CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 69]

PROPOSITION 18.

Problem.

161. *With a uniform force present acting in the downwards direction, to determine the time of the ascent or the descent through any arc of a circle EA (*Fig.23*), ending at the lowest point A.*

Solution. [p. 70]

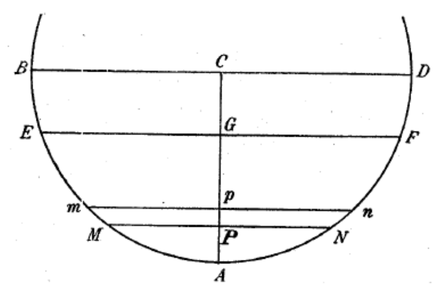

Fig. 23.

Let C be the centre of the circle, CA is the radius of the vertical or the line parallel to the direction of the force g. Putting $AC = a$ and the arc AE equal to the height $AG = b$, the speed at the lowest point A corresponds to the height gb, since the body descending from E has such a speed when it arrives at A . And the body must have such a speed at A, in order that it can rise as far as E. Some element Mm of the arc AE is considered and calling $AP = x$; then $PM = \sqrt{(2ax - x^2)}$ and $Mm = \frac{adx}{\sqrt{(2ax-x^2)}}$. Now the speed at M corresponds to the height $g.GP = gb - gx$ (93). Therefore the time in which the element Mm is traversed either in the ascent or in the descent is equal to $\frac{adx}{\sqrt{g(b-x)(2ax-x^2)}}$.

Which, since it cannot be integrated, we express the integral by a series. Moreover, with putting $2a = c$:

$$\frac{1}{\sqrt{(b-x)(2ax-x^2)}} = \frac{1}{\sqrt{bc}}\left\{ x^{-\frac{1}{2}} + \frac{x^{\frac{1}{2}}(b+c)}{2bc} + \frac{x^{\frac{3}{2}}(3b^2+2bc+3c^2)}{8b^2c^2} + \frac{x^{\frac{5}{2}}(5b^3+3b^2c+3bc^2+5c^3)}{16b^3c^3} + \text{etc.} \right\}.$$

Hence this is multiplied by $\frac{adx}{\sqrt{g}}$ and the integration gives the time, in which the arc *AM* is completed, to equal :

$$\frac{\sqrt{2ax}}{\sqrt{gb}}\left(1+\frac{x(b+c)}{6bc}+\frac{x^2(3b^2+2bc+3c^2)}{40b^2c^2}+\frac{x^3(5b^3+3b^2c+3bc^2+5c^3)}{112b^3c^3}+\text{etc.}\right).$$

Now the time in which the whole arc *EA* is traversed can be produced, if we put $x = b$ and the ratio of the periphery to the diameter $= \pi : 1$, with which in place there is obtained : [p. 71]

$$\frac{\sqrt{2a}}{\sqrt{g}}\left(\frac{\pi}{2}+\frac{\pi b}{8c}+\frac{9\pi b^2}{128c^2}+\text{etc.}\right)=\frac{\pi\sqrt{2a}}{2\sqrt{g}}\left(1+\frac{b}{4c}+\frac{9b^2}{64c^2}+\text{etc.}\right).$$

Where the coefficients $1, \frac{1}{4}, \frac{9}{64}$, etc are the squares of the coefficients $1, \frac{1}{2}, \frac{3}{8}$, which is produced if $(1-z)^{-\frac{1}{2}}$ in resolved into a series. Now the time can therefore be found approximately from this series. Q.E.I.

Corollary 1.

162. Therefore where the arc *EA* is made larger, then the time too is greater, in which it is traversed. Indeed on putting $b = 2a = c$, the time is infinite, since the body in descending is by no means able to complete the semicircle.

Corollary 2.

163. Therefore if the body in an oscillatory motion is moving in the arc EAF of the circle, then the time of one to or fro motion is twice as great as the time of one ascent or one descent, since the time to pass through ANF is equal to the time to pass through AME. Whereby the time of one to or fro motion, or the time for half an oscillation, is equal to

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}\left(1+\frac{b}{4c}+\frac{9b^2}{64c^2}+\text{etc.}\right).$$

Truly the time for one oscillation to be completed is twice as great.

Scholium 1.

164. The series expressing this time can at once be found in this way. An element of time can be resolved into these factors :

$$\frac{adx}{\sqrt{g(bx-xx)}} \times \frac{1}{\sqrt{(2a-x)}}$$

and of these only the latter should be converted into a series, clearly this :

$$\frac{1}{\sqrt{c}}+\frac{1\cdot x}{2\cdot c\sqrt{c}}+\frac{1\cdot 3\cdot x^2}{2\cdot 4\cdot c^2\sqrt{c}}+\text{etc.}$$

with $2a = c$. Moreover, because after the integration, on placing $x = b$, then [p. 72]

$$\int \frac{dx}{\sqrt{(bx-x^2)}} = \pi, \quad \int \frac{xdx}{\sqrt{(bx-x^2)}} = \frac{1\cdot \pi b}{2}, \quad \int \frac{x^2dx}{\sqrt{(bx-x^2)}} = \frac{1\cdot 3\cdot \pi b^2}{2\cdot 4},$$

$$\int \frac{x^3dx}{\sqrt{(bx-x^2)}} = \frac{1\cdot 3\cdot 5\cdot \pi b^3}{2\cdot 4\cdot 6} \text{ etc.}$$

From which the whole descent time can be gathered together to be equal to :

$$\frac{\pi\sqrt{2a}}{2\sqrt{g}}\left(1+\frac{1b}{4c}+\frac{9b^2}{64c^2}+\frac{225b^3}{2304c^3}+\text{etc.}\right).$$

Scholium 2.

165. From which it is apparent that the summation of the series depends on the construction of the equation

$$1+\frac{1b}{4c}+\frac{9b^2}{64c^2}+\text{etc.}$$

I put

$$\frac{b}{c}=\frac{tt}{1+tt}$$

and the sum of the series is equal to $e^{\int\frac{qdt}{t}}$ with *e* denoting the number of which the log is equal to1. With these in place, the series is to be summed by my method explained in Comment. Acad. Petrop. Tom. VII [1740, p. 123; Opera Omnia series I, vol. 14; E41 is translated in this series; however, this appears to be a misquote by Paul Stackel, as this paper does not present the method used to sum the present series. One should look instead in E025 perhaps], and the following equation is found from the exposition:

$$dq+\frac{q^2dt}{t}=\frac{tdt}{(1+tt)^2}.$$

From which equation, if it can be solved, q is found in terms of *t* and hence the sum itself is found in terms of *t* or $\frac{b}{c}$. Moreover since the construction of the equation does not follow from inspection, it is yet apparent that it can be done, since the sum of the series for the time can be assigned with the help of quadrature. Indeed the given sum of the series is found to follow from the construction of that equation.

Corollary 3.

166. If the arc *AE*, in which the descent or the ascent is completed, is put infinitely small, yet the time for that motion is not infinitely small. For in the expression for the time, only *b* vanishes, and the time in which a vanishing arc *AE* is completed is equal to $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$.

[*i. e.* the radius of the circle *a* remains unchanged, while the distance fallen *b* tends towards zero.]

Corollary 4. [p. 73]

167. With the other part *AF* of the circle joined with *AE* the oscillations through the arc *EAF* can be made indefinitely small; still with a finite completion time. Clearly the time for one 'to' or 'fro' motion, or the time for half an oscillation, is equal to $\frac{\pi\sqrt{2a}}{\sqrt{g}}$.

Corollary 5.

168. Therefore the times of this kind of infinitely small oscillations are in the square root ratio directly as the radius and inversely as the force [of gravity] acting.

Corollary 6.

169. These same formulae prevail, if the force acting should not be uniform. For whatever variable force is put in place, yet while the body driven along an infinitely small arc, it has the same constant value.

Corollary 7.

170. It is to be understood that even if the curve *EAF* is not a circle, but any curve, then also these results reported here pertain to infinitely small oscillations on this curve. Then indeed in place of the radius the radius of osculation of this curve is to be taken at the lowest point *A*.

Corollary 8.

171. Oscillations upon an infinitely small arc of the curve *EAF* are effected with the aid of a pendulum, the length of which is the radius *AC*. [p. 74]Therefore the times of indefinitely small oscillations of the pendulum vary directly as the square root of the length of the pendulum and inversely as the square root of the force acting.

Corollary 9.

172. If the curve *ANF* is not equal to the curve *AME*, [*i. e*. no longer circular arcs, and each with its own radius of curvature] it is still sufficient to consider the radius of osculation at the point *A* for infinitely small oscillations. Let this length be equal to α, then the ascent time through the indefinitely small arc *AF* is equal to $\frac{\pi\sqrt{2\alpha}}{2\sqrt{g}}$, and since the descent time through the vanishing arc *EMA* is $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$, then the time for one journey or half an oscillation on the composite curve *EAF* $\frac{\pi(\sqrt{a}+\sqrt{\alpha})}{2\sqrt{g}}$.

Corollary 10.

173. If the oscillations are not indefinitely small on the circle *BAD*, the oscillation times are greater, as the arcs of the oscillations are greater. And if the oscillations are yet definitely small, the time of such an oscillation to the time of an indefinitely small oscillation to is as the square of the diameter of the circle increased by the versed sine of the arc traversed to the square of the diameter itself.

Corollary 11.

174. The height, from which a body descends in the same time by the same force g acting, as it descends along an indefinitely small arc EMA , is equal to $\frac{\pi^2 a}{8}$, or is to the eighth part of the radius as the square of the circumference to the square of the diameter ; [p. 75] this height is hence approximately equal to $\frac{5}{4}a$.

Corollary 12.

175. Moreover the body descends along the chord of the arc *EMA* in the same time that it descends along the diameter of the circle (102). Whereby the descent time along an indefinitely small arc is to the descent time along the corresponding arc is as $\frac{2\sqrt{2a}}{\sqrt{g}}$ to $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$, *i. e*. as the diameter to the fourth part of the circumference. And the descent time from the diameter or from twice the length of the pendulum is to the time of one whole indefinitely small oscillation composed from a to and fro motion is as the diameter to the circumference.

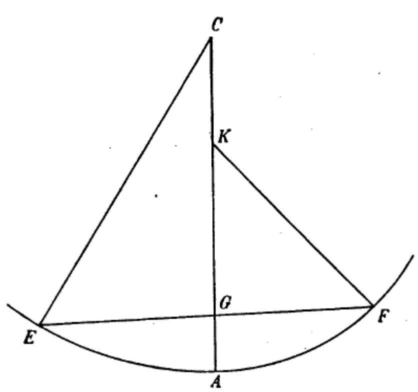

Fig. 24.

Scholium 3.

176. If two circular arcs *AE* and *FA* (Fig. 24), upon which connected oscillations are carried out, are not equal, these oscillations can be made with the aid of a pendulum, if in the centre of *K* of the arc *AF* a nail is driven in, in order that the thread *CA*, after it has described the arc *EA* about the centre, is retained at *K*, and describes the arc *AF* about the centre *K*.

PROPOSITION 19.

Problem.

177. *For a given force acting, to find the length of the pendulum making indefinitely small oscillations, which completes a to and fro motion in a time of one second.*

Solution. [p. 76]

With the length of the pendulum a sought and the force acting g, with the force of gravity denoted by one, the time of one indefinitely small oscillation is equal to $\frac{\pi\sqrt{2a}}{\sqrt{g}}$. Now this has to be expressed in seconds of minutes, with the length a expressed in thousandth parts of Rhenish feet and the formula $\frac{\pi\sqrt{2a}}{\sqrt{g}}$ is to be divided by 250, as is apparent from the first book (221). On account of which the time of one half oscillation is obtained $\frac{\pi\sqrt{2a}}{250\sqrt{g}}$ seconds. Whereby, since the time has to be one second, that is $\pi\sqrt{2a} = 250\sqrt{g}$ and $a = \frac{31250g}{\pi^2} = 3166\frac{1}{4}g$ thousandth parts of Rhenish feet. Therefore, this is the length of the pendulum completing a semi-oscillation in a time of one second. Q.E.I.

Corollary 1.

178. Hence the lengths of the pendulums executing oscillations in the same time, but with different forces acting, are in the ratio of the forces.

Corollary 2.

179. If the force acting g is equal to the force of gravity 1, which case agrees with oscillations on the surface of the earth, the length of the pendulum which makes a single to and fro journey [*i. e.* half an oscillation in one second] is equal to 3.16625 Rhenish feet, or three and one sixth feet. [p. 77]

Scholium 1.

