CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 178]

PROPOSITION 41.

Problem.

367. *If a body is always drawn downwards by some force, the find the brachistochrone line AMC* (Fig. 46), *upon which the body descends the most quickly from A to C.*

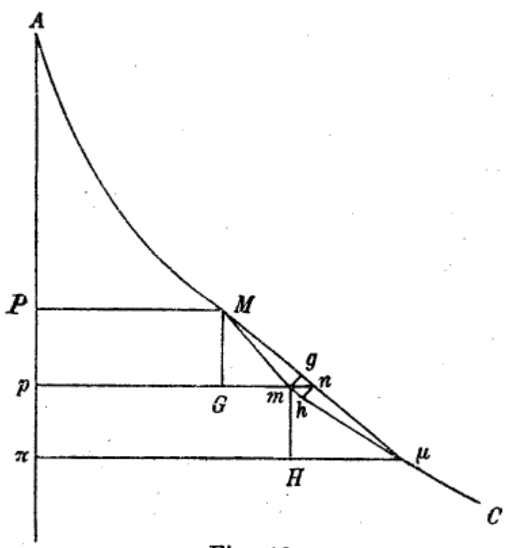


Fig. 46.

Solution. [p. 179]

On putting $AP = x$, $PM = y$ and with the $AM = s$, let the force which pushes the body downwards at M be equal to P; then $v = \int Pdx$ and with this integral thus taken so that it vanishes on putting $x = 0$, if the body indeed is put to begin its motion from rest at A, and $dv = Pdx$. Hence

$$du = Pdx = dv \text{ and } ddw = 0,$$

since du remains invariant in going from m to n. Therefore this equation is obtained (362) :

$$2vd.\frac{dy}{ds} = \frac{dydv}{ds},$$

the integral of which is :

$$l\frac{v}{a} = 2l\frac{dy}{ds} \text{ or } vds^2 = ady^2,$$

and hence :

$$dx^2\int Pdx = ady^2 - dy^2\int Pdx.$$

On account of which this equation is produced for the brachistochrone line sought :

$$dy = \frac{dx\sqrt{\int Pdx}}{\sqrt{(a-\int Pdx)}},$$

in which the indeterminates x and y have been separated from each other. Moreover the length of this curve is found from this equation : $ds = \frac{dx\sqrt{a}}{\sqrt{(a-\int Pdx)}}$.

Q.E.I.

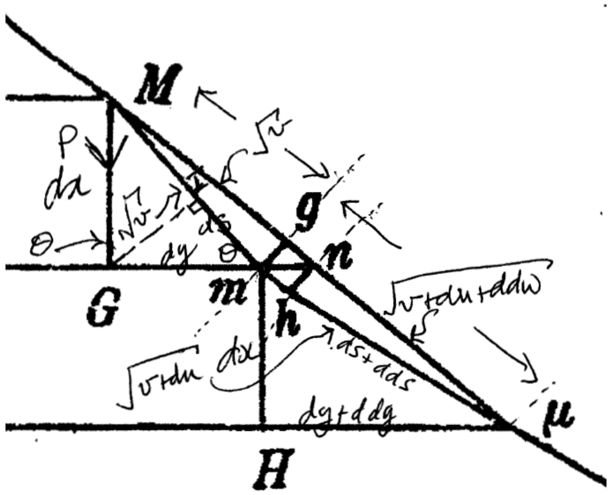
Corollary 1.

368. Therefore at A, where the speed of the body disappears, either $\int Pdx = 0$, and $dy = 0$, or the tangent to the curve at A is incident along the vertical at AP. But when it becomes $\int Pdx = a$, there the tangent to the curve is horizontal [as $dy/ds = 1$ at A].

Corollary 2.

369. Because $ddw = 0$ and $du = Pdx$, the [fundamental] equation (363) becomes :

$$\frac{2v}{r} = \frac{Pdy}{ds}.$$



Now $\frac{Pdy}{ds}$ is the normal force[; in the figure here, the normal force is given by $GI = P\cos\theta = \frac{Pdy}{ds}$; p. 180] acting upon the curve at M, along the normal drawn towards the axis AP. Consequently, the normal force is equal to the centrifugal force and acting in the same direction. On account of which the brachistochrone curve has this property, that the total force exerted on the curve is twice as great as the normal force alone. Now in the following we will demonstrate that all brachistochrone lines have this property in place, either in a vacuum or in a medium with resistance. [Thus, the centrifugal force is the force of the body pressing on the curve due to the motion, which here has the same magnitude as the normal component of the weight $\frac{Pdy}{ds}$ along IG; thus the curve supplies a total normal force equal to $2 \times \frac{Pdy}{ds}$ along GI to produce the required centripetal force.]

Corollary 3.

370. On account of the arbitrary nature of a an infinite number of brachistochrone curves are given all having the starting point A. And this letter a can be used, so that a curve from A passes through the point C, which is the line between A and C, upon which the time is a minimum.

Corollary 4.

371. Because the curve *AMC* (Fig. 47) has a horizontal tangent somewhere, let that be *BC* and the other vertical axis *CQ* is taken at *C*. Let *CQ* = *X*, *QM* = *Y* and *CM* = *S*; then $dX = -dx, dY = -dy, dS = -dS$ and $\int Pdx = a - \int PdX$ with the integral $\int PdX$ thus taken, so that it vanishes on putting *X* = 0. If the curve is referred to this axis *CQ*, this equation is found :

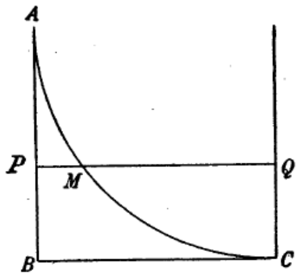


Fig. 47.

$$dY = \frac{dX\sqrt{(a - \int PdX)}}{\sqrt{\int PdX}} \text{ or } dS = \frac{dX\sqrt{a}}{\sqrt{\int PdX}}.$$

Corollary 5.

333. Therefore all these curves have two similar and equal arcs on each side of the axis *CQ*. [p. 181] In a similar manner the curve is equally disposed on each side of the axis *AB*. On account of which curves of this kind have infinite diameters parallel to each other and places at a distance *BC*, unless perhaps the force acting is therefore taken, so that above *A* it is negative, in which case the curve *CMA* can tend upwards and the concave part is changed to downwards.

Example 1.

373. Let the force acting be or *P* = *g*; the integral becomes $\int Pdx = gx;$ thus with *gb* put in place of *a*, this equation is obtained for the brachistochrone for this hypothesis of the force acting :

$$dy = \frac{dx\sqrt{x}}{\sqrt{(b-x)}} \text{ or } ds = \frac{dx\sqrt{b}}{\sqrt{(b-x)}}.$$

But if the equation is referred to the axis *CQ*, it becomes

$$dY = \frac{dX\sqrt{(b-X)}}{\sqrt{X}} \text{ or } dS = \frac{dX\sqrt{b}}{\sqrt{X}},$$

the integral of which is $S = 2\sqrt{bX}$. From which equation it is clear that the curve is a cycloid upon a horizontal base described by a circle of diameter *b* and turned downwards, as thus found by the most celebrated Johan Bernoulli and by other outstanding geometers now some time ago. Therefore if any two points *A* and *M* are given, the line, upon which a body descends from *A* to *M* the most quickly, can be found, if a cycloid is described having the same cusp at A and having the horizontal base passing through the point *M* ; that is easily brought about from a single described cycloid, since all cycloids are similar curves. Moreover the time, at which the body reaches *M* from *A*, is a minimum, equal to

$$\int \frac{dx\sqrt{b}}{\sqrt{g(bx-x^2)}}$$

and the length of the curve *AM* is equal to :

$$\int \frac{dx\sqrt{b}}{\sqrt{(b-x)}} = 2b - 2\sqrt{b(b-x)}.$$

Moreover, when

$$PM = y = \int \frac{dx}{\sqrt{(bx-xx)}},$$

[p. 182] the time to traverse *AM* is equal to

$$\frac{2y+2\sqrt{(bx-xx)}}{\sqrt{gb}}$$

equal to the arc in the circle of diameter *b*, the versed sine of which is equal to *x*, times by $\frac{2}{\sqrt{gb}}$.

[G.G. Leibniz, *Cummunicatio suae pariter duarumque alienarum ad edendum sibi primum a Dn. Io. Bernoullio, deinde a Dn. Marchione Hospitalio communcatarum solutionum problematis curvis celerrimi descensus a Dn. Io. Bernoullio geometris publice propositi, una cum solutione sua problematis alterius ab eodem postea*, Acta erud. 1697, p. 201; *Mathematische Schriften*, herausgegeben von C. I. Gerhardt, 2. Abteilung, Band 1, Halle 1858, p. 301.
Iac. Bernoulli, *Solutio problematum fraternorum ....una cum propositione aliorum*, Acta erud. 1697, p. 211; *Opera*, Genevae 1744. p. 768.
G. De L'Hospital, *Solutio problematis de linea celerrimi descensus*, Acta erud. 1697, p. 217.
I. Newton, *Epistola missa ad praenobilem virum D. Carolum Montague, in qua solvuntur duo problema mathematicis a Johanne Bernoulli math. cel. proposita*, Phil. trans. (London) 1697; Acta erud. 1697, p. 223; *Opuscula*, Tom. I, Lausannae et Genevae 1744, p. 280.
R. Sault, *Analytical investigation of the curve of quickest descent*, Phil. trans. (London) 1698, p. 425.
I. Craig, *The curve of quickest descent*, Phil. trans. (London) 1701, p. 746.
P. St.]

Example 2.

374. If the force acting *P* is as some power of the abscissa *CQ*, clearly $P = \frac{X^n}{f^n}$, then $\int PdX = \frac{X^{n+1}}{(n+1)f^n}$. Consequently, the brachistochrone curve AMC is expressed by this equation :

$$dY = \frac{dX\sqrt{((n+1)af^n - X^{n+1})}}{X^{\frac{n+1}{2}}} \text{ or } dS = \frac{dX\sqrt{(n+1)af^n}}{X^{\frac{n+1}{2}}},$$

thus so that :

$$S = \frac{2X^{\frac{1-n}{2}}}{1-n}\sqrt{(n+1)af^n}.$$

Whereby if $n = 1$ or $n > 1$, the curve *CM* or the line *BC* becomes infinitely great. Moreover the cusp of the curve A, or the place at which the motion starts, has to be taken:

$$CQ = BA = \sqrt[n+1]{(n+1)af^n}.$$

Algebraic curves are produced if

$$n = \frac{1-2m}{1+2m}$$

with *m* denoting some positive number. Therefore in these cases *n* is a negative number less than one, yet thus, so that $n + 1$ is a positive number. Let $m = 1$, then $n = -\frac{1}{3}$.

Whereby it makes

$$dY = \frac{dX}{X^{\frac{1}{3}}}\sqrt{\left(\frac{2}{3}af^{-\frac{1}{3}} - X^{\frac{2}{3}}\right)},$$

the integral of which is [p. 183]

$$Y = \frac{2a\sqrt{2a}}{3\sqrt{3f}} - \left(\frac{2}{3}af^{-\frac{1}{3}} - X^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

Which equation free from irrationalities is of the sixth order. In a similar way other algebraic curves can be found, which under certain hypotheses are brachistochrones.

Scholium 1.

375. From the solution of the given problems the solution of inverse problems follow, in which the force acting is sought directed downwards, such that the given curve is a brachistochrone. Moreover this curve must have the lowest point *C* as a horizontal tangent and for some point *A*, where the motion starts, the tangent is vertical. As if this equation $dY = RdX$ is given for the motion, then

$$R^2\int PdX = a - \int PdX \text{ and } \int PdX = \frac{a}{R^2+1}.$$

Hence it is found that

$$P = \frac{-2aRdR}{(R^2+1)^2dX} = \frac{-2adXdYddY}{dS^4}.$$

If therefore the radius of osculation at *M* is put as *r*, since $r = \frac{dS^3}{-dXddY}$ there is obtained :

$$P = \frac{2adY}{rdS}.$$

Whereby the problem is solved by this single ratio: so that the radius of osculation of the curve at *M* to a given line, is thus as the sine of the angle that the tangent to the curve makes with the vertical, to the force acting, which is sought. Now the height corresponding to the speed that the body has at *M* is given by :

$$a - \int PdX = \frac{aR^2}{R^2+1} = \frac{adY^2}{dS^2},$$

from which it follows that the speed of the body is proportional to the sine of that angle that the tangent to the curve makes with the vertical. As if the curve *CMA* is a circle described with radius *c*, then $r = c$ and

$$dY = \frac{cdX - XdX}{\sqrt{(2cX - XX)}} \text{ and } dS = \frac{cdX}{\sqrt{(2cX - XX)}},$$

from which the force becomes :

$$P = \frac{2a(c - X)}{c^2} = \frac{2a \cdot AP}{cc}.$$

Therefore the force pulling the body down must be proportional to the abscissa *AP*, to which also the speed is proportional.

Scholium 2.

376. The order requires that the brachistochrone line is found for the hypothesis of a force acting pulling downwards, so that we can now determine the brachistochrone lines for the hypothesis of centripetal forces. But the fundamental proposition (361) thus has been prepared, so that elements of the curve *Mm* and $m\mu$ (Fig. 46) and the orthogonal ordinates *MP* and *mp* are referred to the axis *AP*, which in the case of centripetal forces is not easily squared. Indeed it is seen that the elements *MG* and *mH* can be considered as converging to the centre of force; but this error arises from this, since the elements *MG* and *mH* cannot be parallel, as required by the fundamental proposition, and which it is wrong to disregard. It is evident that the situation might be restored by determining the radius of curvature, which, if *MG* and *mH* are parallel to each other, is equal to $MG : d.\frac{mG}{Mm};$ [recall in the inserted diagram above, that $\frac{ds}{dx} = \frac{1}{\sin\theta} = \frac{ds}{d\theta}.\frac{d\theta}{dx}$, or $r = \frac{1}{\sin\theta}.\frac{dx}{d\theta} = dx / d(dy / ds)$, ignoring signs;] which expression cannot be put in place, if *MG* and *mH* converge to a centre of force. Whereby, before we approach brachistochrones under the hypothesis of centripetal forces, [p. 185] we will derive a property from the fundamental proposition to accommodate the hypothesis of whatever the forces are acting. From which it is observed that the most celebrated Hermann in *Phorononia* and others, who gave the brachistochrones for centripetal forces, are in error, since they have used a principle not consistent with the truth, as will soon be indicated.

