Chapter One

Concerning Motion Which In General Is Not Free.

[p. 1]

DEFINITION 1.

1. *A body is said not to move freely,* [i.e. *to be constrained*] *when external obstacles impede its progress, and in a like manner its motion in that direction is less than it should be moving, by reason of the absolute forces acting on it.*

Scholium 1.

2. In the motion of free points that we set out in the first part, the space in which we assume the body moves is a vacuum free from all obstacles; now truly we put in place a space for comparison, so that it is not permitted for the body to progress in any direction because of solid walls across which it is not allowed to pass.

[p. 2]

Corollary 1.

3. Therefore when a body finds an obstacle to its own motion it is not able to keep moving in that direction which it held, then either it comes to rest or the motion must continue in another direction.

Corollary 2.

4. Moreover, in what direction the body progresses after meeting the obstacle must be determined from the circumstances both of the motion and of the position of the obstacle.

Scholium 2.

5. It seems that this is relevant to the theory of the collisions of bodies, in which the body is not yet free to move in this way or that. Truly in this book we assume obstacles of other kinds, which do not require that acquaintance. These are continuous obstacles that restrict the motion of points and neither do they allow any turning back; and a pipe or channel which is either straight or curved is an obstacle of this kind, along which the motion of a small body must continue. In this case the path inside is prescribed in which the body is to progress, and it is not able to escape because of the firmness of the pipe [or tube]. Whereby, since here in place of a body we consider a point, a point on a given line must be moving from this position, and it is unable to leave this line.

Scholium 3.

6. Moreover in this book we deal with the two motions of the impeded or restricted kind, [p. 3] of which the first we have made mention includes the motion of points on a given line or curve. The other kind restricts the freedom of the motion less ; for it only prescribes a surface on which the body must always be moving. And we are to explain these two kinds of impediments to the motion in this book.

Corollary 3.

7. Therefore these properties are sought for the first kind of motion which are : the speed of the body or rather of a point in the position of any prescribed line; the force on this line; and the time in which a given point traverses a portion of the path.

Corollary 4.

8. Concerning the motion of the other kind, more than the motion on this line has to be found, as the body describes the motion upon a given surface. Concerning which we uncover the principles in this first chapter.

Scholium 4.

9. Truly in this first chapter we investigate both kinds of motions, for which the body is acted on by no forces, where we show with what speed it should be progressing, and what force is must exert everywhere not only on a given line but also on a given surface. But if only a surface is given, then in addition we must determine the path along which the body moves when acted on by no forces. Then we set out the principles, by which it can be determined, what changes in the path arise with forces acting, both absolute and relative, [p. 4] and from which in the following chapters we can deduce particular individual cases.

Scholium 5.

10. Moreover, for both motion on a given line as for motion on a given surface, we imagine that all friction has been removed and we put no retardation to the motion in place. On this account the lines and the surfaces upon which the points are placed to be moved, are considered to be the smoothest and free of all asperities, least the motion should be liable to be slowed down on that account. All rotational motion also we imagine to be removed everywhere, which is to be explain at length later. Because of this, a point is considered to be moving as if by creeping along, in order that any part of this, if in this manner a point can be considered as made up of parts of points [this is an idea introduced in Ch. I of Book I and not yet used], then they have the same motion.

Scholium 6.

11. Therefore what has been treated in the preceding book, and what is to be treated concerning the motion of points in this book, can be adapted to bodies of finite size also, but only if the movement of these is always parallel to themselves, and all the parts of the body are provided with equal motion. This indeed will become clearer from the following books, for which there is no disagreements between the case of the motion of finite bodies from the motion of points. On which account therefore in these books we consider only points since, as they do not have different parts, thus also they are unable to have parts with different motions.

PROPOSITION 1.

[p. 5]

Theorem.

12. *A body or a point, which is moving on a given line and is acted on by no forces, always keeps the same speed, only if any two adjoining elements of this line nowhere constitute a finite angle.*

Demonstration.

Since the body, while it is moving on the line *AM* (Fig. 1), is acted on by no force and neither does the curve have any friction, and the motion of the body is unable otherwise to be changed, except in as much as it is impeded by the line *AM* , on which a small body is able to move freely; from which any change in the speed which should arise is to be investigated. Let the speed which the body has at *M* be equal to *c*; here therefore with this speed, the body progresses along the tangent *Mv*, if it could move freely ; which now, since the body is unable to leave the curve *AM*, is unable to happen, for the body is forced to progress along *Mm*. Hence for this reason the motion of the body along *Mv* can be considered to be resolved into the motion along *Mm* and the motion along *Mn*, with the right-angled parallelogram *Mmvn* arising. It is evident that this motion along *Mn*, the direction of which is normal to the element of the curve *Mm*, is to be absorbed by the curve unless there is no change in the motion. Therefore the body progresses with another motion along *Mm* with a speed, which is to the former speed as *Mm* to *Mv*; whereby the speed, with which the body describes the element of the curve *Mm*, is equal to $\frac{Mm.c}{Mv}$. Since truly *Mvm* is the triangle for the rectangle *mn,* thus $Mm < Mv$, [p. 6] this speed is less than the previous speed *c* and the decrease in the speed is equal to $\frac{(Mv-Mm)c}{Mv}$. For the value of this can be found *MO*, the radius of osculation of the curve at *M* which is equal to *r* and the element *Mm* is equal to *ds*; and this is, on account of the angle *O* = angle *mMv*, $MO : Mm = Mm : mv$, from which there comes about $mv = \frac{ds^2}{r}$ and

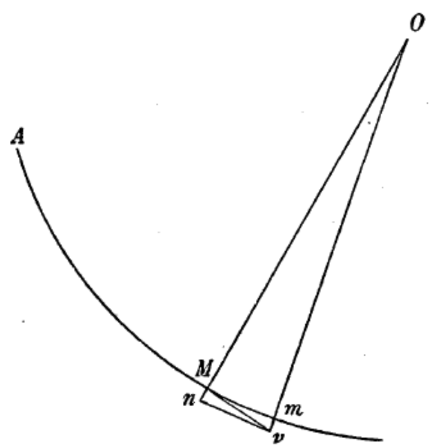


Fig. 1.

$$Mv = \sqrt{\left(ds^2 + \frac{ds^4}{r^2}\right)} = \frac{ds}{r}\sqrt{(r^2 + ds^2)} = ds + \frac{ds^3}{2r^2}.$$

From this the decrement in the speed can now be obtained, while the element of the curve *ds* is traversed, equal to $\frac{cds^2}{2r^2}$, of which with the whole gives the decrement of the speed,

while the body traverses the finite portion of the curve AM. But the expression $\frac{cds^2}{2r^2}$ is equivalent to a differential of the second order ; therefore the integral of this is a differential of the first order. On account of which the decrease of the speed, after the body has traversed an arc of some given size, is infinitely small and the body is carried with a uniform motion along the whole curve *AM*, only if the radius of osculation *r* was nowhere infinitely small. Q.E.D.
[We meet this kind of geometrical argument time after time in Euler's work, where we would now refer to the principle of conservation of energy; in the present case, no work is done on the particle.]

Corollary 1.

13. Therefore in any curve, in which the radius of osculation is nowhere infinitely small, the body moves uniformly, if indeed it is not allowed to be acted on by any forces or friction.

Corollary 2.

14. If the radius osculation is infinitely small, then $\frac{cds^2}{2r^2}$ is either a finite quantity or is a differential of the first order. [p. 7] In the first case the body parts with a finite change in the speed, in the other truly it is infinitely small.

Corollary 3.

15. Moreover since points of this kind are rare in all curves and are widely scattered between each other, and the body still travels uniformly along the arc intercepted by two such points.

Scholium 1.

16. The case, in which the body suffers a sudden finite decrease in speed, is only possible where the curve has cusps. For with these in place the body is forced to turn back directly and normally on the point of the cusp it strikes. Therefore the body then not only loses a finite step in its speed, but it must lose all of its motion entirely, except perhaps is put to be elastic, in which case it may be reflected with the speed with which it arrived, and thus the uniform motion is conserved. In a cusp, two elements of an infinitely acute angle are put in place.

Scholium 2.

17. Truly as well as cusps, other points can be given on curves, in which the radius of curvature is infinitely small ; because any two touching elements are placed in almost the same direction and following this the angle is infinitely small, but it cannot happen that the body suffers a finite decrease in the speed, as the above demonstration shows. On account of which, since points of this kind are rare, the body nevertheless moves with a uniform motion. [p. 8]

Corollary 4.

18. Therefore if the motion of the body were elastic, then the motion is always carried on uniformly on any curve ; but if it is not elastic, then the cusps only upset the motion, while evidently they destroy it.

Scholium 3.

19. In order that these may be made clearer, let two elements of the curve be *AB* and *BC* (Fig. 2) and following the angle *ABC* that they make, *CBD* is next put in place infinitely small, and the sine of this angle is *dz,* with the total sine put equal to 1. Because the body, after it describes the element *AB*, by the force of inertia, [*vi insita*] tries to progress along *BD* with the previous speed which was *c*, this motion is taken in two parts, the one in the direction *BC*, and the other in the direction normal to *BC*, which cannot be effected. Therefore by sending the perpendicular *DC* from *D* to *BC,* the body moves along *BC* with another motion, with a speed which is to the first speed as *BC* to *BD*, i. e. as $\sqrt{(1-dz^2)}$ to 1. Therefore the speed along BC is equal to $c\sqrt{(1-dz^2)}$ or $c-\frac{cdz^2}{2}$; whereby the decrement of the speed is $\frac{cdz^2}{2}$, which is equivalent to a differential of the second order. From which it is understood, as long as the angle CBD of the curve is infinitely small, the motion of the body progresses at a steady rate. But on the curve the angle CBD is either infinitely small or the angle ABC itself is infinitely small, when the point falls on cusps. Consequently only cusps disturb the uniformity of the motion, unless the body was elastic, in which case the motion nevertheless is conserved. [p. 9]

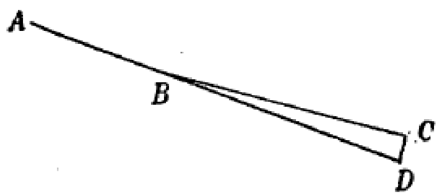


Fig. 2.

PROPOSITION 2.

Theorem.

20. *While the body is moving uniformly along the curve AM* (Fig. 1)*, it pressed the curve normally with a force at the individual points M with a force which is to the force of gravity as the height corresponding to that speed is to half the radius of osculation.*

Demonstration.

If the body on the curve *AM* must be moving freely with a uniform motion, then it is necessary for the force to be present everywhere only acts normally along *MO*, which has the ratio to itself to the force of gravity as the height corresponding to the speed of the body to half the radius of osculation *MO*, as appears from the demonstrations of the preceding book (165, 209, 552). For unless such a force is present then the body travels in a straight line. Moreover in this case the channel *AM*, in which the body is considered to be enclosed, impedes the motion, in which the body progresses less in a straight line. On account of which the body presses the channel normally with so great a force following the direction *Mn*. If indeed the body is pressed by such a normal force present, then it can move freely in the channel *AM* ; and neither might it press the channel; but truly with this force missing, as we put here, it is necessary in order that the body presses the channel itself with such a force. [Newton's third law.]Q.E.D.

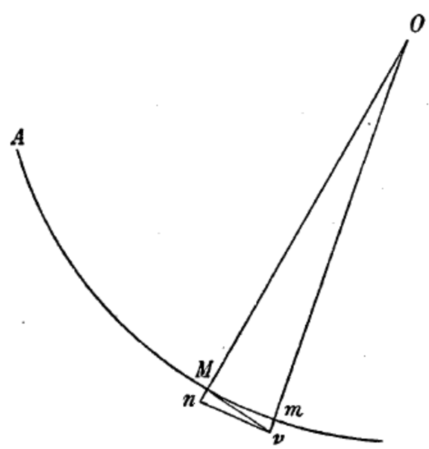


Fig. 1.

Corollary 1.

21. Therefore if the height corresponding to the speed of the body is put as v and the radius of osculating *MO* is equal to r and the gravity on the body is equal to 1, as clearly it has if it has been put on the surface of the earth, the force will be by which the body pressing on the channel at *M* along *Mn*, is equal to $\frac{2v}{r}$.[p. 10]

Corollary 2.

22. If the body moves with a greater or smaller speed along the curve AM, then the force pressing at M is greater or less in the square ratio of the speed, since the height v is proportional to the square of the speed.

Corollary 3.

23. The direction of this force is normal to the curve and is in the opposite direction to the position of the radius of osculation *MO*. Whereby the radius of osculation in the other part of the curve produced gives the direction of this force.

Corollary 4.

24. If the body is moving in a straight line, this force is zero on account of the infinite radius of osculation. This is also evident from the nature of the motion. For the body moves in a straight line uniformly spontaneously and on this account is not pressed by the channel.

Corollary 5.

25. If the curve AM is a circle, the force is the same everywhere. Truly with that to be greater as the radius of the circle is made less. Indeed with the same speed present, the force varies inversely as the radius of the circle.

Scholium 1.

26. Where the body is able to move freely along the curve AM uniformly, it is necessary that [p. 11] it is drawn along the normal MO by a force equal to $\frac{2v}{r}$. From which it is to be understood that the body struggles with such a force in the opposite region, otherwise the body cannot be kept on the curve by that force. Therefore while the body is forced to move along the channel AM it is being carried along by the struggle with this force, and this force is exercised by the channel itself. On account of which the channel must have such firmness, in order that it is able to sustain such force.

Corollary 6.

27. Therefore the body is able to carry out the motion without any expenditure of the speed, which clearly is consistent with the definition of the force.

Corollary 7.

28. Therefore the force arises from the motion alone. On account of which, just as motion is generated from forces, thus forces can arise from the motion.

Scholium 2.

29. Hence it is understood, as now in the first book above we have agreed with the notion (102), that it is unclear whether motion is owing to forces or whether forces to motion. For we see each in the world, truly forces and motion to arise; therefore one is the cause of the other, the question is to be decided from reasoning as well as from observation. Indeed there seems to be hardly any agreement about forces that arise on bodies that remain at rest, with much less forces being decided to arise form these. Besides truly, that everything can be shown to arise from motion, is considered to be the natural cause to be given of all phenomena. [p. 12] For motion once in existence must always to be conserved we have clearly shown above (63); this we have truly elaborated upon, as forces arise from motion. As truly forces without motion that are able either to be present or to be conserved cannot be conceived. On account of which we can conclude that all forces, which are observed in the world, arise from motion; and it falls upon the diligent investigator, that for each and every motion of bodies from any kind of force in the world their origin has to be observed.

Scholium 3.

30. Since it is difficult to understand how such an effect clearly of a continuous force can act on a body without any change arising in the speed, it is a worthwhile task to inquire about the cause of this effect. We have seen in the preceding proposition that the motion of the body was not exactly uniform, but the speed is actually allowed to decrease, while the body is moving through singular elements in the curve. Truly these decrements are equivalent to second order differentials, as also only infinitely small changes in the speed repeated infinitely often are able to diminish the speed. Therefore I declare that the force acting is ascribed to an infinitely small decrement in the speed; and I become more confirmed in this belief, because when the decrement in the speed becomes greater, so also does the force present increase. Since the force at M is equal to $\frac{2v}{r}$ and while it is being traversed, the whole element Mm is acted upon by this force [p. 13], it is permitted to express the effect of this force on the element $Mm = ds$ by $\frac{2vds}{r}$. Truly the above decrement in the speed, while the element Mm has been traversed, is found to be $\frac{cds^2}{2r^2}$ (12). But because this is equal to c there, which here we have as $\sqrt{v}$; hence the equation arises : $-dc = \frac{cds^2}{2r^2}$, and it becomes

$$-\frac{dv}{2\sqrt{v}} = \frac{ds^2\sqrt{v}}{2r^2} \quad \text{or} \quad -dv = \frac{vds^2}{r^2}.$$

Therefore we have $-4vdv = \frac{4v^2ds^2}{2r^2}$ equal to the square of the force that supports the element Mm.

