CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 97]

PROPOSITION 25.

Problem.

224. *If a body is drawn downwards by some constant force, to find the curve AM* (Fig. 32)*, that a body descending on that curve presses upon equally everywhere.*

Solution.

Let *AM* be the curve sought ; with the abscissa to the vertical called $AP = x$, the applied line $PM = y$ and the curve $AM = s$. Again let the force acting on the body at *M* be equal to *P* and the height corresponding to the speed at $A = b$; [p. 98] the height corresponding to the speed at M = $b + \int Pdx$, with the integral $\int Pdx$ thus taken in order that it vanishes on making $x = 0$. With these in place the compression force that the curve sustains along the normal *MN* is equal to

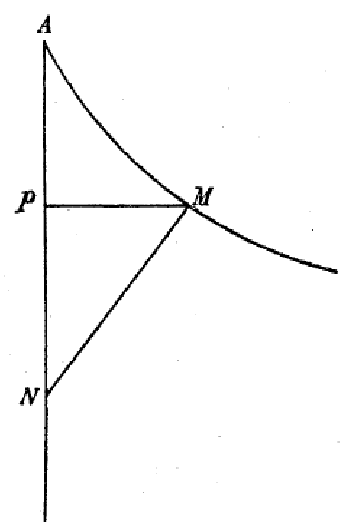

Fig. 32.

$$\frac{Pdy}{ds} + \frac{2(b+\int Pdx)dxddy}{ds^3}$$

(83) with the element *dx* taken as constant [*i. e. x* is the independent variable]. Now since this force has to be constant, it is put equal to *k*, then we have :

$$kds^3 = Pds^2dy + 2bdxddy + 2dxddy\int Pdx.$$

But if *ds* is made constant, then we have

$$kdsdx = Pdxdy + 2bddy + 2ddy\int Pdx,$$

[which amounts to the starting condition where $k = \frac{Pdy}{ds}$, as the centrifugal effect on the curve is zero when *y* is incremental initially. The different ways of expressing the radius

of curvature should also be noted : In this case, if *ds* is constant, then from $ds^2 = dx^2 + dy^2$ we have $0 = dxddx + dyddy$, and in an obvious notation, we have $\cos\theta = \frac{dx}{ds}$ from which $-\sin\theta d\theta = \frac{ddx}{ds} = -\frac{dyddy}{dxds}$ and hence $d\theta = \frac{dyddy}{dxds} \times \frac{1}{\sin\theta} = \frac{dyddy}{dxds} \times \frac{ds}{dy} = \frac{ddy}{dx}$. Hence $R = \frac{ds}{d\theta} = \frac{dsdx}{ddy}$ is the required radius of curvature.]
the integral of which is :

$$\frac{2dy\sqrt{(b+\int Pdx)}}{ds} = \int \frac{kdx}{\sqrt{(b+\int Pdx)}}.$$

This [differential] equation can be constructed, as *P* is given in terms of *x*, and since *y* is not present in this equation, but only *dy*. Q.E.I.

Corollary 1.

225. The integral

$$\int \frac{dx}{\sqrt{(b+\int Pdx)}}$$

expresses the time in which the body starts from *A*, with the same initial speed as that with which it is moved along *AM*, and falls straight down through the height *AP*, and $\sqrt{(b+\int Pdx)}$ gives the speed at the same place. Whereby this speed at *P* divided by the time to traverse *AP* gives $\frac{kds}{2dy}$, from which property the curve *AM* can be determined. [Thus the quantity $\frac{kds}{2dy}$, related to the angle of projection at A, is determined from special values of the time and speed in the above equation, which is used further.]

Corollary 2.

226. Moreover the time to traverse the distance *AP* can be increased by any constant quantity, for example $\sqrt{c}$, [*i. e.* in the ratio $\frac{\sqrt{b}}{\sqrt{c}}$]. And from this constant magnitude the angle which the curve at *A* makes with *AP* is determined. [The sine of the angle between the vertical axis and the curve at A is of course $\frac{dy}{ds}$, and $\frac{kds}{2dy} = \frac{\sqrt{b}}{\sqrt{c}}$; thus, increasing $\sqrt{c}$ makes the angle greater, up to a certain allowable value] Clearly the sine of the angle is equal to $\frac{k\sqrt{c}}{2\sqrt{b}}$ with the total sine put equal to 1. Whereby $\sqrt{c}$ cannot thus be taken greater than $\frac{2\sqrt{b}}{k}$, and if the body starts at rest from *A*, then *c* must be equal to 0.

Example. [p. 99]

227. Let the force be uniform or $P = g$; then we have :

$$\int \frac{kdx}{\sqrt{(b+gx)}} = \frac{2k\sqrt{(b+gx)} - 2k\sqrt{b} + 2k\sqrt{c}}{g} = \frac{2dy\sqrt{(b+gx)}}{ds},$$

[Note that the time has been increased by the added constant $\frac{2k\sqrt{c}}{g}$] hence we have :

$$\frac{dy}{ds} = \frac{k}{g} + \frac{k(\sqrt{c}-\sqrt{b})}{g\sqrt{(b+gx)}}$$

and

$$gdy\sqrt{(b+gx)} = kds\sqrt{(b+gx)} + kds(\sqrt{c}-\sqrt{b}).$$

From which the following equation arises, [noting that $\frac{dx}{ds} = \sqrt{1-(\frac{dy}{ds})^2}$]:

$$dy = \frac{kdx(\sqrt{(b+gx)}+\sqrt{c}-\sqrt{b})}{\sqrt{(g^2(b+gx)-k^2(\sqrt{(b+gx)}+\sqrt{c}-\sqrt{b})^2)}}.$$

Let $\sqrt{(b+gx)} = t$ and $-\sqrt{c}+\sqrt{b} = h;$ then

$$x = \frac{t^2-b}{g} \text{ and } dx = \frac{2tdt}{g}.$$

Therefore with these substituted, we have :

$$dy = \frac{2ktdt(t-h)}{g\sqrt{(g^2t^2-k^2t^2+2k^2ht-k^2h^2)}}.$$

This equation allows the integration for three cases, the first of which occurs for $k = 0$; for then the curve is found that a body describes freely projected from A. The second case is when $h = 0$ or $\sqrt{b} = \sqrt{c}$; for then we have $\frac{dy}{ds} = \frac{k}{g}$ or the line satisfying the equation is an inclined straight line. If in the third case $k = g$ or the total compression force is everywhere equal to the force g acting on the body, then the equation becomes :

$$dy = \frac{2ttdt-2htdt}{g\sqrt{(2ht-h^2)}},$$

the integral of this is :

$$gy = \frac{(2tt - 2ht - 2h^2)}{5h}\sqrt{(2ht - h^2)} + \text{const.}$$

This constant, since with $x = 0$ or $t = \sqrt{b}$ makes $y = 0$, must be equal to :

$$\frac{(2h^2 + 2h\sqrt{b} - 2b)}{5h}\sqrt{(2h\sqrt{b} - h^2)}.$$

Therefore with $\sqrt{(b+gx)}$ restored in place of t and by placing $\sqrt{b} - \sqrt{c} = h = \sqrt{a}$, it is found that :

$$\frac{5gy\sqrt{a}}{2} = (b + gx - a - \sqrt{a(b+gx)})\sqrt{(2\sqrt{a(b+gx)} - a)} + (a - b + \sqrt{ab})\sqrt{(2\sqrt{ab} - a)}.$$

Which is the equation of the curve sought, in which a must be a number less than b. [p. 100]

If the body must fall from rest, any other line besides a straight line is not satisfactory. Indeed it it must be the case that $c = 0$, in order that the angle at A is real, and therefore the equation is put in place :

$$y = \frac{kx}{\sqrt{(g^2 - k^2)}}.$$

Corollary 3.

228. An algebraic equation found, if it is to be free from irrationality, can be made of order five. If $a = b$ is put in this equation, in which case the tangent of the curve at A is vertical, and gives :

$$\frac{5gy\sqrt{a}}{2} = (gx - \sqrt{(a^2 + gax)})\sqrt{(2\sqrt{(a^2 + gax)} - a)} + a\sqrt{a}.$$

Corollary 4.

229. If in general the tangent at A is to be vertical, then $\sqrt{c} = 0$ and thus this equation arises :

$$dy = \frac{kdx(\sqrt{(b+gx)} - \sqrt{b})}{\sqrt{\left(g^2(b+gx) - k^2(\sqrt{(b+gx)} - \sqrt{b})^2\right)}}.$$

If the tangent at A is placed horizontal, then $k\sqrt{c} = g\sqrt{b}$ and this equation arises :

$$dy = \frac{dx(k\sqrt{(b+gx)} + (g-k)\sqrt{b})}{\sqrt{\left(g^2(b+gx) - (k\sqrt{(b+gx)} + (g-k)\sqrt{b})^2\right)}}.$$

Scholium.

230. This curve is called the line of uniform pressing [above we have called this the compression force, and of course it refers to the normal reaction force exerted on the body by the fixed curve having constant magnitude, although the reaction of this force is usually considered in the text; clearly it is inappropriate to use modern terms, while 'pressing' or 'squeezing together' seems too vague. Again, the derived word 'pressure' has a different meaning now, and cannot be used. Remember that many of the latin words recruited by Euler are given different mathematical meanings from those originally found in the dictionary. Even the word 'expression' seems to relate to extracting the juice of the grape. All part of the fun of being Euler, I suppose, and although he was a person of great piety - see some of his *Letters to a German Princess*, he nevertheless had a sense of humour.] and the solution of this problem is set out in the Comment. Acad. Paris., which agrees uncommonly well with our solution.
(G. F. De L'Hospital (1661 - 1704), *Solution d'un problème physico-mathématique, proposé par Iean Bernoulli*, Mém. de l'acad. d. sc. de Paris 1700, p. 9. See also the study by P. Varignon (1654 - 1722), *Usage d'une intégrale donnée par G. F. De L'Hospital, ou sur les pressions des courbes en général, avec la solution de quelques autres questiones approchantes de la sienne,* Mém. de l'acad. d. sc. de Paris 1710, p. 158. References by P. S.)
The other case agrees with this solution, if the force is not constant, but for whatever the variable P, the equation found is nevertheless integrable, if the compression on the curve should be proportional to *P*. [p. 101] For it becomes $k = mP$ and the following equation is produced for the curve sought :

$$\frac{2dy\sqrt{(b+\int Pdx)}}{ds} = \int \frac{mPdx}{\sqrt{(b+\int Pdx)}},$$

the integral of which is :

$$dy\sqrt{(b+\int Pdx)} = mds\sqrt{(b+\int Pdx)} + mds\sqrt{c}.$$

This equation, if $c = 0$, is for a straight line inclined to the horizontal. But the angle is defined by $\sqrt{c}$, which the curve makes with the vertical at *A*; indeed the sine of this angle is $m+\frac{m\sqrt{c}}{\sqrt{b}}$. Whereby if we take $\sqrt{c} = -\sqrt{b}$, the curve is a tangent to the vertical at *A*. Besides, this curve has the property that the time in which the arc *AM* is traversed is proportional to $m \bullet AM - PM$. Finally from the solution of this proposition flows the solution of the following, in which from the given curve and the equal compression of the curve, the magnitude of the force acting downwards is sought.

PROPOSITION 26.

Problem.

231. *With the curve AM* (Fig. 32) given, *and with the initial speed at A corresponding to the height b, to find the size of the force always acting downwards, which arises in order that a body descending along the curve AM exerts the same force everywhere on the curve.*

Solution.

Let the force acting sought be equal to P, and with these lengths named : $AP = x$, $PM = y$ et $AM = s$ and the force that the curve sustains is equal to k , this equation is put in place :

$$kdsdx = Pdxdy + 2bddy + 2ddy\int Pdx$$

(224), in which ds is made a constant element. Therefore from this equation the quantity P has to be elicited. [p. 102] Moreover, the equation multiplied by dy and integrated gives :

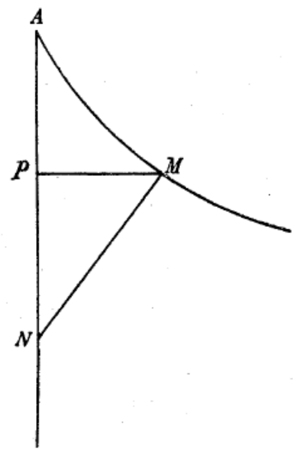

Fig. 32.

$$kds\int dxdy = dy^2\int Pdx + bdy^2,$$

from which there arises :

$$\int Pdx + b = \frac{kds}{dy^2}\int dxdy;$$

which differentiated gives :

$$P = \frac{kds}{dy} - \frac{2kdsddy}{dxdy^3}\int dxdy.$$

But the integral $\int dxdy$ thus has to be taken, in order that with $x = 0$ it becomes $\frac{kds}{dy^2}\int dxdy = b.$ Moreover , so that this integration can advance easier, putting $dy = pdx$; then we have

$$ds = dx\sqrt{(1+p^2)} \text{ and } \int dxdy = ds\int\frac{pdx}{\sqrt{(1+pp)}}$$

and thus,

$$\frac{kds}{dy^2}\int dxdy = \frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

From which equation there is produced :

$$P = \frac{k\sqrt{(1+pp)}}{p} - \frac{2kdp}{p^3dx}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

Q.E.I.

Corollary 1.

232. From this equation the speed of the body at the individual points is also found at once; for the height corresponding to the speed at M is

$$b + \int Pdx = \frac{kds}{dy^2}\int dxdy = \frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

Now the time, in which the arc *AM* is completed, is equal to

$$\frac{1}{\sqrt{k}}\int pdx : \sqrt{\int\frac{pdx}{\sqrt{(1+p^2)}}}.$$

Corollary 2.

