CHAPTER SIX (*Part a*).

CONCERNING THE CURVILINEAR MOTION OF A FREE POINT IN A RESISTIVE MEDIUM

[p. 369]

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PROPOSITION 104.

THEOREM.

860. *If a body is moving in a medium with resistance acted on by some number of absolute forces, the resistive force does not disturb the action of the other absolute forces in any way, except that the tangential force arising from that is diminished.*

DEMONSTRATION.

From the last chapter it has been explained well enough that all absolute forces can be resolved into two forces, the tangential and the normal, if the motion is to be in the same plane. But if the body does not move in the same plane, then three equivalent forces can be assigned in place of any number of forces acting, of which one is the tangential and two are normal. But the force that the resistance exerts on the body, is always put to agree with the direction of the body (117). On account of which the resistive force has to be referred to the tangential force that it diminishes, [p. 370] since the motion of the body is slowed down, and indeed it does not in short affect the normal forces. Therefore it is evident that the resistance has no effect on the absolute forces, except in as much as the tangential force arising from these is diminished by the resistance. Q.E.D.

Corollary 1.

861. Therefore the whole effect of the resistance is consistent with changing the speed of the body and leaves the direction unchanged, except in as much as the action of the normal force varies with the variation of the speed.

Corollary 2.

862. Therefore except besides by the aid of [normal] absolute forces, it is not possible for the body to move along a curve, but always to progress along a straight line, while it loses its motion.

Scholium 1.

863. Therefore in this chapter, in which we treat curvilinear motion, it is necessary that we consider absolute forces likewise and these are of such a kind that they can be resolved to give a normal force, and which are different from what we discussed in Chapter IV. On this account the first force we consider pulls towards a point at an infinite or the direction of this force is always kept parallel to itself. From there we progress to centripetal forces and to other forces set out in whatever manner. And hence also we submit to our analysis motions not contained in the same plane, [p. 371] of such kinds as arise from motion in a resistive medium.

Corollary 3.

864. If the tangential force is T, and either the one normal is N, or the two normals are N and M, and the force of resistance is R, then the rules containing the effect of these forces that we gave in the previous chapter are also to be applied here, except that we put $T-R$ in place of T for these.

Scholium 2.

865. As the force of the resistance is made to depend on the speed of the body, it is necessary to be explained by a rule for the resistance, and the explanation has been widely set out in Chapter IV. Truly in this chapter a wide range of resistances are uncovered to be treated, which did not find a place in that previous chapter. Besides, this treatment thus has been subdivided, in order that at first we can determine the curve described and the motion of the body, from the given absolute forces and the resistance. Then if the curve and the absolute force is given, from these we deduce the resistance. Following this, in the third place, from the given curve and the resistance the absolute force in a given direction is to be investigated. Then from the given curve, with both the speed of the body at individual points and the resistive force, the absolute force and its direction can be found. But in the first part of this chapter [p. 372] the division of motion in coplanar and non-coplanar parts is agreed upon.

PROPOSITION 105.

PROBLEM.

866. *If a body is moving in a medium with some resistance and is acted upon by some absolute forces, yet thus, so that the motion is completed in the same plane, to define the rules that the body observes in its motion.*

SOLUTION.

The body describes the curve AMB on account of the forces acting (Fig.81); let the speed of this at the point M correspond to the height v and the element of the curve $Mm = ds$. Again the normal force is put equal to N, and hence the direction of this force is along the normal MN to the curve, truly the tangential force arising from the same absolute forces is equal to T, the direction of which is MT, the tangent of the curve at M. And the force of the resistance at M is equal to R. With these put in place, the motion of the body can be defined from the normal force N and from the force along the tangent $T-R$ (864). Now let the radius of osculation at M be equal to r and so

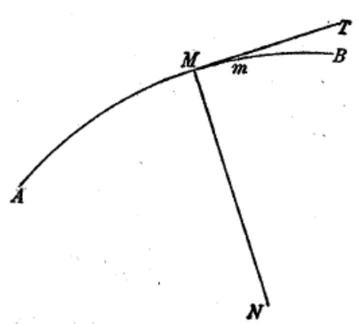


Fig. 81.

$$N = \frac{2v}{r} \text{ and } dv = (T - R)ds$$

(552). From these two equations if v is eliminated, the equation is produced expressing the nature of the curve, and likewise the speed of the body at individual points can be observed from the equation $N = \frac{2v}{r}$. Q.E.I. [p. 373]

Corollary 1.

867. Therefore there arises $v = \frac{Nr}{2}$. Hence there is found : $dv = \frac{Ndr+rdN}{2}$. Which value, if placed in the equation $dv = (T - R)ds$ in place of dv and in R is put $\frac{Nr}{2}$ in place of v, and the equation for the curve described by the body is produced.

Corollary 2.

868. If in R v should have a single dimension, since that comes about if the resistance is proportional to the square of the speed, then the equation $dv = (T - R)ds$ is able to be separated and each force can be determined from that. And this equation solved with $v = \frac{Nr}{2}$ gives a simpler equation for the curve described.

Scholium.

869. Besides this case, in which v has a single dimension in R, many others are given, for which the equation $dv = (T - R)ds$ can be integrated; but there is no need to explain these, as v is eliminated in any case. Truly we have noted the case for this particular idea, since in fact it pertains to the resistance of fluids and which therefore we will examine with more care before the others.

PROPOSITION 106.

PROBLEM.

870. *The force acts normally everywhere to the given line in the position AP* (Fig.82) *and the body moves in a medium with some resistance* [p. 374] *; it is required to determine the curve AMB in which the body moves, and the motion of the body.*

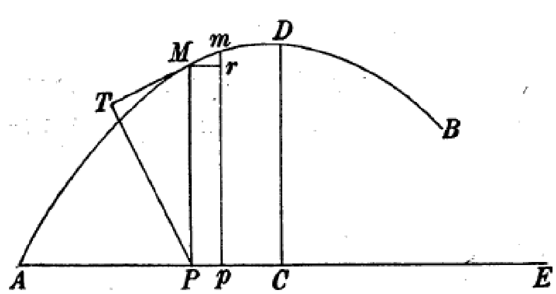


Fig. 82.

SOLUTION.

Let the force which acts on the body at M be equal to P, the direction of which is therefore MP. The speed of the body at M corresponds to the height v and the force of the resistance there is equal to R. The element Mm is taken, and with mp drawn then $AP = x$, $PM = y$ and $Mm = ds$. Then we have $Pp = Mr = dx$ and $mr = dy$. Again the tangent MT is drawn, and

on that the perpendicular *PT* from *P* is drawn. With these done, the force *P* is resolved into the normal $\frac{P.PT}{PM} = \frac{Pdx}{ds}$ and the tangential $\frac{P.PT}{PM} = \frac{Pdy}{ds}$ components.
Moreover since this tangential force retards the motion of the body, the negative value of this must be taken. Therefore with the radius of osculation at *M* put equal to *r* then we have $\frac{Pdx}{ds} = \frac{2v}{r}$ and $dv = -Pdy - Rds$ (866). From which equations the curve itself as well as the motion of the curve can be found. Q.E.I.

Corollary 1.

871. With *dx* placed constant, the radius of osculation is given by $r = -\frac{ds^3}{dxddy}$. On this account we have the following equation :

$$P = -\frac{2vddy}{ds^2}.$$

Which value of P substituted in the other equation gives the equation :
$dv = \frac{2vdyddy}{ds^2} - Rds$ or, since $dyddy = dsdds$, this equation : $dv = \frac{2vdds}{ds} - Rds$.
[The result $dyddy = dsdds$ can be shown in a straight–forward manner on differentiating $dy = ds\cos\theta$ and $dx = ds\sin\theta$, and noting that $d\theta = \frac{dds}{ds}\frac{dx}{dy}$, as $ddx = 0$.]
Whereby the equation is put in place, for whatever the force *P* might have been, only the direction of this is along *MP*.

Corollary 2. [p. 375]

872. If the law of the resistance is some ratio of the multiple of the speed, and the exponent of the resistance is some quantity of the variable *q*, thus in order that $R = \frac{v^m}{q^m}$, then we have this equation :

$$dv = \frac{2vdds}{ds} - \frac{v^m ds}{q^m},$$

of which the integral is :

$$v^{1-m} = \frac{(m-1)dx^{2m-2}}{ds^{2m-2}} \cdot \int \frac{ds^{2m-1}}{q^m dx^{2m-2}}.$$

[This can be shown in an inductive manner by differentiation of the result.]

Corollary 3.

873. If according to the same hypothesis, $m = 1$, then $\frac{dv}{v} = \frac{2dds}{ds} - \frac{ds}{q}$, the integral of which is :

$$lv = 2l\frac{ds}{dx} - \int \frac{ds}{q} \quad \text{or} \quad e^{\int \frac{ds}{q}} v = \frac{a ds^2}{dx^2}.$$

If in addition the resistance is constant or $q = c$, then

$$e^{\frac{s}{c}}v = \frac{ads^2}{dx^2} \text{ or } v = \frac{ae^{-\frac{s}{c}}ds^2}{dx^2}.$$

Therefore in this case the time in which the arc AM is completed is equal to :

$$\int \frac{e^{\frac{s}{2c}}dx}{\sqrt{a}}.$$

PROPOSITION 107.

PROBLEM.

874. *If both the force and the resisting medium is constant and the direction of the force is along the direction MP* (Fig.82) *normal to the given line AP, and the medium has a resistance in the ratio of the square of the speed, to determine the motion of the projected body.*

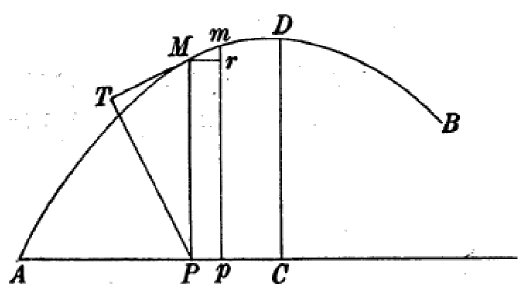


Fig. 82.

SOLUTION. [p. 376]

The force pulling the body towards *AP* is always equal to *g,* and the exponent of the resistance is equal to *c*, and the rest of the denominated terms remain as before. Therefore we have : $P = g$ and $R = \frac{v}{c}$. Hence there arises the following equations :

$$\frac{gdx}{ds} = \frac{2v}{r} \text{ or } gds^2 + 2vddy = 0$$

with *dx* taken as constant, and

$$dv = -gdy - \frac{vds}{c}.$$

From solving these equations together we now find: $e^{\frac{s}{c}}v = \frac{ads^2}{dx^2}$ (873).

Whereby, when it becomes :

$$v = -\frac{gds^2}{2ddy},$$

the equation is produced on eliminating *v* :

$$ge^{\frac{s}{c}}dx^2 = -2addy,$$

which contains the nature of the curve described. Since *dsdds = dyddy*, it can also become :

$$ge^{\frac{s}{c}}dx^2dy = -2adsdds.$$

On putting *dx = pds*, then

$$dds = -\frac{dpds}{p} \text{ and } dy = ds\sqrt{(1-pp)}.$$

With these put in place there arises this equation :

$$ge^{\frac{s}{c}}ds = \frac{2adp}{p^3\sqrt{(1-pp)}},$$

which suffices to construct the described curve. Indeed the equation of this integral is :

$$gce^{\frac{s}{c}} = C - \frac{a\sqrt{(1-pp)}}{p^2} - a\,l\frac{1+\sqrt{(1-p^2)}}{p}.$$

Truly with $\frac{dx}{ds}$ restored in place of p it is found that:

$$gce^{\frac{s}{c}} = C - \frac{adyds}{dx^2} - a\,l\frac{ds+dy}{dx}.$$

Which is a differential equation of the first order, and which cannot be reduced to any simpler form. Q.E.I.

Corollary 1.

875. A differential equation of the third order can be immediately found for the described curve. For on account of $v = -\frac{gds^2}{2ddy}$ then we have $dv = -gdy + \frac{gds^2d^3y}{2ddy^2}$, from the values of which substituted into the equation [p. 377] $dv = -gdy - \frac{vds}{c}$ there is produced the third order equation $dsddy = cd^3y.$

Corollary 2.

