CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 211]

PROPOSITION 49.

Problem.

430. *If a body is acted on by any forces, to find the curve AM* (Fig. 53), *upon which all the descents are made in equal times as far as to the point A.*

Solution.

Whatever the forces should be acting, all these can be reduced to two forces, of which the first now always pulls the body downwards along MQ, and the other pulls the body horizontally along *MP*. Let the force which pulls along *MQ* be equal to *P*, and the force which pulls along *MP* be equal to *Q*; call $AP = x$, $PM = y$, $AM = s$; let the speed at the point *A* correspond to the height *b*, and the speed at *M* correspond to the height *v*. With these in place, the equation arises $v = b - \int Pdx - \int Qdy$. Whereby if we put $h = b$ and $\int Pdx + \int Qdy = z$, then *v* is a function of one dimension of *h* and *z*, and therefore $m = 1$ (408). On this account, this equation is obtained for the curve sought :

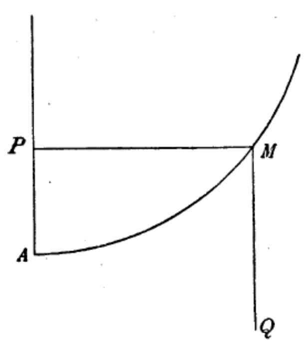

Fig. 53.

$$s = 2\sqrt{a\,z} = 2\sqrt{a(\int Pdx + \int Qdy)}$$

or

$$ds = \frac{aPdx + aQdy}{\sqrt{a(\int Pdx + \int Qdy)}}.$$

But since we have $ds = \sqrt{(dx^2 + dy^2)}$, then

$$\frac{dy}{dx} = \frac{aPQ \pm \sqrt{(\int Pdx + \int Qdy)(aP^2 - \int Pdx - \int Qdy + aQ^2)}}{\int Pdx + \int Qdy - aQ^2}.$$

Therefore at the starting point A, where $\int Pdx + \int Qdy = 0$, then $Pdx + Qdy = 0$ or $dy : dx = -P : Q.$ And it is understood from the preceding proposition, that the time of this descent is equal to the time, [p. 212] according to the hypothesis of gravity equal to 1, in which a pendulum of length 2*a* completes the descent. Q.E.I.

Scholium.

431. If a curve is obtained, upon which all the descents are made in the same time, it is easy to give the curves, upon which all the oscillations are performed in the same time. For since in a vacuum the ascents are similar to the descents, every curve, which is a tautochrone for the descents is such too for the ascents. Whereby two tautochrone curves joined at the point A give a curve, upon which all the oscillations are isochrones. But yet by this reason the other problem, in which all the isochronous oscillations produced are required, is not perfectly resolved ; for it is possible to give an infinitude of curves satisfying this question, yet the parts of which are not suited to bringing about isochrones in the descents alone. Moreover the problem can be proposed in this manner : given any curve to find another, which joined with that curve produces all oscillations of equal times. [E012 : *De innumerabilibus tautochronis in vacuo*, Comment. acad. sc. Petrop. 6 (1729), 1735; *O.O.* series II, vol. 4] Now before we advance to this problem, we bring forwards another problem, in which a curve is sought adjoined to a given curve, so that all the descents on this composite curve are completed in equal times. Which problem has given me the most difficulty since it was proposed to me by the most celebrated Dan. Bernoulli. [E024 : *Solutio singularis casus circa tautochronismum*, Comment. acad. sc. Petrop. 6 (1732/33), 1738; *O.O.* series II, vol. 4] Yet this problem can also be solved by this method, which I use in the investigation of tautochrones.

PROPOSITION 50.

[p. 213]

Problem.

432. *According to the hypothesis of gravity acting uniformly downwards, if the curve ANB is given* (Fig. 54), *to find the curve BMF adjoined to that curve, so that all the descents upon the composite curve as far as A are completed in equal times, the descent starting from any point of the curve BMF.*

Solution.

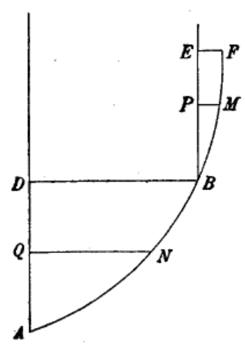

Fig. 54.

If the descent starts from the lowest point *B* of the curve sought, the descent is made along the given curve *BNA* only; hence the time of that, which is also given, must be equal to the times of all the descents. Let *AD* = *a*, *AQ* = *u*, *AN* = *t* and the equation is given between *u* and *t*. Moreover, for the curve sought, let *BP* = *x*, and *BM* = *s*. Now in any descent, let the speed at the point *B* correspond to the height *b*; the speed at the point *M* corresponds to the height $b-x$ and the speed at *N* corresponds to the height $a+b-u$. Therefore the descent time along the unknown curve is equal to $\int\frac{ds}{\sqrt{(b-x)}}$, thus integrated, so that it vanishes on putting *x* = 0, and on putting *x* = *b* in place after the integration. Now the time along the known curve *BNA* is given by $\int\frac{dt}{\sqrt{(a+b-u)}}$, thus integrated, so that it vanishes on putting *u* = 0, and putting *u* = *a* after the integration. [p. 214] Therefore the expression $\int\frac{ds}{\sqrt{(b-x)}}$, after making *x* = *b*, must thus be arranged in order that, if it is added to the expression of the time along *BNA*, then the letter *b* is not present within the sum ; for then the total time of the descent is a constant quantity and does not depend on *b*, or on the point of the curve *BMF* at which the descent started. Let the integral $\int\frac{dt}{\sqrt{(a+b-u)}}$, after putting *u* = *a*, be equal to this series :

$$k+\alpha b+\beta b^2+\gamma b^3+\delta b^4+\text{etc.}+\zeta\sqrt{b}+\eta b\sqrt{b}+\theta b^2\sqrt{b}+\iota b^3\sqrt{b}+\text{etc.}$$

Whereby, if the descent starts from the point *B*, the time of the whole descent is equal to *k*, as *b* vanishes. Therefore *k* itself must be equal to the time of the whole descent along the composite curve, the start of the descent being taken from any point of the curve *BMF*. Now let the nature of the curve sought *BMF* be expressed by the following series :

$$ds=-Adx\sqrt{x}-Bxdx\sqrt{x}-Cx^2dx\sqrt{x}-Dx^3dx\sqrt{x}-\text{etc.}$$
$$-Fdx-Gxdx-Hx^2dx-Ix^3dx-\text{etc.}$$

The ratio of the periphery to the diameter is put as $\pi:1$, which in fact is $l-1:-\sqrt{-1}$, thus so that it becomes :

$$\pi = -\frac{l-1}{\sqrt{-1}} = \sqrt{-1} \bullet l - 1.$$

Now after integration, on putting $x = b$,

$$\int \frac{Adx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1}{2}\pi Ab, \int \frac{Bxdx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1.3}{2.4}\pi Bb^2, \int \frac{Cx^2dx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1.3.5}{2.4.6}\pi Cb^3 \quad \text{etc.}$$

and

$$\int \frac{Fdx\sqrt{x}}{\sqrt{(b-x)}} = 2F\sqrt{b}, \int \frac{Gdx\sqrt{x}}{\sqrt{(b-x)}} = \frac{2}{3}2Gb\sqrt{b}, \int \frac{Hx^2dx}{\sqrt{(b-x)}} = \frac{2.4}{3.5}2Hb^2\sqrt{b} \quad \text{etc.}$$

[We give an elementary working of the first integration here : $\int \frac{Adx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1}{2}\pi Ab$.
$\int \frac{dx\sqrt{x}}{\sqrt{(b-x)}} = \int \frac{xdx}{\sqrt{(bx-x^2)}} = \int \frac{xdx}{\sqrt{\frac{b^2}{4}-(x-\frac{b}{2})^2}}$. Let $X = x - A$ and $A = \frac{b}{2}$, then
$\int \frac{xdx}{\sqrt{\frac{b^2}{4}-(x-\frac{b}{2})^2}} = \int \frac{(X+A)dX}{\sqrt{A^2-X^2}} = \int \frac{XdX}{\sqrt{A^2-X^2}} + \int \frac{AdX}{\sqrt{A^2-X^2}}$. The second integral is just the arc length, since $\int \frac{dX}{\sqrt{A^2-X^2}} = \arcsin(\frac{X}{A})$, for on substituting $X = A\sin\theta$, and noting that the limits are 0 and b, or $-A$ and A, the integral becomes $A\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{b\pi}{2}A$, on inserting the limits; note that Euler specified the limits each time, as the notion of a definite integral had not yet been formed, and the process of integration was viewed by him as the solution of a first order differential equation, which of course is correct. The first integral $\int \frac{XdX}{\sqrt{A^2-X^2}}$ is zero as it is an odd function, and the other integrations follow on integrating by parts successively. It is also possible, of course, to write the second integral as $\int \frac{AdX}{\sqrt{A^2-X^2}} = \sqrt{-1}\int \frac{AdX}{\sqrt{X^2-A^2}} = \sqrt{-1}.\log\left|X + \sqrt{X^2 - A^2}\right|$ This has been done, as Euler has performed the integration in this way using logarithms; which can be examined below in Example 1, (438), in which it is necessary to make the connection $\pi = -\frac{l-1}{\sqrt{-1}} = \sqrt{-1} \bullet l - 1$ to relate the logarithmic result with the elementary derivation. Whether or not this is how Euler came upon this formula for π is still an open question, but here he has at least used the result to obtain the correct value of an integral, which is highly suggestive.]
Since therefore the time for BNA is equal to k, from the expression of these terms taken jointly with these other terms, the homogeneous terms involving b must be removed. Therefore the series becomes :

$$\frac{1}{2}\pi A = \alpha \text{ or } A = \frac{2}{1} \bullet \frac{\alpha}{\pi}$$

in a like manner,

$$B = \frac{2.4}{1.3} \bullet \frac{\beta}{\pi}, C = \frac{2.4.6}{1.3.5} \bullet \frac{\gamma}{\pi}, \text{etc.}$$

and

$$F=\frac{\xi}{2}, G=\frac{3}{2}\cdot\frac{\eta}{2}, H=\frac{3.5}{2.4}\cdot\frac{\theta}{2}, I=\frac{3.5.7}{2.4.6}\cdot\frac{\iota}{2}\text{ etc.}$$ [p. 215]

On this account, since $\alpha, \beta, \gamma, \delta$ etc, ξ, η, θ, ι etc are known quantities as the curve *ANB* is given, this equation is obtained for the curve sought *BMF* :

$$ds=\frac{-dx}{\pi}\left(\frac{2}{1}\alpha\sqrt{x}+\frac{2.4}{1.3}\cdot\beta x\sqrt{x}+\frac{2.4.6}{1.3.5}\gamma x^2\sqrt{x}+\text{etc.}\right)$$
$$-\frac{dx}{2}\left(\xi+\frac{3}{2}\eta x+\frac{3.5}{2.4}\cdot\theta x^2+\frac{3.5.7}{2.4.6}\iota x^3+\text{etc.}\right)$$

the integral of which is

$$s=\frac{-2}{\pi}\left(\frac{2}{3}\alpha x\sqrt{x}+\frac{2.4}{3.5}\cdot\beta x^2\sqrt{x}+\frac{2.4.6}{3.5.7}\gamma x^3\sqrt{x}+\text{etc.}\right)$$
$$-\frac{1}{2}\left(\xi x+\frac{3}{4}\eta x^2+\frac{3.5}{4.6}\cdot\theta x^3+\frac{3.5.7}{4.6.8}\iota x^4+\text{etc.}\right)$$

I give the construction of this series: [The first integral is the time to fall the distance BN from rest at B; the second integral is the time to fall the total distance MBN starting from rest at M.]

$$\int\frac{dt}{\sqrt{(a-u)}}-\int\frac{dt}{\sqrt{(a+b-u)}},$$

the integral is taken thus, so that it vanishes on setting $u = 0$ [For the time to slide down *AB* is the same as the time to slide down the whole curve *AB + BM*, as the curve is a tautochrone]; then [after integrating] on making $u = a$ a certain function of *b* is produced. Now $x(1-z)$ is put in place of *b*, and what is produced is to be called *R*. Then integrate $\frac{Rdz}{\sqrt{z}}$, while *x* is considered as constant, thus as by putting $z = 0$. Then put $z = 1$ and a function of *x* is produced, which is equal to $\frac{\pi s}{\sqrt{x}}$. And in this way an equation is produced for the curve sought. Q.E.I.

Scholium 1.

433. Clearly this singular but yet easy construction follows from that method which I have used in solving a former proposition by C. Riccati, [E 31: *Constructio aequationis differentialis* $ax^n dx = dy + y^2 dx$, Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 231; *O.O.*, series I, vol. 22. P. St.] and this previous solution gives the greatest joy, since, whatever the curve given should be, that sought can always be constructed with the help of this method, even if the equation itself, for which the curve is found, often can barely be handled. Besides it gives at once a finite equation, that otherwise would be found from the sum of the series.

