CHAPTER FIVE (*Part e*).
CONCERNING THE CURVILINEAR MOTION OF FREE POINTS ACTED ON BY ABSOLUTE FORCES OF ANY KIND.

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PROPOSITION 98. [p. 339]

THEOREM.

802. *There are three principal forces which can be put in place, and into which other forces must be resolved, in order that a body can move on a curve that does not exist in a plane; these individual forces are normal to each other; and of these one is the tangential force, and the remaining two are normal to that force, of which one lies in the given plane, and the direction of the other is normal to this plane, and nothing remains of the original forces to change the actions of these forces.*

DEMONSTRATION.

With the plane *APQ* assumed fixed [in space] (Fig. 75) and relative to that axis *AP,* an element *Mm* is described by a body. From the points *M* and *m* the perpendiculars *MQ* and *mq* are sent to the fixed plane and from the points *Q* and *q,* perpendiculars *QP* and *qp* are sent to the axis *AP*. Now, if the body is not acted on by any force, then it progresses along the line *Mm* produced with the speed that is has in *Mm* ; therefore in an equal small interval of time, equal to the time in which it traversed *Mm*, it arrives at *n*, with the element *mn* described equal to and in the same direction as that put for the element *Mm*. Whereby by also sending a perpendicular *nr* from *n* to the plane *APQ*, then the elements *Qq* and *qr* are also equal to each other and placed in the same direction ; because of this, the perpendicular *rπ* sent from *r* to the axis *AP* cuts off the element $p\pi = Pp$.

Let the speed in which the first element *Mm* is described, correspond to the height *v*, and in the first place the tangential force is considered, which has a direction along *mn* and the whole force is taken up with changing the speed. This tangential force *T* is put in place with the existing force of gravity equal to 1, and we have [p. 340]

$$dv = T.Mm,$$

and the element *mn* is completed with a speed corresponding to the height $v + dv$.

Following this, in the plane *Mr* there is considered a force having a direction *ms* normal to the direction *Mm* of the body. This therefore has the effect that the body can deflect from *mn* and progress along the element *mν* placed in the same plane as *Mr*. Let this normal force be equal to *N*, and since the radius of osculation of the elements *Mm* and $m\nu$, with the perpendicular $\nu\varepsilon$ sent along from ν to *mn* is equal to $\frac{m\nu^2}{\nu\varepsilon}$ [In triangle $m\varepsilon\nu$, we can set $ds = Rd\theta$, where $ds = m\nu$ and $d\theta = \varepsilon\nu / m\nu$, giving the required result for the radius R], hence

$$\frac{2\text{v}.\nu\varepsilon}{m\nu^2}=N$$

(561). Truly $\frac{\nu\varepsilon}{m\nu}$ is the sine of the angle $nm\nu$. On account of which,

$$2\text{v}\sin.nm\nu = N.mv = N.Mm,$$

and thus

$$\sin.nmv=\frac{N.Mm}{2\text{v}}.$$

[Note that Fig. 75 is drawn in perspective, so that the elements $Mm = mn = m\nu$, and the force normal to *mn* acts in this manner as previously shown for centripetal forces in a plane.]

The third force is normal to each of those set up on *mn* and *ms*, thus in order that its

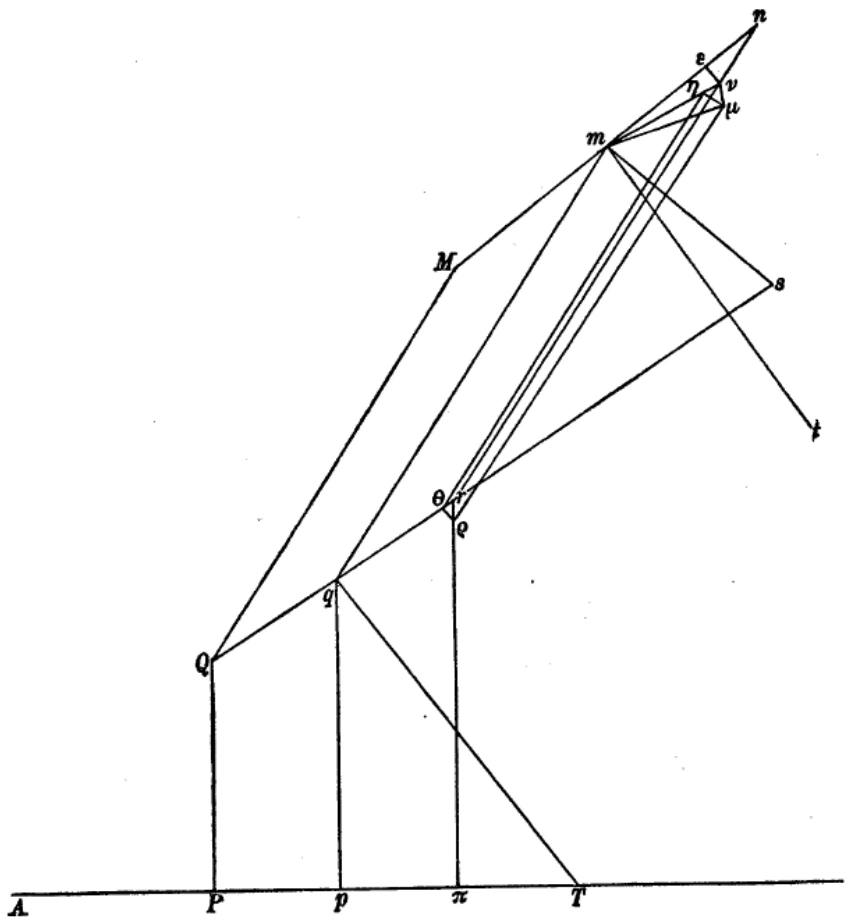


Fig. 75.

direction is along the normal *mt* to the plane *Mr*. This force neither impedes the actions of the preceding forces, nor is it allowed itself to be impeded by their actions. Therefore the whole effect of this force is to draw the body away from the plane *Mr*; the body is drawn from that plane from ν to μ, thus so that the plane $\nu m\mu$ is normal to the plane *Mr*, and the angle $\nu m\mu$ is the result of this force. Therefore from the same argument that we put in place for the preceding force, by evaluating the force in the same way, if this force is M, then it is given by :

$$\sin.vm\mu = \frac{M.Mm}{2v}.$$

Therefore these three forces likewise have the effect that the body, after it has described the element *Mm* moves to the element $m\mu$, with an increase in speed clearly corresponding to the height $v + T.Mm.$ Moreover, any other forces acting on the body can also be resolved in this way into the forces, the directions of which lie along the directions *mn, ms, mt*. As we have determined the effect of these forces on the body, so [p. 341] likewise the effect of any forces can also become known. Q.E.D.

Corollary 1.

803. By taking $v\mu$ in the plane $nr\pi$, and by sending a perpendicular $\mu\rho$ from μ to the plane APQ, $\mu\rho$ is parallel to *rn* itself. Therefore the three coordinates for the points *M, m* and μ are *AP, PQ, QM; Ap*, *pq*, *qm*, and $A\pi$, $\pi\rho$, $\rho\mu$.

Corollary 2.

804. Whereby if from μ the perpendicular $\mu\eta$ is sent to *mv*, it is normal to the plane *Mr*; and likewise in a similar manner $\rho\theta$, which is perpendicular to *qr*, is normal to the same plane. On account of which, as ρ and μ on the line $\rho\mu$ put parallel to this plane, then we have $\rho\theta = \mu\eta$ and $\theta\eta = \rho\mu$.

Corollary 3.

805. If the normal *qT* is drawn to *Qq* in the fixed plane *APQ*, this line *qT* is normal to the plane *Mr*. Therefore since *mt* is also normal to the same plane, then *mt* is parallel to *qT*; and between these the distance is the height *mq*.

Corollary 4.

806. The three coordinates are called $AP = x, PQ = y$ and $QM = z$.
And we have :

$$Pp = p\pi = dx,\ pq = y + dy,\ qm = z + dz$$

and

$$\pi\rho = y + 2dy + ddy \text{ and } \rho\mu = z + 2dz + ddz = \theta\eta.$$

But

$$Qq = \sqrt{(dx^2 + dy^2)} = qr,$$

$$q\rho = \sqrt{\left(dx^2 + (dy + ddy)^2\right)} = q\theta = \sqrt{(dx^2 + dy^2)} + \frac{dyddy}{\sqrt{(dx^2 + dy^2)}};$$

[from a binomial expansion on extracting the equivalent of *qr*]
and hence:

$$r\theta = \frac{-dyddy}{\sqrt{(dx^2 + dy^2)}}.$$

Again we have [p. 342]:

$\pi r = y + 2dy$ and $rn = z + 2dz$.

Then

$$Mm = \sqrt{(dx^2 + dy^2 + dz^2)} = mn$$

and

$$m\mu = \sqrt{\left(dx^2 + (dy + ddy)^2 + (dz + ddz)^2\right)} = \sqrt{(dx^2 + dy^2 + dz^2)} + \frac{dyddy + dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

[Thus, the length $m\mu$ is compounded from the lengths $p\pi$ or dx in the x-direction, $\pi\rho - pq = y + 2dy + ddy - y - dy = dy + ddy$ in the y-direction, and $\rho\mu - qm = z + 2dz + ddz - z - dz = dz + ddz$, in the z-direction; from which the length corresponding to $m\mu$ is extracted by a binomial expansion.]

Corollary 5.

