CHAPTER SIX (*Part b*).

CONCERNING THE CURVILINEAR MOTION OF A FREE POINT IN A RESISTIVE MEDIUM

[p. 396]

.

PROPOSITION 113.

PROBLEM.

925. *As before, with the uniform absolute force g put in place and pulling downwards, to find the force of the resistance which is effective in order that the body can move along the hyperbola NAM* (Fig.85) *freely, with the axis CAQ vertical.*

SOLUTION.

Let C be the centre of the hyperbola and the transverse semi-axis AC is equal to a; and the conjugate semi-axis is equal to c. Putting $CQ = t$ and $QM = AP = x$ then from the nature of the hyperbola :

$$c^2t^2 = a^2x^2 + a^2c^2.$$

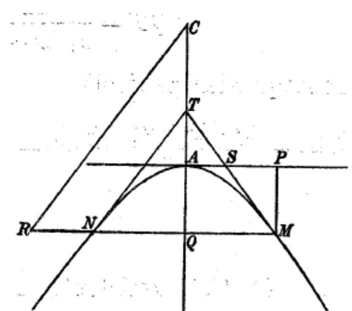

Fig. 85.

Moreover taking $PM = AQ = y$ then $y = t - a$ and $dy = dt, \quad d^2y = d^2t$ and $d^3y = d^3t.$

Truly from the equation we have:

$$t = \frac{a\sqrt{(x^2+c^2)}}{c} \text{ and } dt = \frac{axdx}{c\sqrt{(x^2+c^2)}}$$

and hence

$$ds = \frac{dx\sqrt{(c^4+c^2x^2+a^2x^2)}}{c\sqrt{(x^2+c^2)}}.$$

Again there arises :

$$ddt = ddy = \frac{acdx^2}{(x^2+c^2)^{\frac{3}{2}}} \text{ and } d^3t = d^3y = -\frac{3acxdx^3}{(x^2+c^2)^{\frac{5}{2}}}.$$

From which there becomes :

$$\frac{d^3y}{ddy} = -\frac{3xdx}{x^2+c^2} \text{ and } \frac{d^3y}{ddy^2} = -\frac{3x\sqrt{(x^2+c^2)}}{acdx}.$$

Consequently the resistance becomes [from (908)]:

$$R = -\frac{3gx\sqrt{(c^4+c^2x^2+a^2x^2)}}{2ac^2} \text{ and } v = \frac{g(c^4+c^2x^2+a^2x^2)\sqrt{(x^2+c^2)}}{2ac^3}.$$

Truly with the resistance made equal to the square of the velocity the exponent of the resistance is given by :

$$q = -\frac{\sqrt{(x^2+c^2)(c^4+c^2x^2+a^2x^2)}}{3cx}.$$

Corollary 1.

926. With the tangent MT drawn, there arises :

$$QT = \frac{ax^2}{c\sqrt{(x^2+c^2)}} \text{ and } MT = \frac{x\sqrt{(c^4+c^2x^2+a^2x^2)}}{c\sqrt{(x^2+c^2)}}.$$

Consequently there is produced : [p. 397]

$$R = -\frac{3g\cdot MT\cdot\sqrt{(x^2+c^2)}}{2ac} = -\frac{3g\cdot CQ\cdot MT}{2a^2} \text{ or } R:g = -3CQ\cdot MT:2AC^2.$$

Corollary 2.

927. Since the resistance *R* is found to be negative, from this it is indicated that it is not possible for a body to descend along the hyperbola *AM* in a resisting medium, for it is required to be moved by the medium. But while the body ascends the arc *NA*, since *ds* becomes negative, then the resistance *R* is positive. On account of this, if the body is at *N*, the resistance is given by $R = \frac{3g.CQ.NT}{2AC^2}$.

Corollary 3.

928. From the properties of the hyperbola, we have $CQ : AC = AC : CT$. Thus the resistance at *N*, or *R*, $= \frac{3g.NT}{2CT}$. Or the resistance *R* to the absolute force *g* as 3*NT* to 2*CT*. At the vertex *A* the resistance hence disappears and then increases, as *N* is more distant from *A*. [p. 398]

Corollary 4.

929. The height corresponding to the speed of the body at *M* or *N* is equal to $\frac{g.CQ^2.MT^2}{2AC^2.QT}$, as can be easily deduced from the value of *v* and from the properties of the hyperbola. Moreover since $MT = NT$ and $CQ.NT = \frac{2R.AC^2}{3g}$, we have $v = \frac{2R^2.AC^2}{9g.QT}$.

Corollary 5.

930. If the resistance is put in proportion to the speeds of the body and the exponent of the resistance is *q*, then $R = \frac{\sqrt{v}}{\sqrt{q}}$ and $v = R^2q$. Whereby it is found that $q = \frac{2AC^2}{9g.QT}$. Thereby by this hypothesis the exponent varies inversely as the subtangents QT.

Corollary 6.

931. But if the resistance is put in proportion to the square of the speeds and the exponent of the resistance is q, with the body present at N there is the exponent $q = \frac{NT.CQ}{3TQ}$. Or with *CR* drawn parallel to the tangent *NT* from *C*, crossing the applied line QN produced at R, then $q = \frac{CR}{3}$.

Scholium.

931. Since previously with the circle and now with the hyperbola we have noted that the resistance in the one arc to be made positive and in the other arc to be negative, that is obtained for all curves put in place with two equal arcs *AN* and *AM* around the maximum point A. [p. 399] Since in general for the arc AM the resistance is given by $R = \frac{gdsd^3y}{2ddy^2}$, since in the arc AN *ds* is negative then at N resistance is given by $R = -\frac{gdsd^3y}{2ddy^2}$, thus in order that the resistance at N is the negative of the resistance at M. Whereby, when by the nature of things being such that the resistance cannot be made negative, by which the body is accelerated, then it cannot happen that the body describes a curve in a medium with resistance, which has two similar and equal branches around the maximum point *A*. Truly the height generating the speed at *M* is the same as at *N*; for the value of this, $\frac{gds^2}{2ddy}$ is not changed, even if *ds* becomes negative. Therefore with curves of this kind thus considered and due to Newton, so that some might be elicited for which the density of the resisting medium does not vary much, or which can be treated by our method in which the exponent of the resistance has everywhere almost the same value and such a curve can be taken for the trajectory in a medium of uniform resistance without sensible error; in addition we consider other curves without the vertical diameter, also provided by Newton, and hyperbolas having vertical asymptotes are curves of this kind, and which clearly fall closer to the logarithmic curve which is described by a body in a uniform medium with the resistance in the simple ratio of the speed. In other cases indeed Newton could not determine the trajectories from the simple hypotheses of resistance, but was content to assign approximate values. Which arrangement we follow, since the true trajectories to be given by us are so complicated that hardly any can be deduced in practice. [p. 400]
[Newton's propositions that correspond to those of Euler, and others, are set out in Book II, Sections 2 and 3 of the *Principia*.]

PROPOSITIO 114.

PROBLEM.

933. *Let the curve NM* (Fig.86) *be a hyperbola of any other kind having a vertical asymptote CP, and it is required to determine the resistance which is effective, in order that the body, always acted on by a downwards force, moves on this hyperbola.*

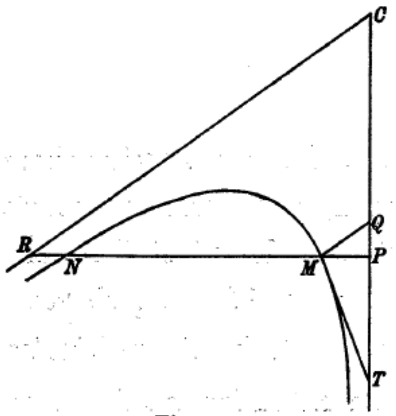

Fig. 86.

SOLUTION.

The asymptote *CP* is considered as the axis to the hyperbola, and from *M* the normal *MP* is drawn. By taking *CP* = *y* and *MP* = *x*, which we have generally taken to be the case above, then these are in position if *dx* is taken as negative everywhere. Let *RC* be the other asymptote and *C* the centre of the hyperbola, and the sine of the angle RCP = α and the cosine of this angle is $\sqrt{(1-\alpha^2)} = \beta$. Therefore with *PM* produced at *R*, $PR = \frac{\alpha y}{\beta}$ and $CR = \frac{y}{\beta}$. From *M* the line *MQ* is drawn parallel to the asymptote *CR*, and

$$MQ = \frac{x}{\alpha} \text{ and } PQ = \frac{\beta x}{\alpha}, \text{ and hence } CQ = \frac{\alpha y - \beta x}{\alpha}.$$

But from the nature of hyperbolas,

$$a^n = \frac{x^{n-1}(\alpha y - \beta x)}{\alpha^n}$$

and hence

$$y = \frac{\beta x}{\alpha} + \frac{\alpha^{n-1}a^n}{x^{n-1}} \text{ and } dy = \frac{\beta dx}{\alpha} - \frac{(n-1)\alpha^{n-1}a^n dx}{x^n}.$$

With the tangent MT drawn, we have PT = $-\frac{xdy}{dx} = -\frac{\beta x}{\alpha} + \frac{(n-1)\alpha^{n-1}a^n}{x^{n-1}}, MT = \frac{xds}{dx}$,

[p. 401] thus in order that $ds = \frac{MT.dx}{x}$. This is indeed the case, in which *x* is taken in the other part, if MT is negative. Again on account of *dx* being constant,

$$d^2y = \frac{n(n-1)\alpha^{n-1}a^n dx^2}{x^{n+1}} \text{ and } d^3y = -\frac{(n+1)(n-1)n\alpha^{n-1}a^n dx^3}{x^{n+2}}.$$

From these there is produced

$$\frac{gdsd^3y}{2ddy^2} = -\frac{g(n+1)x^{n-1}\cdot MT}{2n(n-1)\alpha^{n-1}a^n},$$

which with *MT* made negative is equal to the resistance *R*. Thus the resistance is given by

$$R = \frac{g(n+1)x^{n-1}\cdot MT}{2n(n-1)\alpha^{n-1}a^n}$$

(908) and the height corresponding to the speed at *M*, or *v*, is equal to

$$\frac{gx^{n-1}\cdot MT^2}{2n(n-1)\alpha^{n-1}a^n}.$$

If the medium is put to resist in the ratio of the $2m^{\text{th}}$ multiple of the speeds, [i. e. the m^{th} multiple of v] and the exponent of the resistance is q, then $R=\frac{v^m}{q^m}$ and thus $q=\frac{v}{R^{\frac{1}{m}}}$.

From which it becomes

$$q=\frac{g^{\frac{m-1}{m}}x^{\frac{(m-1)(n-1)}{m}}MT^{\frac{2m-1}{m}}}{2^{\frac{m-1}{m}}(n^2-n)^{\frac{m-1}{m}}(n+1)^{\frac{1}{m}}\alpha^{\frac{(m-1)(n-1)}{m}}a^{\frac{n(m-1)}{m}}}$$

or

$$q^m=\frac{g^{m-1}x^{(m-1)(n-1)}MT^{2m-1}}{2^{m-1}(n^2-n)^{m-1}(n+1)\alpha^{(m-1)(n-1)}a^{n(m-1)}}.$$

Q.E.I.

Corollary 1.

934. Since $x^{n-1}=\frac{\alpha^n a^n}{\alpha y-\beta x}$, the resistance is given by

$$R=\frac{g(n+1)\alpha\cdot MT}{2n(n-1)(\alpha y-\beta x)}=\frac{g(n+1)\cdot MT}{2n(n-1)\cdot CQ}.$$

Hence the resistance R to the force g is as $(n+1)MT$ to $2n(n-1)CQ$. [p. 402] In a similar manner with this value in place of x^{n-1} there is produced :

$$v=\frac{g\cdot MT^2}{2n(n-1)\cdot CQ} \text{ and } q^m=\frac{g^{m-1}\cdot MT^{2m-1}}{2^{m-1}(n^2-n)^{m-1}(n+1)\cdot CQ^{m-1}}.$$

Corollary 2.

935. With the body descending to infinity, $x = 0$ and

$$MT=PT=\frac{(n-1)\alpha^{n-1}a^n}{x^{n-1}}$$

with x vanishing. Therefore in the infinite depths, $R=\frac{g(n+1)}{2n}$ and thus this is of finite magnitude, but $v=\frac{g(n-1)\alpha^{n-1}a^n}{2nx^{n-1}}$. Whereby, since by necessity $n > 1$, with x vanishing the speed of the body becomes infinitely great.

Corollary 3.

936. Therefore by putting the resistance $R=\frac{v^m}{q^m}$ in the infinite depths should also make q infinitely great; and thus with these in place the body will be moving in a vacuum. From which it follows, by however more the body descends, the resistance acting on it becomes smaller or rather the medium becomes rarer.

Corollary 4.

937. In the Apollonian hyperbola $n = 2$. Therefore for this curve it is found that [p. 403]

$$R = \frac{3gx \cdot MT}{4\alpha a^2} = \frac{3g \cdot MT}{4\,CQ} \text{ and } v = \frac{g \cdot MT^2}{4\,CQ} \text{ and } q^m = \frac{g^{m-1} \cdot MT^{2m-1}}{3 \cdot 2^{2m-2}\,CQ^{m-1}}.$$

Corollary 5.

938. If the resistance is put to be in proportion to the speeds, in all these hyperbolas the exponent of the resistance varies directly as CQ, since $m = \frac{1}{2}$ in this case. Therefore for this hypothesis of the resistance, the body is free to describe all the hyperbolas.

Corollary 6.

939. If the resistance is put in the square ratio of the speeds, since $m = 1$, then the exponent of the resistance at M = $\frac{MT}{n+1}$. From which therefore the more MT is varied, the more also the medium is different.

Corollary 7.

940. In addition the time in which the element Mm is described, or $\frac{ds}{\sqrt{v}}$, is equal to $\frac{\sqrt{2}ddy}{\sqrt{g}}$.
Therefore the time taken for the body to reach M, is as

$$\int \frac{dx}{x^{\frac{n+1}{2}}}, \text{ i. e. as } C - \frac{1}{x^{\frac{n-1}{2}}} \text{ or rather as } \frac{1}{x^{\frac{n-1}{2}}} - C.$$

Whereby the time, in which the body reaches M from N, is as

$$\frac{1}{MP^{\frac{n-1}{2}}} - \frac{1}{NP^{\frac{n-1}{2}}}.$$

[Again, the section numbers have increased by 10 in the original. One can presume that Euler decided to remove some sections for some reason at a late stage.]