180. This length agrees extremely well with that found by Huygens from experiment; from which it is apparent that we have assumed correctly the number in the preceding book (220) of 15625 scruples of Rhenish feet that a body falls, acted on by the force of gravity, for a time of one second from rest; for indeed this number departs from the number 250, by which the expressions for the time must be divided, in order that a time of one second is presented. [Recall that the number 250 was just a useful number introduced by Euler as a memory aid, and so was only approximately correct.] Therefore since it may be generally wished to have the sixth part of a foot for the Huygens length of the pendulum of 3.166, clearly the length of this must be determined by observations everywhere on the surface of the earth, from which it generally consists of 1055 thousandth parts of a Rhenish foot.

Scholion 2.

181. Now from observations the universal foot can be determined in the following manner. A pendulum of length f is taken, which is set in motion to make the smallest oscillations, and let the number of these counted in a time of one hour be n, thus in order that a single semi-oscillation is completed in a time of $\frac{3600}{n}$ seconds. Now let the length of the pendulum completing semi-oscillations in one second be z. Whereby, since the times of oscillations of different pendulums acted on by the same force are as the square root ratio of the lengths of the pendulums (171), then the ratio is $\frac{3600}{n} : 1 = \sqrt{f} : \sqrt{z}$ and thus $z = \frac{n^2 f}{12960000}$, [p. 78] and consequently the universal foot is equal to $\frac{n^2 f}{38880000}$.

Corollary 3.

182. Therefore a pendulum four times longer than $3166\frac{1}{4}$ scruples of Rhenish feet completes semi-oscillations in two seconds, since the times of the oscillations are in the square root ratio of the lengths of the pendulums.

Corollary 4.

183. Since the radius of the earth is 20382230 Rhenish feet., if a pendulum of such a length is conceived, a single semi-oscillation of this will last for 2536 sec. Whereby in 24 hours almost 17 whole oscillations are completed.

Corollary 5.

185. [There is no section 184.] Since the time of half an oscillation is $\frac{\pi\sqrt{2a}}{\sqrt{g}}$, the time of whole oscillations is $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$. But this time is equal to the time of the revolution performed on the periphery of a circle of radius a by a body in free motion, which is drawn towards the centre by a force equal to g, as from the preceding book is apparent (612). On this account the time of a whole oscillation of the pendulum equal to the radius of the earth is equal to the time that a body projected on the surface carries out a complete revolution. Now Huygens also showed that a body completes almost 17 revolutions in a time of 24 hours in performing this motion.[p. 79]

Corollary 6.

186. Since the force of gravity shall be to the force that a body on the surface of the sun is urged towards the centre of the sun, as 41 to 1000, the length of the pendulum which on the surface of the sun performs a semi-oscillation in a time of one second is equal to 77.226 Rhenish feet. In a similar manner on account of gravity on the surface of Jupiter being equal to $\frac{167}{82}$, for such a pendulum the length is 6.448 feet. And on the surface of Saturn on account of gravity equal to $\frac{105}{82}$, the length of such a pendulum is 4.054 feet.

PROPOSITION 20.

Problem.

187. *If the curve BAD (*Fig. 25*), upon which oscillations are made, is a cycloid described by the circle with diameter AC on the horizontal base BD, to determine the time of the oscillation through each arc EAF, with a uniform force acting downwards.*

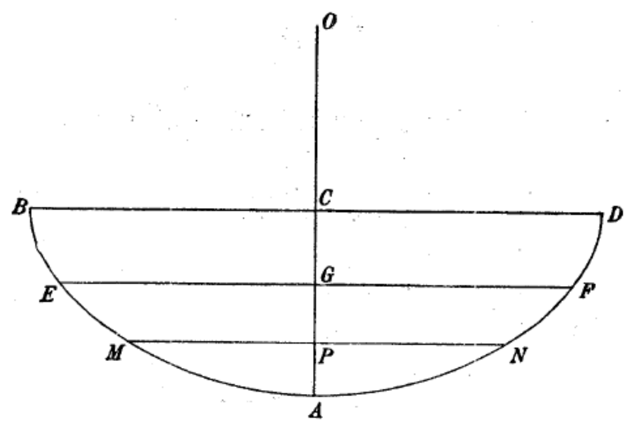

Fig. 25.

Solution.

Let the radius of osculation at A, truly AO, $= a$, which is twice the diameter of the generating circle AC; hence $AC = \frac{1}{2}a$ and with the abscissa $AP = x$ and with the corresponding arc $AM = s$, from the nature of the cycloid, we have $s^2 = 2ax$. Now let the abscissa for the arc EAF, which is traversed in the oscillatory motion correspond to $AG = b$; the speed at the lowest point A corresponds to the height gb and the speed at M corresponds to the height $g(b-x)$. [In the sense that the ratio of the speeds is as the square root of the ratio of the heights.] Whereby, since $ds = \frac{adx}{\sqrt{2ax}}$, [p. 80] the time in which the arc AM is traversed, is equal to :

$$\int \frac{dx\sqrt{2a}}{2\sqrt{g(bx-x^2)}} = \frac{\sqrt{2a}}{2\sqrt{g}} \int \frac{dx}{\sqrt{(bx-xx)}}.$$

Now, if after integration on putting $x = b$, then the time in which the whole arc AE is travelled through, is produced :

$$\int \frac{dx}{\sqrt{(bx-x^2)}} = \pi$$

or the circumference of the circle divided by the diameter. Whereby the time of a single ascent or descent is equal to $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$ and the time of one journey along the arc EAF is equal to $\frac{\pi\sqrt{2a}}{\sqrt{g}}$. And the time for a complete oscillation is equal to $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$. Q.E.I.

Corollary 1.

188. Since in this expression of the time the letter b which determines the magnitude of the arc EAF is not present, all the times of the oscillations which are performed on the same cycloid are equal to each other.

Corollary 2.

189. Therefore the time of any one oscillation is equal to the time of the oscillation through an indefinitely small arc. But the indefinitely small arclet agrees with the arc of the circle with radius OA to be described. Whereby the time of any oscillations on the cycloid BAD is equal to the time in which a pendulum of length a completes the smallest oscillation. It has also been made evident in the previous proposition that the time of one of the smallest oscillations of the pendulum a is equal to $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$ (167), in which we have found the time of a single whole oscillation by the same formula. [p. 81]

Corollary 3.

190. Therefore if the pendulum is thus adjusted, in order that the oscillating body is moving on the cycloid, all the oscillations of this, whether they are large or small, are completed in equal intervals of time. [One may recall that Huygens had to resort to *reductio ad absurdum* arguments to prove this in the *Horologium*.] Whereby if AO is $3166\frac{1}{4}g$ scruples of Rhenish feet, individual semi-oscillations are completed in times of one second.

Corollary 4.

191. Therefore all the descents to the lowest point A on the cycloid are of equal times or isochronous, and likewise all the ascents from the lowest point A, until the speed is spent. Truly the time of one ascent or descent is $\frac{2\pi\sqrt{2a}}{2\sqrt{g}}$.

Scholium 1.

192. On account of this property the cycloid is usually given the name tautochrone, since all the oscillations are completed on these in the same time. Huygens first uncovered this extraordinary property of the cycloid and understood at once that the cycloid could be substituted in place of the circle, that he effected in clocks. Yet now the clockmakers have abandoned this way of making oscillations, as they have learned almost nothing of this use. And surely in a vacuum with any curve, isochronous oscillations are produced, since they are always present with the same magnitude. Now in a resisting medium, in which the oscillations decrease, the cycloid loses this property and thus there is no advantage in the use.

Scholium 2.

193. Also it is understood, if two dissimilar cycloids *AE* and *AF* (Fig. 24) are joined at the lower points, the oscillations upon the composite curve *EAF* are completed in equal times. For since both times of ascent or descent are constant quantities, also the sum of these, clearly the times of half an oscillation and the whole oscillation are equal to each other. Let twice the diameter of the circle generating the cycloid be $AF = \alpha$, then the time of one ascent or descent on $AF = \frac{\pi\sqrt{2\alpha}}{2\sqrt{g}}$. Whereby the to and fro journey on the composite curve *EAF* is completed in a time equal to $\frac{\pi(\sqrt{2a}+\sqrt{2\alpha})}{2\sqrt{g}}$, and now with the time for the whole oscillation equal to $\frac{2\pi(\sqrt{2a}+\sqrt{2\alpha})}{\sqrt{g}}$.

Scholium 3.

194. Order requires that before we can progress to forces acting in different directions, we should explain the effect of forces acting parallel in the same direction, but [in which the magnitudes] are variable, and we should investigate the motion of bodies acted on by forces of this kind upon given curves. But since the contents worthy of note in the examples of motion hitherto set out may lie hidden from us, the principles shall now be explained with the help of which the motion on any curve can be understood, while we defer these other considerations to a fuller treatment, and here we take curves to be investigated, [p. 83] upon which a body is acted on by some kind of force, and advances according to a given law.

PROPOSITION 21.

Problem.

195. *If a body is always drawn towards a fixed centre C by some force (*Fig. 25*) and it is moving on a given curve AM, to determine the motion of the body on this curve, and the force it exerts on individual points of the curve.*

Solution.

Let the initial speed of the body at A corresponds to the height *b* and the distance of the point *A* from the centre *C* be $AC = a$. Now the speed of the body at any place on the curve *M* must correspond to the height *v* and the force, by which the body at *M* is attracted towards *C*, is equal to *P* with the force of gravity arising for the motion the body put equal to 1. The distance *MC* is called *y* and the arc *AM s*; the element $Mm = ds$ and $Mn = -dy$. With centre *C* the circular arcs *MP* and *mp* are described ; then $AP = a - y,\ Pp = Mn = -dy.$ Now with the perpendicular *CT* drawn to that tangent *MT*, we have $MC : MT = Mm : Mn$ and $MC : CT = Mm : mn$, [see vol. 1, (911) Cor. 3 for a similar argument] hence this becomes :

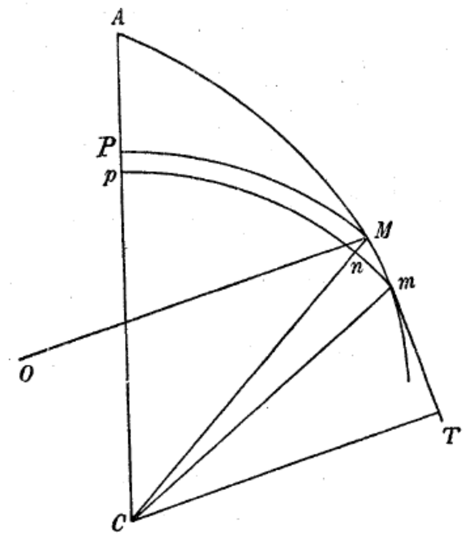

Fig. 26.

$$MT = \frac{-ydy}{ds} \text{ and } CT = \frac{y\sqrt{(ds^2 - dy^2)}}{ds}.$$

From which, if the centripetal force is resolved into the tangential component along *MT* and the normal component along *MO*, then the tangential force is equal to $-\frac{Pdy}{ds}$ and the normal force is equal to $-\frac{P\sqrt{(ds^2-dy^2)}}{ds}$. Hence from the tangential force there is had : $dv = -Pdy.$ [p. 84] Putting the interval $AP = x$, in which the body approaches closer to the centre; then $a - y = x$ and $dx = -dy.$ Whereby $dv = Pdx$, and if *P* depends on the distance *MC*, then $\int Pdx$ can be found. Therefore with the $\int Pdx$ thus accepted, in order that it vanishes on putting $x = 0$, then $v = b + \int Pdx$. From which the time to traverse the arc *AM* is equal to

$$\int \frac{ds}{\sqrt{(b + \int Pdx)}}.$$

The total normal force $\frac{P\sqrt{(ds^2-dy^2)}}{ds}$ is taken up in exerting a force on the curve along *MO*. Therefore since this force can be shown more conveniently, and since the centrifugal force can likewise be shown, I put the perpendicular $CT = p$; then the normal force is equal to $\frac{Pp}{y}$. Then the radius of osculation *MO* is equal to $\frac{ydy}{dp}$, from which the centrifugal force is obtained :

$$\frac{2vdp}{ydy} = \frac{2dp(b+\int Pdx)}{ydy},$$

and the effect of this is contrary to the effect of the normal force. On account of which the curve at *M* is pressed towards *MO* by a force equal to :

$$\frac{Ppdy - 2bdp - 2dp\int Pdx}{ydy}.$$

Q.E.I.

