $n = \frac{z}{x}$, put $\frac{z^{\frac{1}{m}}}{x^{\frac{1}{m}}}$ in place of q and there is produced :

$$z^{\frac{1-m}{m}} x^{\frac{m-1}{m}} dz + (m-1) z^{\frac{1}{m}} x^{\frac{-1}{m}} dx = \frac{bk^m z^{\frac{1-m}{m}} x^{\frac{m-1}{m}} dz - bk^m z^{\frac{1}{m}} x^{\frac{-1}{m}} dx}{gk^m x - b^m z}.$$

Which multiplied by $x^{\frac{1}{m}} z^{\frac{m-1}{m}}$ goes to this :

$$xdz + (m-1)zdx = \frac{bk^m xdz - bk^m zdx}{gk^m x - b^m z}.$$

Moreover the construction of the curve easily follows from the equation :

$$qx = \int \frac{bk^m dq}{gk^m - b^m q^m}.$$

Moreover above we had the series equal to *qx*, from which it is apparent, if we put $q^m = \frac{gk^m}{b^m}$, then $\frac{gx}{b}$ is to be equal to the harmonic series [p. 248]

$$1 + \frac{1}{m+1} + \frac{1}{2m+1} + \text{etc.}$$

and thus *x* is to be infinite. If therefore x is infinite, then $q^m = \frac{z}{x} = \frac{gk^m}{b^m}$, from which it is evident that the straight line *AE* is an asymptote to the curve, and the cosine of the angle *CAE* to be $\frac{b^m}{gk^m}$. But the distance of the vertex of the curve *C* is at a distance *AC* from the point *A*, and is equal to this series :

$$\frac{b}{g} + \frac{b^{m+1}}{(m+1)g^2 k^m} + \frac{b^{2m+1}}{(2m+1)g^3 k^{2m}} + \text{etc.}$$

Moreover by necessity, it must be that $b^m < gk^m$; for otherwise the vertex C stands at an infinite distance from the point A. Q.E.I.

Corollary 1.

490. If the applied line *PM* is called *y*, then

$$z = \sqrt{(x^2 + y^2)} \text{ and } dz = \frac{xdx + ydy}{\sqrt{(x^2 + y^2)}};$$

with which values substituted in the equation, we have :

$$xydy + mx^2 dx + (m-1)y^2 dx = \frac{bk^m y(xdy - ydx)}{gk^m x - b^m \sqrt{(x^2 + y^2)}}.$$

Corollary 2.

491. $y = px$ is placed in this equation; and the equation is changed into this :

$$xpdp + mdx + mp^2dx = \frac{bk^m pdp}{gk^m - b^m\sqrt{(1+pp)}},$$

which equation multiplied by $(1+pp)^{\frac{1-2m}{2m}}$ is integrable; for it gives : [p. 249]

$$m(1+pp)^{\frac{1}{2m}}x = \int \frac{bk^m pdp(1+pp)^{\frac{1-2m}{2m}}}{gk^m - b^m\sqrt{(1+pp)}},$$

which expression can be effected by quadrature.

Corollary 3.

492. If the resistance vanishes and the body is moving in a vacuum, then k becomes infinite; and from the series given above there is found : $qx = \frac{bq}{g}$ or $x = \frac{b}{g}$, thus the line CM is known to become horizontal.

Scholium 1.

493. Moreover since from the general equation little can be concluded about the nature of the curve,, we will pursue this inquiry further with specific examples. Moreover we assume such examples, in which the formula $\frac{bk^m dq}{gk^m - bmq^m}$ even admits to integration by logarithms, from which we arrive at finite expressions, from which the nature of the curve is easily seen . [From this paragraph it can be concluded that Cotes' formula was not known by Euler in the year 1736. P. St.]

Example 1.

494. Therefore let the resistance be proportional to the speeds ; then $m = \frac{1}{2}$. Putting AC = a; since the speed, that the body acquires on falling through AC, must correspond to the height b, then [p. 250]

$$a = -2\sqrt{bk} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k} - \sqrt{b}}$$

(486). Now this gives

$$\sqrt{q} = \frac{z}{x} \text{ or } q = \frac{z^2}{x^2}$$

and

$$qx = \int \frac{bdq\sqrt{k}}{g\sqrt{k} - \sqrt{bq}},$$

which integral is thus to be taken, in order that on putting $z = x$ or $q = 1$ makes $x = a$ or to the assigned value of this. Hence this integral becomes :

$$qx = -2\sqrt{bkq} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k} - \sqrt{bq}},$$

which equation with $\frac{z^2}{x^2}$ substituted in place of q goes over into this equation :

$$z^2 = -2z\sqrt{bk} + 2gkx\, l\frac{gx\sqrt{k}}{gx\sqrt{k} - z\sqrt{b}}.$$

If $q = 1$, then this becomes :

$$x = a = -2\sqrt{bk} + 2gk\, l\frac{g\sqrt{k}}{g\sqrt{k} - \sqrt{b}}.$$

Moreover on putting $q = 1 + dq$; there is obtained :

$$dx + xdq = -dq\sqrt{bk} + \frac{gkdq\sqrt{b}}{g\sqrt{k} - \sqrt{b}} = \frac{bdq\sqrt{k}}{g\sqrt{k} - \sqrt{b}}.$$

But since $\frac{1}{\sqrt{q}}$ is the cosine of the angle *MAC*, then $\sqrt{dq}$ is equal to the sine of this angle. On which account the increment of x is infinitely less than the increment of the angle *MAC* with *MA* incident on *CA*, from which it follows that the tangent of the curve at *C* is horizontal; the tangent of this curve at infinity or the asymptote is *AE*, with the cosine of the angle *EAC* becoming $\frac{\sqrt{b}}{g\sqrt{k}}$. Moreover this curve on the other side of the vertical *AC* has an arc similar and equal to *CMD*.

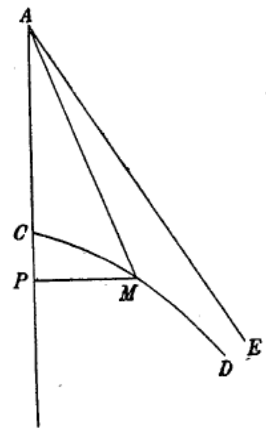

Fig. 60.

Scholium 2.

495. Indeed generally it is also possible to show that the tangent of the curve at C must be horizontal. For on putting $n = 1$ in the series expressing x there is obtained :

$$AC = \frac{b}{g} + \frac{b^{m+1}}{(m+1)g^2k^m} + \text{etc.}$$

[p. 251] n in increased by the element dn; and there is obtained the momentary increment of *AC* equal to

$$\frac{b^{m+1}dn}{(m+1)g^2k^m} + \frac{2b^{2m+1}ndn}{(2m+1)g^3k^{2m}} + \text{etc.}$$

Now let $\frac{1}{n}$ be the cosine of the angle *MAC* and thus the sine is equal to $\frac{\sqrt{(n^2-1)}}{n} = \sqrt{2}dn$ on putting $1 + dn$ in place of n. On account of which the increment of *AC* is infinitely less than the increment of the angle; and thus *AC* is normal to the curve *DMC*.

Example 2.

496. Let the resistance be proportional to the square of the speed, then $m = 1$ and on putting $AC = a$; the equation becomes :

$$a = kl\frac{gk}{gk - b} \text{ and } b = gk\left(1 - e^{\frac{-a}{k}}\right).$$

Now it is then the case that $q = n = \frac{z}{x}$

and

$$qx = z = \int \frac{bkdq}{gk - bq} = k\,l\,\frac{gk}{gk - bq} = k\,l\,\frac{gkx}{gkx - bz}.$$

Hence there is obtained :

$$e^{\frac{z}{k}} = \frac{gkx}{gkx - bz},.$$

and hence it follows that :

$$x = \frac{e^{\frac{z}{k}}bz}{gk\left(e^{\frac{z}{k}} - 1\right)} = \frac{e^{\frac{z}{k}}z\left(e^{\frac{a}{k}} - 1\right)}{e^{\frac{a}{k}}\left(e^{\frac{z}{k}} - 1\right)}$$

on putting this value of *a* in place of *b*. Moreover this equation is used to construct the curve most conveniently :

$$z = k\,l\frac{gk}{gk - bq},$$

in which *z* is *AM* and *q* is the secant of the angle *MAC*.

Scholium 3.