876. Let the sine of the angle, that the curve at *A* makes with the axis *AP* be equal to μ, and the cosine of this is equal to $\sqrt{(1-\mu^2)} = \nu$ and the height corresponding to the speed at *A* is equal to *b*. Therefore by taking $s = 0$ and $ds : dx = 1 : \nu$ there becomes $v = b$; therefore there is had from $e^{\frac{s}{c}}v = \frac{ads^2}{dx^2}$ this equation $\nu^2 b = a$, thus recognised to be a constant *a*.

Corollary 3.

877. Again in the final equation of the curve by taking $s = 0$ and $ds : dx = 1 : \nu$ and $ds : dy = 1 : \mu$ the constant is found : $C = gc + \mu b + \nu^2 b\, l\, \frac{1+\mu}{\nu}$. Whereby for the curve described there arises this equation :

$$\frac{gc}{b}\left(e^{\frac{s}{c}} - 1\right) = \frac{\mu dx^2 - \nu^2 dyds}{dx^2} + \nu^2 l\frac{(1+\mu)dx}{\nu(dy+ds)}.$$

Indeed in order that the speed can be found, this equation is brought into use:

$$e^{\frac{s}{c}}v = \frac{\nu^2 bds^2}{dx^2}.$$

Corollary 4.

878. If D is the maximum point, then there $dx = ds$ and $dy = 0$. Therefore:

$$\frac{gc}{b}\left(e^{\frac{s}{c}}-1\right)=\mu+\nu^2 l\frac{1+\mu}{\nu} \quad \text{or} \quad e^{\frac{s}{c}}=\frac{b\mu+gc+\nu^2 b\, l\frac{1+\mu}{\nu}}{gc}.$$

From which the equation is found [p. 378]

$$\text{arc } AMD = c\, l\frac{b\mu+gc+\nu^2 b\, l\frac{1+\mu}{\nu}}{gc}.$$

Truly the height corresponding to the speed that the body has at D, is equal to

$$\frac{\nu^2 gbc}{gc+b\mu+\nu^2 b\, l\frac{1+\mu}{\nu}}.$$

Corollary 5.

879. If the curve at B is put to have the same inclination to the axis AP, that is has at A, then the ratio is $ds : dx : dy = 1 : \nu : -\mu.$ Hence therefore this equation comes about :

$$\frac{gc}{b}\left(e^{\frac{s}{c}}-1\right)=2\mu+\nu^2 l\frac{1+\mu}{1-\mu},$$

from which there is produced:

$$\text{arc } ADB = c\, l\frac{2\mu b+gc+\nu^2 b\, l\frac{1+\mu}{1-\mu}}{gc}.$$

Corollary 6.

880. From the construction of the curve it is also easy to deduce from the equation : $dsddy = cd^3y$. For on putting $dy = pdx$, there arises $dpdx\sqrt{(1+pp)} = cddp$. Again there becomes $dx = \frac{dp}{q}$, since $ddx = 0$

$$ddp = \frac{dpdq}{q},$$

hence this equation is produced:

$$dp\sqrt{(1+pp)} = cdq \text{ and } q = \frac{1}{c}\int dp\sqrt{(1+pp)}.$$

Therefore on taking the abscissa :

$$x = \int\frac{cdp}{\int dp\sqrt{(1+pp)}}$$

then we have

$$y = \int\frac{cpdp}{\int dp\sqrt{(1+pp)}}.$$

And to these there corresponds :

$$v = -\frac{cg(1+pp)}{2\int dp\sqrt{(1+pp)}}.$$

Hence it is understood that a negative quantity has to be taken for g.

Corollary 7.

881. If the body at A is projected along the direction of AP (Fig.83) and the force acts downwards, [p. 379] then the whole curve AM described by the body falls below AP and y or PM becomes negative ; and since $\mu = 0$ and $\nu = 1$, this equation is had for the curve AM (877) :

$$\frac{gc}{b}\left(e^{\frac{s}{c}} - 1\right) = \frac{dyds}{dx^2} - l\frac{dx}{ds+dy}.$$

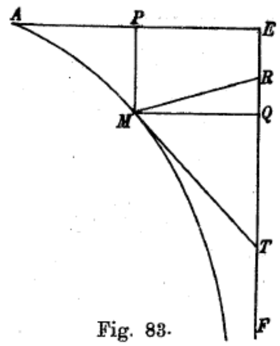


Fig. 83.

Corollary 8.

882. Therefore given the tangent at some point M for the inclination to the line AP or the ratio between ds, dy, and dx, it is possible from this equation to find the length of the arc AM.

Scholium.

883. Experiments have shown that air resists bodies as the square of the speed. Therefore since the force of gravity is uniform and air at not very great heights keeps almost the same density, the case of bodies projected in air is first referred to this proposition. We therefore determine first the curve that a ball describes projected by a gun or cannon, or by any other means. Commonly a parabola is taken for this curve, which clearly is the trajectory in a vacuum, and the air is supposed to be so fine a fluid that it does not merit to be included in the calculation. Indeed, at any rate, a large body projected with a small speed is insensitive to the resistance of the air. But the trajectory departs the greatest distance from a parabola, if a very small body is projected by a large force. Moreover in these cases, even if here the trajectory has been correctly assigned, it is very painful, as the equation is so involved that hardly any practical use can be extracted from it. [p. 380] Newton in the *Phil. Princ.* did not touch this problem[1] and no one after him attempted it, then John Keil [Sav. Prof. Astronomy Oxford at the time] challenged Johan Bernoulli to this problem, although he had not been able to find a solution himself. Moreover a solution was given, not only by Johan Bernoulli[2] in Acta Eruditorum Lips. 1719 in the May issue, but also at nearly the same time Jacob Hermann[3] had inserted a solution in his *Phoronomiae*. But the following problem, in which the resistance is put proportional to the speeds, had been solved, first by Newton[4] in *Phil. Princ.* and then by Huygens in *Tractatu de causa gravitatis*.

[1. Indeed it was presented by Newton, in *Philosophiae naturalis*, Book II, Section 2, Concerning the motion of a body, in which the resistance is proportional to the square of

the speed of the body, but the problem in which the orbit was to be found, had been omitted.
2. Ioh. Bernoulli, *Responsio ad nonneminis provocationem eiusque solutio quaestionis ipsi ad eodem propostitae de invenienda linea curva, quam describit proiectile in medio resistente*, Acta erud. 1719, p. 216; *Opera omnia*, Tom. 2, Lausannae et Genevae 1742, p. 393. See also Acta erud. 1721, p. 228; *Opera omnia*, Tom. 2, p. 513; and the dissertation in the 1742 edition *Problema ballisticum; Opera omnia*, Tom. 4, p. 354.
3. Iac. Hermann, *Phoronomia seu de viribus et motibus corporum solidorum et fluidorum,* Amstelodami 1716.
4. I. Newton, *Philosophiae naturalis principia mathematica*, Londini 1687, Lib. II Sectio 1; Concerning the motion of bodies, in which there is resistance in the ratio of the speed.
5. Chr. Huygens, *Traite de la lumiere, avec un discours sur la cause de la pesanteur*, Leyden 1691.
References supplied by Paul Stackel.]

PROPOSITION 108.

PROBLEM.

884. *If the resistance of the medium were as the speed of the body and the direction of the force along MP* (Fig.82), *and in addition the force is uniform as is the resistance, it is required to determine the curve that the body describes and the speed at individual points.*

SOLUTION.

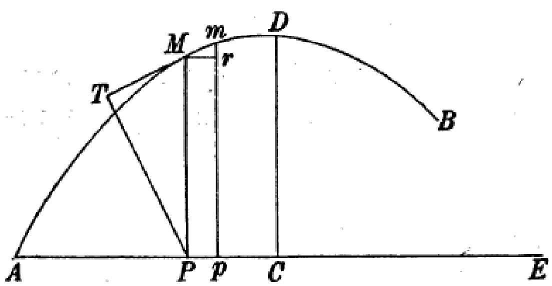


Fig. 82.

With the force put uniform and equal to g as in the preceding proposition, with the exponent of the medium resistance equal to c, with the height corresponding to the speed at M equal to v, $AP = x$, $PM = y$ and the arc $AM = s$, the normal force is as before :

$$gds^2 + 2vddy = 0,$$

with dx taken as constant. For the tangential force, on account of the resistance truly in this case equal to $\frac{\sqrt{v}}{\sqrt{c}}$, this equation is had :

$$dv = -gdy - \frac{ds\sqrt{v}}{\sqrt{c}}. \text{ [p. 381]}$$

With these equations solved together as in (872), where we put $q = c$ and $m = \frac{1}{2}$, it is found that :

$$\sqrt{v} = -\frac{ds}{2dx}\int\frac{dx}{\sqrt{c}} = \frac{ds\sqrt{a}}{2dx} - \frac{xds}{2dx\sqrt{c}} = \frac{ds(\sqrt{ac}-x)}{2dx\sqrt{c}}$$

and hence :

$$v = \frac{ds^2(\sqrt{ac}-x)^2}{4cdx^2}.$$

Thus for the curve described there comes about this equation :

$$2gcdx^2 = -ddy(\sqrt{ac}-x)^2 \text{ or } -\frac{ddy}{dx} = \frac{2gcdx}{(\sqrt{ac}-x)^2},$$

the integral of which is

$$\frac{dy}{dx} = -\frac{2gc}{\sqrt{ac}-x} + k.$$

and again by integration :

$$y = kx - 2gcl\frac{\sqrt{ac}}{\sqrt{ac}-x}.$$

Thus likewise v is agreed upon from the equation :

$$v = \frac{ds^2(\sqrt{ac}-x)^2}{4cdx^2}. \quad \text{Q. E. I.}$$

Corollary 1.

885. This differential equation of the third order can immediately be produced for the curve described, if from the equation $v = -\frac{gds^2}{2ddy}$, and with the differential of this :

$$dv = -gdy + \frac{gds^2d^3y}{2ddy^2}$$

the values are substituted in the equation $dv = -gdy - \frac{ds\sqrt{v}}{\sqrt{c}}$. For there arises :

$$d^3y\sqrt{gc} = -ddy\sqrt{-2ddy}.$$

Corollary 2.

886. If the body at A is projected with a speed corresponding to the height b and the sine of the angle that the tangent at A makes with AP is equal to μ, and the cosine of this is $\sqrt{(1-\mu^2)} = \nu,$ then at the point A, $b = \frac{a}{4\nu^2},$ or $a = 4\nu^2 b,$ from which the indefinite constant a is determined.

Corollary 3.

887. Again the equation

$$\frac{dy}{dx} = -\frac{2gc}{\sqrt{ac}-x} + k$$

[p. 382] translated to the point A gives

$$\frac{\mu}{\nu} = -\frac{2g\sqrt{c}}{\sqrt{a}} + k = -\frac{g\sqrt{c}}{\nu\sqrt{b}} + k$$

as $a = 4\nu^2 b.$ Hence the indefinite quantity is found :

$$k = \frac{\mu\sqrt{b} + g\sqrt{c}}{\nu\sqrt{b}}.$$

Corollary 4.

888. Therefore this differential equation is found for the curve described :

$$dy = \frac{2\mu\nu b dx\sqrt{c} - \mu x dx\sqrt{b} - gxdx\sqrt{c}}{2\nu^2 b\sqrt{c} - \nu x\sqrt{b}} = \frac{\mu dx}{\nu} - \frac{gxdx\sqrt{c}}{2\nu^2 b\sqrt{c} - \nu x\sqrt{b}}.$$

From which is deduced :

$$\frac{ds^2}{dx^2} = \frac{(2\nu b\sqrt{c} - x\sqrt{b})^2 - 2\mu gx(2\nu bc - x\sqrt{bc}) + g^2cx^2}{\nu^2(2\nu b\sqrt{c} - x\sqrt{b})^2}.$$

Whereby, since

$$v = \frac{ds^2(2\nu\sqrt{bc} - x)^2}{4cdx^2},$$

there is finally found :

$$v = \frac{(2\nu b\sqrt{c} - x\sqrt{b})^2 - 2\mu gx(2\nu bc - x\sqrt{bc}) + g^2cx^2}{4\nu^2 bc}.$$

Corollary 5.