233. It is evident from the equation found that the magnitude of the force P is therefore to be greater where k is greater from the other terms; for the value of this variable has been multiplied by the compression k.

Corollary 3.

234. Although now the force P is not seen to depend on the initial speed b, [p. 103] because b is not present in the expression, yet P depends on b on account of the integral $\int\frac{pdx}{\sqrt{(1+pp)}}$, which must thus be taken, in order that with $x = 0$ it becomes $\frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}} = b.$ Hence with the initial speed varying other forces acting are produced, even if the the proposed curve remains the same.

Exemplum 1.

235. Let the curve *AM* be a parabola having the vertex at A and the axis horizontal, thus so that is is given by $ay = x^2$. Therefore we have $dy = \frac{2xdx}{a}$ and hence $p = \frac{2x}{a}$ and

$$\int\frac{pdx}{\sqrt{(1+pp)}} = \int\frac{2xdx}{\sqrt{(a^2+4x^2)}} = \frac{1}{2}\sqrt{(a^2+4x^2)} + C.$$

Whereby

$$\frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}} = \frac{k(a^2+4x^2)^{\frac{3}{2}}}{8x^2} + \frac{kC(a^2+4x^2)}{4x^2}.$$

Since this quantity must be equal to *b*, if *x* = 0, then $C = \frac{-a}{2}$ or

$$\int \frac{p\,dx}{\surd(1+pp)} = \frac{\surd(a^2+4x^2)-a}{2}.$$

From which it is found :

$$P = \frac{ka^3}{4x^3} - \frac{k(a^2-2x^2)\surd(a^2+4x^2)}{4x^3}.$$

Therefore at the point *A* the force *P* is indefinitely small; as the numerator as well as the denominator vanish the value of this expression is equal to zero. Now the speed at *A* cannot be made arbitrary, also if the the constant *C* is seen to be determined from *b*. For *C* has only such a value which the expression

$$b + \int P\,dx = \frac{k\,ds}{dy^2}\int dx\,dy$$

returns of finite magnitude. Therefore *b* depends on *a* and the value of this can be found, if we put *x* = 0 in the expression

$$\frac{k(a^2+4x^2)^{\frac{3}{2}} - ka(a^2+4x^2)}{8x^2}$$

Moreover, there is produced then $b = \frac{ka}{4}$. [p. 104] Therefore the descent must begin with this speed, in order that the compression that arises is everywhere equal to that found from the force *P*.

Example 2.

236. Let the curve *AM* be a circle of radius *a* touching the line *AP* at *A*; then it is given by $y = a - \sqrt{(a^2 - x^2)}$ and $p = \frac{x}{\sqrt{(a^2-x^2)}}$, and also $\sqrt{(1+pp)} = \frac{a}{\sqrt{(a^2-x^2)}}$.

Therefore the integral

$$\int \frac{p\,dx}{\surd(1+pp)} = \int \frac{x\,dx}{a} = \frac{x^2}{2a},$$

to which it is not required to add a constant, because $\frac{1+pp}{pp} = \frac{a^2}{x^2}$ becomes infinite with *x* vanishing. Therefore the equation becomes :

$$k\frac{(1+pp)}{pp}\int \frac{p\,dx}{\surd(1+pp)} = \frac{ka}{2} = b + \int P\,dx;$$

whereby the speed of the bvody is uniform and thus the force acting vanishes. It is evident that the body is progressing uniformly on the circumference of the circle with no force acting and the centrifugal force everywhere is of the same magnitude.

Example 3.

237. Let the curve *AM* be a cycloid having the base horizontal and with the cusp a tangent to the vertical *AP* at *A*, thus so that the equation becomes :

$$dy = \frac{dx\sqrt{2ax}}{\sqrt{(a^2 - 2ax)}}.$$

Therefore we have :

$$p = \frac{\sqrt{2ax}}{\sqrt{(a^2 - 2ax)}} \text{ and } \sqrt{(1+pp)} = \frac{a}{\sqrt{(a^2 - 2ax)}}.$$

Whereby the equation becomes :

$$\int \frac{pdx}{\sqrt{(1+pp)}} = \int \frac{dx\sqrt{2ax}}{a} = \frac{2x\sqrt{2x}}{3\sqrt{a}} + C$$

and

$$\frac{k(1+pp)}{pp} = \frac{ka}{2x}.$$

Therefore with the constant C taken of finite magnidute making $b = \infty$; whereby making $C = 0$;then

$$b + \int Pdx = \frac{k\sqrt{2ax}}{3}$$

and $b = 0$. Therefore there is produced :

$$P = \frac{k\sqrt{a}}{3\sqrt{2x}}.$$

Thus if the body descends on the cycloid *AM* from *A* from rest and is acted on by a downwards force [p. 105], which varies as the square root of the abscissa *AP*, the body is everywhere acted on by a constant force.

Scholium.

238. Therefore cases are given, in which the speed $\sqrt{b}$ cannot be assumed at will, as has come about in these examples. For as often as $\frac{1+pp}{pp}$ is made infinitely large by making $x = 0$, a constant in the integraton of $\frac{pdx}{\sqrt{(1+pp)}}$ has to be added, and generally this is itself determined because the initial speed cannot be infinitely large. But always, if the curve is a tangent to the line *AP* at *A*, then $\frac{1+pp}{pp}$ becomes infinite with $x = 0$, since that is also the reason in other examples considered why the initial speed cannot be made arbitrary.

PROPOSITION 27.

Problem.

239. *If a body is always drawn downwards by some force, to find the curve AM* (Fig. 32), *upon which the body is thus moving, in order that the total compression force sustained by the curve has a given ratio to the compression arising from the normal force.*

Solution.

The body descends from A with a speed corresponding to the height b and on placing $AP = x$, $PM = y$ and $AM = s$ and let the force acting on the body at M be equal to P; [p. 106] the height corresponding to the speed, that the body has at M, is equal to $b+\int Pdx$, now the total force of compresseion that the curve sustains at M following the direction of the normal MN, is equal to

$$\frac{Pdy}{ds}+\frac{2ddy(b+\int Pdx)}{dxds}$$

on taking ds for the constant element. Now this compression force is in the ratio m to 1 to the normal force $\frac{Pdy}{ds}$; hence

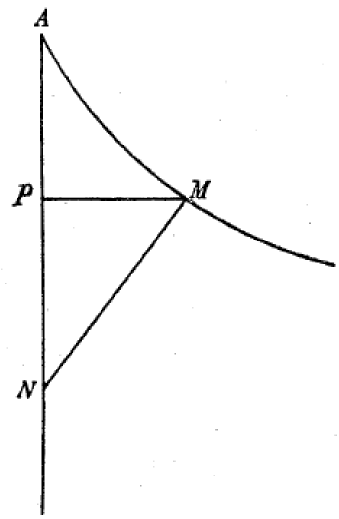

Fig. 32.

$$(m-1)Pdxdy=2ddy(b+\int Pdx),$$

which is the equation of the curve sought. Now this can be reduced by putting v in the place of $b+\int Pdx$, to this form :

$$\frac{(m-1)dv}{v}=\frac{2ddy}{dy},$$

which integrated gives :

$$2l\frac{dy}{ds}=(m-1)l\frac{v}{a} \text{ or } v^{\frac{m-1}{2}}ds=a^{\frac{m-1}{2}}dy.$$

From which there is obtained [as $\frac{dy}{ds}=\sin\theta=\frac{v^{\frac{m-1}{2}}}{a^{\frac{m-1}{2}}}$, etc.]:

$$dy=\frac{v^{\frac{m-1}{2}}dx}{\sqrt{(a^{m-1}-v^{m-1})}}=\frac{dx(b+\int Pdx)^{\frac{m-1}{2}}}{\sqrt{(a^{m-1}-(b+\int Pdx)^{m-1})}},$$

which is the equation of the curve sought. Q.E.I.

Corollary 1.

240. The speed of the body is zero there, where $\frac{dy}{ds}=0$ or where the tangent of the curve is vertical, if indeed $\frac{m-1}{2}$ is a positive number or *m* is greater than one. Therefore in these cases we put the curve to be a tangent to the line *AP* at *A* and the initial speed or $b = 0$. [p. 107]

Corollary 2.

241. Whereby if $m > 1$, or if the total compression force is greater than the normal force arising, the curve sought is given by this equation :

$$dy=\frac{dx(\int Pdx)^{\frac{m-1}{2}}}{\sqrt{(a^{m-1}-(\int Pdx)^{m-1})}},$$

in which $\int Pdx$ thus must be taken so that it vanishing with $x = 0$.

Corollary 3.

242. Whereby if $m = 1$, the centifugal force vanishes and therefore the line sought is straight. Moreover from the equation we have $ddy = 0$, which is the property of a straight line.

Corollary 4.

243. If $m = 0$, then the total pressing force vanishes; whereby there is then produced the curve, that the body freely described, projected with a speed corresponding to its height *b*. Therefore for this curve this equation is found :

$$dy=\frac{dx\sqrt{a}}{(b-a+\int Pdx)}.$$

Corollary 5.

244. If *m* is less than one, then the centrifugal force is in the opposite direction to the normal force and therefore the curve AM is concave downwards. Therefore we put the curve to be normal to *AP* at *P* ; and $b = a$. Therefore with $b = a$, this equation is found for the curve sought :

$$dy=\frac{a^{\frac{1-m}{2}}dx}{\sqrt{((a+\int Pdx)^{1-m}-a^{1-m})}}.$$

Corollary 6. [p. 108]

245. Therefore for the free motion, in which case $m = 0$, this curve described is found for the body, if it is projected horizontally from A with a speed corresponding to the height a, from this equation :

$$dy = \frac{dx\sqrt{a}}{\sqrt{\int Pdx}}.$$

Example 1.

246. Let the uniform force acting or $P = g$; then we have $\int Pdx = gx$. Hence in the cases in which $m > 1$ and the body descends from being at rest at A, the equation for the curves sought with gc written in place of a is this :

$$dy = \frac{x^{\frac{m-1}{2}}dx}{\sqrt{(c^{m-1} - x^{m-1})}}.$$

But if $m < 1$ and the body is projected from A horizontally with a speed corresponding to the height a, the curve upon which the body must be moving, with gc written in place of a, is shown by this equation :

$$dy = \frac{c^{\frac{1-m}{2}}dx}{\sqrt{((c+x)^{1-m} - c^{1-m})}}.$$

These curves are therefore algebraic, if either $\frac{3-m}{2m-2}$ or $\frac{m}{1-m}$ is a positive whole number. Now this comes about if m is a term of either this series $3, \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$ etc., or of this series $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ etc. [e. g. on setting the expression squared equal to x^m]

Corollary 7.

247. Therefore if the total compression force is three times greater than the normal force, then the curve is a circle touching the line AP at A. [p. 109]
For it becomes the equation of the circle of radius c :

$$dy = \frac{xdx}{\sqrt{(c^2 - x^2)}} \quad \text{or} \quad y = c - \sqrt{(c^2 - x^2)},$$

Corollary 8.

248. If the total compression force is twice as great as the normal force or the centrifugal force is equal to the normal force acting in the same way ; then the curve is a cycloid with the vertical cusp a tangent at *A*. For the equation is given by :

$$dy = \frac{dx\sqrt{x}}{\sqrt{(c-x)}}.$$

Example 2.

249. Whatever the force acting *P* should be, curves of the same kind are required, in order that the total compression that the curve sustains, is twice as large as the normal force or as the centrifugal force, which in this case that will be the equation. Therefore putting $m = 2$, and this equation is found for the curve sought :

$$dy = \frac{dx\sqrt{\int Pdx}}{\sqrt{(a-\int Pdx)}}.$$

Or by calling $\int Pdx = X$ then

$$dy = dx\sqrt{\frac{X}{a-X}} = \frac{Xdx}{\sqrt{(aX - X^2)}}.$$

We have brought up this example, since in the following curves with this property are likewise lines of the quickest descent.

Corollary 9.

250. Therefore it is evident that there are endless curves satisfying the question, on account of the quantity *a* being arbitrary. And all these boundless curves have a tangent to the line *AP* at *A*.

Scholium 1. [p. 110]

251. It is apparent from the solution of this problem, how the inverse problem can be solved, in which the curve and the ratio between the total compression force and the normal force is given, and the magnitude of the force acting downwards is sought. Since indeed it is

$$v^{\frac{m-1}{2}} ds = a^{\frac{m-1}{2}} dy$$

or putting $dy = pdx$

$$v^{\frac{m-1}{2}} \sqrt{(1+pp)} = a^{\frac{m-1}{2}} p,$$

it becomes

$$v = \frac{ap^{\frac{2}{m-1}}}{(1+pp)^{\frac{1}{m-1}}} = b + \int Pdx$$

and hence by differentiation :

$$Pdx = \frac{2ap^{\frac{3-m}{m-1}}dp}{(m-1)(1+pp)^{\frac{m}{m-1}}}.$$

Consequently it is found that

$$P = \frac{2ap^{\frac{3-m}{m-1}}dp}{(m-1)(1+pp)^{\frac{m}{m-1}}dx}.$$

Where it is to be noted that the initial speed must now be given ; for the formula

$$\frac{ap^{\frac{2}{m-1}}}{(1+pp)^{\frac{1}{m-1}}},$$

gives b if we put $x = 0$.

Scholium 2.