497. In the solution of the problem in finding the equation of the curve *CMD* we have been using a certain sum of that series ; [p. 252] now it is possible to elicit the same equation without the series in the following manner. Since

$$x = \int \frac{k^m dv}{gk^m - nv^m},$$

this equation expresses the nature of the curve sought, if we put $v = b$ after the integration and $\frac{z}{x}$ in place of *n*. On account of which

$$\int \frac{k^m dv}{gk^m - nv^m}$$

is differentiated, on putting not only *v*, but also *n* to be variable, and then *v* is put equal to the constant *b* and $\frac{z}{x}$ in place of *n*, the differential equation is obtained for the curve sought. So that this can be effected, I put $n = \frac{1}{p^m}$, when there is produced

$$x = \int \frac{k^m p^m dv}{gk^m p^m - v^m}.$$

For the sake of brevity we put

$$\frac{k^m p^m}{gk^m p^m - v^m} = P$$

and the differential equation is equal to this :

$$dx = Pdv + Qdp,$$

if *p* is also taken as a variable. But since *P* is a function of zero dimensions of *v* and *p*, then [see E45 in this series of translations]

$$x = Pv + Qp$$

and thus

$$Q = \frac{x}{p} - \frac{Pv}{p}.$$

Therefore with this value substituted in place of *Q* there is produced :

$$pdx = \frac{k^m p^{m+1} dv - k^m p^m v dp}{gk^m p^m - v^m} + xdp.$$

Now $n^{\frac{-1}{m}}$ is restored in place of *p* and there arises :

$$mndx + xdn = \frac{mk^m ndv + k^m vdn}{gk^m - nv^m},$$

in which equation *n* has been taken equally variable with both *v* et *x*. Now put $v = b$, $dv = 0$ and $n = \frac{z}{x}$ and this equation is had : [p. 253]

$$xdz + (m-1)zdx = \frac{bk^m(xdz - zdx)}{gk^m x - b^m z},$$

which agrees with the equation found above.

PROPOSITION 57.

Problem.

498. *If the resistance were in some multiple ratio of the speed, to find the curve* (Fig.61), *which has this property, that a body descending on any chord AM arrives at M from A in a given constant time.*

Solution.

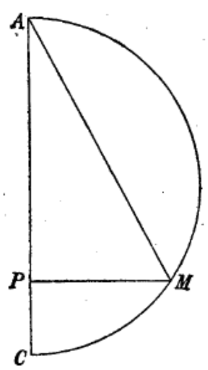

Fig. 61.

With the vertical *AC* drawn, put $AP = x$, $AM = z$ and let $n = \frac{z}{x}$. With the speed at *M* corresponding to the height *v*, and the resistance equal to $\frac{v^m}{k^m}$ let the time be *t* in which the body descends along *AM* , which must be a constant quantity. From the preceding we have

$$x = \int \frac{k^m dv}{gk^m - nv^m} \text{ and } t = \int \frac{nk^m dv}{(gk^m - nv^m)\sqrt{v}}$$

(485). It is needed in finding the nature of the curve *AMC* that either equation is to be integrated, if it is possible, and the value of *v* from the one equation is substituted into the other equation and then $\frac{z}{x}$ is written in place of *n*, with which accomplished an equation is had between *x* and *z* expressing the nature of the curve. [p. 254] But if the integration cannot be performed conveniently, either equation is to be differentiated on putting *n* to be a variable, and afterwards on putting $dt = 0$, from the two equations found *v* must be eliminated, from which an equation is produced containing only n and x, which as $n = \frac{z}{x}$ shows the nature of the curve sought. Towards doing this, putting $n = \frac{1}{p^m}$, from which we have :

$$x = \int \frac{k^m p^m dv}{gk^m p^m - v^m} \text{ and } t = \int \frac{k^m dv}{(gk^m p^m - v^m)\sqrt{v}}.$$

Of the former of these equations, by taking p to be a variable also, the differential has now been found :

$$pdx - xdp = \frac{k^m p^{m+1} dv - k^m p^m v dp}{gk^m p^m - v^m}$$

(497). To the other equation to be differentiated I put :

$$\frac{k^m}{(gk^m p^m - v^m)\sqrt{v}} = P$$

and it becomes :

$$dt = Pdv + Qdp.$$

Moreover, since P is a function of v and p of dimensions $-m - \frac{1}{2}$, then

$$(\tfrac{1}{2} - m)t = Pv + Qp$$

and hence

$$Q = \frac{(1-2m)t}{2p} - \frac{Pv}{p}.$$

With which value substituted in place of Q there is produced :

$$pdt = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}} + \frac{(1-2m)tdp}{2}.$$

Now let $t = 2\sqrt{c}$ and $dt = 0$; we have :

$$(2m-1)dp\sqrt{c} = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}}.$$

From these two equations dv is eliminated and there comes about :

$$pdx - xdp = (2m-1)p^m dp\sqrt{cv}$$

or [p. 255]

$$\sqrt{v} = \frac{pdx - xdp}{(2m-1)p^m dp\sqrt{c}} = \frac{dr}{(2m-1)p^{m-2}dp\sqrt{c}}$$

on putting $x = rp$. This value is substituted in place of v in the equation

$$(2m-1)dp\sqrt{c} = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}}$$

or in this

$$\frac{dr}{p^{m-2}} = \frac{k^m(pdv - vdp)}{gk^m p^m - v^m}.$$

Indeed in the case is which $m = \frac{1}{2}$ or the resistance is proportional to the speeds, then the equation becomes $pdv = vdp$ or $v = \alpha p$ and

$$p = \frac{x}{a} = \frac{1}{n^2} = \frac{xx}{zz},$$

thus it follows to be $zz = ax$; as in this hypothesis of the resistance the curve *AMC* is a circle everywhere as in a vacuum. According to other hypotheses of the resistance, unless the equation for either variable can be integrated, on eliminating *v* a difference of the differentials [*i. e.* a second – order differential equation] equation is had expressing the nature of the curve between *z* and *x*. Q.E.I.

Corollary 1.

499. If we put $v = up$, then:

$$pdv - vdp = ppdu.$$

And hence this arises

$$\sqrt{u} = \frac{dr}{(2m-1)p^{m-\frac{3}{2}}dp\sqrt{c}},$$

which value substituted in the equation

$$dr = \frac{k^m du}{gk^m - u^m}$$

gives an equation between *p* et *r*, from which an equation is formed between *x* and *z*. [p. 256]

Corollary 2.

500. Hence in the medium, which offers resistance in the simple ration of the speeds, it is apparent that the curve *AMC* is a circle. And thus according to this hypothesis of the resistance the times of the descents along particular chords drawn from the point *A* are equal to each other.

Example 1.

501. Let the resistance be proportional to the square of the speeds ; then $m = 1$ and

$$x = \frac{k}{n} l \frac{gk}{gk - nv}$$

or

$$v = \frac{gk}{n}\left(1 - e^{\frac{-nx}{k}}\right).$$

Now besides the time

$$t = 2\sqrt{c} = \frac{\sqrt{nk}}{\sqrt{g}} l \frac{\sqrt{gk} + \sqrt{nv}}{\sqrt{gk} - \sqrt{nv}}$$

or

$$e^{\frac{2\sqrt{gc}}{\sqrt{nk}}} = \frac{\sqrt{gk}+\sqrt{nv}}{\sqrt{gk}-\sqrt{nv}},$$

thus this becomes :

$$v = \frac{gk\left(e^{\frac{2\sqrt{gc}}{\sqrt{nk}}}-1\right)^2}{n\left(e^{\frac{2\sqrt{gc}}{\sqrt{nk}}}+1\right)^2}.$$

Hence on eliminating *v* and with $\frac{z}{x}$ put in place of *n* there is obtained:

$$\frac{e^{\frac{z}{k}}-1}{e^{\frac{z}{k}}} = \frac{\left(e^{\frac{2\sqrt{gcx}}{\sqrt{kz}}}-1\right)^2}{\left(e^{\frac{2\sqrt{gcx}}{\sqrt{kz}}}+1\right)^2}$$

or

$$\frac{2\sqrt{gcx}}{\sqrt{kz}} = l\,\frac{1+\sqrt{\left(1-e^{\frac{-z}{k}}\right)}}{1-\sqrt{\left(1-e^{\frac{-z}{k}}\right)}}.$$

In this case the curve is

$$AC = k\,l\,\frac{\left(e^{\frac{2\sqrt{gc}}{\sqrt{k}}}+1\right)^2}{4e^{\frac{2\sqrt{gc}}{\sqrt{k}}}} = 2k\,l\,\frac{e^{\frac{\sqrt{gc}}{\sqrt{k}}}+e^{\frac{-\sqrt{gc}}{\sqrt{k}}}}{2};$$

if therefore putting *AC* = *a*, it becomes [p. 257]

$$2e^{\frac{a}{2k}} = e^{\frac{\sqrt{gc}}{\sqrt{k}}} + e^{\frac{-\sqrt{gc}}{\sqrt{k}}},$$

thus it is

$$e^{\frac{\sqrt{gc}}{\sqrt{k}}} = e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}$$

or

$$\sqrt{\frac{gc}{k}} = l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}\right).$$