Corollary 8. [p. 404]

950. Therefore in the Apollonian hyperbola, in which $n = 2$, the time taken for the body to travel from N to M, is as $\sqrt{NP} - \sqrt{MP}$ since $NP.MP$ is constant, clearly equal to $\frac{a\alpha^2}{\beta}$.

Scholium.

951. From these examples it is clear that a body cannot describe hyperbolas in a medium with a uniform resistance of this kind, since the exponent of the resistance varies too much, and which clearly is finally indefinitely large. On account of which Newton's custom cannot be approved, in which he desired to substitute these hyperbolas in place of the true trajectories in mediums with uniform resistances. Furthermore, in a medium with the resistance following the square of the speeds, the exponent varies as the tangent MT, which on descending as far as the point N is to be returned as strongly varying according to the calculation. Also this other inconvenience must be understood for the speed, since on descending it becomes infinitely large, when however in a uniform medium it cannot increase beyond a given final speed. Besides clearly, it is not resolved for the hypothesis with the resistance proportional to the square of the speed, how the curve described can have a vertical asymptote, as happens in the case of a medium with the resistance in the simple ratio of the speed. For this resistance, even if the body has no force acting on it, the whole motion is along a finite line; which indeed becomes infinite if the resistance is put as the square of the speed. [p. 405] From which also it follows that the trajectory in this medium does not have an asymptote. It is not even certain that this curve has a hyperbolic asymptote [i. e. becomes hyperbolic for large values of the abscissa.]. Yet meanwhile it can have a vertical asymptote of the other parabolic kind of curve, which is determined from the quadrature of the curve by rectification of the given parabola. But leaving the hypothesis of uniform force we may go on to variable force, yet the direction of this is still everywhere parallel to itself. Indeed we do not investigate the curve described by a given curve with resistance, as this can now be done by Prop. 106 (870) ; but with a given curve and with one of the force, the resistance, or the speed given, we can determine the other quantities.

PROPOSITION 115.

PROBLEM.

952. *Let some variable absolute force that acts downwards along MP be given* (Fig.87); *to determine the required resistance for this, in order that the body moves along the given curve AM.*

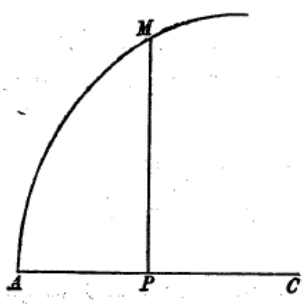

Fig. 87.

SOLUTION.

Let $AP = x$, $PM = y$ and the element of the arc $AM = ds$; while the force shall be P by which the body at M is acted upon, and the height corresponding to the speed at M is v, and the resistance at M is equal to R. With these put in place, we have (870) $dv = -Pdy - Rds$ and $v = -\frac{Pds^2}{2ddy}$ (871) on taking dx constant. [p. 406] From this equation there arises :

$$dv = -Pdy - \frac{dPds^2}{2ddy} + \frac{Pds^2d^3y}{2ddy^2},$$

and thus it produces [recall that $dsdds = dyddy$ as $ddx = 0$]

$$R = \frac{dPds}{2ddy} - \frac{Pdsd^3y}{2ddy^2} \quad \text{or} \quad \frac{2R}{ds} = d.\frac{P}{ddy}.$$

Therefore both v and R can be expressed in terms of the given quantities P, x and y. Q.E.I.

Corollary 1.

953. With the radius of osculation at $M = r$ there is

$$ddy = -\frac{ds^3}{rdx} \text{ and } d^3y = -\frac{3dsdyddy}{rdx} + \frac{ds^3dr}{r^2dx} = \frac{3ds^4dy}{r^2dx^2} + \frac{ds^3dr}{r^2dx}.$$

and hence

$$v = \frac{Prdx}{2ds} \text{ and } R = \frac{-rdPdx - Pdxdr - 3Pdsdy}{2ds^2}.$$

Corollary 2.

954. If the law of the resistance is in the ratio of the square of the speed, and the exponent is equal to q, then $R = \frac{v}{q}$ and $q = \frac{v}{R}$. On which account we have

$$q = \frac{Pdsddy}{Pd^3y - dPddy} \text{ or } q = -\frac{Prdxds}{rdPdx + Pdxdr + 3Pdsdy}.$$

Corollary 3.

955. If the force P to gravity 1 is as y to f, then $P = \frac{y}{f}$ and thus

$$R = \frac{-rdydx - ydxdr - 3ydsdy}{2fds^2} \text{ and } v = \frac{yrdx}{2fds}$$

and

$$q = -\frac{yrdxds}{rdydx + ydxdr + 3ydsdy}.$$

Corollary 4.

956. If the curve AM is the circle of radius $AC = a$, then

$$r = a, \quad dr = 0, \quad dy = \frac{(a-x)dx}{y} \text{ and } ds = \frac{adx}{y}.$$

There is hence produced : $v = \frac{Py}{2}$ and the resistance [p. 407]

$$R = -\frac{y^2dP}{2adx} - \frac{3P(a-x)}{2a} \text{ and } q = -\frac{aPydx}{y^2dP + 3P(a-x)dx}.$$

Example 1.

957. Let the curve *AM* be a circle, the centre of which is *C* and the radius $AC = a$. Moreover the body is always attracted to the axis *AC* in the ratio of the distances, thus so that $P = \frac{y}{f}$; then $v = \frac{y^2}{2f}$ and thus the speed at *M* is as the applied line *MP*. Then the resistance *R* becomes equal to

$$\frac{-y(a-x)-3y(a-x)}{2af} = -\frac{2y(a-x)}{af} = -\frac{2PM\cdot CP}{f\cdot AC} \text{ and } q = -\frac{PM\cdot AC}{4CP}.$$

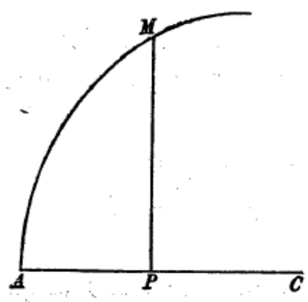
Fig. 87.

Moreover the speed at the point *A* is equal to 0 and the resistance, while the body ascends in the quadrant, is negative or the body is accelerated by the medium. The time, in which travels from A to M, is indefinitely large; for it becomes

$$\int\frac{ds}{\sqrt{v}} = \int\frac{adx\sqrt{2f}}{2ax-x^2} = -\sqrt{\frac{f}{2}}\,l\,0 + \sqrt{\frac{f}{2}}\,l\,\frac{x}{2a-x}.$$

This which can also be understood by itself; for since the speed at A is equal to 0 and here as the force $\frac{y}{f}$ is acting while the force of the medium vanishes, the body must perpetually remain at *A*.

Example 2.

958. With the circle *AM* remaining, if the absolute force varied inversely as the distance *PM* or $P = \frac{f}{y}$, then $v = \frac{f}{2}$. Whereby the speed of the body is the same everywhere, or the body is carried around the periphery of the circle in an equal motion, and the time in which some arc *AM* is completed, is as the arc *AM* itself. But the resistance at *M* is equal to :

$$\frac{f(a-x)}{ay} = -\frac{f\cdot CP}{AC\cdot PM}.$$

Therefore the resistance is negative [p. 408] while the body ascends the quadrant ; moreover while it descends the following quadrant, the resistance becomes positive or is true resistance. Again from the resistance it is found that $q = \frac{AC.PM}{2CP}$. Therefore at the point *A* the force of the resistance moving forwards is indefinitely large in order that it is equal to the force acting.

Example 3.

959. With the circle *AM* remaining, the force acting shall be as some power of the distance *MP* or $P = \frac{y^n}{f^n}$; from which [by (956)] $dP = \frac{ny^{n-2}(a-x)dx}{f^n}$. Therefore there is produced from these, $v = \frac{y^{n+1}}{2f^n}$ and the resistance

$$R = -\frac{(n+3)y^n(a-x)}{2af^n}.$$

Thus the ratio is formed :

$$R : P = -(n+3)\,CP : 2AC.$$

Whereby, if $n = -3$ or the force P varies inversely as the cube of the distance *MP*, then the resistance *R* vanishes and the body from this force acting in a vacuum is able to move in the circle *AM*. Then if $n + 3$ is a positive number, the ascending resistance in the quadrant is negative. But if $n + 3$ is a negative number, then the resistance in this quadrant is positive.

Example 4.

960. If the curve AMB (Fig. 88) is of such a kind that the radius of osculation at M varies as the reciprocal of the applied line PM, [p. 409] that all elastic curves agree on, then

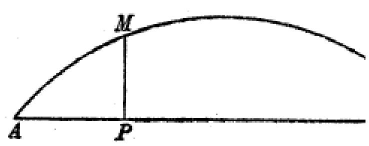

Fig. 88.

$$r = \frac{a^2}{y} \text{ and } dr = -\frac{a^2dy}{y^2}.$$

Therefore we have

$$v = \frac{a^2Pdx}{2yds}$$

and

$$R = \frac{a^2Pdxdy}{2y^2ds^2} - \frac{a^2dPdx}{2yds^2} - \frac{3Pdy}{2ds}.$$

Moreover since $-\frac{ds^3}{dxddy} = \frac{a^2}{y}$ and on integrating $2a^2dx = ds(y^2 + b^2)$, we have

$$dy = \frac{dx\sqrt{(4a^4 - (y^2+b^2)^2)}}{y^2+b^2}.$$

Hence

$$v = \frac{P(y^2+b^2)}{4y}.$$

And

$$R = \frac{P(b^2 - 5y^2)\sqrt{(4a^4 - (y^2+b^2)^2)}}{8a^2y^2} - \frac{dP(y^2+b^2)^2}{8a^2ydx}.$$

Now

$$P = \frac{y}{f} \text{ and } dP = \frac{dx\sqrt{(4a^4 - (y^2+b^2)^2)}}{f(y^2+b^2)},$$

giving

$$v = \frac{y^2+b^2}{4f} \text{ and } R = -\frac{3y\sqrt{(4a^4-(y^2+b^2)^2)}}{4a^2 f} = -\frac{3ydy}{2fds}.$$

Thus $R : P = -3dy : 2ds$. Therefore as the body ascends for a long time, the resistance is negative, and it is positive when it descends.

PROPOSITION 116.

PROBLEM.

961. *If AM is the given curve* (Fig.87) *and the resistance is given by quantities relating to the curve, to find the absolute force P always acting normal to the axis AC, which can be made, so that the body is free to move in this curve,*

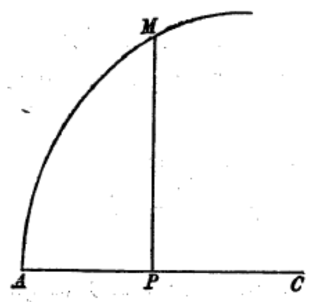

Fig. 87.

SOLUTION.

Let $AP = x$, $PM = y$ and the element of the curve is equal to ds. Then let the resistance at M be equal to R, which is therefore given in terms of x, y and s ; the force sought is equal to P and the speed at M corresponds to the height v. With these in place, we have (871)

$$P = -\frac{2vddy}{ds^2} \text{ and } dv = \frac{2vdyddy}{ds^2} - Rds$$

(cit.). [p. 410] From this equation as dx is constant, it is found by integration that

$$v = \frac{ads^2}{dx^2} - \frac{ds^2}{dx^2}\int \frac{Rdx^2}{ds}.$$

Moreover with v found, P can become known from the equation $P = -\frac{2vddy}{ds^2}$. Q.E.I.

Corollary 1.

962. Therefore we have

$$P = -\frac{2addy}{dx^2} + \frac{2ddy}{dx^2}\int \frac{Rdx^2}{ds}.$$

P can then be determined by separating the quantities.

Corollary 2.

963. Since a constant can be added as you wish, and this can be determined so that the body at A or at some other given place has a given speed.

Corollary 3.

964. If the resistance is put in proportion to the squares of the speeds, and the exponent of the resistance q, then we have :

$$q = \frac{ads^2}{Rdx^2} - \frac{ds^2}{Rdx^2}\int \frac{Rdx^2}{ds}.$$

Corollary 4.

965. The time, in which the arc *AM* is completed, is given by $\int \frac{dx}{\sqrt{v}}$; if the value of v is substituted into this expression, the time to traverse the arc *AM* is equal to

$$\int \frac{dx}{\sqrt{\left(a - \int \frac{Rdx^2}{ds}\right)}}.$$

Corollary 5. [p. 411]

966. If the resistance is put to the force of gravity 1 as the tangent at *M* to the sub tangent or as ds to dx, there arises

$$v = \frac{ds^2}{dx^2}(a - x) \text{ and } P = -\frac{2ddy}{dx^2}(a - x)$$

and the time to complete the arc *AM* is equal to

$$\int \frac{dx}{\sqrt{(a-x)}} = 2\sqrt{a} - 2\sqrt{(a-x)}.$$

But again, we have

$$q = \frac{ds}{dx}(a - x).$$

Example.

967. Let the curve be a circle, the radius of which is $AC = b$, and $y = \sqrt{(2bx - xx)}$ then

$$dy = \frac{dx(b-x)}{y}, \quad ds = \frac{bdx}{y}, \text{ and } ddy = -\frac{b^2dx^2}{y^3}.$$

From these we have

$$v = \frac{b^2}{y^2}\left(a - \int \frac{Rydx}{b}\right) \text{ and } P = \frac{2b^2}{y^3}\left(a - \int \frac{Rydx}{b}\right).$$

With the resistance put equal to
$\frac{ds}{dx}$ or $R = \frac{b}{y}$, then $v = \frac{b^2}{y^3}(a - x)$, and $P = \frac{2b^2}{y^3}(a - x)$, and $q = \frac{b}{y}(a - x)$. The time taken, in which the body arrives at *M* from *A*, is equal to $2\sqrt{a} - 2\sqrt{(a - x)}$. If further, we let $b = a$, then we have :

$$v = \frac{AC^2 \cdot PC}{PM^2}, \text{ and } P = \frac{2AC^2 \cdot PC}{PM^3}, \text{ and } R = \frac{AC}{PM}.$$

Therefore the speed of the body at the highest point of the circle is equal to zero, and the force *P* vanishes at that place. Moreover the body is not able to progress beyond this point, since otherwise the speed would become imaginary ; thence therefore it retreats to the point *A*, as it accelerates due to the negative resistance. Moreover while it arrives at *A*, since here the speed is infinitely great, with this motion of its own it describes the quadrant below *AC*, in which it is it is acted on by the negative force *P* upwards. [p. 412]

PROPOSITION 117.