807. Since $mq, \theta\eta$ and $r\nu$ are parallel to each other, in the same plane and terminated by the lines qr and $m\nu$, then

$$\theta\eta - qm : q\theta = r\nu - mq : qr.$$

[The gradient of the line $m\eta$ is the same as the gradient of the line $m\nu$ in the plane $m\nu rq$, as are $q\theta$ and qr]
For it is the case that :

$$\theta\eta - qm = dz + ddz, \quad q\theta = \sqrt{(dx^2 + dy^2)} + \frac{dyddy}{\sqrt{(dx^2 + dy^2)}}$$

and

$$qr = \sqrt{(dx^2 + dy^2)}.$$

Whereby

$$r\nu - mq = \frac{(dz + ddz)(dx^2 + dy^2)}{dx^2 + dy^2 + dyddy}$$

and hence

$$n\nu = rn - r\nu = -ddz + \frac{dydzddy}{dx^2 + dy^2}.$$

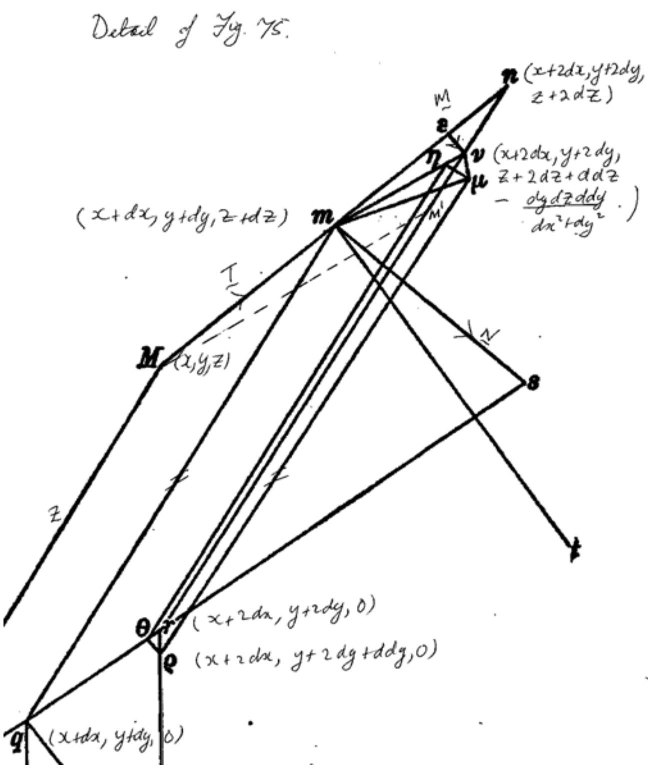
Thus, it is found that :

$$\sin.\ nm\nu = \frac{dydzddy - dx^2ddz - dy^2ddz}{(dx^2 + dy^2 + dz^2)\sqrt{(dx^2 + dy^2)}}.$$

[We apply the sine rule to the triangle $nm\nu$: it is easily found that the sine of the angle $mn\nu$ in this triangle can be found from the right-angled triangle with hypotenuse *Mn* in the plane *Mr* with base *MM'* parallel to *Qr,* to be given by

$$sin.mn\nu = \frac{\sqrt{dx^2+dy^2}}{\sqrt{dx^2+dy^2+dz^2}},$$

while the length $m\nu$ is equal to *mn* = $\sqrt{dx^2+dy^2+dz^2}$, where sums of powers of higher orders are ignored; from these on applying the sine rule, the result quoted emerges. The diagram here shows the coordinates of some of the points, and may be of some assistance to you if you want to establish the result for yourself.]



Corollary 6.

808. Since $r\rho = -ddy$ and $Qq : Pp = r\rho : \rho\theta$, it follows that

$$\theta\varrho = -\frac{dxddy}{\sqrt{(dx^2+dy^2)}} = \mu\eta.$$

On this account we have :

$$\text{sin. } \nu m\mu = -\frac{dxddy}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}.$$

[In triangle $\eta\mu m$, which projects normal to the plane *Mr*, we have from above,

$$m\mu = \sqrt{\left(dx^2+(dy+ddy)^2+(dz+ddz)^2\right)} = \sqrt{(dx^2+dy^2+dz^2)} + \frac{dyddy+dzddz}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

On using the sine rule, and noting that both the large angles are essentially right, we have, $sin.\nu m\mu/\nu\mu = 1/m\mu$, $sin.\nu m\mu = \nu\mu/m\mu$, which gives the result quoted on neglecting the higher order terms in the denominator.]

Corollary 7.

809. Therefore from the three given forces *T, N* and *M* that the body is acted upon, there arises the three following equations :

$$dv = T\sqrt{(dx^2 + dy^2 + dz^2)},$$

$$2vdydzddy - 2vddz(dx^2 + dy^2) = N(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}\sqrt{(dx^2 + dy^2)}$$

and

$$-2vdxddy = M(dx^2 + dy^2 + dz^2)\sqrt{(dx^2 + dy^2)},$$

from which the speed of the body at individual points as well as the curve itself becomes known. [p. 343]

Corollary 8.

810. The two latter equations joined together with *v* eliminated give this equation :

$$\frac{ddz(dx^2 + dy^2)}{dxddy} - \frac{dydz}{dx} = \frac{N\sqrt{(dx^2 + dy^2 + dz^2)}}{M}.$$

For which, the nature of the surface in which the curve described by the body lies, can be assumed to be expressed by the equation,.

Scholium.

811. Therefore from this proposition we can deduce the first rules, from which the motion of a body acted on can be deduced, in order that it does not move in the same plane. Indeed we have shown that all the forces can be resolved in terms of three, the effects of which we have determined ; and thus, whatever forces are proposed to be acting, whatever motion they produce on the body can become known. Indeed it is apparent, if the [second normal] force *M* is not present, then the motion of the body is entirely in its own plane, which does not concern us here. But if the tangential force *T* vanishes with the forces *M* and *N* left, then the body describes a non planar orbit, but yet is still carried around uniformly. From which the position of the orbit is generally known, and it is necessary to find the intersection of this inclined plane, in which the elements *Mm* and $m\mu$ are present, with the plane APQ. [The task of the next proposition.]

PROPOSITION 99.

PROBLEM.

812. *To determine the inclination of the plane, in which the elements Mm and* $m\mu$ (Fig. 76) *described by a body have been placed, relative to the fixed plane APQ and to find the line of intersection of the two planes.* [p. 344]

SOLUTION.

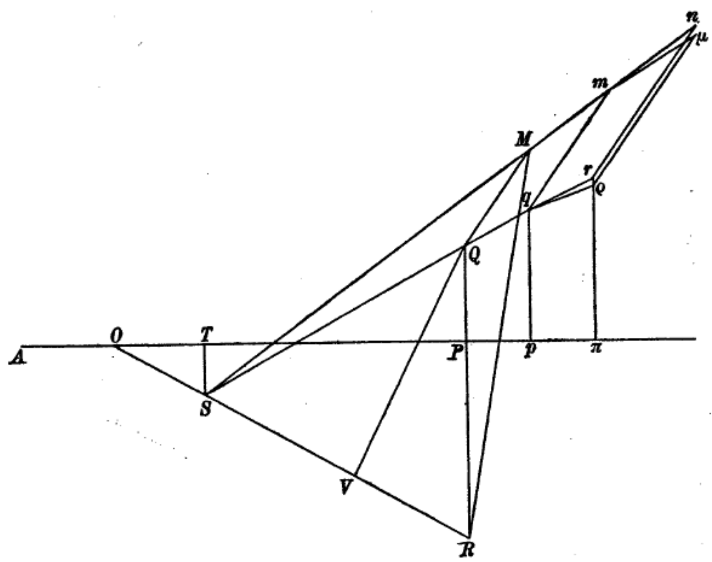


Fig. 76.

In the plane, of which we seek the inclination, the three points *M, m* and μ are given; therefore in this plane a certain line is placed of which the points pass through both planes. [This is the element *mM* ; the plane *APQ* can be considered as horizontal while the eye looks down and sees the plane containing QM, the *z* coordinate, rising vertically, as in the previous proposition. The point μ arises as previously due to the actions of the forces *M* and *N*, and the lines *mM* and $m\mu$ define a plane, which is tangential to the surface on which the body moves.] Whereby if the line *mM* is produced, then it meets the line *qQ* [both in the plane *APQ*] produced in *S*, while the point *S* then lies in the plane $Mm\mu$ as well as in the fixed plane APQ; therefore the line formed by the intersection of the two planes passes through *S*. Therefore with the lengths remaining as before: $AP = x$, $PQ = y$ and $QM = z$ and with the elements of the abscissa *Pp* and $p\pi$ equal to each other, then $qm - QM : Qq = QM : QS$ and hence :

$$QS = \frac{z\sqrt{(dx^2 + dy^2)}}{dz}.$$

For the position of the line *QS* is known from the angle *PQS*, the sine of which is equal to

$$\frac{dx}{\sqrt{(dx^2 + dy^2)}}.$$

In addition, the point n is also situated in the plane $Mm\mu$; on account of this the line passing through n and μ or drawn parallel to this passes through M extant in the same plane. Moreover this line crosses the plane APQ in the point R of the line QP produced, and QR is known from this ratio $rn - \rho\mu : r\rho = QM : QR$; hence we have

$$QR = \frac{zddy}{ddz} \text{ and thus } PR = \frac{zddy}{ddz} - y = \frac{zddy - yddz}{ddz}.$$

[For $rn - r\mu = n\mu = -ddz$; $r\rho = -ddy$, $QM = z$, giving QR as shown, etc.]
For $Qq : Pp = QS : PT$ with ST drawn perpendicular to AP. From which there arises

$$PT = \frac{zdx}{dz}.$$

Furthermore, $Pp : (pq - PQ) = PT : (PQ + ST)$ and thus

$$PQ + ST = \frac{zdy}{dz} \text{ and } ST = \frac{zdy}{dz} - y.$$

The line RS produced cuts the axis AP in O and hence $PR - ST : PT = PR : PO$, from which it is found that:

$$PO = \frac{zdxddy - ydxddz}{dzddy - dyddz}$$

and [p. 345]

$$AO = \frac{xdzddy - xdyddz + ydxddz - zdxddy}{dzddy - dyddz}.$$

Again from these, the tangent of the angle POR is equal to

$$\frac{PR}{PO} = \frac{dzddy - dyddz}{dxddz};$$

[Note that the numerator of PO can be factored to give $dx(zddy - yddz)$, which cancels with the numerator of PR above to give the simpler expression shown];
thus the position of the intersection RO of the plane $Mm\mu$ with the fixed plane APQ is known.[On the annotated diagram, the distance AO along the fixed axis AP is known, and the angle θ to AP has been found.] Moreover, the mutual inclination of these planes is found by sending the perpendicular QV from Q to RS ; for then the tangent of the angle of inclination is equal to $\frac{MQ}{QV}$. Indeed [as the right triangles RVQ and RPO are similar] this becomes :

$$QV = \frac{PO \cdot QR}{RO} = \frac{zdxddy}{\sqrt{(dx^2ddz^2 + (dzddy - dyddz)^2)}}$$

and thus

$$\frac{MQ}{QV} = \frac{\sqrt{(dx^2ddz^2 + (dzddy - dyddz)^2)}}{dxddy},$$

from which the angle of the inclination between the two planes $Mm\mu$ and APQ is determined. Q.E.I.

Corollary 1.

813. If the angle POR is always the same, then

$$\alpha dxddz + dyddz = dzddy$$

and the tangent of the angle is equal to α. This equation on integration gives

$$\alpha dx + dy + \beta dz = 0 \text{ and } \alpha x + y + \beta z = f.$$

[Note that β is defined in the next cor.] From which equation it is known that the whole orbit described by the body lies in the same plane in this case.

Corollary 2.