Corollary 1. [p. 216]

434. If in the equation found for the curve *BMF* , put $x = 0$, then $ds = \frac{-\xi dx}{2}$, hence the inclination of the curve at *B* to the vertical *BP* is known. Since therefore it is apparent how these two curves touch each other, it is also required to determine the position of the tangent to the curve *ANB* at *B*.

Corollary 2.

435. Let $DQ = p$ and $BN = q$ (Fig. 54) ; then $dt = -dq$ and $a - u = p$. Hence the time to traverse *BNA* is equal to $\int \frac{dq}{\sqrt{(b+p)}}$ with $p = a$ put in this integral. Let $dq = Ldp$ be the relation [between *p* and *q*] at the point B, and generally it is of the form $dq = Ldp + Pdp$ with *P* being a function of *p*, such that it vanishes on putting $p = 0$. Therefore we may consider, with *p* vanishing, what kind of terms this equation $\int \frac{dq}{\sqrt{(b+p)}} = \int \frac{Ldp}{\sqrt{(b+p)}}$ produces. Moreover on putting $p = a$ it produces : $2L\sqrt{(b+a)} - 2L\sqrt{b}$, [on integrating between 0 and *a*,] thus in the series taken, at the beginning it produces the term $-2L\sqrt{b}$ (436), which agrees with $\xi\sqrt{b}$; hence this makes $L = -\frac{\xi}{2}$ and $dq = \frac{-\xi dp}{2}$. From which it is understood that the given curve and that sought have a common tangent at the joining point *B*.

Scholium 2.

436. I have said that $2L\sqrt{(b+a)} - 2L\sqrt{b}$ [expanded] in a series gives this term $-2L\sqrt{b}$; for the first term $\sqrt{(b+a)}$ gives these terms $\sqrt{a} + \frac{b}{2\sqrt{a}} + \text{etc.}$ with other comparable terms. [p. 217] Moreover here, only the term *Ldp* gives a term of this form $\xi\sqrt{b}$. Whereby from that alone, the inclination of the curve at *B* can be concluded.

Scholium 3.

437. The construction of the curve sought that I have given, can also be changed in this way : after putting $u = a$ in the [evaluated] integral $\int \frac{dt}{\sqrt{(a-u)}} - \int \frac{dt}{\sqrt{(a+b-u)}}$, and on writing *xz* in place of *b*, and the expression produced is called *R*, then $\frac{Rdz}{\sqrt{(1-z)}}$ is integrated, in which *x* is treated as a constant quantity [*i. e.*, the height of the point on the upper curve is fixed meantime], thus so that it vanishes on putting $z = 0$. Then on putting $z = 1$, that which comes about is equal to $\frac{\pi s}{\sqrt{x}}$; and by this arrangement a more convenient equation is obtained for the curve sought.

Example 1.

438. Let the given curve *ANB* be a cycloid, thus so that the equation of the cycloid becomes $t = 2\sqrt{cu}$ or $dt = \frac{cdu}{\sqrt{cu}}$ [note that this form of the inverted cycloid with upward pointing cusps is given in E001, p. 3 in this series, where *c* here denotes the diameter of the generating circle, is taken as 2*a* there, while $u = y = 2a\sin^2\psi$ or $\sin\psi = \sqrt{\frac{u}{c}}$, and the arc $s = t = 4a\sin\psi = 2\sqrt{uc}$, where 2ψ is the angle turned through by the generating circle, and the origin is taken at the lowest point on the curve; note in the integral below, which can be verified by differentiation, that Euler has changed the sign under the square root to introduce the $\sqrt{-1}$ in the numerator and used a form of logarithmic integration; *a* is simply the constant of integration.]; then [considering the whole curve as a cycloid]

$$\int\frac{dt}{\sqrt{(a-u)}} - \int\frac{dt}{\sqrt{(a+b-u)}} = \int\frac{du\sqrt{c}}{\sqrt{(au-u^2)}} - \int\frac{du\sqrt{c}}{\sqrt{(au+bu-u^2)}}$$

$$= \sqrt{-c}\cdot l\,\frac{a-2u-2\sqrt{(u^2-au)}}{a} - \sqrt{-c}\cdot l\,\frac{a+b-2u-2\sqrt{(u^2-au-bu)}}{a+b}.$$

On putting *u* = *a*, there is obtained :

$$\sqrt{-c}\cdot l-1 - \sqrt{-c}\cdot l\,\frac{b-a-2\sqrt{-ab}}{a+b}$$

$$= \pi\sqrt{c} - 2\sqrt{-c}\cdot l\left(\sqrt{b}-\sqrt{-a}\right) + \sqrt{-c}\cdot l\,(a+b)$$

$$= \pi\sqrt{c} + \sqrt{-c}\cdot l\,\frac{\sqrt{b}+\sqrt{-a}}{\sqrt{b}-\sqrt{-a}}.$$

Put *xz* in place of *b*, then the expression becomes :

$$R = \pi\sqrt{c} + \sqrt{-c}\cdot l\,\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}};$$

which multiplied by $\frac{dz}{\sqrt{(1-z)}}$, gives

$$\frac{\pi dz\sqrt{c}}{\sqrt{(1-z)}} + \frac{dz\sqrt{-c}}{\sqrt{(1-z)}}\cdot l\,\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}},$$

the integral of which is : [p. 218]

$$= -2\pi\sqrt{c(1-z)} - 2\sqrt{c(z-1)}\cdot l\,\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}} - \frac{2\sqrt{-ac}}{\sqrt{x}}\cdot l\,\frac{\sqrt{(z-1)}+\sqrt{z}}{\sqrt{(z-1)}-\sqrt{z}}$$

$$-\frac{2\sqrt{-c(a+x)}}{\sqrt{x}}\cdot l\,\frac{\sqrt{-a(1-z)}-\sqrt{z(a+x)}}{\sqrt{-a(1-z)}+\sqrt{z(a+x)}} + 2\pi\sqrt{c} - 2\sqrt{c}\sqrt{-1}\cdot l-1,$$

which with the two final terms are equal to each other since $\pi = \sqrt{-1}.l-1$.
Now put $z = 1$; and there is obtained :

$$\frac{-2\sqrt{ac}+2\sqrt{c(a+x)}}{\sqrt{x}}\pi ,$$

which must be put equal to $\frac{\pi s}{\sqrt{x}}$. Hence this equation comes about :

$$s=-2\sqrt{ac}+2\sqrt{c(a+x)}$$

or

$$s+ANB=ANBM=2\sqrt{c(AD+BP)}.$$

From which it is apparent that the curve *BMF* is continuous with the given curve *AND*, thus, so that joined together they make a whole cycloid ; which itself follows from the nature of tautochronism, which the cycloid has been found to satisfy.

Example 2.

439. Let the given right line *ANB* be inclined at some angle to the horizontal ; the equivalent to the arc length is given by $dt = ndu$ and

$$\int\frac{ndu}{\sqrt{(a-u)}}-\int\frac{ndu}{\sqrt{(a+b-u)}}=2n\sqrt{a}-2n\sqrt{(a-u)}-2n\sqrt{(a+b)}+2n\sqrt{(a+b-u)}.$$

Placing $u = a$ and $b = xz$; then we have

$$R=2n\sqrt{a}+2n\sqrt{xz}-2n\sqrt{(a+xz)}.$$

On account of which :

$$\int\frac{Rdz}{\sqrt{(1-z)}}=2n\int\frac{dz\sqrt{a}}{\sqrt{(1-z)}}+2n\sqrt{x}\int\frac{dz\sqrt{z}}{\sqrt{(1-z)}}-2n\int\frac{dz\sqrt{(a+xz)}}{\sqrt{(1-z)}}$$
$$=4n\sqrt{a}-4n\sqrt{a(1-z)}+2n\sqrt{x}\int\frac{zdz}{\sqrt{(z-z^2)}}-2n\int\frac{adz+xzdz}{\sqrt{(a-az+xz-xz^2)}}.$$

Now,

$$\int\frac{zdz}{\sqrt{(z-zz)}}=\frac{1}{2}\int\frac{dz}{\sqrt{(z-zz)}}-\sqrt{(z-zz)}=\frac{1}{2}\pi,$$

on integration, putting $z = 1$. But $\int\frac{adz+xzdz}{\sqrt{(a-az+xz-xz^2)}}$, if after integration we put $z = 1$, gives $\sqrt{a}+\frac{a+x}{2\sqrt{x}}A.\frac{2\sqrt{ax}}{a+x}$ with $A.\frac{2\sqrt{ax}}{a+x}$ denoting the arc of a circle of radius equal to 1, the sine of which is $\frac{2\sqrt{ax}}{a+x}$. On this account,

$$\frac{\pi s}{\sqrt{x}}=4n\sqrt{a}+n\pi\sqrt{x}-2n\sqrt{a}-\frac{n(a+x)}{\sqrt{x}}A.\frac{2\sqrt{ax}}{a+x}$$

and hence [p. 219]

$$s=nx+\frac{2n\sqrt{ax}}{\pi}-\frac{n(a+x)}{\pi}A.\frac{2\sqrt{ax}}{a+x}.$$

The differential of this equation is :

$$ds = ndx - \frac{ndx}{\pi} A. \frac{2\sqrt{ax}}{a+x} = \frac{sdx + nadx}{a+x} - \frac{2ndx\sqrt{ax}}{\pi(a+x)}.$$

Moreover this curve cannot ascend beyond a certain height, but only as far as in *F*, where we have $ds = dx$. Therefore on putting $ds = dx$ we have $\frac{n-1}{n} = \frac{1}{\pi} A. \frac{2\sqrt{ax}}{a+x}$.
Hence the ratio becomes : as $n : n-1$ thus the semi periphery of the circle, of which the radius is 1, to the arc of the same circle, of which the cosine is *m*; then $\frac{a-x}{a+x} = m$ and $x = \frac{a(1-m)}{a+m}$. So that, if the *DAB* is 60^0, then $n = 2$ and $m = 0$ and thus $BE = a = AD$. From which it follows, if the angle *DAB* is greater than 60^0, then $x > a$, but if that angle is less than 60^0, then $x < a$. Moreover from the differential equation it is required that now as we have noted at the point *B* to be $ds = ndx$, then now always as far as *F*, to become $ds < ndx$, where it is $ds = dx$.

Corollary 3.

440. If the right line BNA is horizontal, then $n = \infty$ and $a = 0$. Moreover if $n\sqrt{a} = \sqrt{f}$, then $ds = \frac{sdx}{x} - \frac{2dx\sqrt{fx}}{\pi x}$, found from the differential equation, of which the integral is

$$\frac{s}{x} = \frac{-2}{\pi} \int \frac{dx\sqrt{f}}{x\sqrt{x}} = \frac{4\sqrt{f}}{\pi\sqrt{x}}$$

and thus

$$s = \frac{4}{\pi}\sqrt{fx}.$$

Therefore the curve is a cycloid, of which the lowest element of the given curve is kept in place.

Corollary 4. [p. 220]

441. If the differential equation $ds = ndx - \frac{ndx}{\pi} A. \frac{2\sqrt{ax}}{a+x}$ is again differentiated with *dx* put constant, there is produced : $dds = \frac{-nadx^2}{\pi(a+x)\sqrt{ax}}$. From which equation it follows that the radius of osculation of the curve at *B* becomes infinitely small.

Scholium 4.

442. From the general differential equation

$$ds = \frac{-dx}{\pi}\left(\frac{2}{1}\alpha\sqrt{x} + \frac{2\cdot 4}{1\cdot 3}\beta x\sqrt{x} + \text{etc.}\right) - \frac{dx}{2}\left(\zeta + \frac{3}{2}\eta x + \text{etc.}\right)$$

it always follows to be the case that $dds = \infty$ on putting $x = 0$, unless $\alpha = 0$. Therefore as often as α differs from zero, the radius of osculation of the curve sought is equal to zero at *B*. But if it is the case that $\alpha = 0$, then the radius of osculation of the curve *BMF* at the point *B* is found to equal $\frac{\xi^2}{3\eta}\sqrt{(\frac{\xi^2}{4}-1)}$. From which in whatever proposed example, the radius of osculation at the point *B* becomes known at once.

Example 3.

