CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 39]

PROPOSITION 12.

Problem.

83. *A body which is moving on the curve AM* (Fig.12) *is acted on everywhere by a force MF, the direction of which is parallel to the axis AP; to determine the speed of the body at individual points and the time taken for any part of the curve to be described by the body, with the force due to the curve acting on the body at individual points.*

Solution.

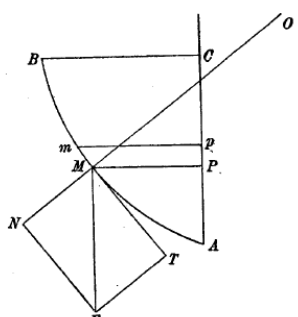


Fig. 12.

Now the body describes the arc *AM* and the speed of the body at *A* corresponds to the height *b* while the speed at *M* corresponds to the height *v*. Now with $AP = x$, $PM = y$ and with the arc $AM = s$ the force *MF* which is called *p* is resolved into the sides, clearly the normal MN and the tangent MT; there is $ds : dx = MF : MT$ and $ds : dy = MF : MN$. Therefore there is hence produced the tangential force $MT = \frac{pdx}{ds}$ and the normal force $MN = \frac{pdy}{ds}$. It is evident here that $dv = -pdx$ and $v = C - \int pdx$ (42). Moreover with the integral thus taken $\int pdx$, in order that it vanishes with x = 0, we have $v = b - \int pdx$; [p. 40] from which equation the speed of the body is known at individual points. From the same equation it is found also the time in which the arc AM is completed; for with the time put as *t*, there results

$$t = \int \frac{ds}{\sqrt{(b - \int pdx)}}.$$

The normal force $MN = \frac{pdy}{ds}$ is completely devoted to pressing the body to the curve along MN (39), therefore it increases the pressing force arising from the centrifugal force, since MN falls in the region opposite to the radius of osculation MO. Whereby, with the radius of osculation put as MO = r , the centrifugal force is equal to $\frac{2v}{r}$ (20), and the total force pressing on the curve next to $MN = \frac{pdy}{ds} + \frac{2v}{r}$. Q.E.I.

Corollary 1.

84. The speed at *M* is therefore of such a size as it would be at *P*, if the body rises with the same initial speed $\sqrt{b}$ along *AP* to the particular heights acted on by the same force.

Corollary 2.

85. Therefore the speed does not depend on the nature of the curve, but only on the height that the body traverses. Clearly if the height of the element were *dx*, then $dv = -pdx$ or $dv = pdx$, as the body either rises or falls.

Corollary 3.

86. Since we have $v = b - \int pdx$, if the abscissa *x* to be used is taken as far as *AC*, for which $\int pdx = b$, then the speed of the body corresponding to that height *B* is equal to zero. Therefore the body rises as far as *B* and there it is at rest ; now continuing by falling from *B* along *BMA*. [p. 41]

Corollary 4.

87. If the ascent along *AMB* is compared with the rectilinear ascent along *APC*, the time to pass along the element *Mm* to the time for *Pp* is as *Mm* to *Pp*, i. e. as *ds* to *dx*.

Corollary 5.

88. Whereby if the line *AMB* is straight, as the ratio *Mm* to *Pp* is constant, the time to pass through *AM* to the time to pass along *AP* is in a constant ratio, surely that which the total sine has to the cosine of the angle *A*, or which the length *AB* has to *AC*.

Corollary 6.

89. With the element *Pp* placed constant, the radius of osculation $r = \frac{-ds^3}{dxddy}$ and thus the centrifugal force is equal to

$$-\frac{2vdxddy}{ds^3} = \frac{-2(b - \int pdx)dxddy}{ds^3}.$$

Whereby the total force pressing on the curve is equal to

$$\frac{pds^2dy - 2(b - \int pdx)dxddy}{ds^3}.$$

Scholium 1.

90. As in this problem, from the given curve and force acting, the speed at individual points, the time to pass through any arc, and the pressing force on any point of the curve, are found : thus from these five things with any two, the remaining three can be found. From which ten problems arise, which all have a solution from the solution of this problem.[p. 42]

Scholium 2.

91. Similarly there can be ten questions of this kind, if the directions of the forces acting are not parallel, but either converge to a centre of force or have their directions determined in some other way. But if also the direction between these sought is put in place, then from the six quantities taken in the computation, from any three the other three can be found; and hence twenty problems are to be found. [One direction is fixed, and there are directions for the other four quantities, leaving 6 variables.]

Scholium 3.

92. Again indeterminate problems may arise, as if in place of the time through which some part of the curve is traversed can only be given by an integral along *AMB* ; for then an infinite number of solutions can be put in place. Besides if more ascents or descents are considered to be integrated upon various parts of the same curve, and the ratio of these is given, then the number of questions is increased much more. To this kind belongs the question of finding the curve, upon which all the ascents and descents to the same point are made in the same time, as these are the most difficult we will handle finally. Moreover now we take the first curve and force acting as given and we solve the problems pertaining to this. Next, we will show how from different given quantities, in what way the others are to be found. [p. 43]

PROPOSITION 13.

Problem.

93. *If the force acting is uniform and acting downwards everywhere, to determine the descent of the body on a given curve AM* (Fig.13), *beginning from A at rest, and to find the force pressing on the curve at individual points M.*

Solution.

With the vertical *AP* or with the parallel in the direction of the force drawn *MF* and with the connected line *MP* at right angles, let $AP = x, PM = y$, with the curve $AM = s$. The force *MF* is put equal to *g* with the force of gravity present equal to 1 and the speed at *M* corresponding to the height *v*. With these in place, the normal force $= \frac{gdy}{ds}$ and the tangential force $= \frac{gdx}{ds}$ (83). Because in this case the tangential force accelerates, we have $dv = gdx$ and $v = gx$ as the speed at $A = 0$. Hence since the radius of osculation along the direction *MO* is equal to $\frac{+ds^3}{dxddy}$ with *dx* placed constant, the centrifugal force $= \frac{+dvdxddy}{ds^3}$, the direction of which is *MN*. Now the normal force $\frac{gdy}{ds}$ acts along the same direction. whereby the total force pressing on the curve along *MN* at *M* is equal to

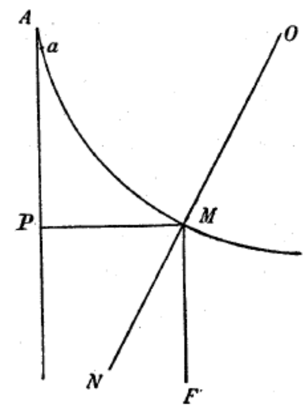


Fig. 13.

$$\frac{gdy}{ds} + \frac{2vdxddy}{ds^3} = \frac{gdy}{ds} + \frac{2gxdxddy}{ds^3}$$

as $v = gx$. Truly the time in which the body traverses the arc AM is equal to $\int \frac{ds}{\sqrt{gx}}$.

Q.E.I. [p. 44]

Corollary 1.

94. Therefore the speed at *M* only depends on the height *AP* through which the body descends, and the body acquires the same amount as falling from *A* to *P* and acted on by the same force *g*.

Corollary 2.

95. Therefore, whatever the curve the body falls along from rest, acted on by the constant force *g*, the speeds are as the square roots from the squares [of the speeds] proportional to the heights traversed; that is as $\sqrt{v}$, i. e. as $\sqrt{gx}$.

Corollary 3.

96. The time, in which the first element *Aa* is traversed, is $\int \frac{ds}{\sqrt{gx}}$ with x evanescent. If therefore the angle *PAa* is less than a right angle, or $s = nx$, then the time to pass through *Aa* is infinitely small and thus the time to pass along *AM* is finite, unless the curve either rises between *A* and *M* beyond *A* or it progresses to infinity. But if the angle *PAa* is right, for the point A, $s^n = ax$ with a number n present greater than one and thus

$$\sqrt{gx} = s^{\frac{n}{2}}\sqrt{\frac{g}{a}} \text{ and } \int \frac{ds}{\sqrt{gx}} = \frac{2s^{\frac{2-n}{2}}}{2-n}\sqrt{\frac{a}{g}}.$$

Whereby if n is less than two, the time to pass along *Aa* is infinitely small and the time to traverse *AM* is finite. But if n is greater than or equal to 2, the time to pass through the first element *Aa* is infinitely great or the body never escapes from A. [p. 45]

Corollary 4.

97. But whenever $n < 2$, to the radius of osculation at *A* is indefinitely small. Whereby in this case, in which the tangent to the curve at *A* is normal to *AP*, the body does not descend in this case, unless the radius of osculation at *A* is infinitely small.

Scholium 1.

98. From which, as the first element is traversed in an infinitely short time, it is correctly concluded that the time to traverse the arc *AM* is finite; for since the body descends along *AM* with an accelerated motion, the following elements are described more quickly and on this account the corresponding time must be finite. Moreover everything is illustrated in the following.

Example 1.

99. Let the line *AM* (Fig. 14) be straight and inclined at some angle to the vertical *AP* and the cosine of the angle $A = n$; then $x = ns$. Therefore the time, in which the body descends along *AM*, is equal to

$$\int \frac{ds}{\sqrt{gns}} = \frac{2\sqrt{s}}{\sqrt{gn}} = \frac{2\sqrt{AM}}{\sqrt{gn}} = \frac{2AM}{\sqrt{g \cdot AP}}$$

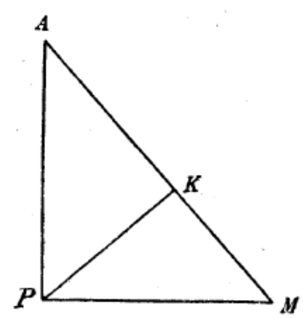


Fig. 14.

or the time to travel along the line at any inclination varies directly as the root of the length of the line and inversely as the root of the cosine of the angle of inclination *MAP*. Moreover the centrifugal force is equal to 0 [*i.e.* infinite radius of curvature], whereby the line *AM* is only acted on by the normal force, which is equal to

$$g\sqrt{(1-n^2)} = \frac{g \cdot PM}{AM}.$$

Corollary 5.

100. Therefore the time to pass along *AM* is to the time to pass along *AK* as $\sqrt{AM}$ to $\sqrt{AK}$. But the time to pass along *AM* is to the time along *AP* as *AM* to *AP* (88). [p. 46] Whereby if it is the case that $AM : AP = \sqrt{AM} : \sqrt{AK}$ or $AM : AP = AP : AK$, which comes about if *PK* is perpendicular to *AM*, then the descent time along *AK* is equal to the descent time along *AP*.

Corollary 6.

101. It is also apparent that the descent time along the perpendicular *PK* is equal to the descent time along *AP*. For the cosine of the angle $APK = \frac{PK}{AP}$. Whereby, since the time along AP to the time along *KP* is as $\frac{\sqrt{AP}}{\sqrt{1}}$ to $\sqrt{KP} : \sqrt{\frac{PK}{AP}}$, this is an equal ratio.

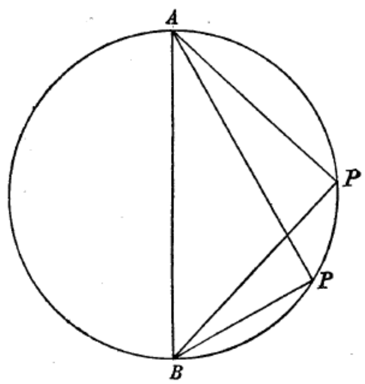


Fig. 15.

Corollary 7.

102. From this it is evident in the circle *APPB* (Fig. 15), that all the descents along the chords *AP* drawn from the uppermost point *A* and all the descents along the chords drawn to the lowest point *B* are to be made in equal times, clearly each in that time in which the body falls freely perpendicularly along the diameter *AB*.

Example 2.

103. If the curve *AMB* (Fig. 16) is a circle and the radius *BC* = *a* and *AP* is a tangent to the circle, then

$$(a-y)^2 + x^2 = a^2 \quad \text{or} \quad y = a - \sqrt{(a^2 - x^2)}.$$

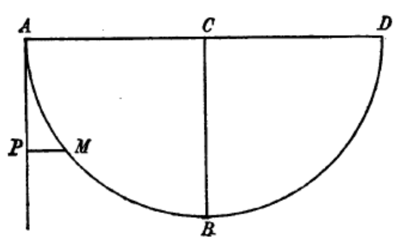


Fig. 16.

Hence we have $ds = \frac{adx}{\sqrt{(a^2-x^2)}}$ and hence $v = gx$ and $r = a$, and the centrifugal force is equal to $\frac{2gx}{a}$ and on this account $dy = \frac{xdx}{\sqrt{(a^2-x^2)}}$, and the total force sustained by the circle at *M* is equal to $\frac{3gx}{a}$.

Therefore the total force pressing on the curve is three times as great as the normal force.

The time then, in which the arc *AM* is traversed, is equal to $\int \frac{adx}{\sqrt{g(a^2x-x^3)}}$, [p. 47] the integration of which neither depends on circular nor hyperbolic quadrature, but with the help of rectification of the elastic curve it is able to be constructed. Meanwhile the time along the quadrant *AB* is equal to

$$2\sqrt{\frac{a}{g}} \times \left(1 + \frac{1}{2\cdot 5} + \frac{1\cdot 3}{2\cdot 4\cdot 9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13} + \text{etc.}\right).$$

Corollary 8.