252. In a like manner, if the motion of the body or the speed of this is given at individual points and the relation between the total pressing force to the normal force is given, from the speed the force acting is found at once. For let v be the height corresponding to the speed at M; then as $b+\int Pdx = v$, $P = \frac{dv}{dx}$ and the equation [p. 111]

$$v^{\frac{m-1}{2}}ds = a^{\frac{m-1}{2}}dy$$

gives the nature of the curve sought. For since v is given, either x or s and constant quantities have to be given, clearly which are used in expressing the nature of the curve. Moreover the same problems proposed under the hypothesis of centripetal forces or of many forces acting do not introduce more difficulties, even if more complex equations may be reached. And since a simple example in a medium cannot be brought forwards as an illustration, this I rather abandon, and more towards that [study] which I am about to set out with great diligence in the following, where the nature of the brachistochrone is worked through, and curves of the same kind are produced. Now therefore I progress to that problem, in which a certain property of the motion is proposed, from which conjointly either the curve is sought with the force acting, or from the curve itself the force acting. Now the exceedingly easy problems as when either the scale of the speeds or of the times is given, I omit, since from the expression of the speed or the force acting either the curve itself comes freely, and an expression of the time to the speed can be easily deduced. Because of these things, we bring forwards questions, in which neither the speeds nor the times are given, but certain relations depending on these.

[p. 112]
PROPOSITION 28.

Problem.

253. A body is acted on by some force acting downwards *; to find the curve AM* (Fig. 33), *on which the body descends in a uniform motion downwards as it is carried forwards or equally as it recedes to the horizontal AB.*

Solution.

On placing $AP = x$, $PM = y$, $AM = s$ and the force acting equal to P, let the initial speed of the body at A correspond to the height b; the speed at M corresponds to the height $b+\int Pdx.$ Whereby the short interval of time, in which the element Mm is traversed, is equal to $\frac{ds}{\sqrt{(b+\int Pdx)}}$. Moreover since the motion along AM must correspond to a uniform motion along AP, it is understood that the motion of the body along AP is with the constant speed corresponding to the height b; the time to pass through Pp must be equal to the time to pass along Mm, hence we have the equation : $\frac{dx}{\sqrt{b}} = \frac{ds}{\sqrt{(b+\int Pdx)}}$ or $dy\sqrt{b} = dx\sqrt{\int Pdx}.$

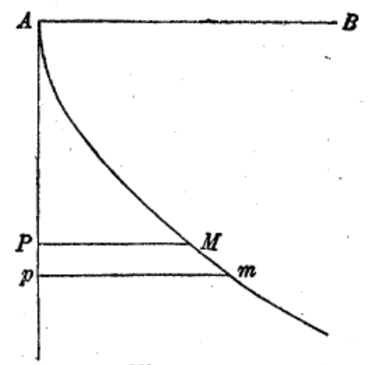

Fig. 33.

Moreover I put the initial speed agreeing with the speed of descent, in order that the curve A is a tangent to the vertical AP and the body in the beginning at first falls in a straight line. For since the motion on account of the acceleration by necessity is one of acceleration, in order that the curve continually becomes more inclined to the horizontal, the initial motion of this is most conveniently taken at A, where the curve is vertical. Therefore the equation $dy\sqrt{b} = dx\sqrt{\int Pdx}.$ is produced for this curve. Q.E.I.

[We can understand this second equation in terms of the squares of the speeds, which add according to the right angle rule : The speed downwards is u always, the speed horizontally is v at the point M, and the speed along ds is V, then $V^2 = u^2 + v^2$. Now u^2 is proportional to b, v^2 is proportional to $\int Pdx$ with the same constant of proportionality, from which the equality of the times gives the above equation.]

Corollary 1. [p. 113]

254. Hence this curve has the property so that when the speed of the body is greater, in that place the more also is the inclination of the curve to the horizontal.

Corollary 2.

255. Therefore in the highest place, where the speed of the body is the least, the inclination of the curve must be a minimum or the tangent to the curve at that place must be vertical.

Corollary 3.

256. Therefore the initial speed $\sqrt{b}$ cannot be equal to zero, since the speed is equal to that respective speed, by which the body progresses downwards as it recedes to the horizontal *AB*.

Scholium 1.

257. This curve is called the *line of uniform descent*, since a body descending on it is progressing with a constant motion downwards. The discovery of this line is set out in the Act. Erud. Lips. A 1689 for the hypothesis of gravity, or of a constant force acting. [G. W. Leibniz, *Concerning isochronous lines, in which a weight falls without gravity,* Acta erud. 1689, p. 234]
Moreover, there it is demonstrated that Neil's cubic parabola satisfies this question, the same as that produced in the following example.

Example 1.

258. Let a uniform force be acting or $P = g$; then $\int Pdx = gx.$ [p. 114] Whereby this equation is obtained for the curve sought : $dy\sqrt{b} = dx\sqrt{gx}$, which on integration gives this equation : $3y\sqrt{b} = 2x\sqrt{gx}$ or $\frac{9by^2}{4g} = x^3$, which is the equation for Neil's parabola with the cusp at *A* tangent to the line *AP* , of which the parameter is $\frac{9b}{4g}$. Therefore for which with another initial speed, another parabola is to be taken.

Example 2.

259. Let the force acting *P* be proportional to some power of the abscissa increased by a given line, in order that $P = \frac{(a+x)^n}{f^n}$; then there arises

$$\int Pdx = \frac{(a+x)^{n+1} - a^{n+1}}{(n+1)f^n}.$$

On account of which for the curve satisfying this equation, we have :

$$dy\sqrt{(n+1)bf^n} = dx\sqrt{((a+x)^{n+1} - a^{n+1})}.$$

If $a = 0$, thus so that the force P acting is proportional to a power n of the distances of the body from the horizontal AB, then :

$$dy\sqrt{(n+1)bf^n} = dx\sqrt{x^{n+1}},$$

of which the integral is :

$$y\sqrt{(n+1)bf^n} = \frac{2x^{\frac{n+3}{2}}}{n+3}$$

or

$$\frac{(n+1)(n+3)^2}{4}bf^n y^2 = x^{n+3}.$$

But $n + 1$ must be a positive number ; otherwise $\int Pdx$ is made infinite, since it must vanish on making $x = 0$. Hence $n + 3 > 2$; whereby the parabolas satisfy the tangents with the verticals at A tangent to AP. For, if $n = 1$ or $P = \frac{x}{f}$, the parabola satisfies the parabola of Apollonius, of which the parameter is $2\sqrt{2}bf$.

Scholium 2.

260. From the solution of this proposition it is evident, how the inverse problem of this can be solved, in which the curve is given, [p. 115] which is a line of equal descent, and it is required to find the force acting. Since indeed it is given by : $dy\sqrt{b} = dx\sqrt{\int Pdx}$, then it becomes $\int Pdx = \frac{bdy^2}{dx^2}$. From which there arises with dx put constant :

$$P = \frac{2bdyddy}{dx^3}.$$

Therefore it is evident that the force P depends on the initial speed $\sqrt{b}$. Now the given curve thus has to be compared, so that it is a tangent with the vertical AP at A. If the radius of osculation at M is called r, then

$$P = \frac{2bds^3dy}{rdx^4}.$$

Whereby if for example the curve AM is a circle tangent to AP at A, the radius of this is equal to a, then

$$r = a, \quad dy = \frac{xdx}{\sqrt{(a^2 - x^2)}} \text{ and } ds = \frac{adx}{\sqrt{(a^2 - x^2)}}.$$

Therefore the circle,

$$P = \frac{2a^2bx}{(a^2 - x^2)^2}.$$

Now the speed at M corresponds to the height

$$b+\int Pdx=\frac{a^2b}{a^2-x^2}.$$

Corollary 4.

261. Likewise it is apparent from the equation, that we have found (253),

$$\frac{dx}{\sqrt{b}}=\frac{ds}{\sqrt{(b+\int Pdx)}}$$

the time, in which the arc *AM* is described, is equal to the time in which the body must move with the constant speed corresponding to the height *b*, traverses the vertical distance *AP*. In this equation clearly the nature of the line of equal descent arises.

[p. 116]

PROPOSITION 29.

Problem.

262. *With a constant force everywhere pulling vertically downwards, to find the curve AM* (Fig. 34)*, upon which a body is progressing uniformly along a given direction AP.*

Solution.

Let *AM* be the curve sought and the tangent *AP* is taken as the axis of this curve, which is directed along the given direction. Hence the problem requires that the body moving on *AM* with a uniform force *g* acting [down] arrives at *M* in the same time as a body moving uniformly, clearly with the speed $\sqrt{b}$, traverses the corresponding width of the abscissa *AP*, and the initial speed at A corresponds to the height *b*. Calling *AP* = *x*, *PM* = *y* and *AM* = *s* and a vertical line *AQ* is drawn at *Q* cutting the horizontal *MQ*. Therefore the speed of the body at *M* is as large as a body acquires from its own speed $\sqrt{b}$ at *Q* by falling along *AQ* ; whereby the speed of the body at *M* corresponds to the height *b* + *gz* on calling *AQ* = *z*. Now from the condition of the problem it follows that :

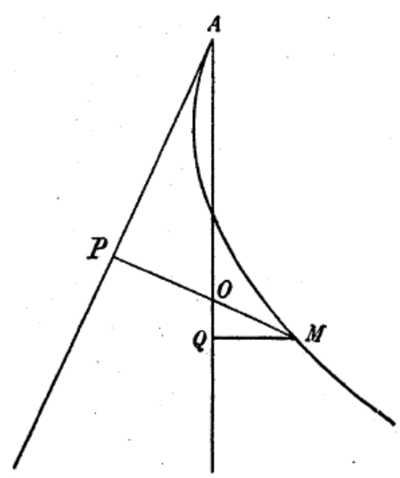

Fig. 34.

$$\int\frac{ds}{\sqrt{(b+gz)}}=\int\frac{dx}{\sqrt{b}} \text{ or } \frac{ds}{\sqrt{(b+gz)}}=\frac{dx}{\sqrt{b}},$$

hence this equation arises : $dy\sqrt{b} = dx\sqrt{gz}$. But z is given in terms of x and y from the *PAQ*; let the sine of this angle be equal to m, and the cosine is equal to $\sqrt{(1-m^2)}$ with the whole sine equal to 1. Now the equation arises :

$$\sqrt{(1-m^2)} : m = AP\ (x) : PO,$$

from which :

$$PO = \frac{mx}{\sqrt{(1-m^2)}} \text{ and thus } MO = \frac{y\sqrt{(1-m^2)} - mx}{\sqrt{(1-m^2)}}.$$

But AO is made equal to $\frac{x}{\sqrt{(1-m^2)}}$. Then as $1 : m = MO : OQ$ it follows that

$$OQ = \frac{my\sqrt{(1-m^2)} - m^2x}{\sqrt{(1-m^2)}}.$$

Consequently $AQ = z = my + x\sqrt{(1-m^2)}$ and hence $dy = \frac{dz}{m} - \frac{dx\sqrt{(1-m^2)}}{m}$.
From which value, substituted into the equation is given :

$$dz\sqrt{b} = dx\sqrt{b(1-m^2)} + m\,dx\sqrt{gz},$$

which is changed into this :

$$dx = \frac{dz\sqrt{b}}{m\sqrt{gz} + \sqrt{b(1-m^2)}}.$$

The integral of which is found : [p. 117]

$$x = \frac{2\sqrt{bz}}{m\sqrt{g}} - \frac{2b\sqrt{(1-mm)}}{m^2 g}\, l\, \frac{m\sqrt{gz} + \sqrt{b(1-m^2)^1)}}{\sqrt{b(1-m^2)}}.$$

[Formula corrected by P.S.]

Which equation with the value $my + x\sqrt{(1-m^2)}$ substituted in place of z gives the nature of the curve sought. Q.E.I.

Corollary 1.

263. Hence the satisfying curve is always a transcendental line, clearly depending on logarithms, unless m is either 0 or 1, *i. e.* unless the line *AP* is either vertical or horizontal.

Corollary 2.

264. Therefore if $m = 0$, the problem agrees with the preceding one; for let $z = x$ and thus the curve is expressed by this equation $dy\sqrt{b} = dx\sqrt{gz}$, which gives the cubical parabola as above.

Corollary 3.

265. Let $m = 1$, and the line *AP* becomes horizontal and $z = y$. Therefore we have :

$$dx = \frac{dy\sqrt{b}}{\sqrt{gy}} \text{ or } x = \frac{2\sqrt{by}}{\sqrt{g}} \text{ or } x^2 = \frac{4by}{g}.$$

Hence this curve is that projection that the body freely describes, projected horizontally with the speed $\sqrt{b}$ at A. For this curve, as it is understood from the above book (567), has that property so that the horizontal motion is uniform.

Corollary 4.

266. If x and y and consequently z is very small, then

$$l\left(1 + \frac{m\sqrt{gz}}{\sqrt{b(1-m^2)}}\right) = \frac{m\sqrt{gz}}{\sqrt{b(1-m^2)}} - \frac{mmgz}{2b(1-m^2)} + \frac{m^3gz\sqrt{gz}}{3b(1-m^2)\sqrt{b(1-m^2)}}$$

as an approximation. [p. 118] Hence the beginning of the curve *AM* is expressed by the equation :

$$x = \frac{z}{\sqrt{(1-m^2)}} - \frac{2mz\sqrt{gz}}{3(1-m^2)\sqrt{b}}$$

or as $z = my + x\sqrt{(1-m^2)}$ by this :

$$y = \frac{2(my + x\sqrt{(1-m^2)})\sqrt{g(my + x\sqrt{(1-mm)})}}{3\sqrt{b}(1-m^2)},$$

which is reduced to this :

$$\frac{9b(1-m^2)y^2}{4g} = (my + x\sqrt{(1-m^2)})^3.$$

Corollary 5.