814. If indeed it is the case that $\alpha x + y + \beta z = f,$ then

$$\alpha dx + dy + \beta dz = 0 \text{ and } ddy + \beta ddz = 0.$$

Hence we have :

$$AO = x + \frac{ydx + \beta zdx}{-dy - \beta dz},$$

[on substituting $\beta = -\frac{ddy}{ddz}$ into the equation for *PO* above, and simplifying,]
and because $-dy - \beta dz = \alpha dx,$ then

$$AO = x + \frac{y}{\alpha} + \frac{\beta z}{\alpha} = \frac{f}{\alpha}$$

and likewise *AO* is constant.

Corollary 3. [p. 346]

815. Again with the angle POR remaining constant or $\alpha x + y + \beta z = f,$ then the tangent of the angle of inclination of the planes $Mm\mu$ and APQ is equal to :

$$\frac{\sqrt{(dx^2 + (-dy - \beta dz)^2)}}{-\beta dx} = \frac{\sqrt{(1 + \alpha^2)}}{-\beta}.$$

Whereby this angle itself is constant.
[This is readily seen on substituting $\beta = -\frac{ddy}{ddz}$ into

$$\frac{MQ}{QV} = \frac{\sqrt{(dx^2 ddz^2 + (dzddy - dyddz)^2)}}{dxddy},$$

and using the result of Cor. 2.]

Corollary 4.

816. For neither can the point of intersection O be put as invariable, unless likewise the orbit described by the body becomes a plane. For let $AO = f$ and putting $x - f = t$ [$= OP$] and $dx = dt,$ then: [from the expression

$$PO = \frac{zdxddy - ydxddz}{dzddy - dyddz}$$

set equal to t, we have]

$$tdzddy - tdyddz = zdtddy - ydtddz$$

and hence

$$\frac{ddy}{tdy - ydt} = \frac{ddz}{tdz - zdt}.$$

This is multiplied by t, from which it is found that :

$$\frac{tddy}{tdy - ydt} = \frac{tddz}{tdz - zdt}.$$

Which equation can be integrated, in account of dt being constant; for it becomes

$$tdy - ydt = \alpha tdz - \alpha zdt.$$

This divided by tt and integrated gives :

$$\frac{y}{t} = \frac{\alpha z}{t} + \beta \text{ or } y = az + \beta x - \beta f.$$

Which is seen to be a plane surface. [The first result follows by setting $\frac{tddy}{tdy-ydt} = \frac{d(tdy-ydt)}{tdy-ydt}$ and likewise $\frac{tddz}{tdz-zdt} = \frac{d(tdz-zdt)}{tdz-zdt}$ and noting that $ddt = 0$; while the second case follows directly from $\frac{tdy-ydt}{t^2} = \frac{\alpha(tdz-zdt)}{t^2}$ on integrating by parts, where integrals cancel.]

Corollary 5.

817. But if the tangent of the angle of inclination of the planes $Mm\mu$ and APQ is constant, an equation of the kind $\alpha x + y + \beta z = f$ is not produced. And elsewhere it has been shown that the orbit described by the body is then not by necessity a plane.

Corollary 6.

818. Whereby, with the curve described by the body not being planar, then neither the point O nor the angle POR can be take as invariable. Moreover if these are variables, then no more can the angle between the planes $Mm\mu$ and APQ be considered as constant. [p. 347]

Corollary 7.

819. The line of intersection *RO*, which is called the line of the nodes in astronomy, if it does not have a constant position, turns about the point *S*. For the element *mMS* has been put in the plane of the elements *Mm* from the preceding argument. Whereby the intersection of *RO* and the preceding line cross over each other at *S*.

Corollary 8.

820. Therefore this point *S* is at that place, where [$AP - TP =$]

$$AT = \frac{xdz - zdx}{dz} \text{ and } ST = \frac{zdy - ydz}{dz}.$$

From which the position of the point *S* is known.

Corollary 9.

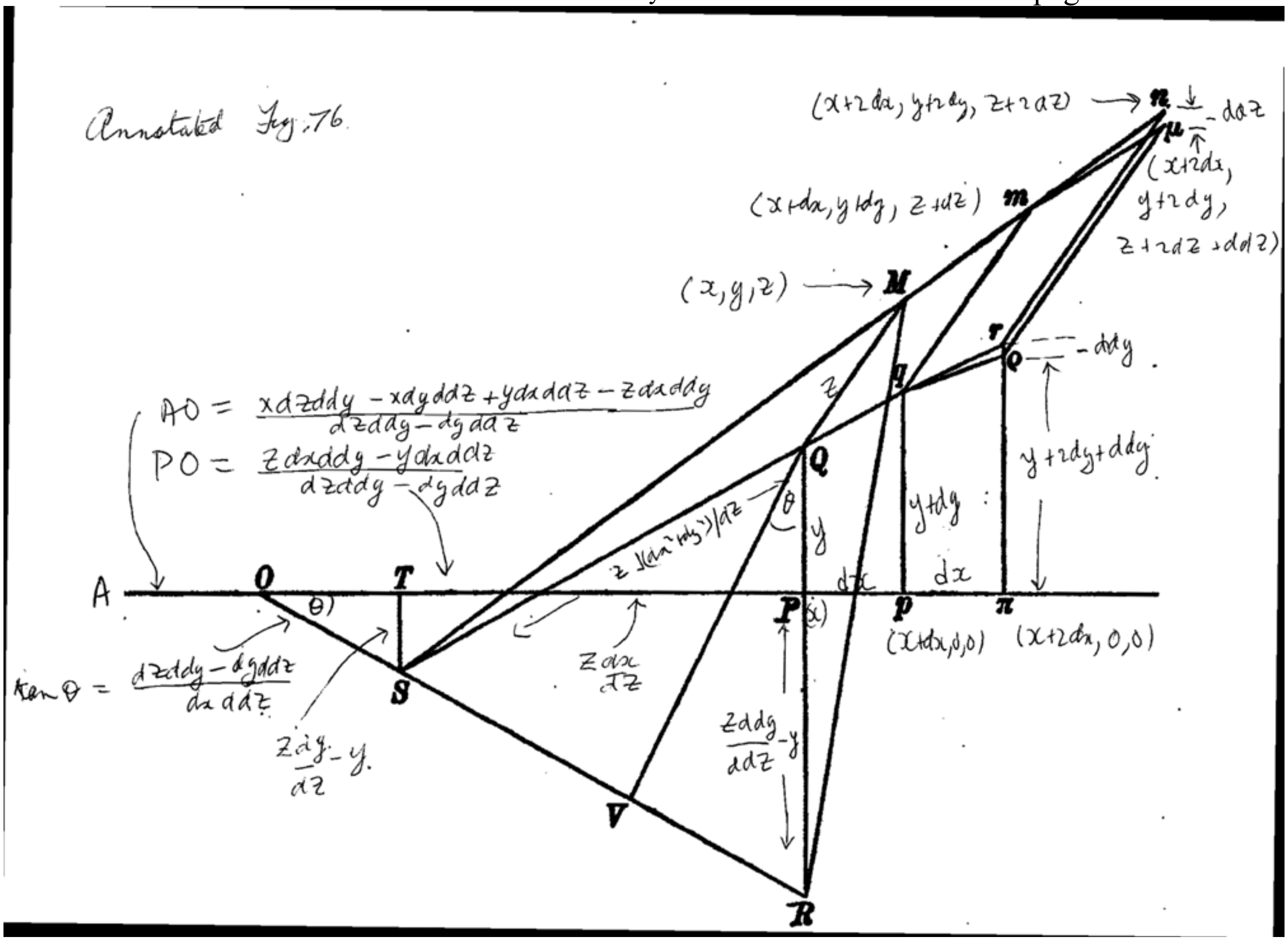
821. If the point S is put invariable, then

$$xdz - zdx = adz \text{ and } zdy - ydz = bdz,$$

hence it is found that $x - a = \alpha z$ and $y - b = \beta z$. Therefore in this case the orbit described by the body is not only planar, but also it is a straight line, since the projection of $Qq\rho$ gives a straight line in the x-y plane, and since $y - b = \beta z$, $Mm\mu$ is also a straight line in the y-z plane.

Scholion.

822. Now from the principles established, which are concerned with the non planar motion of a body on a surface, the description itself can be divided into two parts as before with regard to motion in a plane. In the first of these we instruct how to find the curve described by a body from given forces, and in the second truly it is shown, if the curve is given which the body describes, what kind of forces are required to do this. Here only the curve itself needs to be given, or likewise also the speed of the body at individual points. [p. 348]


Annotated Fig. 76.
(x+2dx, y+2dy, z+2dz)
n
μ
-ddz
(x+2dx, y+2dy, z+2dz+ddz)
(x+dx, y+dy, z+dz)
m
(x, y, z)
M
r
ρ
-ddy
q
Z
AO = xdzddy - xdyddz + ydxddz - zdxddy / dzddy - dyddz
PO = zdxddy - ydxddz / dzddy - dyddz
Q
θ
y
y+dy
y+2dy+ddy
dx
dx
A
O
θ
T
P
p
π
(x+dx,0,0)
(x+2dx, 0,0)
tan θ = dzddy - dyddz / dxddz
S
zdx / dz
zdy / dz - y
zddy / ddz - y
V
R

PROPOSITION 100.

PROBLEM.

823. *If a body is acted on by three forces, of which the directions Mf, Mg and MQ* (Fig. 77) *are parallel to the three coordinate axes AP, PQ and QM, to determine the motion of the body and the orbit in which it moves.*

SOLUTION.

Since *Mf* and *Mg* are parallel to *AP* and *PQ*, the plane *fMg* is parallel to the plane *APQ*. In this plane, *Mi* is drawn parallel to the element *Qq*; and this line *Mi* is also placed in the [vertical] plane *Mq*. A perpendicular *Qd* is sent from *Q* to the element *mM* produced; and from *f* and *g* the perpendiculars *fi* and *gk* are sent to *Mi*. Then from *i* and *k* the perpendiculars *ib* and *kc* fall on *Md*. Moreover *fi* and *gk* are perpendicular to the plane *Mq*, since the plane *fMg* is normal to the plane *Mq*. Now with *AP = x, PQ = y* and *QM = z* remaining as before, let the force pulling the body along *Mf* be equal to *P*, the force which pulls along the body along *Mg* is equal to *Q,* and the force pulling along *MQ* is equal to *R*. Therefore these forces, in order that their effect can become known, must be resolved into forces acting along the tangent *Mm*, placed along the normal to *Mm* in the plane *Mq*, and normal to the plane *Mq*.