104. When the body reaches the lowest point *B,* there it has the speed corresponding to the height *ga*. Therefore this ascends in the other quadrant *BD* and reaches *D*, where its speed vanishes, and thus again it descends to *B* and then it re-ascends to *A* along *BA*. Now in a similar way is the ascent and descent along the sides of a square, since the body either in the ascent or in the descent, has the same speed at the same points.

Scholium 2.

105. We will not offer other examples now, as in what follows, where we consider more descents on a given line, we are to report on more examples. Now truly we disclose in the first place these questions which pertain to the motion on a given line starting from rest at a given point, and the following problem is of this kind.

PROPOSITION 14.

Problem.

106. *If there is an infinite family of similar curves AM, AM, etc.* (Fig.17) *beginning from the fixed starting point A, to find a curve CMM from these other curves cutting the arcs AM, AM, etc, which are traversed in equal times by a body descending along these arcs, as before, with a uniform force present acting downwards everywhere.* [p. 48]

Solution.

From the infinite number of curves given one is taken *AM*, the parameter of which is *a*. Putting in place $AP = x$, $PM = y$ and with the arc $AM = s$ and as before the force acting being equal to *g* , the body descends on the given curve *AM*; the speed at *M* corresponds to the height *gx*. Hence the descent time on *AM* is equal to $\int \frac{ds}{\sqrt{gx}}$. Hence from all these curves AM, AM, etc., so many arcs are to be cut, in order that from these the quantity $\int \frac{ds}{\sqrt{gx}}$ is constant. But $\int \frac{ds}{\sqrt{gx}}$ is referring to other curves, if besides *s* and *x*, the parameter *a* also is made a variable. Therefore with *a* made variable as well in $\int \frac{ds}{\sqrt{gx}}$, the quantity $\int \frac{ds}{\sqrt{gx}}$ is indeed constant for that time in which all the descents are to become the same. Let this time be equal to *k*, and $k = \int \frac{ds}{\sqrt{gx}}$ for individual curves. Whereby if $\int \frac{ds}{\sqrt{gx}}$ is thus differentiated, as also *a* is placed variable, this differential is put equal to zero. In order that the differential of this integral can be found, let $ds = pdx$ and *p* is a function in which *a* and *x* likewise make numbers with zero dimensions [such as a function of *x/a*]. [p. 49] Hence we have the integral $\int \frac{pdx}{\sqrt{gx}}$; this differential with the variable *a* in place also, gives

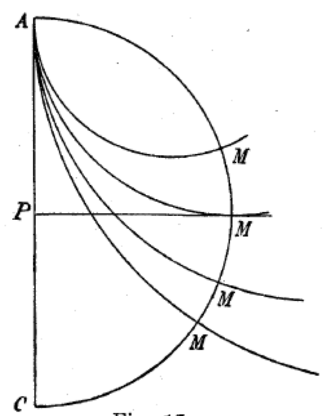


Fig. 17.

$$\frac{pdx}{\sqrt{gx}} + qda,$$

which must be equal to 0. Now the quantity *q* is to be found in the following way. Since $k = \int \frac{pdx}{\sqrt{gx}}$, in the quantity *k* [or in *dk*], the variables *a* and *x* produce a number having the dimensions $\frac{1}{2}$. Moreover this is shown elsewhere, in Vol. VII Comment.

[E044 : *Concerning an infinite number of curves of the same family. Or a method for finding an infinite number of curves of the same kind*, Commen. acad. sc. Petrop. 7 (1734/5) 1740. We now present §23 of this paper, in the original and in translation :

'Sin vero fuerit *u* functio *m* dimensionum ipsarum *a* et *x*, atque $du = Rdx + Sda;$ erit $\frac{u}{x^m}$ functio ipsarum *a* et *x* nullius dimensionis. Differentietur igitur $\frac{u}{x^m}$ et prodibit $\frac{xdu - mudx}{x^{m+1}}$ seu $\frac{Rxdx - mudx + Sxda.}{x^{m+1}}$ Quod cum sit differentiale functionis nullius dimensionis erit $Rx^2 - mux + Sax = 0,$ seu $Rx + Sa = mu.$ Quare si fuerit *u* functio *m* dimensionum ipsarum *a* et *x*; atque ponatur $du = Rdx + Sda;$ erit $Rx + Sa = mu$ ideoque $du = Rdx + \frac{da}{a}(mu - Rx)$ seu $adu = Radx - Rxda + muda.$ '

'However, if *u* is a function of *m* dimensions of *a* and *x*, and $du = Rdx + Sda;$ then $\frac{u}{x^m}$ is a function of *a* and *x* zero dimension. Therefore $\frac{u}{x^m}$ is differentiated, and there is produced $\frac{xdu - mudx}{x^{m+1}}$ or $\frac{Rxdx - mudx + Sxda}{x^{m+1}}$. Because it arises from the differentiation of a function of zero dimensions, $Rx^2 - mux + Sax = 0,$ or $Rx + Sa = mu.$ Whereby if *u* is a function of dimension *m* of *a* and *x* themselves; and the equation $du = Rdx + Sda$ is put in place, then $Rx + Sa = mu$ and thus $du = Rdx + \frac{da}{a}(mu - Rx)$ or $adu = Radx - Rxda + muda.$' Thus, in the present case, $R = \frac{p}{\sqrt{gx}}$, $S = q$, $m = \frac{1}{2}$, and $u = k$. In addition, the establishment of functions of zero dimensions is discussed in E012 in these translations.],
then the equation arises

$$\frac{px}{\sqrt{gx}} + qa = \frac{k}{2}.$$

From which it is found that [there is a misprint present here in the *O. O.*, but not in the original text.]

$$q = \frac{k}{2a} - \frac{p\sqrt{x}}{a\sqrt{g}}.$$

Therefore we have

$$\frac{pdx}{\sqrt{gx}} + qda = \frac{pdx}{\sqrt{gx}} + \frac{kda}{2a} - \frac{pda\sqrt{x}}{a\sqrt{g}} = 0$$

which is the equation for the curve sought. But if the equation between the coordinates *x* and *y* for the curve *CMM* is desired, from the equation for each of the curves *AM*, the value of *a* found must be substituted into the equation between *x* and *y*. Q.E.I.

Corollary 1.

107. Also the equation found at first,

$$\frac{pdx}{\sqrt{gx}} = \frac{pda\sqrt{x}}{a\sqrt{g}} - \frac{kda}{2a}$$

is sufficient for the curve *CMM* to be found. As for any given abscissa $AP = x$, from that the parameter a of this curve *AM* is found, of which the corresponding point *M* of the assumed abscissa x lies on the curve *CMM* sought.

Corollary 2.

108. Moreover since this is a differential equation, and thus to which more curves pertain according to the added constant, it is to be noted that with the addition of the constants, only that solution is to be agreed upon for which the given curve is completed in the time of descent k, for the given value of a that gives the abscissa x only of the required arc *AM* to be cut off. [p. 50]

Corollary 3.

109. If the time of descent k must be equal to the time of descent through the vertical distance $AC = b$, then $k = \frac{2\sqrt{b}}{\sqrt{g}}$. With which value put in place, we have the equation :

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a}.$$

In the integration of this equation, it has to be arranged that the curve passes through the point C.

Scholium 1.

110. Moreover it is always the case that the vertical line *AC* arises as a kind of curve *AM*, if the parameter a is taken to be indefinitely large or small. Whereby the constant time k is most conveniently expressed by the descent through the vertical *AC*, clearly a kind of curve *AM*. And in the construction of the equation found

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a}$$

a constant of such a size is to be added, as by putting $x = b$, a is made infinite or zero, as one or the other value corresponds to the value of the line *AC*.

Scholium 2.

111. If it is possible to integrate $\frac{ds}{\sqrt{gx}}$ itself, without the aid of any equation, by means of which q can be found. For if the integral $\frac{ds}{\sqrt{gx}}$ is itself again differentiated with respect so some variable a also, q is again found; and it is only necessary to put this differential equal to zero. Now most conveniently in these cases the problem is solved, if the integral of $\frac{ds}{\sqrt{gx}}$ is at once put equal to k or to $\frac{2\sqrt{b}}{\sqrt{g}}$ and in place of a the value is substituted in terms of x and y from the given equation for the curve. [p. 51] And in this way the solution is established not only for similar curves, but also for dissimilar ones, but only if the descent times can be expressed by a finite quantity.

Exemplum 1.

112. If all these curves *AM* are straight lines inclined in different ways to the vertical *AC*, then

$$y = nx \text{ and } s = x\sqrt{(1+n^2)},$$

where n is to be considered as a parameter. Hence the time becomes

$$\int \frac{ds}{\sqrt{gx}} = \int \frac{dx\sqrt{(1+n^2)}}{\sqrt{gx}} = \frac{2\sqrt{x(1+n^2)}}{\sqrt{g}},$$

which must be placed equal to $\frac{2\sqrt{b}}{\sqrt{g}}$ itself. Thus there becomes

$$x(1+n^2) = b.$$

Moreover since n is a variable quantity, put the value $\frac{y}{x}$ for that from the equation $y = nx$; with which accomplished, this equation is produced for the curve *CMM* between the orthogonal coordinates x and y :

$$y^2 + x^2 = bx,$$

which is the equation for the circle, the diameter of which is the line $AC = b$.

Scholium 3.

113. This case is the one examined before (102); indeed there it was shown that the body descends in equal intervals of time by all the chords drawn in the circle from the uppermost point. Here indeed the case does not concern similar curves, but we report on this example as a case illustrating scholium 2, because for straight lines and for these, the descent times are expressed by finite quantities. Now the following examples will include similar curves, as the proposition postulates. [p. 52]

Example 2.

114. Let all the curves *AM, AM* be circles tangent to the vertical *AC* at *A*. The radius of each of these is equal to a, and it is given by

$$y = a - \sqrt{(a^2 - x^2)} \text{ and } a = \frac{y^2 + x^2}{2y}.$$

Now these circles are all similar curves, because a, y and x in the equation keep a number of the same dimension, or they complete the homogeneity alone. Therefore the radius a must be handled as a variable parameter. Moreover, there is obtained from that equation $ds = \frac{adx}{\sqrt{(a^2 - x^2)}}$, whereby we have $p = \frac{a}{\sqrt{(a^2 - x^2)}}$ and thus it has the prescribed property, as the dimension of the number formed from *a* and *x* is zero. On this account we have this equation for the curve *CMM* :

$$\frac{adx}{\sqrt{(a^2x-x^3)}}=\frac{da\sqrt{x}}{\sqrt{(a^2-x^2)}}-\frac{da\sqrt{b}}{a}$$

or this

$$\frac{da\sqrt{b}}{a}=\frac{xda-adx}{\sqrt{(a^2x-x^3)}}.$$

Which equation can be solved; for on putting $x = au$ there is produced

$$\frac{da\sqrt{b}}{a\sqrt{a}}=\frac{-du}{\sqrt{(u-u^3)}},$$

in which the indeterminates are separated from each other. Moreover, in which the equation is obtained for the curve *CMM* between the coordinates *x* and *y*, in place of *a* is put the value $\frac{y^2+x^2}{2y}$, and in place of da the differential of this $\frac{y^2dy+2yxdx-x^2dy}{2y^2}$. With which put in place the following differential equation is obtained

$$-xdy+ydx=\frac{(y^2dy+2yxdx-x^2dy)\sqrt{bx}}{y^2+x^2}.$$

Which thus must be integrated, so that with $x = b$ it makes $y = 0$, which curve must pass through the point *C*.

Corollary 4.

115. From this equation the tangent of the curve *CMM* is known at individual points and from the position of the tangent the angle *AMM* is known, [p. 53] in which whatever of the given curves is divided. Clearly the tangent of the angle $AMM=\frac{y}{x-\sqrt{bx}}$. Therefore here the angle at *C* is right on account of $x = b$, or the curve *CMM* is normal at *C* to *AC*.

Corollary 5.

116. If *b* is taken either greater or less, the curve *CMM* is different also and in this way an infinity of isochronous curves arise being cut from the circular arc. And these curves are all similar between themselves on account of the parameter *b*, which constitute a homogeneous equation with *x* and *y*. Hence with one given curve *CMM* innumerable others can be constructed from that, clearly with the *x* and *y* coordinates of the curve *CMM* augmented or diminished in the same ratio as *AC* or *b* is augmented or diminished.

Example 3.

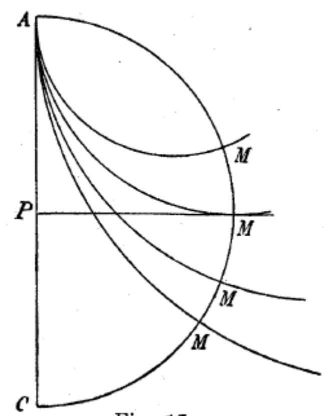


Fig. 17.