[Iac. Hermann, *Phoronomia, seu de viribus et motibus corporum solidorum et fluidorum,* Amstelodami 1716, p. 81.
Ioh. Machin, *Inventio curvae, quam corpus descendens brevissimo tempore describeret, urgente vi centripeta ad datum punctum tendente, quae crescat vel decrescat iuxta quamvis potentiam distantiae a centro; dato nempe imo curvae puncto et altitudine in principio casus,* Phil. trans. (London) 1718, p. 860. P. St.]

PROPOSITION 42.

Theorem.

377. *Whatever the forces acting should be, that line is a brachistochrone, upon which the moving body presses with a force that is twice as great, as either the centrifugal force alone or the normal force alone.*

Demonstration.

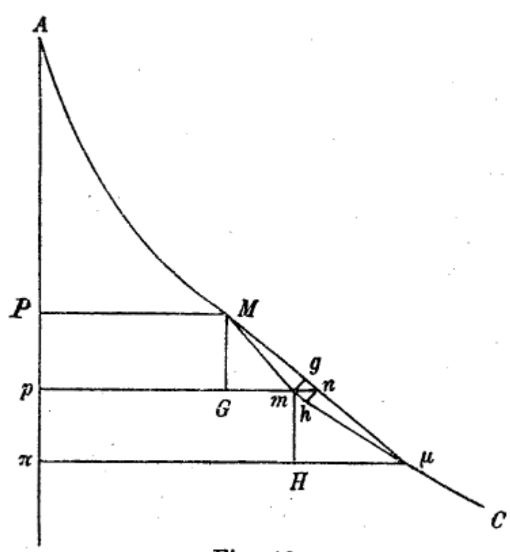


Fig. 46.

Whichever and however many the forces acting should be, all these can be resolved into pairs, of which one pulls along the line *MG* (Fig. 46), and the other along *MP*. Let that one pulling along *MG* be equal to *P* and the one that pulls along *MP* be equal to *Q*, and calling *AP* = *x*, *PM* = *y* and *AM* = *s* and likewise the height corresponding to the speed at M is equal to *v*. From these two forces the tangential force is equal to $\frac{Pdx-Qdy}{ds}$ and the normal force is equal to $\frac{Pdy+Qdx}{ds}$. On this account we have $dv = Pdx - Qdy$. When this expression is compared with that produced above (364), where we put $dv = Pdx + Qdy + Rds$; [p. 186] *Q* is negative [as it acts in the opposite direction, *i.e.* inwards towards the axis *AP*;], and *R* = 0. Hence it follows from this to be the case that $\frac{2v}{r} = \frac{Pdy+Qdx}{ds}$. But $\frac{2v}{r}$ is the centrifugal force, by which the curve at *M* is pressed, and $\frac{Pdy+Qdx}{ds}$ is the normal force. Whereby since the centrifugal force is equal to the normal force, the total compressing force, which acts on the curve, is twice as great as either the centrifugal force alone, or the normal force alone. Q.E.D. [See (369)]

Scholium 1.

378. In the following chapter we demonstrate that this same proposition is in place for a medium with some resistance; which we could demonstrate here by the same labour; but since resistance has been designated to the following chapter, it has been considered rather to transfer this theorem there.

Corollary 1.

379. Therefore from this proposition it is easy to determine the brachistochrone for the hypothesis of any forces acting. And now we can present this from the other part above, where we determined the curves in which the total force given had the ratio to the centrifugal force.

Corollary 2.

380. Since $dv = Pdx - Qdy$, then $v = \int Pdx - \int Qdy$ with these integrals taken so that they vanish on making x and y = 0, [p. 187] if indeed the motion should start from rest at A.

Corollary 3.

381. If therefore here the value for v is substituted, this equation is found for the brachistochrone curve :

$$\frac{2\int Pdx - 2\int Qdy}{r} = \frac{Pdy + Qdx}{ds}.$$

Now we have $r = \frac{ds^3}{dxddy}$ (363) with dx taken constant, since r is put to lie on the part opposite the axis AP; hence this equation is obtained :

$$\frac{2dxddy}{ds^2}\left(\int Pdx - \int Qdy\right) = Pdy + Qdx.$$

Corollary 4.

382. Since this equation is a differential of the second order and thus requires to be integrated twice, whatever the constant that has to be added for the one integration, the other must be effected, so that on making x = 0 also makes y = 0. Therefore an infinity of brachistochrone curves are produced for the hypothesis of the forces acting. And with an arbitrary constant it is able to be effected that the curve can pass through a give point.

Corollary 5.

383. The time, in which a body from A arrives at M , is equal to

$$\int \frac{ds}{\sqrt{\left(\int Pdx - \int Qdy\right)}} = \int \sqrt{\frac{2dxddy}{Pdy + Qdx}},$$

[p. 188] which expression indeed before been found for the equation of the curve; now this time must be the minimum amongst all the other times for all the curves connecting the points A and M.

Scholium 2.

384. Again as in whatever hypothesis of the forces acting these curves are described freely, in which the centrifugal force is equal and opposite to the normal force, thus these curves are brachistochrones, in which the normal force is also equal to the centrifugal force, but acting towards the same region. And as that property is common to all the curves freely described in a resistive medium also, thus this property is extended too to all the brachistochrone lines in a resistive medium.

PROPOSITION 43.

Problem.

385. *If the body is always drawn towards the centre of forces C* (Fig. 48), *to find the brachistochrone line AM upon which the body reaches M from A the quickest.*

Solution.

The line *AC* is drawn from the point *A*, at which the motion begins, to the centre of forces *C*, [p. 189] likewise *MC* is drawn with the perpendicular *CT* to the tangent *MT*. Putting *AC* = *a*, *CM* = *y*, *CT* = *p*, the centripetal force at *M* = *P* and the speed at *M* corresponds to the height *v*. With these in place, then we have $dv = -Pdy$ and $v = -\int Pdy$, with this integral thus taken, so that it vanishes on placing *y* = *a*. Moreover the normal force is equal to $\frac{Pp}{y}$; to this the centrifugal force must be equal and acting in the same direction ; for then the brachistochrone curve comes about, as we have shown in the preceding proposition. Therefore the curve must be convex towards the centre *C* and the radius of osculation falls on the side away from the centre *C*. Whereby, since this expression $\frac{ydy}{dp}$ shows the radius of osculation, in as much as it falls towards the centre, now the expression for the radius of osculation in our case is $\frac{-ydy}{dp}$. Therefore the centrifugal force is equal to $\frac{-2vdp}{ydy} = \frac{2dp\int Pdy}{ydy}$, to which the normal force $\frac{Pp}{y}$ must be put equal ; from which this equation arises : $\frac{2dp}{p} = \frac{Pdy}{\int Pdy}$, the integral of this is :

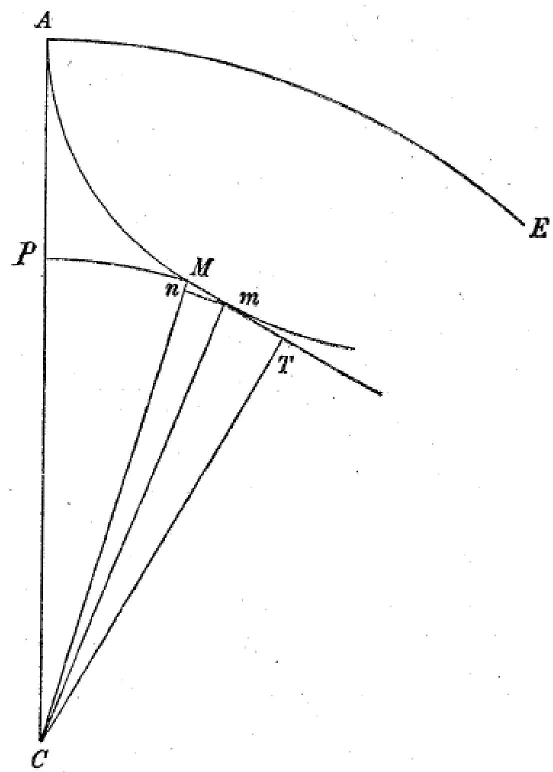


Fig. 48.

$$\frac{pp}{b} = -\int Pdy \quad \text{or} \quad p = \sqrt{-b\int Pdy};$$

which is the equation for the curve sought between *y* and *p*. But if from the centre *C* the arc *MP* is drawn, and this is called $\frac{ys}{a}$, then it becomes :

$$nm = \frac{yds}{a} \text{ and } pp = \frac{y^4 ds^2}{a^2 dy^2 + y^2 ds^2}.$$

Hence the equation becomes :

$$ds = \frac{-apdy}{y\sqrt{(y^2 - p^2)}}$$

and with the value of *p* substituted from the above equation, there is obtained :

$$ds = \frac{-ady\sqrt{-b\int Pdy}}{y\sqrt{(y^2 + b\int Pdy)}},$$

which is the equation between *y* and the arc of the circle *s* described with the radius *a*, which is expressed by the angle *ACM*, from which the construction of the curve sought follows. Q.E.I. [p. 190]

Corollary 1.

386. Since the height corresponding to the speed is given by : $v = -\int Pdy = \frac{pp}{b}$, the speed of the body at some place is as the perpendicular from *C* sent to the tangent, and in a similar manner, where in free motion the speed is inversely proportional to this perpendicular.

Corollary 2.

387. Let the radius of osculation at $M = r$; then $\frac{2v}{r} = \frac{Pp}{b}$ from the condition of the problem. Hence there is obtained $r = \frac{2yv}{Pp} = \frac{2py}{bP}$. Moreover since at the start of the curve at *A*, we set $p = 0$, or *AC* is the tangent to the curve, and the radius of osculation at *A* is equal to zero also, except perhaps the centripetal force *P* likewise vanishes at *A*.

Corollary 3.

388. The body has the maximum speed at the place where $dp = 0$; moreover there from the equation for the curve it makes $dy = 0$. Whereby the body moves the quickest in that place where the straight line *CM* is normal to the curve. Therefore the curve beyond this point recedes from the centre *C*.

Scholium 1.

389. Therefore the speed of the body at the individual points of the brachistochrone is not proportional to the sine of the angle, which the tangent to the curve makes with the direction of the centripetal force ; [p. 191] for the sine of this angle *TMC* is $\frac{p}{y}$, now the speed has been found to be proportional to p. Indeed this property is in place if the centre of forces is at an infinite distance and the directions of the forces acting are parallel to each other, as is understood from Prop. 41, where the speed is as $\frac{dy}{ds}$, i. e. as the sine of the angle, which the element of the curve makes with the direction of the force acting. Moreover the most celebrated Hermann in *Comm. Acad. Petrop. A* 1727 has attributed this common property to all the brachistochrones in vacuo as in a resistive medium. And on this account not only these lines, which he gave in a medium with resistance for brachistochrones, are not such, but also those in vacuo found for centripetal forces. Moreover in this case he found the equation $\frac{-\int Pdy}{b} = \frac{p^2}{y^2}$ and now clearly in disagreement with our equation.

[Recall that Johannes Bernoulli's solution had relied on Fermat's least time principle from optics, whereby one can derive the law of refraction at a plane interface between refracting media, and which can be applied for the uniform force case in mechanics, but which it was unwise to extend to other cases.]

Example 1.

390. Let the centripetal force be in proportion to the distance of the body from the centre of force ; the formula for the force becomes $P = \frac{y}{f}$. Whereby this gives

$$\int Pdy = \frac{y^2 - a^2}{2f} \text{ and } v = \frac{a^2 - y^2}{2f} = \frac{p^2}{b}.$$

Which is the equation for the brachistochrone according to this hypothesis of the centripetal force between *p* and *y*. Now the other equation between the arc described with the radius, which is a measure of the angle ACM, and *y* is this :

$$ds = \frac{-ady\sqrt{b(a^2 - y^2)}}{y\sqrt{(2fy^2 + by^2 - a^2b)}}.$$

The lowest point of this curve or nearest to the centre is obtained by putting either [p. 192] $dy = 0$ or $p = y$; moreover then it becomes :

$$y = \frac{a\sqrt{b}}{\sqrt{(b + 2f)}};$$

therefore this is the minimum distance of the curve from the centre *C*. The radius of osculation of this curve at any point is equal to :

$$\frac{2py}{bP} = \sqrt{\frac{2f(a^2 - y^2)}{b}}.$$

Therefore at the point nearest the centre the radius of osculation is a maximum, clearly equal to :

$$\frac{2af}{\sqrt{b(b + 2f)}}.$$

The tangent of the angle *ACM* is put equal to *t* , with the total sine equal to 1; then we have $\frac{ds}{a} = \frac{dt}{1+tt}$; again on putting

$$\frac{\sqrt{b}(a^2 - y^2)}{\sqrt{(2fy^2 + by^2 - a^2b)}} = q;$$

this equation is found :

$$\frac{dt}{1+tt} = \frac{dq}{1+qq} - \frac{dq}{1+\frac{b+2f}{b}qq}.$$

From which it is understood that the curves are algebraic, as long as $\frac{b}{b+2f}$ is a square number. Again, the length of the curve *AM* is generally equal to :

$$\int \frac{-ydy}{\sqrt{(y^2+b\int Pdy)}};$$

and in this case it is given by :

$$AM = \int \frac{-ydy\sqrt{2f}}{\sqrt{(2fy^2+by^2-a^2b)}} = \frac{2af-\sqrt{2f(2fyy+byy-a^2b)}}{2f+b}.$$

From which equation it follows that the brachistochrone curve *AM* is a hypocycloid, which is generated by the rotation of a circle, the diameter of which is equal to :

$$\frac{a\sqrt{(b+2f)}-a\sqrt{b}}{\sqrt{(b+2f)}},$$

described on the concave side of the periphery *AE* with centre *C* and with radius *AC*. Therefore since *b* can be taken as you wish, it is apparent that all the hypocycloids arising on the periphery *AE* are brachistochrones.

Example 2.

391. Let the centripetal force be proportional to the square of the distances, so that the force is given by $P=\frac{y^2}{f^2}$; hence it becomes :

$$\int Pdy = \frac{-f^2}{y} + \frac{f^2}{a} = \frac{f^2(y-a)}{ay} \text{ and } v = \frac{f^2(a-y)}{ay} = \frac{p^2}{b},$$

[p. 193] Which is the equation for the brachistochrone according to this hypothesis of the centripetal force. Now the other equation between the arc *s* and *y* is this :

$$ds = \frac{-ady\sqrt{bf^2(a-y)}}{y\sqrt{(ay^3+bf^2y-abf^2)}}.$$

Therefore the lowest point of this curve is determined by putting $dy = 0$ with the aid of this cubic equation $ay^3+bf^2y=abf^2$. Moreover this equation between *s* and *y* is sufficient for the curve sought to be constructed.