PROBLEM.

968. *If the medium is uniform and the resistance in the ratio of the square of the velocity, to determine the absolute force acting downwards which can be made, in order that the body in the medium with this resistance describes the given curve AM* (Fig.87).

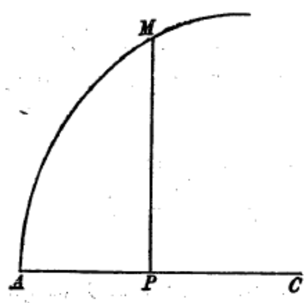

Fig. 87.

SOLUTION.

By putting $AP = x$, $PM = y$, with the element of the arc $AM = ds$, with the speed at M = $\sqrt{v}$, with the exponent of the resisting medium equal to c and the absolute force equal to P, the resistance $R = \frac{v}{c}$. With these in place there are the equations :

$$P = -\frac{2vddy}{ds^2} \text{ and } dv = \frac{2vdyddy}{ds^2} - \frac{vds}{c}$$

(871). Whereby we have :

$$\frac{dv}{v} = \frac{2dyddy}{ds^2} - \frac{ds}{c}$$

and on integrating :

$$lv = l\frac{ads^2}{dx^2} - \frac{s}{c} \text{ or } v = \frac{ae^{-\frac{s}{c}}ds^2}{dx^2}.$$

Therefore with the value of *v* found, there is

$$P = -\frac{2ae^{-\frac{s}{c}}ddy}{dx^2}.$$

Therefore from the given curve, *v* and then *P* can be found. Q.E.I.

Corollary 1.

969. The time in which the body can complete the arc AM or $\int \frac{ds}{\sqrt{v}}$, is equal to

$$\int \frac{e^{\frac{s}{2c}}\,dx}{\sqrt{a}}.$$

Therefore with the curve given or the equation between *s* and *x*, [p. 413] the time can also be obtained, even by quadrature. Therefore these curves are the most convenient to solve, for which there is a given equation between *s* and *x*.

Corollary 2.

970. Since *a* can be taken as a constant of the integration taken as you please, it can be used to determine either the speed at a given point on the curve or the force acting.

Corollary 3.

971. If the curve *AM* is concave towards the axis *AP*, then *ddy* is negative; therefore in these cases the force draws the body towards the axis *AP*. But if the curve is convex towards *AP*, since then *ddy* is positive, then the force *P* is negative, or the body is repelled from the axis *AP*.

Example 1.

972. Let the curve *AM* be a parabola having the axis standing normally to the right line *AP*, of such a kind that is described by a body projected obliquely in a vacuum from *A*, it is given by $by = fx - x^2$ and thus $dy = \frac{fdx}{b} - \frac{2xdx}{b}$, and $ddy = -\frac{2dx^2}{b}$. With these put in place, the force acting is given by :

$$P = \frac{4a}{be^{\frac{s}{c}}}.$$

From which it is understood, that the longer the motion is continued, [p. 414] there the force *P* decreases more. Moreover with *c* made indefinitely large, which is the case in a vacuum, then $e^{\frac{s}{c}} = 1$ and the force $P = \frac{4a}{b}$ is thus constant.

Example 2.

973. If the curve *AM* is such that its equation is given by $y = \alpha x - \beta x^2 - \gamma x^3$, then $dy = \alpha dx - 2\beta x dx - 3\gamma x^2 dx$ and $d^2y = -2\beta dx^2 - 6\gamma x dx^2$.
Hence we have :

$$P = \frac{4\beta a + 12\gamma a x}{e^{\frac{s}{c}}}.$$

At the point A we have:

$$dy = \alpha dx \text{ and } \frac{ds^2}{dx^2} = 1 + \alpha^2.$$

Therefore the height corresponding to the initial speed at A is $a(1+\alpha^2)$; which if called b, then $a=\frac{b}{1+\alpha^2}$, and α is the tangent of the angle below which the body is projected from A. Then from the value of dy, it is found that

$$ds = dx\sqrt{(1+\alpha^2-4\alpha\beta x+4\beta^2x^2-6\alpha\gamma x^2+12\beta\gamma x^3+9\gamma^2x^4)}$$
$$= dx\sqrt{(1+\alpha^2)} - \frac{2\alpha\beta x dx}{\sqrt{(1+\alpha^2)}} \quad \text{etc.}$$

Therefore with the remaining terms ignores the arc length becomes :

$$s = x\sqrt{(1+\alpha^2)} - \frac{\alpha\beta x^2}{\sqrt{(1+\alpha^2)}}$$

and

$$e^{\frac{s}{c}} = 1 + \frac{x\sqrt{(1+\alpha^2)}}{c} - \frac{\alpha\beta x^2}{c\sqrt{(1+\alpha^2)}} + \frac{x^2(1+\alpha^2)}{2c^2} \quad \text{etc.,}$$

and which boundaries can also be pushed back if c is made very large. Whereby, when P is made approximately constant, truly equal to g, it is the case that

$$4\beta a = g \quad \text{and} \quad \frac{3\gamma}{\beta} = \frac{\sqrt{(1+\alpha^2)}}{c}.$$

And if this equation is assumed : $y=\alpha x-\beta x^2-\gamma x^3-\delta x^4$,
it produces

$$\frac{6\delta}{\beta} = \frac{(1+\alpha^2)}{2c^2} - \frac{\alpha\beta}{c\sqrt{(1+\alpha^2)}}.$$

Therefore we have :

$$\beta = \frac{g(1+\alpha^2)}{4b}, \quad \gamma = \frac{g(1+\alpha^2)^{\frac{3}{2}}}{12bc} \text{ and } \delta = \frac{g(1+\alpha^2)^2}{48bc^2} - \frac{\alpha g^2(1+\alpha^2)^{\frac{3}{2}}}{96b^2c}.$$

[p. 415] Therefore this curve of the fourth order is very close to the trajectory in a very rare uniform medium, which has resistance in the ratio of the square of the velocity, and with a uniform force g acting downwards.

Corollary 4.

974. Since the air resistance is proportional to the square of the speeds, if in air composed of heavy globules [Euler had the belief at this time that air consisted of globules : see E002 *De Sono* in these translations] and the body is projected with a great force, then *b et c* are maximum quantities. Whereby it is necessary to take this equation for the projection of this body :

$$y = \alpha x - \frac{g(1+\alpha^2)}{4b}x^2 - \frac{g(1+\alpha^2)^{\frac{3}{2}}}{12bc}x^3,$$

since the curve differs little from the true trajectory.

Corollary 5.

975. Let *AMDB* (Fig. 89) be this trajectory ; in which in order that the point *B* can be found, in which the projected body is incident on the horizontal *AB*, I put $y = 0$ and the equation becomes

$$x^2 = -\frac{3cx}{\sqrt{(1+\alpha^2)}} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}$$

and hence

$$= -\frac{3c}{2\sqrt{(1+\alpha^2)}} + \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)}.$$

and thus

$$AB = \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)} - \frac{3c}{2\sqrt{(1+\alpha^2)}}.$$

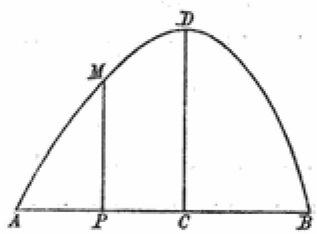

Fig. 89.

$$x^2 = -\frac{3cx}{\sqrt{(1+\alpha^2)}} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}$$

and hence

$$-\frac{3c}{2\sqrt{(1+\alpha^2)}} + \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)}.$$

Thus

$$AB = \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)} - \frac{3c}{2\sqrt{(1+\alpha^2)}}.$$

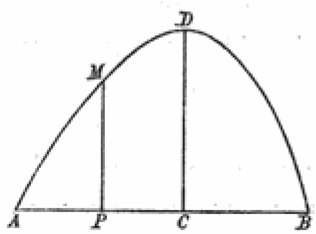

Fig. 89.

[p. 416] Therefore the distance thrown can become known from the given initial speed and the inclination.

Corollary 6.

976. The maximum height of the trajectory *D* can be found by making $dy = 0$. Moreover it becomes :

$$0 = \alpha - \frac{g(1+\alpha^2)}{2b}x - \frac{g(1+\alpha^2)^{\frac{3}{2}}}{4bc}x^2$$

or

$$x^2 = -\frac{2cx}{\sqrt{(1+\alpha^2)}} + \frac{4\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}.$$

And from this equation

$$AC = \sqrt{\left(\frac{c^2}{1+\alpha^2} + \frac{4\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)} - \frac{c}{\sqrt{(1+\alpha^2)}}.$$

Corollary 7.

977. The longest throw, which is produced with the same initial speed $\sqrt{b}$ is produced if the tangent α of the angle of inclination is found from this equation :

$$3\alpha c\sqrt{(1+\alpha^2)}-\frac{8b(1+\alpha^2)}{g}+\frac{24\alpha^2 b}{g}=\alpha\sqrt{\left(9c^2(1+\alpha^2)+\frac{48\alpha bc\sqrt{(1+\alpha^2)}}{g}\right)}$$

or from this :

$$4b(1-2\alpha^2)^2-3\alpha gc(1-2\alpha^2)\sqrt{(1+\alpha^2)}=3\alpha^3 gc\sqrt{(1+\alpha^2)}$$

or thus more simply

$$4b(1-2\alpha^2)^2=3\alpha gc(1-\alpha^2)\sqrt{(1+\alpha^2)}.$$

Corollary 8. [p. 417]

978. If the sine of the angle, that the curve makes at *A* with the horizontal *AC* is set equal to ε with the whole sine equal to 1, then the equation can be written as :

$$9\varepsilon^4-6\varepsilon^2+1=\frac{3gc\varepsilon}{4b}(1-2\varepsilon^2).$$

From which equation the value of ε elicited gives the direction for the longest throw. Moreover from this equation it is found as an approximation [original formula corrected by Paul St.]

$$\varepsilon=\sqrt{\frac{5b+\frac{3}{2}gc\sqrt{2}}{12b+3gc\sqrt{2}}}$$

Corollary 9.

979. Therefore the angle, which produces the longest throw, is a little less than half a right angle, which in a vacuum it makes satisfactorily. For if it becomes

$$\varepsilon=\sqrt{\frac{6b+\frac{3}{2}gc\sqrt{2}}{12b+3gc\sqrt{2}}}$$

then it produces $\varepsilon=\sqrt{\frac{1}{2}}$ and thus the half right angle. But as here we have only 5*b* in the numerator, the distance becomes less for a short while.

Corollary 10.

980. If the body at *A* (Fig. 90) is projected horizontally with a speed $\sqrt{a}$, then $\alpha = 0$ and *y* becomes negative. On this account on putting $AP = x$ and $PM = y$ the kind of this trajectory can be expressed by this equation :

$$y = \frac{gx^2}{4b} + \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2}.$$

Moreover for the curve *AN*, in which the body rises, is given by:

$$y = \frac{gx^2}{4b} - \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2}.$$

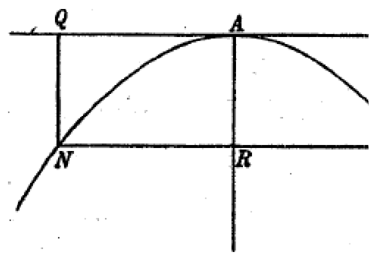

Fig. 90.

Corollary 11.

981. If more than four terms are taken, this equation is produced for the curve *AM* : [p. 418]

$$y = \frac{gx^2}{4b} + \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2} + \frac{gx^5}{240bc^3} + \text{etc.};$$

which terms are in agreement with the summable series, as it differs little from the true sum, if *y* is put equal to the sum of this series. Moreover, it becomes :

$$2by = c^2g\left(e^{\frac{x}{c}} - 1\right) - cgx.$$

Truly for the ascending arc *AN* it becomes :

$$2by = cgx - c^2g\left(1 - e^{-\frac{x}{c}}\right).$$

Corollary 12.

982. The time in which the arc AM is traversed, is equal to

$$\int \frac{ds}{\sqrt{v}} = \int \frac{\sqrt{2ddy}}{\sqrt{g}},$$

as $g = \frac{2vddy}{ds^2}$, the above equation becomes :

$$2bdy = cge^{\frac{x}{c}}dx - cgdx \text{ and } 2bddy = ge^{\frac{x}{c}}dx^2.$$

Therefore we have :

$$\int \frac{\sqrt{2ddy}}{\sqrt{g}} = \int \frac{e^{\frac{x}{2c}}dx}{\sqrt{b}} = \frac{2c}{\sqrt{b}}\left(e^{\frac{x}{2c}} - 1\right).$$

And if *b* and *c* are expressed in scruples of Rhenish feet, then by (222) the time to traverse AM is equal to

$$\frac{c}{125\sqrt{b}}\left(e^{\frac{x}{2c}}-1\right) \text{ min. sec.}$$

Scholium.

983. Therefore from this reasoning we have determined approximately the trajectory described in air for projected bodies, which can be put in place without too much difficulty rather than the parabola, which generally is accustomed to be used. Indeed we can deduce this same equation from the true equation $dsddy = cd^3y$ (875) of these trajectories found above. [p. 419] But since there this reduction was omitted, and here we have preferred to present this material, particularly since here it is evident that the later terms of the equation are strongly decreasing. Finally in a like manner too, the curves for trajectories in mediums with other hypothesis of resistance can be conveniently found; but since these other hypotheses do not find a place in the world, we will not tarry here about finding them.

PROPOSITION 118.

PROBLEM.

984. *To find the resistance at individual points M* (Fig.87) *for an absolute force acting downwards along MP, which can be put in place in order that a body can move on a given curve AM, and can move with a given speed at the individual points M.*

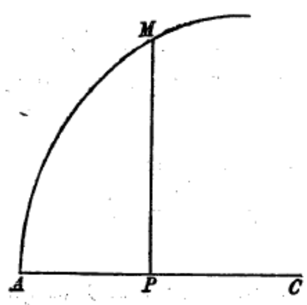

Fig. 87.