267. If $m = 1$, or if the line *AP* is horizontal, and that series equal to the logarithm is continued to infinity and this infinite series is substituted in place of the approximation, all the terms before the indefinitely small ones to $\propto$ vanish. Moreover it gives the infinitesimal $z = 0$ or $y = 0$, which indicates that in this case the horizontal straight line is also is satisfactory. That which by itself is evident, for the body on the horizontal straight line is progressing uniformly, and thus the horizontal motion of this is uniform.

Scholium 1.

268. Therefore miraculously it is seen that the differential and integral equation too that arise if $m = 1$, only produce a parabola, and the horizontal line is seen to be excluded. But it is to be noted that the horizontal straight line is also satisfactory for all the directions *AP*, since the motion on that is uniform and thus progresses uniformly towards all directions. Moreover it is evident our general equation cannot include this straight line, since the line *AP* is never a tangent except in the case $m = 1$, so that it agrees with that. [p. 119] And this is the reason why for the case $m = 1$ also, the straight line is not able to be found directly.

Scholion 2.

269. It is also evident also that from the same labour the problem can be solved in a wider sense, clearly if the force acting is not uniform, but in some manner a variable force is put in place. In so much as, by putting P in place of g and $\int Pdz$ in place of gz in the differential equation, this equation is produced

$$dy\sqrt{b} = dx\sqrt{\int Pdz}$$

for the curve sought. Now z has the same value as before. Whereby if P only depends on the altitude and on constants, then $\int Pdz$ can be provided either by integration or by quadrature. And then the equation for the curve can be constructed; for this equation is arrived at :

$$dx = \frac{dz\sqrt{b}}{m\sqrt{\int Pdz} + \sqrt{b(1-m^2)}},$$

in which the variables x and z are separated from each other in turn. But I do not wish to effect confusion by making the problem more general. Indeed when the problem is presented with wider significance, neither does it have more difficulty nor is it unsuited to particular uses, but for that part remaining it is set up to handle only a particular problem. For the same reason, I resolve the following paracentric isochronous problems under the hypothesis of uniform forces acting in the downwards direction only.

[p. 120]

PROPOSITION 30.

Problem.

270. *Under the hypothesis of a force acting uniformly and tending downwards, to find the curve AM* (Fig. 35), *upon which a body descends uniformly receding from a given point Ct.*

Solution.

Let *AM* be the curve sought; the tangent CA is taken, which passes through the given point *C* ; the speed of the body at A is a minimum. Indeed since the total speed at *C* is devoted to moving away, in other elements of the curve it is necessary, since the speed is greater, that only a part of this is taken for receding. The point *A* is therefore the highest point of the curve sought. Therefore let the speed of the body at *A* correspond to the height *b* and with this speed the body begins to move uniformly along *AP* ; and thus this motion with the descent of the body agrees with the motion along the curve *AM* , so that at any point *P* and *M* equally distant from *C* is reached at the same time. With the speed at *M* corresponding to the height *b* and taking *CP* = *CM* = *x* and let the sine of the angle *PCM* = *t* with the total sine taken as 1. The circular arcs *PM* and *pm* are drawn with centre *C*; then *Mn* = *Pp* = *dx* and the sine of the angle *pCm* is equal to *t* + *dt*. Whereby we have :

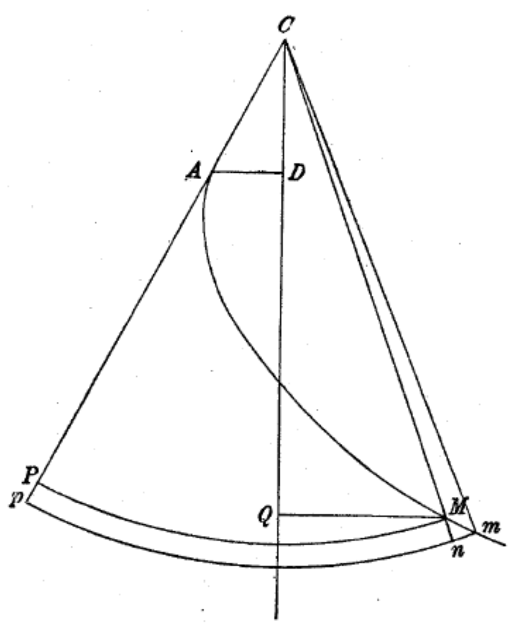

Fig. 35.

$$\text{sine angle } mCn = \frac{dt}{\sqrt{(1-tt)}} = \frac{mn}{x}.$$

Therefore we have [p. 121] :

$$mn = \frac{xdt}{\sqrt{(1-tt)}} \text{ and } Mm = \sqrt{\left(dx^2 + \frac{x^2dt^2}{1-tt}\right)}.$$

Hence since the element *Mm* with the speed $\sqrt{v}$ must be described in the same time as the element *Pp* with the speed $\sqrt{b}$, it becomes :

$$\frac{dx}{\sqrt{b}} = \sqrt{\left(\frac{dx^2}{v} + \frac{x^2 dt^2}{(1-tt)v}\right)}$$

or

$$dx\sqrt{(1-tt)(v-b)} = xdt\sqrt{b}.$$

Therefore it is required that *v* is determined. In order to do this, the vertical CQ is drawn from *C* and the horizontals *AD* and *MQ*; therefore after the body as descended from *A* to *M*, it has fallen by the interval *DQ*. Whereby with the force acting put as *g*, then we have :

$$v = b + g \cdot DQ = b + g \cdot CQ - g \cdot CD.$$

Let *AC* = *a*, the sine of the angle *ACD* = *m*; and the cosine is equal to $\sqrt{(1-m^2)}$, thus $CD = a\sqrt{(1-m^2)}$ and

$$\text{cosine angle } MCQ = mt + \sqrt{(1-m^2)(1-t^2)}.$$

On this account :

$$CQ = mtx + x\sqrt{(1-m^2)(1-t^2)}.$$

From which is constructed :

$$v = b - ga\sqrt{(1-m^2)} + mgtx + gx\sqrt{(1-m^2)(1-t^2)}.$$

In which with the value *v* substituted there is produced this equation :

$$dx\sqrt{(1-tt)\left(mgtx + gx\sqrt{(1-m^2)(1-t^2)} - ga\sqrt{(1-m^2)}\right)} = xdt\sqrt{b}$$

or this :

$$\frac{dx}{x}\sqrt{\left(mgtx + gx\sqrt{(1-m^2)(1-t^2)} - ga\sqrt{(1-m^2)}\right)} = \frac{dt\sqrt{b}}{\sqrt{(1-t^2)}}.$$

Which equation expresses the nature of the curve sought, and if the indeterminates *x* and *t* can be separated from each other in turn, then the curve can be constructed. Q.E.I.

Corollary 1.

271. Therefore it is evident from the equation found that there are innumerable curves to satisfy the question, on account of the three quantities : clearly the angle *ACD*, the distance *AC* and the speed $\sqrt{b}$, from which the body recedes from the fixed point C, which can be varied as it pleases.

Corollary 2. [p. 122]

272. And of these three quantities, any two can be assumed arbitrarily and from the third variable only an infinite number of curves are produced satisfying the question. But since this equation cannot be constructed generally, all the satisfying curves cannot be shown.

Corollary 3.

273. Because it restrains the figure of these curves, it is understood that all these must have the same cusp at *A*, because *A* is the highest point. Otherwise indeed a branch of the curve from *A* must descend to another part of the line *AP*, with the exceptional case in which CAP becomes a horizontal line ; for then this argument comes to an end.

Corollary 4.

274. Now another branch put equal to another part of the line *CP* solves the problem and gives rise to that *AM*. Indeed it is found from the same equation, but if *t* or the angle *PCM* is taken negative.

Corollary 5.

275. But from the single equation found, by inspection it is evident that for two cases the indeterminates can be separated, of which the one is, if $a = 0$, and the other if $m = 1$. Clearly in that case the distance *AC* vanishes and the point *A* is incident on *C*; now in this case the straight line *CP* is made horizontal. [p. 123] We explain both these cases in the following two examples.

Example 1.

276. Hence the point *A* is incident on *C*, or the descending body begins from the point *C* itself; making $a = 0$. Therefore in this case the equation for the curve sought changes to this :

$$\frac{dx\sqrt{g}}{\sqrt{bx}} = \frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}},$$

in which the indeterminates are separated from each other. Therefore the construction of the curve sought can be made by quadrature ; indeed it becomes :

$$\frac{2\sqrt{gx}}{\sqrt{b}} = \int \frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}},$$

which integration can thus be completed, in order that on making $t = 0$, then $x = 0$. And for the general equation thus to be integrated, as by putting $t = 0$ makes $x = a$. Therefore in this case the integral

$$\int \frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}}$$

thus can be taken so that it vanishes by making $t = 0$. Now in the construction of this integral it is observed to be better if I put the cosine of the angle *MCQ* or

$$mt + \sqrt{(1-m^2)(1-tt)} = q,$$

with which done it becomes the sine of the angle *MCm* or

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-dq}{\sqrt{(1-qq)}}.$$

With these substituted this equation is obtained :

$$\frac{2\sqrt{gx}}{\sqrt{b}} = \int \frac{-dq}{\sqrt{(q-q^3)}},$$

which integral can thus be accepted, as on making $q = \sqrt{(1-m^2)}$ it becomes $x = 0$.

Corollary 6.

277. If different values of *b* are given, all the curves which arise are similar to each other ; [p. 124] for with the angle *MCP* maintained, the proportional distance *CM* is taken for *b*, the height generating the initial speed.

Corollary 7.

278. Therefore whatever the angle *ACQ* may be, the construction is not changed, but only a constant is to be added. Whereby the construction serving one case can accommodate all the cases.

Scholium 1.

279. This problem concerned with uniform recession from a fixed point previously was proposed and solved in the Act. Lips. A. 1694 and the solutions presented there agree extremely well with the case of this example; indeed the general solution was not given in that place. On account of which the case of the following example is clearly seen to give anew curves satisfying this equation. But since the following construction agrees with that, though the curves are clearly different, yet also the following case for these, which are treated here concerning this, is considered to be contained. Moreover, curves of this kind are called *paracentric isochrones*, since the motion upon these is uniform from a fixed centre.
(Iac. Bernoulli, *Solutio problematis Leibnitiani, de curva accessus et recessus aequabilis a puncto dato, mediante rectificatione curvae elasticae*, Acta erud 1694, p. 276; *Opera*, Genevae 1744, p. 601.

Iac. Bernoulli, *Constructio curvae access et recessus aequabilis, ope rectificationis curvae cuiuseam algebraicae,* Acta erud 1694, p. 336; *Opera*, Genevae 1744, p. 608.
G. W. Leibniz, *Constructio propria problemis de curva isochrona paracentrica*, Acta erud 1694, p. 364; *Mathematische Schriften,* herausgegeben von C. I. Gerhardt, 2. Abteilung, Band 1, Halle 1858, p. 309; see also Iac. Bernoulli, *Opera omnia*, Genevae 1744, p. 627.
Ioh. Bernoulli, *Constructio facilis curvae recessus aequabilis a dato puncto, per rectificationem curvae algebraicae*, Acta erud 1694, p. 394; *Opera* Omnia, Tom. I, Luasannae et Genevae 1742, p. 119.)

Example 2.

280. Let the line *CAP* be horizontal; put $m = 1$ and the term $ga\sqrt{(1-m^2)}$ vanishes in the general equation. [p. 125] Therefore in this case the equation becomes separable as before ; for the general equation is transformed into this :

$$\frac{dx\sqrt{g}}{\sqrt{bx}} = \frac{dt}{\sqrt{(t-t^3)}} \text{ or } \frac{2\sqrt{gx}}{\sqrt{b}} = \int\frac{dt}{\sqrt{(t-t^3)}},$$

which integral can thus be taken, as on placing $t = 0$ it becomes $x = a$. Whereby $\int\frac{dt}{\sqrt{(t-t^3)}}$ thus on integration, as it vanishes with $t = 0$, hence :

$$\frac{2\sqrt{gx}-2\sqrt{ga}}{\sqrt{b}} = \int\frac{dt}{\sqrt{(t-t^3)}}.$$

Which construction hence agrees with the preceding.

Scholium 2.

281. Whether besides these two cases others are able to be found, that admit separation of the indeterminates, I doubt very much. Certainly no one, as far as I know, has elicited another, on account of which I judge that it is not necessary to tarry longer over this material.

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 97]

PROPOSITIO 25.

Problema.

224. *Si corpus a quacunque vi perpetuo deorsum trahatur, invenire curvam AM* (Fig. 32), *quam corpus super ea descendens ubique aequaliter premit.*

Solutio.