Corollary 8.

31. Therefore the square of the force acting on Mm is equivalent to the decrement of $2v^2$. And if this decrement is equal to ds^2, then the force is equal to the force of gravity, from which the comparison of these forces is known.

Corollary 9.

32. Therefore it is conceded that an infinite number of infinitely small decrements in the speed suffices in the production of a finite force. For as long as the decrement v^2 itself is the homogeneous ds^2, the force is finite; but truly if that infinite number of infinities becomes greater than ds^2, then the force also becomes infinitely large.

Definition 2.

33. This force, which the body exercises on the body in the line of the curve is called the centrifugal force, since the direction of this pulls from the centre of osculation O. [p. 14]

Corollary 1.

34. *The centrifugal force is therefore to the force of gravity as the height corresponding to the speed to half the radius of osculation.*

Corollary 2.

35. Therefore when the body is forced to move along the line of the curve the centrifugal force presses against the curve, even if no external force is acting.

Scholium.

36. Therefore when the body is acted on by some external forces, a force also arises in the channel from these forces as well as from the channel itself, both pressing in a two-fold ratio, truly partially from the external forces and partially from the centrifugal force. Now therefore, what the force shall be that prevails on the constrained body is to be found.

PROPOSITION 3.

Theorem.

37. *If the body, which is moving in the channel AM* (Fig. 3)*, is acted on at M by the force MN, the direction of which is normal to the curve AM, then the speed is neither increased or decreased and the whole force is taken up in pressing against the channel.*

Demonstration.

From the first book (164) it was shown that the force, the direction of which is normal to the direction of the motion, neither increases nor decreases the speed. Though indeed there for free motion it has a place for stiffness, [p. 15] since the normal force neither before or after pulls on the body. Truly in free motion the direction of the normal force does not change, as it is not able to have an effect in this situation. Therefore the body is pressed by this force on the channel and consequently only the force of the channel presses in the direction *MN*. Q.E.D.

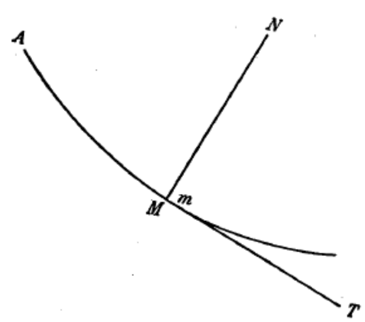


Fig. 3.

Corollary 1.

38. Therefore the direction of such a normal force is either incident in the direction of the centrifugal force or is in the direction contrary to that. In the first case the centrifugal force is increased, and in the other it is diminished.

Corollary 2.

39. Because the direction of the centrifugal force falls on the convex part of the curve, the effect of this is to increase the force, if the normal force falls in the same place; but if the normal force is directed to a concave part of the curve, the effect is diminished the force.

Corollary 3.

40. If the normal force is equal to N and the centrifugal force as before is equal to $\frac{2v}{r}$, the curve is pressed either by the force $\frac{2v}{r}+N$, if the forces act together, or by the force $\frac{2v}{r}-N$, if they act in the opposite directions.

Corollary 4.

41. If the normal force is equal and opposite to the centrifugal force, then the curve sustains no force, or the body does not try to escape from the curve. Therefore in this case the body is free to describe the same curve; [p. 16] it is also evident that the normal force is equal to $\frac{2v}{r}$; for it is brought about here, in order that the body is free to move on any curve.

PROPOSITION 4.

Theorem.

42. *If the body, which is moving in the channel AM* (Fig. 3)*, is acted on at M by a force, the direction of which is along the tangent MT, the effect of this is consistent with this, that the speed of the body is either increased or diminished in the same way as for free motion.*

Demonstration.

Since the direction of this force is the tangent *MT* of the channel, the effect of the channel cannot impede the effect of this force ; nor can this force exercise any effect on the channel. On account of which the force either augments or diminishes the speed of the body, according as the direction of this either acts in the same direction as the body or in the opposite direction, and clearly if the body is moving freely. And with the height corresponding to the speed at *M* equal to *v*, the element $Mm = ds$ and with the force $MT = T$, there is $dv = Tds$ with the accelerating force T; but with retardation that becomes $dv = -Tds$. Q.E.D.

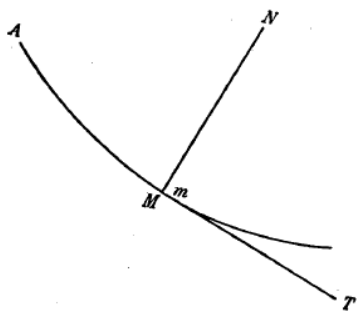


Fig. 3.

Corollary 1.

43. Therefore in the motion of bodies on given lines, the normal force only generates a force on these lines, and the tangential force truly only affects the speed.

Corollary 2.

44. Since the force of the retarding resistance may be greater than the tangential force, [p. 17] it acts in the same manner in the motion of bodies on given lines as in the case of free motion. If therefore as well as the accelerating tangential *T* there is the resistance *R* present, then with both joined together we have $dv = Tds - Rds$.

PROPOSITION 5.

Problem.

45. *If a body is moving on some given line AM* (Fig. 4) *in some medium with resistance and in addition it is acted on by some absolute force, the direction of which is MP, to determine the effect of the absolute force as well as of the resistance as well as the force supported by the curve AM.*

Solution.

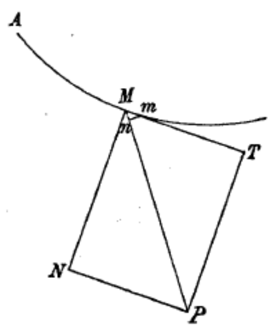


Fig. 4.

Let the height corresponding to the speed at *M* be equal to *v*, the force of the resistance is equal to *R* and the absolute force $MP = P$, the direction of which is such that, as with the element *Mm* taken equal to ds, the perpendicular *mn* from *m* sent to *MP* is equal to dx and $Mn = dy = \sqrt{(ds^2 - dx^2)}$. The force *P* is resolved into these two forces along the normal *MN* to the curve and the force pulling along the tangent MT ; on this account the triangles *MPT* and *Mmn* are similar and the normal force MN or $PT = \frac{Pdx}{ds}$ and the tangential force $MT = \frac{Pdy}{ds}$ increased the speed. Since truly the force of resistance decreases the speed, the speed is only increased by the excess $\frac{Pdy}{ds} - R$; on this account there is (42)

$$dv = Pdy - Rds.$$

The normal force truly is effected by $\frac{Pdx}{ds}$, as the curve is pressed just as much at *M* along the direction *MN* to the convex part of the curve in place. [p. 18] Whereby, since the centrifugal force acting at the same place is equal to $\frac{2v}{r}$, with the radius of curvature designated by *r* the radius of osculation at *M*, the total force by which the curve is pressed on normally at *M,* along *MN,* is equal to $\frac{Pdx}{ds} + \frac{2v}{r}$. Hence the motion of the body on the given curve as well as the force acting on the curve can be found at individual points. Q.E.I.

Corollary 1.

46. Therefore from these two formulas both the acceleration and the force on the wall can be expressed can all be deduced from the expressions, which pertain to the motion on the given lines.

Scholium 1.

47. Here indeed we have put in place a single absolute force; yet nevertheless from that it is understood how the effect of many forces can be understood. Of course as we made in free motion, thus also here the individual forces are to be resolved into two parts, truly the normal and the tangential, from which by gathering these together a single normal force and a single tangential force arises; the effect of which can be determined by Propositions 3 and 4.

Scholion 2.

48. Therefore up to the present we have set out the fundamentals, from which in the following it is permitted to determine the motion of bodies on given. But before we treat the similar principles of the motion on given surfaces, it is expedient that we consider a few cases in which the motion on a given line in effect can be deduced.[p. 19] In as much as with the help of channels, in which the body is contained, it is of minimal use to produce such motion on account of friction and other obstacles, which by no means are able to be removed. Moreover constrained motion of this kind is most conveniently brought into being with the aid of pendulums, as was first done by Huygens[Original reference presumably used by Euler: Chr. Huygens, *Horologium oscillatorum sive de motu pendulorum ad horologia aptato demonstrationes geometricae*. Paris 1673; *Opera varia*, Vol. 1, Lugduni Batavorum 1724, p. 89. See the English translation in this series.]; why we arrange matters to make use of pendulums we explain in the following proposition.

PROPOSITION 6.

Problem.

49. *How a body is able to move on a given line with the help of a pendulum.*

Construction.

Let *AMB* (Fig. 5) be the proposed curve, in which the body must move; the evolute AOC of this curve is constructed and a plate is curved following this figure and set in place. Then a thread is led around this plate, which has one end fixed to the plate, and the body *A* to be moved is fastened to the other end. Therefore when the body begins, it is evident that it must move on the curve *AMB*, because the thread, as it then separates from the place, describes the evolute of this curve.
Q.E.F.

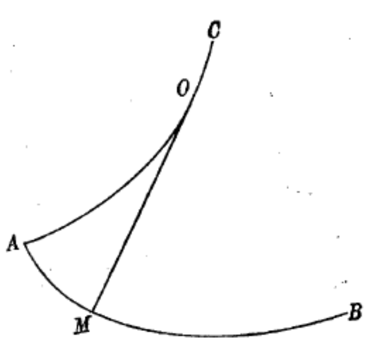


Fig. 5.

Corollary 1.

50. Therefore for this reason the body progresses along the given curve and is not liable to friction. Whereby in this manner such motion along curves as are found in theories can be conveniently put to the test. [p. 20]

Corollary 2.

51. From the theory of evolutes it is understood that the separated part of the thread *MO* is normal to the curve *AMB* and is the radius of osculation of this curve.

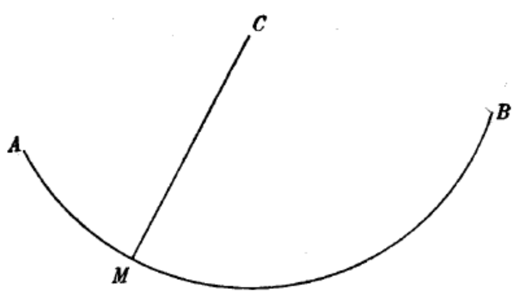


Fig. 6.

Corollary 3.

52. When the body is moving on the periphery of the circle *AMB* (Fig. 6), the curved plate is not needed, but the thread has only to be fastened at the end *C* to the centre of the *C* of the periphery.

Corollary 4.

53. Since the thread MO (Fig. 5) is the radius of osculation, the total centrifugal force is devoted to stretching this thread. Whereby this thread has enough strength not to be liable to be extended. For unless the length is always kept the same, the desired curve is not described.

Corollary 5.

54. By adding the absolute force besides the centrifugal force the normal force is obtained, which also pulls the thread, if the centrifugal force is to be added. But if it acts in the opposite direction, it diminishes the tension in the thread, indeed also, if this force is greater, then the thread is compressed, in which case it is of no use as an evolute. For since the thread must be flexible, it is not able to resist compression and neither does it offer any impediment, in which case the body recedes from the curve AMB towards the evolute.

Scholium 1.

55. Besides this difficulty, the generation of curves by evolutes also labours under this weakness, because the straight line is unable to be produced [p. 21] ; indeed for that to be generated the thread is requires to be infinitely long. In a similar manner this evolute cannot be adapted for curves that have an infinitely great radius of curvature somewhere. Then also neither curves with cusps nor with contrary bends can be described in this manner. Thus on this account the practice only has a place with curves having a finite curvature everywhere, to which it must be added, so that the total force acting on the curve is anywhere directed to the concave part of the curve.

Scholium 2.

56. Huygens, who first developed the principle of the evolute, at once put it to this use, that is apparent from the unusual need of the swinging pendulum clock. For since it can be shown that the swings on a cycloid are all isochronous, he wished to bring about cycloid motion in clocks, which is effected by the pendulum swinging between cycloidal plates. For since the evolute of a cycloid is a cycloid, for this reason it was obtained that a body tied to the end of a thread moves in a cycloid.

Scholium 3.

57. Moreover in this motion of pendulums it is appropriate to note specially that besides the motion of the body, the thread too must be moving, but as it is the custom in this book in which only the motion of points is to be carried out, it is too small to be a concern. In addition, neither do we touch on the motion of the body attached as the pendulum which is not parallel to itself, but rather circular, and which is permitted around the centre of any circle of osculation on the curve. [p. 22] Therefore in this book we submit to be examined only the motion of a point in a given line or surface, and we do not consider either the motion of the thread or the case of circular motion. Moreover in the following motion of pendulums, where both the motion of the thread and circular motion are deduced from a computation, we reduce the motion to that of points only, thus in order that these which are treated in this book, are nevertheless found to have a practical use. On which account, as we have now advised, the point [acting as the pendulum] is considered to be always carried by moving parallel to itself either on a curve or surface without any friction.

PROPOSITION 7.

Theorem.

58. *If a body under the action of no forces is moving in a vacuum or in a medium without resistance on some surface ABC* (Fig. 7), *then it is carried in a uniform motion, with all friction removed from the air.*

Demonstration.

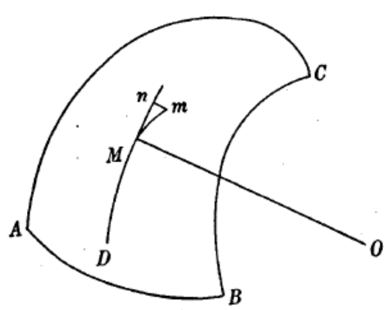


Fig. 7.

When a body moving on a given line is able to continue to be pressed, it is able to move much more on a given surface because there the freedom is less restricted. Therefore, let *DMm* be a line on which the body is progressing ; this is either a straight line or a curve. If this is a straight line, then there is no doubt that the body progresses with a uniform motion. But if it were a curve that could be expressed by an equation, and two adjoining elements of this are either situated nearly in the same direction, or they constitute an acute angle because cusps occur. [p. 23]
In the above case it has been shown that the body suffers no decrease of the motion (12). Now with cusps indeed all the motion is lost, unless [the collision] is elastic. On account

of which, if the motion is made on a curve, or on the part of a curve with the cusps missing, then the motion of the body is uniform. Q.E.D.

Corollary 1.

59. For a decrease in the speed of the body is permitted, as often as the direction is forced to change, now this is therefore equivalent to a second order differential, and even if it is integrated then an infinitely small decrement is produced.

Corollary 2.

60. Clearly if the speed of the body is c and the radius osculation $MO = r$, then the decrease in the speed while the body traverses the element ds is equal to $\frac{cds^2}{2r^2}$ (12).

Scholium.

61. The demonstration of this proposition clearly agrees with the above proposition except for this difference, in the former case the body is forced to move along a given line, while now in the above case it is free to have any path on the given surface. On account of which all the notes that have been made for the first proposition prevail here too. [p. 24] Therefore we will see what path on any given surface the body should traverse.

PROPOSITION 8.

Theorem.

62. *The path DMm* (Fig. 7), *which the body moving on some given surface ABC describes, is the shortest line that can be drawn between the two terminal points D and M, clearly if the body is moving in a vacuum and is acted on by no forces.*

Demonstration.