443. Let the given line *ANB* have this equation, so that it becomes $dt = Cu^n du$, then the time to pass along *NA* is equal to $\int\frac{Cu^n du}{\sqrt{(a+b-u)}}$. Putting

$a+b = f$ and $f-u = r^2$; then we have $u = f-r^2$ and

$$u^n = f^n - \frac{n}{1}f^{n-1}r^2 + \frac{n\cdot n-1}{1\cdot 2}f^{n-2}r^4 - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3}f^{n-3}r^6 + \text{etc.}$$

Now since [p. 221]

$\frac{du}{\sqrt{(a+b-u)}} = -2dr$, then the integral $\int\frac{Cu^n du}{\sqrt{(a+b-u)}} =$

$$\text{Const.} - 2C\left(f^n r - \frac{n}{1\cdot 3}f^{n-1}r^3 + \frac{n\cdot n-1}{1\cdot 2\cdot 5}f^{n-2}r^5 - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7}f^{n-3}r^7 + \text{etc.}\right).$$

Moreover since this quantity must vanish on putting $u = 0$ or $r = \sqrt{f}$, the constant quantity must be added equal to

$$2Cf^{n+\frac{1}{2}}\left(1 - \frac{n}{1\cdot 3} + \frac{n\cdot n-1}{1\cdot 2\cdot 5} - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7} + \text{etc.}\right).$$

Now on putting $u = a$ or $r = \sqrt{b}$ and in place of the series $1-\frac{n}{1.3}+\frac{n.n-1}{1.2.5}-\text{etc.}$ there is placed N; the total time of the descent along *BNA* is equal to : $2CNf^{n+\frac{1}{2}}$

$$-2C\left(f^n\sqrt{b} - \frac{n}{1\cdot 3}f^{n-1}b\sqrt{b} + \frac{n\cdot n-1}{1\cdot 2\cdot 5}f^{n-2}b^2\sqrt{b} - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7}f^{n-3}b^3\sqrt{b} + \text{etc.}\right).$$

Restoring $a + b$ in place of f and this time is given :

$$2CN\left\{a^{n+\frac{1}{2}} + \frac{(2n+1)}{2}a^{n-\frac{1}{2}}b + \frac{(2n+1)(2n-1)}{2\cdot 4}a^{n-\frac{3}{2}}b^2 + \frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6}a^{n-\frac{5}{2}}b^3 + \text{etc.}\right\}$$

$$-2C\left\{a^n\sqrt{b} + \frac{2n}{1\cdot 3}a^{n-1}b\sqrt{b} + \frac{2n(2n-2)}{1\cdot 3\cdot 5}a^{n-2}b^2\sqrt{b} + \frac{2n(2n-2)(2n-4)}{1\cdot 3\cdot 5\cdot 7}a^{n-3}b^3\sqrt{b} + \text{etc.}\right\}.$$

Therefore this series compared with the assumed series for expressing this time gives :

$$k = 2CNa^{n+\frac{1}{2}},\quad \alpha = \frac{2n+1}{2}2CNa^{n-\frac{1}{2}},\quad \beta = \frac{(2n+1)(2n-1)}{2\cdot 4}2CNa^{n-\frac{3}{2}},$$

$$\gamma = \frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6}2CNa^{n-\frac{5}{2}} \text{ etc.},$$

$$\zeta = -2Ca^n,\quad \eta = \frac{-2n}{1\cdot 3}2Ca^{n-1},\quad \theta = \frac{-2n(2n-2)}{1\cdot 3\cdot 5}2Ca^{n-2} \text{ etc.}$$

Hence there arises :

$$ds = \frac{-2CNa^n dx}{\pi}\left\{\begin{aligned}&\frac{(2n+1)\sqrt{x}}{1\cdot\sqrt{a}} + \frac{(2n+1)(2n-1)x\sqrt{x}}{1\cdot 3\cdot a\sqrt{a}}\\ &+\frac{(2n+1)(2n-1)(2n-3)x^2\sqrt{x}}{1\cdot 3\cdot 5\cdot a^2\sqrt{a}} + \text{etc.}\end{aligned}\right\}$$
$$+ Ca^n dx\left(1 + \frac{nx}{1.a} + \frac{n(n-1)x^2}{1\cdot 2\cdot a^2} + \frac{n(n-1)(n-2)x^3}{1\cdot 2\cdot 3\cdot a^3} + \text{etc.}\right).$$

Now the sum of this latter series is $Cdx(a+x)^n$, the integral of which is $\frac{C(a+x)^{n+1}}{n+1}$. [p. 222] Whereby after integration there is obtained :

$$s = \frac{C(a+x)^{n+1} - Ca^{n+1}}{n+1}$$
$$-\frac{4CNa^n\sqrt{x}}{\pi\sqrt{a}}\left(\frac{(2n+1)x}{3} + \frac{(2n+1)(2n-1)x^2}{3\cdot 5\cdot a} + \frac{(2n+1)(2n-1)(2n-3)x^3}{3\cdot 5\cdot 7\cdot a^2} + \text{etc.}\right).$$

Which is the equation for the curve sought BMF, which as often as it is constructed from a finite number of terms, so n becomes the terminus of this series $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ etc. Now $N = \int dp(1-pp)^n$, if p is put equal to 1 after the integration. And with this factor substituted it becomes :

$$\int dp(1-pp)^{n+1} = \frac{2(n+1)}{2n+3}\int dp(1-pp)^n.$$

Whereby if $n = -\frac{1}{2}$, since the integral is $\int\frac{dp}{\sqrt{(1-pp)}} = \frac{\pi}{2}$,

then $N = \frac{\pi}{2}$; if $n = \frac{1}{2}$, then $N = \frac{\pi}{4}$; if $n = \frac{3}{2}$, then $N = \frac{3.\pi}{4.4}$; if $n = \frac{5}{2}$, then $N = \frac{3.5.\pi}{4.6.4}$; if $n = \frac{7}{2}$, then $N = \frac{3.5.7.\pi}{4.6.8.4}$ etc. But since, if $n = 0$, then $N = 1$; then, if $n = 1$, then $N = \frac{2}{3}$; if $n = 2$, then $N = \frac{2.4}{3.5}$; if $n = 3$, then $N = \frac{2.4.6}{3.5.7}$ etc.

For if the curve is a cycloid, then $n = -\frac{1}{2}$ and thus it becomes

$$s = 2C\sqrt{(a+x)} - 2C\sqrt{a},$$

as we found above (438).

Scholium 5.

444. Therefore, when $dt = Cu^n du$, in this case the value for *s* is found and from the nature of the method it is understood, if *dt* is equal to the sum of some number of terms of this kind, then *s* is equal to the sum of the series of the individual terms produced. Therefore for this reason, if some curve is given, [p. 223] the series of terms of the form $Cu^n du$ is sought equal to *dt*. And from all these the corresponding value of *s* is obtained. For if this is the nature of the given line *ANB* : $dt = \frac{du\sqrt{c}}{\sqrt{u}} + \frac{du\sqrt{u}}{\sqrt{c}}$, the first term gives $C = \sqrt{c}$ and $n = -\frac{1}{2}$, thus the arc becomes $s = 2\sqrt{c(a+x)} - 2\sqrt{ac};$ the next term gives $C = \frac{1}{\sqrt{c}}$ and $n = \frac{1}{2}$ and $\mathrm{N} = \frac{\pi}{4}$, hence there arises :

$$s = \frac{2(a+x)^{\frac{3}{2}} - 2a\sqrt{a} - 2x\sqrt{x}}{3\sqrt{c}}.$$

Hence the curve sought is expressed by the following equation :

$$s = \frac{2(a+3c+x)\sqrt{(a+x)} - 2(a+3c)\sqrt{a} - 2x\sqrt{x}}{3\sqrt{c}}.$$

Example 4.

445. Let the given curve be a circle of diameter *c*; then the equation is

$$dt = \frac{\frac{1}{2}c\,du}{\sqrt{(cu-u^2)}} = \frac{1}{2}c\,du\left(\frac{1}{\sqrt{cu}} + \frac{1\cdot\sqrt{u}}{2\cdot c\sqrt{c}} + \frac{1\cdot 3\cdot u\sqrt{u}}{2\cdot 4\cdot c^2\sqrt{c}} + \text{etc.}\right).$$

Now with some term taken separately and the value of *ds* found; there is obtained, on collecting all the terms :

$$ds = \frac{c\,dx}{2}\left\{\begin{array}{l}\frac{1}{\sqrt{c(a+x)}} + \frac{1\cdot\sqrt{(a+x)}}{2\cdot c\sqrt{c}} + \frac{1\cdot 3\cdot(a+x)^{\frac{3}{2}}}{2\cdot 4\cdot c^2\sqrt{c}}\ \text{etc.}\\ -\frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \frac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}} - \frac{3\cdot 1\cdot 3\cdot a\sqrt{x}}{2\cdot 2\cdot 4\cdot c^2\sqrt{c}}\ \text{etc.}\end{array}\right\}$$

From which the following equation arises :

$$\frac{2ds}{cdx} = \frac{1}{\sqrt{(a+x)(c-a-x)}} - \left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}}\right)$$
$$-\frac{a}{1\cdot dx} d\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}}\right)$$
$$-\frac{a^2}{1\cdot 2\cdot dx^2} dd\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \frac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}}\right)$$
$$-\frac{a^3}{1\cdot 2\cdot 3\cdot dx^3} d^3\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \frac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}} - \frac{1\cdot 3\cdot 5\cdot x^2\sqrt{x}}{2\cdot 4\cdot 6\cdot c^3\sqrt{c}}\right) - \text{etc.}$$

Which expression can be changed into many other forms. [p. 224]

PROPOSITION 51.

Problem.

446. *According to the hypothesis of gravity acting uniformly downwards, if the curve AM* (Fig. 55) *is given, to find a curve AN of this kind, such that the oscillations which are performed on the composite curve MAN are all isochronous to each other.* [E012]

Solution.

Let the abscissa of the given curve *AM* be $AP = u$, the corresponding $AM = t$; on account of the given curve there is an equation between *u* et *t*. Then on the curve sought *AN*,

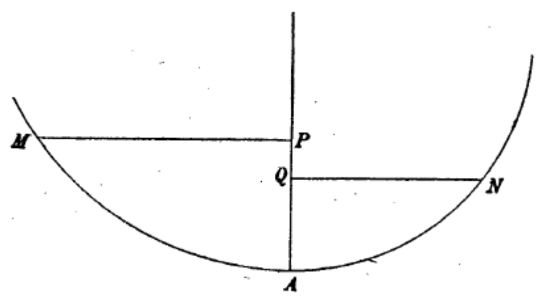

Fig. 55.

putting the abscissa $AQ = x$ and the arc $AN = s$. Now in some oscillation the speed at the point *A* corresponds to the height *b* and the time to traverse *MAN* is equal to

$$\int\frac{dt}{\sqrt{(b-u)}} + \int\frac{ds}{\sqrt{(b-u)}}.$$

And if in this expression on putting $u = b$ and $x = b$, the time of one semi-oscillation is produced , which since it has to be constant, the letter *b* evidently must disappear from the formula expressing it. Putting

$$dt = \frac{du\sqrt{f}}{\sqrt{u}} + Pdu \text{ and } ds = \frac{dx\sqrt{h}}{\sqrt{x}} - Qdx$$

and the time of one semi-oscillation is equal to

$$\int \frac{du\sqrt{f}}{\sqrt{(bu-u^2)}} + \int \frac{dx\sqrt{h}}{\sqrt{(bx-x^2)}} + \int \frac{Pdu}{\sqrt{(b-u)}} - \int \frac{Qdx}{\sqrt{(b-x)}},$$

after putting $u = b$ and $x = b$. [p. 225] Moreover the two first terms of this expression are thus to be compared, in order that from these b vanishes on making $u = b$ and $x = b$; clearly they give $\pi\sqrt{f} + \pi\sqrt{h}$ with π denoting the periphery of the circle of which the diameter is equal to one. Whereby if the later terms are thus to be compared, so that they destroy each other on making $u = b$ and $x = b$, that which is required is found ; but it is necessary that P and Q are such quantities, which do not involve b, since they emerge in the equations of the curves. But the equation becomes

$$\int \frac{Pdu}{\sqrt{(b-u)}} - \int \frac{Qdx}{\sqrt{(b-x)}} = 0$$

on making $u = b$ and $x = b$, if Q is such a function of x as P is of u. Or, when there is no impediment, where it may be less possible to set $x = u$, making $x = u$ and it is required that $Q = P$. Now given P from the equation of the curve AM given, obviously $P = \frac{dt}{du} - \frac{\sqrt{f}}{\sqrt{u}}$. On account of which this equation is obtained for the curve sought :

$$ds = \frac{du\sqrt{h}}{\sqrt{u}} - dt + \frac{du\sqrt{f}}{\sqrt{u}}$$

or

$$s + t = 2\sqrt{hu} + 2\sqrt{fu};$$

from which equation the nature of the curve sought AN can be determined. Q.E.I.

Corollary 1.

447. Therefore on taking $AP = u = x$ (Fig. 56), since $AM = t$ and $AN = s$, then

$$NA + MA = t + s = 2(\sqrt{f} + \sqrt{h})\sqrt{AP},$$

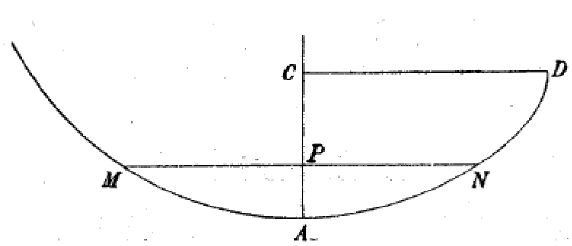

Fig. 56.

or the sum of the arcs corresponding to the same abscissa is proportional to the square root of the abscissa AP.

Corollary 2. [p. 226]

448. Therefore the curve sought *AND* thus ought to be compared, so that the sum of the arcs *AM* + *AN* is equal to the arc of the corresponding cycloid for the same abscissa *AP*. From which property it follows at once that all the oscillations are isochrones.