Therefore

$$\sqrt{x}\,l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}\right) = \sqrt{z}\,l\left(e^{\frac{z}{2k}} + \sqrt{\left(e^{\frac{z}{k}}-1\right)}\right)$$

is the equation for the curve *AMC*. If the resistance should be very small, then the quantity *k* is extremely large and thus :

$$e^{\frac{z}{2k}} + \sqrt{\left(e^{\frac{z}{k}}-1\right)} = 1 + \frac{z}{2k} + \sqrt{\left(\frac{z}{k}+\frac{z^2}{2k^2}\right)} = 1 + \sqrt{\frac{z}{k}} + \frac{z}{2k} + \frac{z\sqrt{z}}{4k\sqrt{k}}$$

and the logarithm of this is equal to :

$$\frac{\sqrt{z}}{\sqrt{k}} + \frac{z\sqrt{z}}{12k\sqrt{k}}.$$

In a similar way,

$$l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}} - 1\right)}\right) = \frac{\sqrt{a}}{\sqrt{k}} + \frac{a\sqrt{a}}{12k\sqrt{k}}.$$

And on this account this equation is obtained for the curve *AMC* :

$$\sqrt{ax} + \frac{a\sqrt{ax}}{12k} = z + \frac{z^2}{12k}$$

or

$$ax\left(1 + \frac{a}{6k} + \frac{a^2}{144k^2}\right) = z^2 + \frac{z^3}{6k} + \frac{z^4}{144k^2}.$$

Hence it is observed, if the resistance completely vanishes, or *k* is made infinitely great, the equation becomes $ax = z^2$ and thus the curve is the circle *AMC*. But if the medium were the rarest, then

$$ax(a + 6k) = 6kz^2 + z^3$$

and on differentiation,

$$adx(a + 6k) = 12kzdz + 3z^2dz.$$

If now this becomes $zdz = xdx$, then the applied line *PM* has a maximum value or it is the place where the tangent to the curve is vertical, clearly

$$a^2 + 6ak = 12kx + 3xz \text{ or } x = \frac{a^2 + 6ak}{12k + 3z},$$

thus the equation becomes :

$$\frac{(6ak + a^2)^2}{3} = 24k^2z^2 + 10kz^3 + z^4,$$

from which equation the value of *z* is approximately

$$\frac{a}{\sqrt{2}} + \frac{(4\sqrt{2} - 5)a^2}{48k}$$

and [p. 258]

$$x = \frac{a}{2} - \frac{(3\sqrt{2} - 4)a^2}{48k} \text{ and } PM = \frac{a}{2} + \frac{(2 - \sqrt{2})a^2}{24k}.$$

Therefore the curve is widest above the middle and is wider everywhere than the height AC.

Corollary 3.

502. Therefore if [another] straight line or curve is a tangent to this curve *AMC* at *M*, thus so that it is situated wholly outside the curve AMC, the body released from A arrives faster by descending along the chord AM than upon any other straight line drawn from A to that other curve.

Example 2.

503. Let *m* be a positive number and the resistance is very small; then *k* is a very large quantity and hence

$$\frac{k^m}{gk^m - nv^m} = \frac{1}{g} + \frac{nv^m}{g^2k^m}.$$

On account of which:

$$x = \frac{v}{g} + \frac{nv^{m+1}}{(m+1)g^2k^m} \text{ and } t = 2\sqrt{c} = \frac{2n\sqrt{v}}{g} + \frac{2n^2v^{m+\frac{1}{2}}}{(2m+1)g^2k^m}$$

and hence there is produced :

$$v = gx - \frac{ng^m x^{m+1}}{(m+1)k^m} \text{ and } \sqrt{v} = \sqrt{gx} - \frac{ng^{m-\frac{1}{2}}x^{m+\frac{1}{2}}}{2(m+1)k^m}$$

and

$$v^{m+\frac{1}{2}} = g^{m+\frac{1}{2}}x^{m+\frac{1}{2}} - \frac{n(2m+1)g^{2m-\frac{1}{2}}x^{2m+\frac{1}{2}}}{2(m+1)k^m}.$$

Moreover with these values substituted, this equation is produced : [p. 259]

$$\frac{\sqrt{gc}}{\sqrt{x}} = n + \frac{n^2g^{m-1}x^m}{(2m+1)(2m+2)k^m} - \frac{n^3g^{2m-2}x^{2m}}{(2m+2)k^{2m}}.$$

Since now $n = \frac{z}{x}$, this equation is obtained :

$$\sqrt{gcx} = z + \frac{g^{m-1}x^{m-1}z^2}{(2m+1)(2m+2)k^m} - \frac{g^{2m-2}x^{2m-2}z^3}{(2m+2)k^{2m}}$$

or

$$gcx = z^2 + \frac{g^{m-1}x^{m-1}z^3}{(m+1)(2m+1)k^m},$$

if the medium is very rare. Thus it is apparent, if the resistance vanishes completely, for the equation be become $z^2 = gcx$ or the curve *AMC* is a circle of diameter *AC*.

Scholium.

504. Therefore if the curve *AMC* is known and any line is given, it is possible to determine the straight line *AM*, upon which the body descending from *A* arrives at the given line the quickest. Cleary the curve *AMC* has to be constructed, which the given line touches at some point *M*; and the right line *AM* is that line, upon which the body by descending from A quickest arrives at the given line. And in a like manner in the preceding problem, if the right line or curve touches the curve *CMD* (Fig. 60) at *M*, the body by descending from *A* along *AM* as far as to the tangent line to the curve *CMD* acquires a greater speed than by descending along another right line drawn from *A* to that line. Therefore from these it is possible to solve problems, in which the right line is required drawn from *A* to the given line, upon which the body on descending either acquires the maximum speed or arrives the quickest at that line. [p. 260] On account of which we will not tarry longer with these problems, but we progress to the consideration of ascents upon given right lines.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 233]

PROPOSITIO 53.

Problema.

465. *Si corpus sollicitetur deorsum a potentia uniformi g in medio quocunque resistente, determinare motum corporis descendis super data curva AM* (Fig.57) *et pressionem, quam curva in singulus punctis sustinet.*

Solutio.

Ponatur in verticali *AP* abscissa *AP* = *x*, applicata *PM* = *y* et arcus *AM* = *s* sitque altitudo celeritati corporis in *M* debita = *v* et resistentia in *M* = *R*. Manifestum iam est ex capite praecedente [(93)], si nulla esset resistantia, fore

$$dv = gdx.$$

Resistentia vero minuit hoc celeritatis incrementum et aequipollet vi tangentiali = R; eiusque solius effectus in hoc consisteret, ut foret

$$dv = -Rds.$$

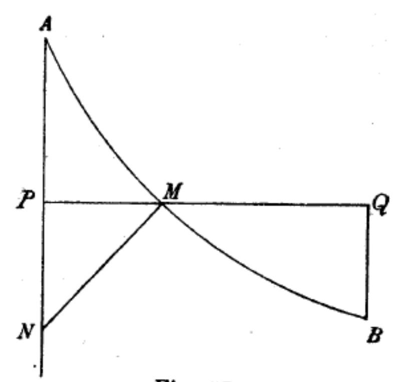

Fig. 57.

Quamobrem si et potentia sollicitans *g* et resistantia ambae simul in corpus agunt, erit

$$dv = gdx - Rds,$$

ex qua aequatione celeritas corporis in quovis puncto *M* est eruenda. [p. 234]
Atque si corpus in *A* ex quiete descendat, integratio ita est instituenda, ut facto *x* = 0 prodeat quoque *v* = 0. Verum si data cum celeritate corpus in *A* descensum inceperit, in integratione effici debet, ut positio *x* = 0 fiat *v* aequalis altitudini debitae illi celeritati initiali. Cum autem inventa fuerit celeritas corporis, habebitur simul tempus, quo quivis arcus *AM* absolvitur, sumendo $\int \frac{ds}{\sqrt{v}}$. Quod ad pressionem, quam curva in *M* sustinet, spectat, curva in *M* duplici vi premitur, vi centrifuga scilet et vi normali. Ponamus curvam esse convexam deorsum et elementum dx constans; erit longitudo radii osculi in contrariam partem normalis *MN* directi

$$= \frac{ds^3}{dxddy},$$

unde vis centrifuga erit

$$= \frac{2vdxddy}{ds^3},$$

qua curva secundum directionem *MN* premitur. Secundum eandem vero directionem curva premetur a vi normali, quae est

$$= \frac{gdy}{ds};$$

vis normalis enim a potentia absoluta *g* tantum oritur, quia directio vis resistentiae est in tangente sita ideoque nullum vim normalem generat. Consequenter tota vis, qua curva in *M* secundum directionem normalis *MN* premitur, est

$$= \frac{gdy}{ds} + \frac{2vdxddy}{ds^3}.$$

Q.E.I.

Corollarium 1.

466. Expressio ergo vis curvam prementis congruit cum ea, quam in vacuo invenimus (83). Neque tamen curva in medio resistente eadem vi premitur qua in vacuo ob celeritatem, a qua vis centrifuga pendet, quae a medio resistente variatur. [p. 235]

Corollarium 2.