The body now describes the curve DM; it is evident that the body will be moving along the tangent Mn from M unless it is forced to persevere on the surface. Therefore, since the motion along Mn cannot be made, it is resolved into two components, of which the one is set out on the surface, and the other now is in a direction perpendicular to the surface, and thus removed from the surface it is not in effect possible to be deduced. On this account, the perpendicular nm is sent from n to the surface; Mm is the element of the line along which the body progresses from M . Hence the plane nMm is normal to the surface, in which are placed both the element mM and that previous element which has just been described. But the shortest line drawn on any surface has this property, that the plane on which any two contiguous elements are placed is normal to the surface. On account of which the line DMm, which is described by the body, is the shortest line on the surface ABC. Q.E.D. [p.

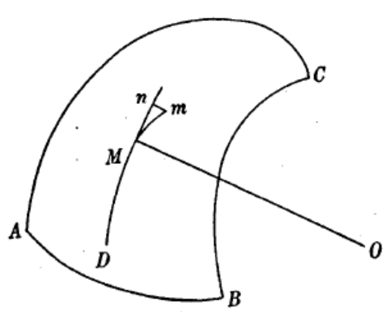


Fig. 7.

24] [Note that *Dn* is in effect the previous element transposed, according to Newton's First Law, and the whole proposition is a generalization of the linear case presented above.]

Corollary 1.

63. Hence if from the point *A*, at which the motion begins, the shortest line *ABC* is drawn following the direction of the motion, the path is obtained, along which the body is moving uniformly.

Corollary 2.

64. Since the tense thread on the surface designates the shortest path, the tense thread also shows the path along which a body on that surface will move uniformly.

Corollary 3.

65. Therefore if the proposed surface were a plane, then the line described by the body would be straight, since in the plane this is the shortest line. And on the surface of a sphere the body moves on a great circle.

Corollary 4.

66. Since the plane in which the two adjoining elements of the curve *DMm* are placed is normal to the surface, then the normal to this plane lies on the surface, and the radius of osculation *MO* of the described curve is now put in the same plane, normal to the surface.

Scholion.

67. As the shortest line to be found on any given surface has been demonstrated by me in Book III Comment. Acad. Imp. Petrop. [*Concerning the shortest line joining any two given points on any surface: linea brevissima in superficie quacunque duo quaelibet puncta iungente*. See E09 in this series of translations.] Moreover there I determined the shortest length from another principle, and the elements of that matter shall not yet be put in place: the shortest line or that which is described by a body that I have decided to determine in the following proposition. [p. 26]
[We should note that the motion of a body on a surface is a dynamics problem, while the shortest distance between two points on a given surface is a purely geometrical problem.]

PROPOSITION 9.

Problem.

68. *On any given surface, to determine the line described by a body moving on the surface, acted on by no forces.*

Solution.

In order that the nature of the proposed surface can be expressed the fixed plane *APQ* is taken, and in that plane the line (Fig. 8) *AP* it taken for the independent axis . Then from some point *M* on the surface the perpendicular *MQ* sent to this plane and from *Q* the perpendicular *QP* is sent to the axis *AP*. Now on putting $AP = x$, $PQ = y$ and $QM = z$ the nature of the surface is given an equation between these three variables *x, y*, *z* and constants. Let the differential equation of this surface be given by :

$dz = Pdx + Qdy,$

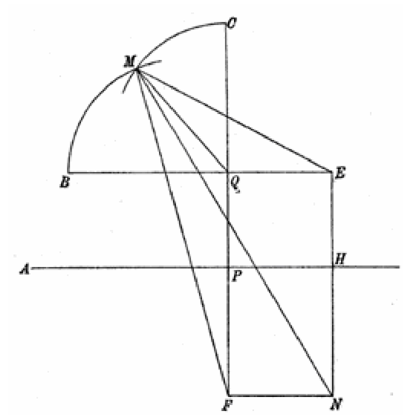


Fig. 8.

from which the shortest line on this surface or the line that the body describes must be determined. Now this line is determined from this consideration, that the radius of osculation is declared to be perpendicular to the surface. On this account, first the normal to the surface is drawn, and then we determine the radius of osculation of that curve drawn in this plane, in which later from the coincidence of these lines the nature of the line sought can be inferred. [p. 27]

In order to find the normal to the surface, first the surface is cut by the plane *MQB*, with the line *BQ* proving to be in the plane *APQ* [the *xy* plane] parallel to the axis *AP*, and the curve *BM* is produced by this section; the nature of this curve is expressed by this equation $dz = Pdx,$ which arises from the surface $dz = Pdx + Qdy$, with *y* constant or $dy = 0$ put in place. To this curve *BM* the normal *ME* [in the *xz* plane] is drawn crossing the line *BQ* produced in *E*; the subnormal $QE = \frac{zdz}{dx} = Pz.$

[since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = Pdx + Qdy$, in modern notation.]

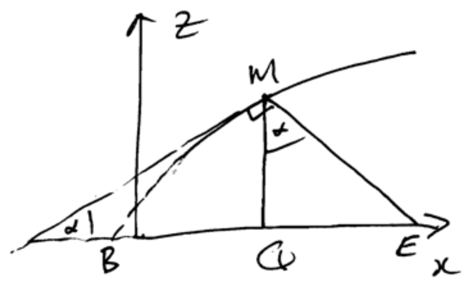
Now with the line *EN* drawn perpendicular to *BE,* any line *MN* drawn from *M* to *NE* is normal to the curve *BM*.

[To understand this statement, consider the line MN, for any N along EH, to rotate about an element of MB centred on M as axis, keeping the same right angle to the element as the coplanar line ME, as the rotation is about an axis normal to the element. Similarly for the other case treated below.]

In a like manner, the surface is cut by the plane *PQM* and the curve *CM* is produced by the section, the nature of which is expressed by the equation between *z* and *y* by keeping *x* constant, which is given by $dz = Qdy$. Let *MF* be the normal to this curve [in the *yz* plane]; the subnormal is given by $QF = \frac{-zdz}{dy} = -Qz$; I use this with the negative sign, since the subnormal *QF* I put in place falls towards *P*. Now with the line *FN* drawn parallel to the axis *AP*, any line drawn from *M* to *FN* is normal to the curve *CM* [as above]. Therefore the line *MN*, which falls at the intersection *N* of the lines *FN* and *EN*, is perpendicular to each curve *BM* and *CM* and on account of this is perpendicular to the surface. Hence the locus of the normal is found [a basic necessity for Ch.4] by taking

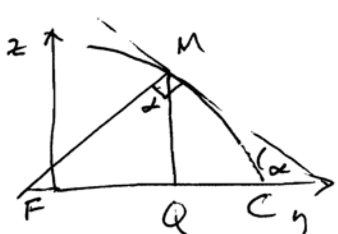
$$AH = x + Pz \text{ and } HN = -Qz - y.$$

[The use of the subtangents is set out in the following sketches :



$$\tan\alpha = \frac{dz}{dx} = \frac{QE}{MQ} = \frac{QE}{z}$$

$$\text{or } QE = z\frac{dz}{dx} = Pz$$



$$\tan\alpha = \frac{dz}{dy} = -\frac{FQ}{MQ} = -\frac{FQ}{z} \quad \therefore \quad FQ = -z\frac{dz}{dy} = -Qz$$

]

Now in the determination of the radius of osculation of any curve on the given surface let the position for two elements of the curve be *Mm* and $m\mu$ (Fig. 9), to which

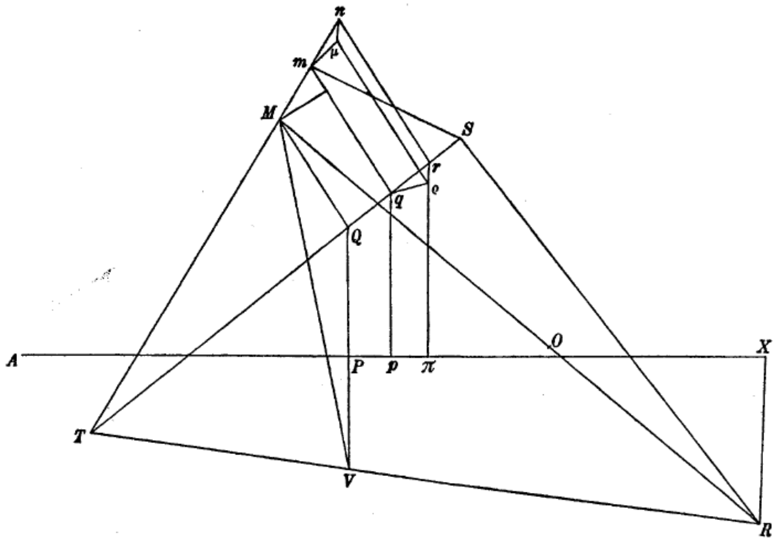


Fig. 9.

there corresponds the elements *Qq* and $q\rho$ in the plane *APQ* , and on the axis *AP,* I assume that the elements *Pp* and $p\pi$ are equal. Therefore we have :

$$Pp = p\pi = dx, \quad pq = y + dy, \quad \pi\varrho = y + 2dy + ddy, \quad Qq = \sqrt{(dx^2 + dy^2)},$$

$$q\varrho = \sqrt{(dx^2 + dy^2)} + \frac{dyddy}{\sqrt{(dx^2 + dy^2)}},$$

$$qm = z + dz, \quad \varrho\mu = z + 2dz + ddz, \quad Mm = \sqrt{(dx^2 + dy^2 + dz^2)}$$

and

$$m\mu = \sqrt{(dx^2 + dy^2 + dz^2)} + \frac{dyddy + dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Qq and *Mm* are produced in both directions, of which the first meets $\pi\rho$ in *r*, and the other crossed the normal *rn* to the plane *APQ* in *n*, [p. 28] and as $Pp = p\pi$

$$qr = Qq \text{ and } mn = Mm, \text{ also } \pi r = y + 2dy, \text{ and } rn = z + 2dz.$$

Now the normal *mS* is drawn to the element *Mm* in the plane *Qm,* crossing *Qq* produced at *S*; hence we have

[Note : The triangles with hypotenuse *Mm* and *mS* are similar; the original equation below has *QS* rather than *qS* as the subject of the formula, which is incorrect; this mistake is perpetuated in the *O.O*. version as well. If QS is required, then it is given by :
$QS = \frac{dx^2 + dy^2 + dz^2 + zdz}{\sqrt{dx^2 + dy^2}}$; all quantities to first order in the elements.]:

$$qS = \frac{(qm - QM)\,QM}{Qq} = \frac{zdz}{\sqrt{(dx^2 + dy^2)}}.$$

Now with *SR* drawn in the plane *APQ* perpendicular to *QS,* then all the lines drawn from *m* to *SR* are normal to the element *Mm*. Therefore the radius of osculation of the curve $Mm\mu$ is one of these normals. Now that one of these normals agrees with the radius of osculation, which lies in that plane, in which the elements *Mm* and $m\mu$ have been placed. On account of which it is required to determine this plane. Now the elements *mn* and $n\mu$ are in this plane; thus each produced as far as the plane *APQ* gives the intersection of that plane with the plane *APQ*. But *nm* or *mM* crosses with the plane *APQ* at *T*, where it crosses the element *Qq* produced. Therefore we have :

$$QT = \frac{z\sqrt{(dx^2 + dy^2)}}{dz}.$$

$n\mu$ itself is parallel to MV, situated in the plane $mn\mu$; now this line *MV* is incident at *V* in the plane *APQ* and from this ratio gives *QV*:

$$(rn - \rho\mu) : r\rho = QM : QV;$$

and thus *QV* becomes

$$QV = \frac{zddy}{ddz}.$$

[since $rn = z + 2dz;\ r\rho = dy + ddy;\ \rho\mu = z + 2dz + ddz;$ then $QV = z.\frac{r\rho}{rn - \rho\mu} = \frac{z(-ddy)}{ddz}$][p. 29] On account of this, the line *TV* produced is the intersection of the plane $mn\mu$ with the plane *APQ*, whereby the line *MR*, which is drawn from the crossing of the lines *SR* and *TV* is likewise normal to *Mm* and lies in the plane $mn\mu$, and therefore the line *MR* is put as the radius of osculation of the curve at M.

[Thus, above it is shown that all lines drawn from *m* intersecting *SR* are normal to the element *Mm*, and now we have the intersection of the plane containing the adjoining element $m\mu$ with this plane as the line *TV* extended to *R*. Euler asserts that *MR* is also normal to *Mm* , at the other end of the element, as *RM* lies in the plane of the curve $mn\mu$, and is normal to one element; for in the limiting process, we can presume that *m* and *M* coalesce.]

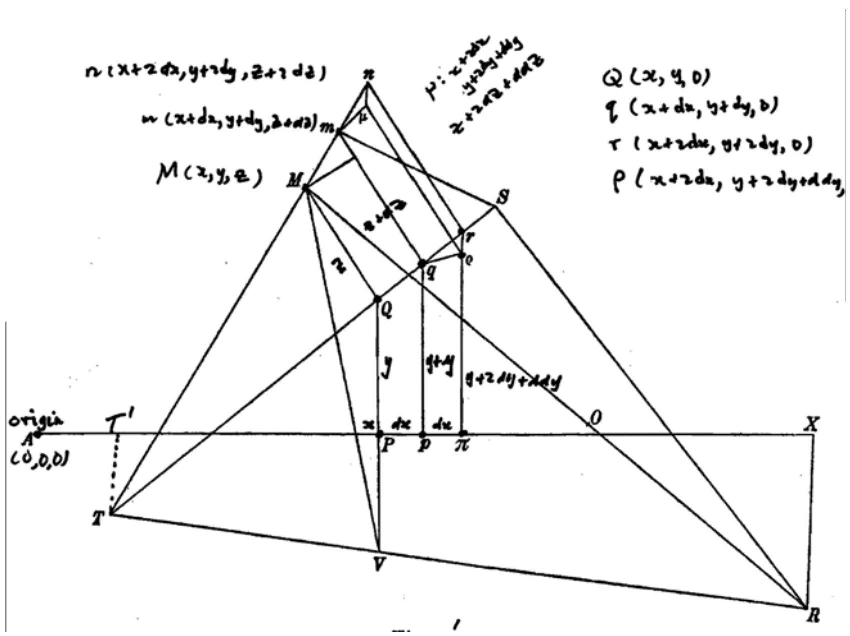
From these, the point R can be found in this manner : that is, with *RX* drawn perpendicular to AP produced,

$$AX = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy} + x$$

and

$$XR = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2 + dy^2)ddz - dydzddy} - y.$$

[***Digression*** : It seems that Euler used an *ab initio* cross product method twice to get these results, as we now demonstrate. We have annotated Fig. 9 and put some of the coordinates on this diagram. Note that derivatives are always put positive by Euler in his diagrams, and we have followed this rule, even though they must be negative in the diagram; thus, the restrictions of the diagram does not interfere with the mathematics.



Mn and *nμ* are elements on the surface; they are small enough compared to the local curvature that they can be considered as straight, and according to Euler's habit, all the 'bending' occurs at the beginning of the second element *nμ*. The first element extended an equal length can be considered as an element of a tangent line at *n* in the direction *Mn*. The normal to the plane *Mmμ* is the vector $\underline{N}$ formed from the cross product of the vectors representing *Mn* and *mμ* :

$$\underline{N} = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ dx & dy + ddy & dz + ddz \end{vmatrix} = i(dyddz - dzddy) - jdxddz + kdxddy$$, where *i, j*, and *k* are unit vectors along the *x, y*, and *z* axes. Now, the direction of the radius of curvature *RM* is perpendicular to both $\underline{N}$ and to the line *TM*, which acts in the direction (*dx, dy, dz*). Thus, the direction ratios of RM, or *d.r.RM* are found from the cross product of $\underline{N}$ with (*dx, dy, dz*) : i. e. $d.r.RM = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ dyddz - dzddy & -dxddz & dxddy \end{vmatrix}$.