Since the angle *Mfi* =*Qqp* it is the case that

$$\sqrt{(dx^2+dy^2)} : dy = P : \frac{Pdy}{\sqrt{(dx^2+dy^2)}}$$

[p. 349] and $\frac{Pdy}{\sqrt{(dx^2+dy^2)}}$ expresses the force arising from *P* acting along *if* , and if *P* alone acts, then by (802) :

$$M = -\frac{Pdy}{\sqrt{(dx^2+dy^2)}}.$$

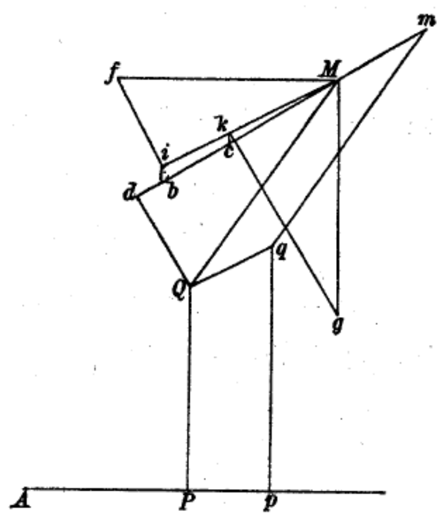


Fig. 77.

Following this, the force pulling along *Mi* is equal to :

$$\frac{Pdx}{\sqrt{(dx^2+dy^2)}}.$$

Again this force can be resolved into a force pulling along *bi* equal to :

$$\frac{Pdxdz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

and a force pulling along *Mb* equal to :

$$\frac{Pdx}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Therefore from P we have the contributions :

$$N = -\frac{Pdxdz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}} \text{ and } T = -\frac{Pdx}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

In a like manner the force Q, the direction of which is Mg, can be resolved into the force acting along kg equal to :

$$\frac{Qdx}{\sqrt{(dx^2+dy^2)}}$$

and the force along Mk equal to :

$$\frac{Qdy}{\sqrt{(dx^2+dy^2)}}.$$

This is finally resolved into the force along ck equal to :

$$\frac{Qdydz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

and the force along Mc equal to :

$$\frac{Qdy}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Whereby, if this force alone acts, we have :

$$T = -\frac{Qdy}{\sqrt{(dx^2+dy^2+dz^2)}},\ N = -\frac{Qdydz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}} \text{ and } M = \frac{Qdx}{\sqrt{(dx^2+dy^2)}}.$$

Finally the force R, having the direction MQ, is resolved into the force along Md equal to :

$$\frac{Rdz}{\sqrt{(dx^2+dy^2+dz^2)}}$$

and the force along dQ equal to :

$$\frac{R\sqrt{(dx^2+dy^2)}}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Therefore from the force R there becomes :

$$T = -\frac{Rdz}{\sqrt{(dx^2+dy^2+dz^2)}} \text{ and } N = \frac{R\sqrt{(dx^2+dy^2)}}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Therefore from these three forces P, Q, and R acting at the same time, the tangential force arising from all :

$$T = \frac{-Pdx - Qdy - Rdz}{\sqrt{(dx^2+dy^2+dz^2)}},$$

the normal force in the plane Mq which is in place :

$$N = \frac{-Pdxdz - Qdydz + Rdx^2 + Rdy^2}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

and the second normal force :

$$M = \frac{-Pdy + Qdx}{\sqrt{(dx^2+dy^2)}}.$$

With these values *T, N* and *M* substituted in place in the equations (809), the three following equations are produced :

$$dv = -Pdx - Qdy - Rdz,$$

$$\frac{2vdydzddy - 2vddz(dx^2 + dy^2)}{dx^2 + dy^2 + dz^2} = -Pdxdz - Qdydz + R(dx^2 + dy^2)$$

and

$$\frac{2vdxddy}{dx^2 + dy^2 + dz^2} = Pdy - Qdx.$$

Which equations determine the equations of motion of the body. Q.E.I. [p. 350]

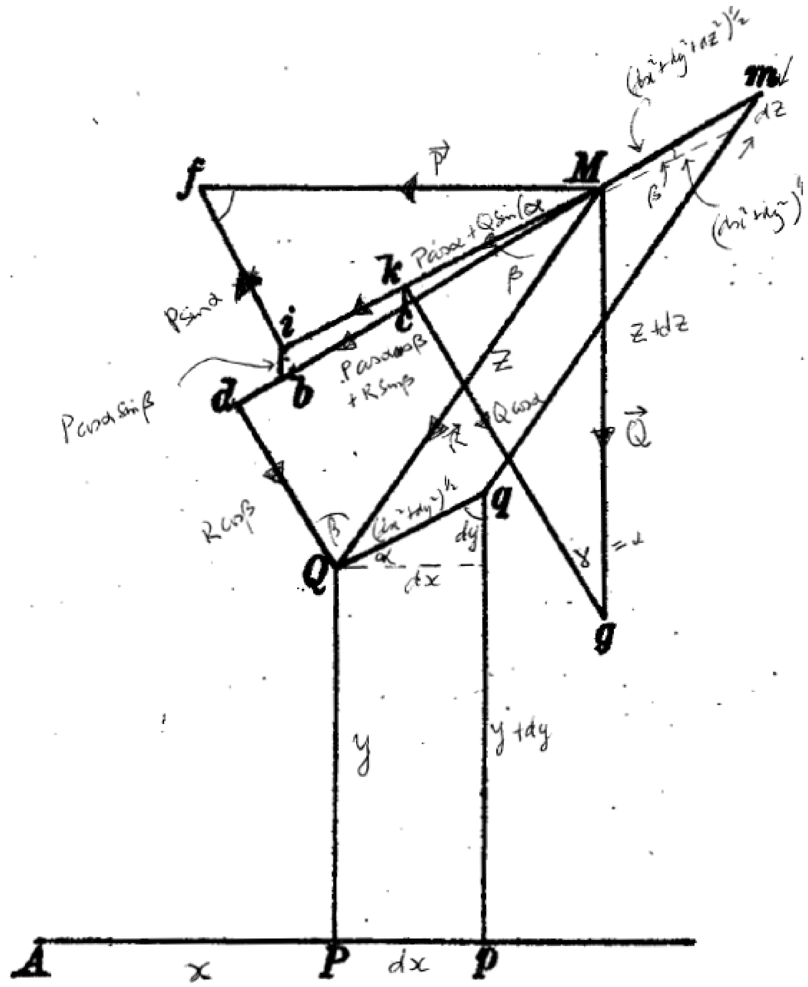


Fig. 77.

Corollary 1.

824. These two last equations give that ratio:

$$dydzddy - ddz(dx^2 + dy^2) : dxddy = -Pdxdz - Qdydz + R(dx^2 + dy^2) : Pdy - Qdx.$$

From which it is found :

$$ddy : ddz = Pdy - Qdx : Pdz - Rdx.$$

Corollary 2.

825. Therefore the plane $Mm\mu$ (Fig. 76), in which the elements Mm and $m\mu$ are present, can be defined in this manner. Let

$$AO = x + \frac{-Pydz + Pzdy - Qzdx + Rydx}{Qdz - Rdy} = \frac{-Pydz + Pzdy + Qxdz - Qzdx + Rydx - Rxdy}{Qdz - Rdy}$$

and the tangent of the angle $POR = \frac{-Qdz+Rdy}{Pdz-Rdx}$

and the tangent of the angle, which the plane $Mm\mu$ makes with the plane APQ,

$$= \frac{\sqrt{((Pdz - Rdx)^2 + (Qdz - Rdy)^2)}}{Pdy - Qdx}$$

Corollary 3.

826. If the force P vanishes, there is found :

$$Q = -\frac{2vddy}{dx^2 + dy^2 + dz^2} \quad \text{and} \quad R = -\frac{dv}{dz} + \frac{2vdyddy}{dz(dx^2 + dy^2 + dz^2)}$$

from the two equations found above. And from the third there is produced :

$$\frac{dv}{2v} = \frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2},$$

the integral of which is

$$vdx^2 = a(dx^2 + dy^2 + dz^2) \text{ or } dx\sqrt{v} = \sqrt{a}(dx^2 + dy^2 + dz^2).$$

Corollary 4.

827. Therefore from this hypothesis the time, or $\int \frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}$,

$$= \int \frac{dx}{\sqrt{a}} = \frac{x}{\sqrt{a}}.$$

From which it is understood that the motion of the body progressing along parallel to the axis *AP* is uniform. [p. 351]

Corollary 5.

828. Again from the same hypothesis, we have :

$$Q = -\frac{2addy}{dx^2} \quad \text{and} \quad R = -\frac{2addz}{dx^2}$$

since $ddy : ddz = Q : R$ (824). From which equations the curve itself described by the body can be determined.

Scholium.

829. From the resolution of forces into three forces that we have considered in this proposition, it is easily seen how all forces which can be devised can generally be reduced. Whereby, with these forces given, it is not difficult to find the curve described by the body, also for whatever cases are proposed this theorem has the maximum usefulness.

PROPOSITIO 101.

PROBLEM.

830. *If a body is always acted on by a force towards the axis AP along the perpendicular MP dropped to the body* (Fig. 78)*, then it is required to find the motion of the body.*

SOLUTION.

With the coordinates drawn as before $AP = \mathbf{x}$, $PQ = \mathbf{y}$ and $QM = \mathbf{z}$ then $MP = \sqrt{(y^2 + z^2)}$. Let the force acting along *MP* be equal to *V* and that is resolved into two components acting along *MQ* and *Mg*, where *Mg* is parallel and equal to *PQ* [p. 352]. Therefore the force acting along MQ is equal to $\frac{Vz}{\sqrt{(y^2+z^2)}}$. With these compared with the preceding proposition, we have

$P = 0, Q = \frac{Vy}{\sqrt{(y^2+z^2)}}$ and $R = \frac{Vz}{\sqrt{(y^2+z^2)}}$.

Whereby we have :

$ddy : ddz = Q : R = y : z$ (828) and

$yddz = zddy$ or $yddz - zddy = 0,$

the integral of this equation is :

$ydz - zdy = bdx.$ [on integrating by parts, and recalling that $ddx = 0$]

Again we have $\frac{Vy}{\sqrt{(y^2+z^2)}} = -\frac{2addy}{dx^2}$, from (828).

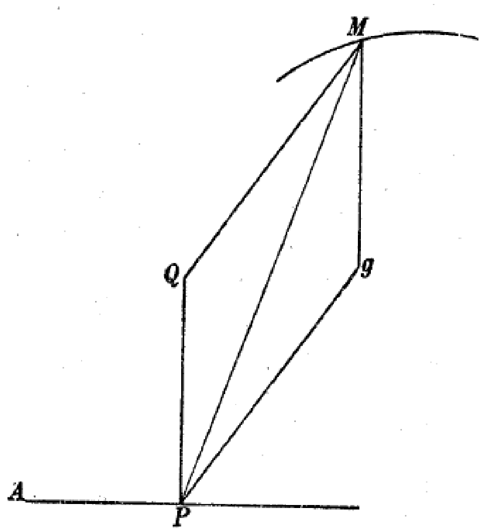


Fig. 78.