Corollary 1.

196. Therefore if the force *P* depends only on the distance *y*, thus in order that the body is acted on equally at equal distances from the centre, then the speed of the body also depends only on the distance, and the body moving on the curve *AM* at equal distances from the centre has equal speeds.

Corollary 2.

197. And at any point *M* the speed has such a size, as the same body acquires if it falls from A with the same initial speed $\sqrt{b}$ through the interval *AP,* [p. 85] clearly with *CP* = *CM* arising.

Corollary 3.

198. Therefore even if the curve *AM* is itself unknown, yet it is possible to assign the speed of the motion at each point at a distance *C* from the centre. Clearly for the distance *y*, $v = b + \int Pdx$ with $x = a - y$ arising.

[The reader will no doubt have long since noted the implicit use of a type of potential energy function in Euler's analysis, where unit mass is assumed; this invention thus relieving him of the task of finding the speed as a function of the time, while making calculations much easier as the speed is a function of a height. At the time there was no system of units to which all physical quantities could be referred; hence comparisons of the work done under uniform gravity and as in this case under a varying force, is found by integration. [Thus, we find that the only units are the second, and the acceleration of gravity, taken as 1.] These can then be compared as a ratio if needed, and the square root taken to give the speed. Thus, each speed corresponds to the body falling from rest from the height evaluated in the comparison. Only occasionally does Euler take the calculation to the extent of getting an actual speed in units such as Rhenish feet per second. The method has its origin in the work of Galileo rather than Newton, whose calculus involved extensive use of time derivatives.]

Corollary 4.

199. If the curve *AM* is such that the compressive force exerted by the body on the curve is zero, then the curve is that described by the body itself beginning to move freely from *A* with speed $\sqrt{b}$. Thus for the free motion, there is the equation :
$Ppdy = 2bdp + 2dp\int Pdx,$ or as $dx = -dy$, it is found that $Ppdy + 2dp\int Pdy = 2bdp.$ The integral of which is $p^2\int Pdy = bp^2 - bh^2$ with the perpendicular arising *h* sent from *C* to the tangent at *A*. From these equations it is found that $P = \frac{2bh^2dp}{p^3dy}$, as we found in the preceding book for free motion (587).

Corollary 5.

200. Therefore in the above motion, the compressive force for any curve *AM*, which the curve sustains at the point *M* along *MO* is equal to :

$$\frac{-\text{diff.}\, p^2(b+\int Pdx)}{pydy} = \frac{\text{diff.}\, p^2(b+\int Pdx)}{pdx(a-x)} = \frac{\text{diff.}\, p^2v}{pdx(a-x)}.$$

[Note that in this differentiation, $dy = -dx$.]

Example 1.

201. Let the curve *AM* be a circle having centre *C*, the motion of the body is uniform on account of this always having the same distance from the centre of force C[p. 86]. Whereby we have $v = b$ and $\int Pdx = 0$ and the time to traverse $AM = \frac{s}{\sqrt{b}} = \frac{AM}{\sqrt{b}}$. Then on putting $y = a$, we have $p = a$ and $dp = dy$. On account of which the compression, which the curve sustains along *MO* or towards the centre *C*, produced is equal to $P - \frac{2b}{a}$. From which it is evident, if $b = \frac{Pa}{2}$, that the body is free to move in this circle.

Example 2.

202. Let the centripetal force *P* be proportional to some power of the distance *y* or the curve *AM* a logarithmic spiral around the centre *C*, thus in order that $p = my$ and $dp = mdy$ and $ds = \frac{dy}{\sqrt{(1-m^2)}}$. Hence we have :

$$v = b + \int \frac{y^n dx}{f^n} = \frac{a^{n+1} - y^{n+1} + (n+1)bf^n}{(n+1)f^n}$$

and the time to complete the arc *AM* is equal to :

$$\frac{\sqrt{(n+1)f^n}}{\sqrt{(1-m^2)}}\int \frac{dy}{\sqrt{(a^{n+1} - y^{n+1} + (n+1)bf^n)}}.$$

Now the compression, that the curve sustains along *MO*, is equal to :

$$\frac{m(n+3)y^n}{(n+1)f^n}-\frac{2mb}{y}-\frac{2ma^{n+1}}{(n+1)f^n y}.$$

Corollary 6. [p. 87]

203. Therefore the body, when it arrives at the centre *C*, has a finite speed, if $n+1$ is a positive number, for the height corresponding to this speed is $\frac{a^{n+1}}{(n+1)f^n}+b$. But if $n+1$ is a negative number and also if it is equal to zero, the speed at *C* becomes infinitely great.

Corollary 7.

204. Now the body is pressed upon by a force tending away from the centre, or the centrifugal force prevails, if $n>-3$. But if $n<-3$, then the normal force prevails, and the curve is pressed upon by an infinite force towards the centre.

PROPOSITION 22.

Problem.

205. *If a body is always drawn towards a centre of force by a centripetal force C (*Fig. 27*) and let the curve EAF be suited to oscillations, the determine the oscillatory motion of the body on this curve.*

Solution.

Let the centripetal force be proportional to some function of the distance from the centre *C*, and the speed of the body at equal distances from the centre *C* such as *M* and N is the same. Now at *E* and *F* the speed of the body is zero; [p. 88] and indeed it is a maximum at the point on the curve *A* nearest to the centre *C*; and the line *CAO* is drawn. Hence the body completes oscillations along the arc *EAF*, to which it is sufficient to investigate the motion to be defined on each curve *AE* and *AF*. Let the maximum speed of the body which it has at *A*, correspond to the height *b* and the speed at some other point *M* correspond to the height *v*. The distance *CM*, which is equal to *CP*, is equal to *y* and the centripetal force at *M* is equal to *P*. Let $CA = a$ and $AP = x$ and $AG = k$ taking $CG = CE$; then $y = a + x$ and

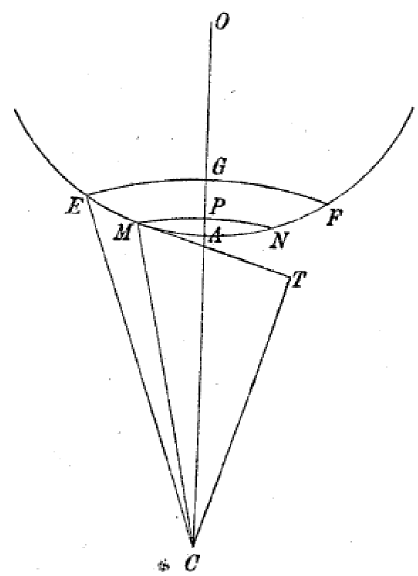

Fig. 27.

$CG = CE = a + k$. With the arc $AM = s$, let the tangent *MT* be equal to $\frac{ydy}{ds}$ as determined by the perpendicular sent from C to that and thus the tangential force is equal to $\frac{Pdy}{ds}$, since with increasing y is opposite to the motion of the body; hence we have the equation : $dv = -Pdy = -Pdx$ and $v = b - \int Pdx$ with the integral $\int Pdx$ thus taken, so that it vanishes at the position $x = 0$. If therefore on putting $v = 0$, the value of x is elicited from the equation $b = \int Pdx$ and the interval AG or k is given. Therefore the time, in which the arc AM is traversed, is equal to $\int \frac{ds}{\sqrt{(b-\int Pdx)}}$, from which the time for the whole arc AE is produced, if after the integration it is thus put in place, in order that the integral vanishes with $x = 0$, $x = k$ or $\int Pdx = b$. In a similar manner the time to complete the arc AF can be found, and therefore from the sum of these times the time of one semi-oscillation is given. Q.E.I.

Corollary 1.

206. If the curve AF is similar and equal to the curve AE, then the times to pass through each are equal, [p. 89] and thus the time of one semi-oscillation is equal to twice the time to pass through AE.

Example 1.

207. If the arc EAF is indefinitely small, the force P acting on account of the invariable distance from the centre C is constant and equal to g. Let the radius of osculation of the curve at A or $AO = h$; then the arc of the circle AE is described by this radius. But from the nature of the circle it follows that $CT = \frac{a^2+2ah-y^2}{2h}$ and

$$MT = \frac{\sqrt{(4h^2y^2 - a^4 - 4a^3h - 4a^2h^2 + 2a^2y^2 + 4ahy^2 - y^4)}}{2h}.$$

But since $y = a + x$ and x is indefinitely small with respect to a and h then $MT = \frac{\sqrt{2ahx(a+h)}}{h}$ and

$$ds = \frac{hydy}{\sqrt{2ahx(a+h)}} = \frac{h(a+x)dx}{\sqrt{2ahx(a+h)}} = \frac{dx\sqrt{ah}}{\sqrt{2(a+h)x}}.$$

But since $v = b - gx$ and thus $b = gk$, we have $v = g(k - x)$ and the element of time is equal to

$$\frac{dx\sqrt{ah}}{\sqrt{2g(a+h)(kx-x^2)}}.$$

But $\int \frac{dx}{\sqrt{(kx-x^2)}}$ on putting $x = k$ is equal to π, for the periphery of the circle arising from the diameter 1.

Consequently the time to pass along the indefinitely small arc AE is equal to

$$\frac{\pi\sqrt{ah}}{\sqrt{2g(a+h)}} = \frac{\pi\sqrt{2ah}}{2\sqrt{g(a+h)}}.$$

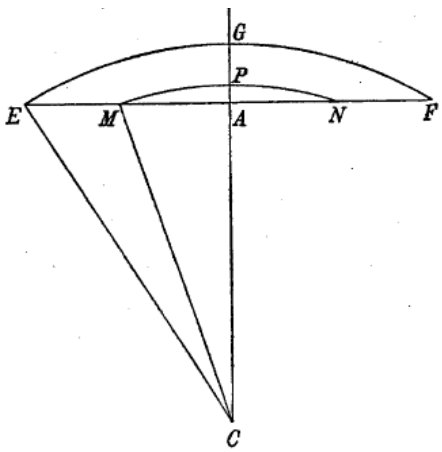

Fig. 28.

Corollary 2.

208. If the centre of force is infinitely distant, in order that $a = \infty$, the direction of the force is parallel to a direction and thus the above time, in which the arc is completed, is equal to $\frac{\pi\sqrt{2h}}{2\sqrt{g}}$.
But if the arc of the circle *EA* is a straight line (Fig. 28) or $h = \infty$, then the time to traverse *EA* is equal to $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$.

Corollary 3. [p. 90]

209. Therefore, if the case is compared likewise with the oscillations of a pendulum acted on by some force g, but in a direction parallel to itself, then the length of the isochronous pendulum is equal to $\frac{ah}{a+h}$. For the time of one descent or ascent of the pendulum is equal to $\frac{\pi\sqrt{2ah}}{2\sqrt{g(a+h)}}$. (166)

Example 2.

210. Now let the centripetal force (Fig. 28) be proportional to some power of the distance or $P = \frac{y^n}{f^n}$ and the line *EF* is straight. Then $AM = s = \sqrt{(y^2 - a^2)}$ and $x = y - a$. Moreover again we have :

$$v = b - \int \frac{y^n dy}{f^n} = b + \frac{a^{n+1} - y^{n+1}}{(n+1)f^n}$$

and with $v = 0$ this becomes :

$$y^{n+1} = a^{n+1} + (n+1)bf^n = (a+k)^{n+1}.$$

Or with the said $CE = c$ then

$$b = \frac{c^{n+1} - a^{n+1}}{(n+1)f^n} \text{ and } v = \frac{c^{n+1} - y^{n+1}}{(n+1)f^n}.$$

Consequently on account of $ds = \frac{ydy}{\sqrt{(y^2 - a^2)}}$, the time to traverse *AM* is equal to

$$\int \frac{ydy\sqrt{(n+1)f^n}}{\sqrt{(y^2-a^2)(c^{n+1}-y^{n+1})}}.$$

Which integration is thus to be taken, in order that it is equal to zero on putting $y = a$. And then, on making $y = c$ the time is had for the line *EA*. Now a semi-oscillation or the motion along *EAF* is equal to twice this time.