MR hence has the direction ratios
$dx(dyddy + dzddz), dydzddz - (dx^2 + dz^2)ddy, (dx^2 + dy^2)ddz - dydzddz.$
Any point on the line MR can hence be represented by :
$x + t.dx(dyddy + dzddz); y + t[dydzddz - (dx^2 + dz^2)ddy]$, and $z + t[(dx^2 + dy^2)ddz - dydzddz]$.
for different values of the parameter *t*. The point R is taken to lie in the z = 0 plane, and hence $t = z/[(dx^2 + dy^2)ddz - dydzddz]$, from which the corresponding *x* and *y* values can be found, as in Euler's equations above. We now return to the text.]

In which therefore the normal *MN* (Fig. 8) falling in the direction of the radius of curvature, must have $AH = AX\ [= x + Pz]$ and $XR = HN\ [= -y - Qz]$; Hence,

$$P(dx^2 + dy^2)ddz - Pdydzddy = dxdyddy + dxdzddz$$

and

$$-Q(dx^2 + dy^2)ddz + Qdydzddy = dx^2ddy + dz^2ddy - dzdyddz.$$

Which equations indeed agree with each other ; for with these solved together, we obtain

$$Pdx + Qdy = dz,$$

which is the equation setting out the nature of the surface itself. Therefore either of these equations solved with this equation $dz = Pdx + Qdy$ gives the curve traversed by the body on the proposed surface. Q.E.I.

Corollary 1.

69. Therefore for the proposed line described on the surface, we have from the above equations :

$$ddz : ddy = Pdydz + dxdy : Pdx^2 + Pdy^2 - dxdz.$$

But since $dz = Pdx + Qdy$, this becomes

$$ddz : ddy = Pdz + dx : Pdy - Qdx$$

or

$$Pdyddz - Qdxddz = Pdzddy + dxddy.$$

Corollary 2. [p. 30]

70. If the other equation is taken and from both sides is subtracted

$$Qdz^2ddz - dy^2ddy,$$

there is obtained

$$-Q(dx^2 + dy^2 + dz^2)ddz + Qdydzddy + dy^2ddy$$
$$= (dx^2 + dy^2 + dz^2)ddy - Qdz^2ddz - dzdyddz.$$

Which is the actual equation given for the shortest line described in any surface in the Transactions of the St. Petersburg Academy of Sciences [Comm. Acad. Petr. E009], Vol. III.

Scholium 1.

71. As in this case in which the body is not acted on by any forces, the direction of the radius of curvature must agree with the normal to the surface, thus in the other cases, when the body is acted on by forces, these lines must constitute a given angle :

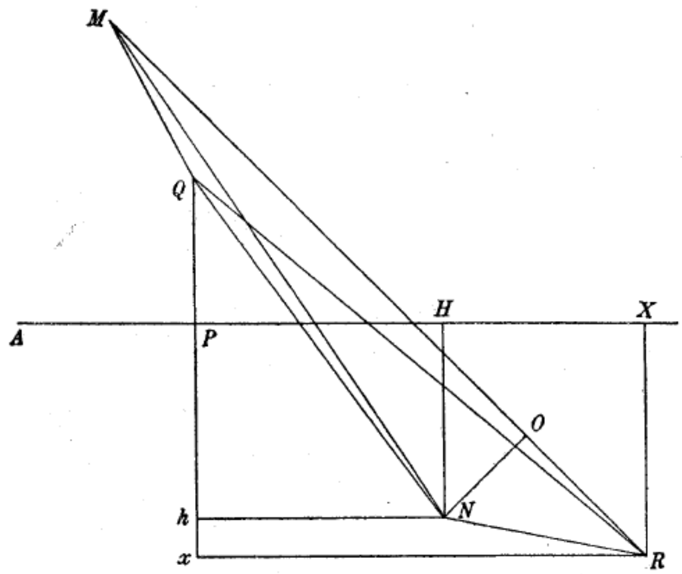


Fig. 10.

On account of which the angle is generally found between *MN* the normal to the surface and *MR* the direction of the radius of osculation ; that is, as we may now either put in place or as we have found (Fig. 10),

$$PQ = y, \quad QM = z, \quad PH = hN = Pz, \quad Qh = -Qz,$$

$$PX = Rx = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy}$$

and

$$Qx = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2 + dy^2)ddz - dydzddy}.$$

With *NR* drawn from *N* and the perpendicular *NO* is sent to *MR* ; there is produced

$$MO = \frac{MR^2 + MN^2 - NR^2}{2MR} = \frac{MQ^2 + Rx \cdot Nh + Qx \cdot Qh}{MR}$$

and

$$NO = \frac{\sqrt{(MR^2 \cdot MN^2 - (MQ^2 + Rx \cdot Nh + Qx \cdot Qh)^2)}}{MR}$$

$$= \frac{\sqrt{(MQ^2(Qx - Qh)^2 + MQ^2(Rx - Nh)^2 + (Rx \cdot Qh - Qx \cdot Nh)^2)}}{MR}.$$

[p. 31] Now the tangent of the angle RMN is equal to $\frac{NO}{MO}$ with the whole sine put equal to 1. Moreover with the assumed variables substituted above and with the equation called upon

$$dz = Pdx + Qdy$$

the tangent of the angle *NMR* becomes equal to

$$\frac{ddy(dx + Pdz) - ddz(Pdy - Qdx)}{(ddz - Qddy)\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Hence with the angle vanishing the equation becomes

$$ddz : ddy = Pdz + dx : Pdy - Qdx$$

as above (69).

Scholium 2.

72. Now the length of the radius of osculating *MO* (Fig. 9) is found from the angle $nm\mu$ with the aid of this ratio : as the sine of the angle $nm\mu$ is to the total sine, thus *Mm* is to *MO*. Now we have

$$n\mu = \sqrt{(ddy^2 + ddz^2)} \text{ and } mn - m\mu = \frac{-dyddy - dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}},$$

hence the perpendicular from *n* in $m\mu$ produced is equal to

$$\frac{\sqrt{(dx^2ddy^2 + dz^2ddy^2 + dx^2ddz^2 + dy^2ddz^2 - 2dydzddyddz)}}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Whereby this perpendicular is to $\sqrt{(dx^2 + dy^2 + dz^2)}$ as $\sqrt{(dx^2 + dy^2 + dz^2)}$ is to MO, and thus the radius of osculation is produced :

$$MO = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2 + ddz^2) + (dyddz - dzddy)^2)}}.$$

Moreover this formula for the radius of osculation is of help in the following proposition, in which we investigate the force that the body exercises on the surface. [p. 32]

Scholium 3.

73. From this general expression for the radius of curvature there arises an expression for the shortest line, if it is solved with this equation :

$$ddz = \frac{ddy\,(Pdz + dx)}{Pdy - Qdx} \text{ and in place } dz = Pdx + Qdy.$$

Moreover the radius of osculation is produced :

$$\frac{(dx^2 + dy^2 + dz^2)\,(Pdy - Qdx)}{dx\,ddy\sqrt{(P^2 + Q^2 + 1)}} = \frac{(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2 + 1)}}{ddz - Qddy}$$
$$= \frac{(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2 + 1)}}{dPdx + dQdy}.$$

And this expression gives the radius of osculation of the curve described on the proposed surface by a body acted on by no forces.

PROPOSITION 10.

Theorem.

74. *The force that a body moving on a surface under the action of no external forces exercises on the surface is made normally to this surface towards the convex side and has the ratio to the force of gravity as the height corresponding to the speed of the body is to half the radius of osculation of the curve described by the body.*

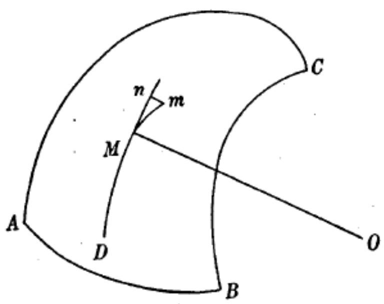


Fig. 7.

Demonstration.

Let *DMm* (Fig. 7) be the curve on the surface *ABC* described by the body, the height corresponding to the speed of the body is equal to v and the radius of osculation of the curve *MO* is equal to r. Because the body is able to move freely from *M*, it can progress along the element *Mn*, now the surface brings it about that the body advances along the element *Mm,* with the distance *nm* being perpendicular to the surface, and the surface is required to exert a force of such a size along *mn* that the body follows the direction of *Mm* along the surface, departing from the direction *Mn*. [p. 33] Now this is performed by the force $\frac{2v}{r}$ acting normally to the surface along the direction of the radius of osculation *MO*. On this account the force of the body is normal to the surface , clearly acting along *mn*, and is equal to $\frac{2v}{r}$ with the force of gravity acting on the body taken as equal to 1. Q.E.D.

Corollary 1.

75. This is therefore the centrifugal force that the body exerts on a surface in a similar way to that when it is forced to move on a given line.

Scholium 1.

76. The force acting on the surface must necessarily be normal. For unless it is normal, it is possible to resolve it into two components, of which one is normal and the other is placed along the surface. Now of these only the normal devotes itself to pressing on the surface, while the other changes the motion of the body.

Corollary 2.

77. We find that the length of the line of the radius of osculation *r*, that the body describes with no forces acting on a proposed surface (73). With this assumed, the centrifugal force is equal to :

$$\frac{2v(ddz-Qddy)}{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}=\frac{2v(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}.$$

Scholion 2. [p. 34]

78. This centrifugal force acting on the surface and the above centrifugal force acting on a given curve that has been discussed above, have the same place in the equation; see Prop. 2 (20) with the adjoining corollaries and scholium. For the shortest line that the body can describe on the surface can be considered to resemble the channels along which a body can move, and then all that has been said concerning the motion in these channels prevails, which have been produced above for the motion upon a given line with no external forces acting.

PROPOSITION 11.

Problem.

79. *To determine the effect of any kind of force that acts upon a body on a given surface either in a vacuum or in a resisting medium.*

Solution.

For any body, the direction of the external force acting can be resolved into three parts : the first of which that we call *M*, the normal direction to the surface; secondly, we designate by *N* that direction normal to the motion of the body as well as being normal to *M*, and the direction of this is in the tangent plane of the surface; and the direction of the third force called *T* agrees with the direction of the motion, which is therefore the tangential force ; surely the first two are the normal forces. Now since the directions of these three forces are mutually normal to each other, neither is able to disturb the others. [p. 35] Whereby, we investigate the effect that each can produce.

The first external force *M*, the direction of which is normal to the surface, has no effect in changing the motion of the body, as the whole is expended in pressing upon the surface. Therefore *M* either diminishes or increases the force arising from the centrifugal force, as the direction of this falls on convex or concave parts of the curve. For that force acting towards an inner part of the curve ; the total force acting on the surface towards the outside is equal to

$$\frac{2v(dPdx + dQdy)}{(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2 + 1)}} - M$$

(77). For the force arising from the centrifugal force is diminished in this case by the force M.

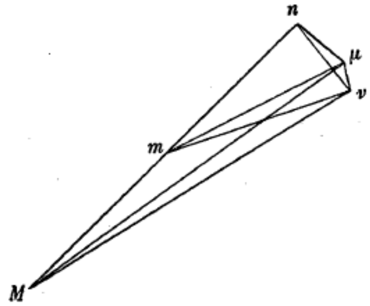


Fig. 11.

The second force *N*, since the direction of this is put both normal to the direction of the surface and to the direction in which the body moves, can only change the direction of the body and neither increase nor decrease the speed. Therefore this force makes the body move along the shortest line deduced, so that it no longer moves in a plane normal to the surface; therefore it is necessary to find the inclination of this plane of the shortest line in which the body moves, to the normal to the surface. Now this angle of inclination is equal to the angle that the radius of osculation of the line described makes with the normal to the curve, and which we have determined previously in general (71). After the body describes the element *Mm* with a speed corresponding to the height *v* [Italic v], [p. 36] is progressing, unless acted upon by the force *N* , along the element *mv* (Fig. 11) to *v* [Italic Greek 'nu' : Microsoft seems to think these two letters are the same.], thus as *Mm* and *mv* are two elements of the shortest line and placed in a plane normal to the surface; the direction of the normal force *N* in the plane; the direction of the normal force in the plane of the paper is reduced to that above, if indeed we put this force *N* above to be put in this position of the elements, as represented in the figure. Therefore this force has the effect, that the body is moving along the $m\mu$ and by the angle $vm\mu$ deflected by the direction *mv*. For this angle corresponding to the radius of osculation is equal to = $\frac{mv^2}{\mu v}$. Whereby when the force *N* generates this angle and the speed of the curve corresponds to the height *v*, from the effect of the normal force it becomes

$$N = \frac{2v \cdot \mu\nu}{m\nu^2} \text{ and thus } \mu\nu = \frac{N \cdot m\nu^2}{2v}.$$

Now where the inclination of the plane $Nm\mu$, in which the body actually moves, to the plane *Mmv*, which with the normal in the surface is found, the perpendicular *vn* is sent from *v* to the element *Mm* produced ; μn is also in the perpendicular *mn* and thus the angle μnv is the angle of inclination of the plane μmM to the plane *vmM*; and since μv is normal to *vn*, the tangent of this angle is equal to $\frac{\mu v}{nv} = \frac{N.mv^2}{2v.nv}$. But nv is determined from the inclination of the elements *Mm* and *mv* or the radius of osculation of the shortest line, of which *Mm* and *mv* are elements. Here the radius of osculation is *r*, it is $\frac{mv^2}{nv} = r$ and thus the tangent of the angle μnv is equal to

$$\frac{Nr}{2v} = \frac{N(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}{2v(dPdx+dQdy)}$$

with the value found substituted in place of *r* (73). [p. 37] For now this angle is equal to the angle, which the radius of osculation of the elements *Mm* and $m\mu$ actually described by the body agrees with the radius of osculation of the elements *Mm* and *mv* or with the normals on the surface. Moreover we have found the tangent of the above angle (71). Whereby with the equation made we have

$$\frac{ddy(dx+Pdz)-ddz(Pdy-Qdx)}{(ddz-Qddy)\sqrt{(dx^2+dy^2+dz^2)}} = \frac{N(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}{2v(dPdx+dQdy)},$$

from which equation the effect of the force *N* is determined. Or since it is the case that

$$ddz - Qddy = dPdx + dQdy,$$

this equation is found

$$ddy(dx+Pdz)-ddz(Pdy-Qdx) = \frac{N(dx^2+dy^2+dz^2)^{\frac{3}{2}}\sqrt{(P^2+Q^2+1)}}{2v}.$$

The third force *T*, since it is placed in the direction of the body, either only increases or decreases the speed. We can put this force to be an acceleration, the effect of this is expressed by this equation :

$$dv = T\sqrt{(dx^2+dy^2+dz^2)}.$$

And if the motion is made in a medium with resistance and the resistance is equal to *R*, only the tangential force *T* is to be diminished by the resistance *R*. On which account we have :

$$dv = (T-R)\sqrt{(dx^2+dy^2+dz^2)}.$$

Q.E.I.

Corollary.

80. Therefore from the two equations, from one of which *v* is determined, and from the other *dv*, the one containing *v* solved with the position of the surface $dz = Pdx + Qdy$ [p. 38] determines the curve that the body describes on the above proposed surface.

Scholium 1.