Which equations can be solved together to determine the curve described by the body.
Moreover the speed of the body is given by the equation :

$$vdx^2 = a(dx^2 + dy^2 + dz^2), \text{ from (826).}$$

or the speed itself is equal to $\frac{\sqrt{a(dx^2+dy^2+dz^2)}}{dx}$.

Q.E.I.

Corollary 1.

831. Putting $dx = pdy$, on account of constant dx :

$$0 = pddy + dpdy \text{ or } ddy = -\frac{dpdy}{p}.$$

With these substituted these equations are obtained in order that the curve described can become known :

$$ydz - zdy = bpdy \text{ and } \frac{Vy}{\sqrt{(y^2+z^2)}} = \frac{2adp}{p^3dy}.$$

Corollary 2.

832. If again on putting $z = qy$, these equations are transformed into :

$$y^2dq = bpdy \text{ and } \frac{V}{\sqrt{(1+qq)}} = \frac{2adp}{p^3dy}.$$

Which also contain three variables.

Scholium 1.

833. It is appropriate to use examples in order that this can be explained most clearly. On account of which we produce some, in which the force V is made to depend on the distance MP, and that we put proportional to some power of the distance MP, for which it is possible to compare this motion with the motion in a plane [p. 353] arising from a centripetal force in proportion to a certain power of the distance. There is a great similarity between these cases [and those previously considered], as if the centre of force in the plane has its place taken by the axis of the force to which the body is attracted. And if the body is thus initially projected, so that it does not progress along the axis AP, then the motion of the body is in the plane PQM, and the body is always attracted to the point P, the centre of force.

Example 1.

834. Let the force V be directly proportional to the distance MP and put $V = \frac{\sqrt{(y^2+z^2)}}{f}$.

Hence we have :

$\frac{y}{f} = \frac{2adp}{p^3dy}$ (831) and on integrating :

$$\frac{y^2}{2f} = c - \frac{a}{p^2} \text{ or } \frac{1}{pp} = \frac{2cf-yy}{2af} \text{ and } p = \frac{\sqrt{2af}}{\sqrt{(2cf-y^2)}}.$$

Since moreover we have, by (832) :

$$dq = \frac{bpdy}{y^2} \text{ giving } \frac{bdy\sqrt{2af}}{y^2\sqrt{(2cf-y^2)}} = dq.$$

The integral of which is :

$$q = \alpha - \frac{\beta\sqrt{(2cf-y^2)}}{y} \text{ with } \beta = \frac{b\sqrt{2af}}{2cf}.$$

Whereby we have :

$$z = \alpha y - \beta\sqrt{(2cf-y^2)},$$

which equation expresses the projection of the described curve on the *y, z* plane normal to the axis *AP*, which is therefore seen to be an ellipse, the centre of which lies on the axis *AP*. Then since $dx = pdy$, it is the case that $dx = \frac{dy\sqrt{af}}{\sqrt{(2cf-y^2)}}$, which equation expresses the projection of the curve sought in the *x, y* plane *APQ*. And thus this is the curve of the sine of Leibniz, with the abscissa *x* taken as the arc, of which the sine is the applied line *y*.

Example 2.

835. If the force V varies inversely as the square of the distance MP or

$$V = \frac{ff}{y^2+z^2} = \frac{ff}{yy(1+q^2)} \text{ since } z = qy. \text{ [p. 354]}$$

On account of which we have :

$$\frac{ff}{y^2(1+q^2)^{\frac{3}{2}}} = \frac{2adp}{p^3dy}.$$

Moreover since $dy = \frac{y^2dq}{bp}$ (832), then we have

$$\frac{f^2dq}{(1+q^2)^{\frac{3}{2}}} = \frac{2abdp}{p^2};$$

the integral of which is :

$$\frac{ffq}{\sqrt{(1+qq)}} = C - \frac{2ab}{p} = C - \frac{2ab^2dy}{y^2dq}$$

with the value $\frac{y^2dq}{bdy}$ put in place of *p*. Hence this becomes :

$$\frac{f^2qdq}{\sqrt{(1+q^2)}} = Cdq - \frac{2ab^2dy}{y^2}$$

and on integrating :

$$f^2\sqrt{(1+q^2)} = Cq + \frac{2ab^2}{y} + D.$$

Whereby, when $q = \frac{z}{y}$, there is produced :

$$f^2\sqrt{(y^2+z^2)} = Cz + Dy + 2ab^2.$$

Which is the equation of the projected curve described in plane normal to the axis *AP* ; that can therefore be recalled as the section of a cone, of which either focus is in the position of the axis *AP*.

Scholium 2.

836. From these it is understood that the projections of the curves described in the plane normal to the axis *AP* are congruent with the curves that bodies describe in that plane with the same force acting. Moreover, neither is this wonderful ; for the motion, that we consider in place of this, can be reduced to the motion in a plane made normal to the axis *AP* by a body always attracted to the axis, provided the plane is considered to move with a uniform speed along the axis *AP*. And for this progressive motion, since it is made uniformly in direction, it is unable to disturb the motion of the body in the plane. Now from this arrangement it can be deduced from (827) where, if the force *P* vanishes, the motion of the body [p. 355] has been shown to progress along the axis uniformly. On account of which just as the force *P* becomes zero, then also the motion sought can be reduced to the motion in the plane. Clearly this can only happen, if a backwards motion of such a size can be imparted to the plane normal to the axis *AP*, as that found for the progressive motion along AP (827).

Scholium 3.

837. Yet meanwhile this fact does merit the most attention, that we have found with so much ease in the examples proposed, the equations between the orthogonal coordinates for the projections of the curve on the plane normal to the axis AP, and thus for the curves described by the body if the progressive motion should disappear. For in the first part of this chapter, in which we considered the motion generated in the plane by a centripetal force, the labour involved much more work and the comparison of the arcs of circles, in order that we could arrive at the ordinary equations for the curves described. Therefore with the greater generality, which most often results in increased difficulty in finding the solutions to questions, it is not an impediment in this case, but rather the solutions can be easier found in particular cases that had been felt more difficult to solve.

Corollary 3.

838. In the case of this proposition, the plane of the elements Mm and $m\mu$ (Fig. 76) is easily found. For since $ddy : ddz = y : z$ then $PO = 0$ and $AO = x$, and O falls on P. [p. 356] Again the tangent of the angle $PQR = \frac{ydz-zdy}{zdx} = \frac{b}{z}$ since $ydz - zdy = bdx$. Therefore the cotangent of the angle POR varies as QM. And then the tangent of the angle of inclination of the plane $Mm\mu$ to the plane APQ is equal to $\frac{\sqrt{(z^2+b^2)}}{y}$.

Scholium 4.

839. Since the case in which the force *P* vanishes can be reduced to motion in a plane, so also the case in which either Q or R disappears can be reduced to motion in a plane. For if the axis is taken on a line normal to *AP* in the plane *APQ*, then the force *Q* lies in a direction parallel to the axis and the remaining forces *P* and *R* now act as *Q* and *R* before. But if the axis is taken normal to *AP* and to the plane *APQ*, then the force *R* takes the place of the force *P* parallel to the axis. Clearly as the coordinates *x, y* and *z* are able to be commuted with respect to each other, so also similarly the order of the forces *P, Q* and *R* can be declared.

PROPOSITION 102.

PROBLEM.

840. *If a body is acted on at individual points M* (Fig. 79) *by two forces, the first in the direction MA, and the other, the direction of which is MQ along the normal sent from M to the plane APQ, then it is required to determine the motion of the body M and its orbit.*

SOLUTION.

With *MP* drawn, which is the normal to *AP*, the force *MA* is resolved into forces acting along *Mf* parallel to *AP* and the other along *MP*. Truly the force along *MP* can be resolved into forces acting along [p. 357] *MQ* and *Mg*. Therefore with $AP = x$, $PQ = y$ and $QM = z$ put as before, and the force pulling along *MA* equal to *V* and the force along *MQ* equal to *W* and with the resolution of the forces put in place and with a comparison made with Prop. 100 (823) it is found that

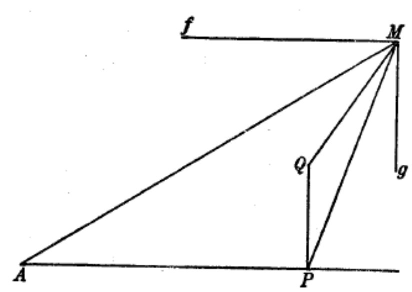


Fig. 79.

$$P = \frac{Vx}{\sqrt{(x^2+y^2+z^2)}}, \quad Q = \frac{Vy}{\sqrt{(x^2+y^2+z^2)}}$$

and

$$R = W + \frac{Vz}{\sqrt{(x^2+y^2+z^2)}}.$$

And from the equations of the same proposition there are produced :

$$V = \frac{2v\,dx\,ddy\sqrt{(x^2+y^2+z^2)}}{(xdy-ydx)(dx^2+dy^2+dz^2)} \quad \text{and} \quad W = \frac{2v\,ddy(xdz-zdx)-2v\,ddz(xdy-ydx)}{(xdy-ydx)(dx^2+dy^2+dz^2)}.$$

From these it is found :

$$\frac{dv}{2v} = \frac{dyddy+dzddz}{dx^2+dy^2+dz^2} - \frac{xddy}{xdy-ydx}$$

and hence on integration :

$$l\sqrt{v} = l\frac{\sqrt{(dx^2+dy^2+dz^2)}}{xdy-ydx} + la\sqrt{a}$$

or

$$\sqrt{v} = \frac{a\sqrt{a}(dx^2+dy^2+dz^2)}{xdy-ydx} \quad \text{or} \quad v = \frac{a^3(dx^2+dy^2+dz^2)}{(xdy-ydx)^2}.$$

This value substituted in place of *v* gives these equations :

$$V = \frac{2a^3dx\,ddy\sqrt{(x^2+y^2+z^2)}}{(xdy-ydx)^3} \quad \text{and} \quad W = \frac{2a^3ddy(xdz-zdx)-2a^3ddz(xdy-ydx)}{(xdy-ydx)^3}.$$

From which the described curve can be determined. Q.E.I.

Corollary 1.

841. The time in which the body comes as far as M, is equal to

$$\int\frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}.$$

Moreover since

$$\sqrt{v} = \frac{a\sqrt{a}(dx^2+dy^2+dz^2)}{xdy-ydx},$$

then that time is given by :

$$\frac{\int xdy-\int ydx}{a\sqrt{a}},$$

which becomes known from the quadrature of the given curve in the plane *APQ*.

Corollary 2.

842. If we put $y = px$ and $z = qx$, the following equations are produced [p. 358]

$$V = \frac{2a^3dx(xddp + 2dxdp)\sqrt{(1+p^2+q^2)}}{x^5dp^3} \text{ and } W = \frac{2a^3dqddp - 2a^3dpddq}{x^3dp^3}.$$

Which are helpful in making the curve known.