Corollary 4. [p. 91]

211. The centripetal force is put in proportion to the distance or $n = 1$;the time to pass along *AM* is equal to :

$$\int \frac{ydy\sqrt{2f}}{\sqrt{(y^2-a^2)(c^2-y^2)}}$$

or on putting $AE = i$ as $c^2 = a^2 + i^2$ and $y^2 = a^2 + s^2$, then the time to traverse *AM* is equal to

$$\int \frac{ds\sqrt{2f}}{\sqrt{(i^2-s^2)}},$$

hence the time to traverse AE is given by $\frac{\pi\sqrt{2f}}{2}$. Therefore all the oscillations on this line are completed in the same time; clearly made in half the time of the oscillation $\pi\sqrt{2f}$.

Corollary 5.

212. If the oscillation is indefinitely small, the time of one semi-oscillation on the line is also $\pi\sqrt{2f}$; but since the centripetal force while it can be considered to be constant, let this be equal to *g*; then $\frac{a}{f} = g$ and thus the time of one semi-oscillation is equal to $\frac{\pi\sqrt{2a}}{\sqrt{g}}$ as above (208).

Corollary 6.

213. Since the directions of gravity actually converge towards the centre of the earth, a body on the surface of the earth on a perfectly horizontal line is able to perform oscillations, unless resistance and friction act as impediments. Moreover the time of one such semi-oscillation shall be (on account of a = radius of the earth and $g = 1$) 2536 seconds (183).

PROPOSITION 23. [p. 92]

Problem.

214. *If a body is acted on by any two forces, of which the direction of one is along the vertical MQ (*Fig. 29*), and the other MP is horizontal ,to define the motion of the body from these forces acting on a given curve AMB.*

Solution.

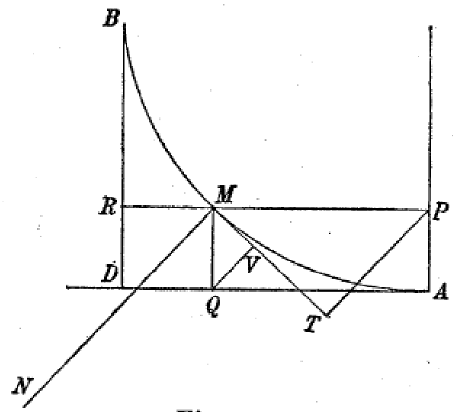

Fig. 29.

Let the speed at *B* be zero, and at *M* it corresponds to the height *v*. The force acting along *MQ* is equal to *P* and that along *MP* is equal to *Q*. Put *BR = t, RM = z,* the arc BM = w, which letters we use for the descent of the body from rest at B. But for the ascent from A with any initial speed, which motion is referring to oscillations, let *AP = x = QM, PM = AQ =y* and the arc *AM* = *s*; now the speed of the body at A corresponds to the height *b*; hence *t + x* = const., likewise *z + y* = const. and *w + s* = const., thus *dt + dx* = 0 [, *dz + dy* = 0] and *dw + ds* = 0. With the forces *P* and *Q* resolved into normal and tangential components, the tangential force that arises from *P* is equal to $\frac{Pdt}{dw}$ and the normal force that arises from *P* is equal to $\frac{Pdz}{dw}$ pulling along *MN*. Then the tangential force arising from *Q* is equal to $\frac{Qdz}{dw}$ and the normal force from *Q* is equal to $\frac{Pdt}{dw}$, which is contrary to that normal force. And beyond that the tangential force along *BM* accelerates the motion and thus

$$dv = Pdt + Qdz \text{ and } v = \int Pdt + \int Qdz$$

[p. 93], with these integrations thus made, in order that they vanish with *t* and *z* = 0. And for the ascent from *A* there is :
$v = b - \int Pdx - \int Qdy$, with these integrations thus made, in order that they vanish with *x* and *y* put equal to 0. Therefore with *t* = *BD* and *z* = *AD* put in this equation :
$v = \int Pdt + \int Qdz$, we find that *v = b*. Whereby the time to traverse *BM* is equal to :

$$\int \frac{dw}{\sqrt{(\int Pdt + \int Qdz)}}$$

and the time for *AM* is equal to :

$$\int \frac{ds}{\sqrt{(b - \int Pdx - \int Qdy)}}.$$

With the element dt or dx taken as constant, the radius of osculation at $M = \frac{dw^3}{dtddz}$ and thus the centrifugal force, the direction of which is along MN, is equal to :

$$\frac{2dtddz(\int Pdt + \int Qdz)}{dw^3}.$$

Hence the total force, by which the curve is pressed upon at M along MN, is equal to :

$$\frac{Pdz - Qdt}{dw} + \frac{2dtddz(\int Pdt + \int Qdz)}{dw^3}.$$

Q.E.I.

Corollary 1.

215. But if P is some function of x or t and Q some function of y or z, so that Pdx as well as Qdy can be integrate; and thus the speed v can be exhibited and with the help of the equation for the curve the time too.

Corollary 2.

216. Because whatever and however many forces are acting, but if the directions of these are in that plane as the curve *AMB*, these forces can be resolved into two forces of this kind, and this proposition extends widely and embraces all the cases in which the directions of the forces and the curve are in the same plane.

Scholium. [p. 94]

217. Also it is apparent that this proposition is of wider applicability if a few cases are added on and examined, in which not all the directions of the forces are in the plane of the curve. For then these forces are to be resolved into two components, of which the one are in the plane of the curve itself, and the other normal to this plane. Therefore these situated in the plane of the curve, so that in the proposition we have used, the analysis gives the acceleration of the body and the compression force along MN ; the other forces, because they are normal to the curve, are only devoted to pressing on the curve. Whereby hence a twofold compression arises, which the curve sustains, the one directed along MN, and the other normal to the plane of the curve. Therefore of these two compressions, if the direction of the mean is taken, there is produced, and there is produced the direction of the equivalent force by which the curve is pressed. On this account there is no need for us to explain cases of this kind, but we will mention briefly a few in which the motion of bodies are on a curve which is not itself placed in the plane in which the forces act, which we take as constant and in the downwards direction.

PROPOSITION 24.

Problem.

218. *With a uniform force acting in a downwards direction, to determine the motion of a body on some curve AM (*Fig. 30*)not set up in the same plane.*

Solution. [p. 95]

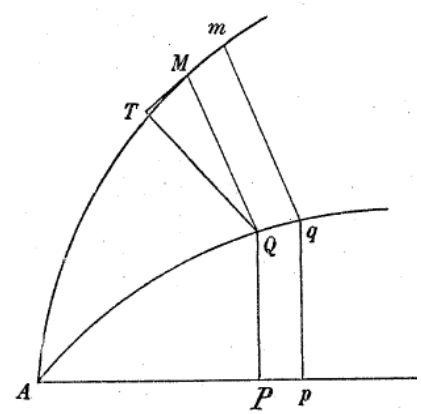
Fig. 30.

Let the projection of the curve *AM* be the curve *AQ* in the horizontal plane, and with the perpendiculars *MQ* and *mq* sent from some nearby points *M* and *m* to this plane, and there are drawn to the axis *AP* taken as you please, the normals *QP* and *qp* and put *AP = x, PQ = y* and *QM = z*. Let the speed of the body at *A* correspond to the height *b*, and the speed at *M* correspond to the height *v*. Now the force is equal to *g*, by which the body at *M* is acted on along *MQ*. With the tangent *MT* drawn, and in that from *Q* to the perpendicular *QT* the force *g* in is resolved into tangential and normal [components]. Since

$$MQ : MT = \sqrt{(dx^2 + dy^2 + dz^2)} : dz$$

the tangential force is equal to :

$$\frac{gdz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

And since

$$MQ : QT = \sqrt{(dx^2 + dy^2 + dz^2)} : \sqrt{(dx^2 + dy^2)}$$

the normal force is equal to :

$$\frac{g\sqrt{(dx^2 + dy^2)}}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Moreover since the tangential force slows the motion, then we have $dv = -gdz$ and $v = b - gz$, hence the time in which the arc *AM* is completed, is made equal to :

$$\int \frac{\sqrt{(dx^2 + dy^2 + dz^2)}}{\sqrt{(b - gz)}}.$$

Now the normal force brings about a compression of the curve by the body at *M* with so much force along a direction normal to *Mm* and situated in the plane *QMmq*. Now the curve is acted on in addition by the centrifugal force along the opposite direction of the position of the radius of osculation by a force equal to $\frac{2(b-gz)}{r}$, with *r* designating the radius of osculation at *M*. Moreover we found above (71) the position of the radius of osculation, from which the direction of the centrifugal force can hence become known.

Now the magnitude of the centrifugal force is given from the radius of osculation, which has been found (72) ; clearly it is given by :

$$r = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2 + ddz^2) + (dyddz - dzddy)^2)}}.$$

Q.E.I.

Corollary 1. [p. 96]

219. Therefore the speed of the body in this case also depends only on the height. And the speed at *M* is of such as magnitude as the body has ascending through QM, when it has a speed corresponding to the height *b* at *Q*.

Corollary 2.

220. Hence the body is unable to ascend to a greater height than to $\frac{b}{g}$. For if we set $b - gz = 0$, the body has lost all its speed at that height and begins to descend again.

Corollary 3.

221. Also it is understood, if the force cannot be taken as constant, but is the variable *P*, then the speed at *M* can be found corresponding to the height $b - \int Pdz$.

Scholium 1.

222. If the curve *AM* is considered to be in the vertical plane (Fig. 31) and with the curve related to the horizontal axis *AQ,* and *AQ* is equal to the curve *AQ* in the previous figure, and *QM* = *QM* in the preced. fig., then the curve AM is also equal to the preceding curve *AM*. If now the body on the curve *AM* ascends with an initial speed at *A* corresponding to the height *b* and acted on by the same force *g*, then it also has the speed at *M* corresponding to the height $b - gz$.

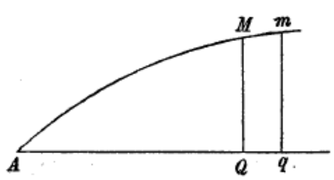

Fig. 31.

And thus the time of ascent along *AM* also agrees with the times of ascent along *AM* in the preced. fig. Therefore by this reason the motion of the body not in the same plane can be reduced to motion on a curve placed in the same plane. [p. 97] For it is not possible to distinguish between the motions ; but the forces acting on these two curves are different. On account of which this compression can be varies as it pleases, with the motion on the curve remaining the same.

Scholium 2.

223. Up to the present we have put the curve in place upon which the body is moving, and the force acting in one given direction, and from these we have deduced the motion of the body and the compression of the curve. Therefore now, since these should suffice, we progress to other questions, in which other quantities are taken as given, and the remaining quantities are to be found. First indeed the compression is given at individual points on the curve and the force acting; from which the curve itself and the motion on the curve must be found. Then from other combinations made from these things, which are come upon in the computation, we will form other questions.

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 69]

PROPOSITIO 18.

Problema.

161. *Existente potentia sollicitante uniformi et deorsum directa determinare tempus ascensus seu descensus per quemvis circuli arcum EA (*Fig.23*)in puncto circulo infimo A terminatum.*

Solutio. [p. 70]

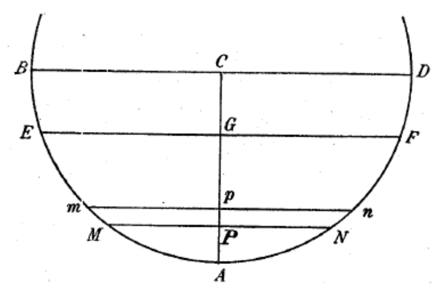

Fig. 23.