889. Moreover the equation of the integral for the curve sought is :

$$y = \frac{\mu x\sqrt{b} + gx\sqrt{c}}{\nu\sqrt{b}} - 2gc\, l\frac{2\nu\sqrt{bc}}{2\nu\sqrt{bc} - x}.$$

From which the construction of the curve by the logarithm is easily completed.

Corollary 6.

890. Also the time, in which the arc *AM* is completed, is easily defined. For since $\frac{ds}{\sqrt{v}} = \frac{2dx\sqrt{c}}{2\nu\sqrt{bc} - x},$ the time for the arc AM is equal to :

$$2\sqrt{c}\, l\frac{2\nu\sqrt{bc}}{2\nu\sqrt{bc} - x}.$$

Corollary 7.

891. It is also apparent from the equation for the curve to have that asymptote *AM* . For since *x* is unable to be greater than $2v\sqrt{bc}$, if we take $AE = 2v\sqrt{bc}$, the applied line [i. e. the *y* co-ordinate] at $E = -\infty$ and likewise this is taken as the asymptote of the curve *AMDB*. This is also understood from the time, [p. 383] since it will be made infinite, before the body arrives at the perpendicular drawn through *E*.

Corollary 8.

892. The maximum point *D* is found by making *dy* = 0. Moreover, then it is found that

$$x = \frac{2\mu\nu b\sqrt{c}}{\mu\sqrt{b} + g\sqrt{c}} = AC.$$

Then, since at *D*, *ds* = *dx* and $2v\sqrt{bc} - x = \frac{2vgc\sqrt{b}}{\mu\sqrt{b}+g\sqrt{c}}$, the height corresponding to the speed at the point is equal to $\frac{v^2g^2bc}{(\mu\sqrt{b}+g\sqrt{c})^2}$ and the speed at D is equal to

$$\frac{\nu g\sqrt{bc}}{\mu\sqrt{b} + g\sqrt{c}}.$$

Corollary 9.

893. Truly the applied line *CD* of the distance of the point *D* from the axis *AP* is equal to

$$2\mu\sqrt{bc} - 2gc\,l\,\frac{\mu\sqrt{b} + g\sqrt{c}}{g\sqrt{c}}$$

and the time, in which the body arrives at *D* from *A*, is equal to

$$2\sqrt{c}\,l\,\frac{\mu\sqrt{b} + g\sqrt{c}}{g\sqrt{c}}.$$

Scholium.

894. Therefore it is possible to reduce the case in which the body is projected obliquely from *A*, to the case, in which it is projected normally to the force from *D*. For with the speed known, with which the body is projected from A, and from the direction of projection, the point D can be found at which the tangent is parallel to AC, and the speed of the body at D. Whereby to improve our understanding of this motion, it is expedient to consider the motion as beginning at D, that we have added as the following last proposition. [p. 384]

PROPOSITION 109.

PROBLEM.

884. *If the body is everywhere attracted downwards equally, and it is projected along the horizontal direction at A* (Fig.83) *with a given velocity in a uniform medium that offers resistance in the simple ratio of the speed, then to determine the curve AM that the body describes and to find the motion of the body on this curve.*

SOLUTION.

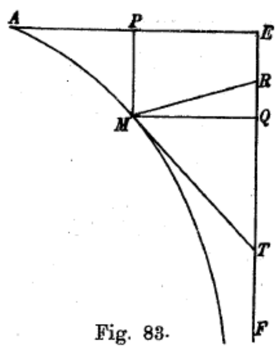


Fig. 83.

Since this proposition is a special case of the preceding, all the derivations remain as before. Moreover *y* or the applied line *PM* becomes negative, since the curve *AM* falls below *AP*, and we have $\mu = 0$ and $\nu = 1$. Therefore with the force *g* acting, with *c* the exponent of the resistance, with *b* the height corresponding to the speed at *A* and *AP* = *x* and *AM* = *s*, this differential equation is found for the curve *AM* :

$$dy = \frac{gxdx\sqrt{c}}{2b\sqrt{c} - x\sqrt{b}}$$

(888) and this integral:

$$y = -\frac{gx\sqrt{c}}{\sqrt{b}} + 2gcl\frac{2\sqrt{bc}}{2\sqrt{bc} - x}$$

(888). And the time, in which the arc AM is traversed, is equal to

$$2\sqrt{c}\,l\,\frac{2\sqrt{bc}}{2\sqrt{bc} - x}$$

(890). Which equations determine the curve AM, and also the motion on the curve. Q.E.I.

Corollary 1.

896. If $l\,\frac{2\sqrt{bc}}{2\sqrt{bc}-x}$ is converted into a series, it gives :

$$\frac{x}{2\sqrt{bc}}+\frac{x^2}{2\cdot 4bc}+\frac{x^3}{3\cdot 8bc\sqrt{bc}}+\frac{x^4}{4\cdot 16b^2c^2}+\text{ etc.}$$

On account of which, we have [p. 385]

$$y=\frac{gx^2}{4b}+\frac{gx^3}{12b\sqrt{bc}}+\frac{gx^4}{32b^2c}+\text{ etc.}$$

and the time, in which the arc AM is traversed, is equal to

$$\frac{x}{\sqrt{b}}+\frac{x^2}{4b\sqrt{c}}+\frac{x^3}{12bc\sqrt{b}}+\frac{x^4}{32b^2c\sqrt{c}}+\text{ etc.}$$

Corollary 2.

897. In a vacuum therefore, when *c* is made infinitely great, the equation becomes $y=\frac{gx^2}{4b}$; in which case therefore the curve AM goes into a parabola, the parameter of which is $\frac{4b}{g}$, and the time in which the arc *AM* is completed is equal to $\frac{x}{\sqrt{b}}$ and $v=b+\frac{gx^2}{4b}=b+gy,$ as it is clear to recollect from Prop. 72 (564).

Corollary 3.

898. By taking $AE=2\sqrt{bc}$ the vertical line EF is the asymptote of the curve AM. Whereby the perpendicular *MQ* is sent from *M* to *EF*, and we have $MQ=PE=2\sqrt{bc}-x$ and $EQ=y.$ On putting $MQ=z$, we have

$$y=-2cg+\frac{gz\sqrt{c}}{\sqrt{b}}+2gc\,l\,\frac{2\sqrt{bc}}{z}$$

and the time in which the arc *AM* is traversed is equal to $2\sqrt{c}\;l\,\frac{2\sqrt{bc}}{z}$.

Corollary 4.

899. Therefore the point *E* through which the asymptote *EF* passes is as far as the body that has been sent from the point *A* can reach, if there is no aid to the force *g* acting, before all the motion has been made available. And likewise in a similar way it is apparent that the time to pass through *AM* is equal to the time to pass along *AP* with the force *g* vanishing. [p. 386] Hence it is understood from this that *g* is not present in the expression for the time.

Corollary 5.

900. The tangent *MT* drawn from M is given by

$$QT = -\frac{zdy}{dz} = 2gc - \frac{gz\sqrt{c}}{\sqrt{b}}.$$

Through M draw MR making an angle with EF, the tangent of which is equal to $\frac{\sqrt{b}}{g\sqrt{c}}$.

With which accomplished,

$$QR = \frac{gz\sqrt{c}}{\sqrt{b}} \text{ and thus } RT = 2gc.$$

Corollary 6.

901. Therefore if *MR* is considered as an oblique applied line [y-axis] of the curve *AM* to the axis *EF*, then on account of the constant sub tangent *RT,* the curve *AM* is logarithmic with the oblique-angled sub tangent equal to *2gc*, and the tangent of the angle of inclination of the applied line *MR* to the asymptote $EF = \frac{\sqrt{b}}{g\sqrt{c}}$. [i. e. the tangent of the angle *MRQ* = *z*/*RQ* = $\frac{\sqrt{b}}{g\sqrt{c}}$, which is constant, and hence *MR* can be considered as an oblique axis; as this axis slides along *EF* as the point *M* varies, the length *RT* remains constant. A curve that has, no doubt, other interesting properties.]

Scholium.

902. The trajectory in this resisting medium and with the force acting under this hypothesis can be constructed not only with the aid of logarithms, but has been examined by Johan Bernoulli in Act. Erud. Lips. 1719 to be an oblique-angled logarithmic curve, the solution of which agrees uncommonly well with our solution. (See note 2 above.)

PROPOSITION 110.

PROBLEM.

903. *With the absolute uniform force put acting along the vertical direction MP* (Fig.82) *and the medium, that is also put as uniform,* [p. 387] *with the resistance in some ratio of the multiple of the speeds, to determine the curve AM described by the projected body.*

SOLUTION.

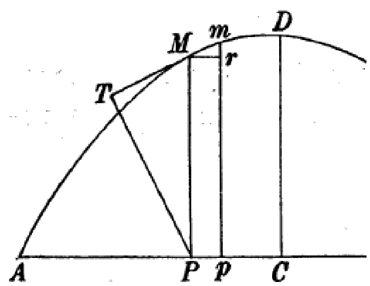


Fig. 82.

With $AP = x$, $PM = y$, $Mn = ds$, the speed at $M = \sqrt{v}$, and the force equal to g as before, with the exponent of the resisting medium equal to c, with the medium resisting in the ratio of the $2m$ th power of the speeds, and given by $R = \frac{v^m}{c^m}$ and $P = g$ (870). Whereby these equations are to be had :

$$gds^2 = -2vddy \text{ and } dv = -gdy - \frac{v^m ds}{c^m},$$

from which the curve AM as well as the motion of the body on the curve can be determined. Moreover, the equation $v = -\frac{gds^2}{2ddy}$ gives

$$dv = -gdy + \frac{gds^2 d^3y}{2ddy^2} \text{ and } v^m = \frac{g^m ds^{2m}}{2^m(-ddy)^m}.$$

Hence with v eliminated, this equation is arrived at for the nature of the curve :

$$c^m d^3y = -\frac{g^{m-1}ds^{2m-1}}{2^{m-1}(-ddy)^{m-2}}$$

For the construction of the curve, put $dy = pdx$ and there arises :

$$ddy = dxdp, \quad d^3y = dxddp \text{ and } ds = dx\sqrt{(1+p^2)}.$$

From which on substituting there becomes

$$2^{m-1}c^m(-dp)^{m-2}ddp = -g^{m-1}dx^m(1+pp)^{\frac{2m-1}{2}}.$$

Again put $dx = \frac{dp}{q}$ and then

$$ddp = \frac{dpdq}{q}.$$

Hence there arises

$$2(-2)^{m-2}c^m q^{m-1}dq = -g^{m-1}dp(1+p^2)^{\frac{2m-1}{2}}$$

and on integrating [p. 388]

$$q^m = -\frac{mg^{m-1}}{2(-2)^{m-2}c^m}\int dp(1+p^2)^{\frac{2m-1}{2}}.$$

From which equation q is given in terms of p, from which it is found on taking the abscissa $x=\int\frac{dp}{q}$, there corresponds the applied line $y=\int\frac{pdp}{q}$. And with the height corresponding to the height

$$v = -\frac{g(1+pp)}{2q}$$

and the time in which the arc AM is completed, i. e. $\int\frac{ds}{\sqrt{v}}$, is equal to

$$\int\frac{dp\sqrt{2}}{\sqrt{-gq}}$$

Q.E.I.

Corollary 1.

904. It is evident that whenever 2*m* is either a positive or negative odd number, the value of q can be shown algebraically in terms of p.

Corollary 2.

905. If the resistance is constant or $m = 0$ and the body initially at A is projected with a speed $\sqrt{b}$ along the horizontal AP, the applied line PM or y likewise taken as negative, has these equations

$$gds^2 = 2vddy \text{ and } dv = gdy - ds \text{ or } v = b + gy - s.$$

Hence this equation is produced :

$$\frac{gds^2}{2ddy} = b + gy - s.$$

Corollary 3.