Sit *AM* curva quaesita; dicatur super axe verticali abscissa $AP = x$, applicata $PM = y$ et curva $AM = s$. Sit porro vis corpus in *M* sollicitans = *P* et altitudo debita celeritati in $A = b$; [p. 98] erit altitudo debita celeritati in M = $b + \int Pdx$ integrali $\int Pdx$ ita sumto, ut evanescat facto $x = 0$. His positis erit pressio, quam curva secundum normalem *MN* sustinet, =

$$\frac{Pdy}{ds} + \frac{2(b+\int Pdx)dxddy}{ds^3}$$

(83) sumto elemento *dx* pro constante. Iam cum haec pressio debeat esse constans, ponatur ea = *k*, erit

$$kds^3 = Pds^2dy + 2bdxddy + 2dxddy\int Pdx.$$

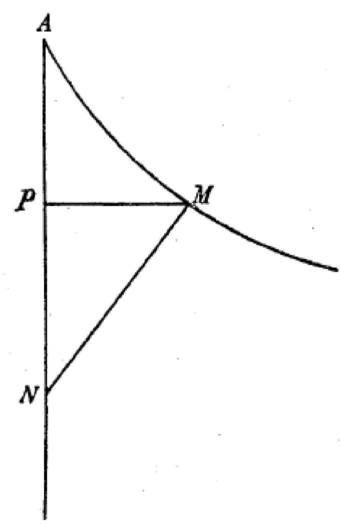

Fig. 32.

At si ponatur *ds* constans, habebitur

$$kdsdx = Pdxdy + 2bddy + 2ddy\int Pdx,$$

cuius integralis est

$$\frac{2dy\sqrt{(b+\int Pdx)}}{ds} = \int \frac{kdx}{\sqrt{(b+\int Pdx)}}.$$

Quae aequatio, cum *P* per *x* detur, construi potest, quia *y* in eam non ingreditur, sed tantum *dy*. Q.E.I.

Corollarium 1.

225. Exprimit

$$\int \frac{dx}{\sqrt{(b+\int Pdx)}}$$

tempus, quo corpus ex *A* celeritate initiali eadem, qua per *AM* movetur, per altitudinem *AP* delabitur, et $\sqrt{(b+\int Pdx)}$ dat celeritatem in eodem loco. Quare celeritas haec in *P* per tempus per *AP* divisa dat $\frac{kds}{2dy}$, ex qua proprietate curva AM determinatur.

Corollary 2.

226. Tempus autem per *AP* quantitate constante quacunque, puta $\sqrt{c}$, potest augeri. Hacque quantitate constante angulus, quem curva in *A* cum *AP* constuit, determinatur. Erit scilicet sinus huius anguli = $\frac{k\sqrt{c}}{2\sqrt{b}}$ posito sinu toto = 1. Quare $\sqrt{c}$ maior non potest accipi quam $\frac{2\sqrt{b}}{k}$ ideoque, si motus in *A* a quiete incipit, *c* debet esse = 0.

Exemplum. [p. 99]

227. Sit potentia uniformis seu $P = g$; erit

$$\int \frac{kdx}{\sqrt{(b+gx)}} = \frac{2k\sqrt{(b+gx)} - 2k\sqrt{b} + 2k\sqrt{c}}{g} = \frac{2dy\sqrt{(b+gx)}}{ds},$$

unde habetur

$$\frac{dy}{ds} = \frac{k}{g} + \frac{k(\sqrt{c}-\sqrt{b})}{g\sqrt{(b+gx)}}$$

atque

$$gdy\sqrt{(b+gx)} = kds\sqrt{(b+gx)} + kds(\sqrt{c}-\sqrt{b}).$$

Ex qua orietur sequens aequatio

$$dy = \frac{kdx(\sqrt{(b+gx)}+\sqrt{c}-\sqrt{b})}{\sqrt{(g^2(b+gx)-k^2(\sqrt{(b+gx)}+\sqrt{c}-\sqrt{b})^2)}}.$$

Sit $\sqrt{(b+gx)} = t$ et $-\sqrt{c}+\sqrt{b} = h;$ erit

$$x = \frac{t^2-b}{g} \quad \text{et} \quad dx = \frac{2tdt}{g}.$$

His igitur substitutis habebitur

$$dy = \frac{2ktdt(t-h)}{g\sqrt{(g^2t^2 - k^2t^2 + 2k^2ht - k^2h^2)}}.$$

Haec aequatio tribus casibus integrationem admittit, quorum primus est, si $k = 0$; tum enim invenitur curva, quam corpus in *A* proiectum libere describit. Alter est casus, quando $h = 0$ seu $\sqrt{b} = \sqrt{c}$; tum enim habetur $\frac{dy}{ds} = \frac{k}{g}$ seu linea satisfaciens erit recta inclinata. Si tertio $k = g$ seu tota pressio aequatur ubique vi sollicitanti corpus *g*, erit

$$dy = \frac{2ttdt - 2htdt}{g\sqrt{(2ht - h^2)}},$$

cuius integralis est

$$gy = \frac{(2tt - 2ht - 2h^2)}{5h}\sqrt{(2ht - h^2)} + \text{const.}$$

Constans haec, quia posito $x = 0$ seu $t = \sqrt{b}$ fit $y = 0$, debet esse =

$$\frac{(2h^2 + 2h\sqrt{b} - 2b)}{5h}\sqrt{(2h\sqrt{b} - h^2)}.$$

Restituto ergo $\sqrt{(b+gx)}$ loco *t* et posito $\sqrt{b} - \sqrt{c} = h = \sqrt{a}$ habebitur

$$\frac{5gy\sqrt{a}}{2} = (b + gx - a - \sqrt{a(b+gx)})\sqrt{(2\sqrt{a(b+gx)} - a)} + (a - b + \sqrt{ab})\sqrt{(2\sqrt{ab} - a)}.$$

Quae est aequatio pro curva quaesita, in qua *a* debet esse numerus minor quam *b*. [p. 100]

Si corpus ex quiete cadere debet, alia linea praeter rectam non satisfacit. Debet enim esse $c = 0$. Debet enim esse $c = 0$, ut angulus ad *A* sit realis, et propterea habetur

$$y = \frac{kx}{\sqrt{(g^2 - k^2)}}.$$

Corollarium 3.

228. Aequatio algebraica inventa, si ab irrationalitate liberetur, fit ordinis quinti. Si in ea ponatur $a = b$, quo casu curvae tangens in *A* est verticalis, prodibit

$$\frac{5gy\sqrt{a}}{2} = (gx - \sqrt{(a^2 + gax)})\sqrt{(2\sqrt{(a^2 + gax)} - a)} + a\sqrt{a}.$$

Corollarium 4.

229. Si generaliter tangens in *A* debeat esse verticalis, erit $\sqrt{c} = 0$ atque ideo prodibit ista aequatio

$$dy = \frac{kdx(\sqrt{(b+gx)} - \sqrt{b})}{\sqrt{\left(g^2(b+gx) - k^2(\sqrt{(b+gx)} - \sqrt{b})^2\right)}}.$$

Si tangens in *A* ponatur horizontalis, erit $k\sqrt{c} = g\sqrt{b}$ habebitur haec aequation

$$dy = \frac{dx(k\sqrt{(b+gx)}+(g-k)\sqrt{b})}{\sqrt{\left(g^2(b+gx)-(k\sqrt{(b+gx)}+(g-k)\sqrt{b})^2\right)}}.$$

Scholion.

230. Vocatur haec curva linea aequabilis pressionis eiusque solutio extat in Comment. Acad. Paris., quae cum hac nostra egregie convenit. (G. F. De L'Hospital (1661 - 1704), *Solution d'un problème physico-mathématique, proposé par Iean Bernoulli*, Mém. de l'acad. d. sc. de Paris 1700, p. 9. Vide etiam Commentationem P. Varignon (1654 - 1722), *Usage d'une intégrale donnée par G. F. De L'Hospital, ou sur les pressions des courbes en général, avec la solution de quelques autres questiones approchantes de la sienne*, Mém. de l'acad. d. sc. de Paris 1710, p. 158.) Ceterum ex solutione constat, si potentia non fuerit constans, sed utcunque variabilis P, aequationem inventam nihilominus integrationem admittere, si pressio in curvam ipsi P debeat proportionalis. [p. 101] Erit enim $k = mP$ atque pro curva quaesita prodibit sequens aequatio

$$\frac{2dy\sqrt{(b+\int Pdx)}}{ds} = \int \frac{mPdx}{\sqrt{(b+\int Pdx)}},$$

cuius integralis est

$$dy\sqrt{(b+\int Pdx)} = mds\sqrt{(b+\int Pdx)} + mds\sqrt{c}.$$

Haec aequatio, si fuerit $c = 0$, erit pro linea recta ad horizontalem inclinata. At per $\sqrt{c}$ definitur angulus, quem curva in *A* cum verticali constituit; eius enim sinus est $m + \frac{m\sqrt{c}}{\sqrt{b}}$. Quare si sumatur $\sqrt{c} = -\sqrt{b}$, curva tangat in *A* verticalem. Praeterea haec curva hanc habet proprietatem, ut tempus, quo arcus *AM* per curritur, proportionalis sit $m \cdot AM - PM$. Denique ex solutione huius propositionis fluit solutio sequentis, in qua ex data curva et pressione aequabili quaeritur quantitas potentiae deorsum tendentis.

PROPOSITIO 26.

Problema.

231. *Data curva AM* (Fig. 32), *et celeritate initiali in A debita altitudini b invenire quantitatem potentiae perpetuo deorsum tendentis, quae faciat, ut corpus super curva AM descendens curvam ubique aequaliter premat.*

Solutio.

Sit potentia sollicitans quaesita = P, dicitisque $AP = x$, $PM = y$ et $AM = s$ atque pressione, quam curva sustinet, = k habebitur ista aequatio

$$kdsdx = Pdxdy + 2bddy + 2ddy\int Pdx$$

(224), in qua ds est elementum constans. Ex hac igitur aequatione quantitatem P erui oportet. [p. 102] Aequatio autem per dy multiplicata et integrata dat

$$kds\int dxdy = dy^2\int Pdx + bdy^2,$$

ex qua prodit

$$\int Pdx + b = \frac{kds}{dy^2}\int dxdy;$$

quae differentiata dat

$$P = \frac{kds}{dy} - \frac{2kdsddy}{dxdy^3}\int dxdy.$$

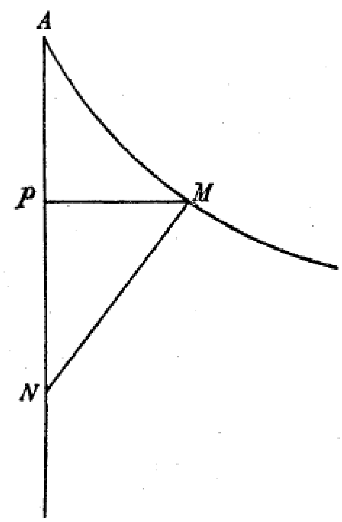

Fig. 32.

At integrale $\int dxdy$ ita est sumendum, ut posito $x = 0$ fiat $\frac{kds}{dy^2}\int dxdy = b.$ Quo autem haec integratio facilius succedat, ponatur $dy = pdx$; erit

$$ds = dx\sqrt{(1+p^2)} \quad \text{et} \quad \int dxdy = ds\int\frac{pdx}{\sqrt{(1+pp)}}$$

ideoque

$$\frac{kds}{dy^2}\int dxdy = \frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

Ex qua aequatione prodibit

$$P = \frac{k\sqrt{(1+pp)}}{p} - \frac{2kdp}{p^3dx}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

Q.E.I.

Corollarium 1.

232. Ex hac aequatione quoque statim celeritas corporis in singulis punctis habetur; altitudo enim debita celeritati corporis in *M* est

$$b+\int Pdx=\frac{kds}{dy^2}\int dxdy=\frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}.$$

Tempus vero, quo arcus *AM* absolvitur, est =

$$\frac{1}{\sqrt{k}}\int pdx:\sqrt{\int\frac{pdx}{\sqrt{(1+p^2)}}}.$$

Corollarium 2.

233. Perspicuum est ex aequatione inventa quantitatem potentiae *P* eo fore maiorem, quo maior sit *k* ceteris paribus; variabilis enim eius valor ductus est in pressionem *k*.

Corollarium 3.

234. Etsi vero non videatur potentia *P* a celeritate initiali *b* pendere, [p. 103] quia in expressone *b* non inest, tamen pendet *P* ab *b* ob integrale $\int\frac{pdx}{\sqrt{(1+pp)}}$, quod ita est accipiendum, ut posito x = 0 fiat $\frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}=b.$ Variata ergo celeritate initiali alia prodit potentia sollicitans, tametsi curva proposita eadem maneat.

Exemplum 1.

235. Sit curva *AM* parabola in *A* verticem ex axem horizontalem habens, ita ut sit $ay=x^2$. Erit ergo $dy=\frac{2xdx}{a}$ hincque $p=\frac{2x}{a}$ et

$$\int\frac{pdx}{\sqrt{(1+pp)}}=\int\frac{2xdx}{\sqrt{(a^2+4x^2)}}=\frac{1}{2}\sqrt{(a^2+4x^2)}+C.$$

Quare erit

$$\frac{k(1+pp)}{p^2}\int\frac{pdx}{\sqrt{(1+pp)}}=\frac{k(a^2+4x^2)^{\frac{3}{2}}}{8x^2}+\frac{kC(a^2+4x^2)}{4x^2}.$$

Quae quantitas cum debeat esse = *b*, si fit *x* = 0, erit $C=\frac{-a}{2}$ seu

$$\int\frac{pdx}{\sqrt{(1+pp)}}=\frac{\sqrt{(a^2+4x^2)}-a}{2}.$$

Ex quibus invenietur

$$P = \frac{ka^3}{4x^3} - \frac{k(a^2 - 2x^2)\sqrt{(a^2 + 4x^2)}}{4x^3}.$$

In ipso ergo puncto *A* potentia *P* erit infinita parva; tam numerator enim quam denominator evanescunt fitque valor istius expressionis = 0. Celeritas vero in *A* non potest esse arbitraria, etiam si constans *C* videatur ex b determinata. Nam *C* talem tantum habet valorem, qui expresionem

$$b + \int Pdx = \frac{kds}{dy^2}\int dxdy$$

reddat finitae magnitudinis. Pendebit ergo *b* ab *a* eiusque valor invenietur, si in expressione

$$\frac{k(a^2 + 4x^2)^{\frac{3}{2}} - ka(a^2 + 4x^2)}{8x^2}$$

ponatur $x = 0$. Tum autem prodibit $b = \frac{ka}{4}$. [p. 104] Hac ergo celeritate descensus incipere debet, ut pressio ubique aequalis a potentia P inventa oriatur.