Example.

843. If the force V is proportional to the distance MA, and the force W is proportional to the perpendicular MQ, on putting

$$V = \frac{\sqrt{(x^2+y^2+z^2)}}{f} \text{ and } W = \frac{z}{g},$$

and thus

$$V = \frac{x\sqrt{(1+p^2+q^2)}}{f} \text{ and } W = \frac{qx}{g}.$$

Whereby from the equations of the preceding corollary, these equations become :

$$x^6dp^3 = 2a^3fdx(xddp + 2dxdp) \text{ and } x^4qdp^3 = 2a^3gdqddp - 2a^3gdpddq.$$

The integral of this equation is $x^2 = C - \frac{2a^3fdx^2}{x^4dp^2}$,

[Euler has not indicated how he solved this equation. However, if we set $t = \frac{dx}{dp}$ then $dp = \frac{dx}{t}$ and $ddp = \frac{ddx}{t} - \frac{dx}{t^2}dt = -\frac{dx}{t^2}dt;$ on substituting these into the above equation, we find that : $x^6dx = 4a^3ft^2dx - 2a^3fxtdt$. This can be rearranged to give the integrals :

$\int xdx = 4a^3f\int\frac{t^2dx}{x^5} - 2a^3f\int\frac{tdt}{x^4};$ and which gives the above result on integration, and treating t and x as independent variables,] from which there arises [on setting $C = c^2$,]

$$dp = \frac{adx\sqrt{2af}}{x^2\sqrt{(c^2-x^2)}},$$

and the integral of this :

$$p = \frac{y}{x} = C - \frac{a\sqrt{2af(c^2-x^2)}}{c^2x} \quad \text{or} \quad y = nx - \frac{a\sqrt{2af(c^2-x^2)}}{c^2}$$

for the equation of the projected curve described in the plane APQ, which therefore is an ellipse, the centre of which has been put at A. From the value of dp found, again

$$ddp = -\frac{adx^2(2c^2-3x^2)\sqrt{2af}}{x^3(c^2-x^2)^{\frac{3}{2}}};$$

from which with the values of dp and ddp substituted into the other equation, there arises :

$$afqxdx^2 = -2accgdxdq + 3agx^2dxdq - ac^2gxddq + agx^3ddq.$$

On putting $q = e^{\int rdx}$, there is produced :

$$fxdx = -c^2gxdr + gx^3dr - 2ccgrdx + 3gx^2rdx - c^2gxr^2dx + gx^3r^2dx.$$

On setting $r=\frac{u}{x^2\sqrt{(c^2-x^2)}}$, there comes into being :

$$fxdx=-\frac{gdu\sqrt{(cc-x^2)}}{x}-\frac{gu^2dx}{x^3} \text{ or } du+\frac{u^2dx}{x^2\sqrt{(c^2-x^2)}}+\frac{fx^2dx}{g\sqrt{(c^2-x^2)}}=0.$$

Which on putting $t=\frac{\sqrt{(c^2-x^2)}}{x}$ or $x=\frac{c}{\sqrt{(1+tt)}}$, the equation is transformed into :

$$du-\frac{u^2dt}{cc}=\frac{ccfdt}{g(1+tt)^2};$$

of which we will show the integral later.

In order to know the plane, in which the elements *Mm* and $m\mu$ (Fig. 76) are situated, on account of : [p. 359]

$$ddy:ddz=gxdy-gydx:gxdz-hzdx$$

with *h* in place of *f* + *g* it is found that :

$$AO=\frac{fz(xdy-ydx)}{hzdy-gydz},$$

$$\text{tang. ang. } POR=\frac{hzdy-gydz}{gxdz-hzdx}$$

and the tangent of the angle of the plane $Mm\mu$ with the plane *APQ* is equal to

$$\frac{\sqrt{((hzdy-gydz)^2+(gxdz-hzdx)^2)}}{g(xdy-ydx)}.$$

Then the time, which the body takes to arrive at M, since it is equal to $\frac{\int xxdp}{a\sqrt{a}}$, is given by :

$$\int\frac{dx\sqrt{2f}}{\sqrt{(c^2-x^2)}}.$$

Thus this time is proportional to the angle, of which the sine is the abscissa *x*, with the total sine taken equal to *c*, or of which the sine is $\frac{x}{c}$, if the total sine is taken as equal to 1. From which it is evident that the motion of the body projected on the plane *APQ* makes equal angles around A in equal times, and the time of one revolution is proportional to $\sqrt{f}$.

Corollary 3.

844. For the projection of the curve described in the plane *APQ* makes a circle, if $n=0$ and $-a\sqrt{2af}=cc$; the centre of which is at *A* and the radius = *c*. Therefore we have $y=\sqrt{(c^2-x^2)}$. And *z* is given from this equation: $\frac{gddz}{dx}=\frac{gxdz-hzdx}{cc-xx}$, which extends to the case of the preceding example equally widely, even if only this particular case is considered.

Corollary 4.

845. In order to discover the value of z from the equation $\frac{gddz}{dx}=\frac{gxdz-hzdx}{cc-xx}$, I put $z=e^{\int rdx}$. From which being done, this differential equation of the first degree is produced:

$$gdr+gr^2dx=\frac{grxdx-hdx}{cc-xx}.$$

Put $r=\frac{u}{\sqrt{(c^2-x^2)}}$ and there is produced :

$$gdu+\frac{gu^2dx}{\sqrt{(c^2-x^2)}}+\frac{hdx}{\sqrt{(c^2-x^2)}}=0.$$

[p. 360] With $\frac{h}{g}$ or $\frac{f+g}{g}=m^2$ this equation arises :

$$\frac{du}{u^2+m^2}+\frac{dx}{\sqrt{(c^2-x^2)}}=0,$$

in which the indeterminates can now be separated from each other in turn.

Corollary 5.

846. Truly,

$$\int\frac{dx}{\sqrt{(c^2-x^2)}}=+\sqrt{-1}\,l\left(\sqrt{(cc-xx)}-x\sqrt{-1}\right)$$

and

$$\int\frac{du}{u^2+m^2}=\frac{1}{2m\sqrt{-1}}\,l\frac{u-m\sqrt{-1}}{u+m\sqrt{-1}}.$$

Therefore with the constant added and with the given numbers, we have then:

$$\left(\frac{\sqrt{(cc-xx)}-x\sqrt{-1}}{b}\right)^{2m}=\frac{u-m\sqrt{-1}}{u+m\sqrt{-1}}$$

and hence

$$u=\frac{b^{2m}+\left(\sqrt{(c^2-x^2)}-x\sqrt{-1}\right)^{2m}}{b^{2m}-\left(\sqrt{(c^2-x^2)}-x\sqrt{-1}\right)^{2m}}\cdot m\sqrt{-1}.$$

[These integrals are easily shown to be true, on differentiation.]

Corollary 6.

847. Since truly $lz=\int rdx$ and $r=\frac{u}{\sqrt{(c^2-x^2)}}$, then we have

$$lz=\int\frac{mdx\left(b^{2m}+(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m}\right)\sqrt{-1}}{\left(b^{2m}-(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}.$$

On putting $\int \frac{dx}{\sqrt{(c^2-x^2)}} = s,$ then there is

$$s = \sqrt{-1}\, l \frac{\sqrt{(c^2-x^2)} - x\sqrt{-1}}{b} \quad \text{and} \quad e^{\frac{s}{\sqrt{-1}}} b = \sqrt{(c^2-x^2)} - x\sqrt{-1}$$

and

$$lz = \int \frac{mds\left(1 + e^{\frac{2ms}{\sqrt{-1}}}\right)\sqrt{-1}}{1 - e^{\frac{2ms}{\sqrt{-1}}}},$$

or on putting $e^{\frac{2ms}{\sqrt{-1}}} = t,$ so that $ds = \frac{dt\sqrt{-1}}{2mt}$, then we have

$$lz = \int \frac{-dt(1+t)}{2t(1-t)} = l\frac{(1-t)k}{\sqrt{t}},$$

and hence it becomes :

$$z = \frac{\left(1 - e^{\frac{2ms}{\sqrt{-1}}}\right)k}{e^{\frac{ms}{\sqrt{-1}}}} = \frac{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)k}{b^m\left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^m}.$$

Corollary 7. [p. 361]

848. Now from the value of z we have

$$dz = \frac{mzdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}$$

and

$$\frac{gxdz - hzdx}{z}$$
$$= \frac{mgdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)x\sqrt{-1} - m^2gdx\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}.$$

And again putting $y = \sqrt{(c^2-x^2)}$ there will be

$$\frac{hzdy - gydz}{z} = \frac{m^2gxdx}{\sqrt{(c^2-x^2)}} - \frac{mgdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}}.$$

Corollary 8.

849. Hence from these there is found :

$$AO = \frac{fcc\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)}{m^2gx\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right) + mg\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}\,(c^2-x^2)}$$

and tang. PQR =

$$\frac{mx\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right) + \left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}\,(c^2-x^2)}{m\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)} - x\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}.$$

In a similar manner, from these the angle of inclination between the plane is found in which the body is moving and the plane *APQ*. [p. 362]

Scholium.

850. The application to finding the value of z and the inclination of the orbit is very difficult on account of the imaginary quantities occurring in turn. On this account we are unwilling to tarry longer with the intersections of the curve described by the body with the plane *APQ*, to be determined. Moreover since this is the great question of the moment in astronomy in finding the motion of the nodes, the following proposition has been designated to this business, in which, we investigate when the body by its own motion shall arrive in the plane *APQ*. For the body completes part of its motion above the plane, and part below; and whether it is above or below, the body is always drawn from the other towards this plane by a force W in the direct ratio of the distance from this plane. Truly the point in the plane *APQ*, through which the body passes from the upper part to the lower part, is called the descending node, and the point in which it reverts to the upper, is called the ascending node.

PROPOSITION 103.

PROBLEM.

851. *If the body is always attracted partially to some fixed point A* (Fig. 79) *in the ratio of the distances from the same, and partially normally to the plane APQ in the ratio of the given distances from this plane as well; it is required to determine the nodes or the points in which the body arrives at this plane, and besides also the points,* [p. 363] *at which the body is at a maximum distance from the plane.*

SOLUTION.