467. In isto descensu corpus non ut in vacuo maximum habet celeritatem in puncto *B*, in quo tangens est horizontalis, sed posito $dv = 0$ locus, in quo corpus maximum habet celeritatem, invenitur ex hac aequatione

$$gdx = Rds \text{ seu } \frac{dx}{ds} = \frac{R}{g}$$

in eo puncto, ubi sinus anguli, quem tangens curvae cum linea horizontali constituit, est ad sinum totum ut potentia absolutae *g* ad resistentiam *R* in eo loco.

Corollarium 3.

468. Celeritas corporis igitur augetur usque ad hoc punctum, in quo celeritas est maxima; ultra vero hoc punctum celeritas iterum decrescit, quia tum *Rds* excedit *gdx* et hanc ob rem fit *dv* negituvum.

Corollarium 4.

469. Si resistentia fuerit ut potestas quaecunque celeritatum, cuius exponens est 2*m*, et si medium resistens fuerit uniforme, cuius exponens sit *k*, ubi *k* est altitudo celeritati debita, quacum corpus movetur, resistentiam patitur vi gravitait aequalem; hoc ergo casu erit

$$R = \frac{v^m}{k^m}$$

atque ista habebitur aequatio ad motum definiendum [p. 236]

$$dv = gdx - \frac{v^m ds}{k^m}.$$

Corollarium 5.

470. Sin autem abscissae in axe *BQ* capiantur fueritque *BQ = x, QM = y* et *BM = s*, propter harum quantitatum differentialia negativa respectu priorum habebitur $dv = -gdx + Rds$. Quae aequatio ita est integranda, ut posito *x* = 0 fiat *v = b*, si quidem celeritas in *B*, quam corpus in hoc puncto obtinet, huic altitudiini fuerit debita. At pressio secundum *MN*, quam curva sustinet, est

$$= \frac{gdy}{ds} - \frac{2vdxddy}{ds^3}.$$

Corollarium 6.

471. Si medium fuerit uniforme, cuius exponens sit *k*, resistentia vero functioni cuicunque ipsius *v*, quae sit *V*, proportionalis, sumatur *K* talis functio ipsius *k*, qualis *V* est ipsius *v*; erit resistantia $R = \frac{V}{K}$ ideoque habebitur ista aequatio

$$dv = -gdx + \frac{Vds}{K}$$

sumto axe *BQ*.

Scholion 1.

472. Formulam hic duplicem incrementum celeritas exhibentem dedi pro duobus axibus *AP* et *BQ*, quia in sequentibus mox illa utemur. [p. 237] Scilicet quando descensus semper fit ex fixo puncto *A*, utemur priore formula *AP* pro axe sumente. At si in eadem curva plures descensus ad punctum fixum usque *B* sint considerandi, ut in motu oscillatorio usu venit, posteriore formula utemur, in qua *BQ* pro axe habetur.

Scholion 2.

473. Quia formula, ex qua motus corporis super data curva determinari debet, ita est comparata, ut indeterminatae paucis casibus a se invicem separari, saepe ex ea nihil, quod ad motum spectat, concludi licet. Quamobrem eos tantum casus evolvere convenit, quibus aequatio $dv = \pm gdx \mp \frac{Vds}{K}$ vel separari vel integrari potest. Hi autem casus omnino ad tres casus generales reducentur. Primus est, quando linea, super qua corpus movetur, est recta; tum enim ob $dx = ndx$ aequatio transit in hanc

$$\frac{\pm Kdv}{gK - nV} = dx,$$

in qua indeterminatae sunt a se invicem separatae. Secundus casus est, quando in *V* unicam tantum obtinet dimensionem *v*; tum enim aequatio integrationem admittit. Tertius casus est, quando tam *v* quam aequatio pro curva ita est comparata, ut in aequatione *v* et *x* ubique eundem dimensionum numerum constituant; tum enim per regulam notam Bernoullianam indeterminatae a se invicem possunt separari. [*De integrationibus aequationum differentialium, ubi traditue methodi alicuius specimen integrandi sine praevia separatione indeterminatarum*. Comment.acad.sc.Petrop. 1 (1726), 1728, p. 168; *Opera Omnia* Tom. 3, Lausannae et Geverae 1742, p. 108.] Hoc autem evenit, si in *Vds* unica fuerit dimensio ipsarum *v* et *x*. Praeter hos quidem casus essent duo alii integrationem admittentes, sed qui huc non pertinent. [p. 238] Primus est, si resistentia evanescit, qui vero casus in praecendente capite iam sufficienter est pertractatus. Alter casus est, si potentia sollicitans g evanescit; de quo autem non est opus, ut agamus, quia motus super quacunque linea congruit cum motu super linea recta, de quo in praecedente libro iam satis est dictum. Praeterea quoque multis casibus aequatio separationem admittit, si fuerit $V = v^2$, quoties scilicet aequatio pro curva ita est comparata, ut aequatio ad casum aequationis, quam quondam Com. Riccati proposuit, potest reduci. Generaliter vero etiam potest in hoc casu celeritas per seriem exhiberi atque finita expressione definiri, quemadmodum ego generalem aequationem Riccatianae dedi constructionem. [E31] Quoties igitur natura rei requiret, praeter tres casus expositos subinde quoque hunc casum, in quo resistentia biquadrato celeritatis est proportionalis, evolvemus.

Scholion 3.

474. Quia haec de motu in medio resistente tractio per se est difficilis et intricata, non ad plures vis sollicantis hypothesis, ut capite praecedents fecimus, eam accommodabimus, sed nobis perpetuo potentia sollicitans erit uniformis et deorsum directa neque de viribus centripetis multum erimus solliciti. Atque cum potentia sollicitans ponatur uniformis, [p. 239] medium resistens quoque tale poni conveniet; fluidum enim, quod resistentiam generat, ipsam corporis gravitatem minuit, et si id non esset uniforme, potentia absoluta non recte uniformi poneretur. Deinde etiam propter eandem rationem curvam, in qua corpus movetur, totam in eodem plano positam assumemus, quo multas difficultates nullam utilitatem afferentes removeamus.

PROPOSITIO 54.

Problema.

475. *Si corpus perpetuo sollicitetur deorsum a potentia uniformi g in medio quocunque resistente, determiare motum corporis super data curva AM* (Fig.58) *ascendentis et pressionem, quam curva in singulus punctis M sustinet.*

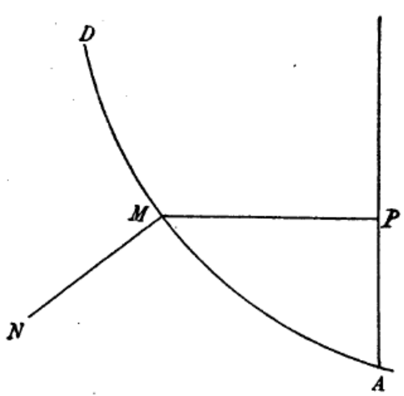

Fig. 58.

Solutio.

In verticali *AP* posita abscissa *AP* = *x*, *PM* = *y* et *AM* = *s* sit atitudo celeritati corporis in *A* debita = *b* et in *M* debita = *v* atque resistentia in *M* = *R*. Erit igitur, dum corpus ascendit, tam potentia sollicitans g quam resistentia R motui contraria. Hanc ob rem erit simili modo quo in praecendente propositione

$$dv = -gdx - Rds.$$

Ex qua aequatione v ita debet determinari, ut facto *x* = 0 fiat *v* = *b*. Deinde cum resistentia in pressionem, quam curva patitur, non ingrediatur, erit ut supra pressio tota, [p. 240] quam curva in *M* secundum directionem normalis *MN* sustinet,

$$= \frac{gdy}{ds} - \frac{2vdxddy}{ds^3}$$

posito dx constante; ubi $\frac{gdy}{ds}$ denotat vim normalem et $-\frac{2vdxddy}{ds^3}$ vim centrifugam, utramque iuxta *MN* directam. Q.E.I.

Corollarium 1.

476. In ascensu corporis ergo super quacunque curva celeritas corporis perpetuo imminuitur atque punctum curvae *D* reperietur, in quo corporis ascendentis celeritas evanescit, si in aequatione

$$dv = -gdx - Rds$$

post integrationem ponatur $v = 0$.

Corollarium 2.

477. Si corpus super curva *DMA* descenderet, haberetur ista aequatio

$$dv = -gdx + Rds \ \ (470),$$

ex qua intelligitur ascensum non esse similem descendsui ut in vacuo. Sed si resistentia fieret negativa seu accelerans, tum ascensus similis foret descensui. Quare descensus in medio resistente congruet cum ascensu in medio tantundem accelerante et vicissim.

Corollarium 3.

478. Quoniam aequatio pro ascensu hoc tantum differt ab aequatione pro descensu, quod resistentia *R* valorem induat negativum, intelligitur iisdem casibus, [p. 241] quibus aequatio pro descensu separari vel integrari potest, iisdem quoque aequationem pro asensu simili modo tractari posse.

Corollarium 4.

479. Si fuerit $R = \frac{V}{K}$, erit pro ascensu super curva *AM* haec aequatio

$$dv = -gdx - \frac{Vds}{K}.$$

At pro descensu habetur

$$dv = -gdx + \frac{Vds}{K}.$$

Quare si illa aequatio poterit integrari, simul quoque aequationis habebitur integrale ponendo tantum –K loco K.