Exemplum 2.

236. Sit curva *AM* circulus radii *a* tangens rectam *AP* in *A*; erit

$y = a - \sqrt{(a^2 - x^2)}$ et $p = \frac{x}{\sqrt{(a^2 - x^2)}}$ atque $\sqrt{(1 + pp)} = \frac{a}{\sqrt{(a^2 - x^2)}}$.

Fiet ergo

$$\int \frac{pdx}{\sqrt{(1 + pp)}} = \int \frac{xdx}{a} = \frac{x^2}{2a},$$

ad quod constantem addere non licet, quia $\frac{1+pp}{pp} = \frac{a^2}{x^2}$ fit infinitum evanescente *x*. Erit ergo

$$k\frac{(1 + pp)}{pp}\int \frac{pdx}{\sqrt{(1 + pp)}} = \frac{ka}{2} = b + \int Pdx;$$

quare celeritas corporis erit uniformis ideoque potentia sollicitans evanescit. Perspicuum est corpus a nulla potentia sollicitatem in peripheria circuli aequabilite progredi eiusque vim centrifugam esse ubique eiusdem magnitudinis.

Exemplum 3.

237. Sit curva *AM* cyclois basin habens horizontalem et cuspide tangens verticalem *AP* in *A*, ita ut sit

$$dy = \frac{dx\sqrt{2ax}}{\sqrt{(a^2 - 2ax)}}.$$

Habetur ergo

$$p = \frac{\sqrt{2ax}}{\sqrt{(a^2 - 2ax)}} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{a}{\sqrt{(a^2 - 2ax)}}.$$

Quare erit

$$\int \frac{pdx}{\sqrt{(1+pp)}} = \int \frac{dx\sqrt{2ax}}{a} = \frac{2x\sqrt{2x}}{3\sqrt{a}} + C$$

et

$$\frac{k(1+pp)}{pp} = \frac{ka}{2x}.$$

Sumto ergo constante C finitae magnitudinis fit $b = \infty$; quare fiat $C = 0$;erit

$$b + \int Pdx = \frac{k\sqrt{2ax}}{3}$$

et $b = 0$. Prodibit igitur

$$P = \frac{k\sqrt{a}}{3\sqrt{2x}}.$$

Si itaque corpus super cycloide AM ex A descendat ex quiete et sollicitetur deorsum a potentia [p. 105], quae reciproce est ut radix quadrata ex abscissa AP, corpus ubique curvam aequali vi premet.

Scholion.

238. Dantur igitur casus, quibus celeritatem $\sqrt{b}$ non pro lubitu assumere licet, quemadmodum in his exemplis evenit. Quoties enim $\frac{1+pp}{pp}$ fit infinite magnum facto $x = 0$, constans in integratione ipsius $\frac{pdx}{\sqrt{(1+pp)}}$ addenda plerumque hoc ipso determinatur, quod celeritas initialis non debeat esse infinite magna. Semper autem, si curva in A tangit rectam AP, fit $\frac{1+pp}{pp}$ infinitum positio $x = 0$, id quod in causa etiam est, quod in exemplis allatis celeritas initialis non sit arbitraria.

PROPOSITIO 27.

Problema.

239. *Si corpus a quacunque vi perpetuo deorsum trahatur, invenire curvam AM* (Fig. 32)*, super qua corpus ita movetur, ut tota pressio, quam curva sustinet, datam habeat rationem ad pressionem a vi normali ortam.*

Solutio.

Descendat corpus ex A celeritate debita altitudini b et posito $AP = x$, $PM = y$ et $AM = s$ sit potentia corpus in M sollicitans $= P$; [p. 106] erit altitudo debita celeritati, quam corpus in M habet, $= b + \int Pdx$, tota vero pressio, quam curva in M secundum directionem normalis MN sustinet, =

$$\frac{Pdy}{ds} + \frac{2ddy(b+\int Pdx)}{dxds}$$

sumto ds pro elemento constante. Iam habeat se haec pressio ad vim normalem $\frac{Pdy}{ds}$ ut m ad 1 ; erit

$$(m-1)Pdxdy = 2ddy(b+\int Pdx),$$

quae est aequtio pro curva quaesita. Haec vero reducetur ponendo v loco $b+\int Pdx$ ad hanc formam

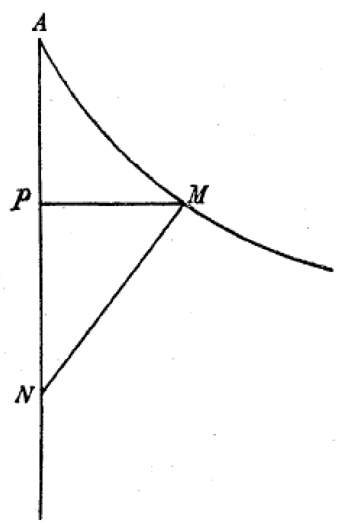

Fig. 32.

$$\frac{(m-1)dv}{v} = \frac{2ddy}{dy},$$

quae integrata dat

$$2l\frac{dy}{ds} = (m-1)l\frac{v}{a} \quad \text{seu} \quad v^{\frac{m-1}{2}}ds = a^{\frac{m-1}{2}}dy.$$

Ex qua habebitur

$$dy = \frac{v^{\frac{m-1}{2}}dx}{\sqrt{(a^{m-1}-v^{m-1})}} = \frac{dx(b+\int Pdx)^{\frac{m-1}{2}}}{\sqrt{(a^{m-1}-(b+\int Pdx)^{m-1})}},$$

quae est aequatio pro curva quaesita. Q.E.I.

Corollarium 1.

240. Celeritas corporis ibi est nulla, ubi $\frac{dy}{ds} = 0$ seu ubi curvae tangens est verticalis, si quidem $\frac{m-1}{2}$ fuerit numerus positivus seu m maiore fuerit unitate. In his igitur casibus curvam in A tangere ponemus rectam AP et celeritatem initialem seu $b = 0$. [p. 107]

Corollarium 2.

241. Quare si $m > 1$ seu si pressio tota maior est quam pressio a vi normali orta, curvam quaesitam dabit ista aequatio

$$dy = \frac{dx(\int Pdx)^{\frac{m-1}{2}}}{\sqrt{(a^{m-1} - (\int Pdx)^{m-1})}},$$

in qua $\int Pdx$ ita debet accipi, ut evanescat posito $x = 0$.

Corollarium 3.

242. Quare si $m = 1$, vis centrifuga evanescet et propterea linea quaesita erit recta. Fit autem ex aequatione $ddy = 0$, quae est proprietas lineae rectae.

Corollarium 4.

243. Si $m = 0$, tum tota pressio evanescit; quare tum prodibit curva, quam corpus celeritate sua altitudini b debita proiectum libere describit. Pro hac igitur curva habebitur ista aequatio

$$dy = \frac{dx\sqrt{a}}{(b - a + \int Pdx)}.$$

Corollarium 5.

244. Si m est unitate minor, tunc vis centrifuga erit contraria vi normali et propterea curva AM erit concava deorsum. Ponamus igitur in A curvam esse normalem ad AP; erit $b = a$. Posito igitur $b = a$ habebitur pro curva quaesita haec aequatio

$$dy = \frac{a^{\frac{1-m}{2}}dx}{\sqrt{((a + \int Pdx)^{1-m} - a^{1-m})}}.$$

Corollarium 6. [p. 108]

245. Pro motu libero igitur, quo casu est $m = 0$, invenietur curva a corpore descripta, si in A horizontaliter celeritate altitudini a debita proiiciatur, ex hac aequatione

$$dy = \frac{dx\sqrt{a}}{\sqrt{\int Pdx}}.$$

Exemplum 1.

246. Sit vis sollicitans uniformis seu $P = g$; erit $\int Pdx = gx$. Casibus ergo, quibus $m > 1$ et corpus in A ex quiete descendit, aequatio pro curvis quaesitis scripto gc loco a erit haec

$$dy = \frac{x^{\frac{m-1}{2}}dx}{\sqrt{(c^{m-1} - x^{m-1})}}.$$

At si sit $m < 1$ et corpus in A celeritate altitudini a debita proiiciatur horizontaler, curva, super qua corpus moveri debebit, scripto gc loco a exponetur hac aequatione

$$dy = \frac{c^{\frac{1-m}{2}}dx}{\sqrt{((c+x)^{1-m} - c^{1-m})}}.$$

Hae ergo curvae erunt algebraicae, si vel $\frac{3-m}{2m-2}$ vel $\frac{m}{1-m}$ fuerit numerus integer affirmativus. Hoc vero evenit, si m fuerit terminus vel ex hac series $3, \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$ etc. vel ex hac serie $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ etc.

Corollarium 7.

247. Si igitur tota pressio triplo debeat esse maior quam vis normalis, curva erit circulus tangens rectam AP in A. [p. 109]
Namque erit

$$dy = \frac{xdx}{\sqrt{(c^2 - x^2)}} \quad \text{seu} \quad y = c - \sqrt{(c^2 - x^2)},$$

aequatio ad circulum radii c.

Corollarium 8.

248. Si tota pressio duplo maior quam vis normalis seu vis centrifuga aequalis vi normali cum eaque conspirans; erit curva cyclois cuspide verticalem in A tangens. Aequatio enim erit

$$dy = \frac{dx\sqrt{x}}{\sqrt{(c-x)}}.$$

Exemplum 2.

249. Quaecunque fuerit potentiae sollicitans P, requirantur curvae eiusmodi, ut pressio tota, quam curva sustinet, sit duplo maior quam vis normalis seu quam vis centrifuga, quae hoc casu illi aequalis erit. Fiat igitur $m = 2$ et pro curva quaesita haec habebitur aequatio

$$dy = \frac{dx\sqrt{\int Pdx}}{\sqrt{(a - \int Pdx)}}.$$

Seu dicto $\int Pdx = X$ erit

$$dy = dx\sqrt{\frac{X}{a - X}} = \frac{Xdx}{\sqrt{(aX - X^2)}}.$$

Hoc exemplum ideo attulimus, quod in sequentibus demonstrabitur curvas huius proprietatis esse simul lineas celerrimi descensus.

Corollarium 9.

250. Perspicitur ergo infinitas esse curvas quaestioni satisfacientes propter quantitatem a arbitrariam. Atque infinitae hae curvae omnes tangent rectam *AP* in *A*.

Scholion 1. [p. 110]

251. Ex solutione huius problematis apparet, quomodo problema inversum, quo curva et ratio inter totam pressionem et vim normalem datur, at quantitas vis sollicitans deorsum tendentis quaeritur, solvi debeat. Cum enim sit

$$v^{\frac{m-1}{2}} ds = a^{\frac{m-1}{2}} dy$$

seu positis $dy = pdx$

$$v^{\frac{m-1}{2}} \sqrt{(1 + pp)} = a^{\frac{m-1}{2}} p,$$

erit

$$v = \frac{ap^{\frac{2}{m-1}}}{(1 + pp)^{\frac{1}{m-1}}} = b + \int Pdx$$

hincque differentiando

$$Pdx = \frac{2ap^{\frac{3-m}{m-1}} dp}{(m-1)(1 + pp)^{\frac{m}{m-1}}}.$$

Consequenter invenitur

$$P = \frac{2ap^{\frac{3-m}{m-1}} dp}{(m-1)(1 + pp)^{\frac{m}{m-1}} dx}.$$

Ubi notandum celeritatem initialem iam esse datam; nam formula

$$\frac{ap^{\frac{2}{m-1}}}{(1+pp)^{\frac{1}{m-1}}},$$

si in ea ponatur $x = 0$, dat b.

Scholion 2.

252. Simili modo, si motus corporis seu celeritas eius in singulis locis detur atque relatio pressionis totius ad vim normalem, invenietur ex celeritate statim potentia sollicitans. Ut sit v altitudino debita celeritati in *M*; erit ob $b+\int Pdx = v$, $P = \frac{dv}{dx}$ atque aequatio [p. 111]

$$v^{\frac{m-1}{2}}ds = a^{\frac{m-1}{2}}dy$$

dabit naturam curvae requisitae. Cum enim *v* sit data, dari debet vel in *x* vel s et constantibus quantitatibus, scilicet quae ad curvae naturam exprimendam adhibentur. Ceterum eadem problemata in hypothesi virium centripetarum vel plurium potentiarum sollicitantium proposita non habent plus difficultatis, etiamsi ad magis perplexas aequationes perveniatur. Atque cum simplicia exempla in medium proferre non liceat ad illustrandum, ea potius relinquo, hocque eo magis, quod in sequentibus, ubi de brachystochronis agetur, eiusdem naturae curvae prodeant, quas ibi diligentius expositurus sum. Nunc igitur ad ea progredior problema, in quibus motus quaedam proprietas proponitur, ex qua coniuncta vel cum potentia sollicitante curva quaeritur vel cum curva ipsa potentia sollicitans. Problemata vero nimis facilia, ut quando vel scala celeritatum vel scala temporum daretur, praetermitto, cum ex expressione celeritatis vel potentia sollicitans vel ipsa curva sponte fluat atque temporis expressio facillime ad celeritatem deducat. Hanc ob rem huiusmodi afferemus quaestiones, in quibus non ipsae celeritates vel tempora dantur, sed relationes quaedam ab iis pendentes.