81. The force *N* needs to be attended to well, in which place it acts, that is whether it inclines either to the right or to the left hand region of the motion of the body. For this indeed the different tangent of the angle μmv either positive or negative has to be taken. Concerning which we will not now be concerned, but defer further inquiry of this to the last chapter of this book.

Scholium 2.

82. Therefore we progress to the following chapter, in which we examine the motion of the body upon a given line in a vacuum. In the third chapter we investigate the motion of a body on a given line with a resisting medium. Finally in the fourth chapter we carefully examine motion on a given surface both for the vacuum and resistive medium cases.

CAPUT PRIMUM

DE MOTU NON LIBERO IN GENERE.

[p. 1]

DEFINITIO 1.

1. *Corpus non libere moveri dicitur, quando externa obstacula impediunt, quo minus iuxta eam directionem progrediatur, iuxta quam cum ratione motus insiti tum ratione potentiarum sollicitantium moveri deberet.*

Scholion 1.

2. In motu puncti libero, quem parte prima exposuimus, spatium, in quo corpus movebatur, ab omnibus obstaculis vacuum assumsimus; nunc vero spatium ita comparatum ponemus, ut corpori non liceat in quaque directione progredi propter firmos parietes transitum non permittentes.

[p. 2]

Corollarium 1.

3. Quando itaque corpus in motu suo obstaculum invenit ideoque eam directionem, secundum quam tendit, conservare non potest, tum vel quiescere vel in alia directione motum continuare debebit.

Corollarium 2.

4. In quanam autem directione corpus progrediatur post occursum obstaculi, ex circumstantiis tum motus tum positionis obstaculi iudicari debet.

Scholion 2.

5. Videtur haec doctrina ad motum corporum ex percussione pertinere, qua de re tamen hoc tamen hoc libro non agetur. Hoc vero libro alius generis obstacula assumimus, quae illam notitiam non requirunt. Sunt haec obstacula continua, quae motum puncti restringunt neque ullam reflexionem admittunt; cuiusmodi est tubus vel canalis sive rectus sive incurvatus, in quo corpusculum motum continuare debet. Hoc casu via penitus praescribitur, in qua corpus progredietur, neque propter tubi firmitatem inde egredi poterit. Quare cum hic loco corporis punctum consideremus, hac positione punctum in data linea moveri debebit neque ex ea excedere poterit.

Scholium 3.

6. Duas autem hoc libro pertractabimus motus impediti seu restricti species, [p. 3] quarum primae modo mentionem fecimus quaeque complectitur motus punctorum super data linea sive recta sive curva. Altera species minus restringit motus libertatem; superficiem enim tantum praescribit, in qua corpus perpetuo versari debeat. Atque has duas motus impediti species isto libro sumus expondituri.

Corollarium 3.

7. Quae igitur in prima specie sunt inquirenda, sunt corporis seu potius puncti celeritas in quovis lineae praescriptae loco, pressio in hanc lineam et tempus, quo punctum datam viae portionem percurrit.

Corollarium 4.

8. Circa motus alterius speciei autem praeter haec inveniri debet ipsa linea, quam corpus super superficie data describit. Quarum rerum fontes hoc primo capite aperiemus.

Scholion 4.

9. Hoc vero capite primum investigabimus motus utriusque speciei, si corpus a nullis potentiis sollicitetur. ubi ostendimus, qua celeritate id progredi debeat et quanta vi ubique tam lineam datam quam superficiem datam premat. Sed si superficies tantum data fuerit, praeterea viam determinabimus, in qua corpus movebitur a nullis potentiis sollicitatum. Deinde vero principia exponemus, ex quibus iudicari licebit, quae mutationes a potentiis sollicitantibus tam absolutis quam relativis oriantur, [p. 4] quo in sequentibus capitibus singula distincte deducere queamus.

Scholion 5.

10. In his autem motibus tam super lineis quam superficiebus datis animum ab omni frictione abstrahimus neque ullam motus retardationem ponemus. Quamobrem lineae et superficies, super quibus puncta moveri ponuntur, levissimae concipi debent et omni asperitate destitutae, ne motus retardationi propter eam sit obnoxius. Motum rotatorium quoque omnino ex animo profligari oportet, cum ex eo mutationes in motu oriantur, quae demum in sequentibus explicari possunt. Hanc ob rem punctum quasi rependo moveri concipiendum est, ut eius pars quaeque, si modo in puncto partes concipi possunt, eundem habeat motum.

Scholion 6.

11. Quae igitur in praecedente libro traditae sunt et in hoc de motu punctorum tradentur, ad corpora finitae magnitudinis quoque accommodari possunt, si modo eorum motus sibi sit perpetuo parallelas et omnes partes corporis aequali motu sint praeditae. Hoc vero ex sequentibus libris clarius apparebit, quibus casibus finitorum corporum motus a motu punctorum non discrepet. Quocirca in his libra ideo punct tantum consideramus, quia, ut partibus destituuntur, ita etiam in partibus diversi motus inesse nequeunt.

PROPOSITIO 1.
[p. 5]
Theorema.

12. *Corpus seu punctum, quod super linea data movetur et a nullis potentiis sollicitatur, perpetuo eandem celeritatem conservabit, si modo illius lineae duo quaeque elementa contigua nusquam finitae magnitudinis angulum constituant.*

Demonstratio.

Quia corpus, dum in linea *AM* (Fig. 1) movetur, a nulla potentia sollicitatur neque frictioni ullus conceditur locus, motus corporis aliter variari nequit, nisi quatenus linea *AM* impedit, quo minus corpus libere moveri possit; ex quo, quae celeritatis immutatio oriri debeat, investigandum est. Sit celeritas, quam corpus in *M* habet, = *c*; hac igitur celeritate corpus, si libere moveretur, in tangente *Mv* progrederetur; quod vero, quia corpus curvam *AM* deserere non potest, fieri nequit, sed corpus cogitur per *Mm* progredi. Hanc ob causam concipiatur motus corporis secundum Mv resolutus in motum per Mm et motum per *Mn*, existente *Mmvn* parallelogrammo rectangulo. Perspicuum hic est motum per *Mn*, cuius directio est normalis in curvae elementum *Mm*, penitus absorberi neque ullum effectum in celeritate immutanda habere posse. Corpus igitur altero motu progredietur in *Mm* celeritate, quae est ad pristinam celeritatem ut *Mm* ad *Mv*; quare celeritatas, qua corpus elementum *Mm* describit, erit = $\frac{Mm.c}{Mv}$. Quoniam vero *Mvm* est triangulum ad *m* rectangulum ideoque $Mm < Mv$, [p. 6] celeritas haec minor erit quam prior *c* atque celeritatis decrementum erit $= \frac{(Mv-Mm)c}{Mv}$. Ad huius valorem inveniendum sit *MO*, radius osculi curvae in *M*, = *r* et elementum *Mm* = *ds*; eritque, ob ang. *O* = ang. *mMv*, $MO : Mm = Mm : mv$, ex quo prodit $mv = \frac{ds^2}{r}$ atque

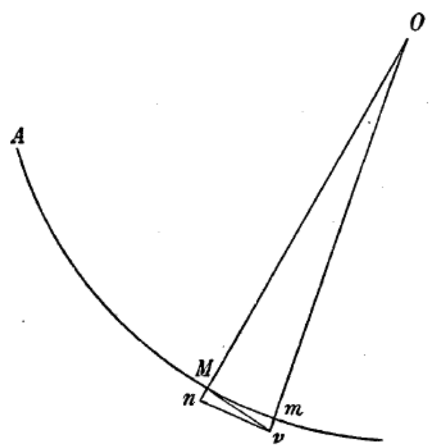


Fig. 1.

$$Mv = \sqrt{\left(ds^2 + \frac{ds^4}{r^2}\right)} = \frac{ds}{r}\sqrt{(r^2 + ds^2)} = ds + \frac{ds^3}{2r^2}.$$

Ex hoc iam obtinebitur decrementum celeritatis, dum corpus curvae elementum ds percurrit, $= \frac{cds^2}{2r^2}$, cuius integrale dabit decrementum celeritatis, dum corpus finitam curvae AM portionem percurrit. At expressio $\frac{cds^2}{2r^2}$ aequavalet differentiali secundi gradus; eius ergo integrale erit differentiale primi gradus. Quamobrem decrementum celeritatis, postquam corpus quantumvis arcum curvae datae percurrit, erit infinite parvam atque corpus motu uniformi feretur per totam curvam *AM*, si modo radius osculi *r* nusquam fuerit infinite parvus. Q.E.D.

Corollarium 1.

13. In omni igitur curva, in qua radius osculi nusquam est infinite parvus, corpus movebitur uniformiter, siquidem a nullis potentiis sollicitatur neque frictionem patitur.

Corollarium 2.

14. Si radius osculi est infinite parvus, tum $\frac{cds^2}{2r^2}$ vel est quantits finita vel differentiale primi gradus. [p. 7] Illo casu corpus finitum celeritatis gradum amittet, hoc vero tantum infinite parvum.

Corollarium 3.

15. Cum autem istius modi puncta in omnibus curvis sint rara et a se invicem dissita, corpus tamen arcum inter duo talia puncta interceptum motu uniformi percurret.

Scholion 1.

16. Casus, quibus corpus celeritatis finitum decrementum subito patitur, alii non esse possunt, nisi ubi curva habet cuspides. His enim in locis corpus directe reverti cogitur et normaliter in punctum cuspidis impingit. Tunc igitur corpus non solum finitum celeritatis gradum amittet, sed omnino omnem motum amittere debebit; nisi forte corpus ponatur elasticum, quo casu celeritate, qua incurrit, reflectetur atque ita motum uniformem conservabit. In cuspide enim duo elementa angulum infinite acutum constituunt.

Scholion 2.

17. Praeter cuspides vero alia dari possunt in curvis puncta, in quibus radius curvedinis est infinite parvus; quia vero duo quaeque elementa contigua fere in directum sunt posita et angulus deinceps positus est infinite parvus, fieri non potest, ut ex demonstratione apparet, ut corpus finitum celeritatis decrementum patiatur. Quamobrem, cum istiusmodi puncta sint rara, corpus nihilominus motu aequabili movebitur. [p. 8]

Corollarium 4.

18. Si igitur corpus motum fuerit elasticum, in quacunque curva semper motu aequabili feretur; at si non sit elasticum, cuspides tantum motum turbabunt, dum eum prorsus tollunt.

Scholion 3.

19. Ut haec clarius percipiantur, sint duo curvae elementa *AB, BC* (Fig. 2) et anguli *ABC*, quem constituunt, deinceps positus *CBD* infinite parvus, cuius sinus sit *dz* posito sinu toto = 1. Quia corpus, postquam elementum *AB* descripsit, vi insita in *BD* progredi conatur celeritate priore, quae sit c, eius motus duplex concipiatur, alter in directione *BC*, alter in directione ad *BC* normali, qui in effectum duci non potest. Demisso igitur ex *D* in *BC* perpendiculo *DC* corpus altero motu per *BC* movebitur celeritate, quae est ad priorem ut *BC* ad *BD*, i. e. ut $\sqrt{(1-dz^2)}$ ad 1. Per BC idcirco habebit celeritatem = $c\sqrt{(1-dz^2)}$ seu $c-\frac{cdz^2}{2}$; quare celeritatis decrementum erit $\frac{cdz^2}{2}$, quod aequivalet differentiali secundi gradus. Ex quo intelligitur, quamdiu in quaque curva angulus CBD fuerit infinite parvus, corpus motu aequabili esse progressurum. At in omni curva angulus CBD vel est infinite parvus vel angulus ABC ipse, quod in cuspidibus accidit. Consequenter cuspides tantum motus uniformitatem perturbant, nisi corpus fuerit elasticum, quo casu nihilominus motus uniformitas conservatur. [p. 9]

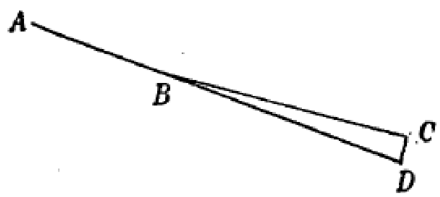


Fig. 2.

PROPOSITIO 2.

Theorema.

20. *Dum corpus motu uniformi in curva AM* (Fig. 1) *movetur, in singulis punctis M premet curvam normaliter vi, quae est ad corporis vim gravitatis ut altitudo eius celeritati debita ad dimimium radium osculi.*

Demonstratio.

Si corpus in curva *AM* libere moveri deberet motu aequabili, tum ubique vim adesse oporteret normalem corpus secundum *MO* trahentem tantum, quae se haberet ad corporis gravitatem ut altitudo celeratati corporis debita ad dimidium radium osculi *MO*, ut ex demonstatis libri praecedentis (165, 209, 552) apparet. Nisi enim talis vis adesset, corpus in linea recta progrederetur. Hoc autem casu canalis *AM*, in quo corpus inclusum concipitur, impedit, quo minus corpus in recta progrediatur. Quamobrem corpus tanta vi canalem normaliter premet secundum directionem *Mn*. Si enim talis vis normalis adesset, corpus in canali *AM* libere moveretur neque illum premeret; hac vero vi absente, ut hic ponimus, necesse est, ut corpus ipsum canalem tanta vi premat. Q.E.D.

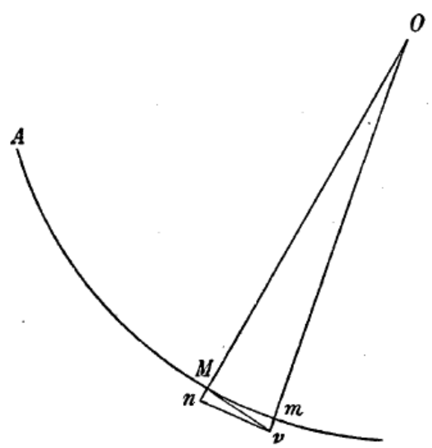


Fig. 1.

Corollarium 1.

21. Si igitur altitudo celeritati corporis debita ponatur v et radius osculi $MO = r$ atque gravitas corporis = 1, quam scilicet haberet, si in superficie terrae esset positum, erit vis, qua corpus canalem in *M* secundum Mn premet, = $\frac{2v}{r}$.[p. 10]

Corollarium 2.

22. Si corpus maiore vel minore celeritate moveretur in curva AM, tum pressio in M maior vel minor esset in duplicata celeritatis ratione, quia altitudino v quadrato celeritatis est proportionalis.

Corollarium 3.

23. Directio huius pressionis est normalis in curvam et directe contratia est positioni radii osculi *MO*. Quare radius osculi in alteram curvae partem productus dabit directionem huius pressionis.

Corollarium 4.

24. Si corpus in linea recta movetur, haec pressio erit nulla ob radium osculi infinitum. Hoc quoque ex ipsa motus natura perspicuum est. Corpus enim motum in recta uniformiter sponte progreditur et hanc ob rem canalem rectum non premit.

Corollarium 5.

25. Si curva AM fuerit circulus. pressio ubique erit eadem. Eo vero maior erit, quo minor est radius circuli. Existente enim celeritate eadem, pressio erit reciproce ut radius circuli.

Scholion 1.

26. Quo corpus in curva AM libere moveri possit uniformiter [p. 11], necesse est, ut secundum normalem MO trahatur vi = $\frac{2v}{r}$. Ex quo intelligi licet corpus tanta vi in plagam oppositam niti; alioquin enim illa vi non esset opus ad corpus in curva conservandum. Dum igitur corpus in canali AM moveri cogitur eius nisus a vi normali tollitur, hunc nisum re ipsa in canalem exercebit. Quamobrem talis canalis tantam firmitatem habere debebit, ut hanc pressionem sustinere queat.

Corollarium 6.