SOLUTION.

Putting in place as before : *AP = x, PM = y*, with the element of arc $AM = ds$, and with the height corresponding to the speed at *M* equal to *v*, which are therefore all given. Then let the force acting downwards on the body at *M* be equal to *P* and the resistance is equal to *R*. With these in place, *P* is immediately found from the equation :

$$P = -\frac{2vddy}{ds^2}$$

(871). But the resistance *R* is found from the equation $dv = \frac{2vdyddy}{ds^2} - Rds$(cit.). Whereby it is given by :

$$R = \frac{2vdyddy}{ds^3} - \frac{dv}{ds}.$$

[p. 420] If the law of the resistance is put in the square ratio of the speed, and the exponent of this is equal to *q*, then we have $R = \frac{v}{q}$, from which there is produced :

$$q = \frac{vds^3}{2vdyddy - dvds^2}$$

with *dx* taken as a constant. Q.E.I.

Corollary 1.

985. Since *dx* is constant, then we have $dyddy = dsdds$. On account of this, we have

$$R = \frac{2vdds}{ds^2} - \frac{dv}{ds} \text{ or } R = dsd. - \frac{v}{ds^2}.$$

And hence,

$$q = \frac{vds^2}{2vdds - dvds} \text{ or } q = \frac{v}{dsd. - \frac{v}{ds^2}}.$$

Corollary 2.

986. If the body must be carried by a uniform motion along the curve *AM*, thus as $v = b$, it arises that

$$P = -\frac{2bddy}{ds^2},\ R = \frac{2bdyddy}{ds^3} = \frac{2bdds}{ds^2}, \text{ and } q = \frac{ds^2}{2dds}.$$

Corollary 3.

987. Therefore in uniform motion, while the body ascends the curve *AM*, the resistance *R* is always negative or the motion of the body accelerates. But when the body descends again, the medium actually resists.

Corollary 4.

988. With the radius of osculation at *M* equal to *r*, since $r = -\frac{ds^2}{dxddy}$, then $ddy = -\frac{ds^2}{rdx}$. On account of this it is found that

$$P = \frac{2vds}{rdx},\ R = -\frac{2vdy}{rdx} - \frac{dv}{ds}, \text{ and } q = -\frac{rvdxds}{2vdyds + rdxdv}.$$

Corollary 5. [p. 421]

989. Again putting $v = b$ and $dv = 0$ there arises

$$P = \frac{2bds}{rdx},\ R = -\frac{2bdy}{rdx}, \text{ and } q = -\frac{rdx}{2dy}.$$

Therefore at the maximum point, since we make $dy = 0$ and $ds = dx$, then $P = \frac{2b}{r}$ and the resistance vanishes there, if perhaps the curvature is infinitely large there or $r = 0$.

Example.

990. Let the curve *AM* be a circle, the centre of which is at *C*, which must be described with a uniform speed $\sqrt{b}$. With the radius put as $AC = a$ we have :

$$dy = \frac{dx(a-x)}{y}, \quad ds = \frac{adx}{y} \text{ and } r = a.$$

From these is found the absolute force acting downwards, $P = \frac{2b}{r}$ or which varies inversely as the distance PM. Therefore the resistance is equal to $\frac{2b(a-x)}{ay}$. Therefore while the body ascends, the resistance is negative proportional to the reciprocal of the tangent of the arc *AM*. And if the resistance is proportional to the square of the speed, then the exponent of this is given by :

$$q = -\frac{ay}{2(a-x)} = -\frac{AC \cdot PM}{2PC}.$$

Thus q is negative and equal to half the tangent of the arc AM. Moreover when the body approaches the horizontal AC, then the resistance *R* and q become positive, or the medium actually offers resistance.

PROPOSITION 119.

PROBLEM.

991. *If the medium is uniform and offers resistance in the ratio of some multiple of the speed, and it is given besides that the body progresses uniformly along the horizontal AP at a constant speed* (Fig.90), *to find the force acting downwards and the curve that the body describes.* [p. 422]

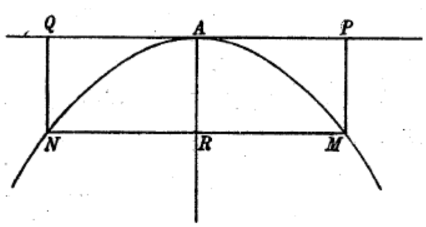

Fig. 90.

SOLUTION.

With $AP = x$, $PM = y$, with the arc $AM = s$, let the horizontal speed of the body while it is at *M* correspond to the height *u*; the height corresponding to the true speed of the body at M is equal to $\frac{uds^2}{dx^2} = v$. Again let the exponent of the resistance of the medium be equal to *c* and the ratio law of the 2*m*-multiple of the speed is the resistance

$$R = \frac{v^m}{c^m} = \frac{u^m ds^{2m}}{c^m dx^{2m}}.$$

With these in place we have :

$$P = \frac{2vddy}{ds^2} = \frac{2uddy}{dx^2}$$

(871); for I put *ddy* in place of *–ddy*, since in our case *y* is falling. Then we have :

$$dv = \frac{2vdyddy}{ds^2} - Rds$$

(cit.). But it is the case that :

$$dv = \frac{duds^2}{dx^2} + \frac{2udsdds}{dx^2}.$$

Hence on account of $dsdds = dyddy$ we have :

$$\frac{duds^2}{dx^2} = -\frac{u^m ds^{2m+1}}{c^m dx^{2m}} \text{ or } c^m dx^{2m-2}du + u^m ds^{2m-1} = 0.$$

Moreover *u* is given in terms of *x* and on account of this *ds* can be determined from *x* only; clearly it is given by :

$$ds = \frac{c^{\frac{m}{2m-1}} dx^{\frac{2m-2}{2m-1}} (-du)^{\frac{1}{2m-1}}}{u^{\frac{m}{2m-1}}}.$$

Thus it is permitted to find the equation for the curve to be found. With this found, likewise *P* can become known from the equation : $P = \frac{2uddy}{dx^2}$. Q.E.I. [p. 423]

Corollary 1.

992. Therefore the horizontal motion cannot be uniform; for on account of $du = 0$, $ds = 0$, except in a vacuum when $c = \infty$, and where it is always by necessity uniform. The acceleration of the horizontal motion must also be much less also in a resisting medium; for then *ds* is either given a negative or imaginary value, which is absurd in each case. Therefore the horizontal motion must be one of deceleration, by which *du* is made negative.

Corollary 2.

993. If the resistance is proportional to the speed itself, then we have $m = \frac{1}{2}$ and the equation for the curve becomes $du\sqrt{c} + dx\sqrt{u} = 0.$ Moreover which, since it does not contain *s* or *y*, cannot pertain to the curve. Moreover this equation itself determines the horizontal motion. It is evident in this hypothesis of the resistance that not any horizontal motion can be taken as you please, but by necessity that has to be accepted, which is determined by this equation.

Corollary 3.

994. Moreover that horizontal motion agrees with the horizontal motion along AP in the same resisting medium, but with no force acting. From which it is understood, if there were any force acting on the body, then the direction of this force is everywhere in the downwards direction only [p. 424], in a medium with the resistance in the simple ratio of the speed always to be the same. Concerning which motion in this hypothesis of the resistance is similar to the motion in a vacuum, in which the horizontal motion is always uniform, however the downwards force acting may vary.

Corollary 4.

995. Therefore with an assumed value for this horizontal motion, the other equation $P=\frac{2uddy}{dx^2}$ can determine the described curve, in which for *P* we are allowed to assume some quantity. Therefore in this case of the hypothesis of the resistance, this problem is generally soluble : in order that the curve is found, as the body acted upon by some downwards force describes.

Corollary 5.

996. Moreover since $-\frac{du\sqrt{c}}{\sqrt{u}}=dx,$ then $2\sqrt{bc}-2\sqrt{cu}=x$ with the initial speed at *A* put equal to $\sqrt{b}$. Therefore the equation becomes :

$$2\sqrt{cu}=2\sqrt{bc}-x \text{ and } u=\frac{(2\sqrt{bc}-x)^2}{4c}.$$

Hence for the curve described this equation is found :

$$2cPdx^2=ddy\left(2\sqrt{bc}-x\right)^2.$$

Corollary 6.

997. Therefore all these curves have a vertical asymptote at the distance $2\sqrt{bc}$ from the vertex A. For *x* cannot be greater than $2\sqrt{bc}$, and when the body moves horizontally [p. 425] is unable to progress beyond this boundary, also the body in the wide part of the curve cannot progress beyond this line.

Corollary 7.

998. If in other hypotheses of the resistance too, the horizontal motion along the curve *AM* is taken to agree with the horizontal motion along the line *AP* with the same hypothesis of resistance, thus in order that :

$$du=-\frac{u^m dx}{c^m},$$

and this equation $ds = dx$ is produced for the curve *AM*, taking $P = 0$. Therefore that agreement does not therefore find a place with other hypotheses of resistance.

Corollary 8.

999. Therefore let the resistance vary as the square of the speed, or $m = 1$. Whereby $ds = -\frac{cdu}{u}$ and by integrating, with the height b put to correspond to the speed at A, $s = c\, l\, \frac{b}{u}$. Therefore in this hypothesis of the resistance, the arc AM divided by $2c$ is equal to the logarithm of the horizontal speeds at A and M.

Corollary 9.

1000. Therefore in this hypothesis of the resistance, we have $u = e^{-\frac{s}{c}}b.$ From which there is produced :

$$P = \frac{2bddy}{e^{\frac{s}{c}}dx^2}.$$

[p. 426] Or since $ds = -\frac{cdu}{u}$, this becomes

$$dds = \frac{-cuddu + cdu^2}{u^2} = \frac{dyddy}{ds}.$$

But

$$dy = \frac{\sqrt{(c^2du^2 - u^2dx^2)}}{u}.$$

Hence we have :

$$ddy = \frac{c^2ududdu - c^2du^3}{u^2\sqrt{(c^2du^2 - u^2dx^2)}}.$$

Consequently we have:

$$P = \frac{2c^2ududdu - 2c^2du^3}{udx^2\sqrt{(c^2du^2 - u^2dx^2)}}.$$

Corollary 10.

1001. If this equation is taken between u and x

$$du = -\frac{u^n dx}{f^n} \text{ or } \frac{1}{(n-1)u^{n-1}} - \frac{1}{(n-1)b^{n-1}} = \frac{x}{f^n},$$

then

$$dx = -\frac{f^n du}{u^n} \text{ and } ddu = \frac{ndu^2}{u}.$$

With these substituted, there is produced in the medium with the resistance varying in the ratio of the square of the speed :

$$P = \frac{2(n-1)c^2u^{3n-2}}{f^{2n}\sqrt{(c^2u^{2n-2}-f^{2n})}}.$$

Moreover, this equation is obtained for the curve sought :

$$dy = \frac{du\sqrt{(c^2u^{2n-2}-f^{2n})}}{u^n}.$$

Corollary 11.

1002. Therefore in this case u^{2n-2} must be greater than $\frac{f^{2n}}{c^2}$ or u greater than

$$\frac{f^{\frac{n}{n-1}}}{c^{\frac{1}{n-1}}}.$$

Whereby, if the horizontal motion is able to become less than this quantity, the motion on the curve does not correspond to the total horizontal motion given. For if the curve is stretched further, then it could be that dy as well as P become imaginary. [p. 427]

Corollary 12.

1003. To avoid this inconvenience n must be made smaller than unity; therefore making $n-1=-k$ or $n=1-k$. With this in place, we have $u^k = b^k - f^{k-1}kx.$ Truly with AP the tangent to the curve at A here we have $ds = dx$, and there $u = b$. Hence we have

$$b^{n-1}c = f^n \text{ , or } \frac{c}{b^k} = f^{1-k} \text{, and thus } u^k = b^k\left(1-\frac{kx}{c}\right).$$

Moreover again we obtain

$$P = -\frac{2kb^{3k}u^{1-2k}}{c\sqrt{(b^{2k}-u^{2k})}} \text{, and } dy = -\frac{dx}{u^k}\sqrt{(b^{2k}-u^{2k})} = -\frac{dx\sqrt{(2kcx-k^2x^2)}}{c-kx}.$$

Scholium.

1004. Therefore none of the hypotheses of this kind of the horizontal motion of trajectories in fluids can be integrated. Indeed whatever the value of k, the force at A, where we put $u = b$, is indefinitely great; then truly is decreases for ever. All these curves also have vertical tangents, where $x = \frac{c}{k}$, which is an asymptote to the curve. Moreover we assign this problem to the end of the first part of the tract, in which we have put the direction of the force to be always parallel among themselves, and we progress to examination of centripetal forces that must soon be considered, in what manner the resisting medium disturbs the motion of bodies attracted to a fixed point. [p. 428]

CAPUT SEXTUM

DE MOTU CURVILINEO PUNCTI LIBERI IN MEDIO RESISTENTE

[p. 396]

.

PROPOSITIO 113.

PROBLEMA.

925. *Posita ut ante vi absoluta g uniformi et deorsum trahente invenire vim resistentiae, qua efficetur, ut corpus in hyperbola NAM* (Fig.85) *axem CAQ verticalem habente libere moveri possit.*

SOLUTIO.

Sit *C* centrum hyperbolae et semiaxis transversus *AC* = *a*; semiaxis vero coniugatus sit = *c*. Ponatur *CQ* = *t* et *QM* = *AP* = *x* eritque ex natura hyperbolae

$$c^2t^2 = a^2x^2 + a^2c^2.$$

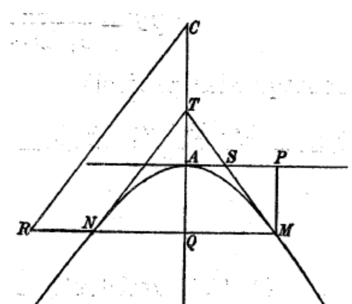

Fig. 85.

Sumta autem *PM* = *AQ* = *y* erit $y = t - a$

et $dy = dt, \quad d^2y = d^2t$ atque $d^3y = d^3t$.