Scholium 2.

392. Therefore from these, which have been presented in this and in the preceding proposition, it is understood, how under any hypothesis of the forces acting, that line can be found upon which a body arrives at a given point in the shortest possible time from a given point. Therefore it is now necessary to determine that line, upon which a body arrives the quickest, not to a given point, but to a given line from a given point, which curve reasonably is one from an infinite number of brachistochrones; but which we will indicate in the following proposition.

PROPOSITION 44.

Theorem.

393. *A body from a given point A* (Fig. 49) *arrives the quickest at some given line BM on the brachistochrone AM, which crosses the given line BM at right angles, and this under the hypothesis of some force acting.* [p. 194]

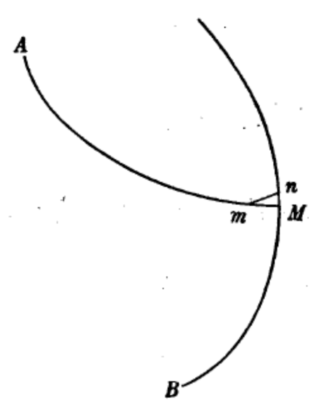


Fig. 49.

Demonstration.

Let *AM* be that line on which the body from *A* arrives the quickest at the line *BM* ; it is evident in the first place that this line is a brachistochrone; for if the line is given, upon which the body the body arrives more quickly at *M* from *A*, by that the question is rather satisfied. Besides this line *AM* crosses at the point *M* of the curve *BM* at right angles; for unless it crosses at right angles, by drawing a shorter normal *mn*, as *mn* < *mM*, the body can arrive quicker at the curve *BM* along *Amn* rather than along *AmM*. Whereby without exception it is necessary that the place be found, so that the given curve *AM* stands normally to the curve. Consequently, the body upon that curve of an infinitude of brachistochrones drawn from A to the curve BM arrives at the curve BM the quickest which crosses curve BM at right angles. Q.E.D.

Corollary 1.

394. Therefore if an infinity of curves are sought, upon which arrives at BM from A in a given time, then it is necessary that the given time is greater than the time for the brachistochrone *AM*; for otherwise the problem becomes impossible.

Corollary 2.

395. If it should happen that more brachistochrone curves are normal to the curve *BM*, [p. 195] the more maximum and minimum times are produces also. For this method indicates minima as well as maxima times.

Corollary 3.

396. Since the time for a brachistochrone curve *AM* is a minimum, it is understood from the method of maxima and minima, if two nearby brachistochrones are considered standing normal to the curve *BM*, the times for these are equal to each other.

Corollary 4.

397. It is hence again evident, if the curve *BM* is of such a kind that all the brachistochrones drawn from the point *A* cut at right angles, the times for all the brachistochrones drawn as far as the curve *BM* must be equal to each other.

Corollary 5.

398. On account of which the curve, which for all the brachistochrone curves drawn from the point A, which is the isochronous arc, or the curve that cuts the curves at the same time of travel, that too cuts all the brachistochrones at right angles, or it is the orthogonal trajectory of these curves.

Corollary 6.

399. And it is evident in turn too, if the curve [p. 196] which from the infinitude of curves the isochronous arc cuts, is the orthogonal trajectory of this, then this infinitude of curves are all brachistochrones.

Scholium.

400. It is easily understood that this proposition has a place too in a medium with resistance ; for in a like manner it is apparent that the time to pass through the normal element *mn* on the curve *BM* to be less that the time to pass through the element *mM*, which is not perpendicular ; moreover in the whole of this demonstration the force has been in place. Whereby if from the infinitude of curves drawn from the point A the law of the forces acting and the resistance can be found, in which these curves are all brachistochrones, likewise the orthogonal trajectory of these curves can also be shown by seeking only the curve the isochronous arc cuts from these curves. And this method of finding orthogonal trajectories now, the most celebrated Johan Bernoulli used in Act. Lips. A 1697. [see the note after (366)]

PROPOSITION 45.

Problem.

401. *Between all the curves joining the points A and C* (Fig. 50) *and equal in length, to determine that curve AMC, upon which a body arrives the quickest from A to C, according to the hypothesis that a uniform force g is acting directed downwards.* [p. 197]

Solution.

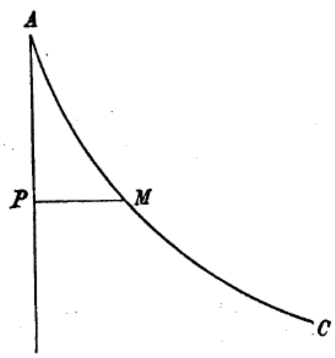


Fig. 50.

With the vertical *AP* and the horizontal *PM* drawn, and calling *AP* = *x*, *PM* = *y* and *AM* = *s*, the time in which the arc AM is completed is equal to $\int \frac{ds}{\sqrt{gx}}$. Now by the method of isoperimetrics, concerning which I gave a singular dissertation with general formulas, from which any problems are able to be easily resolved, in the Comment. Acad. Petr. 1733, [*Problematis isoperimetrici in latissimo sensu accepti solutio generalis*, p. 123; *Opera Omnia*, series I, vol. 25; E027], two quantities are to be considered, the arc $AM = s = \int ds$ and the time to pass through *AM,* equal to $\int \frac{ds}{\sqrt{gx}}$, of which the one with respect to the other has to be a minimum or a maximum. For the same is returned, either if between all the curves of equal length, that curve is sought which has the shortest descent time, or if between all the curves for which the descent is made in the same time that curve is sought which is the shortest. Moreover, from my formulas, $\int ds$ gives this quantity $diff.\frac{dy}{ds}$, and $\int \frac{ds}{\sqrt{gx}}$ gives $diff.\frac{dy}{ds\sqrt{gx}}$, of which the one can be put equal to any multiple of the other. Hence this equation is obtained to be integrated :

$$\frac{dy}{ds} = \frac{dy\sqrt{a}}{ds\sqrt{x}} - m$$

or

$$dy(\sqrt{a} - \sqrt{x}) = mds\sqrt{x}.$$

Now by taking the square the equation becomes :

$$dy^2(\sqrt{a} - \sqrt{x})^2 = m^2xdx^2 + m^2xdy^2,$$

hence it becomes :

$$dy = \frac{mdx\sqrt{x}}{\sqrt{(a - 2\sqrt{ax} + (1 - m^2)x)}} \text{ and } ds = \frac{dx(\sqrt{a} - \sqrt{x})}{\sqrt{(a - 2\sqrt{ax} + (1 - m^2)x)}}.$$

From which the curve sought can be found. Q.E.I.

Corollary 1.

402. In the equation found there are two arbitrary quantities in there *a* and *m*, from which it can be put into effect, that the curve passed through a given point *C* and that likewise it is of the given length[p. 198]. But then this curve is completed the quickest among all the other curves of the same length passing through *A* and *C*.

Corollary 2.

403. If *a* and *m* are put infinitely large, a cycloid is produced, which not only among all the curves of the same length, but among all the curves entirely, is completed the quickest.

Corollary 3.

404. If *m* is put equal to zero, then there is produced *dy* = 0 or a vertical straight line. But if *a* becomes equal to zero, then there arises some straight line drawn through the point *A*. For it is the straight line among all the lines which is completed in the same time, the minimum or the shortest.

Corollary 4.

405. If *m* is put equal to 1, an algebraic curve is produced ; for the equation becomes :

$$dy = \frac{dx\sqrt{x}}{\sqrt{(a - 2\sqrt{ax})}},$$

the integral of which is :

$$y = \frac{-(4a + 4\sqrt{ax} + 6x)\sqrt{(a - 2\sqrt{ax})}}{15\sqrt{a}} + \frac{4a}{15}.$$

This curve is indeed rectifiable; as it becomes :

$$s = \frac{2a}{5} - \frac{(2a + 2\sqrt{ax} - 2x)}{5\sqrt{a}}\sqrt{(a - 2\sqrt{ax})}.$$

Also as well the time for the algebraic arc *AM* can be expressed ; for it becomes :

$$\int \frac{ds}{\sqrt{x}} = \frac{4\sqrt{a}}{3} - \frac{(4\sqrt{a} - 2\sqrt{x})\sqrt{(a - 2\sqrt{ax})}}{3\sqrt{a}}.$$

Corollary 5. [p. 199]

406. If on this curve $x = \frac{a}{4}$ is taken, there the tangent is horizontal and the vertical straight line at that point of the curve is the diameter of the curve. Moreover at that place $y = \frac{4a}{15}$ and the length of the curve at that point is equal to $\frac{2a}{5}$. And the time, in which the arc is completed here, is equal to $\frac{4\sqrt{a}}{3\sqrt{g}}$. Therefore, in the same time, the body descends through the vertical height $\frac{4a}{9}$.

Scholium.

407. Now with these items concerned with the quickest speed of descent dispatched, we progress to the these curves to be considered, upon which more descents have a given relation between each other to be compared. Here it especially concerns the question of tautochronous curves, upon which either all descents, or the whole oscillations, are made in the same time to the lowest point. Then to this other difficult questions can be added that illustrate the strength of the method that we are using.

PROPOSITION 46.

Problem.

408. *To find the general law of tautochronous curves, upon which all descents to the point A are completed in the same time, with the descent taken to start from any point on the curve AM* (Fig. 51). [p. 200]

Solution.

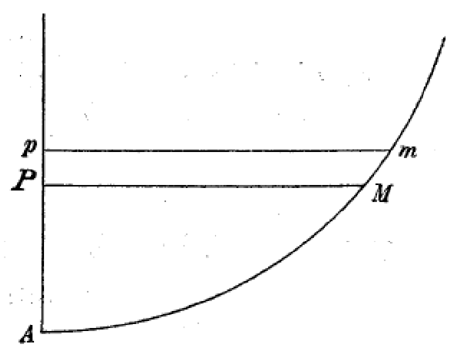


Fig. 51.

The right line *AP* is taken for the axis, calling the part of the curve *AM* = *s*, and let the height corresponding to the speed at *A* be equal to *b*, and the height corresponding to the speed at *M* equal to *v* ; the time in which the arc *AM* is completed is equal to $\int \frac{ds}{\sqrt{v}}$, which integral has thus been taken so that it vanishes on putting s = 0. Then, if *v* is put equal to zero in that integral, the descent time is had from that place in which the speed is zero, as far as the point *A* or the whole descent time ; that has to be expressed by the same amount, whatever the quantity *b* should be. Therefore neither must this quantity *b* be present in the expression for the time nor in the expression for the curve *AM*, since the same descent time must be produced by these same curves, however *b* is changed.

Now let the quantity *v* be composed from the letter *z*, related to the curve and with the curve *AM* only depending on the abscissa *AP* and the applied line *PM* , and not involving *b*, and from the letter *h*, which is composed from *b* and from constant quantities. Moreover let *v* be such a function of of *h* and *z*, that it vanishes on putting $z = h$ and so that it becomes equal to b, if $z = \alpha h$ with α taken as some number. Again on putting $ds = pdz$ for the equation of the curve sought, p must be such a quantity, in which neither *b* nor *h* is contained, since these lettes cannot arise in the equation of the curve. Therefore we have for the expression of the time to complete AM, $\int \frac{pdz}{\sqrt{v}}$ with this integral thus taken, so that it vanishes on putting $v = b$ or $z = \alpha h$. Hence this integral, if $z = h$ is put in that, [p. 201] gives the whole descent time, in which *h* cannot be present. Now this comes about, if $\int \frac{pdz}{\sqrt{v}}$ is a function of *h* and *z* of zero dimensions, or if $\frac{pdz}{\sqrt{v}}$ is a function of zero dimensions. Let *v* be a function of *m* dimensions of *h* and *z*; then it must be that $p = Cz^{\frac{m-2}{2}}$ with *C* denoting a constant quantity not depending on *b*. Hence as often as such a function can be ascertained for *v*, so this equation is obtained for the curve sought:

$$ds = Cz^{\frac{m-2}{2}}dz \text{ or } s = \frac{2Cz^{\frac{m}{2}}}{m} + \text{const.},$$

if the constant is needed, so that *s* vanishes if either *x* or *y* vanishes at *z*. Q.E.I.

Corollary 1.

409. Therefore where this method can be used, it is required that *v* is an expression from finite quantities and that this expression can be changed into a homogeneous function from constant *h* and *z*.

Corollary 2.

410. On account of this it is necessary that $v = b$, if we put $z = \alpha h$, where *h* is not found in the constant quantity added. Therefore it is sufficient, if we had considered making $v = b$ by putting $z = \alpha h$, nor is there a need [to add a constant], as the integration is completed.

Corollary 3.

411. Also it is understood that the curve at *A* must have a tangent normal to the directing of the force acting ; [p. 202] also unless this is the case, the time to descend an infinitely small arc becomes small too.

Scholion.

412. This solution prevails not only if as the figure indicates, the curve is set out by orthogonal coordinates; for nothing is different, for whatever quantities we wish to set out for the nature of the curve, as long as *b* does not enter into *z*. Moreover *z* can contain lines and any quantities depending on the curve. Therefore for this method in a vacuum, under any hypothesis of the forces acting, tautochronous lines can always be found, since the speed can always be expressed by finite quantities. But if, as in mediums with resistance it is customary to happen, the speed cannot be shown by finite quantities, then this method cannot be used, but another way is desired which is successful only if the speed is given by a differential equation.

PROPOSITION 47.

Problem.

413. *If the body is acted on by some downwards force, to find the tautochrone line upon which all the descents are made in the same time.*

Solution. [p. 203]

On putting $AP = x$, $PM = y$ and $AM = s$ (Fig. 51) let the speed at A correspond to the height b and at M it corresponds to the height v. Again let the force acting at $M = P$; then it becomes $v = b - \int Pdx$ with the integral $\int Pdx$ thus taken, in order that it vanishes on making $x = 0$. Now if on putting $b = h$ and $\int Pdx = z$, v is a function of one dimension of h and z and vanishes on making $z = h$ and it becomes $v = b$ by making $z = 0$. Therefore $m = 1$ and thus this equation is found for the curve sought (408) :

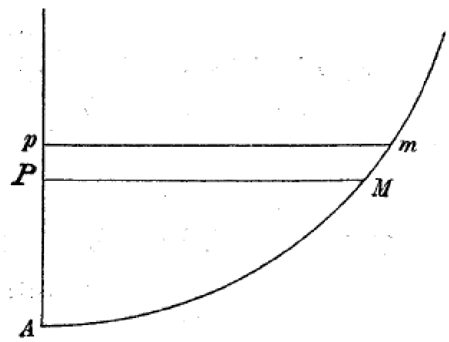


Fig. 51.

$$ds = \frac{Cdz}{\sqrt{z}} \text{ and } s = 2\sqrt{az} = 2\sqrt{a\int Pdx}.$$

If the equation between x and y is desired, then on account of $ds = \frac{aPdx}{\sqrt{a\int Pdx}}$,

$$dy = \frac{dx\sqrt{(aP^2 - \int Pdx)}}{\sqrt{\int Pdx}}.$$

Q.E.I.