[p. 112]
PROPOSITIO 28.

Problema.

253. *Sollicitur corpus a quacunque potentia deorsum tendente ; invenire curvam AM* (Fig. 33)*, super qua corpus descendens motu aequabili deorsum feratur seu aequabiliter a horizontali AB recedat.*

Solutio.

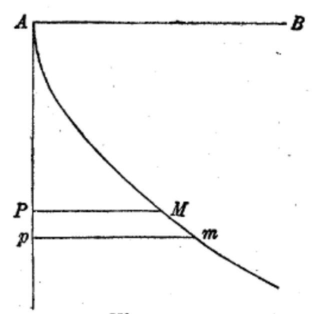

Fig. 33.

Positis $AP = x$, $PM = y$, $AM = s$ et potentia sollicitante = P sit celeritas corporis initialis in A debita altitudini b; erit celeritas in M debita altitudini $b+\int Pdx$. Quare tempusculum, quo elementum Mm percurritur, est $\frac{ds}{\sqrt{(b+\int Pdx)}}$. Quia autem motus per AM respondere debet motui aequabili per AP concipiatur corpus motum super AP celeritate constante debita altitudini b; debebit tempus per Pp aequari tempori per Mm, unde habebitur $\frac{dx}{\sqrt{b}} = \frac{ds}{\sqrt{(b+\int Pdx)}}$ seu $dy\sqrt{b} = dx\sqrt{\int Pdx}$. Pono autem celeritatem initialem congruentem cum celeritate descensus, ut curva in A tangat verticalem AP et corpus primo principio recta descendat. Nam quia propter motum acceleratum necesse est, ut curva continuo magis ad horizontalem inclinetur, eius initium commodissime sumetur in A, ubi curva est verticalis. Prodiitque ergo pro hac curva aequatio $dy\sqrt{b} = dx\sqrt{\int Pdx}$. Q.E.I.

Corollarium 1. [p. 113]

254. Haec ergo curva hanc habet proprietatem, ut, quo maior sit corporis celeritas, eo magis quoque curva in eo loco ad horizontalem sit inclinata.

Corollarium 2.

255. In loco ergo supremo, ubi celeritas corporis est minima, inclinatio curvae debet esse minima seu tangens curvae in eo loco debet esse verticalis.

Corollarium 3.

256. Celeritas igitur initialis $\sqrt{b}$ non potest esse nulla, quia ei aequalis est celeritas respectiva, qua corpus deorsum progreditur seu ab horizontali AB recedit.

Scholion 1.

257. Vocatur haec curva linea aequabilis descensus, quia corpus super ea descendens aequabili motu deorsum progreditur. Lineae huius inventio extat in Act. Erud. Lips. A 1689 pro hypothesi gravitatis seu potentiae sollicitantis uniformis. [G. W. Leibniz, *De linea isochrona, in qua grave sine acceleratione descendit,* Acta erud. 1689, p. 234] Satisfacere autem huic quaestioni demonstratur ibi parabola cubicalis Neiliana, quae eadem in exemplo sequente prodibit.

Exemplum 1.

258. Sit potentia sollicitans uniformis seu $P = g$; erit $\int Pdx = gx.$ [p. 114] Quare pro curva quaesita habebitur ista aequation $dy\sqrt{b} = dx\sqrt{gx}$, quae integrata praebet hanc $3y\sqrt{b} = 2x\sqrt{gx}$ seu $\frac{9by^2}{4g} = x^3$, quae est pro parabola Neiliana cuspide in A verticalem AP tangente, cuius parameter est $\frac{9b}{4g}$. Pro quaque ergo alia celeritate initiali alia est sumenda parabola.

Exemplum 2.

259. Sit potentia sollicitans P potestati cuicunque abscissarum data linea auctarum proportionalis ut $P = \frac{(a+x)^n}{f^n}$; erit

$$\int Pdx = \frac{(a+x)^{n+1} - a^{n+1}}{(n+1)f^n}.$$

Quamobrem pro curva satisfaciente habebitur ista aequatio

$$dy\sqrt{(n+1)bf^n} = dx\sqrt{((a+x)^{n+1} - a^{n+1})}.$$

Si $a = 0$, ita ut potentia sollicitans P sit potestati exponentis n distantiarum corporis a horizontali AB proportionalis, erit

$$dy\sqrt{(n+1)bf^n} = dx\sqrt{x^{n+1}},$$

cuius integralis est

$$y\sqrt{(n+1)bf^n} = \frac{2x^{\frac{n+3}{2}}}{n+3}$$

seu

$$\frac{(n+1)(n+3)^2}{4}bf^n y^2 = x^{n+3}.$$

At $n + 1$ debet esse numerus affirmativus; alioquin $\int Pdx$ fieret infinitum, quia evanescere debet facto $x = 0$. Fit ergo $n + 3 > 2$; quare satisfaciunt parabolae verticibus in *A* verticalem *AP* tangentes. Ut, si $n = 1$ seu $P = \frac{x}{f}$, satisfaciet parabola Apolloniana, cuius parameter est $2\sqrt{2bf}$.

Scholion 2.

260. Ex huius propositionis solutione perspicitur, quomodo eius inversa solvatur, qua data curva, [p. 115] quae sit linea aequabilis descensus, requiritur potentia sollicitans. Cum enim sit $dy\sqrt{b} = dx\sqrt{\int Pdx}$, erit $\int Pdx = \frac{bdy^2}{dx^2}$. Ex qua oritur posito *dx* constante

$$P = \frac{2bdyddy}{dx^3}.$$

Perspicitur ergo potentiam *P* a celeritate initiali $\sqrt{b}$ pendere. Curva vero data ita esse debet comparata, ut in *A* tangat verticalem *AP*. Si curvae radius osculi in *M* dicatur *r*, erit

$$P = \frac{2bds^3dy}{rdx^4}.$$

Quare si ex. gr. curva *AM* fuerit circulus tangens *AP* in *A*, cuius radius = *a*, erit

$$r = a, \quad dy = \frac{xdx}{\sqrt{(a^2 - x^2)}} \quad \text{et} \quad ds = \frac{adx}{\sqrt{(a^2 - x^2)}}.$$

Pro circulo ergo erit

$$P = \frac{2a^2bx}{(a^2 - x^2)^2}.$$

Celeritas vero in *M* debita est altitudini

$$b + \int Pdx = \frac{a^2b}{a^2 - x^2}.$$

Corollarium 4.

261. Patet ceterum ex aequatione, quam invenimus (253),

$$\frac{dx}{\sqrt{b}} = \frac{ds}{\sqrt{(b + \int Pdx)}}$$

tempus, quo arcus *AM* describitur,aequale esse tempori, quo corpus uniformiter celeritate altitudini *b* debita, abscissam *AP* percurrit. In hoc ipso scilicet natura lineae aequabilis descensus nititur.

[p. 116]
PROPOSITIO 29.

Problema.

262. *Trahente uniformi potentia ubique verticaliter deorsum invenire curvam AM* (Fig. 34)*, super qua corpus aequabiliter versus datam plagam AP progreditur*.

Solutio.

Sit *AM* curva quaesita et pro axe sumatur eius tangens *AP*, quae versus datam plagam dirigitur. Problema ergo requiret, ut corpus super *AM* motum a potentia uniformi *g* sollicitatum eodem tempore ad *M* perveniat, quo corpus motu aequabili, nempe celeritate $\sqrt{b}$, latum abscissam respondentem *AP* percurrit, eritque celeritas initialis in A debita altitudini *b*. Dicantur $AP = x$, $PM = y$ et $AM = s$ ducaturque verticalis *AQ* in *Q* secans horizontalem *MQ*. Celeritas igitur corporis in *M* tanta erit, quantam in *Q* cadendo per *AQ* cum sua celeritate $\sqrt{b}$ acquireret; quare celeritas corporis in *M* debita erit altitudini $b + gz$ dicta $AQ = z$. Per conditionem problematis vero debet esse

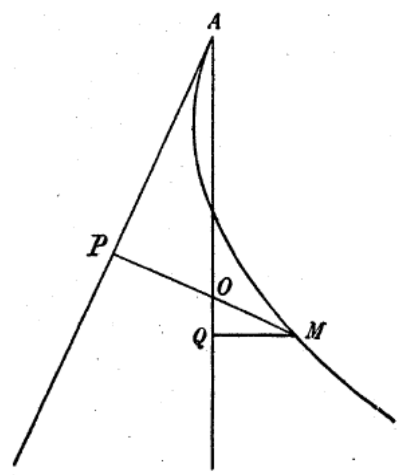

Fig. 34.

$$\int \frac{ds}{\sqrt{(b+gz)}} = \int \frac{dx}{\sqrt{b}} \quad \text{seu} \quad \frac{ds}{\sqrt{(b+gz)}} = \frac{dx}{\sqrt{b}},$$

unde oritur haec aequatio $dy\sqrt{b} = dx\sqrt{gz}$. At *z* in *x* et *y* dabitur ex angulo *PAQ*; sit sinus huius anguli = *m*, erit cosinus = $\sqrt{(1-m^2)}$ posito sinu toto = 1. Nunc erit

$$\sqrt{(1-m^2)} : m = AP\ (x) : PO,$$

ex quo erit

$$PO = \frac{mx}{\sqrt{(1-m^2)}} \quad \text{ideoque} \quad MO = \frac{y\sqrt{(1-m^2)} - mx}{\sqrt{(1-m^2)}}.$$

At AO fiet = $\frac{x}{\sqrt{(1-m^2)}}$. Deinde ob $1 : m = MO : OQ$ erit

$$OQ = \frac{my\sqrt{(1-m^2)} - m^2x}{\sqrt{(1-m^2)}}.$$

Consequenter $AQ = z = my + x\sqrt{(1-m^2)}$ et hinc $dy = \frac{dz}{m} - \frac{dx\sqrt{(1-m^2)}}{m}$.
Quo valore in aequatione inventa substituto prodit

$$dz\sqrt{b} = dx\sqrt{b(1-m^2)} + m\,dx\sqrt{gz},$$

quae transit in hanc

$$dx = \frac{dz\sqrt{b}}{m\sqrt{gz} + \sqrt{b(1-m^2)}}.$$

Cuius integralis invenitur [p. 117]

$$x = \frac{2\sqrt{bz}}{m\sqrt{g}} - \frac{2b\sqrt{(1-mm)}}{m^2 g}\, l\, \frac{m\sqrt{gz}+\sqrt{b(1-m^2)}}{\sqrt{b(1-m^2)}}.$$

[Formula corrected by P.S.]
Quae aequatio loco z valore $my + x\sqrt{(1-m^2)}$ substituto naturam curvae quaesitae. Q.E.I.

Corollarium 1.

263. Curva ergo satisfaciens semper est linea transcendens, nempe a logarithmis pendens, nisi sit m vel 0 vel 1, i. e. nisi recta *AP* vel sit verticalis vel horizontalis.

Corollarium 2.

264. Si igitur $m = 0$, problema cum praecedente convenit; fit enim $z = x$ ideoque curva exprimetur hac aequatione $dy\sqrt{b} = dx\sqrt{gz}$, quae dat parabolam cubicalem ut supra.

Corollarium 3.

265. Si $m = 1$, fit linea *AP* horizontalis et $z = y$. Habetur ergo

$$dx = \frac{dy\sqrt{b}}{\sqrt{gy}} \quad \text{seu} \quad x = \frac{2\sqrt{by}}{\sqrt{g}} \quad \text{seu} \quad x^2 = \frac{4by}{g}.$$

Haec ergo curva est ipsa proiectoria, quam corpus in A celeritate $\sqrt{b}$ horizontaliter proiectum libere describit. Haec enim curva, ut ex superiore libro (567) intelligitur, hanc habet proprietatem, ut motus horizontalis sit aequabilis.

Corollarium 4.

266. Si x et y et consequenter z est valde parvum, erit

$$l\left(1 + \frac{m\sqrt{gz}}{\sqrt{b(1-m^2)}}\right) = \frac{m\sqrt{gz}}{\sqrt{b(1-m^2)}} - \frac{mmgz}{2b(1-m^2)} + \frac{m^3gz\sqrt{gz}}{3b(1-m^2)\sqrt{b(1-m^2)}}$$

quam proxime. [p. 118] Initium ergo curvae *AM* exprimetur hac aequatione

$$x = \frac{z}{\sqrt{(1-m^2)}} - \frac{2mz\sqrt{gz}}{3(1-m^2)\sqrt{b}}$$

seu ob $z = my + x\sqrt{(1-m^2)}$ ista

$$y = \frac{2(my + x\sqrt{(1-m^2)})\sqrt{g(my + x\sqrt{(1-mm)})}}{3\sqrt{b}(1-m^2)},$$

quae reducitur ad hanc

$$\frac{9b(1-m^2)y^2}{4g} = (my + x\sqrt{(1-m^2)})^3.$$

Corollarium 5.

267. Si $m = 1$ seu si linea *AP* est horizontalis et series logarithmo illi aequalis continuatur in infinitum haecque series loco illius substituatur, termini omnes prae infinitesimo ∞ evanescent. Dabit autem infinitesimus $z = 0$ seu $y = 0$, id quod indicat hac casu lineam rectam horizontalem quoque satisfacere. Id quod quidem per se est perspicuum; nam corpus super recta horizontali aequabilite progredietur, ideoque motus eius horizontalis est aequabilis.