906. Moreover this case is easier to handle if $m = 0$ in the differential equation of the third order, for it produces $gd^3y = -\frac{2ddy^2}{ds}$ or on substitution in terms of p and q this equation is made :

$$\frac{gdq}{q} = -\frac{2dp}{\sqrt{(1+p^2)}},$$

the integral of which is

$$glq = 2l\frac{\sqrt{(1+pp)}-p}{a}$$

or from the noted homogeneity [p. 389]

$$aq = (\sqrt{(1+p^2)} - p)^{\frac{2}{g}} = \frac{adp}{dx}.$$

Hence there arises

$$dx = \frac{adp}{(\sqrt{(1+pp)} - p)^{\frac{2}{g}}}.$$

Which again on integration gives :

$$2x = \frac{ga}{(2+g)(\sqrt{(1+pp)} - p)^{\frac{2+g}{g}}} + \frac{ga}{(2-g)(\sqrt{(1+pp)} - p)^{\frac{2-g}{g}}} + k.$$

And hence there is found $y = \int pdx$ and this is completed on integration to give :

$$4y = \frac{ga}{(2+2g)(\sqrt{(1+pp)} - p)^{\frac{2+2g}{g}}} - \frac{ga}{(2-2g)(\sqrt{(1+pp)} - p)^{\frac{2-2g}{g}}} + i.$$

Therefore it is apparent that this curve is algebraic only if g is 1 or 2.

Scholium.

907. And equally general to our solution is the solution given in the Acta Erud. Lipt. in May 1719 by Johan Bernoulli[1] on trajectories in resistive media, where the general construction of these curves was given. But before we leave the constant force hypothesis, we must solve the inverse problems, in which we determine the resistance that is effective, in order that the body describes a given curve acted on by a constant downwards force according to the hypothesis. For this matter has been treated several times, first by Newton[2] in the *Phil. Princ*. then again by Johan Bernoulli[3] in the Act. Lips, A, 1713: where the sharpest of men have noted many interesting things. [p. 390] [Thus, from the observed curve, one can determine whether or not the resistance has a particular form.]

[1 Previous note 2;
2. I. Newton. Philosophae *naturalis principia mathematica*, Londini 1687, Lib. II Sectiones 1, 2, 3;
3 Joh. Bernoulli, *De motu corporum gravium, pendulorum et proiectium*, Acta erud. 1713, p. 77; *Opera omnia*, Tom. 1, p. 514.]

PROPOSITION 111.

PROBLEM.

908. *With an absolute uniform force put in place acting downwards, to determine the resistance at the individual places M* (Fig.84) which can be put in place*, in order that the body describes the given curve BAM.*

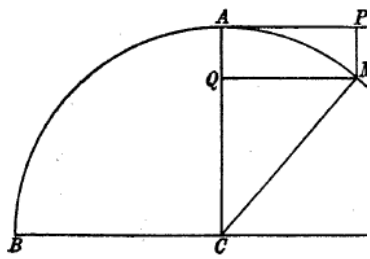


Fig. 84.

SOLUTION.

As before, put $AP = QM = x$, $PM = AQ = y$, and the element of ace $AM = ds$. Then the speed at M is equal to $\sqrt{v}$, and the resistance at M is equal to R. Therefore with these compared with Prop. 106 (870) we have $P = g$ and y must be taken as negative; and there is produced with dx taken as constant (871) : $dv = gdy - Rds$ (870). From this equation, moreover, we have

$$v = \frac{gds^2}{2ddy} \text{ and thus } dv = gdy - \frac{gds^2d^3y}{2ddy^2}.$$

Therefore with these equations solved there is produced :

$$R = \frac{gdsd^3y}{2ddy^2}.$$

Whereby, since the curve is given, from this equation the value of R is finally found, and thus the resistance becomes known. Q.E.I.

Corollary 1.

909. Therefore the force of the resistance at M to the force acting g is in the ratio $ds.d^3y$ to $2ddy^2$. Or, with the radius of curvature at M set equal to r, since $r = \frac{ds^3}{dxddy}$, we have:

$$\frac{dsd^3y}{2ddy^2} = \frac{3dsdy - dxdr}{2ds^2}.$$

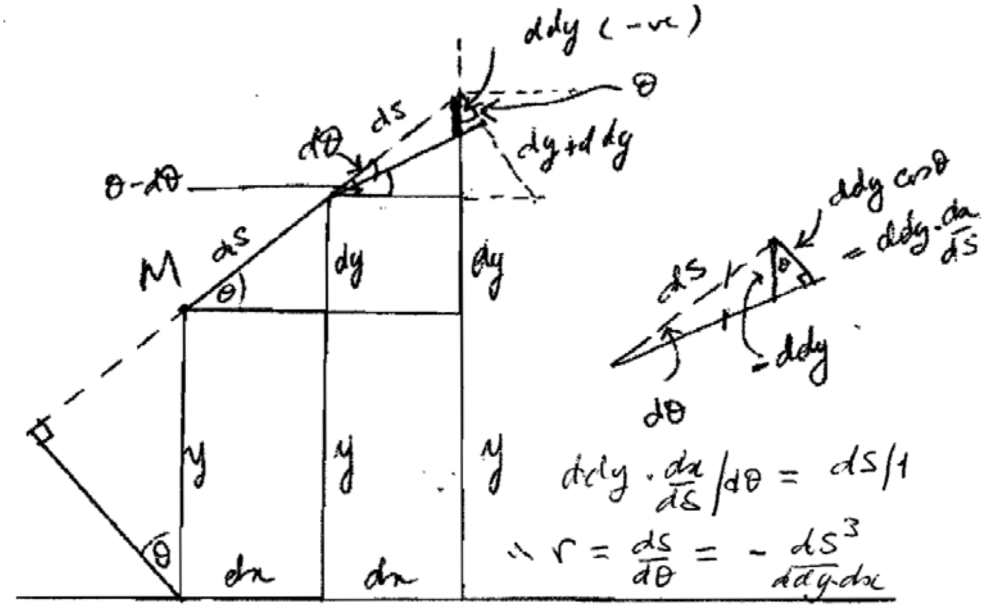
Whereby the ratio becomes:

$$g : R = 2ds^2 : 3dsdy - dxdr.$$

Corollary 2.

910. The height giving rise to the speed at M, surely v, can be determined from this curve; for it is [p. 391] $v = \frac{gd^2s}{2ddy}$. Or by introducing the radius of curvature r from $\frac{d^2s}{ddy} = \frac{rdx}{ds}$ it becomes : $v = \frac{grdx}{2ds}$.

Radius of Curvature



Corollary 3.

911. If the resistance is put in the ratio of the square of the speed, the unknown exponent of the resistance taken as q, then $R = \frac{gdsd^2y}{2ddy^2}$, and the exponent of the resistance of the medium $q = \frac{dsddy}{d^3y}$. And in a like manner the exponents can be found for the other hypotheses of the mediums resistances.

Corollary 4.

912. By introducing the radius of osculation r to the determination of q, we have

$$\frac{d^3y}{ddy} = \frac{ddy(3dsdy - dxdr)}{ds^3} = \frac{3dsdy - dxdr}{rdx}.$$

It consequently becomes :

$$q = \frac{rdxds}{3dsdy - dxdr}.$$

Therefore with a known radius of curvature r then R as well as q can be found from a first order differential equation.

Corollary 5.

912. If the curve AM is a parabola, the axis of which is the vertical AC, since in that case $d^3y = 0$, the resistance produced $R = 0$. From which it is understood that a parabola can be described in a vacuum by a constant force acting downwards, as it agrees with this.

Corollary 6.

914. If the curve is a parabola of some higher order, thus such that is given by $a^{n-1}y = x^n$, then

$$dy = \frac{nx^{n-1}dx}{a^{n-1}}$$

and

$$ds = \frac{dx\sqrt{(a^{2n-2} + n^2x^{2n-2})}}{a^{n-1}}.$$

And again on account of the constant dx :

$$ddy = \frac{n(n-1)x^{n-2}dx^2}{a^{n-1}} \quad \text{and} \quad d^3y = \frac{n(n-1)(n-2)x^{n-3}dx^3}{a^{n-1}}.$$

Whereby the resistance becomes

$$R = \frac{(n-2)g\sqrt{(a^{2n-2} + n^2x^{2n-2})}}{2n(n-1)x^{n-1}}.$$

And with the resistance put as in proportion to the square of the speed , then the exponent of the resistance becomes :

$$q = \frac{x\sqrt{(a^{2n-2} + n^2x^{2n-2})}}{(n-2)a^{n-1}}.$$

Scholium.

915. I will not add other examples of curvature here which can be put in place of AM; but I am resolved to examine these of merit more carefully in the following propositions. Moreover I intend to consider in particular the circle and the hyperbola, that these men cited have treated especially.

PROPOSITION 112.

PROBLEM.

916. *With the absolute force g constant and always acting downwards, to determine the resistance which must be put in place in order that the body is free to move along the periphery of the circle BAMD* (Fig.84). [p. 393]

SOLUTION.

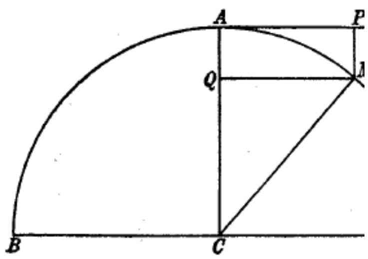


Fig. 84.

Let the radius of the circle $AC = a$, then it is the circle $x^2 = 2ay - y^2$ and the radius of osculation at $M = a = MC$. Whereby since $dr = 0$ we have the ratio

$$g : R = 2ds : 3dy.$$

Indeed $ds : dy = a : x = AC : QM$. On this account,

$$g : R = 2AC : 3QM \text{ or } R = \frac{3g.QM}{2AC}.$$

Indeed the height producing the speed at M is equal to $\frac{gadx}{2ds} = \frac{g.QC}{2}$. But if the resistance is put as the square of the speed, then the exponent of the resistance

$$q = \frac{adx}{3dy} = \frac{a \cdot QC}{3\,QM}.$$

Q.E.I.

Corollary 1.

917. While the body ascends through the arc BA, on account of QM then being negative, the resistance along BA is also negative; or the motion of the body along BA is accelerated by the medium by the tangential force $\frac{3g.QM}{2AC}$.

Corollary 2.

918. Truly the resistance at the point *A* is zero, since *QM* vanishes ; therefore the body at A moves as if in a vacuum. Indeed at the point *D* the resistance is in the three on two ratio to the force g. Indeed at *B* the force is acting up by the same amount.

Corollary 3. [p. 394]

919. Therefore since at *B* and *D* direction of the resistive force agrees with the direction of the force *g*, the body at *B* is urged up by a force equal to $\frac{1}{2}g$; at A it is pulled downwards by the force *g* and at *D* up again by the force $\frac{1}{2}g$.

Corollary 4.

920. Since $\frac{QM}{AC}$ expresses the sine of the angle ACM with the radius put equal to 1, then the resistance at M = $\frac{3}{2}g\,sin.ACM$ or the resistance is everywhere as the sine of the angle *ACM*, by which the body declines from *A*.

Corollary 5.

921. Again $\frac{a.QC}{AM}$ is the tangent of the arc *MD*. Whereby with the resistance put in proportion to the square of the speed, the exponent of the resistance *q* is equal to the third part of the cotangent of the arc *AM*.

Corollary 6.

922. Since the height corresponding to the speed at M is equal to $\frac{g.QC}{2}$, then the speed at $A = \sqrt{\frac{g.AC}{2}}$; and *B* and *D* the speed is equal to zero. Therefore since the body since the body at *B* is actually forced up again by the force $\frac{g}{2}$, it is not a wonder that the body at *B* begins to move up.

Corollary 7.

923. Therefore the body, so in the quadrant *BA* as in the quadrant *AD,* at places equally separated from the point *A,* has equal speeds. [p. 395] Truly the body is not able to reach places below the horizontal *BD* on account of *QC* being negative, and in which case the speed would be imaginary.

Scholium.

924. But the direction in which the body progresses when it arrives at *D* can be easily gathered from the reported events. For when the speed at *D* is equal to 0 and the body at *D* is urged up by the force $\frac{g}{2}$, it is evident that the body must again begin to move up. Moreover, it ascends again through the arc *DMA* in the same manner, in which in the first place it ascends through *BA*, since at *D* as also at *B*, it is urged up by the force $\frac{g}{2}$. Truly this wonderful thing happens in this motion, since the body is at rest at *B* and is urged up again and nevertheless moves on the curve, even if it seems to be unaided by any force, since it is able to change direction as the body accepts to go up at *B*. But to this argument I respond to this argument that the force at *B* is not in a perfectly vertical direction, but strays an infinitely small amount from the true vertical, since that is sufficient for the production of the oblique motion. For the direction of the resistance force at *B* or rather the accelerating force is present on the element of the periphery of the circle at *B*, which is not a perfectly vertical straight line, but inclined at an infinitely small angle towards BC.