Corollary 1.

414. Since the integral $\int Pdx$ vanishes on putting $x = 0$, to tangent to the curve at A is horizontal, unless P vanishes at A. And the curve has a vertical tangent somewhere on the curve and there generally a cusp ; this comes about where the integral becomes $\int Pdx = aP^2$. For there $dy = 0$.

Corollary 2.

415. At the lowest point of this curve at A the radius of osculation is equal to the subnormal $\frac{ydy}{dx} = \frac{sds}{dx}$, since at A, s and y become equal. Whereby at A the radius of osculation is equal to $2aP$, where P denotes the force acting at the point A.

Corollary 3. [p. 204]

416. From the radius of osculation at A and the force acting at A, the time for the ascent or descent is found for the infinitely small arc equal to :

$$\frac{\pi\sqrt{4aP}}{2\sqrt{P}} = \pi\sqrt{a}.$$

(172). And the time of each descent is equal to this time. Hence, under the hypothesis of gravity equal to 1, a pendulum of length 2*a* completes an infinitely small descent in the same time.

Example 1.

417. Let the force acting everywhere be constant, and certainly $P = g$; the integral becomes

$$\int Pdx = gx \text{ and } s = 2\sqrt{gax}$$

likewise

$$dy = \frac{dx\sqrt{(ga-x)}}{\sqrt{x}},$$

thus it is understood that the curve is a cycloid convex downwards, clearly congruent with the brachistochrone line under the same hypotenuse of the force. Moreover since all the descents are made in the same time on the cycloid, that we have shown above. (187).

Example 2.

418. Let the force acting be as some power of *x* ; the force becomes

$$P = \frac{x^n}{f^n} \text{ and } \int Pdx = \frac{x^{n+1}}{(n+1)f^n},$$

if indeed $n + 1$ is a positive number; but if indeed $n + 1$ is a negative number, then the integral $\int Pdx = \infty$. Therefore we have

$$s = \frac{2x^{\frac{n+1}{2}}}{f^{\frac{n}{2}}}\sqrt{\frac{a}{n+1}} \text{ and } dy = \frac{dx\sqrt{((n+1)ax^{n-1}-f^n)}}{\sqrt{f^n}}.$$

From which equation it is understood that the curve is a straight line inclined at some angle to the horizontal if $n = 1$. [p. 205] But if $n > 1$, the curve at the starting point *A* becomes imaginary, clearly until f^n begins to be less than $(n+1)ax^{n-1}$.

Scholium 1.

419. From the general equation it is apparent that the line *AP* is the diameter of the curve. Whereby when it is in a vacuum, the ascents are similar to the descents, all the semi oscillations on the curve *MA* and on the other part as far as produced are also isochrones and consequently also the whole oscillations. Then since on account of the arbitrary *a*, there is an infinite number of tautochronous curves *AM*, and two which are joined at the point *A* so that they have a common horizontal tangent there, so they produce semi-oscillations as well as whole isochronal oscillations, as if a pendulum is permitted thus to be accommodated, so that by oscillating it completes curves of this kind.

Scholium 2.

420. Also it is understood from the solution that these curves we have found are the unique ones which satisfy the question. For in place of *p* another function of *z* cannot be substituted, as in the integral, if *v* is put equal to 0, clearly from the formula *b* or *h* emerge. That which by other methods, in which tautochrones are found, is not clear enough.

Scholium 3.

421. Since the arc is given by $s = 2\sqrt{a\int Pdx}$, then it arises that $P = \frac{sds}{2adx}$. From which it is apparent, that the force acting must be of this kind, in order that the given curve is a tautochrone [p. 206]. Clearly the force acting downwards must be proportional to $\frac{sds}{dx}$ from the given curve taken. Whereby, unless the curve is rectifiable, the value of the force acting cannot be shown to be algebraic.

PROPOSITION 48.

Problem.

422. *If the body is always drawn towards the centre of forces C* (Fig. 52) *by some force, to find the tautochrone line BMA, upon which the body completes all the descend as far as the point A in the same time.*

Solution.

Call *CA* = *c*, *CM* = *y* and the centripetal force acting at *M* is equal to *P*. Again let the speed at A correspond to the height *b* and that at *M* to the height *v*. The integral $\int Pdy$ is taken thus, so that it vanishes on putting *y* = *c*, with which done, the equation becomes $v = b - \int Pdy$. Therefore by taking *b* for *h* and $\int Pdy$ for *z* , there is a function of *h* and *z*, which is equal to *v*, of one dimension ; whereby *m* is equal to 1. If now the arc *AM* is said to be equal to *s*, it gives $s = 2\sqrt{az} = 2\sqrt{a\int Pdy}$ and hence $ds = \frac{aPdy}{\sqrt{a\int Pdy}}$. If now at *M* the tangent is drawn, and to the same line the

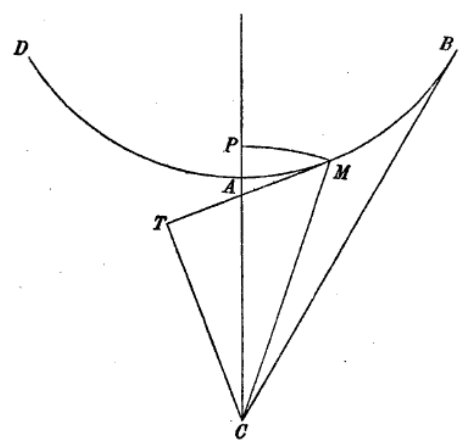


Fig. 52.

perpendicular *CT* is sent from the centre *C*, which is called p, then $\frac{ydy}{\sqrt{(y^2-p^2)}} = ds$.

Whereby the equation is obtained : $y^2\int Pdy = (y^2 - p^2)aP^2$ or $p^2 = y^2 - \frac{y^2\int Pdy}{aP^2}$, the equation for the curve sought. Q.E.I.

Corollary 1. [p. 207]

423. At the point *A*, where $\int Pdy$ vanishes, there $p = y$, or the right line *CA* is normal to the curve , and on that account the speed of the body is a maximum since *A* is the point of the curve closest to the centre *C*.

Corollary 2.

424. If we put $p = 0$, the point of the curve *B* is obtained, at which the right line *CB* is a tangent to the curve. And at that point, which is the maximum, the curve has a cusp. Now the point *B* is found from this equation : $aP^2 = \int Pdy$; and *y* cannot be greater than the value found from this equation.

Corollary 3.

425. Also it is apparent from the equation found : $s = 2\sqrt{a\int Pdy}$, since an ambiguous sign is involved with the square root, that the curve has two branches *AB* and *AD* similar and equal to each other, and on this account the oscillations which occur on the curve *BAD* are equal to each other.

Corollary 4.

426. The radius of osculation at the point *A* is equal to $\frac{2acP}{c-2aP}$. And since $AC = c$, the time in which one infinitely small descent is completed on the portion of the curve *AB*, is equal to $\pi\sqrt{a}$ (207); therefore all the times are equal to this time. [p. 208] On this account, since the oscillations, which are made on the curve *BAD*, are isochronous with the oscillations of a pendulum under the hypothesis of gravity equal to 1, of which the length is equal to $2a$.

Example 1.

427. Let the centripetal force be directly proportional to the distances from the centre, so that it is given by $P = \frac{y}{f}$; the integral becomes

$$\int Pdy = \frac{y^2-c^2}{2f}.$$

Hence the are therefore becomes :

$$AM = s = \sqrt{\frac{2a(y^2-c^2)}{f}} \text{ and } p^2 = yy - \frac{f(yy-c^2)}{2a}.$$

The radius of osculation of this curve at the point *M* which is $\frac{ydy}{dp}$, is found to be

$\frac{2a}{2a-f}\sqrt{\frac{(2a-f)y^2+fc^2)}{2a}}$.

From which it follows, if *2a* < *f*, that the curve has the convex side turned towards the centre *C*, as the figure shows. But if *2a* = *f*, the curve becomes a right line normal at *A* to the line *AC*. Moreover the point *B*, where *CB* is a tangent to the curves, is found from this equation $(f-2a)yy = ccf$, from which it formed :

$BC = \frac{c\sqrt{f}}{\sqrt{(f-2a)}}$. Therefore as often as it happens that $f > 2a$ or the curve is convex towards *C*, the curve has a cusp at *B*. And in these cases the curve is a hypocycloid, which is generated by the rotation of a circle, the diameter of which is equal to $\frac{c\sqrt{f}-c\sqrt{(f-2a)}}{\sqrt{(f-2a)}}$, upon the concave part of a circle with centre *C* described by the radius equal to $\frac{c\sqrt{f}}{\sqrt{(f-2a)}}$. Therefore in this case the tautochrone curves agree with the brachistochrone curves found above (390).

But if *2a* > *f*, in which case the curve is concave towards *C*, then *BC* becomes imaginary and the curve *AM* no further is a hypocycloid. [p. 209] Moreover then we have $p^2 = \frac{(2a-f)yy+ccf}{2a}$, hence *p* everywhere besides *A* is greater than *AC*. Let *c* = 0; then it becomes $p = y\sqrt{\frac{2a-f}{2a}}$, whereby in this case the tautochrone curves are logarithmic spirals described around the centre C. Clearly a body on logarithmic spirals always arrives at the centre C in the same time, from wherever the descent started. Therefore the tautochrones are under this hypothesis of the centripetal force : first all the hypocycloids, then all the right lines drawn in some manner, thirdly all the logarithmic spirals and in the fourth place all the infinitude of other curves contained in this equation :
$p^2 = \frac{(2a-f)yy+ccf}{2a}$, if indeed it should be that $2a > f$ and *c* is not equal to 0. Moreover under this hypothesis for tautochrones Newton in the *Principia* [Book I, Prop. LI, theorem XVIII.] and Hermann in *Phoronomia* and in the Comment. Acad. Petrop. A 1727, [p. 139, see in particular p.150; P.St.] have given only hypocycloids, although here our equally general equation can be had.

Example 2.

428. The centripetal force is put inversely proportional to the square of the distances from the centre, so that $P = \frac{ff}{yy}$; the integral becomes:

$$\int Pdy = \frac{-ff}{y} + \frac{ff}{c} = \frac{ff(y-c)}{cy}.$$

Therefore the curve *AM* is equal to

$$2f\sqrt{\frac{a(y-c)}{cy}} \quad \text{and} \quad pp = \frac{acffyy + cy^5 - y^6}{acff},$$

hence the radius of osculation is found:

$$\frac{ydy}{dp}=\frac{2fy\sqrt{ac(acff+cy^3-y^4)}}{2acff+5cy^3-6y^4}.$$

Therefore where p vanishes, there also the radius of osculation is equal to 0. Moreover the value of BC is itself found from this equation : $y^4=cy^3+acff$. [p. 210] Therefore if we put $BC = k$, then $a=\frac{k^4-ck^3}{cff}$, that which can be assumed, since k is an arbitrary quantity. Moreover it is apparent that this curve between A and B must have a turning point, which arises if the curve at A is concave towards C, clearly if $2aff > c^3$.

Example 3.

429. Let the centripetal force be constant everywhere or $P = 1$; then we have $\int Pdy=y-c$. Whereby if from the centre C with radius CM the arc MP is drawn, it becomes $\int Pdy=AP$ and the arc $AM=2\sqrt{a(y-c)}=2\sqrt{a\cdot AP}$. But again it becomes

$$p^2=y^2-\frac{y^2(y-c)}{a}=\frac{(a+c)y^2-y^3}{a},$$

hence $BC=a+c$ is found and the curve $AB=2a=2(BC-AC)$. Now the radius of osculation at the point M is equal to

$$\frac{2y\sqrt{(a+c-y)a}}{2a+2c-3y}.$$

Therefore at the lowest point A the radius of osculating is equal to $\frac{2ac}{2a-c}$. Therefore the curve at A is concave towards C, if $2a > c$, but convex if $c > 2a$, and the radius of osculation at A is infinitely large, if $2a = c$. In the first case, in which at A the curve is concave towards C, the curve has a turning point, where

$$CM=\frac{2}{3}(a+c)=\frac{2}{3}BC.$$

If $c = 0$ thus so that the point A falls on the centre C, y becomes the chord of this curve and $AM=2\sqrt{ay}$, the radius of osculation of this curve at the centre C is infinitely small, and $p=y\sqrt{\frac{a-y}{a}}$; moreover the curve can itself be constructed from the quadrature of the circle. [p. 211]

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 178]

PROPOSITIO 41.

Problema.

367. *Si corpus perpetuo deorsum trahatur vi quacunque, invenire lineam brachystochronam AMC* (Fig. 46),*super qua corpus citissime ex A ad C descendit.*

Solutio. [p. 179]

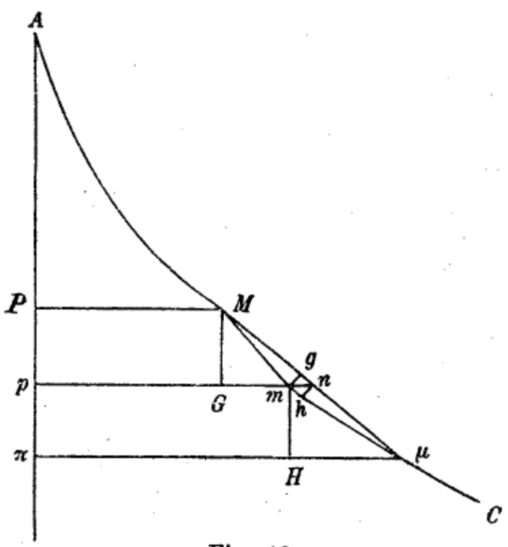


Fig. 46.