Corollary 3.

449. Hence the time of one oscillation is equal to the time of descent on the cycloid, of which at the bottom the radius of osculation is $2(\sqrt{f}+\sqrt{h})^2$. Or a pendulum of this length produces the smallest isochronous semi-oscillations by oscillating on the curve MAN. Now a pendulum of length $\frac{1}{2}(\sqrt{f}+\sqrt{h})^2$ performs whole isochronous oscillations.

Corollary 4.

450. Since the quantity *h* can be taken as you please, an infinite number of curves *AND* can be satisfied, and also *h* can be determined so that the time of an oscillation is a given quantity. For if a one oscillation is to be isochronous to the oscillation of a pendulum of length $\frac{L}{4}$, then we have :

$$L = 2(\sqrt{f}+\sqrt{h})^2 \text{ and thus } \sqrt{h} = \sqrt{\tfrac{L}{2}} - \sqrt{f}\ .$$

Where *L* must be greater than 2*f*.

Corollary 5.

451. If the given curve *AM* is a cycloid or $dt = \frac{du\sqrt{f}}{\sqrt{u}}$, then the other curve AN is also some cycloid ; [p. 227] for it becomes $ds = \frac{dx\sqrt{h}}{\sqrt{x}}$. And upon the two cycloids of this kind not only are the whole oscillations isochronous, but also the individual ascents and descents on whatever cycloid is completed in the same time.

Example 1.

452. Let the given curve be some straight line AM inclined to the horizontal, so that the equation becomes $dt = ndu$; and this equation is produced for the curve sought with $\sqrt{\frac{L}{2}}$ in place of $\sqrt{f}+\sqrt{h}$:

$$ds = \frac{du\sqrt{L}}{\sqrt{2u}} - ndu = \frac{dx\sqrt{L}}{\sqrt{2x}} - ndx\ .$$

Whereby if we call *PN* = *y*, it becomes :

$$dy = dx\sqrt{\left(\frac{L}{2x} - \frac{2n\sqrt{L}}{\sqrt{2x}} + n^2 - 1\right)}\ ,$$

where $\frac{L}{4}$ denotes the length of the isochronous pendulum ; from which equation the equation of the curve sought can be constructed. Moreover, the curve has a turning point

at D and there it has a vertical tangent, which can be obtained by taking $AC = \frac{L}{2(n+1)^2}$.
Indeed the radius of osculation of the curve at the lowest point A is equal to L.

As well, this is to be noted, if $n = 1$, in which case the line AM becomes a vertical line lying on AC, to be the algebraic curve sought ; for it becomes :

$$dy = dx\frac{\sqrt{(L-2\sqrt{2Lx}}}{\sqrt{2x}},$$

the integral of which is

$$y = \frac{L}{3} - \frac{(\sqrt{L}-2\sqrt{2x})\sqrt{(L-2\sqrt{2Lx})}}{3}$$

or

$$9y^2 - 6Ly = -6L\sqrt{2Lx} + 24Lx - 16x\sqrt{2Lx},$$

which free from irrationalities clearly becomes an equation of the fourth dimension. The cusp D of this curve is obtained by taking $AC = \frac{L}{8}$, in which case $CD = \frac{L}{3}$.

Example 2. [p. 228]

453. Let the given curve AM be a circle of radius a; then

$$dt = \frac{adu}{\sqrt{(2au-u^2)}}.$$

Hence with $\sqrt{\frac{L}{2}}$ in place of $\sqrt{f} + \sqrt{h}$ the equation becomes :

$$ds = \frac{du\sqrt{L}}{\sqrt{2u}} - \frac{adu}{\sqrt{(2au-u^2)}} = \frac{dx\sqrt{L}}{\sqrt{2x}} - \frac{adx}{\sqrt{(2ax-x^2)}}.$$

From which the equation follows :

$$dy = dx\sqrt{\left(\frac{L}{2x} - \frac{2a\sqrt{L}}{x\sqrt{(4a-2x)}} + \frac{a^2}{2ax-xx} - 1\right)}.$$

The cusp of the curve AND is where

$$\frac{\sqrt{L}}{\sqrt{2x}} = 1 + \frac{a}{\sqrt{(2ax-x^2)}}$$

or

$$4x^4 + 2Lx^3 - 16ax^3 + L^2x^2 - 8aLx^2 + 24a^2x^2$$
$$-4aL^2x + 12a^2Lx - 16a^3x + 4a^2L^2 - 8a^3L + 4a^4 = 0.$$

Putting $L = a$; this becomes

$$x = 0 \text{ and } 4x^3 - 14ax^2 + 17a^2x - 8a^3 = 0.$$

But if $L = 2a$, this becomes

$$x^4 - 3ax^3 + 5a^2x^2 - 2a^3x - a^4 = 0,$$

thus making $x = a = AC$. And in this case the length of the isochronous pendulum is $\frac{a}{2}$.
[From the equation:

$$\frac{\sqrt{L}}{\sqrt{2x}} = 1 + \frac{a}{\sqrt{(2ax - x^2)}}$$

it follows that

$$4x^4 - 4Lx^3 - 16ax^3 + L^2x^2 + 16aLx^2 + 24a^2x^2$$
$$- 4aL^2x - 12a^2Lx - 16a^3x + 4a^2L^2 - 8a^3L + 4a^4 = 0.$$

On putting $L = a$ this makes

$$x = 0 \quad \text{et} \quad 4x^3 - 20ax^2 + 41a^2x - 32a^3 = 0;$$

but if $L = 2a$, this becomes

$$x^4 - 6ax^3 + 15a^2x^2 - 14a^3x + a^4 = 0,$$

thus it is concluded that the value $x = a$ does not satisfy the problem. Note by P. St.]

Scholium 1.

454. If therefore it can be brought about that a pendulum can perform oscillations on a composite curve of this kind, and equally the oscillations of this pendulum are isochronous, and as if it were moving on a cycloid. And on this account whatever curve can be used for tautochronism. There remains the question concerning this, as to how the curve is to be prepared with the given curve, so that it makes one continuous curve with the given curve, which we set out in the following proposition.

[p. 229]

PROPOSITION 52.

Problem.

455. *According to hypothesis of uniform gravity acting uniformly downwards, to find the continuous curve MAN, upon which all the semi-oscillations can be completed in equal times.*

Solution.

Therefore let *MAN* be the continuous curve (Fig. 56) and on that $AP = x$, $AM = t$, and $AN = s$. A new indeterminate [*i. e.* variable] *z* is assumed, and both *x* and *t* are thus given in terms of z, so that on putting *z* positive the part *AM* of the curve is produced, while on putting *z* negative the negative part *AN* is produced. Now since for the other part, *x*

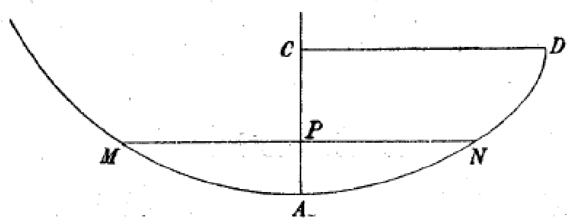

Fig. 56.

maintains the same value, *x* must be such a function of *z*, which remains the same, if z is taken either to be positive or negative, or x must be an even function of *z*. Then *t* must be a function of this kind of *z*, in order that it produces *s*, if $-z$ is put in place of *z*. But since the arc *s* falls on the other side of the axis, the value of this is negative with respect to the curve *AM*; whereby, if in the value *t*, $-z$ is put in place of *z*, it must produce $-s$. Now let *R* be an odd function of *z* and *S* an even function of z, and put $t = R + S$; this becomes $-s = -R + S$; hence $t + s = 2R$. Let the length of the isochronous pendulum be equal to *a*; since this is $\sqrt{2a} = \sqrt{f} + \sqrt{h}$, it follows that $t + s = 2\sqrt{2ax}$ and hence $R = \sqrt{2ax}$ and $x = \frac{R^2}{2a}$. Moreover, since *x* must be an even function of *z*, from this expression, that by itself can be obtained; for since *R* is an odd function, the square of this is an even function. Therefore let $R = z$; then $z = \sqrt{2ax}$ and *S* must be an even function of $\sqrt{2ax}$ or of $\sqrt{x}$. From which done this equation is obtained : $s = \sqrt{2ax} - S$ for all the continuous tautochrone curves. [p. 230] Let $dS = \frac{Tdx}{\sqrt{2ax}}$; then *T* is some odd function of $\sqrt{x}$. Wherefore it becomes $ds = \frac{adx - Tdx}{\sqrt{2ax}}$ and

$$dy = \frac{dx\sqrt{(a^2 - 2aT + T^2 - 2ax)}}{\sqrt{2ax}}$$

on putting $PN = y$. From which equation an infinite number of tautochronous curves are found. Q.E.I.

Corollary 1.

456. Therefore the curve *AN* found in this manner is a tautochrone with the continuous part *AM* of itself. Now by the preceding problem an infinite number of other curves *AM* are given, which joined with the curve *AN* produce isochronous oscillations.

Corollary 2.

457. By the preceding proposition, all the curves *AM*, of which this is the equation :

$$t = \sqrt{2cx} + S \text{ or } dt = \frac{cdx}{\sqrt{2cx}} + \frac{Tdx}{\sqrt{2ax}},$$

produces isochronous oscillations with the curve *AN* . But the length of the isochronous pendulum of these oscillations is equal to $\frac{(\sqrt{a}+\sqrt{c})^2}{4}$.

Corollary 3.

458. Hence among these infinite curves *AM* with *AN* producing isochronous oscillations is that continuous with *AN*, in which $c = a$. And the length of the isochronous pendulum becomes equal to *a*, as we have assumed.

Corollary 4.

459. If we put $c = 0$, then also this curve *AM*, of which the equation is $dt = \frac{Tdx}{\sqrt{2ax}}$ or $t = S$, is tautochronous with the curve *AN* . And in this case the length of the pendulum is $\frac{a}{4}$. [p. 231] Hence as often as $T = \sqrt{2bx}$, so also tautochronism is produced with the right line *AN*, if thus it is inclined at an angle, such that the secant of the angle *MAP* is equal to $\sqrt{\frac{a}{b}}$.

Corollary 5.

460. Since the curve *AN* must be normal to the axis *AP* at *A*, it is required that *T* vanishes on putting $x = 0$. Also likewise it follows from this, since $a - T$ must be a positive quantity, even at the starting point A. For if *T* should become infinite on putting $x = 0$, agreeing with infinite modes, yet thus, as *S* vanishes on putting $x = 0$, the curve *AN* falls on the other part of the axis *AP* and the curve has a cusp at *A* and the body, after descending on *MA*, ascends on reflection on *AN*, which would be contrary to the nature of the oscillations.

Corollary 6.

461. Therefore if T vanishes on putting $x = 0$, the radius of osculation at A, which is $\frac{sds}{dx}$, as $s = y$ in this place is equal to a and thus the oscillations agree with the smallest oscillations of the pendulum of length a, as we assumed.

Corollary 7.

462. The part of the curve AN has a vertical tangent at D and a cusp there ; since the point is found from this equation $a - T = \sqrt{2ax}$ on taking AC equal to the value of x from this equation. The other part too AM has a cusp, if somewhere it becomes : $a + T = \sqrt{2ax}$.

Corollary 8. [p. 232]

463. If it happens that $S = 0$ and $T = 0$, then $s = \sqrt{2ax}$. Whereby the curve is a cycloid and the part AN is equal and similar to the curve AM. Hence it is a continuous cycloid curve, upon which all the oscillations are completed in the same time.

Example.

464. Let $T = \sqrt{2bx}$, in which case the curve AN is also a tautochrone with the right line AC making an angle, the cosine of which is $\sqrt{\frac{a}{b}}$; then the equation arises :

$$ds = \frac{adx - dx\sqrt{2bx}}{\sqrt{2ax}} \text{ and } s = \sqrt{2ax} - \frac{x\sqrt{b}}{\sqrt{a}}.$$

Moreover, there is obtained $dy = \frac{dx\sqrt{(a^2 - 2a\sqrt{2bx} + 2bx - 2ax)}}{\sqrt{2ax}}$.

Which equation also agrees with that, which we found for the curve in the preceding proposition, which constitutes a tautochrone with a straight line (452), if L is written for a and n for $\sqrt{\frac{a}{b}}$. Whereby if $b = a$, an algebraic curve NAM also is found, which is a tautochrone, the equation of which is :

$$dy = dx\sqrt{\frac{a - 2\sqrt{2ax}}{2x}}$$

and the integral of this is :

$$3y = a - (\sqrt{a} - 2\sqrt{2x})\sqrt{(a - 2\sqrt{2ax})}.$$

Which is that curve, which constitutes a tautochrone with the vertical right line, as we found above (452). Now the length of the isochronous pendulum is equal to a, if the body is oscillating on this curve. But if it is moving on the right line AC and on part of the curve AN, the length of the isochronous pendulum is $\frac{a}{4}$. And if D is the cusp of the curve, then $AC = \frac{a}{8}$, now the other root AM rises to infinity. Besides this algebraic tautochrone curve others are easily found.