Moreover these results of ours agree exceedingly well with these, which the celebrated Bernoulli gave in the Act. Lips. A. 1713 and which are present in the later editions of Newton's *Princ. Phil*. [Book II, Sect. 2, Prop. X.] For in the first edition an error has crept into the solution [p. 396], from which it was concluded that the ratio of g to R was established to be equal to the ratio AC to QM. But a word of caution from Nicolas Bernoulli corrected this lapse in the following editions.

[One can almost see Euler's smile as he wrote this, or even hear his gentle laugh echoing down through the centuries.....]

CAPUT SEXTUM

DE MOTU CURVILINEO PUNCTI LIBERI IN MEDIO RESISTENTE

[p. 369]

.

PROPOSITIO 104.

THEOREMA.

860. *Si corpus moveatur in medio resistente a quotcunque potentiis absolutiis sollicitatum, vis resistentiae actionem potentiarum absolutarum aliter non turbat, nisi quod vim tangentialem ex illis ortam minuat.*

DEMONSTRATIO.

Ex capite praecedente satis intelligitur omnes potentias absolutas resolvi posse in duas vires, tangentialem et normalem, si quidem motus fit in eodem plano. At si corpus non in eodem movetur plano, tum tres vires aequivalentes assignari possunt loco quotcunque potentiarum sollicitantium, quarum una est tangentialis et duae normales. Vis autem, quam resistentia in corpus exerit, directio semper congruere ponitur cum directionn corporis (117). Quamobrem vis resistentiae ad vim tangentialem est referenda, quam imminuit, [p. 370] quia motum corporis retardat, vires vero normales prorsus non afficit. Manifestum igitur est resistentiam potentiarum absolutarum effectum aliter non turbare, nisi quatenus vis tangentialis ex iis orta a resistentia minuitur. Q.E.D.

Corollarium 1.

861. Resistentiae igitur effectus totus in alteranda corporis celeritate consistit neque eius directionem immutat, nisi quatenus virium normalium actio variatur variata celeritate.

Corollarium 2.

862. Nisi igitur praeter resistentiam adsint potentiae absolutae, fieri non potest, ut corpus in linea curva moveatur, sed perpetuo in recta moveri perget, quod motum suum perdiderit.

Scholion 1.

863. In hoc igitur capite, quo motus curvilineos tractibimus, necesse est, ut cum resistentia simul potentias absolutas consideremus easque tales, quae resolutione praebeant vim normalem, ne in eandem rem capite IV pertractatam indicamus. Hanc ob rem primo potentiam ad punctum infinite distans tendentem seu directionem sibi perpetuo parallelam conservantem considerabimus. Deinde ad vires centripetas progrediemur aliasque quovis modo dispositas potentias. Denique etiam motus non in eodem plano factos, [p. 371] quales oriantur in medio resistente, examini subiiciemus.

Corollarium 3.

864. Si vis tangentialis sit T et una seu duae normales N seu N et M et vis resistentiae R, conones effectum harum virium continentes, quos in praecedente capite dedimus, etiam hic locum habebunt, si modo in iis loco T ponatur $T - R$.

Scholion 2.

865. Quemadmodum resistentiam, cuius vis a celeritate corporis pendere ponitur, exponi oporteat per legem resistentiae et exponentem, in cap. IV fuse est ostensum. Hoc vero capite varietas resistentiae magnum campum rerum pertractandarum aperiet, quae in capite praecedente locum non inveniebant. Praeterea hanc tractationem ita subdividemus, ut primo ex datis potentiis absolutis et resistentia curvam descriptam et motum corporis in ea determinemus. Deinde si curva et potentia absoluta fuerit data, ex his resistentiam deducimus. Tertio ex data curva et resistentia potentia absoluta datam habens directionem erit investiganda. Denique ex data curva et celeritate corporis in singulis eius punctis et resistentia potentia absoluta eiusque directio poterit inveniri. Primariam autem [p. 372] huius capitis divisionem motus in eodem plano et non in eodem plano factus constituet.

PROPOSITIO 105.

PROBLEMA.

866. *Si corpus moveatur in medio resistente quocunque sollicitatum a potentiis absolutis quibuscunque, ita tamen, ut motum suum in eodem plano absolvat, definere canones, quos in motu suo corpus observat.*

SOLUTIO.

Describat corpus hac ratione sollicitatum curvam *AMB* (Fig.81); sit eius celeritas in M debita altitudini v et curvae elementum $Mm = ds$. Ponatur porro vis normalis $= N$, cuius ergo directio erit MN normalis in curvam, vis vero tangentialis ex iisdem potentiis absolutis orta $= T$, cuius directio est MT, tangens curvae in M. Atque vis resistentiae in M sit $= R$. His positis motus corporis definiri debet ex vi normali N et vi tangentiali $T - R$ (864). Sit iam radius osculi in $M = r$ eritque

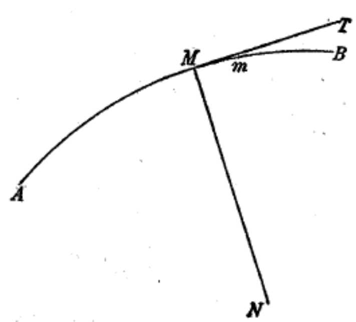


Fig. 81.

$$N = \frac{2v}{r} \text{ et } dv = (T - R)ds$$

(552). Ex his duabus aequationes si eliminetur v, prodibit aequatio naturam curvae AMB exponens simulque corporis in singulis locis celeritas ex aequatione $N = \frac{2v}{r}$ innotescit. Q.E.I. [p. 373]

Corollarium 1.

867. Erit igitur $v = \frac{Nr}{2}$. Unde habebitur $dv = \frac{Ndr+rdN}{2}$. Qui valor si in aequatione $dv = (T - R)ds$ substituatur loco *dv* et in *R* ponatur $\frac{Nr}{2}$ loco *v*, prodibit aequatio pro curva a corpore descripta.

Corollarium 2.

868. Si in *R v* unicam habuerit dimensionem, id quod accidit, si resistentia est quadratis celeritatum proportionalis, aequatio $dv = (T - R)ds$ poterit separari ex eaque determinari. Haecque aequatio cum $v = \frac{Nr}{2}$ coniuncta dabit simpliciorem aequationem pro curva descripta.

Scholion.

869. Praeter hunc casum, quo v unicam habet dimensionem in *R*, plurimi dantur alii, quibus aequatio $dv = (T - R)ds$ potest integrari; sed eos evolvere non est opus, cum nihilominus v possit eliminari. Hunc vero casum idea praecipue notavimus, quia revera ad fluidorum resistentiam pertinet et quem propterea prae aliis diligentius examinabimus.

PROPOSITIO 106.

PROBLEMA.

870. *Tendat vis sollicitans ubique normaliter ad rectam positione datam AP* (Fig.82) *et moveatur corpus* [p. 374] *in medio quocunque resistente; determinare curvam AMB, in qua corpus movebitur, et ipsum corporis motum.*

SOLUTIO.

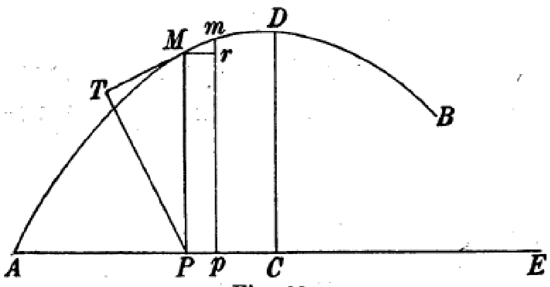


Fig. 82.

Sit vis, quae corpus in *M* sollicitat, = *P*, cuius ergo directio erit *MP*. Celeritas corporis in *M* debeatur altitudini *v* et vis resistentiae ibi sit = *R*. Capiatur elementum *Mm* ductaque mp sit *AP = x*, *PM = y* et *Mm = ds*. Erit *Pp = Mr = dx* et *mr = dy*. Porro ducatur tangens *MT* in eamque ex *P* perpendicularis *PT*. His factis vis *P* resolvetur in normalem $\frac{P.PT}{PM} = \frac{Pdx}{ds}$ et tangentialem $\frac{P.PT}{PM} = \frac{Pdy}{ds}$. Quia autem haec vis tangentialis motum corporis retardat, eius negativum est capiendum. Posito igitur radio osculi in $M = r$ erit $\frac{Pdx}{ds} = \frac{2v}{r}$ et $dv = -Pdy - Rds$ (866). Ex quibus aequationibus tum ipsa curva tum motus corporis poterit inveniri. Q.E.I.

Corollarium 1.

871. Posito dx constante est radius osculi $r = -\frac{ds^3}{dxddy}$. Hanc ob rem habebitur sequens aequatio

$$P = -\frac{2vddy}{ds^2}.$$

Qui valor ipsius P in altera aequatione substitutus dabit aequationem
$dv = \frac{2vdyddy}{ds^2} - Rds$ seu ob $dyddy = dsdds$ hanc $dv = \frac{2vdds}{ds} - Rds$.
Quare aequatio locum habet, quaecunque fuerit potentia P, modo eius directio sit MP.

Corollarium 2. [p. 375]

872. Si resistantiae lex fuerit ratio quaecunque multiplicata celeritatum et exponens resistentiae sit quantitas utcunque variablilis q, ita ut sit $R = \frac{v^m}{q^m}$, tum habebitur ista aequatione

$$dv = \frac{2vdds}{ds} - \frac{v^m ds}{q^m},$$

cuius integralis est

$$v^{1-m} = \frac{(m-1)dx^{2m-2}}{ds^{2m-2}} \cdot \int \frac{ds^{2m-1}}{q^m dx^{2m-2}}.$$

Corollarium 3.

873. Si in eadem hypothesi fuerit $m = 1$, erit $\frac{dv}{v} = \frac{2dds}{ds} - \frac{ds}{q}$, cuius integralis est

$$lv = 2l\frac{ds}{dx} - \int \frac{ds}{q} \quad \text{seu} \quad e^{\int \frac{ds}{q}} v = \frac{ads^2}{dx^2}.$$

Si praeterea resistentia fuerit uniformis seu $q = c$, erit

$$e^{\frac{s}{c}} v = \frac{ads^2}{dx^2} \quad \text{seu} \quad v = \frac{ae^{-\frac{s}{c}} ds^2}{dx^2}.$$

Tempus igitur hoc casu, quo arcus AM absolvitur, erit =

$$\int \frac{e^{\frac{s}{2c}} dx}{\sqrt{a}}.$$

PROPOSITIO 107.

PROBLEMA.

874. *Si et potentia et medium resistens sit uniforme illiusque directio sit MP* (Fig.82) *normalis in rectam datam AP, medium vero resistat in ratione duplicata celeritatum determinare motum corporis proiecti.*

SOLUTIO. [p. 376]

Ponatur potentia corpus perpetuo versus *AP* trahens = *g* et expons resistentiae = *c*, reliquae denominationes vero maneant ut ante. Erit igitur $P = g$ et $R = \frac{v}{c}$. Unde orientur sequentes aequationes

$$\frac{gdx}{ds} = \frac{2v}{r} \quad \text{seu} \quad gds^2 + 2vddy = 0$$

posito *dx* constante atque

$$dv = -gdy - \frac{vds}{c}.$$

Ex his vero aequationibus coniunctis iam invenimus $e^{\frac{s}{c}}v = \frac{ads^2}{dx^2}$ (873).