117. Let all the curves *AM*, *AM* be cycloids having cusps at *A* and vertical tangents *AC* at *A*. With the parameter of any cycloid *AM* put in place, or with twice the diameter of the generating circle equal to *a*, from the nature of the cycloid, we have $s = a - \sqrt{(a^2 - 2ax)}$ and $ds = \frac{adx}{\sqrt{(a^2-2ax)}}$ and hence $dy = \frac{dx\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$. Hence in this case we have the function of *a* and *x* of zero dimension, $p = \frac{a}{\sqrt{(a^2-2ax)}}$, as required. Whereby for the curve *CMM*, this equation is required

$$\frac{adx}{\sqrt{(a^2x-2ax^2)}} = \frac{da\sqrt{x}}{\sqrt{(a^2-2ax)}} - \frac{da\sqrt{b}}{a}$$

or

$$\frac{xda-adx}{\sqrt{(a^2x-2ax^2)}} = \frac{da\sqrt{b}}{a}.$$

[p. 54] If the equation between the orthogonal coordinates *x* and *y* is desired, from the equation

$$y = \int\frac{dx\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$$

or with the differential of this likewise, for the variable *a*, the value of *a* itself must be substituted. Now this differential equation with the variable *q* in place gives

$$ady - yda = \frac{adx\sqrt{2ax} - xda\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$$

or

$$\frac{ady-yda}{\sqrt{2a}} = \frac{axdx-x^2da}{\sqrt{(a^2x-2ax^2)}}.$$

Which goes into this :

$$\frac{ady-yda}{a^2\sqrt{2}} = \frac{axdx-x^2da}{a^2\sqrt{(ax-2x^2)}}.$$

Now the above multiplies by $\frac{1}{4\sqrt{a}}$ provides this :

$$\frac{da\sqrt{b}}{4a\sqrt{a}} = \frac{axda-a^2dx}{4a^2\sqrt{(ax-2x^2)}}.$$

These two added equations give an integrable equation, the integral of which is :

$$\frac{y}{a\sqrt{2}} - \frac{\sqrt{b}}{2\sqrt{a}} = \frac{-\sqrt{(ax-2x^2)}}{2a}.$$

From which the value of *a* elicited becomes :

$$\sqrt{a} = \frac{y\sqrt{2b} \pm \sqrt{(2y^2x - 2bx^2 + 2x^3)}}{b-x}$$

and

$$\sqrt{(ax-2x^2)} = \frac{yx\sqrt{2} \pm \sqrt{(2by^2x - 2b^2x^2 + 2bx^3)}}{b-x}.$$

With which values in the equation :

$$\frac{(xdy - ydx)\sqrt{a} + xdx\sqrt{2b}}{\sqrt{(ax-2x^2)}} = dy\sqrt{b},$$

which arises from the two differential equations with *da* eliminated, on substitution gives

$$\frac{xdy - ydx - bdy}{\sqrt{b}} = \frac{dx\sqrt{(y^2 - bx + x^2)}}{\sqrt{x}},$$

the equation for the curve sought *CMM*.

Corollary 6.

118. From this equation the tangent of the angle is found that the curve CM makes with the applied line PM, truly

$$\frac{dx}{dy} = \frac{-(b-x)\sqrt{x}}{y\sqrt{x} + \sqrt{(by^2 - b^2x + bx^2)}}.$$

Then also the tangent of the angle becomes known, [p. 55] that the cycloid *AM* makes with the applied line *PM*. From the equation of the cycloid is without doubt

$$\frac{dx}{dy} = \frac{\sqrt{(a^2 - 2ax)}}{\sqrt{2ax}} = \frac{\sqrt{(ax-2x^2)}}{x\sqrt{2}}.$$

Now on eliminating *a* the tangent is equal to

$$\frac{y\sqrt{x} + \sqrt{(by^2 - b^2x + bx^2)}}{(b-x)\sqrt{x}}.$$

Whereby, since either of these given angles is the complement of the other, hence with this taken into account, the angle that the curve *CMM* makes with any of the given *AM* is right. Consequently the curve *CMM* is the orthogonal trajectory of all the given cycloids *AM*, *AM*, etc.

Corollary 7.

119. With *AC* taken of another size also other curves *CMM* are produced and thus an infinity of orthogonal trajectories are found, which are all similar to each other. Hence from one easy given, it is possible to construct any number you please.

Scholium 4.

120. All these isochronous curves being cut from the arc, whatever the curves cut, can always be constructed, even if it is not apparent from the equation. For by quadrature the arcs which are completed in a given time of descent can be removed from the given curves, and in this way any points on the curve sought can be found. If certain curves cut are algebraic, then the equation for the curve cut can always thus be compared, as by making a substitution of the indeterminate, which can then be separated in turn from each other. But if the curves cut are expressed by a differential equation, [p. 56] the differential equation for the curve cut most rarely admits to being separable in terms of the indeterminates. Because, in a singular manner, as I have used in the case of the cycloid, the parameter *a* can be eliminated and there the substitution cannot deduce a separation.

Scholium 5.

121. Then it is necessary to observe that all the isochronous curves cut by an arc, the number of which is infinite, are similar to each other, according to the value of the variable *b*, if indeed the curves cut are of such a kind. This is gathered from the general equation

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a},$$

in which, since *p* is a function of zero dimensions of *a* and *x*, and the quantities *a*, *b* and *x* constitute a homogeneous equation. But from the equation of the curve cut, since in that equation *a*, *x*, and *y* are put in place everywhere to make a number of the same dimensions, the value of *a* is a function of *x* and *y* of one dimension. Whereby with that substituted in place of *a*, the equation is obtained for the curve cut, in which *b*, *x*, and *y* everywhere constitute a number of the same dimensions. Consequently, for the variable *b* put in place, there arises an infinite number of curves similar to each other with respect to the point *A*. Hence with a single curve given, the rest can easily be described by reason of the similitudes.

Scholium 6.

122. Now this material concerned with the cutting of isochronous arcs was published in the past [p. 57] in the Act. Erud. Lips. A. 1697 [p.206] [*The radius of curvature in translucent media ....and concerning synchronous curves, or the construction of rays from waves*; *Opera Omnia*, Book I, p. 187] by the Celebrated Johan. Bernoulli and later in the Comment. Acad. Paris by the Cel. Saurino, [1709, p. 257 and 1710, p. 208 ; *General Solution of the problem, ...*] who indeed used another method. Now I have used that method that I have treated in our Comment. pro A. 1734, as the most convenient for the solving of this kind of problems. Now in their works these celebrated men only considered similar curves as I have done, since without doubt for dissimilar curves the solution can be exceedingly difficult and often also too hard to solve. Now these curves are called synchronous in the places they are cited, since arcs traversed in the same time are cut off.

[Note that there was a delay of several years between the writing and the eventual publishing of Euler's works, even at the start of his career at St. Petersburg; thus occasionally he was able to extend his observations from volumes to be published later into earlier ones that had not yet been published either, as below. Thus, one must take the Enestrom Index with a pinch of salt as regards the chronological order of the works, as there was some coming and going, and of course a number of papers did not make it into the index at all in the original assessment, which should be looked at by somebody with an interest. The original Euler Archive is of course at St. Petersburg, and access does not seem to be that simple, and neither is it free.]

Scholium 7.

123. It is apparent, as shown from my dissertation in Vol.VII Comment. Acad. Petrop. that these synchronous curves can be found in a like manner, also if the given curves are not similar, but yet of such a kind that, as with $ds = pdx$ in place, in p the quantities a and x constitute a number of given dimensions; for then it is equally easy to find the value of the letter q ; as if the number of the dimensions of a and x in p is n, this equation for the curve is found :

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{(2n+1)da\sqrt{b}}{a}.$$

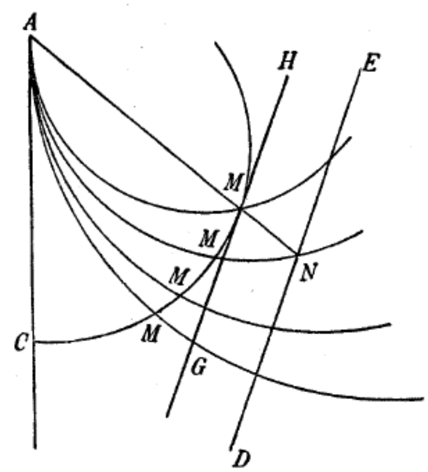
Whereby if $n = -\frac{1}{2}$, so that we can make $p = \frac{\sqrt{ac}}{\sqrt{(a^2-x^2)}}$, then we have $\frac{dx}{x} = \frac{da}{a}$ and thus $x = ma$, or x can be taken in a given ratio to the parameter a ; therefore in which case the construction of the isochrones is most easy. [p. 58] But if p does not have a value of this kind, from my paper cited above it is understood how the required equation has to be sought.

PROPOSITION 15.

Problem.

124. *If as before there is an infinite family of similar curves AM, AM, etc.* (Fig.18) *and the straight line DE in the position given, to find that curve AMN, upon which the body arrives at the line DE in the shortest possible time from A.*

Solution.



By the preceding proposition, with some curve *CMM* described cutting the isochronous arcs *AM*, the tangent *GMH* is drawn parallel to the given line *DE*. It is evident that the body is to arrive in the shortest time along the curve *AM*, which touches the line *GMH* at the point of contact *M*, since all other points of the line *GMH* fall beyond the curve *CMM* and thus a longer time is needed for the body to reach that line. Now, since all the curves [such as *CMM*] cutting isochronous arcs from the curves *AM, AM*, are similar to each other, (121), one is taken from these , which the line *DE* touches; I say that the point of contact is to be at the point *N*, in which the line *AM* drawn through the previous point of contact *M* crosses the line *DE*. From this it then follows from the nature of the similarity of the curves *CMM* with respect to the point *A*, that also it follows, as the arc *AMN* is similar to the arc *AM* and crossed the line *DE* at the same angle that the curve *AM* crosses the line *GH*. Whereby, [p. 59] since the body arrives in the shortest time along *AM* to *GH*, it is necessary that it also arrives in the shortest time on the curve *AMN* to the line *DE*. Q.E.I.

Corollary 1.

125. From this it is understood, that if the line *DE* is horizontal, with the descent along the vertical *AC* , then the body arrives in the shortest possible time, on account of the horizontal tangent to the curve *CMM* at C ; which is indeed evident by itself.

Corollary 2.

126. If therefore the curves *AM, AM* are cycloids, as we put in example 3 of the preceding proposition, the body on that cycloid arrives at the line *DE* the fastest which crosses this line at *N* at right angles, since the angle that any cycloid makes with the curve *CM* is right.

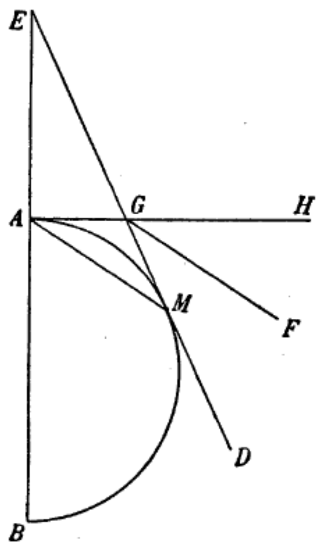


Fig. 19.

Corollary 3.

127. If therefore the line *DE* is vertical or parallel to *AC*, the portion of the cycloid *AMM* is half the cycloid. Whereby the horizontal motion on half the cycloid is the fastest.

Corollary 4.

128. If the curves *AM, AM* are straight lines drawn from the point *A* to the given line position *DE*, the body on that line *AM* (Fig. 19) arrives at *DE*, which is the chord of the circle passing through *A* and having the centre in the vertical line *AB*, and having the tangent line *DE* (112) [p. 60]

Corollary 5.

130. If therefore the angle *DEA* is *n* degrees, the angle *BAM* is $\frac{90+n}{2}$ degrees and the angle *AMG* is $\frac{90-n}{2}$. Or with *AGH* drawn the horizontal, and with the angle *DGH* bisected by the line *GF*, then the sought line *AM* is parallel to *GF*.

Corollary 6.

131. Whereby if the line *DE* is vertical, the body arrives at that line the fastest by descending on the line inclined at 45 degrees to the horizontal [on setting $n = 0$]. Therefore a body inclined at this angle to the horizontal advances the quickest.

Scholium.

132. In a like manner it can be found too, which of an infinite number of similar curves *AM, AM* (Fig. 18), a body can arrive at a given curve by descending the fastest. For if the line *GMH* should be any kind of curve touching the curve *CMM* at *M*, the body on this curve *AM* arrives the quickest at the curve *GMH*, if indeed the whole of the curve *GMH* is placed beyond the curve *CMM*. Also in the same way it might be possible to be determined, if the curves *AM, AM* are not similar, how the above body can arrive the fastest at a given line *GH*. Indeed from the infinity of curves *CMM* the isochronous arc from those being cut is the one sought, which touches the given curve *GMH*, and it is on that curve *AM*, which passes through that point of contact, it is that point which is sought. But since in these cases generally the curves *CMM* are to be found with difficulty and it is much more difficult to determine that curve which is a tangent to the given line, [p. 61] we have restricted the question to similar curves only.

PROPOSITION 16.

Theorem.

132. *The times of the descent, by which a body placed on the curves traverses the curves AM and Am, etc.* (Fig.20) *similar and similarly from the point A, are in the ratio of the square root of the homologous sides.*

Demonstration.