Scholion 1.

268. Mirabile igitur videtur, quod aequatio differentialis et integralis quoque, quae prodit, si ponatur $m = 1$, parabolam tantum praebeat et rectam horizontalem excludere videatur. Sed notandum est lineam rectam horizontalem pro omnibus quoque plagis *AP* satisfacere, cum motus in ea sit aequabilis atque idea versus omnes plagas aequabiliter progrediatur. Perspicuum autem est aequationem nostram generalem hanc rectam comprehendere non posse, quia rectam *AP* nusqum tangit nisi in casu $m = 1$, quo cum ea congruit. [p. 119] Atque haec ipsa ratio quoque est, cur pro casu etiam $m = 1$ recta non directe invenire queat.

Scholion 2.

269. Manifestum quoque est eadem opera problema latiori sensu acceptum solvi potuisse, si scilicet potentia sollicitans non uniformis, sed variabilis utcunque esset posita. Namque substituto P loco g et $\int Pdz$ loco gz in aequatione differentiali prodisset haec aequatio

$$dy\sqrt{b} = dx\sqrt{\int Pdz}$$

pro curva quaesita. Habet vero z eundem valorem quem ante. Quare si P ab altitudine z et constantibus tantum pendeat, poterit $\int Pdz$ vel integrari vel per quadraturas exhiberi. Atque tum aequatio pro curva poterit construi; pervenietur enim ad hanc aequationem

$$dx = \frac{dz\sqrt{b}}{m\sqrt{\int Pdz + \sqrt{b}(1-m^2)}},$$

in qua variabiles x et z sunt a se invicem separatae. Nolui autem problema nimis lata significatione confusum efficere. Quando enim latior significatio neque plus difficultatis habet in se neque ad peculiarem usum accommodari potest, eo relicto particulare tantum problema pertractare constitui. Propter eandem rationem sequens problema isochronae paracentricae in hypothesi tantum potentiae uniformis et deorsum directae resolvo.

[p. 120]

PROPOSITIO 30.

Problema.

270. *In hypothesi potentiae sollicitantis uniformis et deorsum tendentis invenire curvam AM* (Fig. 35)*, super qua corpus descendens aequabiliter a dato puncto C recedit.*

Solutio.

Sit *AM* curva quaesita; eius sumatur tangens *CA*, quae per datum punctum *C* transit; erit corporis in A celeritas minima. Quia enim haec celeritas tota ad recedendum a *C* impenditur, in aliis curvis elementis necesse est, ut celeritas sit maior, eo, quod eius tantum pars ad recessum insumitur. Punctum *A* ergo erit supremum curvae quaesitae. Sit igitur celeritas corporis in *A* debita altitudini *b* hacque celeritate concipiatur corpus per *AP* uniformiter moveri; debebit itaque hic motus cum descensu corporis super curva *AM* ita convenire, ut ad quaeque puncta *P* et *M* aequalite ab *C* distantia simul perveniatur. Posita celeritate in *M* debita altutudini b ducatur *CP* = *CM* = *x* et sit sinus ang. *PCM* = *t* posito sinu toto = 1. Ducantur arcus circulares *PM* et pm centro *C*; erit *Mn* = *Pp* = *dx* et ang. *pCm* sinus = *t* + *dt*. Quare erit

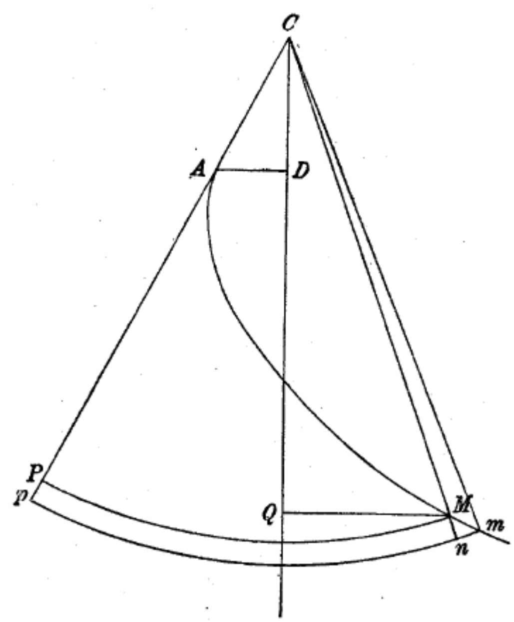

Fig. 35.

$$\text{sinus ang. } mCn = \frac{dt}{\sqrt{(1-tt)}} = \frac{mn}{x}.$$

Erit igitur

$$mn = \frac{xdt}{\sqrt{(1-tt)}} \quad \text{atque} \quad Mm = \sqrt{\left(dx^2 + \frac{x^2dt^2}{1-tt}\right)}.$$

[p. 121]
Cum ergo elementum *Mm* celeritate $\sqrt{v}$ eodem tempore describi debeat, quo elementum *Pp* celeritate $\sqrt{b}$, erit

$$\frac{dx}{\sqrt{b}} = \sqrt{\left(\frac{dx^2}{v} + \frac{x^2dt^2}{(1-tt)v}\right)}$$

seu

$$dx\sqrt{(1-tt)(v-b)} = xdt\sqrt{b}.$$

Requiretur ergo, ut *v* determinatur. Ad hoc ducatur ex *C* verticalis *CQ* et horizontales *AD* et *MQ*; postquam ergo corpus ex *A* ad *M* descendit, deorsum pervenit intervallo *DQ*. Quare posita potentia sollicitante = *g* erit

$$v = b + g \cdot DQ = b + g \cdot CQ - g \cdot CD.$$

Sit *AC* = *a*, sinus anguli *ACD* = *m*; erit eius cosinus = $\sqrt{(1-m^2)}$, unde erit $CD = a\sqrt{(1-m^2)}$ et

$$\text{cosinus ang. } MCQ = mt + \sqrt{(1-m^2)(1-t^2)}.$$

Quam ob rem erit

$$CQ = mtx + x\sqrt{(1-m^2)(1-t^2)}.$$

Ex quibus conficitur

$$v = b - ga\sqrt{(1-m^2)} + mgtx + gx\sqrt{(1-m^2)(1-t^2)}.$$

Quo loco *v* valore substituto prodibit ista aequatio

$$dx\sqrt{(1-tt)\left(mgtx + gx\sqrt{(1-m^2)(1-t^2)} - ga\sqrt{(1-m^2)}\right)} = xdt\sqrt{b}$$

seu haec

$$\frac{dx}{x}\sqrt{\left(mgtx + gx\sqrt{(1-m^2)(1-t^2)} - ga\sqrt{(1-m^2)}\right)} = \frac{dt\sqrt{b}}{\sqrt{(1-t^2)}}.$$

Quae aequatio exprimit naturam curvae quaesitae et, si indeterminatae x et t a se invicem separari possent, ipsa curva construi posset. Q.E.I.

Corollarium 1.

271. Perspicuum igitur est ex aequatione inventa innumerabiles curvas quaesito satisfacere ob tres quantitates, angulum scilicet *ACD*, distantiam *AC* et celeritatem $\sqrt{b}$,qua corpus a fixo puncto C recedit, quae pro lubitu variari possunt.

Corollarium 2. [p. 122]

272. Atque harum trium quantitatum binis quibusque assumtis pro arbitrio tertio sola variabilis infinitas producet curvas quaesito satisfacientes. At quia aequatio haec generaliter construi non potest, omnes curvae satisfacientes exhiberi non possunt.

Corollarium 3.

273. Quod ad figuram curvarum harum attinet, intelligitur eas omnes in *A* cuspidem habere debere, quia *A* est punctum supremum. Aliter enim curvae ramus ex *A* ad alteram partem rectae *AP* descendere debet excepto casu, quo CAP fit linea horizontalis ; tum enim haec ratio cessat.

Corollarium 4.

274. Alter vero ramus ad alteram rectae *CP* partem positus aequae solvit problema ac iste *AM*. Invenitur enim eadem ex aequatione, si modo *t* seu angulus *PCM* acciptur negativus.

Corollarium 5.

275. Ex sola autem aequationis inventae inspectione perspicitur eam duobus casibus separationem indeterminatarum admittere, quorum alter est, si $a = 0$, alter si $m = 1$. Illo scilicet casu evanescit distantia *AC* et punctum *A* in *C* incidit; hoc vero casu recta *CP* fit horizontalis. [p. 123] Hos igitur ambos casus in sequentibus duobus exemplis evolvemus.

Exemplum 1.

276. Incidat ergo punctum *A* in *C*, seu corpus descensum incipiat in ipso puncto *C*; fiet $a = 0$. Hoc ergo casu aequatio pro curva quaesita abibit in hanc

$$\frac{dx\sqrt{g}}{\sqrt{bx}} = \frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}},$$

in qua indeterminatae a se invicem sunt separatae. Constructio igitur curvae quaesitae per quadratas confici poterit ; fiet enim

$$\frac{2\sqrt{gx}}{\sqrt{b}} = \int\frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}},$$

quae integratio ita debet absolvi, ut facto $t = 0$ fiat $x = 0$. Namque generalis aequatio ita debet integrari, ut posito $t = 0$ fiat $x = a$. Hoc igitur casu integrale

$$\int\frac{dt}{\sqrt{(1-tt)(mt+\sqrt{(1-m^2)(1-t^2)})}}$$

ita est accipiendum, ut facto $t = 0$ ipsum evanescat. Ad constructionem huius integralis vero melius perspiciendam pono cosinum anguli *MCQ* seu

$$mt + \sqrt{(1-m^2)(1-tt)} = q,$$

quo facto fiet sinus ang. *MCm* seu

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-dq}{\sqrt{(1-qq)}}.$$

Hisque substitutis habebitur ista aequatio

$$\frac{2\sqrt{gx}}{\sqrt{b}} = \int \frac{-dq}{\sqrt{(q-q^3)}},$$

quod integrale ita est accipiendum, ut facto $q = \sqrt{(1-m^2)}$ fiat $x = 0$.

Corollarium 6.

277. Si ipsi *b* diversi valores attribuantur, omnes curvae, quae oriuntur, erunt inter se similes; [p. 124] manente enim angulo *MCP* distantia *CM* proportionalis est accipienda ipsi *b*, altitudini generanti celeritatem initialem.

Corollarium 7.

278. Quicunque ergo fuerit angulus *ACQ*, constructio non immutatur, sed tantum constans adiicienda. Quare constructio inserviens uni casu ad omnes casus potest accommodari.

Scholion 1.

279. Problema hoc de aequabili recessu a fixo puncto praeterito seculo iam erat propositum et solutum in Act. Lips. A. 1694 atque solutiones, quae ibi extant, conveniunt apprime cum casu huius exempli; universalis enim solutio illo loco non est data. Quamobrem casus exempli sequentis novas prorsus dare videtur curvas huic quaestioni satisfacientes. At quia sequens constructio cum hac convenit, quanquam ipsae curvae sint prorsus differentes, tamen etiam sequens casus in iis, quae hac de re tradita sunt, contineri censendus est. Vocantur autem istiusmodi curvae *isochronae paracentricae*, quia motus super iis a centro fixo fit aequabilis.

(Iac. Bernoulli, *Solutio problematis Leibnitiani, de curva accessus et recessus aequabilis a puncto dato, mediante rectificatione curvae elasticae*, Acta erud 1694, p. 276; *Opera*, Genevae 1744, p. 601.

Iac. Bernoulli, *Constructio curvae access et recessus aequabilis, ope rectificationis curvae cuiuseam algebraicae*, Acta erud 1694, p. 336; *Opera*, Genevae 1744, p. 608.

G. W. Leibniz, *Constructio propria problemis de curva isochrona paracentrica*, Acta erud 1694, p. 364; *Mathematische Schriften*, herausgegeben von C. I. Gerhardt, 2. Abteilung, Band 1, Halle 1858, p. 309; vide etiam Iac. Bernoulli, *Opera omnia*, Genevae 1744, p. 627.

Ioh. Bernoulli, *Constructio facilis curvae recessus aequabilis a dato puncto, per rectificationem curvae algebraicae*, Acta erud 1694, p. 394; *Opera* Omnia, Tom. I, Luasannae et Genevae 1742, p. 119.)

Exemplum 2.

280. Sit linea *CAP* horizontalis; fiet $m = 1$ atque in aequatione generali evanescet terminus $ga\sqrt{(1-m^2)}$. [p. 125] Hoc igitur casu aequatio fit ut ante separabilis; transmutabitur enim generalis aequatio in hanc

$$\frac{dx\sqrt{g}}{\sqrt{bx}} = \frac{dt}{\sqrt{(t-t^3)}} \quad \text{seu} \quad \frac{2\sqrt{gx}}{\sqrt{b}} = \int\frac{dt}{\sqrt{(t-t^3)}},$$

quod integrale ita est accipiendum, ut posito $t = 0$ fiat $x = a$. Quare $\int\frac{dt}{\sqrt{(t-t^3)}}$ ita integrato, ut evanescat posito $t = 0$, erit

$$\frac{2\sqrt{gx}-2\sqrt{ga}}{\sqrt{b}} = \int\frac{dt}{\sqrt{(t-t^3)}}.$$

Quae constructio ergo cum praecedente convenit.

Scholion 2.

281. An praeter hos duos casus alii inveniri queant, qui separationem indeterminatarum admittant, vehementer dubito. A nemine quidem, quantum scio, alius est erutus, quamobrem non necesse esse iudico, ut huic materiae diutius immorer.