Quare, cum sit

$$v = -\frac{gds^2}{2ddy},$$

prodit eliminata *v* aequatio

$$ge^{\frac{s}{c}}dx^2 = -2addy,$$

qua natura curvae descriptae continetur. Cum sit *dsdds* = *dyddy*, erit etiam

$$ge^{\frac{s}{c}}dx^2dy = -2adsdds.$$

Ponatur *dx* = *pds*, erit

$$dds = -\frac{dpds}{p} \text{ et } dy = ds\sqrt{(1-pp)}.$$

His substitutis proveniet ista aequatio

$$ge^{\frac{s}{c}}ds = \frac{2adp}{p^3\sqrt{(1-pp)}},$$

quae ad construendam curvam descriptam sufficit. Aequationis vero huius integralis aequatio est

$$gce^{\frac{s}{c}} = C - \frac{a\sqrt{(1-pp)}}{p^2} - al\frac{1+\sqrt{(1-p^2)}}{p}.$$

Restituto vero $\frac{dx}{ds}$ loco *p* habetur

$$gce^{\frac{s}{c}} = C - \frac{adyds}{dx^2} - al\frac{ds+dy}{dx}.$$

Quae est aequtio differentialis primi gradus atque simplicior reddi non potest. Q.E.I.

Corollarium 1.

875. Pro curva descripta aequatio statim differentialis tertia gradus prodit. Nam ob $v=-\frac{gds^2}{2ddy}$ erit $dv=-gdy+\frac{gds^2d^3y}{2ddy^2}$, quibus valoribus in aequatione [p. 377] $dv=-gdy-\frac{vds}{c}$ substitutis prodibit $dsddy=cd^3y$.

Corollarium 2.

876. Sit sinus anguli, quem curva in A cum axe AP constituit, $=\mu$ eiusque cosinus $=\sqrt{(1-\mu^2)}=\nu$ et altitudo celeritati in A debita $= b$. Facto ergo $s = 0$ et $ds:dx=1:\nu$ fieri debet $v = b$; habebitur ergo ex $e^{\frac{s}{c}}v=\frac{ads^2}{dx^2}$ haec $\nu^2 b=a$, unde constans a cognoscitur.

Corollarium 3.

877. Porro in aequatione curvae ultima facto $s = 0$ et $ds:dx=1:\nu$ et $ds:dy=1:\mu$ invenietur constans $C=gc+\mu b+\nu^2 b\,l\frac{1+\mu}{\nu}$. Quare pro curva descripta haec orietur aequatio

$$\frac{gc}{b}\left(e^{\frac{s}{c}}-1\right)=\frac{\mu dx^2-\nu^2 dyds}{dx^2}+\nu^2 l\frac{(1+\mu)dx}{\nu(dy+ds)}.$$

Ad celeritatem vero inveniendam inservit aequatio

$$e^{\frac{s}{c}}v=\frac{\nu^2 bds^2}{dx^2}.$$

Corollarium 4.

878. Si fuerit D punctum supremum, erit ibi $dx = ds$ et $dy = 0$. Habebitur ergo

$$\frac{gc}{b}\left(e^{\frac{s}{c}}-1\right)=\mu+\nu^2 l\frac{1+\mu}{\nu}\quad\text{seu}\quad e^{\frac{s}{c}}=\frac{b\mu+gc+\nu^2 b\,l\frac{1+\mu}{\nu}}{gc}.$$

Ex qua aequatione reperitur [p. 378]

$$\text{arcus } AMD=c\,l\frac{b\mu+gc+\nu^2 b\,l\frac{1+\mu}{\nu}}{gc}.$$

Altitudo vero debita celeritati, quam corpus habebit in D, erit =

$$\frac{\nu^2 gbc}{gc+b\mu+\nu^2 b\,l\frac{1+\mu}{\nu}}.$$

Corollarium 5.

879. Si in *B* curva eandem habere ponatur inclinationem ad axem *AP*, quam habuit in *A*, erit $ds : dx : dy = 1 : \nu : -\mu$. Hinc ergo emerget ista aequatio

$$\frac{gc}{b}\left(e^{\frac{s}{c}} - 1\right) = 2\mu + \nu^2 l \frac{1+\mu}{1-\mu},$$

ex qua prodit

$$\text{arcus } ADB = c\, l \frac{2\mu b + gc + \nu^2 b\, l \frac{1+\mu}{1-\mu}}{gc}.$$

Corollarium 6.

880. Constructio curvae etiam facilis deduci potest ex aequatione $dsddy = cd^3y$. Namque ponatur $dy = pdx$, erit $dpdx\sqrt{(1+pp)} = cddp$. Porro fiat $dx = \frac{dp}{q}$, erit ob $ddx = 0$

$$ddp = \frac{dpdq}{q},$$

unde prodibit ista aequatio

$dp\sqrt{(1+pp)} = cdq$ atque $q = \frac{1}{c}\int dp\sqrt{(1+pp)}$.

Sumta igitur abscissa

$$x = \int \frac{cdp}{\int dp\sqrt{(1+pp)}}$$

erit

$$y = \int \frac{cpdp}{\int dp\sqrt{(1+pp)}}.$$

Hisque respondebit

$$v = -\frac{cg(1+pp)}{2\int dp\sqrt{(1+pp)}}.$$

Unde cognoscitur pro g accipiendam esse quantitatem negativam.

Corollarium 7.

881. Si corpus in A proiiciatur secundum directionem ipsius AP (Fig.83) et potentia tendat deorsum, [p. 379] tota curva AM a corpore descripta cadet infra AP fietque *y* seu *PM* negativa; atque cum sit $\mu = 0$ et $\nu = 1$, habebitur pro curva *AM* ista aequatio (877)

$$\frac{gc}{b}\left(e^{\frac{s}{c}} - 1\right) = \frac{dyds}{dx^2} - l\frac{dx}{ds+dy}.$$

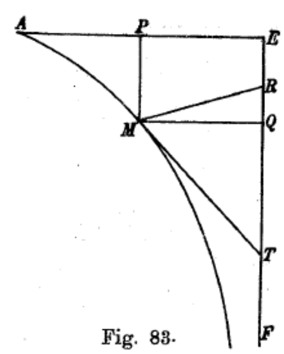


Fig. 83.

Corollarium 8.

882. Data igitur tangentis in quocunque puncto *M* inclinatione ad rectam *AP* seu ratione inter *ds, dy*, et *dx* inveniri potest ex hac aequatione longitudino arcus *AM*.

Scholion.

883. Experimenta docuere aerem corporibus resistere in duplicata celeritatum ratione. Cum igitur vis gravitatis sit uniformis et aer in non nimis altis distantiis eandem fere densitatem servet, casus corporum in aere proiectorum apprime ad hanc propositionem refertur. Determinavimus igitur veram curvam, quam globi ex sclopetis vel tormentis vel alio modo proiecti describunt. Sumitur vulgo pro hac curva parabola, quippe quae in vacuo est proiectoria, et aer tam subtile fluidum esse creditur, ut eius resistantia in computum duci non mereatur. Insensibilis quidem utique est resistentia aeris, si corpus magnum parva celeritate proiiciatur. Sed longissime a parabola aberrabit proiectoria, si exiguum corpus magna vi proiiciatur. His autem in casibus, tametsi hic vera assignata est proiectoria, maxime dolendum est, aequationem tam esse intricatam, [p. 380] ut vix quicquam ad usum practicum ex ea possit deduci. Neutonus in *Phil. Princ*. hoc problema non attigit[1] neque post eum quisquam tentavit, donec Keilius ad hoc problema Ioh. Bernoullium provocaverit, etsi ipse solutionem exhibere non potuerit. Dedit autem solutionem non solum Ioh. Bernoullius[2] in Act. Lips. 1719 mensis Maii, sed eodem fere tempore Iac. Hermannus[3] *Phoronomiae* suae inseruit. Sequaens autem problema, in quo resistentia ipsis celeritatibus proportionalis ponitur, tum a Neutono[4] in *Phil. Princ*. tum a Hugenio in *Tractatu de causa gravitatis* est solutum.

[1. Egit quidem Neutonus in *Philosophiae naturalis* Libri secundi Sectione secunda de motu corporum, quibus resistitur in duplicatata ratione velocitatum, sed problema de invenienda orbita, quae a corpore proiecto describitur, praetermisit.

2. Ioh. Bernoullil, *Responsio ad nonneminis provocationem eiusque solutio quaestionis ipsi ad eodem propostitae de invenienda linea curva, quam describit proiectile in medio resistente*, Acta erud. 1719, p. 216; *Opera omnia*, Tom. 2, Lausannae et Genevae 1742, p. 393. Vide etiam Acta erud. 1721, p. 228; *Opera omnia*, Tom. 2, p. 513; nec non dissertationem anno 1742 editam *Problema ballisticum; Opera omnia*, Tom. 4, p. 354.

3. Iac. Hermann, *Phoronomia seu de viribus et motibus corporum solidorum et fluidorum,* Amstelodami 1716.

4. I. Newton, *Philosophiae naturalis principia mathematica*, Londini 1687, Lib. II Sectio 1; De motu corporum, quibus resistitur in rationis velocitatis.

5. Chr. Huygens, *Traite de la lumiere, avec un discours sur la cause de la pesanteur*, Leyde 1691.

References supplied by Paul Stackel.]

PROPOSITIO 108.

PROBLEMA.

884. *Si media resistentia fuerit ut ipsa corporis celeritas et potentiae directio MP* (Fig.82) *praetereaque tam potentia sit uniformis quam medium resistens, determinare curvam, quam corpus proiectum describit, atque celeritatem in singulis locis.*

SOLUTIO.

Positis ut in praecedente propositione potentia uniformi = *g*, exponente medii resistentis = *c*, altitudine celeritati in *M* debita = *v, AP = x, PM = y* et arcu *AM = s* erit ex vi normali ut ante

$$gds^2 + 2vddy = 0,$$

sumpto *dx* pro constante. Ex vi tangentiali vero ob resistentiam hoc casu $= \frac{\sqrt{v}}{\sqrt{c}}$ habebitur ista aequatio

$$dv = -gdy - \frac{ds\sqrt{v}}{\sqrt{c}}.$$ [p. 381]

Ex his aequationibus coniunctis per (872), ubi fit $q = c$ et $m = \frac{1}{2}$, obtinetur

$$\sqrt{v} = -\frac{ds}{2dx}\int\frac{dx}{\sqrt{c}} = \frac{ds\sqrt{a}}{2dx} - \frac{xds}{2dx\sqrt{c}} = \frac{ds(\sqrt{ac}-x)}{2dx\sqrt{c}}$$

eritque ergo

$$v = \frac{ds^2(\sqrt{ac}-x)^2}{4cdx^2}.$$

Unde pro curva descripta provenit ista aequatio

$$2gcdx^2 = -ddy(\sqrt{ac}-x)^2 \quad \text{seu} \quad -\frac{ddy}{dx} = \frac{2gcdx}{(\sqrt{ac}-x)^2},$$

cuius integralis est

$$\frac{dy}{dx} = -\frac{2gc}{\sqrt{ac}-x} + k.$$

atque iterum integrando

$$y = kx - 2gcl\frac{\sqrt{ac}}{\sqrt{ac}-x}.$$

Unde simul constabit *v* ex aequatione

$$v = \frac{ds^2(\sqrt{ac}-x)^2}{4cdx^2}. \quad \text{Q. E. I.}$$

Corollarium 1.

885. Pro curva descripta statim haec aequatio differentialis tertii gradus prodiisset, si ex aequatione $v = -\frac{gds^2}{2ddy}$ eiusque differentiali

$$dv = -gdy + \frac{gds^2 d^3y}{2ddy^2}$$

valores in aequatione $dv = -gdy - \frac{ds\sqrt{v}}{\sqrt{c}}$ fuissent substituti. Provenisset enim

$$d^3y\sqrt{gc} = -ddy\sqrt{-2ddy}.$$

Corollarium 2.

886. Si corpus in *A* proiiciatur celeritate altitudini *b* debita et sinus anguli, quem tangens in *A* cum *AP* constituit, sit = μ, eius cosinus $\sqrt{(1-\mu^2)} = \nu$, erit in puncto A $b = \frac{a}{4\nu^2}$, seu $a = 4\nu^2 b$, ex quo constans indefinita *a* determinatur.

Corollarium 3.