Scholion.

480. Secundum tres igitur casus supra memoratos, quibus aequatio inventa vel separari vel integrari potest, tam descensum quam ascensum pertractabimus, si scilicet detur curva, super qua motus fieri ponitur. Deinde autem ex datis potentia sollicitante, resistantia et pressione curvam investigabimus. Tertio si motus quaedam proprietas fuerit proposita, curvam determinabimus, quae in data resistentiae hypothesi satisfaciat. Praeterea sequentur alia problemata, in quibus harum quatuor rerum – resistentiae, motus, pressiones et curvae – duae dantur, reliquae duae requiruntur. Habebimuis deinceps quoque problemata indeterminata, quibus omnes curvae requiruntur, super quibus corpus descendens vel eandem celeritatem acquirit vel eas eodem tempore absolvit. [p. 242]

Tum sequetur doctrina de lineis brachystochronis atque tandem caput concludet de motu oscillatorio tractatio.

PROPOSITIO 55.

Problema.

481. *In medio resistente uniformi quocunque et hypothesi gravitatis uniformi g, determiare motum corporis descendentis super linea recta AMB* (Fig.59) *ad horizontalem utcunque inclinata.*

Solutio.

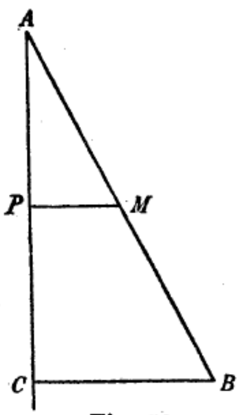

Fig. 59.

Posita $AP = x$ erit $AM = s = nx$; $PM = y$ et quia resistentia medium resistens est uniforma, erit resistentia $R = \frac{V}{K}$. Posita ergo altitudine celerati in M debita = v erit

$$dv = -gdx - \frac{nVdx}{K} \text{ (465).}$$

Unde fit

$$\frac{Kdv}{gK - nV} = dx,$$

in qua aequatione indeterminatae sunt a se invicem separatae; erit ergo

$$x = \int \frac{Kdv}{gK - nV},$$

in qua integratione efficiendum est, ut posito $x = 0$ fiat $v = 0$, si quidem descensu in A ex quiete incipiat. Sin vero habeat celeritatem initialem, haec per integrationem est introducenda. Tempus per spatium AM est

$$= \int \frac{ndx}{\sqrt{v}}.$$

Posito ergo loco dx eius valore in v habebitur tempus per $AM = \int \frac{nKdv}{(gK - nV)\sqrt{v}}$, quod integrale ita est sumendum, ut posita $\sqrt{v}$ = celeritati initiali in A evanescat. Pressio vero, quam linea in quovis puncto M sustinet, est constans, nempe aequalis vi normali

$$\frac{gdy}{ds} = \frac{g\sqrt{(n^2 - 1)}}{n},$$

quia vis centrifuga evanescit ob $ddy = 0$. Q.E.I. [p. 243]

Corollarium 1.

482. Celeritas corporis ergo tam diu acceleratur, quam diu est $gK > nV$. At si semel fuerit $gK = nV$, corpus neque accelerabitur neque retardabitur. Diminuetur vero corporis celeritas, si in initio A fuerit $nV > gK$.

Corollarium 2.

483. Si ergo corpus in A descensum a quiete incipiat, motus perpetuo crescet, ita tamen, ut semper sit $gK > nV$, quippe quae est ultima celeritas, quam descensu per infinitum spatium demum acquiret.

Corollarium 3.

484. Quo maior ergo est angulus *BAC*, eo minor est ultima, quam corpus acquirere potest, celeritas. Maximam vero celeritatem ultimam, qua aequabiliter progreditur, acquirit descensu super recta verticali *AC*.

Corollarium 4.

485. Si resisentia fuerit ut potestas indicis 2*m* celeritatum, erit $V = v^m$ et $K = k^m$, unde ista habebitur aequatio [p. 244]

$$x = \int \frac{k^m dv}{gk^m - nv^m}$$

atque tempus per AM = $\int \frac{Nk^m dv}{(gk^m - nv^m)\sqrt{v}}$.

Exemplum 1.

486. Resistat medium in simplici ratione celeritatum; erit $2m = 1$ atque

$$dv = gdx - \frac{ndx\sqrt{v}}{\sqrt{k}}.$$

Hinc fit

$$x = \int \frac{dv\sqrt{k}}{g\sqrt{k} - n\sqrt{v}} = -\frac{2\sqrt{kv}}{n} + \frac{2gk}{n^2}\, l\frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{v}}$$

vel per seriem

$$x = \frac{2v}{2g} + \frac{2nv\sqrt{v}}{3g^2\sqrt{k}} + \frac{2n^2v^2}{4g^3k} + \frac{2n^3v^2\sqrt{v}}{5g^4k\sqrt{k}} + \text{etc.},$$

si quidem descensus in *A* ex quiete incipiat. Tempus autem per spatium *AM* erit =

$$\int \frac{ndv\sqrt{k}}{g\sqrt{kv} - nv} = 2\sqrt{k}\, l\frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{v}}.$$

Quare si tempus per *AM* ponatur = *t*, erit

$$nx + 2\sqrt{kv} = \frac{gt\sqrt{k}}{n}$$

atque in serie

$$t = \frac{2n\sqrt{v}}{g} + \frac{2n^2v}{2g^2\sqrt{k}} + \frac{2n^3v\sqrt{v}}{3g^3k} + \frac{2n^4v^2}{4g^4k\sqrt{k}} + \text{etc.}$$

Si ergo corpus in descensu per *AB* acquisivit celeritatem altitudini *b* debitam, ex hac reperitur altitudo

$$AC = \frac{-2\sqrt{bk}}{n} + \frac{2gk}{n^2}\, l\frac{g\sqrt{k}}{g\sqrt{k} - n\sqrt{b}}.$$

Exemplum 2.

486. Resistat medium in duplicata ratione celeritatum; erit $m = 1$ et

$$x = \int \frac{kdv}{gk - nv} = \frac{k}{n} l \frac{gk}{gk - nv},$$

si quidem corpus in *A* ex quiete ascensum inchoaverit. Quare si *e* sit numerus, cuius logarithmus est unitas, erit [p. 245]

$$e^{\frac{nx}{k}} = \frac{gk}{gk - nv} \quad \text{atque} \quad v = \frac{gk\left(e^{\frac{nx}{k}} - 1\right)}{ne^{\frac{nx}{k}}} = \frac{gk}{n}\left(1 - e^{\frac{-nx}{k}}\right).$$

Hanc ob rem, si corpus in *B* habuerit celeritatem altitudini *b* debitam, erit $AC = \frac{k}{n} l \frac{gk}{gk - nb}$. Atque si corpus per spatium infinitum descendat, habebit celeritatem altitudini $\frac{gk}{n}$ debitam. Tempus vero per spatium *AM* erit =

$$\int \frac{nkdv}{(gk - nv)\sqrt{v}} = \frac{\sqrt{nk}}{\sqrt{g}} \, l \frac{\sqrt{gk} + \sqrt{nv}}{\sqrt{gk} - \sqrt{nv}}.$$

Per series tam spatiam *x* quam tempus commode exprimitur; id quod generaliter pro quovis valore litterae *m* in sequente exemplo monstrabimus.

Exemplum 3.

488. Sit resistentia ut potestas exponentis 2*m* celeritatum; erit

$$dx = \frac{k^m dv}{gk^m - nv^m};$$

quae expressio in seriem conversa dat

$$dx = \frac{dv}{g} + \frac{nv^m dv}{g^2 k^m} + \frac{n^2 v^{2m} dv}{g^3 k^{2m}} + \text{etc.}$$

Ex qua invenitur

$$x = \frac{v}{g} + \frac{nv^{m+1}}{(m+1)g^2 k^m} + \frac{n^2 v^{2m+1}}{(2m+1)g^3 k^{2m}} + \frac{n^3 v^{3m+1}}{(3m+1)g^4 k^{3m}} + \text{etc.}$$

Atque si ponatur tempus per $AM = t$, quia est

$$dt = \frac{ndx}{\sqrt{v}},$$

erit

$$t = \frac{2n\sqrt{v}}{g} + \frac{2n^2 v^m \sqrt{v}}{(2m+1)g^2 k^m} + \frac{2n^3 v^{2m} \sqrt{v}}{(4m+1)g^3 k^{2m}} + \frac{2n^4 v^{3m} \sqrt{v}}{(6m+1)g^4 k^{3m}} + \text{etc.}$$

[p. 246]

PROPOSITIO 56.

Problema.

489. *Resistat medium in ratione quacunque multiplicata celeritatum datumque sit punctum A* (Fig.60), *ex quo infinitae rectae AM sint eductae; determinare curvam CMD huiusmodi, ut corpus per quamlibet rectam AM descendens in puncto M eandem habeat celeritatem.*

Solutio.