Since the curves *AM*, *Am* are similar, *AM:Am*, *AP:Ap* and *PM:pm* are in a given ratio, clearly that, which the homologous sides hold; let the ratio of the homologous sides be *N:n*. Because the speed at *M* is to the speed at *m* as $\sqrt{AP}$ to $\sqrt{Ap}$, the speeds at *M* and *m* are in the square root ratio of the homologous sides. Now similar elements are taken from *M* and *m*, clearly holding the ratio *N* to *n*, are the times in which these two elements are traversed, in the ratio composed from the direction of the elements, i. e. *N* to *n*, and to the reciprocals of the speeds, i. e. $\sqrt{N}:\sqrt{n}$. From which it follows that the times, in which the homologous elements of the curves *AM*, *Am* are traversed, are in the square root ratio

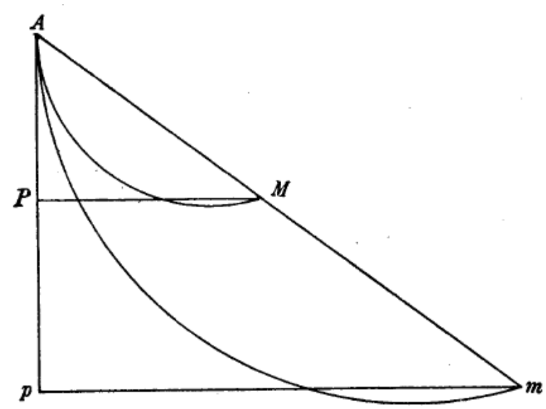


Fig. 20.

of the homologous sides. Whereby, since this ratio is constant, the times in which the whole curves *AM* and *Am* are traversed, keep this same ratio. Q.E.D. [p. 62]

Corollary 1.

133. Therefore the times, in which similar and similarly circular arcs put in place for the descent, are in the square root ratio of the radii.

Corollary 2.

134. Therefore pendulums, which describe similar arcs, complete oscillations in times which are in the square root ratio maintained by the lengths of the pendulums.

Corollary 3.

135. The same ratio of the times is kept in place, if the pendulum bodies do not describe circular orbits, but other curves, provided these are similar to each other and they complete similar arcs.

Scholium.

136. Moreover in these all the forces acting we put to be uniform and to be pulling downwards, even if we have disregarded this condition. For we have put this hypothesis in place to be acted on previously, as we are now about to progress to others.

PROPOSITION 17.

Problem.

137. *With the force present acting uniformly downwards, a body moves on some curve AM* (Fig.21) *with an given initial speed at A; to determine the motion of the body on this curve and the force pressing the body to the curve sustained at individual points.* [p. 63]

Solution.

With the force acting put as g and with the initial speed at A corresponding to the height b and as well, $AP = x$, $PM = y$, $AM = s$ and with the speed at M corresponding to the height v, with these in place, there is $dv = gdx$ (93), and thus $v = b + gx$. And again the time in which the arc AM is completed is equal to $\int \frac{ds}{\sqrt{(b+gx)}}$.

Furthermore, the total force experienced by the curve along the direction of the normal MN, is equal to (93)

$$\frac{gdy}{ds} + \frac{2vdxddy}{ds^3} = \frac{gdy}{ds} + \frac{2(b+gx)dxddy}{ds^3}$$

with the element dx present constant. For this solution only differs from that solution for proposition 13, because there it was $v = gx$, here it is $v = b + gx$. From these formulae therefore the motion as well as the pressing force are known. Q.E.I.

[We note that this is not the reaction force, as that force acts on the body, and which is the force we would now calculate. For some reason, Euler persists with the force the body exerts on the curve throughout this book.]

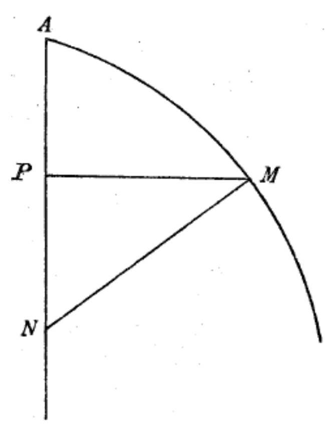


Fig. 21.

Corollary 1.

138. If the line *AM* is straight, now from (88) it is understood that the time to traverse *AM* is to the time to traverse *AP* beginning with the same speed $\sqrt{b}$, is as *AM* to *AP*. Now on account of the centrifugal force vanishing, the force pressing the body on the curve is equal to $\frac{gdy}{ds}$ or constant.

Corollary 2.

139. Also it appears in this case, since the motion does not start from rest, that the speed only depends on the height. Whereby, whatever the curve *AM* shall be, the speed of the body at any point of this is known, also with the kind of curve unknown. [p. 64]

Example 1.

140. Let the curve *AM* be a parabola, having the vertex at *A* and the axis *AP* vertical; hence with the parameter of this is put equal to a , $y^2 = ax$ and

$$dy = \frac{adx}{2\sqrt{ax}} \text{ and } ds = \frac{dx\sqrt{(a^2 + 4ax)}}{2\sqrt{ax}}.$$

Hence the time obtained for the body to pass along *AM* is equal to

$$\int \frac{dx\sqrt{(a+4x)}}{2\sqrt{x(b+gx)}}.$$

Hence, as with *dx* kept constant, then $ddy = \frac{-adx^2}{4x\sqrt{ax}}$, and we have

$$\frac{dxddy}{ds^3} = \frac{-2a}{(a+4x)\sqrt{(a^2+4ax)}}.$$

Consequently the total pressing force is equal to

$$\frac{ga}{\sqrt{(a^2+4ax)}} - \frac{4a(b+gx)}{(a+4x)\sqrt{(a^2+4ax)}} = \frac{ga^2-4ab}{(a+4x)\sqrt{(a^2+4ax)}}.$$

Corollary 3.

141. Therefore if $b = \frac{ga}{4}$, then the force on the curve vanishes. Thus in this case the body is free to move along this parabola, which is also the case treated in the preceding book (564).

Corollary 4.

142. Therefore with the present $b = \frac{1}{4}ga$, the time to traverse the arc *AM* is equal to

$$\int \frac{dx}{\sqrt{gx}} = \frac{2\sqrt{x}}{\sqrt{g}}.$$

Therefore this is equal to the time to descend beginning from rest along the abscissa *AP* .

Corollary 5.

143. If $b > \frac{1}{4}ga$, then the force becomes negative; then the curve is therefore pressed in the direction away from the axis *AP*. But if $b < \frac{1}{4}ga$, the direction of the force is along *MN*. The size of the force pressing at the individual points of the curve varies inversely as the radius of osculation. [p. 65]

Example 2.

144. If the curve *AM* is a circle, the radius of which is equal to *a* and the centre is placed on the vertical line *AP*, then it is given by $y^2 = 2ax - x^2$, hence

$$dy = \frac{adx - xdx}{\sqrt{(2ax - x^2)}} \text{ and } ds = \frac{adx}{\sqrt{(2ax - x^2)}}.$$

Hence the time in which the arc *AM* is traversed is equal to

$$\int \frac{adx}{\sqrt{(2ax - x^2)(b + gx)}}.$$

And since $\frac{dxddy}{ds^3} = \frac{-1}{a}$, the force that the circle undergoes at a point is equal to

$$g - \frac{gx}{a} - \frac{2(b + gx)}{a} = g - \frac{3gx}{a} - \frac{2b}{a}.$$

Corollary 6.

145. The time can be expressed by logarithms, if *b* = 0; moreover it is equal to infinity , or the body perpetually remains at *A*. That is apparent from the above treatment (97). For since the normal to the curve at *A* is *AP* and since neither is the radius of osculation infinitely small, then the body cannot descend.

Corollary 7.

146. If $b = \frac{ga}{2}$ or the initial speed is the same size as the body acquires in falling from a height of half the radius of the circle, the total force acting with the centrifugal force is equal to $\frac{3gx}{a}$; and thus it is in proportion to the height travelled through.

DEFINITION 3.

147. *Oscillatory motion is reciprocal motion in which the body alternately approaches and recedes from the starting point of the motion M* (Fig. 22). *Thus if the body is moving on a given curve MAN* [p. 66], *first it descends on MA, then it ascends on AN, while it loses speed;*

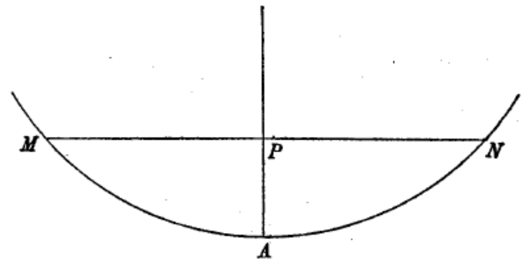


Fig. 22.

then it descends from N again and ascends on the arc AM, with which done it again descends and this periodic motion continues. Such motion is called oscillatory.

Corollary 1.

148. Oscillatory motion hence consists of alternate descents and ascents on a given line; and in the descending motion the body moves with an acceleration, and in the ascents the body truly loses the speeds acquired.

Corollary 2.

149. Hence whatever the previous ascent touched upon, the descent is made on the same part of the curve. Whereby, since the speed of the body depends only on the height in a vacuum, the body at the same point on the curve either in the ascent or in the descent has the same speed.

Corollary 3.

150. From which it follows that the time for the descent along *MA* is equal to the time of the descent along *AM* and likewise in the same manner the time of the ascent along *AN* is equal to the time of the descent along *NA*.

Corollary 4.

151. The body ascending on the arc *AN* until it reaches the point *N*, since the height is equal to that of the point *M* from which it fell. The one follows from the other, since the speed is determined only by the height. [p. 67]

Corollary 5.

152. If the curve *AN* is similar and equal to the curve *AM*, then the motion along *AN* is equal to the motion along *AM*. Whereby all the ascents and descents are made in equal s times.

Corollary 6.

153. If the curves *MA* and *AN* are dissimilar, at least the time along *MAN* is equal to the time along NAM, or the times of approaching and receding are equal to each other.

Corollary 7.

154. Since the body always reaches the same height, clearly this oscillatory motion must last indefinitely.

Corollary 8.

155. Hence any curve is suitable for the production of oscillatory motion if it has two arcs such as *MAN* ascending from the lowest point A.

Scholium 1.

156. Here we have set out the properties of oscillatory motion, such as follow from the exposition of the hypothesis of uniform forces acting, and always pulling downwards. Indeed the same is true also, if the forces depend in some manner on the height, or even if they are directed towards a fixed point, that becomes more apparent in what follows. In a medium with resistance, truly the matter is otherwise, for neither is the ascent along a given curve similar to descent along the same, nor in the ascent does the body reaches an height equal to that from which in the descent it had fallen.

Scholium 2.

157. It is usual to call the motion along *MAN* the going movement, following the returning movement along *NAM* ; hence oscillatory motion consists of alternate goings and comings [we do not have such handy words as the Latin *itus* and *reditus* used here in the English language to express this notion, though to and fro' might be our equivalent]. Truly an oscillation is called by others constant to and fro' motion, as the [term] oscillation is called by others, to and fro'. Here we accept the name oscillation in the basic sense, so that thus one oscillation is agreed to be one to and fro' motion. The to motion and indeed the fro' motion each consists of one descent and one ascent, and thus the whole oscillation includes two descents and two ascents. Therefore since the time for the to motion is the same as the time for the fro' motion, the time of one oscillation is double the time for one to or one fro' motion.

[This may sound pedantic, but that is not the case, as previous writers incl. Newton and Huygens, had timed pendulum swings from one extreme to the other, and not back to the starting point, as is the more logical thing to do. Thus again we see the hand of Euler gently guiding humanity in the right direction.]

Corollary 9.

158. Therefore in this chapter, in which motion is a vacuum is undertaken, if we wish to examine the motion of oscillations, we have a need to consider only either the ascent or the descent upon the two parts of the curve *AM*, *AN*.

Scholium 3.

159. Nothing matters, provided the arcs *AM* and *AN* in succession make one curve, but if they are different curves, then they are connected at A thus so that they have a common tangent [p. 69]; for otherwise the motion is disturbed. Whereby there is only the need in an inquiry about oscillatory motion to define the motion on the curves *AM* and *AN* themselves. This then is sufficient in the determination of oscillations, as the relation between larger and smaller oscillations can then be found. Moreover these oscillations are called larger which are completed by larger arcs, and the smaller by lesser arcs.

Scholium 4.

160. It is evident from Proposition 6 (49), how oscillations are able to be effected with the help of pendulums, clearly with the aid of the evolute of the curves AM and AN, around which the thread is taken. Also the use of pendulums was adapted to oscillations by Huygens, as is apparent from his habit of applying that motion to the perfection of clocks. Truly the same difficulties that we have mentioned in the place cited, they have here in this place. On account of which we only investigate the motion of points upon given lines, and we lead the mind away from all the circumstances of pendulums which are able to disturb our intention.

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 39]

PROPOSITIO 12.

Problema.

83. *Sollicitetur corpus, quod super curva AM* (Fig.12) *movetur, ubique a potentia MF, cuius directio sit parallela axi AP; determinare celeritatem corporis in singulis punctis atque tempus, quo curvae quaevis portio describatur, nec non pressionem, quam curva in singulis punctis patitur.*

Solutio.

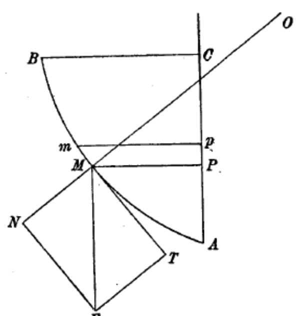


Fig. 12.

Descripserit corpus iam arcum *AM* sitque eius celeritas in *A* debita altitudini *b* atque celeritas in *M* debita altitudini *v*. Positis nunc $AP = x$, $PM = y$ et arcu $AM = s$ resolvatur potentia *MF*, quae sit *p*, in laterales, normalem scilicet MN et tangentialem MT; erit
$ds : dx = MF : MT$ et $ds : dy = MF : MN$.