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 211]

PROPOSITIO 49.

Problema.

430. *Si corpus sollicitur a potentiis quibuscunque, invenire curvam AM* (Fig. 53), *super qua omnes descensus ad punctum A usque fiant aequalibus temporibus.*

Solutio.

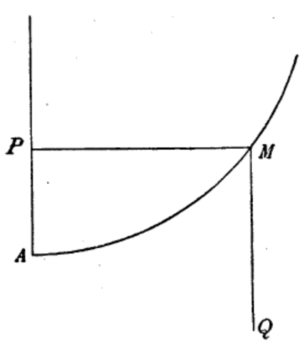

Fig. 53.

Quaecunque fuerint potentiae sollicitantes, eae omnes reduci possunt ad duas, quarum altera corpus perpetuo deorsum trahat secundum MQ, altera vero horizontaliter secundum *MP*. Sit vis, quae secundum *MQ* trahit, = *P* et vis, quae secundum *MP* trahit, = *Q*; dicatur $AP = x$, $PM = y$, $AM = s$ sitque celeritas in puncto *A* debita altitudini *b* et celeritas in *M* debita altitudini *v*. His positis erit $v = b - \int Pdx - \int Qdy$. Quare si ponatur $h = b$ et $\int Pdx + \int Qdy = z$, erit *v* functio unius dimensionis ipsarum *h* et *z* et propterea $m = 1$ (408). Quamobrem habebitur pro curva quaesita ista aequatio

$$s = 2\sqrt{az} = 2\sqrt{a(\int Pdx + \int Qdy)}$$

seu

$$ds = \frac{aPdx + aQdy}{\sqrt{a(\int Pdx + \int Qdy)}}.$$

At quia est $ds = \sqrt{(dx^2 + dy^2)}$, erit

$$\frac{dy}{dx} = \frac{aPQ \pm \sqrt{(\int Pdx + \int Qdy)(aP^2 - \int Pdx - \int Qdy + aQ^2)}}{\int Pdx + \int Qdy - aQ^2}.$$

In ipso ergo principio *A*, ubi est $\int Pdx + \int Qdy = 0$, erit $Pdx + Qdy = 0$ seu $dy : dx = -P : Q$. Atque ut ex praecedentibus intelligitur, tempus cuiusque descensus aequatur tempori, [p. 212] quo in hypothesi gravitatis = 1 pendulum longitudinis $2a$ descensum absolvit. Q.E.I.

Scholion.

431. Si habeatur curva, super qua omnes descensus fiunt eodem tempore, facile erit dare curvas, super quibus oscillationes omnes eodem tempore peragantur. Nam quia in vacuo ascensus similes sunt descensibus, omnis curva, quae est tautochrona pro descensibus, talis quoque pro ascensibus. Quare duae curvae tautochronae coniunctae in puncto A dabunt curvam, super qua omnes oscillationes sunt isochronae. Attamen hac ratione alterum problema, quo curvae omnes oscillationes isochronas producentes requiruntur, non perfecte solvitur; dari enim possunt curvae infinitae huic quaestioni satisfacientes, quarum tamen partes non sint aptae ad descensus solos isochronos efficiendos. Problema autem hoc modo proponi potest : data curva quacunque invenire aliam, quae cum ea coniuncta producat omnes oscillationes aequidiurnas. [E012 : De innumerabilibus tautochronis in vacuo, Comment. acad. sc. Petrop. 6 (1729), 1735; *O.O.* series II, vol. 4] Nunc vero antequam ad haec progrediamur, aliud problema proferemus, in quo quaeritur curva datae curvae adiungenda, ut omnes descensus super hac curva composita absolvantur temporibus aequalibus. Quod problema ut maxime difficillimum mihi quondam erat propositum a Cl. Dan. Bernoulli. [E024 : *Solutio singularis casus circa tautochronismum*, Comment. acad. sc. Petrop. 6 (1732/33), 1738; *O.O.* series II, vol. 4] Attamen hac methodo, qua in investigatione tautochronarum utor, etiam istud problema resolvi potest.

PROPOSITIO 50.

[p. 213]

Problema.

432. *In hypothesis gravitatis uniformis deorsum tendentis si detur curva ANB* (Fig. 54), *invenire curvam BMF ei adiungendam, ut omnes descensus super hac curva composita ad A usquae absolvantur aequalibus temporibus, in quocunque curvae BMF puncto descensus incipiat.*

Solutio.

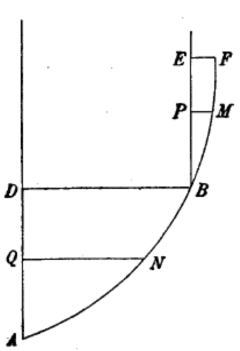

Fig. 54.

Si descensus incipiat in infimo curvae quaesitae puncto B, descensus fiet per datam curvam BNA tantum; eius ergo tempori, quod etiam dabitur, aequalia esse debent omnium descensuum tempora. Sit $AD = a$, $AQ = u$, $AN = t$ dabiturque aequatio inter u et t. Pro curva autem quaesita sit $BP = x$, et $BM = s$. Nunc in descensu quocunque sit celeritas in puncto B debita altitudini b; erit celeritas in M debita altitudini $b-x$ atque celeritas in N debita altitudini $a+b-u$. Tempus ergo descensus per curvam incognitam est

$\int\frac{ds}{\sqrt{(b-x)}}$, ita integratum, ut evanescat posito $x = 0$ et post integrationem posito $x = b$. Tempus vero per curvam cognitam *BNA* erit $\int \frac{dt}{\sqrt{(a+b-u)}}$, ita integratum, ut evanescat posito $u = 0$ et post integrationem posito $u = a$. Expressio ergo $\int \frac{ds}{\sqrt{(b-x)}}$, postquam factum est $x = b$, ita debet esse comparata, [p. 214] ut, si addatur ad expressionem temporis per *BNA*, ex aggregato penitus egrediatur littera *b*; tum enim totius tempus descensus erit quantitas constans neque pendens a *b* seu a puncto curvae *BMF*, in quo descensus incepit. Sit integrale $\int \frac{dt}{\sqrt{(a+b-u)}}$ postquam positum est $u = a$ aequale huic seriei

$$k + \alpha b + \beta b^2 + \gamma b^3 + \delta b^4 + \text{etc.} + \zeta \sqrt{b} + \eta b\sqrt{b} + \theta b^2\sqrt{b} + \iota b^3\sqrt{b} + \text{etc.}$$

Quare si descensus in puncto *B* incipiat, tempus totius descensus erit = *k* ob evanescentem *b*. Ipsi *k* ergo aequale esse debet tempus totius descensus per curvam compositam, in quocunque curvae *BMF* puncto ponatur initium descensus. Sit nunc curvae quaesitae *BMF* natura sequente serie expressa

$$ds = -Adx\sqrt{x} - Bxdx\sqrt{x} - Cx^2dx\sqrt{x} - Dx^3dx\sqrt{x} - \text{etc.}$$
$$- Fdx - Gxdx - Hx^2dx - Ix^3dx - \text{etc.}$$

Ponatur peripheriae ad diametrum ratio $\pi : 1$, quae revera est $l-1 : -\sqrt{-1}$, ita ut sit

$$\pi = -\frac{l-1}{\sqrt{-1}} = \sqrt{-1} \cdot l-1.$$

Est vero post integrationem posito $x = b$

$$\int \frac{Adx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1}{2}\pi\, Ab, \int \frac{Bxdx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1.3}{2.4}\pi\, Bb^2, \int \frac{Cx^2dx\sqrt{x}}{\sqrt{(b-x)}} = \frac{1.3.5}{2.4.6}\pi\, Cb^3 \quad \text{etc.}$$

et

$$\int \frac{Fdx\sqrt{x}}{\sqrt{(b-x)}} = 2F\sqrt{b}, \int \frac{Gdx\sqrt{x}}{\sqrt{(b-x)}} = \frac{2}{3}2Gb\sqrt{b}, \int \frac{Hx^2dx}{\sqrt{(b-x)}} = \frac{2.4}{3.5}2Hb^2\sqrt{b} \quad \text{etc.}$$

Quo igitur horum terminorum cum illis terminis coniunctim tempus per *BNA* exprimentibus aggregatum aequaetur ipsi *k*, termini homogenei *b* involventes sese tollere debent. Fiet igitur

$$\frac{1}{2}\pi\, A = \alpha \text{ seu } A = \frac{2}{1} \cdot \frac{\alpha}{\pi}$$

similique modo

$$B = \frac{2.4}{1.3} \cdot \frac{\beta}{\pi}, C = \frac{2.4.6}{1.3.5} \cdot \frac{\gamma}{\pi}, \text{etc.}$$

atque

$$F = \frac{\xi}{2}, G = \frac{3}{2} \cdot \frac{\eta}{2}, H = \frac{3.5}{2.4} \cdot \frac{\theta}{2}, I = \frac{3.5.7}{2.4.6} \cdot \frac{\iota}{2} \text{ etc.}$$ [p. 215]

Quamobrem, cum $\alpha, \beta, \gamma, \delta$ etc, ξ, η, θ, ι etc sint quantitates cognitae propter curvam *ANB* datam, habebitur pro curva quaesito *BMF* ista aequatio

$$ds = \frac{-dx}{\pi}\left(\frac{2}{1}\alpha\sqrt{x} + \frac{2.4}{1.3} \cdot \beta x\sqrt{x} + \frac{2.4.6}{1.3.5}\gamma x^2\sqrt{x} + \text{etc.}\right)$$
$$\frac{-dx}{2}\left(\xi + \frac{3}{2}\eta x + \frac{3.5}{2.4} \cdot \theta x^2 + \frac{3.5.7}{2.4.6}\iota x^3 + \text{etc.}\right)$$

cuius integralis est

$$s = \frac{-2}{\pi}\left(\frac{2}{3}\alpha x\sqrt{x} + \frac{2.4}{3.5}\cdot\beta x^2\sqrt{x} + \frac{2.4.6}{3.5.7}\gamma x^3\sqrt{x} + \text{etc.}\right)$$
$$\frac{-1}{2}\left(\xi x + \frac{3}{4}\eta x^2 + \frac{3.5}{4.6}\cdot\theta x^3 + \frac{3.5.7}{4.6.8}\iota x^4 + \text{etc.}\right)$$

Cuius seriei hanc do constructionem : sumatur

$$\int\frac{dt}{\sqrt{(a-u)}} - \int\frac{dt}{\sqrt{(a+b-u)}},$$

ita ut evanscat posito $u = 0$; tum fiat $u = a$ et prodibit functio quaedam ipsius b. Ponatur $x(1-z)$ loco b, et quod prodit, sit R. Tum integretur $\frac{Rdz}{\sqrt{z}}$, dum x ut constans consideretur, ita ut posito $z = 0$. Deinde ponatur z = 1 et prodibit functio ipsius x, quae erit = $\frac{\pi s}{\sqrt{x}}$. Hocque modo prodibit aequatio pro curva quaesita. Q.E.I.

Scholion 1.

433. Constructio haec prorsus singularis, set facilis tamen, sequitur ex ea methodo, qua usus sum in aequatione a C. Riccati quondam proposita construenda, [E 31: *Constructio aequationis differentialis* $ax^n dx = dy + y^2 dx$, Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 231; *O.O.,* series I, vol. 22. P. St.] atque hac potissimum gaudet praerogativa, quod, quaecunque fuerit curva data, quaesita eius ope semper possit constui, etiamsi aequatio ipsa, quae pro curva invenitur, minime saepe tractari possit. Dat praeterea statim aequationem finitam eam, quae alias ex summatione serierum inveniretur.

Corollarium 1. [p. 216]

434. Si in aequatione pro curva *BMF* inventa ponatur $x = 0$, erit $ds = \frac{-\xi dx}{2}$, unde inclinatione curvae in *B* ad verticalem *BP* innotescit. Quo igitur appareat, quomodo hae duae curvae invicem cohaereant, oportet quoque positionem tangentis curvae *ANB* in *B* determinare.

Corollarium 2.

435. Sit $DQ = p$ et $BN = q$ (Fig. 54) ; erit $dt = -dq$ et $a - u = p$. Unde tempus per *BNA* erit = $\int\frac{dq}{\sqrt{(b+p)}}$ posito in hoc integrali $p = a$. Sit in ipso puncto B $dq = Ldp;$ erit generatim $dq = Ldp + Pdp$ existente P tali functione ipsius p, quae evanescat posito p = 0. Videamus ergo, evanescente p qualem terminum haec aequatio $\int\frac{dq}{\sqrt{(b+p)}} = \int\frac{Ldp}{\sqrt{(b+p)}}$ producat. Prodit autem posito $p = a$: $2L\sqrt{(b+a)} - 2L\sqrt{b}$, unde in serie initio assumta prodit terminus $-2L\sqrt{b}$ (436), qui convenit cum $\xi\sqrt{b}$; erit ergo

$L=-\frac{\xi}{2}$ et $dq=\frac{-\xi dp}{2}$. Ex quo intelligitur curvam datam et quaesitam in puncto coniunctionis B commumem habere tangentem.