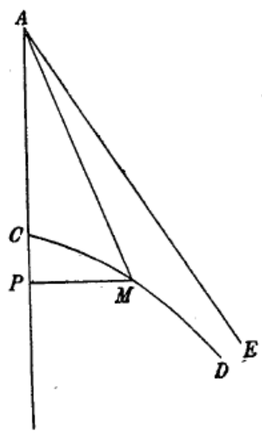

Fig. 60.

Sit $2m$ exponens potestatis celeritatis, cui resistentia est proportionalis, dicaturque $AP = x$ et $AM = z$ et ponatur $z = nx$. Sit altitudo celeritati in M debita $= v$, quae debet esse constans, scilicet $= b$. Erit ergo

$$dx = \frac{k^m dv}{gk^m - nv^m} \ (485)$$

denotante ut supra k exponentem resistentiae et g potentiam sollicitatem deorsum tendentem. Ad naturam curvae CM ergo inveniendam oportet integrare aequationem $dx = \frac{k^m dv}{gk^m - nv^m}$ ita ut posito v = 0 fiat quoque $x = 0$, tum autem poni b loco v atque $\frac{z}{x}$ loco n, hocque modo obtinebitur aequatio inter x et z naturam curvae exponens. Per seriem autem supra aequationem propositam integravimus (488), unde posito b loco v habebimus

$$x = \frac{b}{g} + \frac{nb^{m+1}}{(m+1)g^2k^m} + \frac{n^2b^{2m+1}}{(2m+1)g^3k^{2m}} + \text{etc.}$$

Ponatur q^m loco n et multiplicetur ubique per q; quo facto habebimus

$$qx = \frac{bq}{g} + \frac{b^{m+1}q^{m+1}}{(m+1)g^2k^m} + \frac{b^{2m+1}q^{2m+1}}{(2m+1)g^3k^{2m}} + \text{etc.}$$

Sumantur differentialia et dividatur per bdg; habebitur [p. 247]

$$\frac{qdx + xdq}{bdq} = \frac{1}{g} + \frac{b^m q^m}{g^2k^m} + \frac{b^{2m}q^{2m}}{g^3k^{2m}} + \text{etc.} = \frac{k^m}{gk^m - b^m q^m}$$

seu

$$qdx + xdq = \frac{bk^m dq}{gk^m - b^m q^m}.$$

At quia est $q^m = n$ et $n = \frac{z}{x}$, ponatur $\frac{z^{\frac{1}{m}}}{x^{\frac{1}{m}}}$ loco *q* et prodibit

$$z^{\frac{1-m}{m}} x^{\frac{m-1}{m}} dz + (m-1) z^{\frac{1}{m}} x^{\frac{-1}{m}} dx = \frac{bk^m z^{\frac{1-m}{m}} x^{\frac{m-1}{m}} dz - bk^m z^{\frac{1}{m}} x^{\frac{-1}{m}} dx}{gk^m x - b^m z}.$$

Quae multiplicata per $x^{\frac{1}{m}} z^{\frac{m-1}{m}}$ abit in hanc

$$xdz + (m-1) zdx = \frac{bk^m xdz - bk^m zdx}{gk^m x - b^m z}.$$

Constructio autem curvae facilius sequitur ex aequatione

$$qx = \int \frac{bk^m dq}{gk^m - b^m q^m}.$$

Supra autem habuimus seriem ipsi *qx* aequalem, ex qua patet, si fuerit $q^m = \frac{gk^m}{b^m}$, tum $\frac{gx}{b}$ aequari seriei harmonicae [p. 248]

$$1 + \frac{1}{m+1} + \frac{1}{2m+1} + \text{etc.}$$

ideoque esse *x* infinitum. Si igitur *x* is infinitum, erit $q^m = \frac{z}{x} = \frac{gk^m}{b^m}$, ex quo perspicitur rectam *AE* fore curvae asymtoton et cosinum anguli *CAE* fore $\frac{b^m}{gk^m}$. Verticis autem curvae *C* a puncto *A* distantia *AC* aequalis erit huic seriei

$$\frac{b}{g} + \frac{b^{m+1}}{(m+1)g^2 k^m} + \frac{b^{2m+1}}{(2m+1)g^3 k^{2m}} + \text{etc.}$$

Debet autem esse necessario $b^m < gk^m$; alias enim vertex C a puncto A infinite distaret. Q.E.I.

Corollarium 1.

490. Si applicata *PM* vocetur *y*, erit

$$z = \sqrt{(x^2 + y^2)} \quad \text{et} \quad dz = \frac{xdx + ydy}{\sqrt{(x^2 + y^2)}};$$

quibus valoribus in aequatione inventa substitutis habebitur

$$xydy + mx^2 dx + (m-1) y^2 dx = \frac{bk^m y (xdy - ydx)}{gk^m x - b^m \sqrt{(x^2 + y^2)}}.$$

Corollarium 2.

491. Ponatur in hac aequatione $y = px$; transmutabitur ista aequatio in hanc

$$xpdp + mdx + mp^2dx = \frac{bk^m pdp}{gk^m - b^m\sqrt{(1+pp)}},$$

quae aequatio per $(1+pp)^{\frac{1-2m}{2m}}$ multiplicate sit integrabilis; habebitur enim [p. 249]

$$m(1+pp)^{\frac{1}{2m}}x = \int \frac{bk^m pdp(1+pp)^{\frac{1-2m}{2m}}}{gk^m - b^m\sqrt{(1+pp)}},$$

quae expressio per quadratures effici potest.

Corollarium 3.

492. Si resistentia evanescat corpusque in vacuo moveatur, fit k infinitum; atque ex supra data serie invenitur $qx = \frac{bq}{g}$ seu $x = \frac{b}{g}$, unde cognoscitur lineam CM fieri rectam horizontalem.

Scholion 1.

493. Quia autem ex hac aequatione generali parum ad cognitionem curvae potest concludi, in exemplis specialibus hanc disquisitionem ulterius prosequemur. Talia autem assumemus exempla, in quibus formula $\frac{bk^m dq}{gk^m - bmq^m}$ integrationem saltem per logarithmos admittit, quo ad expressiones finitas perveniamus, ex quibus facile erit curvae naturam perspicere. [Ex hoc paragrapho concludi potest Eulero formulam Cotesianam anno 1736 notam non fuisse. P. St.]

Exemplum 1.

494. Sit igitur resistentia ipsis celeritatibus proportionalis; erit $m = \frac{1}{2}$. Ponatur $AC = a$; quia celeritas, quam corpus per AC cadendo acquirit, debita esse debet altitudini b, erit [p. 250]

$$a = -2\sqrt{bk} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k} - \sqrt{b}}$$

(486). Deinde vero erit

$$\sqrt{q} = \frac{z}{x} \text{ seu } q = \frac{z^2}{x^2}$$

atque

$$qx = \int \frac{bdq\sqrt{k}}{g\sqrt{k} - \sqrt{bq}},$$

quod integrale ita est accipiendum, ut posito $z = x$ seu $q = 1$ fiat $x = a$ vel eius valori assignato. Erit ergo

$$qx = -2\sqrt{bkq} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{bq}},$$

quae aequatio loco q substituto $\frac{z^2}{x^2}$ abit in hanc

$$z^2 = -2z\sqrt{bk} + 2gkxl\frac{gx\sqrt{k}}{gx\sqrt{k}-z\sqrt{b}}.$$

Si q = 1, fit

$$x = a = -2\sqrt{bk} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{b}}.$$

Fiat autem $q = 1 + dq$; habebitur

$$dx + xdq = -dq\sqrt{bk} + \frac{gkdq\sqrt{b}}{g\sqrt{k}-\sqrt{b}} = \frac{bdq\sqrt{k}}{g\sqrt{k}-\sqrt{b}}.$$

At quia $\frac{1}{\sqrt{q}}$ est cosinus anguli *MAC*, erit $\sqrt{dq}$ = sinui huius anguli. Quamobrem incrementum ipsius *x* infinities minus est quam incrementum anguli *MAC* incidente *MA* in *CA*, ex quo sequitur tangentem curvae in *C* esse horizontalem; huiusque curvae tangens in infinito seu asymtota erit *AE* existente anguli *EAC* cosinu $\frac{\sqrt{b}}{g\sqrt{k}}$. Ceterum haec curva ex altera verticalis *AC* parte arcum habebit similem et aequalem ipsi *CMD*.

Scholion 2.