With the three coordinates x, y and z remaining as before, and the force, by which the body is drawn towards A, is equal to $\frac{\sqrt{(x^2+y^2+z^2)}}{f}$; and the force, by which the body is drawn towards the plane APQ, is equal to $\frac{z}{g}$ and on putting $\frac{f+g}{g}=m^2$ it is evident that the body is incident in the plane *APQ*, when $z = 0$. But $z = 0$, as often as

$$b^{2m}-(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m}=0$$

(847). This comes about whenever $u=\infty$, by (846). Therefore since, by (845)

$$\frac{du}{u^2+m^2}+\frac{dx}{\sqrt{(c^2-x^2)}}=0,$$

with centre A the circle BQC is described (Fig. 80) with radius $AB = c$ and the body is moving along the region BQC and in place of this equation, this equivalent equation is taken [multiplying by c changes the angles into arcs]:

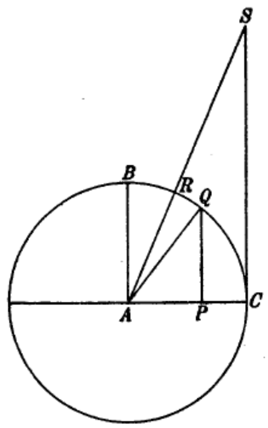


Fig. 80.

$$\frac{\frac{c^3}{m^2}du}{\frac{c^2u^2}{m^2}+c^2}+\frac{cdx}{\sqrt{(c^2-x^2)}}=0 \text{ or } \frac{\frac{1}{m}\cdot cc\cdot\frac{cdu}{m}}{\frac{c^2u^2}{m^2}+c^2}=-\frac{cdx}{\sqrt{(c^2-x^2)}}.$$

The integral of which is

$$\frac{1}{m}\int\frac{cc\cdot\frac{cdu}{m}}{\frac{c^2u^2}{m^2}+c^2}=C-\int\frac{cdx}{\sqrt{(c^2-x^2)}},$$

in which

$$\int\frac{cc\cdot\frac{cdu}{m}}{\frac{ccuu}{m^2}+c^2}$$

expresses the arc, of which the tangent [*SC*] is $\frac{cu}{m}$, and

$$\int\frac{cdx}{\sqrt{(c^2-x^2)}}$$

expresses the arc *BQ*, the sine of which is $AP = x$. [Thus, the integrals are taken to represent the arcs of which the angles arctan and arcsin of add up to one right angle.] Let *C* be the arc BQC or the quadrant of the circle, and the tangent $CS=\frac{cu}{m}$, to which there corresponds the arc *CR*. Hence the above equation is changed into this: $\frac{1}{m}CQR = CQ$ and $CR = m.CQ$. From this, it is evident that $u = \infty$, [p. 364] if the angle *CAR* is right, or equal to 3, or 5, or 7, etc. right angles. And thus as a consequence, the motion of the body has an infinite value of *u*, if the arc *CR* is made successively into the degrees of the following sequence of degrees, with the values [corresponding to up or down in the diagram in turn] :
$90, -90, -270, -450, -630$ etc.
Moreover then the arc CQ contains the degrees :
$\frac{90}{m}, -\frac{90}{m}, -\frac{270}{m}, -\frac{450}{m}, -\frac{630}{m}$, etc. and thus the arc *BQ* :
$90-\frac{90}{m}, 90+\frac{90}{m}, 90+\frac{270}{m}, 90+\frac{450}{m}, 90+\frac{630}{m}$, etc.
Whereby if the body were perchance at a node, then it would arrive at the other nodes successively by the absolute angular motion about *A,* at angles of
$\frac{180}{m}, \frac{360}{m}, \frac{540}{m}, \frac{720}{m}$, etc. degrees.
Therefore two adjacent nodes are separated by an angle of $\frac{180}{m}$ degrees. And the ascending or descending nodes are distant from the following node of the same kind by an angle of $\frac{360}{m}$ degrees, i. e, by the angle $\frac{360.\sqrt{g}}{\sqrt{(g+f)}}$ degrees. Thus demonstrating the first part of the proposition.

It follows that the body is at the greatest distance from the plane *APQ*, when $dz = 0$; and that comes about, whenever

$$b^{2m} + (\sqrt{(c^2 - x^2)} - x\sqrt{-1})^{2m} = 0$$

(848). Moreover now it is the case that $u = 0$. [from (846)]. Therefore with the previous construction kept in place that avoids putting $u = 0$, then u is zero as often as $CR = 0$ or – 180 or – 360, etc. degrees. Then moreover the arc CQ contains the degrees :

$0, -\frac{180}{m}, -\frac{360}{m}, -\frac{540}{m}$ etc.

and thus the arc BQ has the degrees :

$90, 90+\frac{180}{m}, 90+\frac{360}{m}, 90+\frac{540}{m}$, etc.

Therefore the maximum distance of the body from the plane APQ is equally distant from the nearest node on each sides. Thus demonstating the second part of the proposition. [p. 365]

Corollary 1.

852. Therefore the nodes finally return to the same point, if m is a rational number or $\frac{f+g}{g}$ is a square number. But if $\frac{f+g}{g}$ is not a square number, then the body is never incident at the same point in the plane APQ , in which before it was finally incident.

Corollary 2.

853. If g is a positive number or the body is always attracted to the plane by a positive force, then m is a number greater than one. Therefore the interval between the two nodes is less than an angle of 180 degrees. Whereby the nodes are progressing backwards, and the distance of a node from the position of the previous node is distant by the angle $\frac{180(m-1)}{m}$ degrees.

Corollary 3.

854. If the body is always repelled from the plane APQ, then g becomes a negative number $m = \sqrt{\frac{g-f}{g}}$. Whereby if $g > f$, then m is a real number, but less than unity ; therefore then the nodes as a consequence are progressing more quickly by the amount which f is less distant than g. And if $f = g$, then the body escapes from the node and never returns to the surface APQ . But if $f > g$, then the body always departs form this plane. [p. 366]

Corollary 4.

855. Since the body is at a maximum distance from the plane APQ, when it is true that

$$(\sqrt{(c^2 - x^2)} - x\sqrt{-1})^{2m} = -b^{2m},$$

the maximum distance itself can be had, if this is substituted in the value of z found (847). Moreover this maximum distance is equal to $\frac{2k}{\sqrt{-1}}$, which is therefore the same everywhere.

Corollary 5.

856. If this circle *BQC* (Fig. 80) were the projection of the orbit described by the body in the plane *APQ*, then the tangent of the angle of inclination of the plane of the orbit to the plane *APQ* is equal to $\frac{2m^2k}{c\sqrt{-1}}$ in the places where the body is at a maximum distance from this plane, or equal to $\frac{2m^2k}{c}$ with k put in place of $\frac{k}{\sqrt{-1}}$, since the constant k must avoid having imaginary values. [An imaginary value of k has to be assumed initially.]

Corollary 6.

857. With the same hypothesis kept for the places, where the body is incident in the plane *APQ*, the tangent of the angle of incidence is equal to $\frac{dz\sqrt{(c^2-k^2)}}{cdx}$ with the equality

$$(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m}=b^{2m}.$$

Thus this tangent becomes equal to

$$\frac{mz(b^{2m}+(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m})\sqrt{-1}}{(b^{2m}-(\sqrt{(c^2-x^2)}-x\sqrt{-1})^{2m})c}$$

and with the value of z put in place from (847) this is equal to $\frac{2mk\sqrt{-1}}{c}$,

[p. 367] or with a suitable value put in place of k this tangent becomes equal to $\frac{2mk}{c}$.

Corollary 7.

858. Therefore the tangent of the angle of inclination of the orbit described by a body to the plane *APQ*, if the body is at a maximum distance from the plane *APQ*, is to the same tangent, if the body is incident in this plane, as m to 1. Therefore the ratio can be made : as the separation of the two nodes is to 180 degrees, thus the inclination of the orbit, if the body is at a maximum distance from the nodes, to the inclination of the orbit described by the body, if the body changes from one node to the other.

Scholium.

859. Indeed this proposition is seen to have some use in astronomy, because there that force, by which the body is attracted to the fixed point *A*, we can make proportional to the distance, truly for celestial bodies that have in place a force that is inversely proportional to the square of the distance. Yet this is extremely useful, if the orbits of bodies do not depart much from circles; for there is no interest in orbits that depart from circles, in whatever way the centripetal force depends on the distances. On account of which, when the orbits of planets do not depart greatly with circular orbits, this proposition [p. 368] is able to be adapted successfully to this motion. And it then becomes most useful, as the curve corresponding to *f* can be found, which can itself be to the distance from the centre as the force of gravity to the centripetal force. The other force, which draws the body to the given plane, is effective as some proportional of the distance; yet if that does not happen, then the letter *g* must be considered as a variable, from which truly the approximate motions of the nodes can be gathered from all the values of g, by choosing as it were the mean value. In lunar motion the motion of the nodes merits special attention, clearly which will happen to be nearly our preceding determination. Moreover it is observed from the opposition of the preceding node that the nodes differ by nearly 43', thus in order that

$$\frac{180(m-1)}{m}=\frac{43}{60} \text{ and } m=1+\frac{43}{10757}=1\frac{1}{250}.$$

From which the lunar force is known to be always pulling from behind towards the plane of the ecliptic.

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PROPOSITIO 98. [p. 339]

THEOREMA.

802. *Tres principales vires,quae faciunt, ut corpus in curva non in eodem plano existente moveatur, et in quas aliae vires resolvi debent,singulae sunt inter se normales; earum una est tangentialis, reliquae duae normales ad eum, quarum altera directionem habet in dato plano, alterius vero directio est normalis ad hoc planum, harumque virium nulla reliquarum actiones immutature valet .*

DEMONSTRATIO.

Assumto plano fixo *APQ* (Fig. 75) in eoque axe *AP* sit *Mm* elementum a corpore descriptum. Ex punctis *M* et *m* in planum fixum demittantur perpendicula *MQ* et *mq* et ex punctis *Q* et *q* perpendicula in axem *QP, qp*. Iam si corpus a nulla vi sollicitaretur, in recta *Mm* producta progrederetur celeritate, quam habuit in *Mm*; aequali ergo tempusculo, quo *Mm* percurrit, perveniet in *n*, descripto elemento *mn* aequali et in directum posito elemento *Mm*. Quare demisso quoque ex *n* in planum *APQ* perpendiculo *nr*, erunt elementa *Qq* et *qr* inter se quoque aequalia et in directum posita; hanc ob rem perpendiculum *rπ* ex *r* in axem *AP* demissum abscindet elementum $p\pi = Pp$.

Sit celeritas, qua corpus elementum *Mm* describit, debita altitudini *v*, et consideretur primo vis tangentialis, quae directionem habet iuxta *mn* et tota in alteranda celeritate absumitur. Ponatur haec vis tangentialis T existente vi gravitatis = 1, erit [p. 340]

$$dv = T.Mm,$$

et elementum *mn* absolvitur celeritate debita altitudini $v + dv$.