27. Apparet igitur corpus motum sine ullo celeritatis dispendio effectum edere posse, qui scilicet consistit in pressione definita.

Corollarium 7.

28. Ex motu ergo solo pressio oriri potest. Quamobrem, uti ex pressione seu a potentiis motus generatur, ita quoque ex motu pressio oriri potest.

Scholion 2.

29. Intelligitur hinc, quod iam supra inuimus libro primus (102), incertum esse, utrum motus potentiis debeatur an vero potentiae motui. Videmus enim in mundo utrumque, potentias nempe et motum, existere; utrum igitur alterius sit causa, quaestio est tum ex ratione tum ex observationibus decidenda. Rationi quidem minime consentaneum videatur corporibus conatus insitos tribuere, multo minus potentias per se existentes statuere. Praeterea vera is phaenomenorum causas genuinas dedisse censendus est, qui omnia a motu orta demonstraverit. [p. 12] Motum enim semel existentem perpetuo conservari debere clare ostendimus supra (63); hic vero, quemadmodum ex motu potentiae oriantur, exposuimus. Quemadmodum vero potentiae sine motu vel existere vel conservari queant, concipi non potest. Quamobrem concludimimus omnes potentias, quae in mundo conspiciuntur, a motu provenire; atque diligenti scrutatori incumbit investigare, ex quonam quorumque corporum motu quaelibet potentiae in mundo observata ortum suum habeat.

Scholion 3.

30. Cum difficile intellectu sit, quomodo talis effectus, pressio scilicet continua, a corpore moto sine ullo celeritatis dispendio oriatur, operae pretium erit in huius rei causam inquirere. Vidimus in praecedente propositione motum corporis in curva linea non absolute aequabilem esse, sed celeritatem revera decrementum pati, dum corpus per singula elementa curvae movetur. Haec vero decrementa differentialibus secundi gradus aequivalent, ut etiam infinities repetita celeritatem corporis infinite parum tantum minuere queant. Huic igitur infinite parvo celeritatis decremento pressionem adscribe debere iudico ; in hacque sententia eo magis confirmor, quod, quo maius sit hoc celeritatis decrementum, eo maior quoque existat pressio. Cum pressio in M sit $\frac{2v}{r}$ hacque vi totum elementum Mm, [p. 13], dum percurritur, prematur, licebit huius pressonis effectum in $Mm = ds$ exponere per $\frac{2vds}{r}$. Supra vero decrementum celeritas, dum corpus elementum Mm percurrit, inventum est $\frac{cds^2}{2r^2}$ (12). Quod autem ibi erat c, hic nobis est $\sqrt{v}$; ergo cum esset $-dc = \frac{cds^2}{2r^2}$, erit

$$-\frac{dv}{2\sqrt{v}} = \frac{ds^2\sqrt{v}}{2r^2} \quad \text{seu} \quad -dv = \frac{vds^2}{r^2}.$$

Habebitur ergo $-4vdv = \frac{4v^2ds^2}{2r^2}$ = quadrato pressionis, quam sustinet elementum Mm.

Corollarium 8.

31. Quadratum pressionis ergo in Mm exercitae aequivalet decremento ipsius $2v^2$. Atque si hoc decrementum aequale fuerit ipsi ds^2, tum pressio aequalis est vi gravitatis, ex quo comparatio harum pressionum cognoscitur.

Corollarium 9.

32. His ergo concessis istud infinities infinite parvum celeritatis decrementum sufficit ad pressionem finitam producendam. Quamdiu enim ipsius v^2 decrementum homogeneum est ipsi ds^2, pressio est finita; sin vero id decrementum infinities maius existeret quam ds^2, pressio quoque foret infinite magna.

Definitio 2.

33. Pressio haec, quam corpus in linea curva motum exercet in hanc lineam, vocatur vis centrifuga, eo, quod eius directio a centro circuli oscularis O tendit. [p. 14]

Corollarium 1.

34. *Vis centrifuga ergo est ad vim gravitatis ut altitudino celeritati debita ad dimidium radium osculi.*

Corollarium 2.

35. Quando ergo corpus in linea curva moveri cogitur, hanc curvam vi centrifuga premit, etiamsi a nulla potentia sollicetur.

Scholion.

36. Quando vero corpus a potentiis quoque sollicitatur, pressio quoque in canalem ab his potentiis orietur tumque canalis duplici ratione premetur, partim nempe a potentiis, partim a vi centrifuga. Nunc igitur, quid potentiae in corpus non libere motum valeant, investigandum est.

PROPOSITIO 3.

Theorema.

37. *Si corpus, quod in canali AM* (Fig. 3) *movetur, sollicitetur in M a potentia MN, cuius directio normalis est in curvam AM, celeritas neque augebitur neque minueter, sed tota potentia in premendo canali consumetur.*

Demonstratio.

Ex priore libro (164) manifestum est potentiam, cuius directio in directionem motus sit normalis, celeritatem neque augere neque minuere. Quanquam hoc enim ibi de motu libero rigore locum habet, [p. 15] cum potentia normalis corpus neque in consequentia neque in antecedentia trahat. In motu libero vero potentia normalis directionem corporis immutat, quem effectum hoc loco habere non potest. Hac igitur vi apprimetur corpus ad canalem et consequenter tanta vi canalem premet in directionem *MN*. Q.E.D.

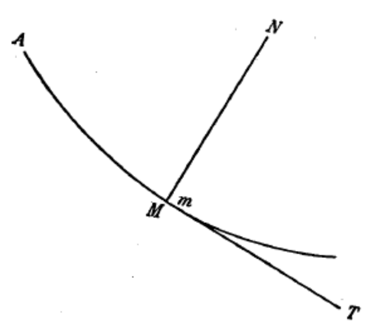


Fig. 3.

Corollarium 1.

38. Directio igitur talis vis normalis vel incidit in directionem vis centrifugae vel ei directe est contraria. Illo casu auget vim centrifugam, hoc casu minuit.

Corollarium 2.

39. Quia directio vis centrifugae in convexam curvae partem incidit, eius effectus augebitur, si normalis vis directio in eandem plagam incidit; at si normalis vis in concavam partem dirigitur, minuetur effectus.

Corollarium 3.

40. Si vis normalis furerit = N et vis centrifuga ut ante = $\frac{2v}{r}$, premetur curva vel vi $\frac{2v}{r}+N$, si hae vires fuerint conspirates, vel vi $\frac{2v}{r}-N$, si fuerent contrariae.

Corollarium 4.

41. Si vis normalis furerit aequalis et contrarius vi centrifugae, curva nullam pressionem sustinebit, seu corpus ex ea egredi non conabitur. Hoc ergo casu eandem curvam corpus libere describeret; [p. 16] id quod perspicuum quoque est ex vi normali, quae tum est $\frac{2v}{r}$; hac enim efficitur, ut corpus aequabiliter in quacunque curva libere moveatur.

PROPOSITIO 4.

Theorema.

42. *Si corpus, quod in canali AM* (Fig. 3) *movetur, in M sollicitetur a potentia, cuius directio sit secundum tangentem MT, huius effectus in hoc consistet, ut celeritatem corporis vel augeat vel diminuat eodem modo quo in moto libero.*

Demonstratio.

Quia huius potentiae directio est ipsa canalis tangens *MT*, canalis effectum huius potentiae impedire non potest; neque etiam in canalem haec potentia ullum effectum exercere poterit. Quamobrem augebit haec potentia vel diminuet celeritatem corporis, prout eius directio directioni corporis vel conspirans vel contraria fuerit, prorsus ac si corpus libere moveretur. Atque posita altitudine celeritati in *M* debita = *v*, elemento *Mm = ds* et vi *MT* = *T*, erit $dv = Tds$ accelerante potentia T; at retardante ea erit $dv = -Tds$. Q.E.D.

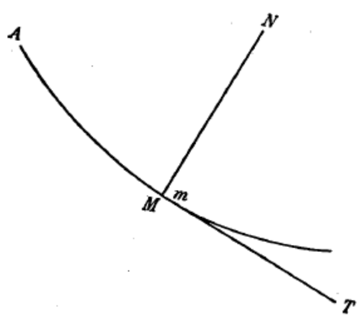


Fig. 3.

Corollarium 1.

43. In motu corporum igitur super lineis datis vis normalis pressionem tantum generat in eas, vis tangentialis vero celeritatem tantum afficit.

Corollarium 2.

44. Cum vis resistentiae effectum vis tangentialis retardantis praestet, [p. 17] eodem quoque modo aget in motum corporum super datis lineis ac in motum liberum. Si igitur praeter vim tangentialem accelerantem T affuerit resistentia R, prodibit ex ambabus coniunctim $dv = Tds - Rds$.

PROPOSITIO 5.

Problema.

45. *Si corpus super linea data AM* (Fig. 4) *moveatur in medio quocunque resistente et insuper sollicitetur a potentia absoluta, cuius directio sit MP, determinare effectum tam potentiae absolutae quam resistentiae nec non pressionem, quam curva AM sustinet.*

Solutio.

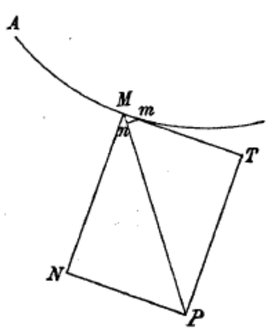


Fig. 4.

Sit altitudo celeritati in M debita = v, vis resistentiae = R et vis absoluta $MP = P$, cuius directio sit talis, ut sumto elemento $Mm = ds$ sit perpendiculum mn ex m in MP demissum = dx et $Mn = dy = \sqrt{(ds^2 - dx^2)}$. Resolvatur potentia P in has duas secundum MN normalem in curvam et MT tangentem trahentes; erit ob triangula MPT et Mmn similia vis normalis MN seu $PT = \frac{Pdx}{ds}$ et vis tangentialis $MT = \frac{Pdy}{ds}$ celeratem augens. Quia vero vis resistantiae celeritatem minuit, augebitur celeritas tantum ab excessu $\frac{Pdy}{ds} - R$; hanc ob rem erit (42) $dv = Pdy - Rds$.

Normalis vis $\frac{Pdx}{ds}$ vero efficit, ut curva in M tantundem prematur secundum directionem MN ad convexam curvae partem sitam. Quare, cum vis centrifuga in eandem plagam urgeat, [p. 18] quae est = $\frac{2v}{r}$ designate r radium osculi in M, erit vis totalis, qua curva in M secundum MN premitur, = $\frac{Pdx}{ds} + \frac{2v}{r}$. Unde tum motus corporis super curva tum curvae pressio in singulis punctis innotescit. Q.E.I.

Corollarium 1.

46. Ex his duabus formulis igitur accelerationem et pressionem exprimentibus omnia deduci possunt, quae ad motum super lineis datis pertinent.

Scholion 1.

47. Hic quidem unicam potentiam absolutam posuimus; nihilominus tamen satis ex eo intelligitur, quomodo plurium potentiarum effectus sit determinandus. Scilicet quemadmodum in motu libero fecimus, ita etiam hic singulae potentiae in binas, normalem nempe et tangentialem, sunt resolvendae, ex quibus colligendis una vis normalis unaque tangentialis oritur; quarum effectus per propositiones 3 et 4 determinari poterunt.

Scholion 2.

48. Hactenus igitur fundamenta exposuimus, ex quibus in sequentibus motum corporum super lineis datis determinare licebit. Antequam autem pro motu superficiebus datis similia principia tradamus, expedit, ut paucis ostendamus, quo modo motus super linea data in effectum deduci possit. [p. 19] Namque ope canalis, in quo corpus contineatur, talis motus minime produci poterit propter frictionem aliaque obstacula, quae tolli neutiquam possunt. Commodissime autem huiusmodi motus non liberi efficiuntur pendulorum ope, uti primum a Hugenio factum est [Chr. Huygens, *Horologium oscillatorum sive de motu pendulorum ad horologia aptato demonstrationes geometricae*. Paris 1673; *Opera varia*, Vol. 1, Lugduni Batavorum 1724, p. 89]; quamobrem hanc pendulorum ad institutum nostrum accommodationem sequenti propositione explicabimus.

PROPOSITIO 6.

Problema.

49. *Ope penduli efficere, ut corpus in data linea moveatur.*

Constructio.

Sit *AMB* (Fig. 5) curva proposita, in qua corpus moveri debeat; huius curvae construatur evoluta *AOC* laminaque secundum eius figurum incurvetur et firmetur. Tum filum huic laminae circumducatur, quod altero termino ad laminam sit affixum, altero vero termino in A annexum habeat corpus movendum. Quando igitur corpus incipit, perspicuum est id in curva *AMB* moveri debere, quia filum, dum a lamina separatur, hanc curvam evolutione describit.
Q.E.Fac.

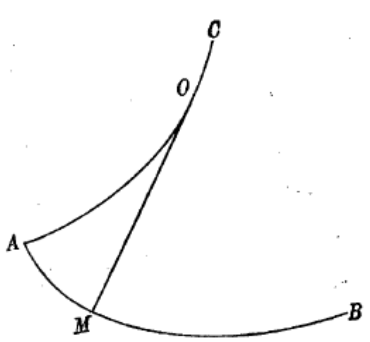


Fig. 5.

Corollarium 1.

50. Hac igatur ratione corpus in data curva progreditur atque frictionibus non est obnoxium. Quare tali motu commodissime per experimenta effici poterunt, quae in theoria inveniuntur. [p. 20]

Corollarium 2.

51. Ex doctrina de evolutionibus intelligitur fili partem *MO* a lamina separatam in curvam *AMB* esse normalem ipsumque eius radium osculi.

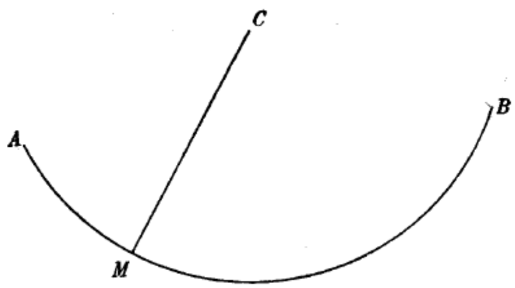


Fig. 6.

Corollarium 3.

52. Quo corpus in peripheria circuli *AMB* (Fig. 6) moveatur, lamina incurvata non est opus, sed filum altero termino *C* tantummodo in centro *C* peripheriae est figendum.

Corollarium 4.

53. Quia filum MO (Fig. 5) est radius osculi, vis centrifuga tota ad tendendum hoc filum impendetur. Quare hoc filum tum satis roboris habere, tum extensioni obnoxium non esse debet. Nisi enim eandem perpetuo longitudinem conservet, curvam desideratam non describet.

Corollarium 5.

54. Accedente potentia absoluta habebitur praeter vim centrifugam vis normalis, quae filum quoque tendet, si vi centrifugae fuerit conspirans. At si contraria fuerit, minuet tensionem fili, imo etiam, si maior fuerit, comprimet, quo casu evolutio nullius erit usus. Nam cum filum debeat esse flexile, compressioni resistere non poterit neque ideo impedire, quo minus corpus a curva AMB versus evolutam recedat.

Scholion 1.

55. Praeter hanc difficultatem ista curvarum per evolutiones generatio hoc quoque laborat defectu, [p. 21] quod linea recta produci nequeat; ad eam enim generandam filum requireretur infinite longum. Simili modo haec evolutio ad curvas accommodari non potest, quae alicubi radium osculi habent infinite magnum. Deinde etiam neque cuspide neque flexu contrario praeditae curvae hoc modo describi possunt. Quamobrem ita praxis locum tantum habet in curvis ubique finitam curvaturam habentibus, ad quod addi debet, ut pressio curvae totalis nusquam in curae concavam partem dirigatur.