Ex aequationem vero habebitur

$$t = \frac{a\sqrt{(x^2+c^2)}}{c} \quad \text{et} \quad dt = \frac{axdx}{c\sqrt{(x^2+c^2)}}$$

ideoque

$$ds = \frac{dx\sqrt{(c^4+c^2x^2+a^2x^2)}}{c\sqrt{(x^2+c^2)}}.$$

Porro fiet

$$ddt = ddy = \frac{acdx^2}{(x^2+c^2)^{\frac{3}{2}}} \quad \text{et} \quad d^3t = d^3y = -\frac{3acxdx^3}{(x^2+c^2)^{\frac{5}{2}}}.$$

Ex quibus erit

$$\frac{d^3y}{ddy} = -\frac{3xdx}{x^2+c^2} \quad \text{et} \quad \frac{d^3y}{ddy^2} = -\frac{3x\sqrt{(x^2+c^2)}}{acdx}.$$

Consequenter proveniet resistentia

$$R = -\frac{3gx\sqrt{(c^4+c^2x^2+a^2x^2)}}{2ac^2} \quad \text{atque} \quad v = \frac{g(c^4+c^2x^2+a^2x^2)\sqrt{(x^2+c^2)}}{2ac^3}.$$

Resistentia vero quadratis celeritatum proportionali posita erit exponens resistentiae

$$q = -\frac{\sqrt{(x^2+c^2)(c^4+c^2x^2+a^2x^2)}}{3cx}.$$

Corollarium 1.

926. Ducta tangente MT erit

$$QT = \frac{ax^2}{c\sqrt{(x^2+c^2)}} \quad \text{et} \quad MT = \frac{x\sqrt{(c^4+c^2x^2+a^2x^2)}}{c\sqrt{(x^2+c^2)}}.$$

Consequenter prodibit [p. 397]

$$R = -\frac{3g\cdot MT\cdot\sqrt{(x^2+c^2)}}{2ac} = -\frac{3g\cdot CQ\cdot MT}{2a^2} \quad \text{seu} \quad R : g = -3CQ\cdot MT : 2AC^2.$$

Corollarium 2.

927. Quia resistentia R invenitur negativa, indicio id est descensum per AM in hyperbola fieri non posse in medio resistente, sed requiri, ut corpus a medio promoveatur. At dum corpus per arcum NA ascendit, quia fit ds negativum, resistentia R erit affirmativa. Hanc ob rem, si corpus est in N, erit resistentia $R = \frac{3g.CQ.NT}{2AC^2}$.

Corollarium 3.

928. Ex natura hyperbola est $CQ : AC = AC : CT$. Itaque erit resistantia in *N*, seu *R*, $= \frac{3g.NT}{2CT}$. Vel erit resistantia *R* ad potentiam absolutam *g* ut 3*NT* ad 2*CT*. In vertice ergo *A* resistentia evanescit crescitque, quo magis *N* ab *A* distat. [p. 398]

Corollarium 4.

929. Altitudo debita celeritati corporis in *M* vel *N* est $= \frac{g.CQ^2.MT^2}{2AC^2.QT}$, uti ex valore ipsius *v* et natura hyperbola facile deducitur. Cum autem sit $MT = NT$ et $CQ.NT = \frac{2R.AC^2}{3g}$, erit $v = \frac{2R^2.AC^2}{9g.QT}$.

Corollarium 5.

930. Si resistentia ponatur celeritatibus proportionalis et exponens resistentiae sit *q*, erit $R = \frac{\sqrt{v}}{\sqrt{q}}$ et $v = R^2q$. Quare invenietur $q = \frac{2AC^2}{9g.QT}$. Hac igitur hypothesi exponens erit reciproce ut subtangens QT.

Corollarium 6.

931. At si resistentia ponatur quadratis celeratatum proportionalis sitque expons resistentiae q, erit existente corpore in N hic exponens $q = \frac{NT.CQ}{3TQ}$. Seu ducta ex C parallela CR tangenti NT, occurrente applicatae QN productae in R, erit $q = \frac{CR}{3}$.

Scholion.

931. Quod ante in circulo et nunc in hyperbola observavimus resistentiam in altero arcu fieri affirmativam, in altero negativam, id in omnibus curvis circa supremum punctum *A* duos arcus similes et aequaeles ut *AN* et *AM* habentibus locum obtenit. [p. 399]
Nam cum generaliter pro arcu AM sit resistentia $R = \frac{gdsd^3y}{2ddy^2}$, quia in arcu AN sit ds negativum, erit in N resistentia $R = -\frac{gdsd^3y}{2ddy^2}$, ita ut resistentia in N sit negativum resistentiae in M. Quare, cum rerum natura non detur resistentia negativa, qua motus corporis acceleratur, fieri non potest, ut corpus in medio resistente curvam describat, quae circa summum punctum A habeat duos ramos similies et aequales. Altitudo vero generans celeritatem tam in M quam in N est eadem; eius enim valor $\frac{gds^2}{2ddy}$ non mutatur, etiamsi *ds* fiat negativum. Cum igitur istius modi curvae a Neutono ideo sint consideratae, ut aliquam erueret, pro qua medii resistentis densitas non multum variaret, seu nostro tractandi modo, in qua exponens resistentiae ubique fere sit eiusdem valoris, quo talem curvam pro proiectoria in medio resistenti uniformi habere posset sine sensibili errore, alias curvas non diametro verticali praeditas cum Neutono considerabimus, cuius modi sunt hyperbolae asymtoton verticalem habentes, quippe quae propius accedunt ad logarithmicam, quae a corpore in medio resistente in simplici celeritatum ratione eoque uniformi describitur. In alia enim resistentiae hypothesi Neutonus proiectorias non determinavit, sed contentus fuit veris proximas assignare. Quod institutum, cum verae proiectoriae a nobis datae tam sint implicatae, ut vix quicquam ex iis ad praxin possit deduci, etiam sequemur. [p. 400]

PROPOSITIO 114.

PROBLEMA.

933. *Sit curva NM* (Fig.86) *hyperbola cuiuscunque gradus alteram habens asymtoton CP verticalem, determinare resistentiam, quae efficiat, ut corpus perpetuo vi a deorsum sollicitatum in hac hyperbola possit moveri.*

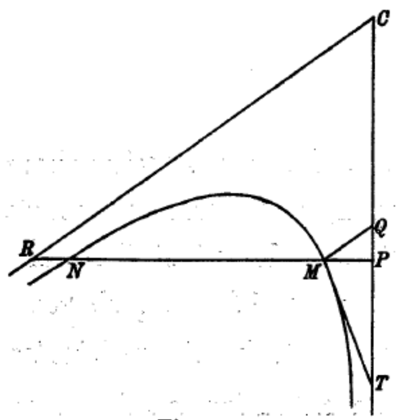

Fig. 86.

SOLUTIO.

Consideretur asymtotos *CP* tanquam axis ad eumque ex *M* normalis ducatur *MP*. Posita *CP* = *y* et *MP* = *x*, quae supra generaliter tradidimus, haec locum habebunt, si modo ibi *dx* sumatur negativum. Sit *RC* altera asymtotos et *C* centrum hyperbolae, sinus ang. RCP = α eiusque cosinus $\sqrt{(1-\alpha^2)} = \beta$. Erit ergo producta *PM* in *R* $PR = \frac{\alpha y}{\beta}$ et $CR = \frac{y}{\beta}$. Ex *M* ducatur *MQ* parallela asymtoto *CR*, erit

$$MQ = \frac{x}{\alpha} \text{ et } PQ = \frac{\beta x}{\alpha} \text{ ideoque } CQ = \frac{\alpha y - \beta x}{\alpha}.$$

At ex natura hyperbolarum erit

$$a^n = \frac{x^{n-1}(\alpha y - \beta x)}{\alpha^n}$$

atque hinc

$$y = \frac{\beta x}{\alpha} + \frac{\alpha^{n-1}a^n}{x^{n-1}} \quad \text{et} \quad dy = \frac{\beta dx}{\alpha} - \frac{(n-1)\alpha^{n-1}a^n dx}{x^n}.$$

Ducta tangente MT erit PT = $-\frac{xdy}{dx} = -\frac{\beta x}{\alpha} + \frac{(n-1)\alpha^{n-1}a^n}{x^{n-1}}, MT = \frac{xds}{dx}$, [p. 401]

ita ut sit $ds = \frac{MT.dx}{x}$. Hoc vero casu, quo *x* in altera parte sumitur, fit MT negativa. Porros ob *dx* constans erit

$$d^2y = \frac{n(n-1)\alpha^{n-1}a^n dx^2}{x^{n+1}} \quad \text{et} \quad d^3y = -\frac{(n+1)(n-1)n\alpha^{n-1}a^n dx^3}{x^{n+2}}.$$

Ex his oritur

$$\frac{gdsd^3y}{2ddy^2} = -\frac{g(n+1)x^{n-1}\cdot MT}{2n(n-1)\alpha^{n-1}a^n},$$

qui valor facto *MT* negativo aequatur resistentiae *R*. Erit itaque

$$R = \frac{g(n+1)x^{n-1}\cdot MT}{2n(n-1)\alpha^{n-1}a^n}$$

(908) atque altitudo debita celeritati in *M*, seu *v*, =

$$\frac{gx^{n-1}\cdot MT^2}{2n(n-1)\alpha^{n-1}a^n}.$$

Si medium resistere ponatur in ratione 2*m*-plicata celeritatum et exponens resistentiae sit *q*, erit $R=\frac{v^m}{q^m}$ ideoque $q=\frac{v}{R^{\frac{1}{m}}}$. Ex quo fiet

$$q=\frac{g^{\frac{m-1}{m}}x^{\frac{(m-1)(n-1)}{m}}MT^{\frac{2m-1}{m}}}{2^{\frac{m-1}{m}}(n^2-n)^{\frac{m-1}{m}}(n+1)^{\frac{1}{m}}\alpha^{\frac{(m-1)(n-1)}{m}}a^{\frac{n(m-1)}{m}}}$$

seu

$$q^m=\frac{g^{m-1}x^{(m-1)(n-1)}MT^{2m-1}}{2^{m-1}(n^2-n)^{m-1}(n+1)\alpha^{(m-1)(n-1)}a^{n(m-1)}}.$$

Q.E.I.

Corollarium 1.

934. Quia est $x^{n-1}=\frac{\alpha^n a^n}{\alpha y-\beta x}$, erit resistentia

$$R=\frac{g(n+1)\alpha\cdot MT}{2n(n-1)(\alpha y-\beta x)}=\frac{g(n+1)\cdot MT}{2n(n-1)\cdot CQ}.$$

Unde erit resistentia *R* ad potentiam *g* ut $(n+1)MT$ ad $2n(n-1)CQ$. [p. 402] Simili modo hoc loco x^{n-1} valore substituto erit

$$v=\frac{g\cdot MT^2}{2n(n-1)\cdot CQ}\quad \text{et}\quad q^m=\frac{g^{m-1}\cdot MT^{2m-1}}{2^{m-1}(n^2-n)^{m-1}(n+1)\cdot CQ^{m-1}}.$$

Corollarium 2.

935. Descendente corpore in infinitum fiet $x=0$ et

$$MT=PT=\frac{(n-1)\alpha^{n-1}a^n}{x^{n-1}}$$

evanescente *x*. In profunditate ergo infinita erit $R=\frac{g(n+1)}{2n}$ ideoque finitae magnitudinis, at erit $v=\frac{g(n-1)\alpha^{n-1}a^n}{2nx^{n-1}}$. Quare, cum necessario sit $n>1$, evanescente *x* fiet corporis celeritas infinite magna.

Corollarium 3.

936. Posita igitur resistantia $R=\frac{v^m}{q^m}$ in profunditate infinita debebit etiam *q* esse infinite magna; his itaque locis corpus in vacuo movebitur. Ex quo sequitur, quo magis corpus descendat, eo minorem fore resistentiam seu potius medium eo rarius.

Corollarium 4.

937. In hyperbola Apolloniana fit $n = 2$. Pro hac igitur curva invenitur [p. 403]

$$R = \frac{3gx \cdot MT}{4\alpha a^2} = \frac{3g \cdot MT}{4\,CQ} \quad \text{et} \quad v = \frac{g \cdot MT^2}{4\,CQ} \quad \text{atque} \quad q^m = \frac{g^{m-1} \cdot MT^{2m-1}}{3 \cdot 2^{2m-2}\,CQ^{m-1}}.$$

Corollarium 5.

938. Si resistentia ponatur celeritatibus proportionalis, erit in omnibus his hyperbolis exponens resistentiae directe ut *CQ*, ob $m = \frac{1}{2}$ hoc casu. Hac igitur resistentiae hypothesi corpus omnes hyperbolas poterit libere describere.

Corollarium 6.

939. Si resistentia ponatur in duplicata celeritatum ratione, ut sit m = 1, erit exponens resistentiae in M = $\frac{MT}{n+1}$. Quo magis igitur MT variatur, eo magis quoque medium erit difforme.

Corollarium 7.

940. Tempus praeterea, quo elementum *Mm* describitur, seu $\frac{ds}{\sqrt{v}}$, erit = $\frac{\sqrt{2}ddy}{\sqrt{g}}$. Tempus igitur, quo corpus in M usqae pervenit, est ut

$$\int \frac{dx}{x^{\frac{n+1}{2}}}, \quad \text{i. e. ut} \quad C - \frac{1}{x^{\frac{n-1}{2}}} \quad \text{seu potius ut} \quad \frac{1}{x^{\frac{n-1}{2}}} - C.$$

Quare tempus, quo corpus ex *N* in *M* pervenit, erit ut

$$\frac{1}{MP^{\frac{n-1}{2}}} - \frac{1}{NP^{\frac{n-1}{2}}}.$$

Corollarium 8. [p. 404]

950. In hyperbola igitur Apolloniana, in qua $n = 2$, erit tempus, quo corpus ab N ad M pervenit, ut $\sqrt{NP} - \sqrt{MP}$ ob $NP.MP$ constans, nempe = $\frac{a\alpha^2}{\beta}$.

Scholion.