Positis $AP = x$, $PM = y$ et arcu $AM = s$ sit vis, quae corpus in M deorsum urget, $= P$; erit $v = \int Pdx$ hoc integrali ita accepto, ut evanescat posito $x = 0$, si quidem corpus motum in A ex quiete inchoare ponitur, atque $dv = Pdx$. Erit ergo

$$du = Pdx = dv \text{ et } ddw = 0,$$

quia du invariatum manet eunte m in n. Quocirca habebitur ista aequatio

$$2vd.\frac{dy}{ds} = \frac{dydv}{ds},$$

cuius integralis est

$$l\frac{v}{a} = 2l\frac{dy}{ds} \text{ seu } vds^2 = ady^2,$$

hincque

$$dx^2\int Pdx = ady^2 - dy^2\int Pdx.$$

Quamobrem pro lineae brachystochrona quaesita habebitur ista aequatio

$$dy = \frac{dx\sqrt{\int Pdx}}{\sqrt{(a-\int Pdx)}},$$

in qua indeterminatae x et y sunt a se invicem separatae. Curvae autem longitudo habetur ex hac aequatione

$$ds = \frac{dx\sqrt{a}}{\sqrt{(a-\int Pdx)}}.$$

Q.E.I.

Corollarium 1.

368. In *A* igitur, ubi celeritas corporis evanescit seu $\int Pdx = 0$, erit $dy = 0$, seu tangens curvae in *A* erit verticalis incidens in *AP*. At ubi fit $\int Pdx = a$, ibi tangens curvae erit horizontalis.

Corollarium 2.

369. Quia $ddw = 0$ et $du = Pdx$, erit

$$\frac{2v}{r} = \frac{Pdy}{ds}$$

(363). Est vero $\frac{Pdy}{ds}$ vis normalis, [p. 180] qua curva in *M* secundum normalem versus axes *AP* ductam premetur. Consequenter vis normalis est aequalis vi centrifugae et eandem plagam tendens. Quocirca linea brachystochrona hanc habet proprietatem, ut tota pressio, qua curva premitur, sit duplo maior quam vis normalis sola. In sequentibus vero demonstrabimus hanc proprietatem in omnibus lineis brachystochronis sive in vacuo sive in medio resistente locum habere.

Corollary 3.

370. Propter arbitrariam *a* dantur infinitae curvae brachystochronae omnes in *A* initium habentes. Atque hac litera a effici potest, ut curva ex *A* per datum punctum *C* transeat, quae erit linea inter *A* et *C*, super qua tempus est minimum.

Corollarium 4.

371. Quia curva *AMC* (Fig. 47) alicubi habet tangentem horizontalem, sit ea *BC* et in *C* sumatur alius axis verticalis *CQ*. Sit $CQ = X$, $QM = Y$ et $CM = S$; erit $dX = -dx, dY = -dy,$ et $dS = -dS$ atque $\int Pdx = a - \int PdX$ integrali $\int PdX$ ita accepto, ut evanescat posito $X = 0$. Ad hunc ergo axem *CQ* si curva referatur, habebitur ista aequation

$$dY = \frac{dX\sqrt{(a - \int PdX)}}{\sqrt{\int PdX}} \text{ seu } dS = \frac{dX\sqrt{a}}{\sqrt{\int PdX}}.$$

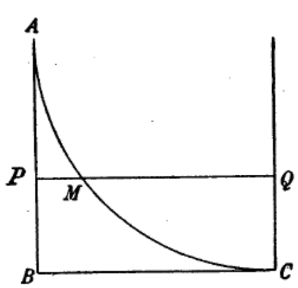


Fig. 47.

Corollarium 5.

333. Hae ergo omnes curvae ad utramque partem axis *CQ* duos arcus habent similes et aequales. [p. 181] Simili modo ad utramque partem axis *AB* curva aequaliter est disposita. Quamobrem huiusmodi curvae infinitas diametros habebunt inter se parallelas et ad distantiam *BC* positis, nisi forte potentia sollicitans ita accipiatur, ut supra *A* sit negativa, quo casu curva *CMA* sursum tendere poterit et partem concavam deorsum convertere.

Exemplum 1.

373. Sit potentia sollicitans uniformis seu $P = g$; erit $\int Pdx = gx;$ unde loco a posito gb pro brachystochrona in hac potentiae sollicitantis hypothesi habebitur ista aequatio

$$dy = \frac{dx\sqrt{x}}{\sqrt{(b-x)}} \text{ seu } ds = \frac{dx\sqrt{b}}{\sqrt{(b-x)}}.$$

At si aequatio ad axem CQ referatur, erit

$$dY = \frac{dX\sqrt{(b-X)}}{\sqrt{X}} \text{ seu } dS = \frac{dX\sqrt{b}}{\sqrt{X}},$$

cuius integralis est $S = 2\sqrt{bX}$. Ex qua aequatione patet curvam esse cycloidem super basi horizontali a circulo diametri b descriptam et deorsum conversam, quemadmodum hoc a Cel. Ioh. Bernoulli aliisque eximiis geometris iam pridem est inventum. Si itaque dentur duo quaecunque puncta A et M, linea, super qua corpus ex A citissime ad M descendit, invenitur, si describatur cyclois cuspidem in A et basem horizontamem habens atque per punctum M transiens; id quod ex eo, quod omnes cycloides sunt curvae similes, ex unica descripta cycloide facile efficitur. Tempus autem, quo corpus ex A ad M pertingit quodque est minimum, erit =

$$\int \frac{dx\sqrt{b}}{\sqrt{g(bx-x^2)}}$$

et curvae AM longitudo erit =

$$\int \frac{dx\sqrt{b}}{\sqrt{(b-x)}} = 2b - 2\sqrt{b(b-x)}.$$

Cum autem sit

$$PM = y = \int \frac{dx}{\sqrt{(bx-xx)}},$$

[p. 182] erit tempus per AM =

$$\frac{2y+2\sqrt{(bx-xx)}}{\sqrt{gb}}$$

= arcui in circulo diametri b, cuius sinus versus est = x, ducto in $\frac{2}{\sqrt{gb}}$.

[G.G. Leibniz, *Cummunicatio suae pariter duarumque alienarum ad edendum sibi primum a Dn. Io. Bernoullio, deinde a Dn. Marchione Hospitalio communcatarum solutionum problematis curvis celerrimi descensus a Dn. Io. Bernoullio geometris publice propositi, una cum solutione sua problematis alterius ab eodem postea,* Acta erud. 1697, p. 201; *Mathematische Schriften,* herausgegeben von C. I. Gerhardt, 2. Abteilung, Band 1, Halle 1858, p. 301.
Iac. Bernoulli, *Solutio problematum fraternorum ....una cum propositione aliorum,* Acta erud. 1697, p. 211; *Opera*, Genevae 1744. p. 768.
G. De L'Hospital, *Solutio problematis de linea celerrimi descensus,* Acta erud. 1697, p. 217.
I. Newton, *Epistola missa ad praenobilem virum D. Carolum Montague, in qua solvuntur duo problema mathematicis a Johanne Bernoulli math. cel. proposita,* Phil. trans.

(London) 1697; Acta erud. 1697, p. 223; *Opuscula*, Tom. I, Lausannae et Genevae 1744, p. 280.
R. Sault, *Analytical investigation of the curve of quickest descent*, Phil. trans. (London) 1698, p. 425.
I. Craig, *The curve of quickest descent*, Phil. trans. (London) 1701, p. 746.
P. St.]

Exemplum 2.

374. Si potentia sollicitans *P* fuerit ut potestas quaecunque abscissae *CQ*, nempe $P=\frac{X^n}{f^n}$, erit $\int PdX=\frac{X^{n+1}}{(n+1)f^n}$. Consequenter curva brachystochrona AMC exprimetur hac aequatione

$$dY=\frac{dX\sqrt{((n+1)af^n-X^{n+1})}}{X^{\frac{n+1}{2}}}\quad\text{seu}\quad dS=\frac{dX\sqrt{(n+1)af^n}}{X^{\frac{n+1}{2}}},$$

ita ut sit

$$S=\frac{2X^{\frac{1-n}{2}}}{1-n}\sqrt{(n+1)af^n}.$$

Quare si fuerit vel *n* = 1 vel *n* > 1, curva *CM* erit infinite magna seu ipsa recta *BC*. Cuspis autem curvae A seu locis, in quo motus incipit, habetur sumendo

$$CQ=BA=\sqrt[n+1]{(n+1)af^n}.$$

Curvae prodibunt algebraicae, si fuerit

$$n=\frac{1-2m}{1+2m}$$

denotante *m* numerum integrum affirmativum quemcunque. His igitur casibus erit *n* numerus negativus unitate minor, ita tamen, ut *n* + 1 sit numerus affirmativus. Sit *m* = 1, erit $n=-\frac{1}{3}$. Quare fiet

$$dY=\frac{dX}{X^{\frac{1}{3}}}\sqrt{\left(\frac{2}{3}af^{-\frac{1}{3}}-X^{\frac{2}{3}}\right)},$$

cuius integralis est [p. 183]

$$Y=\frac{2a\sqrt{2a}}{3\sqrt{3f}}-\left(\frac{2}{3}af^{-\frac{1}{3}}-X^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

Quae aequatio ab irrationalitate liberata fit ordinis sexti. Simili modo aliae curvae algebraicea invenientur, quae in certis hypothesibus sunt brachystochronae.

Scholion 1.

375. Ex data problematis solutione sequitur simul solutio problematis inversi, quo quaeritur potentia sollicitans deorsum directa talis, ut data curva sit brachystochrona. Debet autem haec curva in puncto infimo *C* habere tangentem horizontalem et alicubi in *A*, ubi est motus initium, tangentem horizontalem et alicubi in A, ubi est motus initium, tangentem verticalem. Ut si fuerit aequatio pro curva data haec $dY = RdX$, erit

$$R^2\int PdX = a - \int PdX \quad \text{atque} \quad \int PdX = \frac{a}{R^2+1}.$$

Unde invenitur

$$P = \frac{-2aRdR}{(R^2+1)^2 dX} = \frac{-2adXdYddY}{dS^4}.$$

Si ergo radius osculi in *M* ponatur *r*, propter $r = \frac{dS^3}{-dXddY}$ habebitur $P = \frac{2adY}{rdS}$.

Quare problema hac unica solventur analogia : ut radius osculi curvae in *M* ad lineam datam, ita sinus anguli, quem tangens curvae in *M* cum verticali facit, ad potentiam sollicitantem, quae quaeritur. Altutudo vero debita celeritati, quam corpus in *M* habet, est

$$a - \int PdX = \frac{aR^2}{R^2+1} = \frac{adY^2}{dS^2},$$

ex quo sequitur celeritatem corporis esse illi ipsi sinui anguli, [p. 184] quem tangens curvae cum verticali constituit, proportionalem. Ut si sit curva *CMA* circulus radio *c* descriptus, erit *r* = *c* et

$$dY = \frac{cdX - XdX}{\sqrt{(2cX - XX)}} \quad \text{atque} \quad dS = \frac{cdX}{\sqrt{(2cX - XX)}},$$

ex quibus fit

$$P = \frac{2a(c-X)}{c^2} = \frac{2a\cdot AP}{cc}.$$

Vis ergo corpus deorsum trahens proportionalis esse debet abscissae *AP*, cui etiam celeritas est proportionalis.

Scholion 2.

376. Inventa linea brachystochrona pro hypothesi potentiae sollicitantis deorsum tendentis ordo requireret, ut linaes brachystochronas in hypothesi virium centripetarum determinaremus. At proposito fundamentalis (361) ita est comparata, ut elementa curvae *Mm* et $m\mu$ (Fig. 46) ad axem *AP* et ordinatas orthogonales *MP*, *mp* referantur, quod ad casum virium centripetarum non commode quadrat. Videntur quidem elementa *MG* et mH ut convergentia ad centrum virium considerari posse; sed hic ipse error, qui ex hoc oritur, quod elementa *MG* et *mH* non essent parallela, ut propositio fundamentalis requirit, perperam negligitur. Perspicuum hoc reddi potest determinando radio osculi, qui, si *MG* et *mH* fuerint inter se parallela, est $= MG : d.\frac{mG}{Mm};$ quae autem expressio non locum habet, si *MG* et *mH* ad centrum virium convergunt. Quare antequam ad brachystochronas in hypothesi virium centripetarum accedamus, [p. 185] ex propositione fundamentali generalem derivabimus proprietatem cuicunque potentiarum sollicitantium hypothesi accommodatam. Ex quibus perspicietur Cel. Hermannum in *Phorononia* aliosque, qui brachystochronas pro viribus centripetis dederunt, esse deceptos, dum usi sunt principio cum veritate non consentaneo, ut mox indicabitur.

[Iac. Hermann, *Phoronomia, seu de viribus et motibus corporum solidorum et fluidorum,* Amstelodami 1716, p. 81.
Ioh. Machin, *Inventio curvae, quam corpus descendens brevissimo tempore describeret, urgente vi centripeta ad datum punctum tendente, quae crescat vel decrescat iuxta quamvis potentiam distantiae a centro; dato nempe imo curvae puncto et altitudine in principio casus,* Phil. trans. (London) 1718, p. 860. P. St.]

PROPOSITIO 42.

Theorema.

377. *Quaecunque fuerint potentiae sollicitantes, ea linea erit brachystochrona, quam corpus super ea motum premit vi duplo maiore, quam est vel sola vis centrifuga vel sola vis normalis.*

Demonstratio.

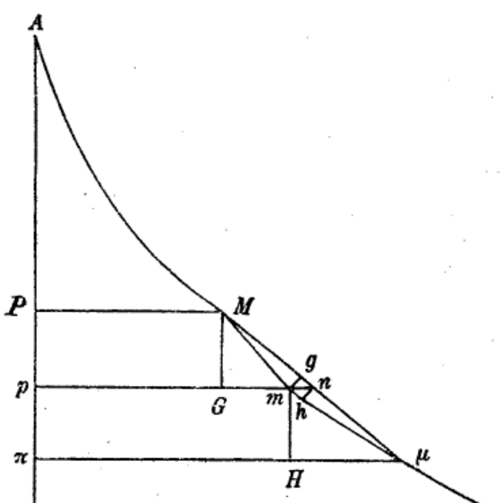


Fig. 46.