887. Aequatio porro

$$\frac{dy}{dx} = -\frac{2gc}{\sqrt{ac-x}} + k$$

[p. 382] ad punctum *A* translata dabit

$$\frac{\mu}{\nu} = -\frac{2g\sqrt{c}}{\sqrt{a}} + k = -\frac{g\sqrt{c}}{\nu\sqrt{b}} + k$$

ob $a = 4\nu^2 b$. Hinc invenietur indefinita quantitas

$$k = \frac{\mu\sqrt{b} + g\sqrt{c}}{\nu\sqrt{b}}.$$

Corollarium 4.

888. Pro curva igitur descripta invenietur ista aequatio differentialis

$$dy = \frac{2\mu\nu bdx\sqrt{c} - \mu xdx\sqrt{b} - gxdx\sqrt{c}}{2\nu^2 b\sqrt{c} - \nu x\sqrt{b}} = \frac{\mu dx}{\nu} - \frac{gxdx\sqrt{c}}{2\nu^2 b\sqrt{c} - \nu x\sqrt{b}}.$$

Ex qua deducitur

$$\frac{ds^2}{dx^2} = \frac{(2\nu b\sqrt{c} - x\sqrt{b})^2 - 2\mu gx(2\nu bc - x\sqrt{bc}) + g^2cx^2}{\nu^2(2\nu b\sqrt{c} - x\sqrt{b})^2}.$$

Quare, cum sit

$$v = \frac{ds^2(2\nu\sqrt{bc} - x)^2}{4cdx^2},$$

reperiatur tandem

$$v = \frac{(2\nu b\sqrt{c} - x\sqrt{b})^2 - 2\mu g x(2\nu bc - x\sqrt{bc}) + g^2 c x^2}{4\nu^2 bc}.$$

Corollarium 5.

889. Aequatio autem integralis pro curva quaesita erit

$$y = \frac{\mu x\sqrt{b} + gx\sqrt{c}}{\nu\sqrt{b}} - 2gc\,l\frac{2\nu\sqrt{bc}}{2\nu\sqrt{bc} - x}.$$

Ex qua constructio curvae per logarithmicam facile perficitur.

Corollarium 6.

890. Tempus etiam, quo arcus *AM* absolvitur, facile definitur. Nam cum sit $\frac{ds}{\sqrt{v}} = \frac{2dx\sqrt{c}}{2\nu\sqrt{bc} - x}$, erit tempus per arcum *AM* =

$$2\sqrt{c}\,l\frac{2\nu\sqrt{bc}}{2\nu\sqrt{bc} - x}.$$

Corollarium 7.

891. Apparet etiam ex aequatio pro curva *AM* eam habere asymtoton. Nam cum *x* non possit esse maior quam $2\nu\sqrt{bc}$, si capiatur $AE = 2\nu\sqrt{bc}$, erit applicata in $E = -\infty$ ideoque asymptotos curvae *AMDB*. Intelligitur hoc etiam ex tempore, [p. 383] quod fit infinitum, antequam corpus ad perpendicularem per *E* ducam pervenit.

Corollarium 8.

892. Punctum summum *D* reperietur, si fiat *dy* = 0. Tum autem invenitur

$$x = \frac{2\mu\nu b\sqrt{c}}{\mu\sqrt{b} + g\sqrt{c}} = AC.$$

Deinde, cum sit in *D* $ds = dx$ et $2\nu\sqrt{bc} - x = \frac{2\nu gc\sqrt{b}}{\mu\sqrt{b} + g\sqrt{c}}$, erit altitudo debita celeritati in puncto D $= \frac{\nu^2 g^2 bc}{(\mu\sqrt{b} + g\sqrt{c})^2}$ et ipsa celeritas in D =

$$\frac{\nu g\sqrt{bc}}{\mu\sqrt{b} + g\sqrt{c}}.$$

Corollarium 9.

893. Applicata vero *CD* seu distantia puncti *D* ab axe *AP* erit =

$$2\mu\sqrt{bc} - 2gc\,l\,\frac{\mu\sqrt{b}+g\sqrt{c}}{g\sqrt{c}}$$

et tempus, quo corpus ex *A* ad *D* pervenit, erit

$$2\sqrt{c}\,l\,\frac{\mu\sqrt{b}+g\sqrt{c}}{g\sqrt{c}}.$$

Scholion.

894. Reduci igitur potest casus, quo corpus ex *A* oblique proiicitur, ad casum, quo ad directionem potentiae normaliter ex *D* proiicitur. Cognita enim celeritate, qua corpus in A proiicitur, et directione inveniri poterit punctum D, in quo tangens ipsi AC est parallela, et celeritas corporis in D. Quare ad meliorem huius motus cognitionem expedit motum tanquam in D incipientem considerari, quem in finem sequentem propositionem adiecimus. [p. 384]

PROPOSITIO 109.

PROBLEMA.

884. *Si corpus ubique aequaliter deorsum attrahutur atque in A* (Fig.83) *secundum directionem horizontalem AP data velocitate proiiciatur in medio uniformi, quod in simplici celeritatum ratione resistat, determinare curvam AM, quam corpus describet, et motum corporis in hac curva.*

SOLUTIO.

Cum proposito haec sit casus specialis praecedentis, maneant omnes denominationes ut ante. Fiet autem *y* seu applicata *PM* negativa, quia curva *AM* infra *AP* cadet, atque erit $\mu = 0$ et $\nu = 1$. Existente ergo *g* potentia sollicitante, *c* exponente resistantiae, *b* altitudine celeritati in *A* debita et *AP* = *x* et *AM* = *s*, habebitur aequatio pro curva *AM* haec differentialis

$$dy = \frac{gx\,dx\sqrt{c}}{2b\sqrt{c}-x\sqrt{b}}$$

(888) atque haec integralis

$$y = -\frac{gx\sqrt{c}}{\sqrt{b}} + 2gc\,l\,\frac{2\sqrt{bc}}{2\sqrt{bc}-x}$$

(888). Atque tempus, quo arcus AM percurritur, =

$$2\sqrt{c}\,l\,\frac{2\sqrt{bc}}{2\sqrt{bc}-x}$$

(890). Quae aequationes tam curvam AM quam motum in hac curva determinant. Q.E.I.

Corollarium 1.

896. Si $l\,\frac{2\sqrt{bc}}{2\sqrt{bc}-x}$ in seriem convertatur, prodibit

$$\frac{x}{2\sqrt{bc}}+\frac{x^2}{2\cdot 4bc}+\frac{x^3}{3\cdot 8bc\sqrt{bc}}+\frac{x^4}{4\cdot 16b^2c^2}+\text{ etc.}$$

Quamobrem erit [p. 385]

$$y=\frac{gx^2}{4b}+\frac{gx^3}{12b\sqrt{bc}}+\frac{gx^4}{32b^2c}+\text{ etc.}$$

et tempus, quo arcus AM percurritur, =

$$\frac{x}{\sqrt{b}}+\frac{x^2}{4b\sqrt{c}}+\frac{x^3}{12bc\sqrt{b}}+\frac{x^4}{32b^2c\sqrt{c}}+\text{ etc.}$$

Corollarium 2.

897. In vacuo igitur, quando c fit infinite magnum, erit $y=\frac{gx^2}{4b}$; quo igitur casu curva AM abit in parabolam, cuius parameter est $\frac{4b}{g}$, et tempus, quo arcus *AM* absolvitur, est = $\frac{x}{\sqrt{b}}$ atque $v=b+\frac{gx^2}{4b}=b+gy,$ quemadmodum ex prop. 72 (564) colligere licet.

Corollarium 3.

898. Sumta $AE=2\sqrt{bc}$ erit verticalis EF curvae *AM* asymotos. Quare se ex *M* in *EF* demittatur perpendiculum *MQ*, erit $MQ=PE=2\sqrt{bc}-x$ et $EQ=y.$ Ponatur $MQ=z$, erit

$$y=-2cg+\frac{gz\sqrt{c}}{\sqrt{b}}+2gc\,l\,\frac{2\sqrt{bc}}{z}$$

et tempus, quo arcus *AM* percurritur, $=2\sqrt{c}\,l\,\frac{2\sqrt{bc}}{z}$.

Corollarium 4.

899. Punctum igitur *E*, per quod transit asymtotos *EF*, tantum distat a puncto *A*, quousque corpus ex *A*, si nulla adesset potentia sollicitans *g*, posset pertingere, antequam motum omnem admitteret. Atque simili modo patet tempus per *AM* aequale esse tempori per *AP* potentia *g* evanescente. [p. 386] Intelligitur hoc ex eo, quod *g* in his expressionibus non inest.

Corollarium 5.

900. Ducatur ex *M* tangens *MT*, erit

$$QT = -\frac{zdy}{dz} = 2gc - \frac{gz\sqrt{c}}{\sqrt{b}}.$$

Per M ducatur MR constituens cum EF angulum, cuius tangens est = $\frac{\sqrt{b}}{g\sqrt{c}}$.

Quo facto erit

$$QR = \frac{gz\sqrt{c}}{\sqrt{b}} \text{ ideoque } RT = 2gc.$$

Corollarium 6.

901. Si igitur *MR* consideretur ut applicata curvae *AM* ad axem *EF* obliquangula, erit curva *AM* ob subtangentem *RT* constantem logarithmica obliquanula subtangentis = *2gc* et tangens anguli inclinationis applicatarum *MR* ad asymtoton $EF = \frac{\sqrt{b}}{g\sqrt{c}}$.

Scholion.

902. Proiectoriam in hac resistentiae et vis sollicitantis hypotheis non solum ope logarithmicae construi posse, sed ipsam esse logarithmicam obliquangulam observavit Ioh. Bernoulli in Actis Lips. 1719. Cuius solutio cum hac nostra egregie convenit. (Vide notam 2.)

PROPOSITIO 110.

PROBLEMA.

903. *Posita potentiae absolutae uniformis directione MP* (Fig.82) *verticali et medio, quod etiam uniforme ponitur,* [p. 387] *resistente in quacunque multiplicata celeritatum ratione, determinare curvam AM, quam corpus proiectum describit.*

SOLUTIO.

Manente *AP = x, PM = y, Mn = ds*, celeritate in *M* = $\sqrt{v}$, potentia = *g*, exponente medii resistentis = *c* resistat medium in ratione 2*m*-plicata celeritatum eritque $R = \frac{v^m}{c^m}$ et *P* = *g* (870). Quare habebuntur istae aequationes

$$gds^2 = -2vddy \quad \text{et} \quad dv = -gdy - \frac{v^m ds}{c^m},$$

ex quibus tum curva AM tum motus corporis in curva determinatur. Aequatio autem $v = -\frac{gds^2}{2ddy}$ dat

$$dv = -gdy + \frac{gds^2 d^3y}{2ddy^2} \quad \text{et} \quad v^m = \frac{g^m ds^{2m}}{2^m(-ddy)^m}.$$

Hinc eliminata v pervenitur ad istam aequationem

$$c^m d^3y = -\frac{g^{m-1}ds^{2m-1}}{2^{m-1}(-ddy)^{m-2}}$$

pro curvae natura. Ad hance construendam ponatur $dy = pdx$ eritque

$$ddy = dxdp, \quad d^3y = dxddp \quad \text{et} \quad ds = dx\sqrt{(1+p^2)}.$$

Quibus substitutis habebitur

$$2^{m-1}c^m(-dp)^{m-2}ddp = -g^{m-1}dx^m(1+pp)^{\frac{2m-1}{2}}.$$

Ponatur porro $dx = \frac{dp}{q}$ eritque

$$ddp = \frac{dpdq}{q}.$$

Unde habebitur

$$2(-2)^{m-2}c^m q^{m-1}dq = -g^{m-1}dp(1+p^2)^{\frac{2m-1}{2}}$$

et integrando [p. 388]

$$q^m = -\frac{mg^{m-1}}{2(-2)^{m-2}c^m}\int dp(1+p^2)^{\frac{2m-1}{2}}.$$

Ex qua aequatione datur q in p, quo invento, sumta abscissa $x = \int\frac{dp}{q}$, est respondens applicata $y = \int\frac{pdp}{q}$. Atque celeritati debita altitudo

$$v = -\frac{g(1+pp)}{2q}$$

et tempus, quo arcus AM absolvitur, i. e. $\int\frac{ds}{\sqrt{v}}$, =

$$\int\frac{dp\sqrt{2}}{\sqrt{-gq}}$$

Q.E.I.