951. Ex his manifestum est corpus in medio resistente uniformi huiusmodi hyperbolas describere non posse, cum exponens resistentiae nimium sit variabilis, quippe que tandem fit infinite magnus. Quamobrem Neutoni institutum, quo has hyperbolas loco verarum proiectoriarum in medio resistente uniformi substituere voluit, probari non potest. In medio exin secundum quadrata celeritatum resistente exponens est ut tangens MT, quae tam descendendo quam ad punctum N regrediendo vehementer variatur. Intellegi etiam haec inconvenientia potest ex celeritate, quae descendendo in infinitum crescit, cum tamen in medio uniformi non ultra datum terminum crescere queat. Preaterea non satis liquet in resistentiae quadratis celeritatum proportionalis hypothesi curvam descriptam habere asymtoton verticalem, quemadmodum in medio resistente in simplice ratione celeritatum. Nam hac resistentia, etiamsi corpus a nulla potentia urgetur, totius motus linea est finita; quae vero, si resistentia quadratis celeritatum proportionalis ponitur, fit infinita. [p. 405] Ex quo etiam consequi videtur proiectoriam in hac resistentia non esse habituram asymtoton. Hoc saltem certum est non habere hanc curvam asymtoton hyperbolicam. Interim tamen habet asymtoton verticalem alius generis, quae ex quadratura curvae per rectificationem parabolae datae determinatur. Sed relicta potentiae uniformi hypothesi pergamus ad potentiam variabilem, cuius tamen directio ubique sibi sit parallela. Ex datis quidem potentia et resistentia curvam descriptam non investigabimus, cum hoc prop. 106 (870) iam sit factum; sed data curva atque vel potentia vel resistentia vel celeritate reliqua determinabimus.

PROPOSITIO 115.

PROBLEMA.

952. *Sit potentia absoluta utcunque variabilis et ubique deorsum tendens iuxta MP* (Fig.87); *determinare resistentiam requisitam ad hoc, ut corpus in data curva AM moveatur.*

SOLUTIO.

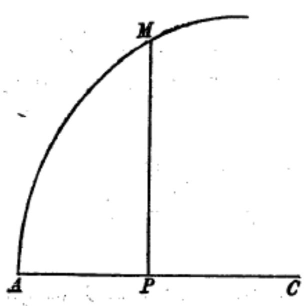

Fig. 87.

Sit $AP = x$, $PM = y$ et elementum arcus $AM = ds$; tum sit vis, qua corpus in M secundum MP sollicitatur, $= P$ et altitudo celeritati in M debita $= v$ atque resistentia in $M = R$. His positis erit (870) $dv = -Pdy - Rds$ et $v = -\frac{Pds^2}{2ddy}$ (871) sumto dx constante. [p. 406] Ex hac ergo aequatione erit

$$dv = -Pdy - \frac{dPds^2}{2ddy} + \frac{Pds^2d^3y}{2ddy^2},$$

unde prodibit

$$R = \frac{dPds}{2ddy} - \frac{Pdsd^3y}{2ddy^2} \quad \text{seu} \quad \frac{2R}{ds} = d.\frac{P}{ddy}.$$

Inveniuntur ergo tam v quam R per datas quanitates P, x et y expressa.
Q.E.I.

Corollarium 1.

953. Posito radio osculi in $M = r$ erit

$$ddy = -\frac{ds^3}{rdx} \quad \text{et} \quad d^3y = -\frac{3dsdyddy}{rdx} + \frac{ds^3dr}{r^2dx} = \frac{3ds^4dy}{r^2dx^2} + \frac{ds^3dr}{r^2dx}.$$

unde prodit

$$v=\frac{Prdx}{2ds}\quad \text{et}\quad R=\frac{-rdPdx-Pdxdr-3Pdsdy}{2ds^2}.$$

Corollarium 2.

954. Si lex resistantiae sit celeritatum ratio duplicata et exponens = q, erit $R=\frac{v}{q}$ et $q=\frac{v}{R}$. Quocirca reperietur

$$q=\frac{Pdsddy}{Pd^3y-dPddy}\quad \text{seu}\quad q=-\frac{Prdxds}{rdPdx+Pdxdr+3Pdsdy}.$$

Corollarium 3.

955. Sit vis P ad gravitatem 1 ut y ad f, erit $P=\frac{y}{f}$ ideoque

$$R=\frac{-rdydx-ydxdr-3ydsdy}{2fds^2}\quad \text{et}\quad v=\frac{yrdx}{2fds}$$

atque

$$q=-\frac{yrdxds}{rdydx+ydxdr+3ydsdy}.$$

Corollarium 4.

956. Si curva AM fuerit circulus radii $AC = a$, erit

$$r=a,\quad dr=0,\quad dy=\frac{(a-x)dx}{y}\quad \text{et}\quad ds=\frac{adx}{y}.$$

Unde prodit $v=\frac{Py}{2}$ et resistantia [p. 407]

$$R=-\frac{y^2dP}{2adx}-\frac{3P(a-x)}{2a}\quad \text{atque}\quad q=-\frac{aPydx}{y^2dP+3P(a-x)dx}.$$

Exemplum 1.

957. Sit curva AM circulus, cuius centrum C et radius $AC = a$. Corpus autem perpetuo ad axem AC attrahatur in ratione distantiarum, ita ut sit $P=\frac{y}{f}$; erit $v=\frac{y^2}{2f}$ ideoque celeritas in M erit ut applicata MP. Deinde resistentia R fiet =

$$\frac{-y(a-x)-3y(a-x)}{2af}=-\frac{2y(a-x)}{af}=-\frac{2PM\cdot CP}{f\cdot AC}\quad \text{et}\quad q=-\frac{PM\cdot AC}{4CP}.$$

Celeritas autem in puncto A erit = 0 et resistentia, dum corpus in quadrante ascendit, negativa seu corpus a medio accelerabitur. Tempus vero, quo corpus ex A in M pervenit, erit infinite magnum; fit enim

$$\int\frac{ds}{\sqrt{v}}=\int\frac{adx\sqrt{2f}}{2ax-x^2}=-\sqrt{\frac{f}{2}}\,l0+\sqrt{\frac{f}{2}}\,l\frac{x}{2a-x}.$$

Id quod etiam per se intelligitur; nam cum celeritas in A sit = 0 et hic tam vis sollicitans $\frac{y}{f}$ quam vis medii evanescat, corpus perpetuo in A debebit perseverare.

Exemplum 2.

958. Manente curva *AM* circulo, si vis absoluta fuerit reciproce ut distantia *PM* seu $P=\frac{f}{y}$, erit $v=\frac{f}{2}$. Quare celeritas corporis ubique erit eadem, seu corpus feretur motu aequabili per circuli peripheriam, et tempus, quo arcus quivis *AM* absolvitur, erit ut ipse arcus *AM*. At resistentia in *M* erit =

$$\frac{f(a-x)}{ay}=-\frac{f\cdot CP}{AC\cdot PM}.$$

Resistentia igitur, [p. 408] dum corpus per quadrantem ascendit, erit negativa; dum autem per sequentem quadrantem descendit, resistentia fiet affirmativa seu erit vera resistentia. Ex resistentia porro invenitur $q=\frac{AC.PM}{2CP}$. In puncto igitur *A* resistentiae vis promovens erit infinite magna et potentia sollicitans, quae resistantiae est aequalis.

Exemplum 3.

959. Manente *AM* circulo sit vis sollicitans ut potestas quaecunque distantiae *MP* seu $P=\frac{y^n}{f^n}$; erit $dP=\frac{ny^{n-2}(a-x)dx}{f^n}$. Ex his igitur prodibit $v=\frac{y^{n+1}}{2f^n}$ et resistentia

$$R=-\frac{(n+3)y^n(a-x)}{2af^n}.$$

Unde fit

$$R:P=-(n+3)CP:2AC.$$

Quare, si fuerit $n=-3$ seu potentia P reciproce ut cubus distantiae *MP*, evanescet resistentia *R* corpusque ab hac potentia sollicitatum in vacuo moveri poterit in circulo *AM*. Deinde si $n+3$ est numerus affirmativus, resistantia per quadrantem ascensus erit negativa. At si $n+3$ est numerus negativus, resistentia per hunc quadrentem fit affirmativa.

Exemplum 4.

960. Si curva AMB (Fig. 88) fuerit talis, ut radius osculi in M sit reciproce ut applicata PM, [p. 409] id quod omnes curvas elasticas competit, erit

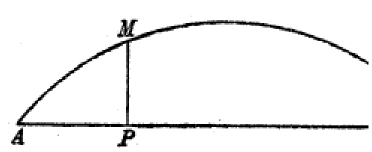

Fig. 88.

$$r=\frac{a^2}{y} \quad \text{et} \quad dr=-\frac{a^2dy}{y^2}.$$

Fiet igitur

$$v=\frac{a^2Pdx}{2yds}$$

et

$$R=\frac{a^2Pdxdy}{2y^2ds^2}-\frac{a^2dPdx}{2yds^2}-\frac{3Pdy}{2ds}.$$

Cum autem sit $-\frac{ds^3}{dxddy}=\frac{a^2}{y}$ atque integrando

$2a^2dx = ds(y^2 + b^2)$, est

$$dy = \frac{dx\sqrt{(4a^4 - (y^2+b^2)^2)}}{y^2+b^2}.$$

Hinc erit

$$v = \frac{P(y^2+b^2)}{4y}.$$

Atque

$$R = \frac{P(b^2 - 5y^2)\sqrt{(4a^4 - (y^2+b^2)^2)}}{8a^2y^2} - \frac{dP(y^2+b^2)^2}{8a^2ydx}.$$

Sit nunc

$$P = \frac{y}{f} \quad \text{et} \quad dP = \frac{dx\sqrt{(4a^4 - (y^2+b^2)^2)}}{f(y^2+b^2)},$$

erit

$$v = \frac{y^2+b^2}{4f} \quad \text{et} \quad R = -\frac{3y\sqrt{(4a^4 - (y^2+b^2)^2)}}{4a^2f} = -\frac{3ydy}{2fds}.$$

Ideoque $R : P = -3dy : 2ds$. Quam diu igitur corpus ascendit, resistentia est negativa, at quando descendo, erit affirmativa.

PROPOSITIO 116.

PROBLEMA.

961. *Si data sit curva AM* (Fig.87) *et resistentia per quantitates ad curvam pertinentes, invenire potentiam absolutam P perpetuo normaliter ad axem AC tendentem, quae faciat, ut corpus in hac curva libere moveri possit.*

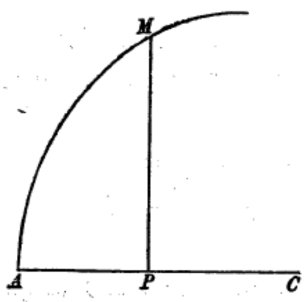

Fig. 87.

SOLUTIO.

Sit $AP = x$, $PM = y$ et elementum curvae $= ds$. Deinde sit resistentia in $M = R$, quae ergo per x, y et s dabitur; potentia quaesita sit $= P$ et celeritas in M debita altitudini v. His positis erit (871)

$$P = -\frac{2vddy}{ds^2} \quad \text{et} \quad dv = \frac{2vdyddy}{ds^2} - Rds$$

(cit.). [p. 410] Ex hac aequatione ob dx constans reperitur integrando

$$v = \frac{ads^2}{dx^2} - \frac{ds^2}{dx^2}\int \frac{Rdx^2}{ds}.$$

Inventa autem v innotescet P ex aequatione $P = -\frac{2vddy}{ds^2}$. Q.E.I.

Corollarium 1.

962. Erit igitur

$$P = -\frac{2a\,ddy}{dx^2} + \frac{2ddy}{dx^2}\int\frac{Rdx^2}{ds}.$$

Per meras igitur quantitas datas determinatur *P*.

Corollarium 2.

963. Quia constans addita a pro lubitu potest accipi, ita ea poterit determinari, ut corpus in puncto *A* vel alio quodam dato puncto datam habeat celeritatem.

Corollarium 3.

964. Si resistentia ponatur quadratis celeritatum proportionalis et exponens resistentiae *q*, erit

$$q = \frac{a\,ds^2}{Rdx^2} - \frac{ds^2}{Rdx^2}\int\frac{Rdx^2}{ds}.$$

Corollarium 4.

965. Tempus, quo arcus *AM* percurritur, est $\int\frac{dx}{\sqrt{v}}$; in qua expressione si valor ipsius *v* substituatur, proveniet tempus per *AM* =

$$\int\frac{dx}{\sqrt{\left(a - \int\frac{Rdx^2}{ds}\right)}}.$$

Corollarium 5. [p. 411]

966. Si resistentia ponatur ad vim gravitatis 1 ut tangens in *M* ad subtangentem seu ut *ds* ad *dx*, erit

$$v = \frac{ds^2}{dx^2}(a - x) \quad \text{et} \quad P = -\frac{2ddy}{dx^2}(a - x)$$

atque tempus per *AM* =

$$\int\frac{dx}{\sqrt{(a-x)}} = 2\sqrt{a} - 2\sqrt{(a-x)}.$$

At porro erit

$$q = \frac{ds}{dx}(a - x).$$

Exemplum.

967. Sit curva circulus, cuius radius $AC = b$, erit $y = \sqrt{(2bx - xx)}$ et

$$dy = \frac{dx(b-x)}{y} \quad \text{et} \quad ds = \frac{bdx}{y} \quad \text{atque} \quad ddy = -\frac{b^2dx^2}{y^3}.$$

Ex his habebitur

$$v = \frac{b^2}{y^2}\left(a - \int \frac{Rydx}{b}\right) \quad \text{atque} \quad P = \frac{2b^2}{y^3}\left(a - \int \frac{Rydx}{b}\right).$$

Ponatur resistentia =

$\frac{ds}{dx}$ seu $R = \frac{b}{y}$, erit $v = \frac{b^2}{y^3}(a-x)$ et $P = \frac{2b^2}{y^3}(a-x)$ atque $q = \frac{b}{y}(a-x)$. Tempus vero, quo corpus ex A in M pervenit, erit $= 2\sqrt{a} - 2\sqrt{(a-x)}$. Si ulterius sit $b = a$, erit

$$v = \frac{AC^2 \cdot PC}{PM^2} \quad \text{et} \quad P = \frac{2AC^2 \cdot PC}{PM^3} \quad \text{et} \quad R = \frac{AC}{PM}.$$

Celeritas igitur corporis in supremo circuli puncto erit = 0 et potentia P ibidem evanescit. Corpus autem ultra hoc punctum non poterit progredi, quia alias celeritas fieret imaginaria; inde igitur revertetur ad punctum A, quia a resistentia, quae in recessu fit negativa, acceleratur. Dum autem pervenerit in A, quia hic celeritas est infinite magna, hoc motu suo describet quadrantem infra AC, in quo ab potentiam P negativam sursum urgibitur. [p. 412]

PROPOSITIO 117.