Hinc igitur prodibit vis tangentialis $MT = \frac{pdx}{ds}$ et vis normalis $MN = \frac{pdy}{ds}$. Perspicuum hic est ergo $dv = -pdx$ and $v = C - \int pdx$ (42). Sumto autem integrali $\int pdx$ ita, ut evanescat posito x = 0, erit $v = b - \int pdx$; [p. 40]
ex qua aequatione corporis celeritas in singulis punctis cognoscitur. Ex eadem aequatione innotescit quoque tempus, quo arcus AM absolvitur; posito enim tempore *t* erit

$$t = \int \frac{ds}{\sqrt{(b - \int pdx)}}.$$

Vis normalis $MN = \frac{pdy}{ds}$ tota impenditur in curvae pressionem secundum MN (39), augebit ergo pressionem a vi centrifuga ortam, quia MN in oppositam radii osculi MO plagam cadit. Quare, cum posito radio osculi MO = r vis centrifuga sit = $\frac{2v}{r}$ (20), erit totalis pressio in curvam iuxta $MN = \frac{pdy}{ds} + \frac{2v}{r}$. Q.E.I.

Corollarium 1.

84. Celeritas in *M* igitur tanta est, quanta foret in *P*, si corpus eadem celeritate initiali $\sqrt{b}$ per *AP* eadem in singulis altitudinibus potentia *p* sollicitatum ascendisset.

Corollarium 2.

85. Celeritas igitur non pendet a natura curvae, sed tantum ab altitudine, quam corpus percurrit. Si nimirum altitudinis elementum fuerit *dx*, erit $dv = -pdx$ vel $dv = pdx$, prout corpus vel ascendit vel descendit.

Corollarium 3.

86. Cum sit $v = b - \int pdx$, si sumatur abscissa *x* tanta uti *AC*, pro qua sit $\int pdx = b$, erit corporis in illa altutudine *B* celeritas = 0. Corpus igitur in *B* usque ascendit ibique quiescit; continuo vero ex *B* descendet per *BMA*. [p. 41]

Corollarium 4.

87. Si ascensus per *AMB* cum ascensu rectilineo per *APC* comparetur, erit tempus per elementum *Mm* ad tempus per *Pp* ut *Mm* ad *Pp*, i. e. ut *ds* ad *dx*.

Corollarium 5.

88. Quare si linea *AMB* fuerit recta, ob rationem *Mm* ad *Pp* constantem erit tempus per *AM* ad tempus per *AP* in constanti ratione, nempe ea, quam habet sinus totus ad cosinus anguli *A*, seu quam habet longitudo *AB* ad *AC*.

Corollarium 6.

89. Posito elemento *Pp* constante est radius osculi $r = \frac{-ds^3}{dxddy}$ ideoque vis centrfuga =

$$-\frac{2vdxddy}{ds^3} = \frac{-2(b-\int pdx)dxddy}{ds^3}.$$

Quare pressio totalis erit =

$$\frac{pds^2dy - 2(b-\int pdx)dxddy}{ds^3}.$$

Scholion 1.

90. Quemadmodum in hoc problemate ex datis curva et potentia sollicitante inventa sunt celeritas in singulis punctis, tempus per quemvis arcum et pressio in singula curvae puncta, ita ex harum quinque rerum duabus quibusque datis reliquae tres possunt inveniri. Ex quo decem nascentur problemata quae omnia solutionem ex huius problematis solutione habebunt. [p. 42]

Scholion 2.

91. Similiter habebuntur decem huiusmodi quaestiones, si directiones potentiae sollicitantis non fuerint parallelae, sed vel convergentes ad centrum virium vel alio modo determinatas directiones habentes. At si etiam directio inter quaesita ponatur, tunc ob sex res in computum ducendas ex ternis quibusque reliquae tres invenientur; hincque viginti orientur problemata.

Scholion 3.

92. Orientur porro problemata indeterminata, ut si loco temporis per quamvis curvae portionem tantum integrum tempus per *AMB* daretur; tum enim infinitae solutiones locum haberent. Praeterea si plures descensus vel ascensus integrari considerentur super eiusdem curvae variis partibus eorumque ratio detur, numerus quaestionum multo magis augebitur. Ad hoc genus pertinet quaestio de invenienda curva, super qua omnes descensus ad datum punctum fiant eodem tempore, quas tanquam difficillimas ultimo pertractabimus. Nunc autem primum curvam et potentiam sollicitatem tanquam datas accipiemus et problemata eo pertinentia solvemus. Deinceps vero ex aliis datis quemmadmodum reliqua sint invenienda, monstrabimus. [p. 43]

PROPOSITIO 13.

Problema.

93. *Si potentia sollicitans fuerit uniformis et ubique deorsum tendat, determinare descensum corporis super data curva AM* (Fig.13) *in A ex quiete incipientem atque pressionem, quam curva in singulis punctis M sustinet.*

Solutio.

Ducta verticali *AP* seu parallela directionibus potentiae *MF* atque applicata rectangula *MP* sit $AP = x$, $PM = y$, curva $AM = s$. Ponatur potentia $MF = g$ existente vi gravitatis = 1 et celeritas in *M* debita altitudini *v*. His positis erit vis normalis = $\frac{gdy}{ds}$ et vis tangentialis = $\frac{gdx}{ds}$ (83). Quia hoc casu vis tangentialis accelerat, erit $dv = gdx$ et $v = gx$ ob celeritatem in $A = 0$. Deinde quia radius osculi in *MO* directus est = $\frac{+ds^3}{dxddy}$ posito *dx* constante, erit vis centrifuga = $\frac{+dvdxddy}{ds^3}$, cuius directio est *MN*. Secundum eandem plagam vero premit vis normalis $\frac{gdy}{ds}$. Quare tota pressio, quam curva in *M* sustinet secundum *MN*, est =

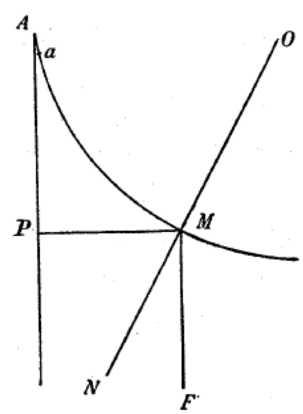


Fig. 13.

$$\frac{gdy}{ds} + \frac{2vdxddy}{ds^3} = \frac{gdy}{ds} + \frac{2gxdxddy}{ds^3}$$

ob $v = gx$. Tempus vero, quo corpus arcum AM percurret, est = $\int \frac{ds}{\sqrt{gx}}$. Q.E.I. [p. 44]

Corollarium 1.

94. Celeritas igitur in *M* tantum ab altitudine *AP*, per quam descendit, pendet atque tanta est, quantam idem corpus ex *A* in *P* delapsum et ab eadem potentia *g* sollicitatum acquiret.

Corollarium 2.

95. In quacunque igitur curva corpus a potentia uniformi *g* sollicitatum ex quiete descendat, celeritates erunt radicibus a quadratis ex altitudinibus percursis proportionales; est enim celeritas ut $\sqrt{v}$, i. e. ut $\sqrt{gx}$.

Corollarium 3.

96. Tempus, quo primum elementum *Aa* percurritur, est $\int \frac{ds}{\sqrt{gx}}$ evanescente *x*. Si igitur angulus *PAa* fuerit recto minor seu $s = nx$, erit tempus per *Aa* infinite parvum ideoque tempus *AM* finitum, nisi curva vel ascendat inter *A* et *M* supra *A* vel in infinitum progrediatur. At si angulus *PAa* fuerit rectus, erit ipso puncto *A* $s^n = ax$ existente n numero unitate maiore ideoque

$$\sqrt{gx} = s^{\frac{n}{2}}\sqrt{\frac{g}{\alpha}} \quad \text{et} \quad \int \frac{ds}{\sqrt{gx}} = \frac{2s^{\frac{2-n}{2}}}{2-n}\sqrt{\frac{\alpha}{g}}.$$

Quare si *n* fuerit binario minor, tempus per *Aa* erit infinite parvum et tempus per *AM* finitum. At si *n* = 2 vel > 2, tempus per primum elementum *Aa* erit infinite magnum seu corpus ex *A* nunquam egredietur. [p. 45]

Corollarium 4.

97. Quoties autem *n* < 2, toties radius osculi in *A* est infinite parvus. Quare in casu, quo tangens curvae in *A* ad *AP* est normalis, corpus non descendet, nisi radius osculi in *A* fuerit infinite parvus.

Scholion 1.

98. Ex quo, quod primum elementum tempore infinite parva percurritur, recte concluditur tempus per arcum *AM* esse finitum; cum enim corpus motu accelerato per *AM* descendat, multo celerius sequentia elementa describentur et hanc ob rem tempus debebit esse finitum. Exemplis autem sequentibus omnia illustrabuntur.

Exemplum 1.

99. Sit linea *AM* (Fig. 14) recta utcunque inclinata ad verticalem *AP* atque cosinus ang. $A = n$; erit $x = ns$. Tempus ergo, quo corpus per *AM* descendit, erit =

$$\int \frac{ds}{\sqrt{gns}} = \frac{2\sqrt{s}}{\sqrt{gn}} = \frac{2\sqrt{AM}}{\sqrt{gn}} = \frac{2AM}{\sqrt{g \cdot AP}}$$

seu tempus per lineam utcunque inclinatam est directe ut radix ex ipsa linea et inversa ut radix ex cosinu anguli inclinationis *MAP*. Vis centrifuga erit autem = 0, quare linea *AM* tantum a vi normali premitur, quae est =

$$g\sqrt{(1-n^2)} = \frac{g \cdot PM}{AM}.$$

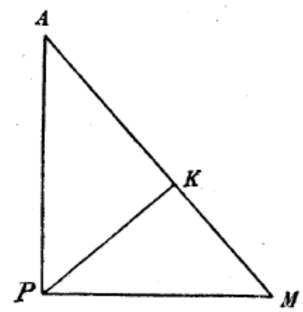


Fig. 14.

Corollarium 5.

100. Tempus ergo per *AM* est ad tempus per *AK* ut $\sqrt{AM}$ ut $\sqrt{AK}$. At tempus per *AM* est ad tempus per *AP* ut *AM* ad *AP* (88). [p. 46] Quare si fuerit $AM : AP = \sqrt{AM} : \sqrt{AK}$ seu $AM : AP = AP : AK$, quod evenit, si *PK* est in *AM* perpendicularis, tum tempus descensus per *AK* aequale est tempori descensus per *AP*.

Corollarium 6.

101. Patet etiam tempus descensus per perpendiculum *PK* aequales esse tempori descensus per *AP*. Est enim cosinus anguli *APK* = $\frac{PK}{AP}$. Quare, cum sit tempus per *AP* ad tempus per *KP* ut $\frac{\sqrt{AP}}{\sqrt{1}}$ ad $\sqrt{KP} : \sqrt{\frac{PK}{AP}}$, erit haec ratio aequalitatis.

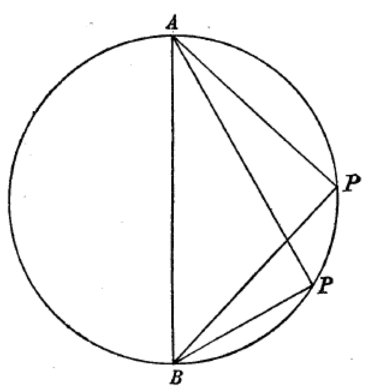


Fig. 15.

Corollarium 7.

102. Ex hoc perspicitur in circulo *APPB* (Fig. 15) omnes descensus per chordas *AP* ex puncto supremo *A* ductus nec non omnes descensus per chordas ad punctum infimum *B* ductus aequalibus fieri temporibus, eo scilicet tempore, quo corpus per diametrum *AB* perpendiculariter delabitur.

Exemplum 2.

103. Si curva *AMB* (Fig. 16) fuerit circulus ac radius $BC = a$ et *AP* tangat circulum, erit

$$(a-y)^2 + x^2 = a^2 \quad \text{seu} \quad y = a - \sqrt{(a^2 - x^2)}.$$

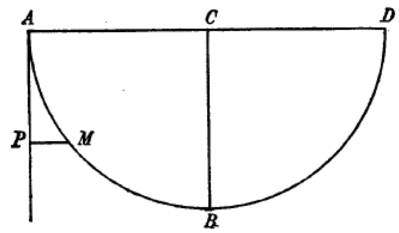


Fig. 16.

Habebitur ergo $ds = \frac{adx}{\sqrt{(a^2-x^2)}}$ et ob $v = gx$ et $r = a$ erit vis centrifuga = $\frac{2gx}{a}$ atque ob $dy = \frac{xdx}{\sqrt{(a^2-x^2)}}$ tota pressio, quam circulus in *M* sustinet, = $\frac{3gx}{a}$. Triplo igitur maior est tota pressio quam sola vis normalis.

Tempus deinde, quo arcus *AM* percurritur, est = $\int \frac{adx}{\sqrt{g(a^2x-x^3)}}$, [p. 47] cuius integratio neque a circuli nec hyperbolae quadratura pendet, sed ope rectificationis curvae elasticae construi potest. Tempus interim per quadrantem *AB* est =

$$2\sqrt{\frac{a}{g}} \times \left(1 + \frac{1}{2\cdot 5} + \frac{1\cdot 3}{2\cdot 4\cdot 9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13} + \text{etc.}\right).$$

Corollarium 8.