Scholion 2.

436. Dixi $2L\sqrt{(b+a)}-2L\sqrt{b}$ in serie dare hunc terminum $-2L\sqrt{b}$; nam $\sqrt{(b+a)}$ dat termions hos $\sqrt{a}+\frac{b}{2\sqrt{a}}+\text{etc.}$ cum aliis comparandos. [p. 217] Hic autem solus terminus Ldp dat terminum huius formae $\xi\sqrt{b}$. Quare ex eo tantum de inclinatione curvae in B concludere licet.

Scholion 3.

437. Constructio curvae quaesitae, quam dedi, etiam hoc modi immutari potest : scribatur, postquam in integrali $\int\frac{dt}{\sqrt{(a-u)}}-\int\frac{dt}{\sqrt{(a+b-u)}}$ positum est $u = a$, loco b hoc xz et vocato eo, quod prodit, R integretur $\frac{Rdz}{\sqrt{(1-z)}}$, in quo x ut quantitas constans tractetur, ita ut evanescat posito $z = 0$. Tum ponatur $z = 1$ atque id, quod provenit, aequetur $\frac{\pi s}{\sqrt{x}}$; hacque ratione plerumque commodius aequatio pro curva quaesita obtinetur.

Exemplum 1.

438. Sit curva data ANB cyclois, ita ut sit $t=2\sqrt{cu}$ seu $dt=\frac{cdu}{\sqrt{cu}}$; erit

$$\int\frac{dt}{\sqrt{(a-u)}}-\int\frac{dt}{\sqrt{(a+b-u)}}=\int\frac{du\sqrt{c}}{\sqrt{(au-u^2)}}-\int\frac{du\sqrt{c}}{\sqrt{(au+bu-u^2)}}$$

$$=\sqrt{-c}\cdot l\,\frac{a-2u-2\sqrt{(u^2-au)}}{a}-\sqrt{-c}\cdot l\,\frac{a+b-2u-2\sqrt{(u^2-au-bu)}}{a+b}.$$

Ponatur $u = a$ et habebitur

$$\sqrt{-c}\cdot l-1-\sqrt{-c}\cdot l\,\frac{b-a-2\sqrt{-ab}}{a+b}$$

$$=\pi\sqrt{c}-2\sqrt{-c}\cdot l\left(\sqrt{b}-\sqrt{-a}\right)+\sqrt{-c}\cdot l\,(a+b)$$

$$=\pi\sqrt{c}+\sqrt{-c}\cdot l\,\frac{\sqrt{b}+\sqrt{-a}}{\sqrt{b}-\sqrt{-a}}.$$

Ponatur xz loco b et habebitur

$$R=\pi\sqrt{c}+\sqrt{-c}\cdot l\,\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}};$$

quo multiplico per $\frac{dz}{\sqrt{(1-z)}}$ habebitur

$$\frac{\pi dz\sqrt{c}}{\sqrt{(1-z)}}+\frac{dz\sqrt{-c}}{\sqrt{(1-z)}}\cdot l\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}},$$

cuius integrale est [p. 218]

$$=-2\pi\sqrt{c(1-z)}-2\sqrt{c(z-1)}\cdot l\frac{\sqrt{xz}+\sqrt{-a}}{\sqrt{xz}-\sqrt{-a}}-\frac{2\sqrt{-ac}}{\sqrt{x}}\cdot l\frac{\sqrt{(z-1)}+\sqrt{z}}{\sqrt{(z-1)}-\sqrt{z}}$$
$$-\frac{2\sqrt{-c(a+x)}}{\sqrt{x}}\cdot l\frac{\sqrt{-a(1-z)}-\sqrt{z(a+x)}}{\sqrt{-a(1-z)}+\sqrt{z(a+x)}}+2\pi\sqrt{c}-2\sqrt{c}\sqrt{-1}\cdot l-1,$$

qui duo ultimi termini sunt inter se aequales ob $\pi=\sqrt{-1}.l-1.$ Ponatur nunc $z = 1$; habebitur

$$\frac{-2\sqrt{ac}+2\sqrt{c(a+x)}}{\sqrt{x}}\pi,$$

quod aequale est ponendum ipsi $\frac{\pi s}{\sqrt{x}}$. Hinc provenit ista aequatio

$$s=-2\sqrt{ac}+2\sqrt{c(a+x)}$$

seu

$$s+ANB=ANBM=2\sqrt{c(AD+BP)}.$$

Ex quo patet curvam *BMF* esse continuationem datae *AND*, ita ut coniunctae totam cycloidem constituant; id quod ex natura tautochronismi, cui cyclois satisfacere inventa est, per se sequitur.

Exemplum 2.

439. Sit linea data *ANB* recta ad horizontem utcunque inclinata; erit $dt = ndu$ atque

$$\int\frac{ndu}{\sqrt{(a-u)}}-\int\frac{ndu}{\sqrt{(a+b-u)}}=2n\sqrt{a}-2n\sqrt{(a-u)}-2n\sqrt{(a+b)}+2n\sqrt{(a+b-u)}.$$

Ponatur $u = a$ atque $b = xz$; erit

$$R=2n\sqrt{a}+2n\sqrt{xz}-2n\sqrt{(a+xz)}.$$

Quamobrem erit

$$\int\frac{Rdz}{\sqrt{(1-z)}}=2n\int\frac{dz\sqrt{a}}{\sqrt{(1-z)}}+2n\sqrt{x}\int\frac{dz\sqrt{z}}{\sqrt{(1-z)}}-2n\int\frac{dz\sqrt{(a+xz)}}{\sqrt{(1-z)}}$$
$$=4n\sqrt{a}-4n\sqrt{a(1-z)}+2n\sqrt{x}\int\frac{zdz}{\sqrt{(z-z^2)}}-2n\int\frac{adz+xzdz}{\sqrt{(a-az+xz-xz^2)}}.$$

Est vero

$$\int\frac{zdz}{\sqrt{(z-zz)}}=\frac{1}{2}\int\frac{dz}{\sqrt{(z-zz)}}-\sqrt{(z-zz)}=\frac{1}{2}\pi,$$

postquam in integrali positum est $z = 1$. At $\int \frac{adz+xzdz}{\sqrt{(a-az+xz-xz^2)}}$, si post integrationem ponatur $z = 1$, dat $\sqrt{a}+\frac{a+x}{2\sqrt{x}}A.\frac{2\sqrt{ax}}{a+x}$ denotante $A.\frac{2\sqrt{ax}}{a+x}$ arcum circuli radii = 1, cuius sinus est $\frac{2\sqrt{ax}}{a+x}$. Quocirca erit

$$\frac{\pi s}{\sqrt{x}} = 4n\sqrt{a} + n\pi\sqrt{x} - 2n\sqrt{a} - \frac{n(a+x)}{\sqrt{x}} A.\frac{2\sqrt{ax}}{a+x}$$

hincque [p. 219]

$$s = nx + \frac{2n\sqrt{ax}}{\pi} - \frac{n(a+x)}{\pi} A.\frac{2\sqrt{ax}}{a+x}.$$

Huius aequationis differentialis est

$$ds = ndx - \frac{ndx}{\pi} A.\frac{2\sqrt{ax}}{a+x} = \frac{sdx+nadx}{a+x} - \frac{2ndx\sqrt{ax}}{\pi(a+x)}.$$

Curva haec autem non ultra datam altitudinem poterit ascendere, ut in *F* usque, ubi erit $ds = dx$. Posito igitur $ds = dx$ erit $\frac{n-1}{n} = \frac{1}{\pi}A.\frac{2\sqrt{ax}}{a+x}$.
Fiat ergo ut $n : n-1$ ita semipheripheria circuli, cuius radius est 1, ad arcum eiusdem circuli, cuius cosinus sit m; erit $\frac{a-x}{a+x} = m$ atque $x = \frac{a(1-m)}{a+m}$. Ut si fuerit angulus *DAB* 60^0, erit $n = 2$ et $m = 0$ ideoque $BE = a = AD$. Ex quo sequitur, si angulus *DAB* fuerit maior quam 60^0, fore $x > a$, at si ille angulus minor fuerit quam 60^0, fore $x < a$. Ceterum ex aequatione differentiali opportet ut iam notavimus in puncto *B* fore $ds = ndx$, tum vero perpetuo fieri $ds < ndx$ usque in *F*, ubi est $ds = dx$.

Corollarium 3.

440. Si linea recta BNA fuerit horizontalis, erit $n = \infty$ et $a = 0$. Si autem fit $n\sqrt{a} = \sqrt{f}$, erit $ds = \frac{sdx}{x} - \frac{2dx\sqrt{fx}}{\pi x}$ ex aequatione differentiali inventa, cuius integrale est

$$\frac{s}{x} = \frac{-2}{\pi}\int \frac{dx\sqrt{f}}{x\sqrt{x}} = \frac{4\sqrt{f}}{\pi\sqrt{x}}$$

ideoque

$$s = \frac{4}{\pi}\sqrt{fx}.$$

Curva ergo erit cyclois, cuius infimum elementum curvae datae locum tenet.

Corollarium 4. [p. 220]

441. Si aequatio differentialis $ds = ndx - \frac{ndx}{\pi} A. \frac{2\sqrt{ax}}{a+x}$ denuo differentietur posito dx constante, prodibit $dds = \frac{-nadx^2}{\pi(a+x)\sqrt{ax}}$. Ex qua aequatione sequitur curvae in B radium osculi fore infinite parvum.

Scholion 4.

442. Ex aequatione generali differentiali

$$ds = \frac{-dx}{\pi}\left(\frac{2}{1}\alpha\sqrt{x} + \frac{2\cdot 4}{1\cdot 3}\beta x\sqrt{x} + \text{etc.}\right) - \frac{dx}{2}\left(\zeta + \frac{3}{2}\eta x + \text{etc.}\right)$$

sequitur semper fore $dds = \infty$ posito $x = 0$, nisi fuerit $\alpha = 0$. Quoties igitur non fuerit $\alpha = 0$, radius osculi curvae quaesitae in B erit = 0. At si fuerit $\alpha = 0$, tum radius osculi curvae BMF in puncto B invenitur = $\frac{\xi^2}{3\eta}\sqrt{(\frac{\xi^2}{4} - 1)}$. Ex quo in quovis exemplo proposito statim radius osculi curvae in puncto B innotescit.

Exemplum 3.

443. Sit linea data ANB hanc habuerit aequationem, ut sit $dt = Cu^n du$, erit tempus per NA = $\int \frac{Cu^n du}{\sqrt{(a+b-u)}}$. Ponatur $a + b = f$ et $f - u = r^2$; erit $u = f - r^2$ et

$$u^n = f^n - \frac{n}{1} f^{n-1} r^2 + \frac{n\cdot n-1}{1\cdot 2} f^{n-2} r^4 - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3} f^{n-3} r^6 + \text{etc.}$$

Cum vero sit [p. 221]
$\frac{du}{\sqrt{(a+b-u)}} = -2dr$, erit $\int \frac{Cu^n du}{\sqrt{(a+b-u)}}$ =

$$\text{Const.} - 2C\left(f^n r - \frac{n}{1\cdot 3} f^{n-1} r^3 + \frac{n\cdot n-1}{1\cdot 2\cdot 5} f^{n-2} r^5 - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7} f^{n-3} r^7 + \text{etc.}\right).$$

Quia autem haec quantitas evanescere debet facto $u = 0$ seu $r = \sqrt{f}$, erit quantitas illa constans addenda =

$$2Cf^{n+\frac{1}{2}}\left(1 - \frac{n}{1\cdot 3} + \frac{n\cdot n-1}{1\cdot 2\cdot 5} - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7} + \text{etc.}\right).$$

Ponatur nunc $u = a$ seu $r = \sqrt{b}$ atque loco seriei $1 - \frac{n}{1.3} + \frac{n.n-1}{1.2.5} - \text{etc.}$ ponatur N; prodibit totum descensus per BNA tempus = $2CNf^{n+\frac{1}{2}}$

$$- 2C\left(f^n\sqrt{b} - \frac{n}{1\cdot 3} f^{n-1} b\sqrt{b} + \frac{n\cdot n-1}{1\cdot 2\cdot 5} f^{n-2} b^2\sqrt{b} - \frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3\cdot 7} f^{n-3} b^3\sqrt{b} + \text{etc.}\right).$$

Restituatur $a + b$ loco f et orietur hoc tempus =

$$2CN\left\{\begin{array}{l} a^{n+\frac{1}{2}}+\frac{(2n+1)}{2}a^{n-\frac{1}{2}}b+\frac{(2n+1)(2n-1)}{2\cdot 4}a^{n-\frac{3}{2}}b^2 \\ \quad +\frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6}a^{n-\frac{5}{2}}b^3+\text{etc.}\end{array}\right\}$$

$$-2C\left\{\begin{array}{l} a^n\sqrt{b}+\frac{2n}{1\cdot 3}a^{n-1}b\sqrt{b}+\frac{2n(2n-2)}{1\cdot 3\cdot 5}a^{n-2}b^2\sqrt{b} \\ \quad +\frac{2n(2n-2)(2n-4)}{1\cdot 3\cdot 5\cdot 7}a^{n-3}b^3\sqrt{b}+\text{etc.}\end{array}\right\}.$$