Quaecunque et quotcunque fuerint potentiae sollicitantes, eae omnes in binas resolvi possunt, quarum altera trahat secundum *MG* (Fig. 46), altera secundum *MP*. Sit illa secundum *MG* trahens = *P* et, quae secundum *MP* trahit, = *Q* et dicantur $AP = x$, $PM = y$ et $AM = s$ itemque altitudo celeritati in M debita = v. Erit ex his duabus viribus vis tangentialis = $\frac{Pdx-Qdy}{ds}$ et vis normalis = $\frac{Pdy+Qdx}{ds}$. Hanc ob rem erit $dv = Pdx - Qdy$. Cum hac expressione comparetur, quod supra (364) est allatum, ubi posuimus $dv = Pdx + Qdy + Rds$; [p. 186] erit *Q* negativum et *R* = 0. Sequitur ergo exinde fore $\frac{2v}{r} = \frac{Pdy+Qdx}{ds}$. At est $\frac{2v}{r}$ vis centrifuga, qua curva in *M* premitur, et $\frac{Pdy+Qdx}{ds}$ est vis normalis. Quare cum vis centrifuga sit aequalis vi normali, tota pressio, quam curva sustinet, duplo maior est quam vel sola vis centrifuga vel sola vis normalis. Q.E.D.

Scholion 1.

378. In sequente capite demonstrabimus hanc eandem propositionem locum etiam habere in medio quocunque resistente; id quod quidem eadem opera hic demonstrare potuissemus ; sed quia resistentiae sequens caput est destinatum, eo potius hoc theorema transferre visum est.

Corollarium 1.

379. Ex hac igitur propositione facile erit in quacunque potentiarum sollicitantium hypothesi brachystochronas determinare. Hocque ipsum iam ex aliqua parte supra praestitimus, ubi curvas determinavimus, in quibus pressio totalis datam habeat rationem ad vim centrifugam.

Corollarium 2.

380. Cum sit $dv = Pdx - Qdy$, erit $v = \int Pdx - \int Qdy$ his integralibus ita accipiendis, ut evanescant factis x et $y = 0$, [p. 187] si quidem motus in A ex quiete incipere debet.

Corollarium 3.

381. Si ergo hic pro v inventus valor substituatur, habebitur aequatio pro curva brachystochrona haec

$$\frac{2\int Pdx - 2\int Qdy}{r} = \frac{Pdy + Qdx}{ds}.$$

Est vero $r = \frac{ds^3}{dxddy}$ (363) sumto dx pro constante, quia in parem oppositam axi AP cadere r ponitur; unde habetur haec aequatio

$$\frac{2dxddy}{ds^2}\left(\int Pdx - \int Qdy\right) = Pdy + Qdx.$$

Corollarium 4.

382. Quia haec aequatio est differentialis secundi gradus atque ideo duplicem integrationem requirit, altera integratione constans quaevis poterit adiici, altera effici debet, ut facto $x = 0$ fiat quoque $y = 0$. Infinitae ergo prodeunt curvae brachystochronae pro eadem potentiarum sollicitantium hypothesi. Atque constante arbitraria effici poterit, ut curva per datum punctum transeat.

Corollarium 5.

383. Tempus, quo corpus ex A ad M pervenit, est =

$$\int \frac{ds}{\sqrt{(\int Pdx - \int Qdy)}} = \int \sqrt{\frac{2dxddy}{Pdy + Qdx}},$$

[p. 188] quae quidem expressio prius ex aequatione curvae est investiganda; minimum vero hoc tempus esse debet inter omnia alia tempora motuum per curvas omnes puncta A et M iungentes.

Scholion 2.

384. Quemadmodum porro in quacunque potentiarum sollicitantium hypothesi eae curvae libere describuntur, in quibus vis centrifuga aequalis est et contraria vi normali, ita eae curvae erunt brachystochronae, in quibus vis normalis quoque aequalis est vi centrifugae, sed in eandem plagam tendens. Atque quemadmodum illa proprietas communis est omnium curvarum libere descriptarum etiam in medio resistente, ita haec quoque propriets ad omnes lineas brachystochronas in medio resistente extenditur.

PROPOSITIO 43.

Problema.

385. *Si corpus in quacunque perpetuo trahatur ad centrum virium* (Fig. 48), *invenire lineam brachystochronam AM, super qua corpus ex A citissime ad M pertinget.*

Solutio.

A puncto *A*, in quo motus initium ponitur, ad centrum virium *C* ducatur recta *AC*, [p. 189] item *MC* et in tangentem *MT* ex *C* perpendiculum *CT*. Ponatur $AC = a$, $CM = y$, $CT = p$, vis centripeta in $M = P$ et celeritas in *M* debita altitudini *v*. His positis erit $dv = -Pdy$ et $v = -\int Pdy$ hoc integrali ita accepto, ut evanescat posito $y = a$. Vis normalis autem erit $= \frac{Pp}{y}$; cui aequalis esse debet vis centrifuga et in eandem plagam tendens; tum enim proveniet curva brachystochrona, ut propositione praecedente demonstravimus. Curva igitur debebit esse convexa versus centrum *C* et radius osculi in partem aversam a centro *C* cadet. Quare, cum haec expressio $\frac{ydy}{dp}$ exhibeat radium osculi, quatenus versus centrum cadit, erit vera radii osculi expressio in nostro casu $\frac{-ydy}{dp}$. Vis igitur centrifuga erit =

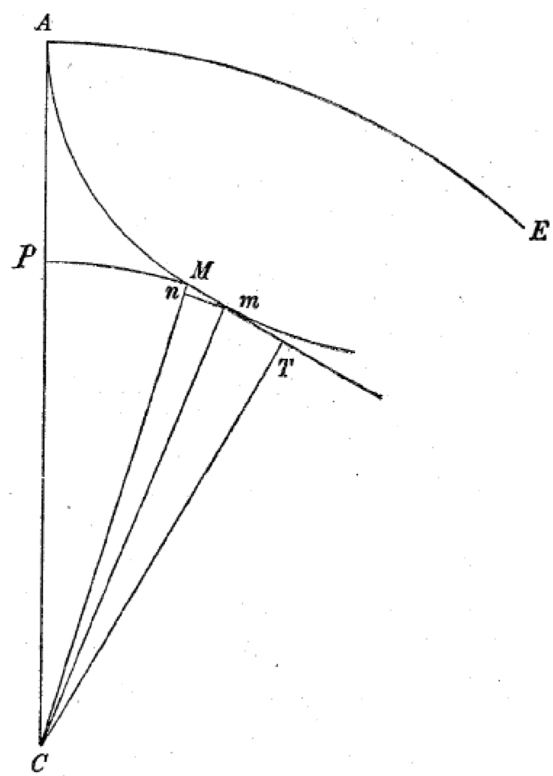


Fig. 48.

$\frac{-2vdp}{ydy} = \frac{2dp\int Pdy}{ydy}$, cui aequalis poni debet vis normalis $\frac{Pp}{y}$; ex quo oritur haec aequatio $\frac{2dp}{p} = \frac{Pdy}{\int Pdy}$, cuius integralis est

$$\frac{pp}{b} = -\int Pdy \quad \text{seu} \quad p = \sqrt{-b\int Pdy};$$

quae est aequatio pro curva quaesita inter *y* et *p*. At si centro *C* ducatur arcus *MP* hicque dicatur $\frac{ys}{a}$, erit

$$nm = \frac{yds}{a} \quad \text{et} \quad pp = \frac{y^4ds^2}{a^2dy^2 + y^2ds^2}.$$

Hinc fiet

$$ds = \frac{-apdy}{y\sqrt{(y^2 - p^2)}}$$

atque valore ipsius *p* ex superiore aequatione substituto habebitur

$$ds = \frac{-ady\sqrt{-b\int Pdy}}{y\sqrt{(y^2 + b\int Pdy)}},$$

quae est aequatio inter *y* et arcum circuli *s* radio *a* descriptum, qui metitur angulum *ACM*, ex qua fluit constructio curvae quaesitae. Q.E.I. [p. 190]

Corollarium 1.

386. Quia altitudo celeritati debita est $v = -\int Pdy = \frac{pp}{b}$, celeritas corporis in quovis loco erit ut perpendiculum ex *C* in tangentem demissum, simili modo, quo in motu libero celeritas est huic perpendiculo reciproce proportionalis.

Corollarium 2.

387. Sit radius osculi in $M = r$; erit $\frac{2v}{r} = \frac{Pp}{b}$ ex conditione problematis. Hinc ergo habebitur $r = \frac{2yv}{Pp} = \frac{2py}{bP}$. Quia autem in initio curvae in *A* est $p = 0$ seu *AC* tangens curvae, erit radius osculi quoque in $A = 0$, nisi forte simul vis centripeta *P* in *A* evanescat.

Corollarium 3.

388. Maximum corpus habebit celeritatem in loco, ubi $dp = 0$; ibi autem ex aequatione pro curva fit $dy = 0$. Quare in eo loco corpus celerrime movetur, ubi recta *CM* in curvam est normalis. Curva ergo ultra hoc punctum a centro *C* recedit.

Scholion 1.

389. Celeritas ergo corporis in singulis brachystochronae punctis non est proportionalis sinui anguli, quem tangens curvae cum directione vis centripetae constituit; [p. 191] huius enim anguli *TMC* sinus est $\frac{p}{y}$, celeritas vero ipsi p inventa est proportionalis. Haec quidem proprietas locum habet, si centrum virium infinitae distat et directiones vis sollicitantis sunt inter se parallelae, ut ex prop. 41 intelligitur, ubi celeritas erat ut $\frac{dy}{ds}$, i. e. ut sinus anguli, quem elementum curvae cum directione potentiae sollicitantis constituit. Hanc autem proprietatem Cel. Hermannus in Comm. Acad. Petrop. A 1727 omnibus brachystochronis tam in vacuo quam in medio resistente commumem esse est arbitratus. Atque hanc ob rem non solum eae lineae, quas in medio resistente pro brachystochronis dedit, tales non sunt, sed etiam quas in vacuo pro viribus centripetis invenit. Hoc autem casu invenit hanc aequationem $\frac{-\int Pdy}{b}=\frac{p^2}{y^2}$ a nostra atque vera aequatione prorsus discrepantem.

Exemplum 1.

390. Sit vis centripeta ipsis distantiis corporis a centro proportionalis; fiet $P=\frac{y}{f}$. Quare erit

$$\int Pdy=\frac{y^2-a^2}{2f} \quad \text{atque} \quad v=\frac{a^2-y^2}{2f}=\frac{p^2}{b}.$$

Quae est aequatio pro brachystochrona in hac vis centripetae hypothesi inter *p* et *y*. Altero vero aequatio inter arcum *s* radio *a* descriptum, que est mensura anguli ACM, et y est haec

$$ds=\frac{-a\,dy\sqrt{b(a^2-y^2)}}{y\sqrt{(2fy^2+by^2-a^2b)}}.$$

Huius curvae punctum infimum seu centro proximum habetur [p. 192] ponendo vel $dy = 0$ vel $p = y$; tum autem erit

$$y=\frac{a\sqrt{b}}{\sqrt{(b+2f)}};$$

haec ergo est minima curvae a centro *C* distantia. Radius osculi huius curvae in quovis puncto est =

$$\frac{2py}{bP}=\sqrt{\frac{2f(a^2-y^2)}{b}}.$$

In puncto ergo centro proximo radius osculi est maximus, quippe =

$$\frac{2af}{\sqrt{b(b+2f)}}.$$

Ponatur tangens anguli *ACM* = *t* posito sinu toto = 1; erit $\frac{ds}{a}=\frac{dt}{1+tt}$; ponatur porro

$$\frac{\sqrt{b(a^2-y^2)}}{\sqrt{(2fy^2+by^2-a^2b)}}=q;$$

habebitur ista aequatio

$$\frac{dt}{1+tt} = \frac{dq}{1+qq} - \frac{dq}{1+\frac{b+2f}{b}qq}.$$

Ex quo intellititur curvam toties esse algebraicam, quoties est $\frac{b}{b+2f}$ numerus quadratus.

Longitudo curvae *AM* est porro generaliter =

$$\int \frac{-ydy}{\sqrt{(y^2+b\int Pdy)}};$$

hoc ergo casu erit

$$AM = \int \frac{-ydy\sqrt{2f}}{\sqrt{(2fy^2+by^2-a^2b)}} = \frac{2af-\sqrt{2f(2fyy+byy-a^2b)}}{2f+b}.$$

Ex qua aequatione sequitur curvam brachystochronam *AM* esse hypocycloidem, quae generatur rotatione circuli, cuius diameter est =

$$\frac{a\sqrt{(b+2f)}-a\sqrt{b}}{\sqrt{(b+2f)}},$$

super concava parte peripheriae *AE* centro *C* radio *AC* descriptae. Cum igitur *b* pro lubitu accipere liceat, apparet omnes hypocycloides super peripheria *AE* natas esse brachystochronas.

Exemplum 2.

391. Sit vis centripeta proportionalis quadratis distantiarum, ut sit $P=\frac{y^2}{f^2}$; unde erit

$$\int Pdy = \frac{-f^2}{y} + \frac{f^2}{a} = \frac{f^2(y-a)}{ay} \quad \text{atque} \quad v = \frac{f^2(a-y)}{ay} = \frac{p^2}{b},$$

[p. 193] Quae est aequatio pro brachystochrona in hac vis centripetae hypothesi. Altero vero aequatio inter arcum *s* et *y* erit ista

$$ds = \frac{-ady\sqrt{bf^2(a-y)}}{y\sqrt{(ay^3+bf^2y-abf^2)}}.$$

Huius ergo curvae punctum infimum posito $dy = 0$ determinabitur ope huius aequationis cubicae $ay^3+bf^2y=abf^2$. Ceterum ista aequatio inter *s* et *y* sufficit ad curvam quaesitam construendam.

Scholion 2.

392. Ex his igitur, quae in hac et praecedentibus propositionibus allata sunt, intelligitur, quomodo in quacunque potentiarum sollicitantium hypothesi ea linea sit invenienda super qua corpus ex dato puncto ad datum punctum citissime perveniat. Nunc ergo etiam determinari oportet eam lineam, super qua corpus a dato puncto citissime non ad datum punctum, set ad datam lineam perveniat. quae sane curva una erit infinitis brachystochronis; at quaenam ea sit, in sequente propositione declaribimus.

PROPOSITIO 44.

Theorema.

393. *Corpus a dato puncto A* (Fig. 49) *ad quamvis lineam datam BM celerimme pervenit super linea brachystochrona AM, quae datae linea BM ad angulos rectos occurrit, hocque in quacunque potentiarum sollicitantium hypothesi.* [p. 194]

Demonstratio.