Sit *C* circuli centrum, erit *CA* radius verticalis seu parallelus directioni potentiae *g*. Ponatur $AC = a$ et arcus *AE* altitudo $AG = b$, erit celeritas in infimo puncto *A* debita altitudini *gb*, quia corpus ex *E* descendens tantam habebit celeritatem, cum in *A* pervenerit. Atque tantam celeritatem corpus in *A* habere debet, ut ad *E* usque ascendere possit. Consideretur quodvis arcus *AE* elementum *Mm* et dicatur $AP = x$; erit $PM = \sqrt{(2ax - x^2)}$ et $Mm = \frac{adx}{\sqrt{(2ax-x^2)}}$. Celeritas vero in *M* erit debita altitudini $g.GP = gb - gx$ (93).
Tempus igitur, quo elementum *Mm* sive ascensu sive descensu percurritur, erit = $\frac{adx}{\sqrt{g(b-x)(2ax-x^2)}}$.
Quod quia integrari non potest, per series eius integrale exprimemus. Est autem posito $2a = c$

$$\frac{1}{\sqrt{(b-x)(2ax-x^2)}} = \frac{1}{\sqrt{bc}}\left\{\begin{array}{l} x^{-\frac{1}{2}} + \frac{x^{\frac{1}{2}}(b+c)}{2bc} + \frac{x^{\frac{3}{2}}(3b^2+2bc+3c^2)}{8b^2c^2} \\ + \frac{x^{\frac{5}{2}}(5b^3+3b^2c+3bc^2+5c^3)}{16b^3c^3} + \text{ etc.} \end{array}\right\}.$$

Hoc ergo per $\frac{adx}{\sqrt{g}}$ multiplicatum et integratum dat tempus, quo arcus *AM* absolvitur, =

$$\frac{\sqrt{2ax}}{\sqrt{gb}}\left(1+\frac{x(b+c)}{6bc}+\frac{x^2(3b^2+2bc+3c^2)}{40b^2c^2}+\frac{x^3(5b^3+3b^2c+3bc^2+5c^3)}{112b^3c^3}+\text{etc.}\right).$$

Totum vero tempus per arcum *EA* prodibit, si fiat $x = b$ et ratio peripheriae ad diametrum $= \pi : 1$, quo posito habebitur [p. 71]

$$\frac{\sqrt{2a}}{\sqrt{g}}\left(\frac{\pi}{2}+\frac{\pi b}{8c}+\frac{9\pi b^2}{128c^2}+\text{etc.}\right)=\frac{\pi\sqrt{2a}}{2\sqrt{g}}\left(1+\frac{b}{4c}+\frac{9b^2}{64c^2}+\text{etc.}\right).$$

Ubi coefficientes $1, \frac{1}{4}, \frac{9}{64}$, etc sunt quadrata coefficientium $1, \frac{1}{2}, \frac{3}{8}$, qui prodeunt, si $(1-z)^{-\frac{1}{2}}$ in seriem resolvitur. Ex hac igitur serie tempus vero proxime potest inveniri. Q.E.I.

Corollarium 1.

162. Quo maior igitur arcus *EA* est, eo maius quoque erit tempus, quo is percurritur. Fit enim posito $b = 2a = c$ tempus infinitum, quia corpus descensu semicirculum nequaquam describere potest.

Corollarium 2.

163. Si igitur corpus oscillatorio motu movetur in arcu circuli EAF, erit tempus unius itus vel reditus duplo maius quam tempus unius ascensus vel descensus, quia tempus per ANF aequale est tempori per AME. Quare unius itus reditusve tempus seu tempus dimidiae oscillationis erit =

$$\frac{\pi\sqrt{2a}}{\sqrt{g}}\left(1+\frac{b}{4c}+\frac{9b^2}{64c^2}+\text{etc.}\right).$$

Integra vero oscillatio tempore duplo maiore absolvetur.

Scholion 1.

164. Series haec tempus exprimens statim hoc modo potest invenire. Temporis elementum in hos factores resolvatur

$$\frac{adx}{\sqrt{g}(bx-xx)}\times\frac{1}{\sqrt{(2a-x)}}$$

horumque posterior tantum in seriem commutetur, scilicet hanc

$$\frac{1}{\sqrt{c}}+\frac{1\cdot x}{2\cdot c\sqrt{c}}+\frac{1\cdot 3\cdot x^2}{2\cdot 4\cdot c^2\sqrt{c}}+\text{etc.}$$

posito $2a = c$. Quia autem post integrationem fit $x = b$, erit [p. 72]

$$\int \frac{dx}{\sqrt{(bx-x^2)}} = \pi, \quad \int \frac{xdx}{\sqrt{(bx-x^2)}} = \frac{1 \cdot \pi b}{2}, \quad \int \frac{x^2dx}{\sqrt{(bx-x^2)}} = \frac{1 \cdot 3 \cdot \pi b^2}{2 \cdot 4},$$

$$\int \frac{x^3dx}{\sqrt{(bx-x^2)}} = \frac{1 \cdot 3 \cdot 5 \cdot \pi b^3}{2 \cdot 4 \cdot 6} \text{ etc.}$$

Ex quibus totum descensus tempus ut ante colligitur =

$$\frac{\pi\sqrt{2a}}{2\sqrt{g}}\left(1 + \frac{1b}{4c} + \frac{9b^2}{64c^2} + \frac{225b^3}{2304c^3} + \text{etc.}\right).$$

Scholion 2.

165. Quo appareat, a cuiusnam aequationis constructione summatio seriei

$$1 + \frac{1b}{4c} + \frac{9b^2}{64c^2} + \text{etc.}$$

pendeat, pono

$$\frac{b}{c} = \frac{tt}{1+tt}$$

et summam seriei $= e^{\int \frac{qdt}{t}}$ denotante e numerum, cuius log. est = 1. His positis ex mea series summandi methodo in Comment. Acad. Petrop. Tom. VII [1740, p. 123; Opera Omnia series I, vol. 14; E41] exposita invenitur sequens aequatio

$$dq + \frac{q^2dt}{t} = \frac{tdt}{(1+tt)^2}.$$

Ex qua aequatione, si construi posset, inveniretur q in t indeque ipsa summa per t seu per $\frac{b}{c}$. Quia autem aequatio constructionem non admittit, in se spectata, apparet eam tamen constui posse, quia summa seriei per tempora in circulo ope quadraturarum assignari potest. Data enim summa seriei ex ea constructio aequationis inventae sequitur.

Corollarium 3.

166. Si arcus AE, in quo descensus vel ascensus absolvitur, ponitur infinite parvus, tempus per eum tamen non fit infinite parvum. Evanescit enim in expressione temporis tantum b eritque tempus descensus vel ascensus per arcum AE evanescentem $= \frac{\pi\sqrt{2a}}{2\sqrt{g}}$.

Corollarium 4. [p. 73]

167. Iuncta altera circuli parte AF cum AE oscillationes per arcum EAF evanescentem fient infinite parvae; tempore tamen absolventur finito. Scilicet tempus unius itus vel reditus seu tempus unius dimidiae oscillationis erit $= \frac{\pi\sqrt{2a}}{\sqrt{g}}$.

Corollarium 5.

168. Tempora igitur huiusmodi oscillationum infinite parvarum sunt in ratione subduplicata composita ex directa radiorum et reciproca potentiarum sollicitantium.

Corollarium 6.

169. Haec eadem valent, si potentia sollicitans non fuerit uniformis. Nam utcunque variabilis ponatur, tamen, dum in corpus super arcu infinite parvo motum agit, constantem habebit valorem.

Corollarium 7.

170. Intelligitur, etiamsi curva *EAF* non fuerit circulus, sed curva quaecunque, tum etiam, quae hic allata sunt, ad oscillationes infinite parvas super hac curva pertinere. Tum vero loco radii a radius osculi huius curvae in puncto infimo A est accipiendus.

Corollarium 8.

171. Huiusmodi oscillationes super arcu infinite parvo *EAF* efficiuntur ope penduli, cuius longitudo est radius *AC*. [p. 74]Tempora igitur oscillationum infinite parvarum pendulorum sunt directe ut radix quadrata ex longitudine penduli et reciproce ut radix quadrata ex potentia sollicitante.

Corollarium 9.

172. Si curva *ANF* non fuerit aequalis curvae *AME*, pro oscillationibus infinite parvis radium osculi in *A* tantum considerare sufficit. Sit is = α, erit tempus ascensus per arcum AF infinite parvum $=\frac{\pi\sqrt{2\alpha}}{2\sqrt{g}}$, atque cum tempus descensus per arcum EMA evanescentem sit $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$, erit tempus unius itus seu dimidiae oscillationis super curva composita EAF $=\frac{\pi(\sqrt{a}+\sqrt{\alpha})}{2\sqrt{g}}$.

Corollarium 10.

173. Si oscillationes non fuerint infinite parvae super circulo *BAD*, tempora oscillationum maiora erunt, quo maiores sint oscillationum arcus. Atque si oscillationes tamen sint valde parvae, erit tempus talis oscillationis ad tempus oscillationis infinite parvae ut quadruplum diametri circuli sinu verso arcus percursi auctum ad quadruplum diametri ipsum.

Corollarium 11.

174. Altitudo, ex qua corpus eodem tempore ab eadem potentia *g* sollicitatem descendit, quo fit descensus per arcum *EMA* infinite parvum, est $=\frac{\pi^2 a}{8}$, seu est ad octavam radii partem ut quadratum peripheriae circuli ad quadratum diametri; [p. 75] quam proxime ergo haec altitudo erit $=\frac{5}{4}a$.

Corollarium 12.

175. Super chorda autem arcus *EMA* corpus descendit tempora eodem, quo per diametrum circuli (102). Quare tempus descensus super chorda infinite parva est ad tempus descensus super arcu respondente ut $\frac{2\sqrt{2a}}{\sqrt{g}}$ ad $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$, i. e. ut diameter ad quartam peripheriae partem. Atque tempus descensus ex diametro seu dupla penduli longitudine est ad tempus unius integrae oscillationis infinite parvae ex itu et reditu compositae ut diameter ad peripheriam.

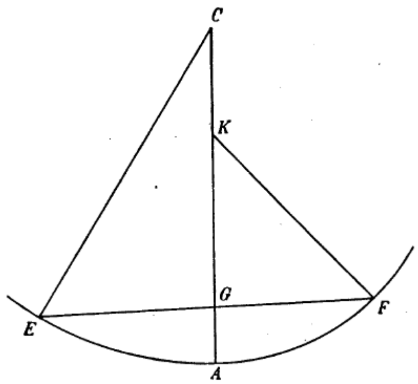

Fig. 24.

Scholion 3.

176. Si duo arcus circulares *AE* et *FA* (Fig. 24), super quibus coniunctis oscillationes peraguntur, non sunt aequales, ope pendul hae oscillationes confici possunt, si in centro *K* arcus *AF* clavus infigatur, ut filum *CA*, postquam arcum *EA* circa centrum descripsit, in *K* retineatur et circa centrum *K* arcum *AF* describat.

PROPOSITIO 19.

Problema.

177. *Data potentia sollicitante invenire longitudinem penduli infinite parvas oscillationes conficiens, quod singulos itus reditusve uno minuto secundo absolvat.*

Solutio. [p. 76]

Existente *a* longitudine penduli quaesita et *g* potentia sollicitante, unitate vim gravitas denotante, est tempus unius dimidiae oscillationis infinite parvae $\frac{\pi\sqrt{2a}}{\sqrt{g}}$. Haec vero expressio ut in minutis secundis habeatur, longitudo *a* in partibus millesimis pedis Rhenani est exprimenda et formula $\frac{\pi\sqrt{2a}}{\sqrt{g}}$ per 250 dividenda, ut ex primo libro (221) apparet. Quamobrem habebitur tempus unius dimidiae oscillationis $\frac{\pi\sqrt{2a}}{250\sqrt{g}}$ min. sec. Quare, cum hoc tempus unum minutum secundum esse debeat, erit $\pi\sqrt{2a} = 250\sqrt{g}$ atque $a = \frac{31250g}{\pi^2} = 3166\frac{1}{4}g$ part. mill. pedis Rhen.

Haec ergo est longitudo penduli semioscillationes uno minuto secundo absolventis. Q.E.I.

Corollarium 1.

178. Longitudines ergo pendulorum eodem tempore oscillationes peragentium, sed a diversis potentiis sollicitatorum, sunt in ipsarum potentiarum ratione.

Corollarium 2.

179. Si potentia sollicitans *g* aequalis est vi gravitatis 1, qui casus in oscillationes in superficie terrae factas competit, erit penduli longitudo, quod itus reditusque singulos uno minuto secundo absolvit, = 3,16625 pedum Rhen. seu trium pedum cum sexta pedis parte. [p. 77]

Scholion 1.

180. Apprime convenit haec longitudo cum ea, quam Hugenius per experimenta invenit; ex quo apparet nos in praecedente libro (220) numerum 15625 scrup. pedis Rhenani recte pro altitudine, ex qua corpus vi gravitatis sollicitatum tempore unius minuti secundi delabitur, assumsisse; ex hoc enim numero fluit numerus 250, per quem temporum expressiones dividi debent, ut minuta secunda praebeant. Cum igitur Hugenius longitudinis 3,166 ped. tertiam partem pro pede universali haberi velit, quippe cuius longitudo ubique terrarum per observationes potest determinari, continebit hic pes universalis 1055 partes millesimas pedis Rhenani.