495. Generaliter quidem etiam ostendi potest curvae tangentem in *C* esse debere horizontalem. Posita enim *n* = 1 in serie *x* exprimente habetur

$$AC = \frac{b}{g} + \frac{b^{m+1}}{(m+1)g^2k^m} + \text{etc.}$$

[p. 251] Augeatur *n* elemento *dn*; habebitur incrementum momentaneum ipseus *AC* =

$$\frac{b^{m+1}dn}{(m+1)g^2k^m} + \frac{2b^{2m+1}ndn}{(2m+1)g^3k^{2m}} + \text{etc.}$$

Est vero $\frac{1}{n}$ cosinus anguli *MAC* ideoque sinus = $\frac{\sqrt{(n^2-1)}}{n} = \sqrt{2dn}$ posito 1 + *dn* loco *n*. Quamobrem incrementum ipsius *AC* infinities est minus quam incrementum anguli; atque ideo *AC* normalis erit in curvam *DMC*.

Exemplum 2.

496. Sit resistentia ipsis quadratis celeritatis proportionalis; erit $m = 1$ positique *AC* = *a*; erit

$$a = kl\frac{gk}{gk-b} \quad \text{atque} \quad b = gk\left(1 - e^{\frac{-a}{k}}\right).$$

Deinde vero est

$$q = n = \frac{z}{x}$$

atque

$$qx = z = \int \frac{bkdq}{gk - bq} = kl\frac{gk}{gk - bq} = kl\frac{gkx}{gkx - bz}.$$

Habebitur ergo

$$e^{\frac{z}{k}} = \frac{gkx}{gkx - bz}, .$$

unde sequitur

$$x = \frac{e^{\frac{z}{k}}bz}{gk\left(e^{\frac{z}{k}} - 1\right)} = \frac{e^{\frac{z}{k}}z\left(e^{\frac{a}{k}} - 1\right)}{e^{\frac{a}{k}}\left(e^{\frac{z}{k}} - 1\right)}$$

posito loco *b* eius valore in *a*. Ad curvam autem constructuendam commodissime adhibetur haec aequatio

$$z = kl\frac{gk}{gk - bq},$$

in qua *z* est *AM* et *q* est secans anguli *MAC*.

Scholion 3.

497. In solutione problematis ad inveniendam aequationem curvae *CMD* usi fuimus seriei cuiusdam summatione; [p. 252] eandem vero aequationem sine seriebus sequenti modo elicere licet. Quia est

$$x = \int \frac{k^m dv}{gk^m - nv^m},$$

haec ipsa aequatio exprimit naturam curvae quaesitae, si post integrationem ponatur $v = b$ et $\frac{z}{x}$ loco *n*. Quamobrem si

$$\int \frac{k^m dv}{gk^m - nv^m}$$

differentietur, posito non solum *v*, sed etiam *n* variabili, atque tum ponatur *v* constans = *b* et $\frac{z}{x}$ loco *n*, habebitur aequatio differentialis pro curva quaesita. Ad hoc efficiendum pono $n = \frac{1}{p^m}$, quo prodeat

$$x = \int \frac{k^m p^m dv}{gk^m p^m - v^m}.$$

Ponamus brevitatis gratia

$$\frac{k^m p^m}{gk^m p^m - v^m} = P$$

sitque aequatio differentialis haec

$$dx = Pdv + Qdp,$$

si etiam *p* variabilis accipiatur. Quia autem *P* est functio nullius dimensionis ipsarum *v* et *p*, erit

$$x = Pv + Qp$$

ideoque

$$Q = \frac{x}{p} - \frac{Pv}{p}.$$

Hoc igitur loco *Q* valore substituto prodibit

$$pdx = \frac{k^m p^{m+1} dv - k^m p^m v dp}{gk^m p^m - v^m} + xdp.$$

Restituatur $n^{\frac{-1}{m}}$ loco *p* et orietur

$$mndx + xdn = \frac{mk^m ndv + k^m vdn}{gk^m - nv^m},$$

in qua aequatione *n* aeque variabilis est assumta ac *v* et x. Nunc ponatur $v = b$, $dv = 0$ et $n = \frac{z}{x}$ atque habebitur ista aequatio [p. 253]

$$xdz + (m-1)zdx = \frac{bk^m(xdz - zdx)}{gk^m x - b^m z},$$

quae cum aequatione supra inventa congruit.

PROPOSITIO 57.

Problema.

498. *Si resistentia fuerit in quacunque multiplicata ratione celeritatum, invenire curvam* (Fig.61), *huius proprietatis, ut corpus descendens super quavis subtensa AM dato tempore ex A ad M perveniat.*

Solutio.

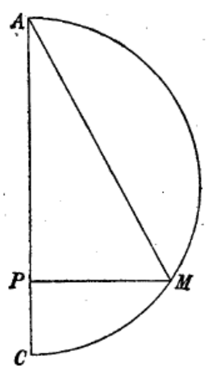
Fig. 61.

Ducta verticali *AC* ponatur $AP = x$, $AM = z$ sitque $n = \frac{z}{x}$.

Posita altitudine celeritati in *M* debita = *v*, et resistentia = $\frac{v^m}{k^m}$ sit tempus, quo corpus per *AM* descendit, = *t*, quod debet esse quantitas constans. Habebimus ergo ex praecedentibus

$$x = \int \frac{k^m dv}{gk^m - nv^m} \quad \text{et} \quad t = \int \frac{nk^m dv}{(gk^m - nv^m)\sqrt{v}}$$

(485). Quocirca ad naturam curvae *AMC* inveniendam opus est, ut utraque aequatio, si fieri potest, re ipsa integretur et valor ipsius *v* ex altera aequatione in altera substituatur atque tum loco *n* scribatur $\frac{z}{x}$, quo facto habebitur aequatio inter *x* et *z* naturam curvae quaesitae exprimens. [p. 254]

At si integrationes non commode perfici poterunt, utraque aequatio est differentianda ponendo quoque *n* variabili, et postquam positum est $dt = 0$, ex duabus aequationibus inventis eliminari debet *v*, quo prodeat aequatio *n* et *x* tantum contens, quae ob $n = \frac{z}{x}$ exhiberet naturam curvae quaesitae. Ad hoc ponatur $n = \frac{1}{p^m}$, quo habeamus

$$x = \int \frac{k^m p^m dv}{gk^m p^m - v^m} \quad \text{et} \quad t = \int \frac{k^m dv}{(gk^m p^m - v^m)\sqrt{v}}.$$

Quarum aequationum illius, sumto quoque *p* variabili, differentialis iam est inventa

$$pdx - xdp = \frac{k^m p^{m+1} dv - k^m p^m v dp}{gk^m p^m - v^m}$$

(497). Ad alteram aequationem differentiandum pono

$$\frac{k^m}{(gk^m p^m - v^m)\sqrt{v}} = P$$

sitque

$$dt = Pdv + Qdp.$$

Quia autem *P* est functio ipsarum *v* et *p* dimensionum $-m-\frac{1}{2}$, erit

$$(\tfrac{1}{2} - m)t = Pv + Qp$$

atque hinc

$$Q = \frac{(1-2m)t}{2p} - \frac{Pv}{p}.$$

Quo valore loco *Q* substituto prodibit

$$pdt = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}} + \frac{(1-2m)tdp}{2}.$$

Sit nunc $t = 2\sqrt{c}$ atque $dt = 0$; habebimus

$$(2m-1)dp\sqrt{c} = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}}.$$

Eliminetur ex his duabus aequationibus *dv* et proveniet

$$pdx - xdp = (2m-1)p^m dp\sqrt{cv}$$

seu [p. 255]

$$\sqrt{v} = \frac{pdx - xdp}{(2m-1)p^m dp\sqrt{c}} = \frac{dr}{(2m-1)p^{m-2}dp\sqrt{c}}$$

posito $x = rp$. Substituator hic valor loco *v* in aequatione

$$(2m-1)dp\sqrt{c} = \frac{k^m(pdv - vdp)}{(gk^m p^m - v^m)\sqrt{v}}$$

vel in hac

$$\frac{dr}{p^{m-2}} = \frac{k^m(pdv - vdp)}{gk^m p^m - v^m}.$$

Casu quidem, quo $m = \frac{1}{2}$ seu resistentia celeritatibus proportionalis, erit $pdv = vdp$ seu $v = \alpha p$ et

$$p = \frac{x}{a} = \frac{1}{n^2} = \frac{xx}{zz},$$

unde sequitur fore $zz = ax$; quamobrem in hac resistentiae hypothesi curva *AMC* est circulus omnino ut in vacuo. In aliis hypothesibus, nisi re ipsa aequatio alterutra integretur, eliminata *v* habebitur aequatio differentio–differentialis inter *z* et *x* naturam curvae exprimens. Q.E.I.

Corollarium 1.

499. Si ponatur $v = up$, erit

$$pdv - vdp = ppdu.$$

Atque hinc erit

$$\sqrt{u} = \frac{dr}{(2m-1)p^{m-\frac{3}{2}}dp\sqrt{c}},$$

qui valor substitutus in aequatione

$$dr = \frac{k^m du}{gk^m - u^m}$$

dabit aequationem inter *p* et *r*, ex qua aequatio inter *x* et *z* formabitur. [p. 256]

Corollarium 2.