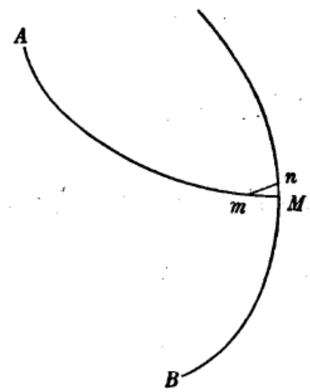


Fig. 49.

Sit *AM* ea linea super qua corpus ex *A* citissime ad lineam *BM* perveniat; perspicuum est primo hanc lineam fore brachystochronam; nam si daretur linea, super qua corpus citius ab *A* ad *M* pervenerit, ea potius quaesito satisfaceret. Praeterea haec linea *AM* ad angulos rectos in *M* curvae *BM* occurrit; nisi enim ad angulos rectos occurret ducta minima normali *mn* ob *mn* < *mM* corpus citius per *Amn* ad curvam *BM* perveniret quam per *AmM*. Quare ne haec exceptio locum invenire possit, necesse est, ut curva AM datae curvae normaliter insistat. Consequenter corpus super ea infinitarum brachystochronarum ex A ad curvam BM ductarum citissime ad curvam BM pervenit, quae curvae BM ad angulos rectos occurrit. Q.E.D.

Corollarium 1.

394. Si ergo infinitae curvae quaerantur, super quibus corpus dato tempore ab *A* ad *BM* perveniat, oportet, ut datum tempus sit maius quam tempus per brachystochronam *AM*; alias enim problem fieret impossibile.

Corollarium 2.

395. Si accidat, ut plures curvae brachystochronae sint normales in curvam *BM*, [p. 195] plura quoque prodibunt tempora minima vel maxima. Haec enim methodus tam minima quam maxima declarat.

Corollarium 3.

396. Quia tempus per curvam brachystochronam *AM* est minimum, intelligitur ex methodo maximorum et minimorum, si duae brachystochronae proximae concipiantur normaliter insistentes curvae *BM*, tempora per eas esse inter se aequalia.

Corollarium 4.

397. Hinc porro perspicitur, si curva *BM* fuerit eiusmodi, ut omnes brachystochronas ex puncto A ductas secet ad angulos rectos, tempora per omnes brachystochronas ad curvam *BM* usque ductas fore inter se aequalia.

Corollarium 5.

398. Quamobrem curva, quae ab omnibus curvis brachystochronis ex puncto *A* ductis arcus isochronos seu eodem tempore percursos abscindit, ea quoque omnes brachystochronas ad angulos rectos secabit seu erit illarum traiectoria orthogonalis.

Corollarium 6.

399. Atque vicissim quoque perspicitur, si curva, [p. 196] quae ab infinitis curvis arcus isochronos abscindit, fuerit earum traiectoria orthogonalis, eas infinitas curvas omnes esse brachystochronas.

Scholion.

400. Facile intelligitur hanc propositionem locum quoque habere in medio resistente; simili enim modo apparet tempus per elementum *mn* normale in curvam *BM* minus esse quam tempus per elementum mM, quod non est perpendiculare; in hoc autem totius demonstrationis vis est sita. Quare si infinitis curvis ex puncto A eductis lex potentiarum sollicitantium et resistentiae poterit inveniri, in qua eae curvae omnes sint brachystochronae, simul harum curvarum traiectoria orthogonalis poterit exhiberin quaerendo tantum curvam ab iis curvis arcus isochronos abscindentem. Atque hanc ipsam traiectorias orthogonales inveniendi methodum iam adhibuit Cel. Ioh. Bernoulli in Act. Lips. A 1697. [vide notam post (366)]

PROPOSITIO 45.

Problema.

401. *Inter omnes curvas puncta A et C* (Fig. 50) *iungentes et aequaliter longas eam determinare AMC, super qua corpus celerrime ex A ad C perveniat, in hypothesi potentiae sollicitantis unformis g et deorsum directae.* [p. 197]

Solutio.

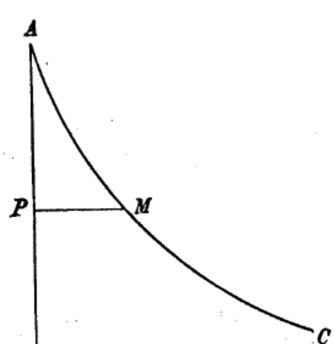


Fig. 50.

Ducta verticali *AP* et horizontali *PM* dicatur *AP* = *x*, *PM* = *y* et *AM* = *s* eritque tempus, quo arcus *AM* absolvitur, = $\int \frac{ds}{\sqrt{gx}}$. Iam per methodum isoperimetricorum, de qua peculiarem dedi dessertationem cum formulis generalibus, ex quibus quaevis problemata facile resolvi possunt, in Comment. Acad. Petr. 1733, [*Problematis isoperimetrici in latissimo sensu accepti solutio generalis*, p. 123; *Opera Omnia*, series I, vol. 25; E027], duae quantitates considerandae sunt; arcus $AM = s = \int ds$ et tempus per *AM* = $\int \frac{ds}{\sqrt{gx}}$, quarum altera alterius respectu debet esse minima vel maxima. Eodem enim redit, sive inter omnes curvas aeque longas quaeratur ea, quae brevissimum habeat descensum, sive inter omnes, super quibus descensus fiunt eodem tempore, ea, quae est brevissima. Per formulas autem meas dat $\int ds$ hanc quantitatem $diff.\frac{dy}{ds}$ et $\int \frac{ds}{\sqrt{gx}}$ dat $diff.\frac{dy}{ds\sqrt{gx}}$, quarum altera alterius multiplicato cuicunque aequalis est ponenda. Habetur ergo integrando

$$\frac{dy}{ds} = \frac{dy\sqrt{a}}{ds\sqrt{x}} - m$$

seu

$$dy(\sqrt{a} - \sqrt{x}) = mds\sqrt{x}.$$

Sumendis vero quadratis erit

$$dy^2(\sqrt{a} - \sqrt{x})^2 = m^2xdx^2 + m^2xdy^2,$$

unde erit

$$dy = \frac{mdx\sqrt{x}}{\sqrt{(a - 2\sqrt{ax} + (1-m^2)x)}} \quad \text{et} \quad ds = \frac{dx(\sqrt{a} - \sqrt{x})}{\sqrt{(a - 2\sqrt{ax} + (1-m^2)x)}}.$$

Ex quo curva quaesita determinabitur. Q.E.I.

Corollarium 1.

402. In aequatione inventa duae insunt quantitates *a* et *m* arbitrariae, quibus effici potest, ut curva per datum punctum *C* transeat et ut simul sit datae longitudinis [p. 198]. Atque tum haec curva celerrime absolvetur inter omnes alias curvas eiusdem longitudinis per *A* et *C* transeuntes.

Corollarium 2.

403. Si ponantur *a* et *m* infinite magna, prodibit cyclois, quae non solum inter omnes curvas eiusdem longitudinis, sed inter omnes omnino citissime absolvitur.

Corollarium 3.

404. Si ponantur $m = 0$, prodit $dy = 0$ seu recta verticalis. At si fiat $a = 0$, oritur recta quaecunque per punctum *A* ducta. Est enim recta linea inter omnes lineas, quae eodem tempore absolvuntur, minima seu brevissima.

Corollarium 4.

405. Si ponantur $m = 1$, prodit curva algebraica; erit enim

$$dy = \frac{dx\sqrt{x}}{\sqrt{(a - 2\sqrt{ax})}},$$

cuius integralis est

$$y = \frac{-(4a + 4\sqrt{ax} + 6x)\sqrt{(a - 2\sqrt{ax})}}{15\sqrt{a}} + \frac{4a}{15}.$$

Haec curva etiam est rectificabilis; namque erit

$$s = \frac{2a}{5} - \frac{(2a + 2\sqrt{ax} - 2x)}{5\sqrt{a}}\sqrt{(a - 2\sqrt{ax})}.$$

Quin etiam tempus per arcum *AM* algebraice poterit exprimi; erit enim

$$\int \frac{ds}{\sqrt{x}} = \frac{4\sqrt{a}}{3} - \frac{(4\sqrt{a} - 2\sqrt{x})\sqrt{(a - 2\sqrt{ax})}}{3\sqrt{a}}.$$

Corollarium 5. [p. 199]

406. Si in hac curva sumatur $x = \frac{a}{4}$, ibi tangens erit horizontalis atque recta verticalis in eo curvae puncto erit diameter curvae. Illo autem loco fit $y = \frac{4a}{15}$ et longitudo curvae ad hoc punctum erit = $\frac{2a}{5}$. Atque tempus, quo hic arcus absolvitur, est = $\frac{4\sqrt{a}}{3\sqrt{g}}$. Eodem ergo tempore corpus recta descendet per altitudinem $\frac{4a}{9}$.

Scholion.

407. Missis nunc hisce de celerrimo descensu progredimur ad eas curvas considerandas, super quibus plures descensus inter se comparati datam teneant relationem. Huc maxime pertinet quaesito de curva tautochronis, super quibus vel omnes descensus ad punctum infimum curvae fiunt eodem tempore vel integrae oscillationes. Ad haec deinde aliae accedere possunt quaestiones cum difficiles tum vim methodi, qua utemur, illustrantes.

PROPOSITIO 46.

Problema.

408. *Invenire legem generalem curvarum tautochronarum, super quibus omnes descensus ad punctum A, initio descensus ubicunque in curva AM* (Fig. 51) *accepto, absolvantur eodem tempore.* [p. 200]

Solutio.

Sumta recta *AP* pro axe dicatur curvae portio *AM* = *s* sitque altitudo celeritati in *A* debita = *b* et altitudo celeritati in *M* debita = *v* ; erit tempus, quo arcus *AM* absolvitur, = $\int \frac{ds}{\sqrt{v}}$, quod integrale ita est capiendum, ut evanescat posito s = 0. Tum, si in eo integrali ponatur *v* = 0, habebitur tempus descensus a loco, in quo celeritas erat nulla, usque ad punctum *A* seu totam descensus tempus; id quod eadem quantitate expressum esse debet, quaecunque fuerit quantitas *b*. Haec igitur

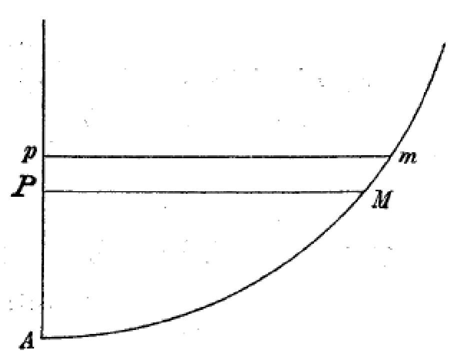


Fig. 51.

quantitas b neque in expressione temporis inesse debet neque in expressione pro curva AM, quia haec eadem curva idem descensus tempus producere debet, utcunque varietur b.

Sit iam v quantitas composita ex littera z, ad curvam pertinente et a curva AM cum abscissa AP et applicata PM tantum pendente neque b involvente, atque ex littera h, quae ex b et quantitatibus constantibus sit composita. Sit autem v talis ipsarum h et z functio, ut evanescat posito $z = h$ atque ut fiat = b, si sit $z = \alpha h$ existente α numero quocunque. Ponatur porro $ds = pdz$ pro aequatione curvae quaesitae; debebit p talis esse quantitas, in qua non contineatur b vel h, quia hae litterae in aequationem curvae ingredi nequeunt. Habebimus ergo pro expressione temporis per AM $\int \frac{pdz}{\sqrt{v}}$ hoc integrali ita accepto, ut evanescat posito $v = b$ seu $z = \alpha h$. Deinde hoc integrale, si in eo ponatur $z = h$, [p. 201] dabit tempus totius descensus, in quo h inesse non poterit. Hoc vero evenit, si $\int \frac{pdz}{\sqrt{v}}$ fuerit functio ipsarum h et z nullius dimensionis seu si $\frac{pdz}{\sqrt{v}}$ fuerit functio nullus dimensionis. Sit v functio m dimensionum ipsarum h et z; debebit esse $p = Cz^{\frac{m-2}{2}}$ denotante C quantitatem constantem a b non pendentem. Quoties ergo pro v talis functio fuerit comperta, habebitur pro curva quaesita haec aequatio

$$ds = Cz^{\frac{m-2}{2}}dz \quad \text{seu} \quad s = \frac{2Cz^{\frac{m}{2}}}{m} + \text{const.},$$

si opus est, quo s evanescat, si in z evanescat vel x vel y. Q.E.I.

Corollarium 1.

409. Quo igitur haec methodus possit adhiberi, oportet, ut v quantitatibus finitis sit expressum atque ut ea expressio transmutari possit in functionem homogeneam ex h et z constantem.

Corollarium 2.

410. Hanc ob rem necesse est, ut fiat $v = b$, si ponatur $z = \alpha h$, quo in quantitate constante adiecta etiam non reperiatur h. Sufficit igitur, si viderimus fieri $v = b$ facto $z = \alpha h$, neque opus est, ut integratio absolvatur.

Corollarium 3.

411. Intelligitur etiam curvam in A habere debere tangentem normalem in directionem vis sollicitantis; [p. 202] nisi etiam hoc fuerit, tempus per arcum descensus infinite parvum foret quoque parvum.

Scholion.

412. Valet haec solutio non solum, si, uti figura indicat, curva exponatur per coordinatas orthogonales; nihil enim interest, quibusnam quantitatibus naturam curvae exponere velimus, dummodo in z non ingrediatur b. Potest autem z continere lineas et quantitates quascunque a curva pendentes. Hac igitur methodo in vacuo, in quacunque potentiarum sollicitantium hypothesi, lineae tautochronae poterunt inveniri, quia semper celeritas per quantitates finitas exprimi potest. At si, ut in mediis resistentibus fieri solet, celeritas non potest exhiberi finitis quantitatibus, haec methodus usum habere nequit, sed alia desideratur, quae succedit, etiam si celeritas per aequationem differentialem tantum detur.

PROPOSITIO 47.

Problema.