Haec igitur series cum serie assumta hoc tempus exprimente comparata dat

$$k=2CNa^{n+\frac{1}{2}},\quad \alpha=\frac{2n+1}{2}2CNa^{n-\frac{1}{2}},\quad \beta=\frac{(2n+1)(2n-1)}{2\cdot 4}2CNa^{n-\frac{3}{2}},$$
$$\gamma=\frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6}2CNa^{n-\frac{5}{2}}\text{ etc.},$$
$$\zeta=-2Ca^n,\quad \eta=\frac{-2n}{1\cdot 3}2Ca^{n-1},\quad \theta=\frac{-2n(2n-2)}{1\cdot 3\cdot 5}2Ca^{n-2}\text{ etc.}$$

Hinc oritur

$$ds=\frac{-2CNa^n dx}{\pi}\left\{\begin{array}{l}\frac{(2n+1)\sqrt{x}}{1\cdot\sqrt{a}}+\frac{(2n+1)(2n-1)x\sqrt{x}}{1\cdot 3\cdot a\sqrt{a}} \\ +\frac{(2n+1)(2n-1)(2n-3)x^2\sqrt{x}}{1\cdot 3\cdot 5\cdot a^2\sqrt{a}}+\text{etc.}\end{array}\right\}$$
$$+Ca^n dx\left(1+\frac{nx}{1.a}+\frac{n(n-1)x^2}{1\cdot 2\cdot a^2}+\frac{n(n-1)(n-2)x^3}{1\cdot 2\cdot 3\cdot a^3}+\text{etc.}\right).$$

Huius vero seriei posterioris summa est $Cdx(a+x)^n$ huiusque integralis $\frac{C(a+x)^{n+1}}{n+1}$. [p. 222] Quare post integrationem habebitur

$$s=\frac{C(a+x)^{n+1}-Ca^{n+1}}{n+1}$$
$$-\frac{4CNa^n\sqrt{x}}{\pi\sqrt{a}}\left(\frac{(2n+1)x}{3}+\frac{(2n+1)(2n-1)x^2}{3\cdot 5\cdot a}+\frac{(2n+1)(2n-1)(2n-3)x^3}{3\cdot 5\cdot 7\cdot a^2}+\text{etc.}\right).$$

Quae est aequatio pro curva quaesita BMF, quae toties ex terminorum numero finito constat, quoties n fuerit terminus huius seriei $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc. Est vero $N=\int dp(1-pp)^n$, si post integrationem ponatur p = 1. Hacque facta substitutione est

$$\int dp(1-pp)^{n+1} = \frac{2(n+1)}{2n+3}\int dp(1-pp)^n.$$

Quare si fuerit $n=-\frac{1}{2}$, quia est $\int \frac{dp}{\sqrt{(1-pp)}} = \frac{\pi}{2}$,

erit $N=\frac{\pi}{2}$; si $n=\frac{1}{2}$, erit $N=\frac{\pi}{4}$; si $n=\frac{3}{2}$, erit $N=\frac{3.\pi}{4.4}$; si $n=\frac{5}{2}$, erit $N=\frac{3.5.\pi}{4.6.4}$; si $n=\frac{7}{2}$,

erit $N=\frac{3.5.7.\pi}{4.6.8.4}$ etc. At quia, si $n=0$, est $N=1$, erit, si $n=1$, $N=\frac{2}{3}$;

si $n=2$, erit $N=\frac{2.4}{3.5}$; si $n=3$, erit $N=\frac{2.4.6}{3.5.7}$ etc.

Ut si curva fuerit cyclois, erit $n=-\frac{1}{2}$ ideoque erit

$$s = 2C\sqrt{(a+x)} - 2C\sqrt{a},$$

ut supra invenimus (438).

Scholion 5.

444. Quando igitur est $dt = Cu^n du$, hic valor pro s invenitur atque ex ipsa methodi natura intelligitur, si *dt* aequetur aggregato aliquot huiusmodi terminorum, tum *s* aequalem fore aggregato serierum a singulis terminis productarum. Hac igitur ratione, si curva dat [p. 223] furerit quaecunque, series est quaerenda terminorum huius formae $Cu^n du$ ipsi dt aequalis. Atque ex iis omnibus debitus ipsius *s* valor obtinebitur. Ut si fuerit natura lineae datae *ANB* haec $dt = \frac{du\sqrt{c}}{\sqrt{u}} + \frac{du\sqrt{u}}{\sqrt{c}}$, primus terminus dat $C=\sqrt{c}$ et $n=-\frac{1}{2}$, unde fit $s = 2\sqrt{c(a+x)} - 2\sqrt{ac}$; alter terminus dat $C=\frac{1}{\sqrt{c}}$ et $n=\frac{1}{2}$ et N $=\frac{\pi}{4}$, unde oritur

$$s = \frac{2(a+x)^{\frac{3}{2}} - 2a\sqrt{a} - 2x\sqrt{x}}{3\sqrt{c}}.$$

Curva quaesita ergo sequente aequatione exprimetur

$$s = \frac{2(a+3c+x)\sqrt{(a+x)} - 2(a+3c)\sqrt{a} - 2x\sqrt{x}}{3\sqrt{c}}.$$

Exemplum 4.

445. Sit curva data circulus diametri *c*; erit

$$dt = \frac{\frac{1}{2}c\,du}{\sqrt{(cu-u^2)}} = \frac{1}{2}c\,du\left(\frac{1}{\sqrt{cu}} + \frac{1\cdot\sqrt{u}}{2\cdot c\sqrt{c}} + \frac{1\cdot 3\cdot u\sqrt{u}}{2\cdot 4\cdot c^2\sqrt{c}} + \text{etc.}\right).$$

Sumto nunc quolibet termino seorsim et inveniatur valor ipsius *ds*; habebitur colligendis omnibus

$$ds = \frac{c\,dx}{2}\left\{\begin{array}{l} \dfrac{1}{\sqrt{c(a+x)}} + \dfrac{1\cdot\sqrt{(a+x)}}{2\cdot c\sqrt{c}} + \dfrac{1\cdot 3\cdot(a+x)^{\frac{3}{2}}}{2\cdot 4\cdot c^2\sqrt{c}} \text{ etc.} \\ -\dfrac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \dfrac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}} - \dfrac{3\cdot 1\cdot 3\cdot a\sqrt{x}}{2\cdot 2\cdot 4\cdot c^2\sqrt{c}} \text{ etc.} \end{array}\right\}$$

Ex quibus sequens aequatio nascitur

$$\begin{aligned} \frac{2ds}{c\,dx} = {} & \frac{1}{\sqrt{(a+x)(c-a-x)}} - \left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}}\right) \\ & - \frac{a}{1\cdot dx} d\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}}\right) \\ & - \frac{a^2}{1\cdot 2\cdot dx^2} dd\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \frac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}}\right) \\ & - \frac{a^3}{1\cdot 2\cdot 3\cdot dx^3} d^3\left(\frac{1}{\sqrt{(cx-x^2)}} - \frac{1}{\sqrt{cx}} - \frac{1\cdot\sqrt{x}}{2\cdot c\sqrt{c}} - \frac{1\cdot 3\cdot x\sqrt{x}}{2\cdot 4\cdot c^2\sqrt{c}} - \frac{1\cdot 3\cdot 5\cdot x^2\sqrt{x}}{2\cdot 4\cdot 6\cdot c^3\sqrt{c}}\right) - \text{etc.} \end{aligned}$$

[p. 224]Quae expressio in multas alias formas transmutari potest.

PROPOSITIO 51.

Problema.

446. *In hypothesis gravitatis uniformis deorsum tendentis si detur curva AM* (Fig. 55), *invenire curvam AN eiusmodi, ut oscillationes, quae peraguntur super curva composita MAN, sint omnes inter se isochronae.* [E012]

Solutio.

Sit datae curvae *AM* abscissa $AP = u$, arcus respondes $AM = t$; dabitur ob curvam datam aequatio inter u et t. Deinde in curva quaesita *AN*

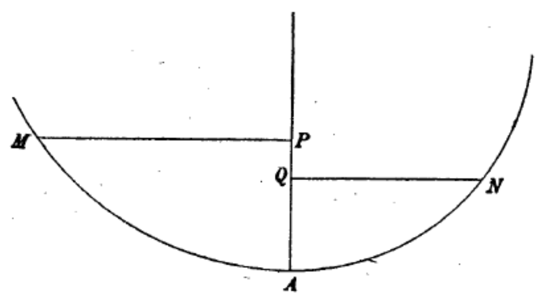

Fig. 55.

ponatur abscissa $AQ = x$ et arcus $AN = s$. Iam in oscillatione quacunque sit celeritas in puncto *A* debita altitudini b eritque tempus per *MAN* =

$$\int \frac{dt}{\sqrt{(b-u)}} + \int \frac{ds}{\sqrt{(b-u)}}.$$

Atque si in hac expressione ponatur $u = b$ et $x = b$, prodibit tempus unius semioscillationis; quod cum debeat esse constans, ex formula id exprimente littera b prorsus evanescere debet. Ponatur

$$dt = \frac{du\sqrt{f}}{\sqrt{u}} + Pdu \text{ et } ds = \frac{dx\sqrt{h}}{\sqrt{x}} - Qdx$$

eritque tempus unius semioscillationis =

$$\int \frac{du\sqrt{f}}{\sqrt{(bu-u^2)}} + \int \frac{dx\sqrt{h}}{\sqrt{(bx-x^2)}} + \int \frac{Pdu}{\sqrt{(b-u)}} - \int \frac{Qdx}{\sqrt{(b-x)}},$$

postquam positum est $u = b$ et $x = b$. [p. 225] Huius autem expressionis duo priores termini iam ita sunt comparati, ut b ex iis evanescat facto $u = b$ et $x = b$; dant nimirum $\pi\sqrt{f} + \pi\sqrt{h}$ denotante π peripheram circuli diametri = 1. Quare si posteriores termini ita fuerint comparati, ut sese destruant facto $u = b$ et $x = b$, habibitur id, quod quaeritur; at P et Q tales necesse est sint quantitates, quae b non involvant, quia in aequationes curvarum ingrediuntur. At erit

$$\int \frac{Pdu}{\sqrt{(b-u)}} - \int \frac{Qdx}{\sqrt{(b-x)}} = 0$$

facto $u = b$ et $x = b$, si Q talis fuerit functio ipsius x, qualis P est ipsius u. Seu, cum nihil impediat, quo minus possit $x = u$, fiat $x = u$ oportebitque esse $Q = P$. Datur vero P ex aequatione curvae AM datae quippe est $P = \frac{dt}{du} - \frac{\sqrt{f}}{\sqrt{u}}$. Quocirca pro curva quaesita haec habebitur aequatio

$$ds = \frac{du\sqrt{h}}{\sqrt{u}} - dt + \frac{du\sqrt{f}}{\sqrt{u}}$$

seu

$$s + t = 2\sqrt{hu} + 2\sqrt{fu}\,;$$

ex qua aequatione determinatur nature curvae quaesitae AN. Q.E.I.

Corollarium 1.

447. Sumta igitur $AP = u = x$ (Fig. 56), sum sit $AM = t$ et $AN = s$, erit

$$NA + MA = t + s = 2(\sqrt{f} + \sqrt{h})\sqrt{AP}\,,$$

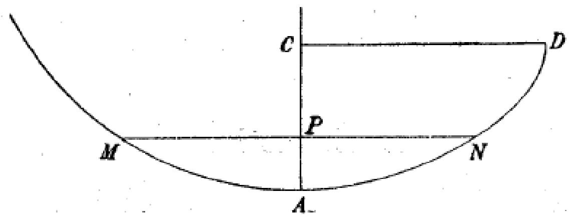

Fig. 56.

seu summa arcuum eidem abscissae respondentium proportionalis est radici quadratae ex abscissa AP.

Corollarium 2. [p. 226]

448. Curva igitur quaesita *AND* ita debet esse comparata, ut summa arcuum $AM + AN$ aequalis sit arcui cycloidis eidem abscissae AP respondentis. Ex qua proprietate sponte fluit omnes oscillationes esse isochronas.

Corollarium 3.

449. Tempus ergo unius oscillationis aequatur tempori descensus super cycloide, cuius in infimo radius osculi est $2(\sqrt{f} + \sqrt{h})^2$. Seu pendulum huius longitudinis producet semioscillationes minimas isochronas oscillatinibus super curva MAN. Pendulum vero longitudinis $\frac{1}{2}(\sqrt{f} + \sqrt{h})^2$ peraget totas oscillationes isochronas.

Corollarium 4.