PROBLEMA.

968. *Si medium fuerit uniforme atque resistat in duplicat ratione celeritatum, determinare potentiam absolutam deorsum tendentem, quae faciat, ut corpus in hoc medio resistente describat curvam datam AM* (Fig.87)

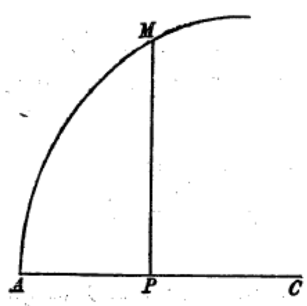

Fig. 87.

SOLUTIO.

Positis $AP = x$, $PM = y$, elemento arcus $AM = ds$, celeritate in M = $\sqrt{v}$, exponente medii resistens = c et potentia absoluta = P, erit $R = \frac{v}{c}$. His positis erit

$$P = -\frac{2vddy}{ds^2} \quad \text{et} \quad dv = \frac{2vdyddy}{ds^2} - \frac{vds}{c}$$

(871). Quare habebitur

$$\frac{dv}{v} = \frac{2dyddy}{ds^2} - \frac{ds}{c}$$

et integrando

$$lv = l\frac{ads^2}{dx^2} - \frac{s}{c} \quad \text{seu} \quad v = \frac{ae^{-\frac{s}{c}}ds^2}{dx^2}.$$

Valore igitur ipsius v invento erit

$$P = -\frac{2ae^{-\frac{s}{c}}ddy}{dx^2}.$$

Data ergo curva tum v tum P inveniuntur. Q.E.I.

Corollarium 1.

969. Tempus, quo corpus arcum AM absolvit, seu $\int \frac{ds}{\sqrt{v}}$, erit =

$$\int \frac{e^{\frac{s}{2c}}dx}{\sqrt{a}}.$$

Data ergo curva seu aequatione inter s et x, [p. 413] habebitur quoque tempus, saltem per quadraturas. Eae igitur curvae ad hoc sunt commodissimae, pro quibus datur aequatio inter s et x.

Corollarium 2.

970. Quia a pro lubitu potest accipi, utpote quantitas integratione adiecta, eius determinatione effici potest,ut vel celeritas in dato curvae loco sit data, vel potentia sollicitans.

Corollarium 3.

971. Si curva *AM* versus axem *AP* est concava, tum est *ddy* negativum; his igitur casibus potentia corpus ad axem *AP* trahet. At si curva erit convexa versus *AP*, quia tum *ddy* fit affirmativum, potentia *P* fit negativa, seu corpus ab axe *AP* repelletur.

Exemplum 1.

972. Sit curva *AM* parabola axem habens normaliter insistentem rectae *AP*, qualis in vacuo a corpore ex *A* oblique proiecto describitur, erit
$by = fx - x^2$ ideoque $dy = \frac{fdx}{b} - \frac{2xdx}{b}$ atque $ddy = -\frac{2dx^2}{b}$. His substitutis erit potentia sollicitans

$$P = \frac{4a}{be^{\frac{s}{c}}}.$$

Ex quo intelligitur, quo diutius motus continuetur, [p. 414] eo magis decrescere potentiam P. Facto autem c infinite magno, id quod fit in vacuo, erit $e^{\frac{s}{c}} = 1$ atque potentia $P = \frac{4a}{b}$ et idcirco constans.

Exemplum 2.

973. Sit curva *AM* talis, ut eius aequatio sit $y = \alpha x - \beta x^2 - \gamma x^3$, erit
$dy = \alpha dx - 2\beta xdx - 3\gamma x^2 dx$ et $d^2y = -2\beta dx^2 - 6\gamma xdx^2$.
Hinc erit

$$P = \frac{4\beta a + 12\gamma a x}{e^{\frac{s}{c}}}.$$

In puncto *A* est

$$dy = \alpha dx \quad \text{et} \quad \frac{ds^2}{dx^2} = 1 + \alpha^2.$$

Altitudo igitur debita celeritati initiali in *A* est $a(1+\alpha^2)$; quae si dicatur *b*, erit $a = \frac{b}{1+\alpha^2}$, et α est tangens anguli, sub quo corpus ex A proiicitur. Deinde ex ipsius *dy* valore reperitur

$$ds = dx\sqrt{(1 + \alpha^2 - 4\alpha\beta x + 4\beta^2 x^2 - 6\alpha\gamma x^2 + 12\beta\gamma x^3 + 9\gamma^2 x^4)}$$
$$= dx\sqrt{(1+\alpha^2)} - \frac{2\alpha\beta x dx}{\sqrt{(1+\alpha^2)}} \quad \text{etc.}$$

Neglectis igitur reliquis terminis foret

$$s = x\sqrt{(1+\alpha^2)} - \frac{\alpha\beta x^2}{\sqrt{(1+\alpha^2)}}$$

atque

$$e^{\frac{s}{c}} = 1 + \frac{x\sqrt{(1+\alpha^2)}}{c} - \frac{\alpha\beta x^2}{c\sqrt{(1+\alpha^2)}} + \frac{x^2(1+\alpha^2)}{2c^2} \quad \text{etc.,}$$

qui termini quoque reiici possunt, si c fuerit valde magnum. Quare, quo P fiat quam proxime constans, nempe = *g*, debebit esse

$$4\beta a = g \quad \text{et} \quad \frac{3\gamma}{\beta} = \frac{\sqrt{(1+\alpha^2)}}{c}.$$

Atque si assumto fuisset haec aequatio $y = \alpha x - \beta x^2 - \gamma x^3 - \delta x^4$, prodiisset

$$\frac{6\delta}{\beta} = \frac{(1+\alpha^2)}{2c^2} - \frac{\alpha\beta}{c\sqrt{(1+\alpha^2)}}.$$

Erit ergo

$$\beta = \frac{g(1+\alpha^2)}{4b}, \quad \gamma = \frac{g(1+\alpha^2)^{\frac{3}{2}}}{12bc} \quad \text{et} \quad \delta = \frac{g(1+\alpha^2)^2}{48bc^2} - \frac{\alpha g^2(1+\alpha^2)^{\frac{3}{2}}}{96b^2c}.$$

[p. 415] Haec igitur curva quarti ordinis erit quam proxime proiectoria in medio valde raro uniformi, quod resistit in duplicata celeritatum ratione, et potentia uniformi g deorsum tendente.

Corollarium 4.

974. Quia aeris resistentia est quadratis celeritatum proportionalis, si in aere valde gravis globus atque magnus ingenti vi proiiciatur, tum b et c erunt quantitaties maximae. Quare pro proiectoria huius corporis accipi poterit haec aequatio

$$y = \alpha x - \frac{g(1+\alpha^2)}{4b}x^2 - \frac{g(1+\alpha^2)^{\frac{3}{2}}}{12bc}x^3,$$

quae curva a vera proiectoria quam minime differet.

Corollarium 5.

975. Sit *AMDB* (Fig. 89) haec proiectoria; in qua ut inveniatur punctum *B*, quo corpus proiectum incidit in horizontalem *AB*, pono $y = 0$ eritque

$$x^2 = -\frac{3cx}{\sqrt{(1+\alpha^2)}} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}$$

hincque

$$x = -\frac{3c}{2\sqrt{(1+\alpha^2)}} + \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)}.$$

Erit itaque

$$AB = \sqrt{\left(\frac{9c^2}{4(1+\alpha^2)} + \frac{12\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)} - \frac{3c}{2\sqrt{(1+\alpha^2)}}.$$

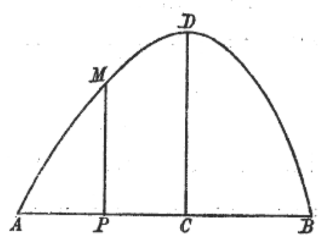

Fig. 89.

[p. 416] Innotescit igitur ex hac aequatione longitudo iactus ex data celeritate initiali et inclinatione.

Corollarium 6.

976. Punctum iactus summum *D* reperietur faciendo $dy = 0$. Fiet autem

$$0 = \alpha - \frac{g(1+\alpha^2)}{2b}x - \frac{g(1+\alpha^2)^{\frac{3}{2}}}{4bc}x^2$$

seu

$$x^2 = -\frac{2cx}{\sqrt{(1+\alpha^2)}} + \frac{4\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}.$$

Atque ex hac aequatione

$$AC = \sqrt{\left(\frac{c^2}{1+\alpha^2} + \frac{4\alpha bc}{g(1+\alpha^2)^{\frac{3}{2}}}\right)} - \frac{c}{\sqrt{(1+\alpha^2)}}.$$

Corollarium 7.

977. Iactus longissimus, qui eadem celeritate initiali $\sqrt{b}$ producitur, prodibit, si anguli inclinationis tangens α ex ista aequatione determinatur

$$3\alpha c\sqrt{(1+\alpha^2)} - \frac{8b(1+\alpha^2)}{g} + \frac{24\alpha^2 b}{g} = \alpha\sqrt{\left(9c^2(1+\alpha^2) + \frac{48\alpha bc\sqrt{(1+\alpha^2)}}{g}\right)}$$

seu hac

$$4b(1-2\alpha^2)^2 - 3\alpha gc(1-2\alpha^2)\sqrt{(1+\alpha^2)} = 3\alpha^3 gc\sqrt{(1+\alpha^2)}$$

seu ista simpliciore

$$4b(1-2\alpha^2)^2 = 3\alpha gc(1-\alpha^2)\sqrt{(1+\alpha^2)}.$$

Corollarium 8. [p. 417]

978. Si sinus anguli, quem curva in *A* cum horizontali *AC* constituit, sit = ε posito sinu toto = 1, erit

$$9\varepsilon^4 - 6\varepsilon^2 + 1 = \frac{3gc\varepsilon}{4b}(1-2\varepsilon^2).$$

Ex qua aequatione valor ipsius ε erutus dabit directionem pro iactu longissimo. Ex hac autem aequationem reperitur quam proxime

$$\varepsilon = \sqrt{\frac{5b + \frac{3}{2}gc\sqrt{2}}{12b + 3gc\sqrt{2}}}$$

Corollarium 9.

979. Angulus igitur, qui iactum longissimum producit, aliquantulum est minor quam semirectus, qui in vacuo satis facit. Nam se esset

$$\varepsilon = \sqrt{\frac{6b + \frac{3}{2}gc\sqrt{2}}{12b + 3gc\sqrt{2}}}$$

foret $\varepsilon = \sqrt{\frac{1}{2}}$ ideoque angulus semirectus. At cum hic in numeratore habeamus tantum 5*b*, parumper erit minor.

Corollarium 10.

980. Si corpus in *A* (Fig. 90) horizontaliter proiiciatur celeritate $\sqrt{a}$, fiet $\alpha = 0$ atque *y* negativa. Quamobrem positis *AP* = *x* et *PM* = *y* istius proiectoriae natura hac exprimetur aequatione

$$y = \frac{gx^2}{4b} + \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2}.$$

Pro curva autem *AN*, in qua corpus ascendit, erit

$$y = \frac{gx^2}{4b} - \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2}.$$

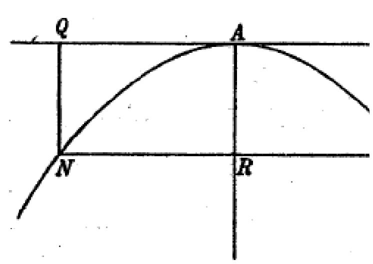

Fig. 90.

Corollarium 11.

981. Si adhuc plures termini quam quatuor accipiantur, prodiret aequatio pro curva *AM* haec [p. 418]

$$y = \frac{gx^2}{4b} + \frac{gx^3}{12bc} + \frac{gx^4}{48bc^2} + \frac{gx^5}{240bc^3} + \text{etc.};$$

qui termini cum seriem summabilem constituant, quam minime a vero aberrabitur, si *y* ponatur aequalis summae huius seriei. Erit autem

$$2by = c^2g\left(e^{\frac{x}{c}} - 1\right) - cgx.$$

Pro arcu vero ascensus *AN* erit

$$2by = cgx - c^2g\left(1 - e^{-\frac{x}{c}}\right).$$

Corollarium 12.

982. Tempus, quo arcus AM percurritur, est =

$$\int \frac{ds}{\sqrt{v}} = \int \frac{\sqrt{2ddy}}{\sqrt{g}},$$

cum sit $g = \frac{2vddy}{ds^2}$. Est vero

$$2bdy = cge^{\frac{x}{c}}dx - cgdx \quad \text{et} \quad 2bddy = ge^{\frac{x}{c}}dx^2.$$

Prodibit igitur

$$\int \frac{\sqrt{2ddy}}{\sqrt{g}} = \int \frac{e^{\frac{x}{2c}}dx}{\sqrt{b}} = \frac{2c}{\sqrt{b}}\left(e^{\frac{x}{2c}} - 1\right).$$

Atque si *b* et *c* in scrupulis pedis Rhenani exprimuntur, erit (222) tempus per *AM* =

$$\frac{c}{125\sqrt{b}}\left(e^{\frac{x}{2c}} - 1\right) \text{ min. sec.}$$

Scholion.

983. Hac igitur ratione vero proxime determinavimus proiectoriam in aere a corporibus proiectis descriptam, quae non difficulter loco parabolae, quae vulgo adhiberi solet, potest substituti. Hanc quidem eandem aequationem deducere potuissemus ex vera aequationem $dsddy = cd^3y$ (875) harum proiectoriarum supra inventa. [p. 419] Sed cum ibi haec reductio esset omissa, hic eam afferre maluimus, praecipue quod hoc loco clarius appareat aequationis assumtae terminos posteriores vehementer decrescere. Denique simili quoque modo curvae cum proiectoriis in aliis medii resistentis hypothesibus proxime convenientes possunt inveniri; sed cum aliae hypotheses in mundo locum non habeant, iis inveniendis hic non immorabimur.

PROPOSITIO 118.

PROBLEMA.