Corollarium 1.

904. Perspicuum est, quoties $2m$ fuerit vel numerus affirmativus impar vel numerus negativus par, valorem ipsius q algebraice per p posse exhiberi.

Corollarium 2.

905. Si resistentia est constans seu $m = 0$ et corpus initio in A proiectum sit celeritate $\sqrt{b}$ secundum horizontalem AP, erit applicata PM, y, negativa ideoque habebuntur hae aequationes

$$gds^2 = 2vddy \quad \text{et} \quad dv = gdy - ds \quad \text{seu} \quad v = b + gy - s.$$

Unde prodibit haec aequatio

$$\frac{gds^2}{2ddy} = b + gy - s.$$

Corollarium 3.

906. Commodius autem hic casus tractibitur, si in aequatione differentiali tertii gradus ponatur $m = 0$; prodit enim $gd^3y = -\frac{2ddy^2}{ds}$ seu substitutionibus per p et q factis haec

$$\frac{gdq}{q} = -\frac{2dp}{\sqrt{(1+p^2)}},$$

cuius integralis est

$$glq = 2l\frac{\sqrt{(1+pp)} - p}{a}$$

seu observata homogeneitate [p. 389]

$$aq = (\sqrt{(1+p^2)} - p)^{\frac{2}{g}} = \frac{adp}{dx}.$$

Hinc habebitur

$$dx = \frac{adp}{(\sqrt{(1+pp)} - p)^{\frac{2}{g}}}.$$

Quae denuo integrata dat

$$2x = \frac{ga}{(2+g)(\sqrt{(1+pp)} - p)^{\frac{2+g}{g}}} + \frac{ga}{(2-g)(\sqrt{(1+pp)} - p)^{\frac{2-g}{g}}} + k.$$

Hincque reperitur $y = \int pdx$ et absoluta integratione

$$4y = \frac{ga}{(2+2g)(\sqrt{(1+pp)} - p)^{\frac{2+2g}{g}}} - \frac{ga}{(2-2g)(\sqrt{(1+pp)} - p)^{\frac{2-2g}{g}}} + i.$$

Patet igitue hanc curvam fore algebraicam, nisi sit g vel 1 vel 2.

Scholion.

907. Atque late ac haec nostra solutio patet Ioh. Bernoulli solutio proiectoriarum in medio resistente, quam dedit in Act. Lips. A. 1719 Mai, ubi etiam constructionem generalem pro his curvis dedit. Antequam autem hanc potentiae uniformis hypothesin relinquamus, problemata inversa solvemus, quibus determinabimus resistentiam, quae efficit, ut corpus in hac potentiae uniformis et deorsum tendentis hypothesi datam curvam describat. Haec enim materia tum a Neutono in Phil. Princ. tum a Ioh. Bernoullio in Act. Lips, A, 1713 pluribus est pertractata : ubi Viri acutissimi multa eximia notaverunt. [p. 390]

PROPOSITIO 111.

PROBLEMA.

908. *Posita potentiae absoluta uniformi et deorsum tendente, determinare resistentiam in singulis locis M* (Fig.84), *qua fiat, ut corpus datam curvam BAM describat.*

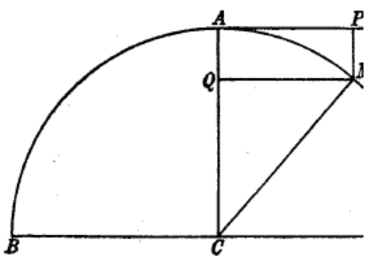


Fig. 84.

SOLUTIO.

Ponatur ut ante $AP = QM = x$, $PM = AQ = y$, et elementum arcus $AM = ds$. Deinde sit celeritas in $M = \sqrt{v}$, et resistantia in $M = R$. His igitur comparatis cum prop. 106 (870) fiet $P = g$ et y negative debet accipi; eritque

$$gds^2 = 2vddy$$

sumto dx constans (871) et $dv = gdy - Rds$ (870). Ex illa aequatione autem est

$$v = \frac{gds^2}{2ddy} \quad \text{ideoque} \quad dv = gdy - \frac{gds^2d^3y}{2ddy^2}.$$

Coniunctis igitur his aequationibus prodibit

$$R = \frac{gdsd^3y}{2ddy^2}.$$

Quare, cum curva sit data, ex eius aequatione reperietur finitus valor ipsius R, ideoque innotescit resistentia. Q.E.I.

Corollarium 1.

909. Erit igitur vis resistentiae in M ad vim sollicetantem g ut dsd^3y ad $2ddy^2$. Seu posito radio osculi in $M = r$ ob $r = \frac{d^3s}{dxddy}$ erit

$$\frac{dsd^3y}{2ddy^2} = \frac{3dsdy - dxdr}{2ds^2}.$$

Quare erit

$$g : R = 2ds^2 : 3dsdy - dxdr.$$

Corollarium 2.

910. Altitudo generans celeritatem in M, nempe v, ex ipsa curva determinatur; est enim [p. 391] $v = \frac{gd^2s}{2ddy}$. Seu introducto radio osculi r ob $\frac{d^2s}{ddy} = \frac{rdx}{ds}$ erit $v = \frac{grdx}{2ds}$.

Corollarium 3.

911. Si resistentia ponatur in duplicat celeritatum ratione, exponens vero resistentiae sumatur incognitus q, erit $R = \frac{gdsd^2y}{2ddy^2}$, invenietur medii resistentis exponens $q = \frac{dsddy}{d^3y}$.

Similique modo pro aliis medii resistentis hypothesibus inveniri potest medii resistentis exponens.

Corollarium 4.

912. Introducto ad q determinandum radio osculi r erit

$$\frac{d^3y}{ddy}=\frac{ddy(3dsdy-dxdr)}{ds^3}=\frac{3dsdy-dxdr}{rdx}.$$

Consequenter fiet

$$q=\frac{rdxds}{3dsdy-dxdr}.$$

Cognito igitur radio osculi r tam R quam q per differentialia primi gradus determinabuntur.

Corollarium 5.

912. Si curva AM fuerit parabola, cuius axis est verticalis AC, quia in ea est $d^3y=0$, prodibit quoque resistentia $R = 0$. Ex quo cognoscitur parabolam in vacuo describi posse a corpore uniformiter deorsum tracto, quemadmodum cuique satis constat. [p. 392]

Corollarium 6.

914. Si curva fuerit parabola quaecunque superioris ordinis, ita ut sit $a^{n-1}y=x^n$, erit

$$dy=\frac{nx^{n-1}dx}{a^{n-1}}$$

et

$$ds=\frac{dx\sqrt{(a^{2n-2}+n^2x^{2n-2})}}{a^{n-1}}.$$

Porroque ob dx constans

$$ddy=\frac{n(n-1)x^{n-2}dx^2}{a^{n-1}} \quad \text{et} \quad d^3y=\frac{n(n-1)(n-2)x^{n-3}dx^3}{a^{n-1}}.$$

Quare fiet

$$R=\frac{(n-2)g\sqrt{(a^{2n-2}+n^2x^{2n-2})}}{2n(n-1)x^{n-1}}.$$

Atque posita resistentia quadratis celeritatum proportionali erit exponens resistentiae

$$q=\frac{x\sqrt{(a^{2n-2}+n^2x^{2n-2})}}{(n-2)a^{n-1}}.$$

Scholion.

915. Alia exempla curvarum, quae loco AM assumi possunt, hic non adiungo; sed iis sequentes propositiones destino, cum diligentius examinari mereantur. Considerabo autem praecipue circulum et hyperbolam, cum hae curvae a Viris citatis maxime sint tractatae.

PROPOSITIO 112.

PROBLEMA.

916. *Posita vi absoluta g uniformi et perpetuo deosum trahente invenire resistantiam, quae faciat, ut corpus libere in peripheria circuli BAMD* (Fig.84) *moveatur.* [p. 393]

SOLUTIO.

Sit radius circuli $AC = a$, erit $x^2 = 2ay - y^2$ et radius osculi in $M = a = MC$. Quare ob $dr = 0$ erit

$$g : R = 2ds : 3dy.$$

Est vero $ds : dy = a : x = AC : QM$. Hanc ob rem erit $g : R = 2AC : 3QM$ seu $R = \frac{3g.QM}{2AC}$.

Altitudo vero generans celeritatem in M erit = $\frac{gadx}{2ds} = \frac{g.QC}{2}$. At si resistentia quadratis celeritatum proportionalis punatur, erit exponens resistentiae

$$q = \frac{adx}{3dy} = \frac{a \cdot QC}{3\,QM}.$$

Q.E.I.

Corollarium 1.

917. Dum corpus per arcum BA ascendit, ob QM tum existentem negativam erit quoque resistantia per BA negativa; seu motus corporis per BA a medio accelerabitur vi tangentiali $\frac{3g.QM}{2AC}$.

Corollarium 2.

918. Resistentia vero in puncto *A*, quia *QM* evanescit, erit nulla; corpus igitur in A tanquam in vacuo movebitur. In puncto vero *D* resistentia erit ad potentiam g in sesquialtera ratione. In *B* vero tantundem sursum a resistentia sollicitabitur.

Corollarium 3. [p. 394]

919. Cum igitur in *B* et *D* directio vis resistentiae cum directione potentiae *g* conveniat, corpus in *B* sursum urgibitur vi $\frac{1}{2}g$; in A deorsum trahetur a vi *g* et in *D* sursum iterum a vi $\frac{1}{2}g$.

Corollarium 4.

920. Quia $\frac{QM}{AC}$ exprimit sinum anguli ACM posito radio = 1, erit resistentia in M = $\frac{3}{2}g\,sin.ACM$ seu resistentia ubique est ut sinus anguli *ACM*, quo corpus ab *A* declinavit.

Corollarium 5.

921. Porro est $\frac{a.QC}{AM}$ tangens arcus *MD*. Quare posita resistentia quadratis celeritatum proportionali erit exponens resistentiae *q* aequalis tertiae parti cotangentis arcus *AM*.

Corollarium 6.

922. Cum altitudo debita celeritati in M sit = $\frac{g.QC}{2}$, erit celeritas in $A = \sqrt{\frac{g.AC}{2}}$; in *B* vero et *D* erit celeritas = 0. Quia igitur corpus in *B* revera sursum pellitur vi $\frac{g}{2}$, mirum non est corpus in *B* sursum moveri incipere.

Corollarium 7.

923. Corpus igitur tam in quadrante *BA* quam *AD* in locis aeque dissitis a puncto *A* aequales habebit celeritates. [p. 395] Infra vero horizontalem *BD* corpus pervenire non potest ob *QC* negativam, quo casu celeritas fit imaginaria.

Scholion.

924. Quorsum autem corpus, cum in *D* pervenerit, sit progressurum, facile ex allatis colligi potest. Nam cum celeritas in *D* sit = 0 et corpus in *D* sursum urgeatur vi $\frac{g}{2}$, perspicuum est corpus iterum sursum moveri debere. Eodem autem modo iterum per arcum *DMA* ascendet, quo initio per *BA* ascendit, quia tam in *D* quam in *B* vi = $\frac{g}{2}$ sursum urgetur. Hoc vero mirabile in hoc motu occurrit, quod corpus in *B* quiescens sursum pellatur et nihilominus in curva moveatur, etiamsi nulla vis adesse videatur, quae corporis directionem, quam in *B* sursum accipit, posset inflectere. Sed ad hoc respondeo vis in *B* directinem non perfecte sursum tendere, sed infinite parum a vera verticali aberrare, id quod sufficit ad motum abliquum producendum. Directio enim vis resistentiae in *B* seu potius vis accelerantis est elementum peripheriae circuli in *B* insistens, quod non perfecte est recta verticalis, sed infinite parum inclinata ad *BC*.

Ceterum haec nostra egregie conveniunt cum iis, quae Cel. Bernoulli dedit in Act. Lips. A. 1713 et quae sunt in Neutoni Princ. Phil. posterioribus editionibus. In prima enim editione error in [p. 396] solutionem irrepsit, quo inductus fuit, ut rationem *g* ad *R* statueret aequalem rationi *AC* ad *QM*. Monitus autem de hoc a Nicolao Bernoullio in sequentibus editionibus hunc lapsum emendavit.