450. Quia quantitatem h pro lubitu accipere licet, infinitae curvae AND satisfaciunt; atque etiam poterit determinari, ut tempus oscillationis sit datae quantitatis. Ut si una oscillatio isochrona esse debeat oscillationi penduli longitudinis $\frac{L}{4}$, erit

$$L = 2(\sqrt{f} + \sqrt{h})^2 \text{ ideoque } \sqrt{h} = \sqrt{\tfrac{L}{2}} - \sqrt{f}\ .$$

Quare *L* maius esse debet quam 2*f*.

Corollarium 5.

451. Si data curva *AM* fuerit cyclois seu $dt = \frac{du\sqrt{f}}{\sqrt{u}}$, altera curva AN erit quoque cyclois quaecunque; [p. 227] fit enim $ds = \frac{dx\sqrt{h}}{\sqrt{x}}$. Atque super duabus huiusmodi cycloidibus non solum integrae oscillationes erunt isochronae, sed etiam singuli ascensus et descensus super qualibet cycloide absolventur eodem tempore.

Exemplum 1.

452. Sit curva data AM recta utcunque ad horizontem inclinata, ut sit $dt = ndu$; prodibit pro curva quaesita posito $\sqrt{\frac{L}{2}}$ loco $\sqrt{f} + \sqrt{h}$ haec aequatio

$$ds = \frac{du\sqrt{L}}{\sqrt{2u}} - ndu = \frac{dx\sqrt{L}}{\sqrt{2x}} - ndx\ .$$

Quare si vocetur $PN = y$, erit

$$dy = dx\sqrt{\left(\frac{L}{2x} - \frac{2n\sqrt{L}}{\sqrt{2x}} + n^2\text{-}1\right)}\ ,$$

ubi $\frac{L}{4}$ denotat longitudinem penduli isochroni; ex qua aequatione curva quaesita poterit construi. Curva autem in *D* habebit punctum reversionis ibique tangentem verticalem, quod habebitur sumendo $AC = \frac{L}{2(n+1)^2}$. Curvae vero in infimo loco *A* radius osculi est $= L$.

Hic praeterea notandum est, si $n = 1$, quo casu linea *AM* fit recta verticalis in *AC* incidens, fore curvam quaesitam algebraecam; erit namque

$$dy = dx\frac{\sqrt{(L - 2\sqrt{2Lx}}}{\sqrt{2x}}\ ,$$

cuius integralis est

$$y = \frac{L}{3} - \frac{(\sqrt{L} - 2\sqrt{2x})\sqrt{(L - 2\sqrt{2Lx})}}{3}$$

seu

$$9y^2 - 6Ly = -6L\sqrt{2Lx} + 24Lx - 16x\sqrt{2Lx},$$

quae ab irrationalitate prorsus liberata fit quatuor dimensionum. Huius curvae cuspis D habebitur sumendo $AC = \frac{L}{8}$, quo casu fit $CD = \frac{L}{3}$.

Exemplum 2. [p. 228]

453. Sit curva data *AM* circulus radii *a*; erit

$$dt = \frac{adu}{\sqrt{(2au-u^2)}}.$$

Hinc posito $\sqrt{\frac{L}{2}}$ loco $\sqrt{f}+\sqrt{h}$ erit

$$ds = \frac{du\sqrt{L}}{\sqrt{2u}} - \frac{adu}{\sqrt{(2au-u^2)}} = \frac{dx\sqrt{L}}{\sqrt{2x}} - \frac{adx}{\sqrt{(2ax-x^2)}}.$$

Ex qua aequatione sequitur

$$dy = dx\sqrt{\left(\frac{L}{2x} - \frac{2a\sqrt{L}}{x\sqrt{(4a-2x)}} + \frac{a^2}{2ax-xx} - 1\right)}.$$

Cuspis curvae *AND* erit, ubi est

$$\frac{\sqrt{L}}{\sqrt{2x}} = 1 + \frac{a}{\sqrt{(2ax-x^2)}}$$

seu

$$4x^4 + 2Lx^3 - 16ax^3 + L^2x^2 - 8aLx^2 + 24a^2x^2$$
$$-4aL^2x + 12a^2Lx - 16a^3x + 4a^2L^2 - 8a^3L + 4a^4 = 0.$$

Ponatur $L = a$; fiet

$$x = 0 \quad \text{et} \quad 4x^3 - 14ax^2 + 17a^2x - 8a^3 = 0.$$

At si $L = 2a$, erit

$$x^4 - 3ax^3 + 5a^2x^2 - 2a^3x - a^4 = 0,$$

unde fit $x = a = AC$. Hocque casu longitudo penduli isochroni est $\frac{a}{2}$.

[Ex aequatione

$$\frac{\sqrt{L}}{\sqrt{2x}} = 1 + \frac{a}{\sqrt{(2ax-x^2)}}$$

sequitur

$$4x^4 - 4Lx^3 - 16ax^3 + L^2x^2 + 16aLx^2 + 24a^2x^2$$
$$-4aL^2x - 12a^2Lx - 16a^3x + 4a^2L^2 - 8a^3L + 4a^4 = 0.$$

Posito $L = a$ fit

$$x = 0 \quad \text{et} \quad 4x^3 - 20ax^2 + 41a^2x - 32a^3 = 0;$$

at si $L = 2a$, erit

$$x^4 - 6ax^3 + 15a^2x^2 - 14a^3x + a^4 = 0,$$

unde concluditur valorem $x = a$ problemati non satisfacere. P. St.]

Scholion 1.

454. Si igitur efficiatur, ut pendulum in huiusmodi curva composita oscillationes peragat, eius oscillationes aeque erunt isochronae, ac si in cycloide movetur. Atque hanc ob rem quaecunque curva ad tautochronismum adhiberi poterit. Restat in hoc negotio ista quaestio, quemadmodum curvam datam comparatam esse oporteat, ut inventa cum data unam curvam continuam constituat, id quod sequente propositione praestabimus.

[p. 229]

PROPOSITIO 52.

Problema.

455. *In hypothesi gravitatis uniformis deorsum tendentis invenire curvam continuam MAN, super qua omnes semioscillationes absolvantur aequalibus temporibus.*

Solutio.

Sit igitur curva *MAN* (Fig. 56) curva continua in eaque $AP = x$ et $AM = t$ et $AN = s$. Assumatur nova indeterminata z atque x et t ita dentur in z, ut posita z affirmativa prodeat curvae pars *AM*, at posita z negativa prodeat curvae pars *AN*. Quia nunc pro ultraque parte x eundem obtinet valorem, debebit x talis esse functio ipsius z, quae eadem maneat, sive z affirmative sumatur sive negative, seu x debet esse functio par ipsius z. Deinde t eiusmodi esse debet functio ipsius z, ut prodeat s, si ponatur $-z$ loco z. At quia arcus s in alteram partem axis cadit, eius valor erit negativus respectu curvae *AM*; quare, si in valore t ponatur $-z$ loco z, prodire debet $-s$. Sit nunc R functio impar ipsius z et S eius functio par et ponatur $t = R + S$; fiet $-s = -R + S$; unde fit $t + s = 2R$. Sit longitudo penduli isochroni $= a$; quia est $\sqrt{2a} = \sqrt{f} + \sqrt{h}$, debebit esse $t + s = 2\sqrt{2ax}$ hincque erit $R = \sqrt{2ax}$ et $x = \frac{R^2}{2a}$. Quia autem x debet esse functio par ipius z, ex hac expressione id per se obtinetur; cum enim R sit functio impar, eius quadratum erit functio par. Sit igitur $R = z$; erit $z = \sqrt{2ax}$ atque S debebit esse functio par ipsius $\sqrt{2ax}$ seu ipsius $\sqrt{x}$. Quo facto habebitur ista aequatio $s = \sqrt{2ax} - S$ pro omnibus curvis continuis tautochronis. [p. 230] Sit $dS = \frac{Tdx}{\sqrt{2ax}}$; erit T functio impar quaecunque ipsius $\sqrt{x}$. Quapropter fiet $ds = \frac{adx - Tdx}{\sqrt{2ax}}$ atque

$$dy = \frac{dx\sqrt{(a^2 - 2aT + T^2 - 2ax)}}{\sqrt{2ax}}$$

posito $PN = y$. Ex qua aequatione infinitae curvae tautochronae continuae reperiuntur. Q.E.I.

Corollarium 1.

456. Curva igitur hoc modo inventa *AN* est tautochrona cum sui ipsius parte continua *AM*. Dantur vero per praecedens problema infinitae aliae curvae *AM*, quae cum *AN* coniunctae oscillationes isochronas producunt.

Corollarium 2.

457. Per praecedentem propositionem omnis curva *AM*, cuius haec iest aequatio

$$t = \sqrt{2cx} + S \quad \text{seu} \quad dt = \frac{cdx}{\sqrt{2cx}} + \frac{Tdx}{\sqrt{2ax}},$$

oscillationes isochronas cum curva *AN* producit. At harum oscillationum longitudo penduli isochroni est = $\frac{(\sqrt{a}+\sqrt{c})^2}{4}$.

Corollarium 3.

458. Inter has ergo infinitas curvas *AM* cum *AN* oscillationes isochronas producentes ea est continua cum *AN*, in quae est $c = a$. Atque longitudo penduli isochroni fit = a, ut assumsimus.

Corollarium 4.

459. Si ponatur $c = 0$, erit cum curva *AN* quoque tautochrona haec curva *AM*, cuius aequatio est $dt = \frac{Tdx}{\sqrt{2ax}}$ seu $t = S$. Hocque casu longitudo penduli est $\frac{a}{4}$. [p. 231] Quoties ergo est $T = \sqrt{2bx}$, toties quoque linea recta cum *AN* tautochronismum producit, si ita fuerit inclinata, ut anguli *MAP* secans sit $= \sqrt{\frac{a}{b}}$.

Corollarium 5.

460. Quia curva *AN* in puncto *A* ad axem *AP* normalis esse debet, oportet, ut *T* evanescat posito $x = 0$. Idem etiam sequitur ex eo, quod $a - T$ debeat esse quantitas affirmativa, saltem in initio *A*. Si enim *T* fieret infinitum posito $x = 0$, id quod infinitis modis accedere potest, ita tamen, ut *S* evanescat posito $x = 0$, curva *AN* in alteram axis *AP* partem caderet curvaque in *A* haberet cuspidem et corpus, postquam super *MA* descendit, per reflexionem super *AN* ascenderet, quod esset contra naturam oscillationum.

Corollarium 6.

461. Si igitur *T* evanescit posito $x = 0$, radius osculi in *A*, qui est $\frac{sds}{dx}$, ob $s = y$ in hoc loco erit = a ideoque oscillationes congruent cum oscillationibus minimis penduli longitudinis a, ut assumsimus.

Corollarium 7.

462. Curvae portio *AN* habebit in *D* tangentem verticalem ibique cuspidem; quod punctum invenitur ex hac aequatione $a - T = \sqrt{2ax}$ sumendo *AC* = valori ipsius x ex hac aequatione. Altera quoque pars *AM* habebit cuspidem, si alicubi fuerit $a + T = \sqrt{2ax}$.

Corollarium 8. [p. 232]

463. Si fuerit $S = 0$ et $T = 0$, erit $s = \sqrt{2ax}$. Quare curva erit cyclois atque portio *AN* aequalis et similis curvae *AM*. Est ergo cyclois curva continua, super qua omnes oscillationes absolvuntur eodem tempore.

Exemplum.

464. Sit $T = \sqrt{2bx}$, quo casu curva AN quoque est tautochrona cum recta AC angulum constituente, cuius cosinus est $\sqrt{\frac{a}{b}}$; erit

$$ds = \frac{adx - dx\sqrt{2bx}}{\sqrt{2ax}} \quad \text{atque} \quad s = \sqrt{2ax} - \frac{x\sqrt{b}}{\sqrt{a}}.$$

Habebitur autem $dy = \frac{dx\sqrt{(a^2 - 2a\sqrt{2bx} + 2bx - 2ax)}}{\sqrt{2ax}}$.

Quae aequatio etiam congruit cum ea, quam in propositione praecedente pro curva invenimus, quae cum recta tautochronam constituat (452), si modo scribatur *L* pro *a* et *n* pro $\sqrt{\frac{a}{b}}$. Quare si fuerit $b = a$, curva quoque algebraica invenitur *NAM*, quae est tautochrona, cuius aequatio est

$$dy = dx\sqrt{\frac{a - 2\sqrt{2ax}}{2x}}$$

et integralis haec

$$3y = a - (\sqrt{a} - 2\sqrt{2x})\sqrt{(a - 2\sqrt{2ax})}.$$

Quae est ea ipsa curva, quae cum recta verticali tautochronam constituit, ut supra invenimus (452). Longitudo vero penduli isochroni est = *a*, si corpus in hac curva oscillatur. At si moveatur super recta *AC* et parte curvae *AN*, longitudo pendul isochroni erit $\frac{a}{4}$. Atque si *D* fuerit cuspis curvae, erit $AC = \frac{a}{8}$, alter vero ramus *AM* in infinitum ascendit. Praeter hanc curvam tautochronam algebraicam aliae vix inveniri poterunt.