Scholion 2.

181. Observationibus vero hic pes universalis sequenti modo commodissime determinatur. Sumatur pendulum longitudinis *f*, quod ad minimas oscillationes faciendas impellatur, numerenturque eius dimidiae oscillationes tempore unius horae earumque numerus sit *n*, ita ut una semioscillatio absolvatur tempore $\frac{3600}{n}$ min. sec. Sit iam longitudo penduli semioscillationes minutis secundis absolventis *z*. Quare, cum tempora oscillationum diversorum pendulorum ab eadem potentia sollicitatorum sint in subduplicata ratione pendulorum (171), erit $\frac{3600}{n} : 1 = \sqrt{f} : \sqrt{z}$ ideoque $z = \frac{n^2 f}{12960000}$ [p. 78] et consequenter pes universalis $= \frac{n^2 f}{38880000}$.

Corollarium 3.

182. Pendulum igitur quadruplo longius quam $3166\frac{1}{4}$ scrup. pedis Rhenani semioscillationes duobus minutis secundis absolvet, quia tempora oscillationum sunt in subduplicata ratione longitudinum pendulorum.

Corollarium 4.

183. Cum semidiameter telluris sit 20382230 ped. Rhen., si tantae longitudinis pendulum concipiatur, durabit eius una semioscillatio 2536 min.sec. Quare in horis 24 prope 17 oscillationes integras absolvet.

Corollarium 5.

185. Quia tempus dimidiae oscillationis est $\frac{\pi\sqrt{2a}}{\sqrt{g}}$, erit tempus integrae oscillationis $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$. At huic tempori aequale est tempus revolutionis in peripheria circuli radii a a corpore motu libero peractae, quod ad centrum circuli urgetur vi = g, ut ex praecedente libro (612) apparet. Hanc ob rem tempus unius oscillationis integrae pendui semidiametro terrae aequalis aequatur tempori, quo corpus proiectum in superficie unam revolutionem perageret. Ostendit vero quoque Hugenius corpus hoc modo motum tempore 24 horarum fere 17 revolutiones esse absoluturum. [p. 79]

Corollarium 6.

186. Cum vis gravitatis sit ad vim, qua corpus in superficie solis ad centrum solis urgetur, ut 41 ad 1000, erit longitudo penduli, quod in superficie solis semioscillationes minuto secundo absolvit, = 77.226 ped. Rhenan. Simili modo ob gravitatem in superficie Iovis $=\frac{167}{82}$ tale pendulum longum erit 6.448 ped. Atque in superficie Saturni ob gravitatem $=\frac{105}{82}$ talis pendul longitudo erit 4.054 ped.

PROPOSITIO 20.

Problema.

187. *Si fuerit curva BAD (*Fig. 25*), super qua fiunt oscillationes, cyclois circulo diametri AC super basi horizontali BD descripta, determinare tempus oscillationis per quemque arcum EAF existente potentia sollicitante uniformi et deorsum tendente.*

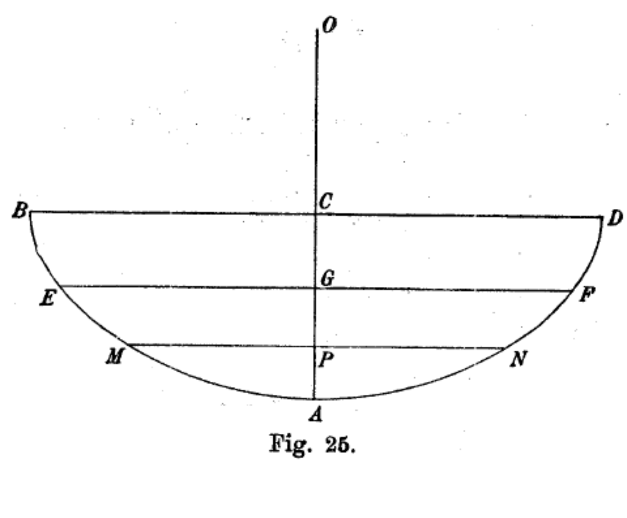

Fig. 25.

Solutio.

Sit radius osculi in *A*, nempe *AO,* = *a*, qui est duplum diametri circuli generatoris *AC*; erit ergo $AC = \frac{1}{2}a$ et posita abscissa *AP* = *x* et arcu respondente *AM* = *s* erit ex natura cycloidis $s^2 = 2ax$. Sit iam abscissa arcui *EAF*, qui motu oscillatorio percurritur, respondens *AG* = *b*; erit celeritas in puncto infimo *A* debita altitudini *gb* et celeritas in *M* debita altitudini $g(b-x)$. Quare, cum sit $ds = \frac{adx}{\sqrt{2ax}}$, [p. 80] erit tempus, quo arcus *AM* percurritur, =

$$\int \frac{dx\sqrt{2a}}{2\sqrt{g(bx-x^2)}} = \frac{\sqrt{2a}}{2\sqrt{g}}\int \frac{dx}{\sqrt{(bx-xx)}}.$$

Est vero, si post integrationem ponatur *x* = *b*, quo tempus per totum arcum *AE* prodeat,

$$\int \frac{dx}{\sqrt{(bx-x^2)}} = \pi$$

seu peripheria circuli per diametrum divisa. Quare tempus unius ascensus vel descensus est = $\frac{\pi\sqrt{2a}}{2\sqrt{g}}$ et tempus unius itus vel reditus per arcum *EAF* erit = $\frac{\pi\sqrt{2a}}{\sqrt{g}}$. Atque tempus unius integrae oscillationis erit = $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$ Q.E.I.

Corollarium 1.

188. Quia in hanc temporis expressionem littera *b*, quae quantitatem arcus *EAF* determinat, non ingreditur, omnium oscillationum tempora, quae super eadem cycloide perficiuntur, sunt inter se aequalia.

Corollarium 2.

189. Tempus ergo uniuscuiusque oscillationis erit aequale tempori oscillationis per arculum infinite parvum. At arculus infinite parvus congruit cum arculo circulo radio *OA* descripti. Quare tempus cuiusque oscillationis super cycloide *BAD* aequale erit tempori, quo pendulum longitudinis *a* oscillationem minimam absolvit. Id quod etiam ex praecedente propositione elucet; tempus enim minimae oscillationis penduli *a* est = $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$ (167), qua eadem formula tempus unius oscillationis integrae super cycloide expressum invenimus. [p. 81]

Corollarium 3.

190. Si igitur pendulum ita adaptetur, ut corpus oscillans in cycloide moveatur, omnes eius oscillationes, sive fuerint magnae sive parvae, aequalibus absolvuntur temporibus. Quare si *AO* fuerit $3166\frac{1}{4}g$ scrup. pedis Rhenani, singulari semioscillationes minuto secundo absolventur.

Corollarium 4.

191. Omnes igitur descensus super cycloide ad punctum infimum *A* sunt aequitemporanei seu isochroni, item omnes ascensus ex puncto infimo *A*, donec celeritas fuerit absumta. Tempus vero unius ascensus vel descensus est $\frac{2\pi\sqrt{2a}}{2\sqrt{g}}$.

Scholion 1.

192. Propter hanc proprietatem cyclois tautochronae nomine appelari solet, quia omnes oscillationes superea eodem tempore absolventur. Hugenius primus hanc eximiam cycloidis proprietatem detexit statimque cogativit de cycloide in locum circuli substituenda in oscillationibus, id quod in horologiis effecit. Nunc tamen horologiorum artifices hunc oscillandi modum rursus deseruerunt, quod eius usum nimis exiguum compererint. Atque certe in vacuo quaelibet curva oscillationes isochronas producit, quia perpetuo eiusdem magnitudinis existunt. In medio resistente vero, quo oscillationes decrescunt, [p. 82] cylcois hanc proprietatem amittit ideoque nullius est utilitatis.

Scholion 2.

193. Intellegitur etiam, si duae cycloides *AE* et *AF* (Fig. 24) dissimiles in punctis infimis iungantur, oscillationes super curva composita *EAF* aequalibus temporibus absolvi. Nam cum super utraque tempora ascensus vel descensus sint constantis quantitatis, etiam summae eorum, nempe tempora semioscillationum et integratum oscillationum, inter se erunt aequalia. Sit duplum diametri circuli generantis cycloidem $AF = \alpha$, erit tempus unius ascensus vel descensus super $AF = \frac{\pi\sqrt{2\alpha}}{2\sqrt{g}}$. Quare itus reditusve super curva composita EAF absolvetur tempore $= \frac{\pi(\sqrt{2a}+\sqrt{2\alpha})}{2\sqrt{g}}$, integra vero oscillatio tempore $= \frac{2\pi(\sqrt{2a}+\sqrt{2\alpha})}{\sqrt{g}}$.

Scholion 3.

194. Ordo requiret, ut, antequam ad alias potentiae sollicitantis directiones progrediamur, effectus potentiae, cuius directiones sint adhuc parallelae, sed variabiles, evolveremus motumque corporis a huiusmodi potentia sollicitati super data curva investigaremus. Sed cum exempla motum notatu dignum continentia nobis adhuc lateant atque principia, quarum ope motus super quaque curva cognoscitur, iam sint exposita, pleniorem tractationem eo differemus, ubi curvas sumus investigaturi, [p. 83] super quibus corpus a huiusmodi potentiis sollicitatum data lege incedat.

PROPOSITIO 21.

Problema.

195. *Si corpus perpetuo vi quacunque ad centrum fixum C (*Fig. 25*) trahatur atque super data curva AM moveatur, determinare motum corporis super hac linea et pressionem quam in singulis punctis sustinet.*

Solutio.

Sit corporis celeritas initialis in *A* debita altitudini *b* et puncti *A* a centro *C* distantia *AC* = *a*. Celeritas vero corporis in quocunque curvae loco *M* debita sit altitudini v et vis, qua corpus in M versus C sollicitatur, sit = P existente vi gravitate corporis moti = 1. Dicatur distantia *MC y* et arcus *AM s*; erit elementum *Mm = ds* et *Mn = – dy*. Centro *C* describantur arcus circulares *MP, mp* ; erit $AP = a - y,\ Pp = Mn = -dy.$ Iam ducta tangente *MT* in eamque perpendiculo *CT* erit $MC : MT = Mm : Mn$ et $MC : CT = Mm : mn$, unde erit

$$MT = \frac{-ydy}{ds} \quad \text{et} \quad CT = \frac{y\sqrt{(ds^2 - dy^2)}}{ds}.$$

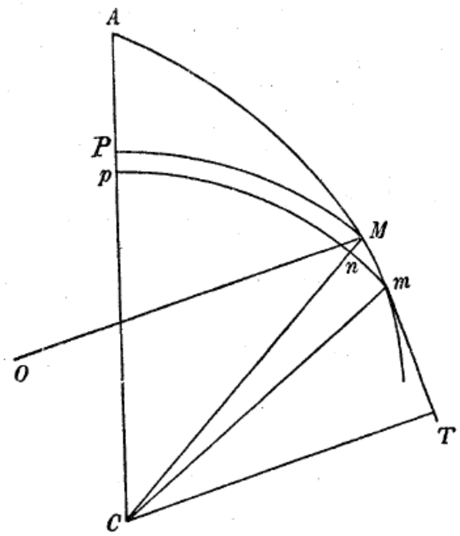

Fig. 26.

Ex quibus, si vis centripeta in tangentialem secundum *MT* et normalem secundum *MT* et normalem secundum *MO* resolvatur, erit vis tangentialis $= -\frac{Pdy}{ds}$ et normalis $= -\frac{P\sqrt{(ds^2-dy^2)}}{ds}$. Ex vi tangentali ergo habebitur $dv = -Pdy.$ [p. 84] Ponatur intervallum *AP = x*, quo corpus propius ad centrum accessit; erit $a - y = x$ et $dx = -dy.$ Quare erit $dv = Pdx$, et si *P* a distantia *MC* pendeat, poterit $\int Pdx$ exhiberi. Ita igitur integrali $\int Pdx$ accepto, ut evanescat posito *x* = 0, erit $v = b + \int Pdx$. Ex quo tempus per arcum *AM* erit =

$$\int \frac{ds}{\sqrt{(b + \int Pdx)}}.$$

Vis normalis $\frac{P\sqrt{(ds^2-dy^2)}}{ds}$ tota in pressione curvae secundum *MO* insumitur. Quo igitur haec commodius exponatur et cum vi centrifuga simul exhibeatur, pono perpendiculum *CT = p*; erit vis normalis $= \frac{Pp}{y}$. Deinde radius osculi *MO* erit = $\frac{ydy}{dp}$, ex quo habetur vis centrifuga =

$$\frac{2vdp}{ydy} = \frac{2dp(b+\int Pdx)}{ydy},$$

cuius effectus effectui vis normalis est contrarius. Quamobrem curva in *M* versus *MO* premetur vi =

$$\frac{Ppdy - 2bdp - 2dp\int Pdx}{ydy}.$$

Q.E.I.