984. *Invenire tam resistantiam in singulis locis M* (Fig.87) *quam potentiam absolutam deorsum secundum MP tendentem, quae faciant, ut corpus in data curva AM et data cum celeritate in singulis punctis M moveri possit.*

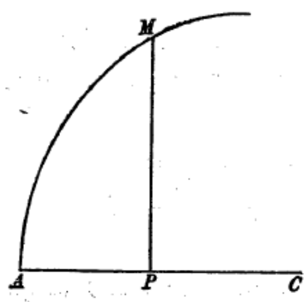

Fig. 87.

SOLUTIO.

Positis ut ante $AP = x$, $PM = y$, elemento arcus $AM = ds$, et altitudine celeritati in M debita = v, quae igitur omnia dantur. Deinde sit potentia corpus in M deorsum trahens = P et resistentia = R. His positis statim reperitur P ex aequatione

$$P = -\frac{2vddy}{ds^2}$$

(871). At resistantia R invenietur ex aequatione $dv = \frac{2vdyddy}{ds^2 - Rds}$ (cit.). Quare erit

$$R = \frac{2vdyddy}{ds^3} - \frac{dv}{ds}.$$

[p. 420] Si resistantiae lex ponatur duplicata celeritatum ratio eiusque exponens = q, erit $R = \frac{v}{q}$, ex quo prodibit

$$q = \frac{vds^3}{2vdyddy - dvds^2}$$

sumto dx pro constante. Q.E.I.

Corollarium 1.

985. Quia dx est constans, erit $dyddy = dsdds$. Hanc ob rem erit

$$R = \frac{2vdds}{ds^2} - \frac{dv}{ds} \quad \text{seu} \quad R = ds\,d. - \frac{v}{ds^2}.$$

Atque hinc erit

$$q = \frac{vds^2}{2vdds - dvds} \quad \text{seu} \quad q = \frac{v}{ds\,d. - \frac{v}{ds^2}}.$$

Corollarium 2.

986. Si corpus debet motu uniformi per curvam AM ferri, ita ut sit $v = b$, proveniet

$$P = -\frac{2bddy}{ds^2} \quad \text{et} \quad R = \frac{2bdyddy}{ds^3} = \frac{2bdds}{ds^2} \quad \text{atque} \quad q = \frac{ds^2}{2dds}.$$

Corollarium 3.

987. In motu igitur uniformi, dum corpus in curva *AM* ascendit, resistentia *R* semper est negativa seu motum corporis accelerat. At quando corpus iterum descendit, medium revera resistet.

Corollarium 4.

988. Posito radio osculi in M = r, quia est $r = -\frac{ds^2}{dxddy}$, erit $ddy = -\frac{ds^2}{rdx}$.

Hanc ob rem habebitur

$$P = \frac{2vds}{rdx} \quad \text{et} \quad R = -\frac{2vdy}{rdx} - \frac{dv}{ds} \quad \text{atque} \quad q = -\frac{rvdxds}{2vdyds + rdxdv}.$$

Corollarium 5. [p. 421]

989. Posito iterum $v = b$ et $dv = 0$ erit

$$P = \frac{2bds}{rdx} \quad \text{et} \quad R = -\frac{2bdy}{rdx} \quad \text{atque} \quad q = -\frac{rdx}{2dy}.$$

In supremo igitur puncto, quo fit $dy = 0$ et $ds = dx$, erit $P = \frac{2b}{r}$ atque resistantia ibi evanescit, nisi forte curvatura ibi sit infinite magna seu $r = 0$.

Exemplum.

990. Sit curva *AM* circulus, cuius centrum in *C*, qui motu aequabili seu celeritate $\sqrt{b}$ debeat describi. Positi eius radio $AC = a$ erit

$$dy = \frac{dx(a-x)}{y}, \quad ds = \frac{adx}{y} \quad \text{et} \quad r = a.$$

Ex quibus invenitur potentia absoluta deorsum tendens $P = \frac{2b}{r}$ seu erit reciproce ut distantia PM. Resistantia vero erit = $\frac{2b(a-x)}{ay}$. Dum igitur corpus ascendit, resistentia erit negativa atque reciproce proportionalis tangenti arcus *AM*. Ac si resistentia sit quadratis celeritatum proportionalis, erit eius exponens

$$q = -\frac{ay}{2(a-x)} = -\frac{AC \cdot PM}{2PC}.$$

Est itaque q negativa atque aequalis dimidiae tangenti arcus AM. Quando autem corpus versus horizontem AC accedit, fiet tum resistentia *R* tum *q* affirmitiva seu medium revera resistet.

PROPOSITIO 119.

PROBLEMA.

991. *Si medium sit uniforme atque resistat in quacunque multiplicata celeritatum ratione deturque praeterea motus corporis progressivus secundum horizontalem AP* (Fig.90), *invenire potentiam deorsum tendentem et curvam, quam corpus describet.* [p. 422]

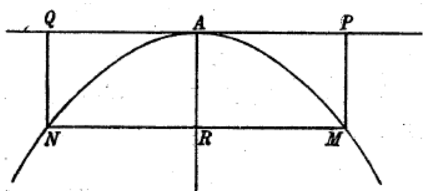

Fig. 90.

SOLUTIO.

Positis $AP = x$, $PM = y$, arcu $AM = s$ sit celeritas horizontalis corporis, dum est in M, debita altitudini u; erit verae celeritati in M altitudo debita $= \frac{uds^2}{dx^2} = v$. Sit porro exponens medii resistentis = c et lex ratio 2m-plicata celeritas, erit resistentia

$$R = \frac{v^m}{c^m} = \frac{u^m ds^{2m}}{c^m dx^{2m}}.$$

His positis erit

$$P = \frac{2vddy}{ds^2} = \frac{2uddy}{dx^2}$$

(871); pono enim ddy loco $-ddy$, quia in nostro casu y deorsum cadit. Deinde erit

$$dv = \frac{2vdyddy}{ds^2} - Rds$$

(cit.). At vero est

$$dv = \frac{duds^2}{dx^2} + \frac{2udsdds}{dx^2}.$$

Hinc ergo ob $dsdds = dyddy$ habebitur

$$\frac{duds^2}{dx^2} = -\frac{u^m ds^{2m+1}}{c^m dx^{2m}} \quad \text{seu} \quad c^m dx^{2m-2} du + u^m ds^{2m-1} = 0.$$

Datur autem u in x et hanc ob rem ds ex hac aequatione per x tantum poterit determinari; erit scilicet

$$ds = \frac{c^{\frac{m}{2m-1}} dx^{\frac{2m-2}{2m-1}} (-du)^{\frac{1}{2m-1}}}{u^{\frac{m}{2m-1}}}.$$

Unde invenire licet aequationem pro curva invenienda. Hac vero inventa simul innotescit P ex aequatione $P = \frac{2uddy}{dx^2}$. Q.E.I. [p. 423]

Corollarium 1.

992. Motus igitur horizontalis non potest esse uniformis; foret enim ob $du = 0$ etiam $ds = 0$, nisi in vacuo, quo $c = \propto$, ubi semper et necessario est aequabilis. Multo minus quoque in medio resistente motus horizontalis poterit esse acceleratus; tum enim ds vel negativum vel imaginarium obtineret valorem, quod utrumque absurdum. Motus ergo horizontalis debebit esse retardatus, quo fiat du negativum.

Corollarium 2.

993. Si resistentia fuerit ipsis celeritatibus proportionalis, fiet $m = \frac{1}{2}$ et aequatio pro curva abit in $du\sqrt{c} + dx\sqrt{u} = 0.$ Quae autem, quia non continet s vel y, ad curvam pertinere non potest. Haec autem aequatio ipsum motum horizontalem determinat. Id quod indicio est in hac resistentiae hypothesi non quemvis motum horizontalem pro lubitu accipe posse, sed necessario eum esse accipiendum, qui hac aequatione determinatur.

Corollarium 3.

994. Iste autem motus horizontalis congruit cum motu corporis horizontali in AP in eodem medio resistentia, sed a nulla potentia sollicitati. Ex quo cognoscitur, quaecunque fuerit potentia corpus sollicitans, modo eius directio [p. 424] ubique deorsum tendat, in medio resistente in simplici celeritatum ratione motum horizontalem perpetuo esse eundem. Qua in re motus in hac resistentiae hypothesi similis est motui in vacuo, in quo motus horizontalis semper est aequabilis, quantumcunque potentia deorsum tendens sit variabilis.

Corollarium 4.

995. Assumto igitur hoc motus horizontalis valore altera aequatio $P = \frac{2uddy}{dx^2}$ curvam descriptam determinabit, in qua pro P vero quamcunque quantitatem assumere licebit. In hac igitur resistentiae hypothesi hoc problema generaliter est solutum; ut inveniatur curva, quam corpus utcunque deorsum sollicitatum describit.

Corollarium 5.

996. Cum autem sit $-\frac{du\sqrt{c}}{\sqrt{u}} = dx,$ erit $2\sqrt{bc} - 2\sqrt{cu} = x$ posita celeritate initiali in $A = \sqrt{b}$. Fiet ergo

$$2\sqrt{cu} = 2\sqrt{bc} - x \quad \text{et} \quad u = \frac{(2\sqrt{bc} - x)^2}{4c}.$$

Pro curva ergo descipta haec habebitur aequatio

$$2cPdx^2 = ddy\,(2\sqrt{bc} - x)^2.$$

Corollarium 6.

997. Hae igitur omnes curvae asymtoton habebunt verticalem in distantia $2\sqrt{bc}$ a vertice A. Namque *x* maius esse nequit quam $2\sqrt{bc}$, et cum corpus horizontaliter [p. 425] motum ultra hunc terminum progredi nequeat, etiam corpus in curva latum non ulta pertingere poterit.

Corollarium 7.

998. Si in aliis quoque resistentia hypothesibus motus horizontalis in curva *AM* cum motu horizontali in resta *AP* in eadem resistentiae hypothesi congruens accipiatur, ita ut ponitur

$$du = -\frac{u^m dx}{c^m},$$

prodit pro curva *AM* haec aequatio $ds = dx$ et $P = 0$. Illa igitur congruentia in aliis resistentiae hypothesibus nequidem locum habet.

Corollarium 8.

999. Sit igitur resistentia ut quadratum celeritas seu $m = 1$. Quare erit $ds = -\frac{cdu}{u}$ et integrando, posita *b* altitudine celeritati in *A* debita, $s = c\, l\, \frac{b}{u}$. In hac igitur resistentiae hypothesi erit arcus *AM* divisus per 2*c* aequalis differentiae logarithmorum celeritatum horizontalium in *A* et *M*.

Corollarium 9.

1000. In hac ergo resistentiae hypothesi erit $u = e^{-\frac{s}{c}}b.$ Ex quo prodibit

$$P = \frac{2bddy}{e^{\frac{s}{c}}dx^2}.$$

[p. 426] Seu cum sit $ds = -\frac{cdu}{u}$, erit

$$dds = \frac{-cuddu + cdu^2}{u^2} = \frac{dyddy}{ds}.$$

At est

$$dy = \frac{\sqrt{(c^2du^2 - u^2dx^2)}}{u}.$$

Unde habebitur

$$ddy = \frac{c^2ududdu - c^2du^3}{u^2\sqrt{(c^2du^2 - u^2dx^2)}}.$$

Consequenter erit

$$P = \frac{2c^2ududdu - 2c^2du^3}{udx^2\sqrt{(c^2du^2 - u^2dx^2)}}.$$

Corollarium 10.

1001. Si aequatio inter *u* et *x* accipiatur ista

$$du = -\frac{u^n dx}{f^n} \quad \text{seu} \quad \frac{1}{(n-1)u^{n-1}} - \frac{1}{(n-1)b^{n-1}} = \frac{x}{f^n},$$

erit

$$dx = -\frac{f^n du}{u^n} \quad \text{et} \quad ddu = \frac{ndu^2}{u}.$$

His substitutis prodibit in medio resistente in duplicata celeritatum ratione

$$P = \frac{2(n-1)c^2u^{3n-2}}{f^{2n}\sqrt{(c^2u^{2n-2} - f^{2n})}}.$$

Pro curva autem quaesita haec habebitur aequatio

$$dy = \frac{du\sqrt{(c^2u^{2n-2} - f^{2n})}}{u^n}.$$

Corollarium 11.

1002. Hoc igitur casu u^{2n-2} maius esse debet quam $\frac{f^{2n}}{c^2}$ seu *u* maius quam

$$\frac{f^{\frac{n}{n-1}}}{c^{\frac{1}{n-1}}}.$$

Quare, si motus horizontalis fieri potest minor quam haec quantitas, motus in curva non toti motui horizontali dato respondebit. Nam si curva ulterius tenderet, foret tam *dy* quam *P* imaginarium. [p. 427]

Corollarium 12.

1003. Ad inconveniens hoc evitandum debebit *n* minus esse unitate; fiat ergo $n-1=-k$ seu $n=1-k$. Hoc posito erit $u^k = b^k - f^{k-1}kx.$ Existente vero *AP* tangente curvae in *A* erit , ubi *ds* = *dx*, ibi *u* = *b*. Hinc erit

$$b^{n-1}c = f^n \quad \text{seu} \quad \frac{c}{b^k} = f^{1-k} \quad \text{ideoque} \quad u^k = b^k\left(1 - \frac{kx}{c}\right).$$

Porro autem fit

$$P = -\frac{2kb^{3k}u^{1-2k}}{c\sqrt{(b^{2k} - u^{2k})}} \quad \text{et} \quad dy = -\frac{dx}{u^k}\sqrt{(b^{2k} - u^{2k})} = -\frac{dx\sqrt{(2kcx - k^2x^2)}}{c - kx}.$$

Scholion.

1004. Nulla igitur huiusmodi hypothesis motus horizontalis in proiectoriam in fluido potest quadrare. Quicquid enim sit *k*, potentia in A, ubi fit *u* = *b*, est infinite magna; deinde vero perpetuo decrescit. Hae curvae etiam omnes habent tangentem verticalem, ubi est $x = \frac{c}{k}$, quae est asytotos curvae. Ceterum hoc problemate finem imponimus huic primae tractationi, qua directionem potentiae sibi semper parallelam posuimus, atque progredimur ad vires centripetas considerandas examinaturi, quomodo medium resistens motum corporum ad fixum punctum attractorum turbet. [p. 428]