413. *Si corpus deorsum sollicitetur vi quacunque, invenire lineam tautochronam super qua omnes descensus fiant eodem tempore.*

Solutio. [p. 203]

Posito $AP = x$, $PM = y$ et $AM = s$ (Fig. 51) sit celeritas in A debita altitudini b et in M debita altitudini v. Sit porro vis sollicitans in $M = P$; erit $v = b - \int Pdx$ integrali $\int Pdx$ ita accepto, ut evanescat facto $x = 0$. Iam si ponatur $b = h$ et $\int Pdx = z$, erit v functio unius dimensionis ipsarum h et z et evanescit facto $z = h$ fitque $v = b$ facto $z = 0$. Erit igitur $m = 1$ ideoque habebitur pro curva quaesita ista aequatio

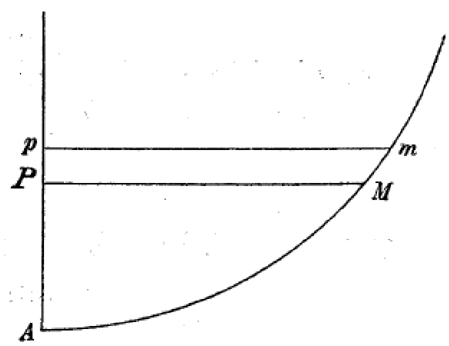


Fig. 51.

$$ds = \frac{Cdz}{\sqrt{z}} \quad \text{et} \quad s = 2\sqrt{az} = 2\sqrt{a\int Pdx}.$$

Si desideretur aequatio inter x et y, erit ob $ds = \frac{aPdx}{\sqrt{a\int Pdx}}$

$$dy = \frac{dx\sqrt{(aP^2 - \int Pdx)}}{\sqrt{\int Pdx}}.$$

Q.E.I.

Corollarium 1.

414. Quia evanescit $\int Pdx$ facto x = 0, tangens curvae in A erit horizontalis, nisi in A evanescat P. Atque curva alicubi habebit tangentem verticalem ibique plerumque cuspidem; hoc evenit, ubi erit $\int Pdx = aP^2$. Ibi enim fit dy = 0.

Corollarium 2.

415. Huius curvae in puncto infimo A radius osculi est aequalis subnormalis = $\frac{ydy}{dx} = \frac{sds}{dx}$, quia in A s et y fiunt aequalia. Quare in A erit radius osculi = $2aP$, ubi P denotat potentiam sollicitantem in puncto A.

Corollarium 3. [p. 204]

416. Ex radio osculi in A et potentia sollicitante in A invenitur tempus ascensu vel descensus per infinite parvum arcum =

$$\frac{\pi\sqrt{4aP}}{2\sqrt{P}}=\pi\sqrt{a}.$$

(172). Huicque tempori tempus cuiusque descensus est aequale. In hypothesi ergo gravitatis = 1 pendulum longitudinis 2*a* descensus infinite parvos eodem tempore absolvet.

Exemplum 1.

417. Sit potentia sollicitans ubique constans, nempe $P = g$; erit

$$\int Pdx = gx \text{ atque } s = 2\sqrt{gax}$$

itemque

$$dy=\frac{dx\sqrt{(ga-x)}}{\sqrt{x}},$$

unde intelligitur curvam esse cycloidem deorsum convexam, prorsus cum linea brachystochrona in eadem potentiae hypothesi congruentem. Quod autem omnes descensus fiant eodem tempore super cycloide, iam supra demonstravimus (187).

Exemplum 2.

418. Sit potentia sollicitans ut potestas quaecunque ipsius x ; erit

$$P=\frac{x^n}{f^n} \quad \text{et} \quad \int Pdx=\frac{x^{n+1}}{(n+1)f^n},$$

si quidem fuerit $n + 1$ numerus affirmativus; sin enim esset $n + 1$ numerus negativus, fieret $\int Pdx = \infty$. Erit igitur

$$s=\frac{2x^{\frac{n+1}{2}}}{f^{\frac{n}{2}}}\sqrt{\frac{a}{n+1}} \quad \text{atque} \quad dy=\frac{dx\sqrt{((n+1)ax^{n-1}-f^n)}}{\sqrt{f^n}}.$$

Ex qua aequatione intellititur curvam fore rectam utcunque inclinatam ad horizontalem si fuerit $n = 1$. [p. 205] At si $n > 1$, curva in initio *A* fit imaginaria, quo usque nimirum f^n incipit minus esse quam $(n+1)ax^{n-1}$.

Scholion 1.

419. Ex aequatione generali apparet rectam *AP* esse diametrum curvae. Quare cum in vacuo ascensus sint similes descensibus, semioscillationes omnes super curva *MA* ad alteram partem usque producta erunt quoque isochronae et consequenter etiam integrae oscillationes. Deinde cum ob arbitrariam *a* infinitae sint curvae tautochronae *AM*, duae quaequo in puncto *A* coniunctae, ut ibi habeant tangentem commumem horizontalem, producent tam semioscillationes quam integras oscillationes isochronas, si scilicet pendulum ita accomodetur, ut oscillando huiusmodi curvas absolvat.

Scholion 2.

420. Intelligitur etiam ex solutione eas curvas, quas invenimus, esse solas, quae quaesito satisfaciunt. Nam loco *p* alia functio ipsius *z* substituti nequit, ut in integrali, si ponatur $v = 0$, prorsus ex formual exeant *b* seu *h*. Id quod aliis methodis, quibus tautochronae sunt inventae, non satis liquet.

Scholion 3.

421. Quia est $s = 2\sqrt{a\int Pdx}$, erit $P = \frac{sds}{2adx}$. Ex quo apparet, cuiusmodi esse debeant potentia sollicitans, [p. 206] ut data curva sit tautochrona. Scilicet potentia deorsum tendens debet esse proportionalis ipsi $\frac{sds}{dx}$ ex data curva desumto. Quare, nisi curva sit rectificabilis, valor potentiae sollicitantis non potest algebraice exhiberi.

PROPOSITIO 48.

Problema.

422. *Si corpus perpetuo trahatur ad centrum virium C* (Fig. 52) *vi quacunque, invenire lineam tautochronam BMA, super qua omnes corpus descensus ad punctum A usque eodem tempore absolvat.*

Solutio.

Dicatur $CA = c$, $CM = y$ et vis centripeta in M sollicitans $= P$. Sit porro celeritas in A debita altitudini b et ea in M altitudini v. Sumatur $\int Pdy$ ita, ut evanescat posito $y = c$, quo facto erit $v = b - \int Pdy$. Sumtis ergo b pro h et $\int Pdy$ pro z erit functio ipsarum h et z, cui v aequatur, unius dimensionis; quare erit $m = 1$. Si nunc arcus AM dicatur $= s$, erit

$$s = 2\sqrt{az} = 2\sqrt{a\int Pdy} \text{ atque hinc } ds = \frac{aPdy}{\sqrt{a\int Pdy}}.$$

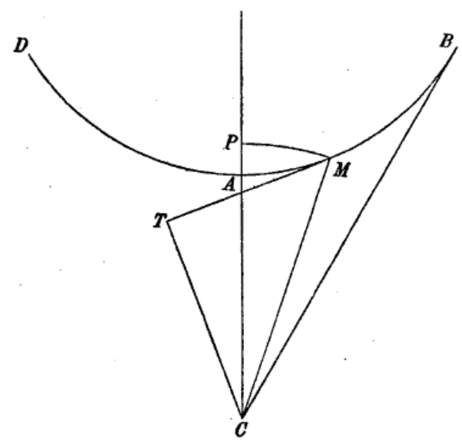


Fig. 52.

Si nunc in M ducatur tangens in eamque ex centro C demittatur per pendiculum CT, quod dicatur p, erit $\frac{ydy}{\sqrt{(y^2 - p^2)}} = ds$. Quare habebitur

$$y^2\int Pdy = (y^2 - p^2)aP^2 \text{ seu } p^2 = y^2 - \frac{y^2\int Pdy}{aP^2},$$ aequatio pro curva quaesita. Q.E.I.

Corollarium 1. [p. 207]

423. In puncto A, ubi $\int Pdy$ evanescit, erit $p = y$, seu recta CA erit normalis in curvam in eoque propterea celeritas corporis erit maxima, quia A est punctum curvae centro C proximum.

Corollarium 2.

424. Si ponatur $p = 0$, habebitur punctum curvae B, in quo recta CB curvam tangit. In eoque puncto, quod erit supremum, curva cuspidem habebit. Invenitur vero punctum B ex hac aequtione $aP^2 = \int Pdy$; atque y non poterit esse maior quam valor ex hac aequatione inventus.

Corollarium 3.

425. Apparet etiam ex aequatione inventa $s = 2\sqrt{a}\int Pdy$, quia signum radicale signum ambiguum involvit, curvam duos habere ramos *AB* et *AD* inter se similes et aequales et hanc ob rem oscillationes, quae super curva *BAD* fiunt, esse inter se aequales.

Corollarium 4.

426. Radius osculi in puncto *A* est = $\frac{2acP}{c-2aP}$. Et quia est $AC = c$, erit tempus, quo unus descensus super curvae *AB* portione infinite parva absolvitur, = $\pi\sqrt{a}$ (207); huic igitur tempori omnes descensus erunt aequales. [p. 208] Hanc ob rem oscillationes, quae fiunt super curva *BAD*, isochronae erunt cum oscillationibus penduli in hypothesi gravitatis = 1, cuius longitudo est = $2a$.

Exemplum 1.

427. Sit vis centripeta directa proportionalis distantiis a centro, ut sit $P = \frac{y}{f}$; erit

$$\int Pdy = \frac{y^2 - c^2}{2f}.$$

Hinc ergo erit

$$AM = s = \sqrt{\frac{2a(y^2 - c^2)}{f}} \text{ atque } p^2 = yy - \frac{f(yy - c^2)}{2a}.$$

Huius curvae radius osculi in puncto *M* qui est $\frac{ydy}{dp}$, invenitur = $\frac{2a}{2a-f}\sqrt{\frac{(2a-f)y^2 + fc^2}{2a}}$.

Ex quo sequitur, si fuerit $2a < f$, curvam versus centrum *C* fore convexam, ut exhibet figura. At si fuerit $2a = f$, curva evadet linea recta in *A* normalis ad rectam *AC*. Punctum autem *B*, ubi *CB* tangit curvam, invenitur ex hac aequatione $(f - 2a)yy = ccf$, ex qua fit $BC = \frac{c\sqrt{f}}{\sqrt{(f-2a)}}$. Quoties ergo accidit, ut sit $f > 2a$ seu curva convexa versus *C*, habebit curva cuspidem in *B*. Atque in his casibus curva erit hypocylois, quae generatur rotatione circuli, cuis diameter est $= \frac{c\sqrt{f} - c\sqrt{(f-2a)}}{\sqrt{(f-2a)}}$, super concava parte circuli centro *C* radio = $\frac{c\sqrt{f}}{\sqrt{(f-2a)}}$ descripti. Hoc ergo casu curvae tautochronae conveniunt cum brachystochronis supra inventis (390).

At si $2a > f$, quo casu curva est concave versus *C*, fit *BC* imaginaria et curva *AM* non amplius est hypocycloις. [p. 209] Tum autem erit $p^2 = \frac{(2a-f)yy + ccf}{2a}$, unde *p* ubique praeter in *A* erit maior quam *AC*. Sit $c = 0$; fiet $p = y\sqrt{\frac{2a-f}{2a}}$, quare hoc casu curvae tautochronae erunt spirales logarithmicae circa centrum C descriptae. Corpus scilicet super spirali logarithmicae perpetuo eodem tempore ad centrum C pertinget, ubicunque descensum inceperit. Hac igitur vis centripetae hypothesi tautochronae erunt: primi omnes hypocycloides, deinde omnes lineae rectae utcunque ductae, tertio omnes spirales logarithmicae et quarto infinitae curvae aliae hac aequatione contentae $p^2 = \frac{(2a-f)yy+ccf}{2a}$, si quidem fuerit $2a > f$ et c non = 0. In hac autem vis centripetae hypothesi pro tautochronis tantum dederunt hypocycloides Newtonus in *Princ*. [Lib. I, Prop. LI, theorema XVIII.] et Hermannus in *Phoronomia* et Comment. Acad. Petrop. A 1727, [p. 139, vide praecipue p. 150; P.St.] quamvis hic aequationem aeque generalem ac nostram habuerit.

Exemplum 2.

428. Ponatur vis centripeta reciproce proportionalis quadratis distantiarum a centro, ut sit $P = \frac{ff}{yy}$; erit

$$\int Pdy = \frac{-ff}{y} + \frac{ff}{c} = \frac{ff(y-c)}{cy}.$$

Curva igitur AM erit =

$$2f\sqrt{\frac{a(y-c)}{cy}} \quad \text{atque} \quad pp = \frac{acffyy + cy^5 - y^6}{acff},$$

unde invenitur radius osculi

$$\frac{ydy}{dp} = \frac{2fy\sqrt{ac(acff + cy^3 - y^4)}}{2acff + 5cy^3 - 6y^4}.$$

Ubi ergo p evanescit, ibi etiam radius osculi fit = 0. Ipsius BC valor autem reperietur ex hac aequatione $y^4 = cy^3 + acff$. [p. 210] Si igitur ponatur $BC = k$, erit $a = \frac{k^4 - ck^3}{cff}$, id quod assumi potest, quia k est quantitas arbitratia. Ceterum apparet hanc curvam intra A et B habere posse punctum flexus contrarii, id quod evenit, si curva in A est concava versus C, nempe si fuerit $2aff > c^3$.

Exemplum 3.

429. Sit vis centripeta ubique constans seu $P = 1$; erit $\int Pdy = y - c$. Quare si centro C radio CM ducatur arcus MP, erit $\int Pdy = AP$ atque arcus $AM = 2\sqrt{a(y-c)} = 2\sqrt{a} \cdot AP$. At erit porro

$$p^2 = y^2 - \frac{y^2(y-c)}{a} = \frac{(a+c)y^2 - y^3}{a},$$

unde invenitur $BC = a + c$ et curva $AB = 2a = 2(BC - AC)$. Radius osculi vero in puncto M erit =

$$\frac{2y\sqrt{(a+c-y)a}}{2a+2c-3y}.$$

In puncto ergo infimo A erit radius osculi = $\frac{2ac}{2a-c}$. Curva igitur in A erit concava versus C, si $2a > c$, at convexa, si $c > 2a$, atque radius osculi in A erit infinite magnus, si $2a = c$. Primo casu, quo curva in A est concave versus C, curva habebit punctum flexus contrarii, ubi est

$$CM = \frac{2}{3}(a + c) = \frac{2}{3}BC.$$

Si est $c = 0$ ita ut punctum A in centrum C cadat, fiet y chorda huius curvae eritque $AM = 2\sqrt{ay}$, cuius curvae radius osculi in ipso centro C est infinite parvus, atque est $p = y\sqrt{\frac{a-y}{a}}$; curva autem ipsa per quadraturam circuli potest constui. [p. 211]