Corollarium 1.

196. Si igitur vis *P* a distantia *y* tantum pendeat, ita ut corpus in aequalibus a centro distantiis aequaliter urgeatur, celeritas corporis a distantia quoque tantum pendebit atque corpus super curva *AM* motum in aequalibus a centro distantiis aequales habebit celeritates.

Corollarium 2.

197. Atque in quovis puncto *M* celeritas tanta erit, quantam idem corpus acquireret, si eadem celeritate initiali $\sqrt{b}$ ex *A* per intervallum *AP* descenderet, [p. 85] existente nimirum *CP* = *CM*.

Corollarium 3.

198. Etiamsi igitur ipsa curva *AM* sit incognita, tamen corporis super ea moti in quaque a centro *C* distantia celeritas potest assignari. Est nempe pro distantia *y*, $v = b + \int Pdx$ existente $x = a - y$.

Corollarium 4.

199. Si curva *AM* fuerit talis, ut pressio, quam corpus in eam exercet, sit nulla, erit curva ea ipsa, quam corpus motum in *A* celeritate $\sqrt{b}$ inchoans libere describeret. Erit itaque pro motu libero $Ppdy = 2bdp + 2dp\int Pdx,$ seu ob $dx = -dy$ habebitur $Ppdy + 2dp\int Pdy = 2bdp.$ Cuius integralis est $p^2\int Pdy = bp^2 - bh^2$ existente *h* perpendiculo ex *C* in tangentem in *A* demisso. Ex his aequationibus invenitur $P = \frac{2bh^2dp}{p^3dy}$, uti praecendente libro (587) pro motu libero invenimus.

Corollarium 5.

200. In motu igitur super quacunque curva *AM* pressio, quam curva in *M* secundum *MO* sustinet, est =

$$\frac{-\text{diff.}\, p^2(b+\int Pdx)}{pydy} = \frac{\text{diff.}\, p^2(b+\int Pdx)}{pdx(a-x)} = \frac{\text{diff.}\, p^2 v}{pdx(a-x)}.$$

Exemplum 1.

201. Sit curva *AM* circulus centrum in *C* habens, erit motus corporis uniformis propter [p. 86] eandem eius perpetuo a centro virium *C* distantiam. Quare erit $v = b$ et $\int Pdx = 0$ atque tempus per $AM = \frac{s}{\sqrt{b}} = \frac{AM}{\sqrt{b}}$. Deinde cum sit $y = a$, erit et $p = a$ et $dp = dy$. Quamobrem pressio, quam curva secundum *MO* seu versus centrum *C* sustinet, prodibit $= P - \frac{2b}{a}$. Ex quo perspicitur, si fuerit $b = \frac{Pa}{2}$, corpus libere per hunc circulum motum iri.

Exemplum 2.

202. Sit vis centripeta *P* potestati cuicunque distantiarum *y* proportionalis seu curva *AM* spiralis logarithmica circa centrum *C*, ita ut sit $p = my$ et $dp = mdy$ atque $ds = \frac{dy}{\sqrt{(1-m^2)}}$.

Erit ergo

$$v = b + \int \frac{y^n dx}{f^n} = \frac{a^{n+1} - y^{n+1} + (n+1)bf^n}{(n+1)f^n}$$

atque tempus per arcum $AM =$

$$\frac{\sqrt{(n+1)f^n}}{\sqrt{(1-m^2)}} \int \frac{dy}{\sqrt{(a^{n+1} - y^{n+1} + (n+1)bf^n)}}.$$

Pressio vero, quam curva secundum *MO* sustinet, erit =

$$\frac{m(n+3)y^n}{(n+1)f^n} - \frac{2mb}{y} - \frac{2ma^{n+1}}{(n+1)f^n y}.$$

Corollarium 6. [p. 87]

203. Corpus igitur, cum in centrum *C* pervenerit, celeritatem habebit finitam, si $n + 1$ est numerus affirmativus; altitudo enim esti celeritati debita est $\frac{a^{n+1}}{(n+1)f^n} + b$. At si $n + 1$ est numero negativus vel etiam = 0, celeritas corporis in *C* erit infinite magna.

Corollarium 7.

204. In ipso vero centro corpus vi infinita premetur directione a centro tendente, seu vis centrifuga praevalebit, si fuerit $n > -3$. At si $n < -3$, tunc vis normalis praevalebit atque curva vi infinita versus centrum premetur.

PROPOSITIO 22.

Problema.

205. *Si corpus perpetuo vi centripeta ad centrum virium C (*Fig. 27*) trahatur dataque sit curva EAF ad oscillandum idonea, determinare motum oscillatorium corporis super hac curva.*

Solutio.

Sit vis centripeta functioni cuicunque distantiarum a centro *C* proportionalis, erit celeritas corporis in aequalibus a centro *C* distantiis, ut *M* et N, eadem. In *E* vero et *F* celeritas corporis sit nulla; [p. 88] maxima vero erit in puncto curvae *A* centro *C* proximo; ducaturque recta *CAO*. Corporis ergo per arcum *EAF* oscillationes absolvet, ad quas definiendas motum corporis super utraque curva *AE* et *AF* investigare sufficit. Sit celeritas corporis maxima, quam habet in *A*, debita altitudini *b* et celeritas in quocunque puncto *M* debita altitudini *v*. Ponatur distantia *CM*, cui aequalis sit *CP*, $= y$ et vis centripeta in $M = P$. Sit $CA = a$ et $AP = x$ atque $AG = k$ sumta $CG = CE$; erit $y = a + x$ et $CG = CE = a + k$. Posito arcu $AM = s$ erit tangens *MT*, quam perpendiculum ex *C* in eam demissum determinat, $= \frac{ydy}{ds}$ ideoque vis

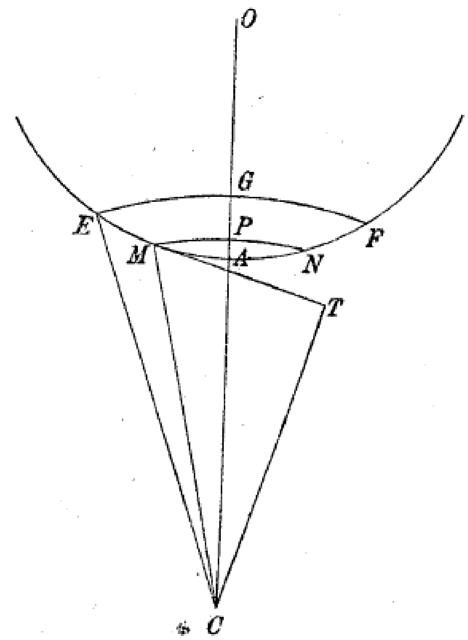

Fig. 27.

tangentialis $= \frac{Pdy}{ds}$, quae motui corporis crescente *y* est contraria; unde habebitur $dv = -Pdy = -Pdx$ et $v = b - \int Pdx$ integrali $\int Pdx$ ita accepto, ut evanescat posito $x = 0$. Si igitur ponatur $v = 0$, dabit ex aequatione $b = \int Pdx$ valor ipsius *x* erutus intervallum *AG* seu *k*. Tempus ergo, quo arcus *AM* percurritur, est $= \int \frac{ds}{\sqrt{(b - \int Pdx)}}$, ex quo tempus per totum arcum *AE* prodibit, si post integrationem ita institutam, ut integrale evanescat posito [$x = 0$, ponatur] $x = k$ seu $\int Pdx = b.$ Simili modo tempus per arcum *AF* invenietur, quo igitur invento summa horum temporum dabit tempus unius semioscillationis. Q.E.I.

Corollarium 1.

206. Si curva AF similis et aequalis fuerit curvae AE, tempora per utramque erunt aequalia, [p. 89] atque ideo tempus unius semioscillationis aequabitur duplo tempori per AE.

Exemplum 1.

207. Si arcus *EAF* fuerit infinite parvus, potentia sollicitans *P* ob distantiam a centro *C* invariabilem erit constant = *g*. Sit radius osculi curvae in *A* seu $AO = h$; erit *AE* arcus circuli hoc radio descriptus. At ex natura circuli erit $CT = \frac{a^2+2ah-y^2}{2h}$ et

$$MT = \frac{\sqrt{(4h^2y^2 - a^4 - 4a^3h - 4a^2h^2 + 2a^2y^2 + 4ahy^2 - y^4)}}{2h}.$$

Sed ob $y = a + x$ et *x* respectu *a* et *h* infinite parvum erit $MT = \frac{\sqrt{2ahx(a+h)}}{h}$ et

$$ds = \frac{hydy}{\sqrt{2ahx(a+h)}} = \frac{h(a+x)dx}{\sqrt{2ahx(a+h)}} = \frac{dx\sqrt{ah}}{\sqrt{2(a+h)x}}.$$

At cum sit $v = b - gx$ ideoque $b = gk$, habebimus $v = g(k-x)$ atque elementum temporis =

$$\frac{dx\sqrt{ah}}{\sqrt{2g(a+h)(kx-x^2)}}.$$

At $\int \frac{dx}{\sqrt{(kx-x^2)}}$ posito $x = k$ fit = π, peripheriae circuli existente diametro 1. Consequenter tempus per arcum AE infinite parvum est =

$$\frac{\pi\sqrt{ah}}{\sqrt{2g(a+h)}} = \frac{\pi\sqrt{2ah}}{2\sqrt{g(a+h)}}.$$

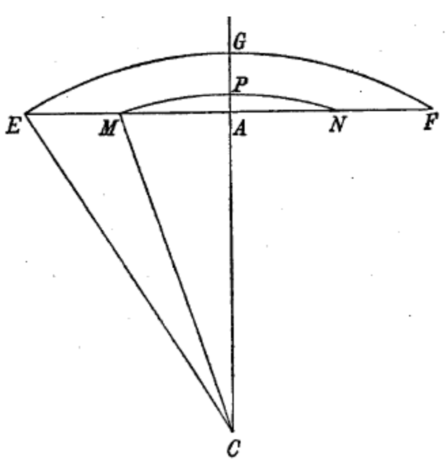

Fig. 28.

Corollarium 2.

208. Si centrum virium infinite distet, ut esset $a = \infty$, erit potentiae directio sibi parallela ideoque ut supra erit tempus, quo arcus *AE* absolvitur = $\frac{\pi\sqrt{2h}}{2\sqrt{g}}$.

At si arcus circuli *EA* fit linea recta (Fig. 28) seu $h = \infty$, erit tempus per $EA = \frac{\pi\sqrt{2a}}{2\sqrt{g}}$.

Corollarium 3. [p. 90]

209. Si ergo casus comparetur cum oscillationibus penduli a potentia g quoque, sed directiones sibi parallelas habente, sollicitati, erit penduli isochroni longitudo $=\frac{ah}{a+h}$.
Tempus enim unius descensus seu ascensus huius penduli est (166) $=\frac{\pi\sqrt{2ah}}{2\sqrt{g(a+h)}}$.

Exemplum 2.

210. Sit iam (Fig. 28) vis centripeta potestati cuicunque distantiarum proportionalis seu $P=\frac{y^n}{f^n}$ et linea *EF* recta. Erit $AM=s=\sqrt{(y^2-a^2)}$ et $x=y-a.$ Erit autem porro

$$v=b-\int\frac{y^n dy}{f^n}=b+\frac{a^{n+1}-y^{n+1}}{(n+1)f^n}$$

positoque $v=0$ fiet

$$y^{n+1}=a^{n+1}+(n+1)bf^n=(a+k)^{n+1}.$$

Vel dicta $CE=c$ erit

$$b=\frac{c^{n+1}-a^{n+1}}{(n+1)f^n}\quad \text{et} \quad v=\frac{c^{n+1}-y^{n+1}}{(n+1)f^n}.$$

Consequenter ob $ds=\frac{ydy}{\sqrt{(y^2-a^2)}}$ habebitur tempus per $AM=$

$$\int\frac{ydy\sqrt{(n+1)f^n}}{\sqrt{(y^2-a^2)(c^{n+1}-y^{n+1})}}.$$

Quod integrale ita est accipiendum, ut fiat = 0 positi $y=a$. Tumque facto $y=c$ habebitur tempus per lineam *EA*. Semioscillatio vero seu motus per *EAF* aequabitur duplo huius temporis.