104. Cum corpus ad infimum punctum *B* pervenerit, ibi habebit celeritatem altitudini *ga* debitam. Hac igitur ascendet in altero quadrante *BD* pertingetque ad *D*, ubi eius celeritas evanescet, ideoque rursus descendet ad *B* tumque ad *A* per *BA* reascendet. Similis vero erit ascensus descensui per quadrantem, quia corpus, sive ascendat sive descendat, in iisdem punctis eandem habet celeritatem.

Scholion 2.

105. Alia exempla non afferimus, cum in sequentibus, ubi plures descensus ad punctum fixum super data linea considerabimus, plura simus allaturi. Nunc vero primum eas quaestiones evolvemus, quae pertinent ad motum super data linea ex dato puncto fixo a quiete inceptum, cuius modi est problema sequens.

PROPOSITIO 14.

Problema.

106. *Si fuerit infinitae curvae similes AM, AM, etc.* (Fig.17) *ex puncto fixo A initium sumentes, invenire curvam CMM ab illis curvis arcus AM, AM, etc. abscindentem, qui a descendente super iis corpore aequalibus temporibus percurrantur existente ut ante potentia sollicitante uniformi et ubique deorsum directa.* [p. 48]

Solutio.

Ex infinitis curvis datis sumatur una quaecunque *AM*, cuius parameter sit *a*. Positioque $AP = x$, $PM = y$ et arcu $AM = s$ et existente ut ante potentia sollicitante $= g$ descendat corpus super curva *AM*; erit celeritas in *M* debita altitudini *gx*. Tempus ergo descensus super *AM* erit $= \int \frac{ds}{\sqrt{gx}}$. Ab omnibus ergo curvis AM, AM etc. tanti arcus sunt abscindendi, ut pro iis sit $\int \frac{ds}{\sqrt{gx}}$ quantitas constans. At $\int \frac{ds}{\sqrt{gx}}$ ad alias curvas referetur, si praeter *s* et *x* etiam parameter *a* ponatur variabilis. Posito igitur in $\int \frac{ds}{\sqrt{gx}}$ etiam *a* variabili quantitas $\int \frac{ds}{\sqrt{gx}}$ ponenda est = constanti, nempe ei tempori, quo omnes descensus fieri debent. Sit hoc tempus = *k*, erit $k = \int \frac{ds}{\sqrt{gx}}$ in singulis curvis. Quare si $\int \frac{ds}{\sqrt{gx}}$ ita differentietur, ut etiam *a* variabile ponatur, hoc differentiale nihilo aequale est ponendum. Ad hoc differentiale inveniendum sit $ds = pdx$ eritque *p*, quia omnes curvae ponuntur similes, functio, in qua *a* et *x* nullum dimensionum numerum simul consituunt. [p. 49]Habebimus ergo $\int \frac{pdx}{\sqrt{gx}}$; hoc differentiatum posito quoque *a* variabili dabit

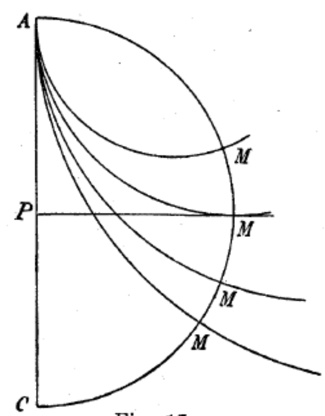


Fig. 17.

$$\frac{pdx}{\sqrt{gx}} + qda,$$

quod fieri debet = 0. Quantitas *q* vero sequenti modo invenietur. Quia est $k = \int \frac{pdx}{\sqrt{gx}}$, in quantitate *k* variabiles a et *x* dimensionum numerum constituent $\frac{1}{2}$. Ostendi autem alibi, in Tom. VII Comment [E044 : *De infinitis curvis eiusdem generis. Seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis,* Commen. acad. sc. Petrop. 7 (1734/5) 1740. Legetur ibi, p.185 : '*Sin vero fuerit u functio m dimensionum ipsarum a et x atque du = Rdx + Sda, erit Rx + Sa = mu*'], tum fore

$$\frac{px}{\sqrt{gx}} + qa = \frac{k}{2}.$$

Ex quo invenitur

$$q = \frac{k}{2a} - \frac{p\sqrt{x}}{a\sqrt{g}}.$$

Habebitur ergo

$$\frac{pdx}{\sqrt{gx}} + qda = \frac{pdx}{\sqrt{gx}} + \frac{kda}{2a} - \frac{pda\sqrt{x}}{a\sqrt{g}} = 0$$

quae est aequatio pro curva quaesita. At si aequatio inter coordinatas *x* et *y* pro curva *CMM* desideretur, ex aequatione pro quaque curvarum *AM* valor ipsius *a* in *x* et *y* inventus substituti debet. Q.E.I.

Corollarium 1.

107. Aequatio etiam primo inventa

$$\frac{pdx}{\sqrt{gx}} = \frac{pda\sqrt{x}}{a\sqrt{g}} - \frac{kda}{2a}$$

sufficit ad curvam *CMM* inveniendam. Nam pro quavis abscissa $AP = x$ ex ea invenitur *a* parameter eius curvae *AM*, cuius punctum *M* respondens assumtae abscissae *x* est in curva quaesita *CMM*.

Corollarium 2.

108. Cum autem haec aequatio sit differentialis ideoque ad plures curvas pro constante, quae adiicitur, pertineat, notandum est in additione constantis eam tantum solutioni esse convenientem, quae pro data curva seu pro dato ipsius *a* valore det abscissam *x* tantum arcum *AM* abscindentem, qui tempore *k* descensu absolvatur. [p. 50]

Corollarium 3.

109. Si tempus *k* aequale esse debeat tempori descensus per verticalem $AC = b$, erit $k = \frac{2\sqrt{b}}{\sqrt{g}}$. Quo valore substituto habebitur aequatio

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a}.$$

In cuius integratione id est faciendum, ut curva per punctum *C* transeat.

Scholion 1.

110. Erit autem semper recta verticalis *AC* species curvarum *AM*; quae oritur, si parameter a vel infinite magna vel infinite parva accipiatur. Quare commodissime tempus constans *k* per descensum per verticalem *AC*, quippe speciem curvarum *AM*, exprimitur. Atque in constructione aequationis inventae

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a}$$

tanta constans est addenda, ut posito $x = b$ fiat a vel infinitum vel nihil, prout ille vel iste valor ipsius a rectae *AC* respondeat.

Scholion 2.

111. Si $\frac{ds}{\sqrt{gx}}$ reipsa potest integrari, ne data quidem aequatione opus est, ad quam inveniendam opus fuit q determinare. Nam si integrale ipsius $\frac{ds}{\sqrt{gx}}$ iterum differentietur positio quoque a variabili, reipsa obtenitur q; atque hoc differentiale tantum nihilo aequale esset ponendum. Commodissime vero his casibus problema solvetur, si integrale ipsius $\frac{ds}{\sqrt{gx}}$ statim ipsi k vel $\frac{2\sqrt{b}}{\sqrt{g}}$ aequale ponatur et loco a eius valor in x et y substituatur ex aequatione pro curvis datis. [p. 51] Atque hoc modo solutio in promtu est non solum pro curvis similibus, sed dissimilibus etiam, si modo tempora descensus per quantitates finitas exprimi possunt.

Exemplum 1.

112. Si omnes hae curvae *AM* fuerint rectae diversimode ad verticalem *AC* inclinatae, erit

$$y = nx \quad \text{et} \quad s = x\sqrt{(1+n^2)},$$

ubi n tanquam parameter est consideranda. Erit ergo

$$\int \frac{ds}{\sqrt{gx}} = \int \frac{dx\sqrt{(1+n^2)}}{\sqrt{gx}} = \frac{2\sqrt{x(1+n^2)}}{\sqrt{g}},$$

quod aequale poni debet ipsi $\frac{2\sqrt{b}}{\sqrt{g}}$. Erit itaque

$$x(1+n^2) = b.$$

Cum autem n sit quantitas variabilis, ponatur pro ea valor $\frac{y}{x}$ ex aequatione $y = nx$; quo facto prodibit pro curva *CMM* aequatio inter coordinatas orthogonales x et y ista

$$y^2 + x^2 = bx,$$

quae est pro circulo, cuius diameter est recta $AC = b$.

Scholion 3.

113. Hic casus est ille ipse casus ante pertractatus (102); ibi enim ostensum est corpus per omnes chordas in circulo ex puncto supremo eductas aequalibus temporibus descendere. Pertinet hic quidem casus non ad curvas similes; sed hoc exemplum attulimus ad casum scholii 2 illustrandum, quia pro rectis hisce tempora descensus finitis quantitatibus exprimuntur. Sequentia exempla vero curvas similes, uti propositio postulat, complectentur. [p. 52]

Exemplum 2.

114. Sint curvae *AM, AM* omnes circuli tangentes verticalem *AC* in *A*. Ponatur radius cuiusque eorum = *a*, erit

$$y = a - \sqrt{(a^2 - x^2)} \quad \text{atque} \quad a = \frac{y^2 + x^2}{2y}.$$

Hi circuli vero omnes sunt curvaqe similes, quia *a, y* et *x* in aequatione eundem dimensionum numerum tenent seu homogeneitatem complent sola. Radius igitur a tanquam parameter variabilis debet tractari. Habetur autem ex illa aequatione $ds = \frac{adx}{\sqrt{(a^2 - x^2)}}$, quare erit $p = \frac{a}{\sqrt{(a^2 - x^2)}}$ ideoque praescriptam habet proprietatem, ut *a* et *x* dimensionum numerus sit nullus. Hanc ob rem pro curva *CMM* haec habebitur aequatio

$$\frac{adx}{\sqrt{(a^2x - x^3)}} = \frac{da\sqrt{x}}{\sqrt{(a^2 - x^2)}} - \frac{da\sqrt{b}}{a}$$

seu haec

$$\frac{da\sqrt{b}}{a} = \frac{xda - adx}{\sqrt{(a^2x - x^3)}}.$$

Quae aequatio construi potest; posito enim $x = au$ prodit

$$\frac{da\sqrt{b}}{a\sqrt{a}} = \frac{-du}{\sqrt{(u - u^3)}},$$

in qua indeterminatae sunt a se invicem separatae. Quo autem aequatio inter coordinatas *x* et *y* pro curva *CMM* obtineatur, ponatur loco *a* valor $\frac{y^2 + x^2}{2y}$ et loco *da* eius differentiale $\frac{y^2dy + 2yxdx - x^2dy}{2y^2}$. Quibus substitutis sequens prodit aequatio differentialis

$$-xdy + ydx = \frac{(y^2dy + 2yxdx - x^2dy)\sqrt{bx}}{y^2 + x^2}.$$

Quae ita integrari debet, ut posito $x = b$ fiat $y = 0$, quia curva per punctum *C* transire debet.

Corollarium 4.

117. Ex hac aequatione tangens curvae *CMM* in singulis punctis cognoscitur et ex positione tangentis innotescit angulus *AMM*, [p. 53] quo curva *CMM* quamlibet datarum intersecat. Erit scilicet tangens anguli $AMM = \frac{y}{x - \sqrt{bx}}$. Hic ergo angulus est rectus in *C* ob $x = b$, seu curva *CMM* in *C* ad *AC* est normalis.

Corollarium 5.

116. Si b vel maior vel minor accipiatur, curva *CMM* alia quoque erit hocque modo infinitae orientur curvae a circulis arcus isochronos abscindentes. Haecque curvae omnes inter se erunt similes ob parametrum b, quae in aequatione cum x et y homogeneitatem constituit. Data ergo una curva *CMM* innumerabiles aliae ex ea construi possunt, abscissis scilicet et applicatis curvae *CMM* in eadem ratione augendis vel diminuendis, in qua *AC* seu b augetur vel diminuitur.

Exemplum 3.