Scholion 2.

56. Hugenius, qui primus evolutionis doctrinam excoluit, statim eam ad hunc ipsum usum adhibuit, uti ex eius egregio opere de horologio oscillatorio apparet. Cum enim invenisset oscillationes super cycloide omnes esse isochronas, motum super cycloide in horologia inferre volebat, quod per pendulum intra cycloides oscillans effecit. Cum enim cycloidis evoluta sit cyclois, hac ratione obtinuit, ut corpus filo annexum in cycloide movetur.

Scholion 3.

57. In hoc autem pendulorum motu maxime notari convenit praeter corpus motum filum quoque moveri debere, id quod ad institutum huius libri, in quo de motu puncti tantum agetur, minime pertinet. Praeterea motus corporis pendulo annexi non est sibi parallelus, sed circularis, circa centrum scilicet circuli curvam osculantis, qui motus pariter hoc loco non attingitur. [p. 22] Hoc igitur libro motum puncti duntaxat super linea vel superficie data examini subiiciemus mentemque tam a motu fili quam a motu circulari abstrahemus. In sequentibus autem motum pendulorum, ubi et motus fili et motus circularis in computum ducetur, ad motum puncti tantum reducemus, ita ut haec, quae hoc libro tractabuntur, nihilominus in praxi usum sint habitura. Quamobrem, ut iam monuimus, punctum motu sibi semper parallelo super curva seu superficie sine ulla frictione ferri est concipiendum.

PROPOSITIO 7.

Theorema.

58. *Si corpus a nullis potentiis sollicitatum moveatur in vacuo seu medio non resistente super superficie quacunque ABC* (Fig. 7), *motu feretur uniformi, animum ab omni frictione abstrahendo.*

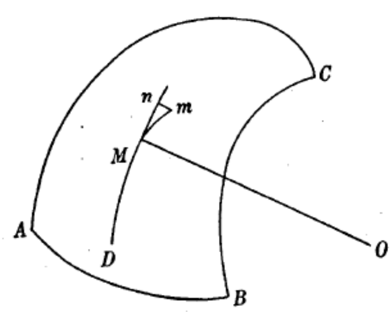


Fig. 7.

Demonstratio.

Cum corpus super linea data motum impressum continuare queat, multo magis super superficie data moveri poterit, eo, quod eius libertas minus est restricta. Sit igitur *DMm* linea, in qua corpus progreditur; haec erit vel recta vel curva. Si ista linea fuerit recta, dubium non est, quin corpus motu aequabili sit progressurum. Sin autem fuerit curva quae aequatione exprimi potest, duo quaeque eius elementa contigua vel proxime in directum erunt sita [p. 23] vel angulum infinite acutum constituent, quod in cuspidibus accidit.
Illo casu supra demonstratum est corpus nullum motus decrementum pati (12). In cuspidibus vero corpus quidem omnem motum amittet, nisi fuerit elasticum. Quamobrem, si motus tantum fiat in curva vel parte curvae cuspidibus carente, motus corporis erit aequabilis. Q.E.D.

Corollarium 1.

59. Patietur quidem corpus celeritatis decrementum, quoties directionem mutare cogitur, hoc vero differentiali secundi gradus aequivalet ideoque, etiamsi integretur, decrementum tamen infinite parvum producit.

Corollarium 2.

60. Si scilicet corporis celeritas fuerit c et radius osculi $MO = r$, erit decrementum celeritatis, dum corpus elementum ds percurrit, $= \frac{cds^2}{2r^2}$ (12).

Scholion.

61. Demonstratio huius propositione prorsus congruit cum demonstratione primae propositionis neque aliud est discrimen, nisi quod corpus illo casu in data linea moveri cogitur, hoc vero casu super superficie data viae quaerendae habeat libertatem. Quamobrem omnes annotationes, quae circa primam propositionem sunt factae, hic quoque valent. [p. 24] Videbimus ergo, quamnam viam corpus in superficie quacunque motum percurrere debeat.

PROPOSITIO 8.

Theorema.

62. *Via DMm* (Fig. 7), *quam corpus super superficie quacunque ABC motum describit, est linea brevissima, qua inter terminos D et M duci potest, si scilicet corpus in vacuo moveatur et a nullis potentiis sollicetetur.*

Demonstratio.

Descripserit corpus iam curvam *DM*; manifestum est corpus ex *M* in tangente *Mn* esse progressurum, nisi in superficie perseverare cogeretur. Quia igitur motus per *Mm* fieri non potest, resolvatur is in duos laterales, quorum alter in ipsa superficie sit dispositus, alterius vero directio in superficiem sit perpendicularis atque ideo penitus non in effectum deduci possit. Hanc ob rem ex *n* in superficiem demittatur perpendiculum nm; erit recta *Mm* elementum, in quo corpus ex *M* progredietur. Planum ergo *nMm*, in quo posita sunt et elementum *mM* et id, quod a corpore immediate ante est descriptum, erit normale in superficiem. At linea brevissima in quavis superficie ducta hanc habet proprietatem, ut planum, in quo posita sunt duo quaque elementa contigua, sit in superficiem normale. Quamobrem linea *DMm*, quae a corpore describitur, est linea brevissima in superficie *ABC*. Q.E.D. [p. 24]

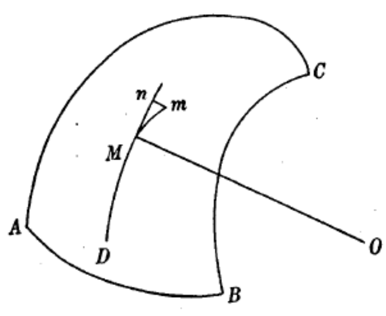


Fig. 7.

Corollarium 1.

63. Si ergo ex puncto *A*, in quo motus incipit, linea brevissima in superficie *ABC* secundum directionem motus ducatur, habibitur via, qua corpus motu uniformi movebitur.

Corollarium 2.

64. Quia filum tensum in superficie lineam brevissimam designat, ostendet filum tensum simul viam, in qua corpus super ea superficie movetur.

Corollarium 3.

65. Si igitur superficies proposita fuerit plana, corpus lineam rectam describet, quia haec in plano est linea brevissima. Atque in superficie sphaerica corpus in circulo maximo movebitur.

Corollarium 4.

66. Quia planum, in quo posito sunt duo curvae *DMm* elementa contigua, normale est in superficiem, radius osculi curvae vero in eodem plano sit positus et in curvam normalis, erit radius osculi curvae descriptae *MO* normalis in superficiem.

Scholion.

67. Quemadmodum in quavis superficie linea brevissima sit invenienda, a me primum ostensum est in Tomo III Comment. Acad. Imp. Petrop. [*De linea brevissima in superficie quacunque duo quaelibet puncta iungente*. See E09 in this series of translations.] Cum autem ibi ex alio principio lineam brevissimam determinaverim atque [p. 26] haec materia elementis nondum sit inserta, sequenti propositione lineam hanc brevissimam seu eam, quae a corpore describitur, determinare constitui.

PROPOSITIO 9.

Problema.

68. *In superficie quacunque determinare lineam, quam corpus a nullis potentiis sollicitatum, quod super ea movetur, describit.*

Solutio.

Ad naturam superficiei propositae exprimendam sumatur pro arbitrio planum *APQ* fixum (Fig. 8) in eoque recta *AP* pro axe . Tum ex quovis superficiei puncto *M* demittatur in hoc planum perpendiculum *MQ* et ex *Q* in axem *AP* perpendicularis *QP*. Positis nunc $AP = x$, $PQ = y$ et $QM = z$ natura superficiei dabitur per aequationem inter has tres variabiles *x, y* et *z* et constantes. Sit huius aequationis differentialis

$dz = Pdx + Qdy,$

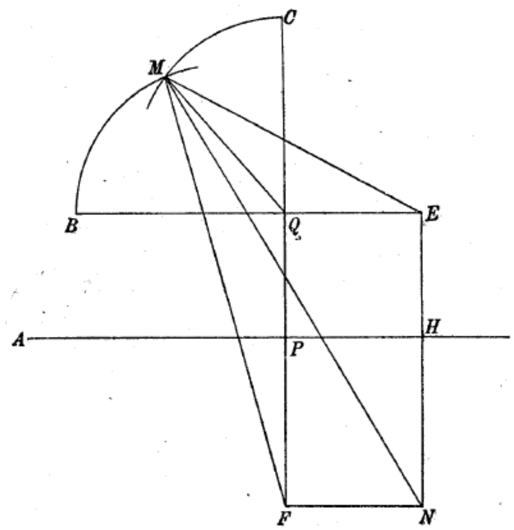


Fig. 8.

ex qua linea brevissima in hac superficie seu linea, quam corpus describit, determinari debet. Haec linea vero ex hoc determinatur, quod eius radius osculi in ipsam superficiei normalem indicat. Quamobrem primo normalem superficiei et deinde cuiusque in ea ductae curvae radium osculi determinabimus, quo postquam ex coincidentia harum linearum natura lineae quaesitae possit concludi. [p. 27]

Ad normalem in superficiem inveniendam secetur primo superficies plano *MQB*, existe *BQ* recta in plano *APQ* parallela axi *AP*, prodeatque ex hac sectione curva BM; cuius natura exprimetur hac aequatione $dz = Pdx,$ quae in locali pro superficie $dz = Pdx + Qdy$ oritur posita *y* constane seu $dy = 0$. Ducatur ad hanc curvam *BM* normalis *ME* rectae *BQ* productae in *E* occurrens; erit subnormalis $QE = \frac{zdz}{dx} = Pz.$ Ducta nunc *EN* perpendiculari ad *BE* quaevis recta *MN* a *M* ad *NE* ducta normalis erit in curvam *BM*. Simili modo superficies secetur plano *PQM* prodeatque sectio *CM*, cuius natura exprimetur aequatione inter *z* et *y* manente *x* constante, quae erit $dz = Qdy$. Sit *MF* normalis in hanc curvam; erit subnormalis $QF = \frac{-zdz}{dy} = -Qz$; signo negativo utor, quia subnormalem *QF* versus *P* cadere pono. Ducta nunc recta *FN* parallela axi *AP* quaevis recta ex *M* ad *FN* ducta normalis erit in curvam *CM*. Recta *MN* ergo, quae in punctum intersectionis *N* rectarum *FN* et *EN* cadit, perpendicularis in utramque curvam *BM* et *CM* et hanc ob rem perpendicularis erit in superficiem. Locus ergo normalis invenitur sumendo

$$AH = x + Pz \text{ et } HN = -Qz - y.$$

Ad determinandam vero radii osculi cuiusvis curvae in superficie data ductae positionem sint duo curvae elementa *Mm* et *mμ* (Fig. 9), quibus

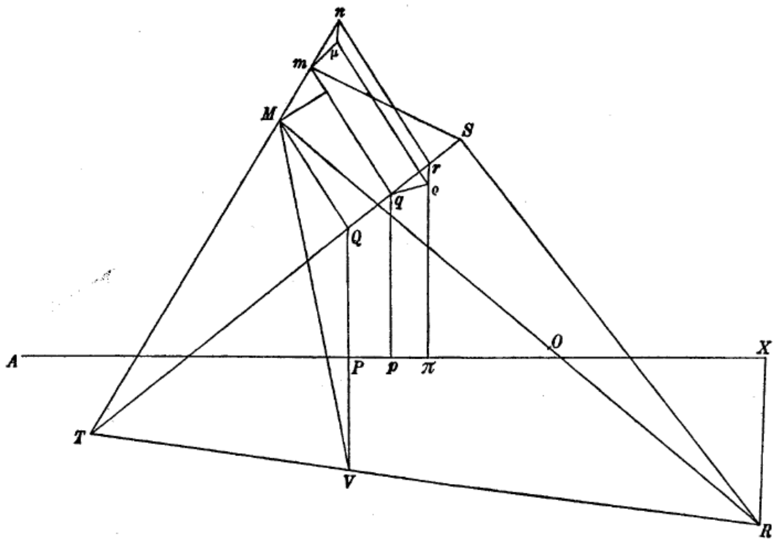


Fig. 9.

respondeant in plano *APQ* elementa *Qq, qρ* atque in axe *AP* assumo elementa *Pp, pπ*, quae sint aequalia. Erit ergo

$$Pp = p\pi = dx, \quad pq = y + dy, \quad \pi\varrho = y + 2dy + ddy, \quad Qq = \sqrt{(dx^2 + dy^2)},$$

$$q\varrho = \sqrt{(dx^2 + dy^2)} + \frac{dyddy}{\sqrt{(dx^2 + dy^2)}},$$

$$qm = z + dz, \quad \varrho\mu = z + 2dz + ddz, \quad Mm = \sqrt{(dx^2 + dy^2 + dz^2)}$$

et

$$m\mu = \sqrt{(dx^2 + dy^2 + dz^2)} + \frac{dyddy + dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Producantur *Qq* et *Mm* utrinque, quarum illa ipsi $\pi\rho$ in *r*, haec vero ipsi *rn* normali in planum *APQ* in *n* occurrat, [p. 28] eritque ob $Pp = p\pi$

$$qr = Qq \text{ et } mn = Mm \text{ atque } \pi r = y + 2dy \text{ ac } rn = z + 2dz.$$

Iam ad elementum *Mm* ducatur in plano *Qm* normalis *mS* occurrens ipsi *Qq* productae in *S*; erit

$$QS = \frac{(qm - QM)\,QM}{Qq} = \frac{zdz}{\sqrt{(dx^2 + dy^2)}}.$$

Ducta iam SR in plano APQ perpendiculari ad QS omnes rectae ex m ad SR ductae normales erunt ad elementum Mm. In his igitur normalibus erit radius osculi curvae

$Mm\mu$. Ea vero harum normalium congruet cum radio osculi, quae in eo sita erit plano, in quo posita sunt elementa *Mm* et $m\mu$. Quamobrem hoc planum determinari oportet. In hoc vero plano sunt elementa *mn* et $n\mu$; ambo itaque usque ad planum *APQ* producta dabunt intersectionem illius plani cum plano *APQ*. At *nm* vel *mM* occurrit plano *APQ* in *T*, ubi cum elemento *Qq* producto concurrit. Est igitur

$$QT = \frac{z\sqrt{(dx^2+dy^2)}}{dz}.$$

Ipsi $n\mu$ parallela *MV* in plano $mn\mu$ erit sita; haec vero *MV* in planum *APQ* incidet in *V* dabiturque *QV* ex analogia hac

$$(rn - \varrho\mu) : r\varrho = QM : QV;$$

erit itaque [p. 29]

$$QV = \frac{zddy}{ddz}.$$

Hanc ob rem recta *TV* producta erit intersectio plani $mn\mu$ cum plano *APQ*, quare recta *MR*, quae in concursum rectarum *SR* et *TV* est ducta, erit simul normalis in *Mm* et posita in plano $mn\mu$ eritque propterea *MR* positio radii osculi curvae in M. Ex his punctum R hoc modo determinabitur: erit, ducta RX perpendiculari in AP productam,

$$AX = \frac{zdx(dyddy + dzddz)}{(dx^2+dy^2)ddz - dydzddy} + x$$

atque

$$XR = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2+dy^2)ddz - dydzddy} - y.$$

Quo igitur normalis in superficiem *MN* in radii osculi curvae directionem incidat, debet esse $AH = AX$ et $XR = HN$; unde erit

$$P(dx^2+dy^2)ddz - Pdydzddy = dxdyddy + dxdzddz$$

et

$$-Q(dx^2+dy^2)ddz + Qdydzddy = dx^2ddy + dz^2ddy - dzdyddz.$$

Quae quidem aequationes inter se congruent; fiet enim ex iis coniunctim

$$Pdx + Qdy = dz,$$

quae est ipsa aequatio naturam superficiei exponens. Harum igitur aequationum alterutra cum hac $dz = Pdx + Qdy$ coniuncta dabit curvam a corpore in proposita superficie percursam. Q.E.I.