500. In medio ergo, quod resistit in simplici celeritatum ratione, apparet curvam *AMC* esse circulum. Atque ideo in hac resistentiae hypothesi tempora descensuum per per singulas circuli chordas ex puncto *A* ductas sunt inter se aequalia.

Exemplum 1.

501. Sit resistentia quadratis celeritatum proportionalis; erit *m* = 1 atque

$$x = \frac{k}{n} l \frac{gk}{gk - nv}$$

seu

$$v = \frac{gk}{n}\left(1 - e^{\frac{-nx}{k}}\right).$$

Praeterea vero erit

$$t = 2\sqrt{c} = \frac{\sqrt{nk}}{\sqrt{g}} l \frac{\sqrt{gk} + \sqrt{nv}}{\sqrt{gk} - \sqrt{nv}}$$

seu

$$e^{\frac{2\sqrt{gc}}{\sqrt{nk}}} = \frac{\sqrt{gk} + \sqrt{nv}}{\sqrt{gk} - \sqrt{nv}},$$

unde fit

$$v = \frac{gk\left(e^{\frac{2\sqrt{gc}}{\sqrt{nk}}} - 1\right)^2}{n\left(e^{\frac{2\sqrt{gc}}{\sqrt{nk}}} + 1\right)^2}.$$

Eliminata ergo *v* et $\frac{z}{x}$ posito loco *n* habebitur

$$\frac{e^{\frac{z}{k}} - 1}{e^{\frac{z}{k}}} = \frac{\left(e^{\frac{2\sqrt{gcx}}{\sqrt{kz}}} - 1\right)^2}{\left(e^{\frac{2\sqrt{gcx}}{\sqrt{kz}}} + 1\right)^2}$$

seu

$$\frac{2\sqrt{gcx}}{\sqrt{kz}} = l\,\frac{1+\sqrt{\left(1-e^{\frac{-z}{k}}\right)}}{1-\sqrt{\left(1-e^{\frac{-z}{k}}\right)}}.$$

In hac curva est

$$AC = k\,l\,\frac{\left(e^{\frac{2\sqrt{gc}}{\sqrt{k}}}+1\right)^2}{4e^{\frac{2\sqrt{gc}}{\sqrt{k}}}} = 2k\,l\,\frac{e^{\frac{\sqrt{gc}}{\sqrt{k}}}+e^{\frac{-\sqrt{gc}}{\sqrt{k}}}}{2};$$

si igitur ponatur $AC = a$, erit [p. 257]

$$2e^{\frac{a}{2k}} = e^{\frac{\sqrt{gc}}{\sqrt{k}}} + e^{\frac{-\sqrt{gc}}{\sqrt{k}}},$$

unde erit

$$e^{\frac{\sqrt{gc}}{\sqrt{k}}} = e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}$$

seu

$$\sqrt{\frac{gc}{k}} = l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}\right).$$

Erit igitur

$$\sqrt{x}\,l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}\right) = \sqrt{z}\,l\left(e^{\frac{z}{2k}} + \sqrt{\left(e^{\frac{z}{k}}-1\right)}\right)$$

aequatio pro curva *AMC*. Si resistentia fuerit valde parva, erit *k* quantitas vehementer magna atque ideo

$$e^{\frac{z}{2k}} + \sqrt{\left(e^{\frac{z}{k}}-1\right)} = 1 + \frac{z}{2k} + \sqrt{\left(\frac{z}{k}+\frac{z^2}{2k^2}\right)} = 1 + \sqrt{\frac{z}{k}} + \frac{z}{2k} + \frac{z\sqrt{z}}{4k\sqrt{k}}$$

huiusquae logarithmus erit =

$$\frac{\sqrt{z}}{\sqrt{k}} + \frac{z\sqrt{z}}{12k\sqrt{k}}.$$

Simili modo erit

$$l\left(e^{\frac{a}{2k}} + \sqrt{\left(e^{\frac{a}{k}}-1\right)}\right) = \frac{\sqrt{a}}{\sqrt{k}} + \frac{a\sqrt{a}}{12k\sqrt{k}}.$$

Atque hanc ob rem habebitur pro curva *AMC* haec aequatio

$$\sqrt{ax} + \frac{a\sqrt{ax}}{12k} = z + \frac{z^2}{12k}$$

seu

$$ax\left(1+\frac{a}{6k}+\frac{a^2}{144k^2}\right) = z^2 + \frac{z^3}{6k} + \frac{z^4}{144k^2}.$$

Unde perspicitur, si resistentia prorsus evanescat seu *k* fiat infinite magnum, fore $ax = z^2$ atque ideo curvam *AMC* circulum. At si medium rarissimum fuerit, erit

$$ax(a+6k) = 6kz^2 + z^3$$

et differentiando

$$adx(a+6k) = 12kzdz + 3z^2dz.$$

Si nunc fiat *zdz = xdx*, habebitur applicata *PM* maxima seu locus, ubi tangens curvae est verticalis, scilicet

$$a^2 + 6ak = 12kx + 3xz \quad \text{seu} \quad x = \frac{a^2+6ak}{12k+3z},$$

unde fit

$$\frac{(6ak+a^2)^2}{3} = 24k^2z^2 + 10kz^3 + z^4,$$

ex qua aequatione ipsius *z* valor quam proxime est

$$\frac{a}{\sqrt{2}} + \frac{(4\sqrt{2}-5)a^2}{48k}$$

atque [p. 258]

$$x = \frac{a}{2} - \frac{(3\sqrt{2}-4)a^2}{48k} \quad \text{et} \quad PM = \frac{a}{2} + \frac{(2-\sqrt{2})a^2}{24k}.$$

Curva ergo latissima est supra medietatem ibique latior est quam altitudo AC.

Corollarium 3.

502. Si igitur linea recta vel curva hanc curvam AMC in M tangat, ita ut tota extra spatium AMC sit sita, corpus ex A ad eam lineam citius perveniet descendendo super chorda AM quam super quavis alia recta ex A ad eam lineam ducta.

Exemplum 2.

503. Sit *m* numerus affirmativus et resistentia valde parva; erit k quantitas vehementer magna atque hinc

$$\frac{k^m}{gk^m - nv^m} = \frac{1}{g} + \frac{nv^m}{g^2k^m}.$$

Quocirca erit

$$x = \frac{v}{g} + \frac{nv^{m+1}}{(m+1)g^2k^m} \quad \text{atque} \quad t = 2\sqrt{c} = \frac{2n\sqrt{v}}{g} + \frac{2n^2v^{m+\frac{1}{2}}}{(2m+1)g^2k^m}$$

hincque prodit

$$v = gx - \frac{ng^mx^{m+1}}{(m+1)k^m} \quad \text{et} \quad \sqrt{v} = \sqrt{gx} - \frac{ng^{m-\frac{1}{2}}x^{m+\frac{1}{2}}}{2(m+1)k^m}$$

atque

$$v^{m+\frac{1}{2}} = g^{m+\frac{1}{2}}x^{m+\frac{1}{2}} - \frac{n(2m+1)g^{2m-\frac{1}{2}}x^{2m+\frac{1}{2}}}{2(m+1)k^m}.$$

His autem valoribus substitutis prodit ista aequatio [p. 259]

$$\frac{\sqrt{gc}}{\sqrt{x}} = n + \frac{n^2g^{m-1}x^m}{(2m+1)(2m+2)k^m} - \frac{n^3g^{2m-2}x^{2m}}{(2m+2)k^{2m}}.$$

Quia vero est $n = \frac{z}{x}$, habebitur ista aequatio

$$\sqrt{gcx} = z + \frac{g^{m-1}x^{m-1}z^2}{(2m+1)(2m+2)k^m} - \frac{g^{2m-2}x^{2m-2}z^3}{(2m+2)k^{2m}}$$

seu

$$gcx = z^2 + \frac{g^{m-1}x^{m-1}z^3}{(m+1)(2m+1)k^m},$$

si medium est rarissimum. Unde patet, si resistentia penitus evanescat, fore $z^2 = gcx$ seu curvam *AMC* circulum diametri *AC*.

Scholion.

504. Si igitur cognita fuerit curva *AMC* et detur linea quaecunque, determinari poterit recta *AM*, super qua corpus ex *A* celerrime ad datam lineam pertingat. Scilicet construenda est curva *AMC*, quae datam lineam tangat v.g.in *M*; eritque recta *AM* ea recta, super qua corpus descendendo ex A citissime ad lineam datam perveniat. Atque simili modo in praecedente problemate, si recta vel curva tangat curvam *CMD* (Fig. 60) in *M*, corpus ex *A* descendendo per *AM* usque ad lineam tangentem curvam *CMD* maiorem acquiret celeritatem quam descendendo super quavis alia recta ex *A* ad eam lineam ducta. Ex his igitur solvi possunt problemata, quibus requiritur recta ex *A* ad datam lineam ducta, super qua corpus descendendo vel maximam acquirat celeritatem [p. 260] vel citissime ad eam lineam pertingat. Quamobrem hisce problematibus non diutus immorabimur, sed ad ascensum superlineis rectis considerandum progrediemur.