115. Sint curvae *AM*, *AM* omnes cycloides cuspides in A habentes et tangentes verticalem *AC* in A. Posita parametro cuiusque cycloidis *AM* seu dupla diametro circuli generatoris $= a$ erit ex natura cycloidis $s = a - \sqrt{(a^2 - 2ax)}$ atque $ds = \frac{adx}{\sqrt{(a^2-2ax)}}$ hincque $dy = \frac{dx\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$. Hoc ergo casu est $p = \frac{a}{\sqrt{(a^2-2ax)}}$ functio ipsarum a et x nullius dimensionis, ut requiritur. Quare pro curva *CMM* reperitur ista aequatio

$$\frac{adx}{\sqrt{(a^2x-2ax^2)}} = \frac{da\sqrt{x}}{\sqrt{(a^2-2ax)}} - \frac{da\sqrt{b}}{a}$$

seu

$$\frac{xda-adx}{\sqrt{(a^2x-2ax^2)}} = \frac{da\sqrt{b}}{a}.$$

[p. 54] Si aequatio inter coordinatas orthogonales x et y desideretur, ex aequatione

$$y = \int \frac{dx\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$$

seu huius differentiali posito quoque a variabili valor ipsius a debet substitui. Haec vero aequatio differentiata posito q quoque variabili dat

$$ady - yda = \frac{adx\sqrt{2ax} - xda\sqrt{2ax}}{\sqrt{(a^2-2ax)}}$$

seu

$$\frac{ady-yda}{\sqrt{2a}} = \frac{axdx - x^2da}{\sqrt{(a^2x-2ax^2)}}.$$

Quae abit in hanc

$$\frac{ady-yda}{a^2\sqrt{2}} = \frac{axdx - x^2da}{a^2\sqrt{(ax-2x^2)}}.$$

Superior vero per $\frac{1}{4\sqrt{a}}$ multiplicata praebet hanc

$$\frac{da\sqrt{b}}{4a\sqrt{a}} = \frac{axda - a^2dx}{4a^2\sqrt{(ax-2x^2)}}.$$

Hae duae aequationes additae dant aequationem integrabilem, cuius integralis est

$$\frac{y}{a\sqrt{2}} - \frac{\sqrt{b}}{2\sqrt{a}} = \frac{-\sqrt{(ax-2x^2)}}{2a}.$$

Ex qua valor ipsius *a* erutus fit

$$\sqrt{a} = \frac{y\sqrt{2b} \pm \sqrt{(2y^2x - 2bx^2 + 2x^3)}}{b-x}$$

et

$$\sqrt{(ax-2x^2)} = \frac{yx\sqrt{2} \pm \sqrt{(2by^2x - 2b^2x^2 + 2bx^3)}}{b-x}.$$

Quibus valoribus in aequatione

$$\frac{(xdy - ydx)\sqrt{a} + xdx\sqrt{2b}}{\sqrt{(ax-2x^2)}} = dy\sqrt{b},$$

quae oritur ex duabus differentialibus eliminato *da*, substitutis prodibit

$$\frac{xdy - ydx - bdy}{\sqrt{b}} = \frac{dx\sqrt{(y^2 - bx + x^2)}}{\sqrt{x}},$$

aequatio pro curva quaesit *CMM*.

Corollarium 6.

118. Ex hac aequatione invenitur tangens anguli, quem curva CM cum applicata PM constituit, nempe

$$\frac{dx}{dy} = \frac{-(b-x)\sqrt{x}}{y\sqrt{x} + \sqrt{(by^2 - b^2x + bx^2)}}.$$

Deinde etiam innotescit tangens anguli, [p. 55] quem cyclois *AM* cum applicata *PM* constituit. Ex aequatione cycloidis erit nimirum

$$\frac{dx}{dy} = \frac{\sqrt{(a^2 - 2ax)}}{\sqrt{2ax}} = \frac{\sqrt{(ax - 2x^2)}}{x\sqrt{2}}.$$

Eliminato vero *a* erit ista tangens =

$$\frac{y\sqrt{x} + \sqrt{(by^2 - b^2x + bx^2)}}{(b-x)\sqrt{x}}.$$

Quare, cum horum angulorum alter alterius sit complementum sumto illius deinceps posito, erit angulus, quem curva *CMM* cum qualibet datarum *AM* constut, rectus. Consequenter curva *CMM* est traiectoria orthogonalis omnium cycloidum datarum *AM*, *AM*, etc.

Corollarium 7.

119. Sumto *AC* alius magnitudinis aliae quoque curvae *CMM* prodibunt et sic infinitae traiectoriae orthogonales inveniuntur, quae omnes inter se sunt similes. Data ergo una facile, quotquot libuerit, construere licebit.

Scholion 4.

120. Omnes hae curvae arcus abscindentes isochronos, quaecunque fuerint curvae secandae, semper construi possunt, etiamsi id ex aequatione non appareat. Per quadraturas enim ex datis curvis arcus possunt abscindi, qui dato tempore descensu absolvantur, hocque modo puncta quotlibet curvae quaesitae inveniuntur. Si quidem curvae secandae sunt algebraicae, aequatio pro curva secante semper ita est comparata, ut factis debitis substitutionibus indeterminatae a se invicm possint separari. At si curvae secandae diffefererentiali aequatione exprimantur, [p. 56] aequatio differentialis pro curva secante rarissime separationem indeterminatarum admittit. Causa est, quod peculiari modo, quo in hoc cycloidum casu usus sum, parameter *a* eliminari debeat eaque substitutio ad separationem non deducat.

Scholion 5.

121. Deinde observandum est omnes curvas arcus isochronos abscindentes, quarum numerus pro vario ipsius *b* valore est infinitus, inter se similes esse, si quidem curvae secandae fuerint tales. Colligitur hoc ex generali aequatione

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{da\sqrt{b}}{a},$$

in qua, cum p sit functio ipsarum *a* et *x* nullius dimensionis, quantitates *a, b* et *x* homogeneitatem constituunt. At ex aequatione curvarum secandarum, quia in ea *a, x,* et *y* ubique eundem dimensionum numerum conficere ponuntur, valor ipsius *a* erit functio ipsarum *x* et *y* unius dimensionis. Quare eo substituto loco *a* habebitur aequatio pro curva secante, in qua *b, x,* et *y* ubique eundem dimensionum numerum constituunt. Consequenter b variabili posito oriuntur infinitae curvae similes inter se respectu puncti *A*. Data ergo unica reliquae facile ex similitudinis ratione describuntur.

Scholion 6.

122. Materia haec de arcubus isochronis abscindendis iam praeterito seculo ist pertractata [p. 57] in Act. Erud. Lips. A. 1697 [p.206][*Curvatura radii in diaphanis non uniformibus....et de curva synchrona, seu radiorum unda construenda*; *Opera Omnia*, Tom I, p. 187] Cel. Ioh. Bernoullio atque postquam in Commet. Acad. Paris a Cel. Saurino, [1709, p. 257 and 1710, p. 208 ; *Solution generale du probleme*, ...] qui vero alia methodo sunt usi. Ego vero eam adibui methodum, quam in nostris Comment. pro A. 1734 tradidi, tanquam commodissimam ad huiusmodi problemata solvenda. In his vero locis Viri Cel. curvas quoque similes tantum ut ego consideraverunt, sine dubio, quia pro curvis dissimilibus solutio fit nimis difficilis et saepe etiam vires superat. Vocantur vero in locis citatis hae curvae synchronae, quia arcus simul percursi abscinduntur.

Scholion 7.

123. Ex mea dissertatione Tomi VII Comment. Acad. Petrop. apparet has curvas synchronas simili modo posse inveniri, si curvae datae etiam non fuerint similes, sed eiusmodi tamen, ut posito $ds = pdx$ in p quantitates a et x datum dimensionum numerum constituant; tum enim aeque facile valor literae q invenitur; ut si numerus dimensionum ipsarum a et x in p fuerit n, aequatio pro curva secante reperietur haec

$$\frac{pdx}{\sqrt{x}} = \frac{pda\sqrt{x}}{a} - \frac{(2n+1)da\sqrt{b}}{a}.$$

Quare si fuerit $n = -\frac{1}{2}$, ut si fuerit $p = \frac{\sqrt{ac}}{\sqrt{(a^2-x^2)}}$, erit $\frac{dx}{x} = \frac{da}{a}$ ideoque $x = ma$, seu x in data ratione ad parametrum a est capiendum; quo igitur casu constructio synchronarum est facillima. [p. 58] At si p non huiusmodi habuerit valorem, ex supra citata dissertatione mea intelligitur, quo modo in aequationem quaesitam sit inquirendum.

PROPOSITIO 15.

Problema.

124. *Si fuerit ut ante infinitae curvae similes AM, AM, etc.* (Fig.18) *et recta positione data DE, invenire eam curvam AMN, super qua corpus tempore brevissimo ex A ad rectam DE descensu pervenit.*

Solutio.

Descripta per propositionem praecedentem quacunque curva *CMM* arcus *AM* isochronos abscindente ducatur tangens *GMH* parallela datae rectae *DE*. Manifestum est super curva *AM*, quae ad punctum contactus *M* tendit, corpus tempore brevissimo ad rectam *GMH* esse venturum, quia quaque alia puncta rectae *GMH* extra curvam *CMM* cadunt ideoque longiore tempore opus est, quo corpus ad ea perveniat. Iam, quoniam omnes curvae a curvis *AM*, *AM* arcus isochronos abscindentes sunt inter se similes (121), concipiatur ex iis una, quae rectam *DE* tangat; dico punctum contactus fore in *N* puncto, quo recta *AM* per prius punctum contactus *M* ducta rectae *DE* occurrit. Sequitur hoc tum ex natura similitudinis curvarum *CMM* respectu puncti *A*, tum etiam ex eo, quod arcus *AMN* similis sit arcui *AM* atque rectae *DE* in eodem angulo occurrat, quo curva *AM* rectae *GH*. Quare, [p. 59] cum corpus per *AM* tempore brevissimo ad *GH* perveniat, necesse est, ut quoque tempore brevissimo super curva *AMN* ad rectam *DE* perveniat. Q.E.I.

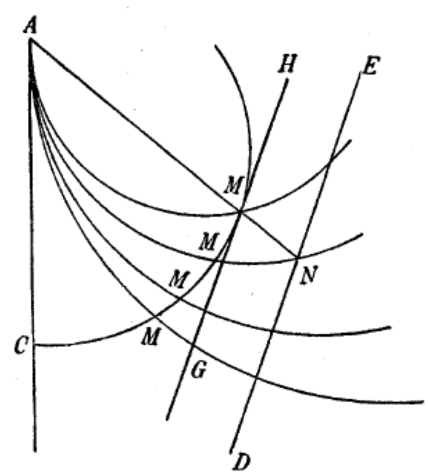


Fig. 18.

Corollarium 1.

125. Ex hoc perspicitur, si recta *DE* fuerit horizontalis, corpus descensu per verticalem *AC* ad eam citissime pervenire ob tangentem curvae *CMM* in *C* horizontalem; id quod quidem per se perspicuum est.

Corollarium 2.

126. Si ergo curvae *AM*, *AM* fuerint cycloides, ut in exemplo 3 propositionis praecedentis posuimus, corpus super ea cycloide celerrime ad rectam *DE* pervenit, quae huic rectae in *N* ad angulos rectos occurrit, quia angulus, quem quaeque cyclois cum curva *CM* constituit, est rectus.

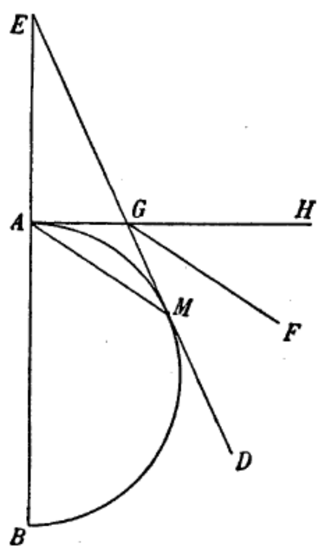


Fig. 19.

Corollarium 3.

127. Si igitur recta *DE* fuerit verticalis seu parallela ipsi *AC*, portio cycloidis *AMM* erit dimidia cyclois. Quare super dimidia cycloide motus horizontalis est celerrimus.

Corollarium 4.

128. Si curvae *AM*, *AM* sint rectae ex puncto *A* ad rectum positionem datam *DE* ductae, corpus super ea *AM* (Fig. 19) citissime ad *DE* perveniet, quae est chorda circuli per *A* transeuntis et centrum in verticali *AB* habentis atque rectam *DE* tangentis (112) [p. 60]

Corollarium 5.

130. Si igitur angulus *DEA* fuerit *n* graduum, erit angulus *BAM* $\frac{90+n}{2}$ graduum et angulis *AMG* graduum $\frac{90-n}{2}$. Seu ducta horizontali *AGH* anguloque *DGH* bisecto recta *GF* erit quaesita linea *AM* parallela ipsi *GF*.

Corollarium 6.

131. Quare si linea *DE* fuerit verticalis, corpus ad eam citissime perveniet descendendo super recta ad horizontem angulo semirecto inclinata. Corpus igitur super recta hoc modo inclinata motu horizontali celerrime progreditur.

Scholion.

132. Simili modo quoque inveniri potest, super quanam infinitarum curvarum similium *AM*, *AM* (Fig. 18) corpus descensu citissime ad datam curvam perveniat. Nam si linea *GMH* fuerit curva quaecunque tangens curvam *CMM* in *M*, corpus super hac curva *AM* celerimme ad curvam *GMH* perveniet, si quidem tota curva *GMH* extra curvam *CMM* fuerit sita. Eodem etiam modo posset determinari, si curvae *AM*, *AM* non fuerint similes, super quanam corpus celerimme ad datam lineam *GH* perveniat. Ex infinitis enim curvis

CMM arcus isochronos abscindentibus ea est quaerenda, quae datam *GMH* tangat, eritque ea curva *AM*, quae per punctum contactus transit, ea, quae quaeritur. Sed cum in his casibus difficile plerumque sit curvas *CMM* invenire multoque difficilius eam [p. 61] determinare, quae datam lineam tangat, quaestionem ad curvas similes tantum restrinximus.

PROPOSITIO 16.

Theorema.

132. *Tempora descensuum, quibus corpus curvas AM et Am, etc.* (Fig.20) *similes similiterque ex puncto A positas percurrit, sunt in ratione subduplicata laterum homologorum.*

Demonstratio.