Corollarium 1.

69. Erit igitur pro linea in superficie proposita descripta

$$ddz : ddy = Pdydz + dxdy : Pdx^2 + Pdy^2 - dxdz.$$

At quia est $dz = Pdx + Qdy$, erit

$$ddz : ddy = Pdz + dx : Pdy - Qdx$$

seu

$$Pdyddz - Qdxddz = Pdzddy + dxddy.$$

Corollarium 2. [p. 30]

70. Si assumatur altera aequatio et utrinque subtrahatur

$$Qdz^2ddz - dy^2ddy,$$

habebitur

$$-Q(dx^2 + dy^2 + dz^2)ddz + Qdydzddy + dy^2ddy$$
$$= (dx^2 + dy^2 + dz^2)ddy - Qdz^2ddz - dzdyddz.$$

Quae est illa ipsa aequatio, quam pro linea brevissima in quacunque superficie dedi in Comm. Acad. Petr. Tom. III.

Scholion 1.

71. Ut in hoc casu, quo corpus a nulla potentia sollicitatur, directio radii osculi cum normali in superficiem congruere debet, ita in aliis casibus, quando corpus sollicitatur a potentiis, hae lineae datum angulum constituere

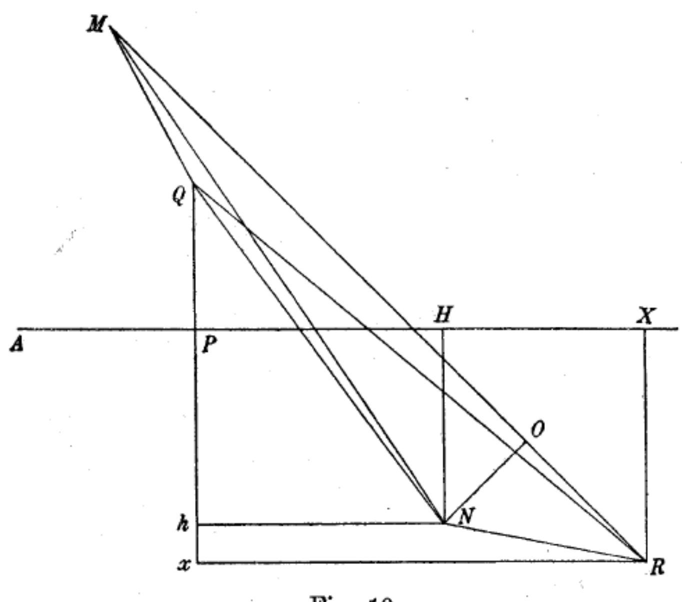


Fig. 10.

debent. Quamobrem ad hunc angulum generaliter inveniendum sit *MN* (Fig. 10) normalis in superficiem et *MR* directio radii osculi ; erit, ut iam vel posuimus vel invenimus,

$$PQ = y, \quad QM = z, \quad PH = hN = Pz, \quad Qh = -Qz,$$

$$PX = Rx = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy}$$

et

$$Qx = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2 + dy^2)ddz - dydzddy}.$$

Ductaq *NR* ex *N* in *MR* demittatur perpendiculum *NO*; erit

$$MO = \frac{MR^2 + MN^2 - NR^2}{2MR} = \frac{MQ^2 + Rx \cdot Nh + Qx \cdot Qh}{MR}$$

et

$$NO = \frac{\sqrt{(MR^2 \cdot MN^2 - (MQ^2 + Rx \cdot Nh + Qx \cdot Qh)^2)}}{MR}$$

$$= \frac{\sqrt{(MQ^2(Qx - Qh)^2 + MQ^2(Rx - Nh)^2 + (Rx \cdot Qh - Qx \cdot Nh)^2)}}{MR}.$$

[p. 31] Anguli vero RMN tangens est = $\frac{NO}{MO}$ posito sinu toto = 1. Substitutis autem supra assumtis symbolis et in subsidium vocata aequatione

$$dz = Pdx + Qdy$$

prodibit tangens anguli *NMR* =

$$\frac{ddy(dx + Pdz) - ddz(Pdy - Qdx)}{(ddz - Qddy)\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Hoc ergo angulo evanescente fit

$$ddz : ddy = Pdz + dx : Pdy - Qdx$$

ut supra (69).

Scholion 2.

72. Ipsa vero radii osculi longitudo *MO* (Fig. 9) invenitur ex angulo $nm\mu$ ope huius analogiae : ut sinus anguli $nm\mu$ ad sinum totum, ita *Mm* ad *MO*. Est vero

$$n\mu = \sqrt{(ddy^2 + ddz^2)} \quad \text{et} \quad mn - m\mu = \frac{-dyddy - dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}},$$

ergo perpendiculum ex *n* in $m\mu$ productum =

$$\frac{\sqrt{(dx^2ddy^2 + dz^2ddy^2 + dx^2ddz^2 + dy^2ddz^2 - 2dydzddyddz)}}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Quare hoc perpendicularum est ad $\sqrt{(dx^2+dy^2+dz^2)}$ ut $\sqrt{(dx^2+dy^2+dz^2)}$ ad *MO*, unde prodit radius osculi

$$MO = \frac{(dx^2+dy^2+dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2+ddz^2)+(dyddz-dzddy)^2)}}.$$

Hoc autem radio osculi opus erit in sequente propositione, in qua pressionem, quam corpus in superficiem exercet, investigabimus. [p. 32]

Scholion 3.

73. Ex hac generali radii osculi expressione orietur ea pro radio osculi linea brevissimae, si coniungatur cum hac aequatione

$$ddz = \frac{ddy\,(Pdz+dx)}{Pdy-Qdx} \quad \text{et locali} \quad dz = Pdx + Qdy.$$

Prodibit autem radius osculi =

$$\frac{(dx^2+dy^2+dz^2)(Pdy-Qdx)}{dxddy\sqrt{(P^2+Q^2+1)}} = \frac{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}{ddz-Qddy}$$
$$= \frac{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}{dPdx+dQdy}.$$

Atque haec expressio dat radium osculi curvae in superficie proposita descriptae a corpore a nullis potentiis sollicitato.

PROPOSITIO 10.

Theorema.

74. *Pressio, quam corpus in superficie motum et a nullis potentiis sollicitatum in ipsam superficiem exercet, fit normaliter in eam versus eius convexitatem et se habet ad vim gravitatis ut altitudo celeritati corporis debita ad dimidium radii osculi curvae a corpore descriptae.*

Demonstratio.

Sit *DMm* (Fig. 7) curva in superficie *ABC* a corpore descripta, altitudo celeritati corporis debita = v et radius osculi curvae $MO = r$. Quia corpus ex *M*, se libere moveri posset, progrederetur in elemento *Mn*, superficies vero efficit, ut per elementum *Mm* incedat existente *nm* perpendiculo in superficiem, superficies a corpore secundum [p. 33] directionem *nm* premetur tanta vi, quanta opus est ad corpus ex directione *Mn* in directionem *Mm* pertrahendum. Hoc vero praestatur a vi $\frac{2v}{r}$ normaliter in superficiem seu secundum directionem radii osculi *MO* agente. Quamobrem pressio corporis in superficiem erit normalis, quippe agens secundum *mn*, et aequalis $\frac{2v}{r}$ existente vi gravitatis corporis = 1. Q.E.D.

Corollarium 1.

75. Haec est igitur vis centrifuga, quam corpus in superficiem simili modo exercet, quo in lineam datam, in qua moveri cogitur.

Scholion 1.

76. Pressio in superficiem necessario debet esse normalis. Nam nisi esset normalis, resolvi posset in duas, quarum altera esset normalis, altera in ipsa superficie posita. Harum vero normalis tantum ad premendam superficiem impenditur, dum altera ipsum corporis motum immutaret.

Corollarium 2.

77. Longitudinem radii osculi r lineae, quam corpus a nullis potentiis sollicitatum super proposita superficie describit, invenimus (73). Ea igitur assumta erit vis centrifuga =

$$\frac{2v(ddz-Qddy)}{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}=\frac{2v(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}.$$

Scholion 2. [p. 34]

78. De hac vi centrifuga in superficiem exercita eadem locum habent, quae supra de vi centrifuga in datam curvam sunt annotata; vide prop. 2 (20) cum annexis coroll. et schol. Linea enim brevissima, quam corpus super superficei percurrit, instar canalis considerari potest, in quo corpus moveatur, atque tum de motu in hoc canali omnia valent, quae supra de motu super data linea nullis agentibus potentiis sunt allata.

PROPOSITIO 11.

Problema.

79. *Determinare effectum cuiusvis potentiae, quem exerit in corpus super data superficie motum tam in vacuo quam in medio resistente.*

Solutio.

Quaecunque sit directio potentiae sollicitantis corpus, ea resolvi potest in tres potentias laterales, quarum primae, quam vocibimus M, directio normalis in superficiem, secundae, quam per N designabimus, directio normalis tam in directionem motus corporis quam in directionem M, cuius igitur directio erit in plano tangente superficiem, tertia potentiae T appellatae directio congruat cum directione motus, quae igitur erit vis tangentialis; priores vero erunt vires normales. Quia nunc harum trium virium directiones sunt inter se normales, nullius effectus a reliquis perturbari poterit. [p. 35] Quare, quem effectum quaeque producat, investigabimus.

Prima potentia M, cuius direcito in superficiem est normalis, nullum habebit effectum in immutando corporis motu, sed tota impendetur in pressionem superficiei. Augebit igitur vel diminuet pressionem a vi centrifuga ortam, prout eius directio in plagam convexae partis superficiei incidit vel in plagam partis concavae. Incidat ea in partem interiorem; erit totalis pressio in superficiem versus partes exteriores =

$$\frac{2v(dPdx + dQdy)}{(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2 + 1)}} - M$$

(77). Pressio enim a vi centrifuga orta minuetur hoc casu potentia M.

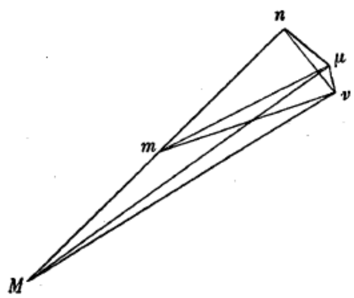


Fig. 11.

Secunda potentia *N*, quia eius directio in ipsa superficie est posita et normalis in directionem corporis, corporis directionem tantum immutabit celeritatem neque augendo neque minuendo. Haec vis igitur corpus a linea brevissima deducet facietque, ut non amplius in plano ad superficiem normali moveatur; huius igitur plani, in quo corpus movebitur, inclinationem ad planum lineae brevissimae normale in superficiem investigari oportet. Huius vero inclinationis angulo aequalis est angulus, quem radius osculi lineae descriptae cum normali in curvam constituit quemque ante generaliter determinavimus (71). Postquam corpus elementum *Mm* celeritate altitudini *v* debita descripsit, [p. 36] progrederetur, nisi a vi *N* sollicitaretur, per elementum *mv* (Fig. 11) in *v*, ita ut *Mm* et *mv* essent duo elementa linea brevissimae et posita in plano ad superficiem normali; erit directio vis *N* normalis in planum; erit directio vis *N* normalis in planum chartae, sit ea sursum reducetur, si quidem ponamus hanc vim *N* sursum esse directam hac elementorum positione, ut in figura repreasentatur. Efficiat ergo haec vis, ut corpus per elementum $m\mu$ moveatur anguloque $vm\mu$ a directione *mv* deflectat. Huic angulo respondet radius osculi = $\frac{mv^2}{\mu v}$. Quare cum vis *N* hunc angulum generet celeritasque curvae debita sit altitudini *v*, erit ex effectu virium normalium

$$N = \frac{2v \cdot \mu v}{mv^2} \quad \text{ideoque} \quad \mu\nu = \frac{N \cdot mv^2}{2v}.$$

Quo nunc inclinatio plani $Nm\mu$, in quo corpus actu movebitur, ad planum *Mmv*, quod in superficiem est normale, inveniatur, demittatur ex *v* in elementum *Mm* productum perpendiculum *vn*; erit μn quoque in mn perpendiculare ideoque angulus μnv erit angulus inclinationis plani μmM ad planum *vmM*; atque cum μv sit normalis ad *vn*, huius anguli tangens erit = $\frac{\mu v}{nv} = \frac{N.mv^2}{2v.nv}$. At *nv* determinatur ex inclinatione elementorum *Mm* et *mv* seu radio oscul lineae brevissimae, cuius *Mm* et *mv* sunt elementa. Sit hic radius osculi *r*, erit $\frac{mv^2}{nv} = r$ ideoque tangens anguli μnv =

$$\frac{Nr}{2v} = \frac{N(dx^2 + dy^2 + dz^2)\sqrt{(P^2 + Q^2 + 1)}}{2v(dPdx + dQdy)}$$

substituto loco *r* valore invento (73). [p. 37] Huic vero angulo aequalis est angulus, quem radius osculi elementorum *Mm*, $m\mu$ a corpore actu descriptorum constituit cum radio osculi elementorum *Mm, mv* seu cum normali in superficiem. Huius autem anguli tangentem supra invenimus (71). Quare facta aequatione habebimus

$$\frac{ddy(dx+Pdz)-ddz(Pdy-Qdx)}{(ddz-Qddy)\sqrt{(dx^2+dy^2+dz^2)}}=\frac{N(dx^2+dy^2+dz^2)\sqrt{(P^2+Q^2+1)}}{2v(dPdx+dQdy)},$$

qua aequatione effectus potentiae *N* determinatur. Seu cum sit

$$ddz-Qddy=dPdx+dQdy,$$

habebitur ista aequatio

$$ddy(dx+Pdz)-ddz(Pdy-Qdx)=\frac{N(dx^2+dy^2+dz^2)^{\frac{3}{2}}\sqrt{(P^2+Q^2+1)}}{2v}.$$

Tertia potentia *T*, quia in directione corporis est posita, celeritatem tantum vel auget vel diminuit. Ponamus eam esse accelerantem, exprimetur eius effectus hac aequatione

$$dv=T\sqrt{(dx^2+dy^2+dz^2)}.$$

Atque si motus in medio fiat resistente resistentiaque sit = *R*, minuends tantum est vis tangentialis *T* resistentia *R*. Quamobrem habebitur

$$dv=(T-R)\sqrt{(dx^2+dy^2+dz^2)}.$$

Q.E.I.

Corollarium.

80. Ex duabus igitur aequationibus, quarum altera *v*, altera *dv* determinat, una conflatur *v* non amplius continens, quae cum locali pro superficie $dz=Pdx+Qdy$ coniuncta [p. 38] determinat curvam, quam corpus super proposita superficie describit.

Scholion 1.

81. De potentia *N* bene est attendendum, in quam plagam tendat, h.e. an ad dextram an ad sinistram regionem corporis moti vergat. Pro hac enim differentia tangens anguli μnv vel affirmativa vel negativa est accipienda. De hoc vero non erimus hic solliciti, sed ulteriorem huius rei disquisitionem in caput ultimum huius libri differemus.

Scholion 2.

82. Ad sequens igitur caput secundum progredimur, in quo motum corporis super data linea in vacuo examinabimus. Capite tertio vero motus super data linea in medio resistente investigabimus. Quarto denique capite motum super data superficie tam in vacuo quam in medio resistente scrutabimur.