Corollarium 4. [p. 91]

211. Ponatur vis centripeta distantiis proportionalis seu $n=1$;erit tempus per $AM=$

$$\int\frac{ydy\sqrt{2f}}{\sqrt{(y^2-a^2)(c^2-y^2)}}$$

seu posito $AE=i$ ob $c^2=a^2+i^2$ et $y^2=a^2+s^2$ erit tempus per $AM=$

$$\int\frac{ds\sqrt{2f}}{\sqrt{(i^2-s^2)}},$$

unde tempus per AE erit $\frac{\pi\sqrt{2f}}{2}$. Omnes igitur oscillationes super hac recta absolvuntur eodem tempore; dimidia nimirum oscillatio tempore $\pi\sqrt{2f}$ conficietur.

Corollarium 5.

212. Si oscillatio est infinite parva, tempus unius semioscillationis super recta erit quoque $\pi\sqrt{2f}$; at cum vis centripeta tum ut constans considerari possit, sit ea = *g*; erit $\frac{a}{f}=g$ ideoque tempus unius semioscillationis $=\frac{\pi\sqrt{2a}}{\sqrt{g}}$ ut supra (208)

Corollarium 6.

213. Quia directiones gravitatis revera convergunt ad centrum terrae, corpus in superficie telluris super recta perfecte horizontali oscillationes peragere posset, nisi resistantia et frictiones impedirent. Tempus autem unius semioscillationis talis foret (ob a = semidiametro terrae et g = 1) 2536 minut. secund. (183).

PROPOSITIO 23. [p. 92]

Problema.

214. *Si corpus sollicitetur a duabus quibuscunque potentiis, quarum alterius directio sit verticalis MQ (*Fig. 29*), alterius horizontalis MP, definire motum corporis ab istis viribus sollicitati super data curva AMB.*

Solutio.

Sit celeritas in B nulla, in M debita altitudini v. Vis sollicitans secundum MQ sit = P et ea secundum $MP = Q$. Ponatur $BR = t$, $RM = z$, arcus BM = w, quas litteras ad descensum corporis ex quiete ex B adhibebimus. At pro ascensu ex A quacunque cum celetate initiali, qui motus ad oscillationes referetur, sit $AP = x = QM$, $PM = AQ = y$ et arcus $AM = s$; celerita vero corporis in A debita sit altitudini b; erit ergo $t + x$ = const., item $z + y$ = const. et $w + s$ = const., unde $dt + dx = 0$ [, $dz + dy = 0$] et $dw + ds = 0$. Resolutis potentiis P et Q in normales et tangentiales erit vis tangentialis ex P orta = $\frac{Pdt}{dw}$ et vis normalis ex P orta = $\frac{Pdz}{dw}$ trahens secundum MN. Deinde erit vis tangentialis ex Q orta = $\frac{Qdz}{dw}$ et normalis ex Q = $\frac{Pdt}{dw}$, quae villi normali est contraria. Ultraque vis tangentialis motum per BM accelerat ideoque erit

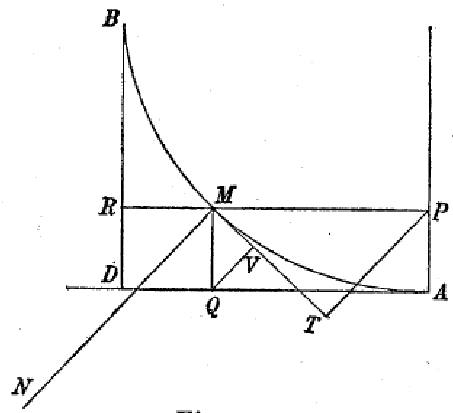

Fig. 29.

$$dv = Pdt + Qdz \text{ et } v = \int Pdt + \int Qdz$$

[p. 93] his integralibus ita acceptis, ut evanescant factist et $z = 0$. Atque pro ascensu ex A erit

$v = b - \int Pdx - \int Qdy$ his integralibus ita sumtis, ut evanescant positis x et $y = 0$. Positis igitur in illa aequatione $v = \int Pdt + \int Qdz$, $t = BD$ et $z = AD$ fiet $v = b$. Quare tempus per BM erit =

$$\int \frac{dw}{\sqrt{(\int Pdt + \int Qdz)}}$$

et tempus per AM =

$$\int \frac{ds}{\sqrt{(b - \int P dx - \int Q dy)}}.$$

Sumto elemento dt vel dx constante erit radius osculi curvae in $M = \frac{dw^3}{dtddz}$ atque vis centrifuga, cuius directio secundum *MN* est, =

$$\frac{2dtddz(\int Pdt + \int Qdz)}{dw^3}.$$

Totalis ergo vis, qua curva in M secundum MN premitur, est =

$$\frac{Pdz - Qdt}{dw} + \frac{2dtddz(\int Pdt + \int Qdz)}{dw^3}.$$

Q.E.I.

Corollarium 1.

215. Si *P* est functio ipsius *x* vel *t* quaecunque et *Q* functio ipsius *y* vel *z* quaecunque, tam *Pdx* quam *Qdy* integrari poterunt; atque ideo celeritas *v* poterit exhiberi et ope aequationis pro curva tempus quoque.

Corollarium 2.

216. Quia quaecunque et quotcunque potentiae sollicitantes, si modo earum directiones sint in eo plano, in quo est curva *AMB*, in huiusmodi duas potentias possunt resolvi, haec propositio latissime patet et omnes casus complectitur, quibus potentiarum directiones et curva sunt in eodem plano.

Scholion. [p. 94]

217. Patet etiam haec propositio latius,si pauca adiiciantur, et comprehendit casus, quibus non omnes potentiarum directiones sunt in plano curvae. Tum enim hae potentiae in binas sunt resolvendae, quarum alterae sint in ipso curvae plano, alterae ad hoc planum normales. Illae igitur in plano curvae sitae eodem modo, quo in propositione usi sumus, tractatae dabunt accelerationem corporis et pressionem secundum *MN*; alterae potentiae, quia normales sunt in curvam, in curva premenda tantum insumentur. Quare hinc duplex nascetur pressio, quam curva sustinet, altera secundum *MN* directa, altera ad planum curvae normalis. Harum igitur duarum pressionum si media sumatur directio, prodibit directio potentiae aequivalentis, in qua curva premitur. Quamobrem non est opus, ut huiusmodi casus evolvamus, sed paucis attingemus motum corporum super curva, quae ipsa non est in plano sita, ubi potentiam sollicitantem constantem et deorsum tendentem ponemus.

PROPOSITIO 24.

Problema.

218. *Existente potentia sollicitante uniformi eiusque directione recta deorsum tendente determinare motum corporis super curva quacunque AM (*Fig. 30*)non in eodem plano constituta.*

Solutio. [p. 95]

Sit curva *AQ* proiectio curvae *AM* in plano horizontali demissisque ex punctis quibusque proximis *M* et *m* in hoc planum perpendiculis *MQ* et *mq* ducantur ad axem pro lubitu assumtum *AP* normales *QP* et *qp* ponaturque $AP = x$, $PQ = y$ et $QM = z$. Sit corporis celeritas in *A* debita altitudini *b*, celeritas in *M* altitudini *v*. Potentia vero sit $= g$, qua corpus in *M* secundum *MQ* sollicitatur. Ducta tangente *MT* et in eam ex *Q* perpendiculari *QT* resolvatur potentia *g* in tangentialem et normalem. Erit ob

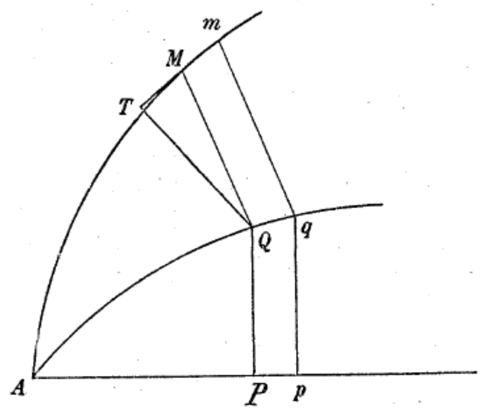

Fig. 30.

$$MQ : MT = \sqrt{(dx^2 + dy^2 + dz^2)} : dz$$

vis tangentialis =

$$\frac{gdz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Atque ob

$$MQ : QT = \sqrt{(dx^2 + dy^2 + dz^2)} : \sqrt{(dx^2 + dy^2)}$$

vis normalis =

$$\frac{g\sqrt{(dx^2 + dy^2)}}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Quia autem vis tangentialis motum retardat, erit $dv = -gdz$ et $v = b - gz$, unde tempus, quo arcus *AM* absolvetur, prodit =

$$\int \frac{\sqrt{(dx^2 + dy^2 + dz^2)}}{\sqrt{(b - gz)}}.$$

Vis normalis vero efficiet, ut curva in *M* a corpore tanta vi prematur iuxta directionem ad *Mm* normalem et in plano *QMmq* sitam. Premitur vero curva praeterea a vi centrifuga secundum directionem positioni radii osculi oppositam vi = $\frac{2(b-gz)}{r}$ designante *r* radiuim osculi in *M*. Invenimus autem supra (71) positionem radii osculi, ex qua proinde directio vis centrifugae innotescit. Quantitas vero vis centrifugae dabitur ex radio osculi, qui (72) est inventus; est nempe

$$r = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2 + ddz^2) + (dyddz - dzddy)^2)}}.$$

Q.E.I.

Corollarium 1. [p. 96]

219. Celeritas igitur corporis hoc quoque casu ab altitudine tantum pendet. Atque celeritas in *M* tanta est, quantam corpus per *QM* ascendens cum celeritate in *Q* altitudini *b* debita in *M* haberet.

Corollarium 2.

220. Non poterit ergo corpus ad maiorem altitudinem ascendere quam ad $\frac{b}{g}$. Nam si est $b - gz = 0$, corpus in ea altitudine omnem celeritatem amisit iterumque descendet.

Corollarium 3.

221. Intelligitur etiam, si potentia non constans fuisset accepta, sed variabilis *P*, tum inventam fuisse celeritatem in *M* debitam altitudini $b - \int Pdz$.

Scholion 1.

222. Si in plano verticali concipiatur curva *AM* (Fig. 31) ad axem horizontalem *AQ* relata fueritque *AQ* = curvae *AQ* praeced. fig. et *QM* = *QM* praeced. fig., erit quoque curva AM aequalis curvae *AM* praeced. fig. Si iam corpus super curva *AM* ascendat celeritate initiali in *A* debita altitudini *b* et ab eadem potentia *g* sollicitatum, habebit in *M* quoque celeritatem altitudini $b - gz$ debitam.

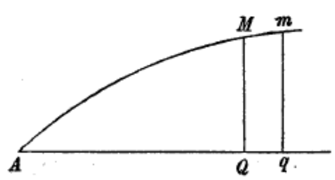

Fig. 31.

Atque ideo tempus quoque ascensus per *AM* congruet cum tempores ascensus per *AM* in praeced. fig. Hac igitur ratione motus corporis super curva non in eodem plano sita reduci potest ad motum super curva in eodem plano posita. [p. 97] Inter motus enim ipsos nullum erit discrimen ; at pressiones, quas hae duae curvae sufferunt, erunt diversae. Quamobrem hoc modo pressio, ut libet, poterit variari manente motu corporis super curva eodem.

Scholion 2.

223. Posuimus hactenus curvam, super qua corpus movetur, et potentiam sollicitantem una cum directione datas ex iisque motum corporis et pressionem curvae deduximus. Nunc igitur, cum haec sufficere possint, ad alias quaestiones progrediemur, in quibus alia pro datis accipiuntur reliquaque sunt invenienda. Et primo quidem data sit pressio in singulis curvae punctis et potentia sollicitans; ex quibus ipsa curva et motus super ea debeat inveniri. Deinde aliis factis combinationibus inter eas res, quae in computum veniunt, alias quaestiones formabimus.