Quia curvae *AM*, *Am* sunt similes, erunt *AM:Am*, *AP:Ap* et *PM:pm* in data ratione, nempe ea, quam latera homologa tenent; sit ratio laterum homologorum *N:n*. Quia celeritas in *M* est ad celeritatem in *m* ut $\sqrt{AP}$ ad $\sqrt{Ap}$, erunt celeritates in *M* et *m* in ratione subduplicata laterum homologorum. Sumantur iam ex *M* et *m* elementa similia, rationem scilicet *N* ad *n* tenentia, erunt tempora, quibus haec duo elementa homologa percurruntur, in ratione composita ex directa elementorum, i. e. *N* ad *n*, et reciproca celeritatum, i. e. $\sqrt{N}:\sqrt{n}$. Ex quo sequitur tempora, quibus curvarum *AM, Am* elementa homologa percurrentur, esse in ratione subduplicata

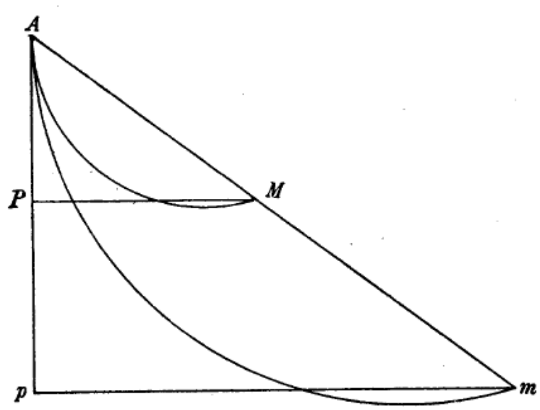


Fig. 20.

laterum homologorum. Quare, cum haec ratio sit constans, tempora, quibus totae curvae AM et Am percurruntur, eandem hanc rationem tenebunt. Q.E.D. [p. 62]

Corollarium 1.

133. Tempora igitur, quibus arcus circulares similes similiterque positi descensu percurruntur, sunt in subduplicata ratione radiorum.

Corollarium 2.

134. Pendula igitur, quae arcus circulares similes describunt, oscillationes absolvent temporibus, quae rationem subduplicatam longitudinum pendulorum tenebunt.

Corollarium 3.

135. Eadem ratio temporum locum habet, si corpora pendula non circulos describant, sed alias curvas, dummodo eae fuerint inter se similes similesque arcus absolvantur.

Scholion.

136. In his autem omnibus potentiam sollicitantem semper ponimus uniformem deorsumque tendentem, etiamsi hanc conditionem omiserimus. Hanc enim hypothesin ante pertractare constituimus, quam ad alias sumus progressuri.

PROPOSITIO 17.

Problema.

137. *Existente potentia sollicitante uniforme tendentque deorsum moveatur corpus super curve quancunque AM* (Fig.21) *cum data celeritate initiali in A; determinare motum corporis super hac curva et pressionem, quam curva in singulis punctis sustinet.* [p. 63]

Solutio.

Posita potentia sollicitante g et celeritate initiali in A debita altitudini b praetereaque $AP = x$, $PM = y$, $AM = s$ et celeritate in M debita altitudini v, his positis erit $dv = gdx$ (93), unde fit $v = b + gx$. Porroque tempus per arcum AM erit $\int \frac{ds}{\sqrt{(b+gx)}}$. Deinde pressio totalis, quam sustinet curva secundum directionem normalis MN, erit (93) =

$$\frac{gdy}{ds} + \frac{2vdxddy}{ds^3} = \frac{gdy}{ds} + \frac{2(b+gx)dxddy}{ds^3}$$

existente dx elemento constante. Hoc enim tantum differt haec solutio a solutione propositione 13, quod ibi esset $v = gx$, hic vero sit $v = b + gx$. Ex his igitur formulis tum motus tum pressio cognoscitur. Q.E.I.

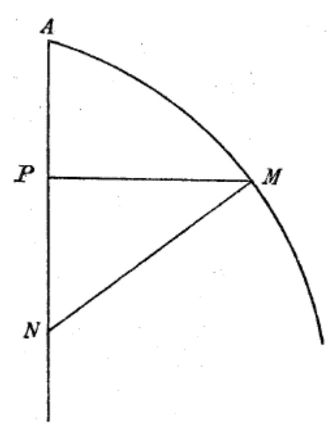


Fig. 21.

Corollarium 1.

138. Si linea *AM* fuerit recta, iam ex (88) intelligitur tempus per *AM* esse ad tempus descensus per *AP* eadem celeritate $\sqrt{b}$ incepti, ut est *AM* ad *AP*. Pressio vero ob evanescentem vim centrifugam erit = $\frac{gdy}{ds}$ seu constans.

Corollarium 2.

139. Patet etiam hoc casu, quo motus non a quiete incipit, celeritatem ab altitudine tantum pendere. Quare, quaecunque fuerit curva *AM*, celeritas corporis in quovis eius puncto innotescit, etiam incognita curvae natura. [p. 64]

Exemplum 1.

140. Sit curva *AM* parabola verticem in *A* et axem verticalem *AP* habens; erit ergo posita eius parametro = a , $y^2 = ax$ et

$$dy = \frac{adx}{2\sqrt{ax}} \quad \text{et} \quad ds = \frac{dx\sqrt{(a^2+4ax)}}{2\sqrt{ax}}.$$

Habebitur ergo tempus per *AM* =

$$\int \frac{dx\sqrt{(a+4x)}}{2\sqrt{x(b+gx)}}.$$

Deinde, cum posito *dx* constante sit $ddy = \frac{-adx^2}{4x\sqrt{ax}}$, erit

$$\frac{dxddy}{ds^3} = \frac{-2a}{(a+4x)\sqrt{(a^2+4ax)}}.$$

Consequenter pressio totalis est =

$$\frac{ga}{\sqrt{(a^2+4ax)}} - \frac{4a(b+gx)}{(a+4x)\sqrt{(a^2+4ax)}} = \frac{ga^2-4ab}{(a+4x)\sqrt{(a^2+4ax)}}.$$

Corollarium 3.

141. Si igitur est $b = \frac{ga}{4}$, pressio curvae evanescit. Corpus ideo hoc casu libere in hac parabola moveri posset, qui est etiam ipse casus praecedente libro (564) pertractus.

Corollarium 4.

142. Existente igitur $b = \frac{1}{4}ga$ erit tempus per arcum *AM* =

$$\int \frac{dx}{\sqrt{gx}} = \frac{2\sqrt{x}}{\sqrt{g}}.$$

Hoc ergo tempus aequatur tempori descensus per abscissam *AP* a quiete incepti.

Corollarium 5.

143. Si $b > \frac{1}{4} ga$, pressio fit negativa; tum igitur curva plagam axi *AP* oppositam premitur. At si $b < \frac{1}{4} ga$, directio pressionis erit in *MN*. Quantitas vero pressionis in singulis curvae punctis erit reciproce ut radius osculi. [p. 65]

Exemplum 2.

144. Si curva *AM* fuerit circulus, cuius radius = a et centrum in verticali *AP* sit positum, erit $y^2 = 2ax - x^2$, unde

$$dy = \frac{adx - xdx}{\sqrt{(2ax - x^2)}} \quad \text{et} \quad ds = \frac{adx}{\sqrt{(2ax - x^2)}}.$$

Erit ergo tempus, quo arcus *AM* percurritur, =

$$\int \frac{adx}{\sqrt{(2ax - x^2)(b + gx)}}.$$

Atque cum sit $\frac{dxddy}{ds^3} = \frac{-1}{a}$, erit pressio, quam circulus in puncto patitur, =

$$g - \frac{gx}{a} - \frac{2(b + gx)}{a} = g - \frac{3gx}{a} - \frac{2b}{a}.$$

Corollarium 6.

145. Tempus per logarithmos exprimi potest, si fuerit $b = 0$; fit autem = ∞, seu corpus perpetuo in *A* manebit. Id quod per supra tradita (97) patet. Nam quia curva in *A* est normalis in *AP* neque radius osculi infinite parvus, corpus descendere non potest.

Corollarium 7.

146. Si est $b = \frac{ga}{2}$ seu celeritas initialis tanta, quantam corpus acquirit cadendo ex altitudine dimidii radii circuli, pressio totalis cum vi centrifuga erit conspirans atque = $\frac{3gx}{a}$; erit itaque altitudine percursae proportionalis.

DEFINITION 3.

147. *Motus oscillatorius est motus reciprocus, quo corpus alternatim accedit et recedit ab initio motus M* (Fig. 22). *Ita si corpus super curva MAN* [p. 66] *moveatur, primo descentet super MA, tum ascendet in AN, donec celeritatem amiserit;*

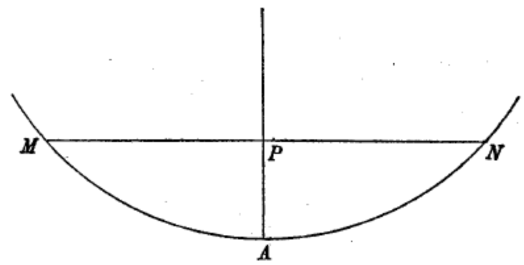


Fig. 22.

deinde ex N iterum descendit ascendetque in arcu AM, quo facto iterum descendet hancque periodum continuabit. Atque talis motus oscillatorius vocatur.

Corollarium 1.

148. Motus oscillatorius ergo consistit in alternis descensibus et ascensibus super linea curva; atque descensu motu accelerato movetur, ascensu vero celeritatem acquisitam rursus perdit.

Corollarium 2.

149. Quilibet ergo descensus super eadem curvae parte fit, super qua praecedens ascensus contigit. Quare, cum celeritas corporis ab altitudine tantum pendeat in vacuo, corpus in eodem curvae puncto sive in ascensu sive in descensu eandem habet celeritatem.

Corollarium 3.

150. Ex quo sequitur tempus descensus per *MA* aequale esse tempori ascensus per *AM* similique modo tempus ascensus per *AN* tempori descensus per *NA*.

Corollarium 4.

151. Corpus in arcu *AN* ascendens ad punctum *N* usque perveniet, quod aeque altum est ac punctum *M*, ex quo erat delapsum. Sequitur hoc ex eo, quod celeritas per altitudinem tantum determinetur. [p. 67]

Corollarium 5.

152. Si curva *AN* similis et aequalis fuerit curvae *AM*, tum motus per *AN* aequalis erit motui per *AM*. Quare omnes ascensus et descensus aequalibus fient temporibus.

Corollarium 6.

153. Si curva *MA*, *AN* fuerint dissimiles, tempus saltem per *MAN* aequale erit tempori per NAM, seu tempora accessionum et recessionum erunt inter se aequalia.

Corollarium 7.

154. Quia corpus semper ad eandem altitudinem pertinget, manifestum est hunc motum oscillatorium perpetuo durare debere.

Corollarium 8.

155. Curva ergo ad motum oscillatorium producendum apta est omnis curva, quae de puncto infimo A duos habet arcus ascendentes, ut MAN.

Scholion 1.

156. Exposuimus hic proprietates motus oscillatorii, quales ex exposita hypothesi potentiae sollicitantis uniformis et perpetuo deorsum tendentis consequuntur. Eaedem vero quoque locum habent, si potentia utcunque ab altitudine pendeat vel etiam ad fixum punctum dirigatur; id quod in sequentibus plenius apparebit. In medio resistente vero res aliter se habet; [p. 68] nam neque ascensus per datam curvam similis est descensui per eandem, neque in ascensu corpus ad aequalem altitudinem pertingit eit, ex qua descensu erat delapsum.

Scholion 2.

157. Vocare solet motus per *MAN* itus, sequens vero motus per *NAM* reditus; consistit ergo motus oscillatorius ex alternis itibus et reditibus. Oscillatio vero ab aliis vocatur motus ex itu et reditu constans, ab aliis tam itus quam reditus oscillatio vocatur. Hic priori sensu oscillationis vocem accipiemus, ita ut una oscillatio ex uno itu unoque reditu constet. Itus vero atque reditus uterquo uno ascensu unoque descensu consistit atque ideo integra oscillatio duos ascensus duosque descensus complectetur. Cum igitur tempus itus aequale sit reditus tempori, erit tempus unius oscillationis duplo maius quam tempus unius itus seu reditus.

Corollarium 9.

158. In hoc ergo capite, in quo de motu in vacuo agitur, si motum oscillatorium examinare velimus, vel ascensus vel descensus solos super duabus curvae partibus *AM*, *AN* considerare opus habebimus.

Scholion 3.

159. Nihil refert, utrum arcus *AM* et *AN* unam curvam continuam constituant, an vero sint diversae curvae, dummodo in A ita sint coniunctae, [p. 69] ut commumem habeant tangentem; alias enim motus perturbaretur. Quare ad motum oscillatorium inquirendum tantum opus est, ut motus super curvis *AM* et *AN* seorsum definiamus. Sufficit enim hoc tum ad oscillationes determinandas tum ad relationem inter maiores minoresque oscillationes inveniendam. Vocantur autem eae oscillationes maiores, quae maioribus arcubus absolvuntur, minores vero, quae minoribus.

Scholion 4.

160. Ex propositione 6 (49) perspicitur, quomodo oscillationes ope pendulorum effici queant, scilicet ope evolutae curvaram AM et AN, circum quas filum circumducitur. Ab Hugenio etiam iste pendulorum usus ad oscillationes accommodatur, ut vel ex eius instituto, quo eo motu ad horologia perficienda utitur, apparet. Eaedem vero difficultates,quas loco citato commemoravimus, hic locum habent. Quamobrem motum puncti super datis lineis hic tantum investigabimus mentemque ab omnibus pendulorum circumstantiis abducemus, quae nostrum institutum turbare possent.