Deinde in plano *Mr* concipiatur vis directionem habens *ms* normalem ad corporis directionem *Mm*. Haec ergo efficiet, ut corpus ab *mn* declinet et in mv elemento in eodem plano Mr posito progrediatur. Sit haec vis = *N*, et cum radius osculi elementorum *Mm* et $m\nu$, demisso ex ν in *mn* perpendiculo $\nu\varepsilon$, sit = $\frac{m\nu^2}{\nu\varepsilon}$, erit

$$\frac{2v.\nu\varepsilon}{m\nu^2} = N$$

(561). Est vero $\frac{\nu\varepsilon}{m\nu}$ sinus anguli $nm\nu$. Quamobrem erit

$$2v\sin.nmv = N.mv = N.Mm,$$

ideoque

$$\sin.nmv = \frac{N.Mm}{2v}.$$

Tertia vis normalis ad utramque expositarum *mn* et *ms*, ita ut eius directio *mt* sit

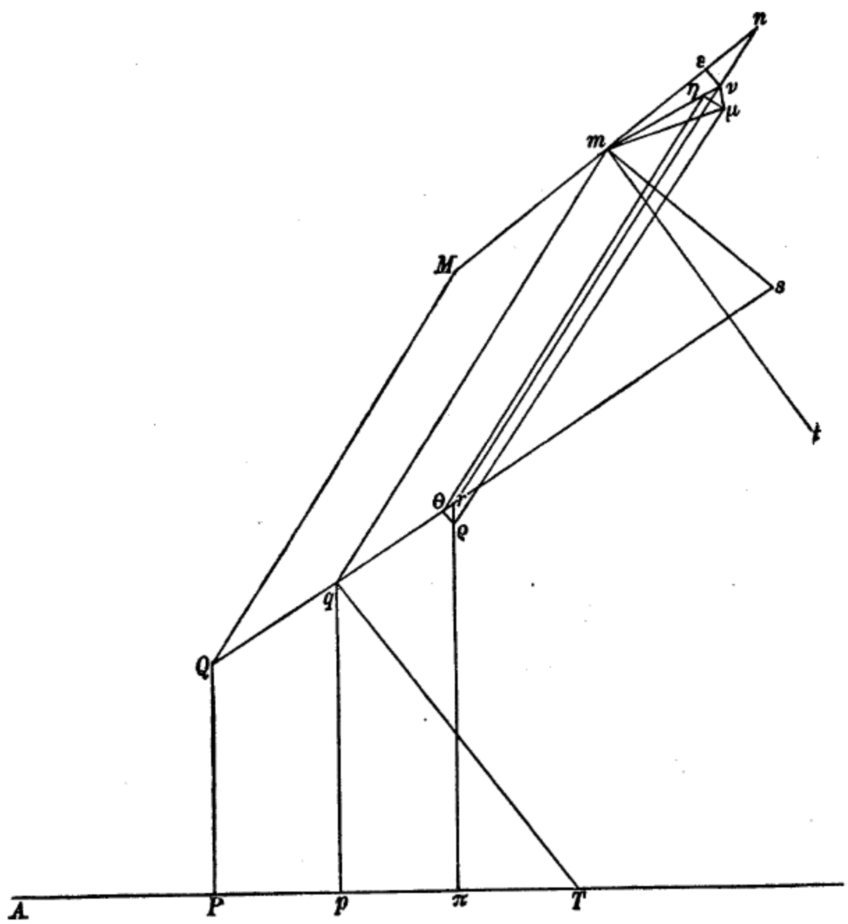


Fig. 75.

normalis in planum *Mr*. Haec igitur vis neque praecedentium actiones impediet neque ab ipsis impedimentum seu immutationem patietur. Tota ergo impendetur ad corpus a plano *Mr* detrahendum; deducat ea corpus ex ν in μ, ita ut planum $\nu m\mu$ sit normale in planum *Mr*, eritque eius effectus angulus $\nu m\mu$. Hoc igitur effectu eodem, quo circa praecedentem vim normalem fecimus, modo aestimato, si fuerit haec vis M, erit

$$\sin.\nu m\mu = \frac{M.Mm}{2v}.$$

Tres ergo hae vires simul efficiunt, ut corpus, postquam elementum Mm descripsit, progrediatur in elemento $m\mu$ aucta celeritate debita scilicet altitudini $v + T.Mm.$ Quaecunque autem aliae vires corpus sollicitent, eae omnes resolvi possunt in huiusmode tres, quarum directiones sunt *mn, ms, mt*. Quarum effectus in corpus cum determinaverimus, [p. 341] simul quarumcunque virium effectus cognoscentur. Q.E.D.

Corollarium 1.

803. Sumta $\nu\mu$ in plano $nr\pi$ et demisso ex μ in planum APQ perpendiculo $\mu\rho$ erit $\mu\rho$ parallela ipsi *rn*. Tres igitur coordinatae pro punctis *M, m* et μ erunt *AP, PQ, QM; Ap*, *pq*, *qm*, et $A\pi, \pi\rho, \rho\mu$.

Corollarium 2.

804. Quare si ex μ in *mv* perpendiculum $\mu\eta$ demittatur, erit id in planum *Mr* normale; similique modo $\rho\theta$, quae est ad *qr* perpendicularis, in idem planum normalis erit. Quamobrem, ob ρ et μ in recta $\rho\mu$ huic plano parallelo posita, erit $\rho\theta = \mu\eta$ et $\theta\eta = \rho\mu$.

Corollarium 3.

805. Si ad *Qq* ducatur normalis *qT* in plano fixo *APQ*, erit haec *qT* normalis in planum *Mr*. Cum igitur *mt* in idem planum sit quoque normalis, erit *mt* parallela ipsi *qT*; inter hasque distantia erit altitudo *mq*.

Corollarium 4.

806. Trium coordinatarum dicantur $AP = x, PQ = y$ et $QM = z$.
Eritque

$$Pp = p\pi = dx, pq = y + dy, qm = z + dz$$

atque

$$\pi\rho = y + 2dy + ddy \text{ et } \rho\mu = z + 2dz + ddz = \theta\eta.$$

At

$$Qq = \sqrt{(dx^2 + dy^2)} = qr,$$
$$q\rho = \sqrt{\left(dx^2 + (dy + ddy)^2\right)} = q\theta = \sqrt{(dx^2 + dy^2)} + \frac{dyddy}{\sqrt{(dx^2 + dy^2)}};$$

ideoque

$$r\theta = \frac{-dyddy}{\sqrt{(dx^2 + dy^2)}}.$$

Porro erit $\pi r = y + 2dy$ et $rn = z + 2dz$.
Denique erit [p. 342]

$$Mm = \sqrt{(dx^2 + dy^2 + dz^2)} = mn$$

et

$$m\mu = \sqrt{\left(dx^2 + (dy + ddy)^2 + (dz + ddz)^2\right)} = \sqrt{(dx^2 + dy^2 + dz^2)} + \frac{dyddy + dzddz}{\sqrt{(dx^2 + dy^2 + dz^2)}}.$$

Corollarium 5.

807. Quare $mq, \theta\eta$ et $r\nu$ sunt inter se parallelae, in eodem plano et rectis qr ac $m\nu$ terminatae, erit

$$\theta\eta - qm : q\theta = r\nu - mq : qr.$$

Est vero

$$\theta\eta - qm = dz + ddz, \quad q\theta = \sqrt{(dx^2+dy^2)} + \frac{dyddy}{\sqrt{(dx^2+dy^2)}} \text{ et } qr = \sqrt{(dx^2+dy^2)}.$$

Qiare est

$$r\nu - mq = \frac{(dz+ddz)(dx^2+dy^2)}{dx^2+dy^2+dyddy}$$

hincque

$$n\nu = rn - r\nu = -ddz + \frac{dydzddy}{dx^2+dy^2}.$$

Unde reperitur

$$\sin.\ nm\nu = \frac{dydzddy - dx^2ddz - dy^2ddz}{(dx^2+dy^2+dz^2)\sqrt{(dx^2+dy^2)}}.$$

Corollarium 6.

808. Cum deinde set $r\rho = -ddy$ et $Qq : Pp = r\rho : \rho\theta$, erit

$$\theta\varrho = -\frac{dxddy}{\sqrt{(dx^2+dy^2)}} = \mu\eta.$$

Hanc ob rem habebitur

$$\sin.\ \nu m\mu = -\frac{dxddy}{\sqrt{(dx^2+dy^2)}(dx^2+dy^2+dz^2)}.$$

Corollarium 7.

809. Ex datis igitur tribus viribus T, N et M corpus sollicitantibus orientur tres sequentes aequationes :

$$dv = T\sqrt{(dx^2+dy^2+dz^2)},$$

$$2vdydzddy - 2vddz(dx^2+dy^2) = N(dx^2+dy^2+dz^2)^{\frac{3}{2}}\sqrt{(dx^2+dy^2)}$$

atque

$$-2vdxddy = M(dx^2+dy^2+dz^2)\sqrt{(dx^2+dy^2)},$$

ex quibus tum celeritas corporis in singulis locis tum ipsa curva cognoscuntur. [p. 343]

Corollarium 8.

810. Duae posteriores aequationes coniunctae et eliminata *v* dant istam aequationem

$$\frac{ddz(dx^2+dy^2)}{dx\,ddy}-\frac{dydz}{dx}=\frac{N\sqrt{(dx^2+dy^2+dz^2)}}{M}.$$

Quae assumi potest pro aequatione naturam superficiei exprimente, in qua curva descripta extat.

Scholion.

811. Hac igitur propositione primarias dedimus regulas, ex quibus motus corporis ita sollicitati, ut in eodem plano moveri nequeat, deduci poterit. Ostendimus enim omnes potentias in ternas, quarum effectus determinavimus, posse resolvi; et idcirco, quaecunque proponantur potentiae sollicitantes, ope talis resolutionis, quem motum eae in corpore producant, cognoscitur. Apparet etiam, si potentia *M* desit, corpus motum suum in plano esse absoluturum, qui igitur casus huc non pertinet. At si potentia tangentialis *T* evanescat manentibus reliquis *M* et *N*, corpus quidem orbitam non planam describet, sed tamen motu uniformi feretur. Quo igitur situs orbitae in universum cognoscatur, inclinationem plani, in quo sunt elementa *Mm* et $m\mu$, ad planum APQ eiusque cum hoc intersectionem investigari oportebit.

PROPOSITIO 99.

PROBLEMA.

812. *Determinare plani, in quo duo elementa Mm et* $m\mu$ (Fig. 76) *a corpore descripta sunt posita, inclinationem ad planum fixum APQ eiusque cum hoc intersectionem.*
[p. 344]

SOLUTIO.

In plano, cuius inclinationem quaerimus, dantur tria puncta *M, m* et μ; in hoc igitur plano posita erit quaevis recta per horum punctorum duo transiens. Quare si recta *mM*

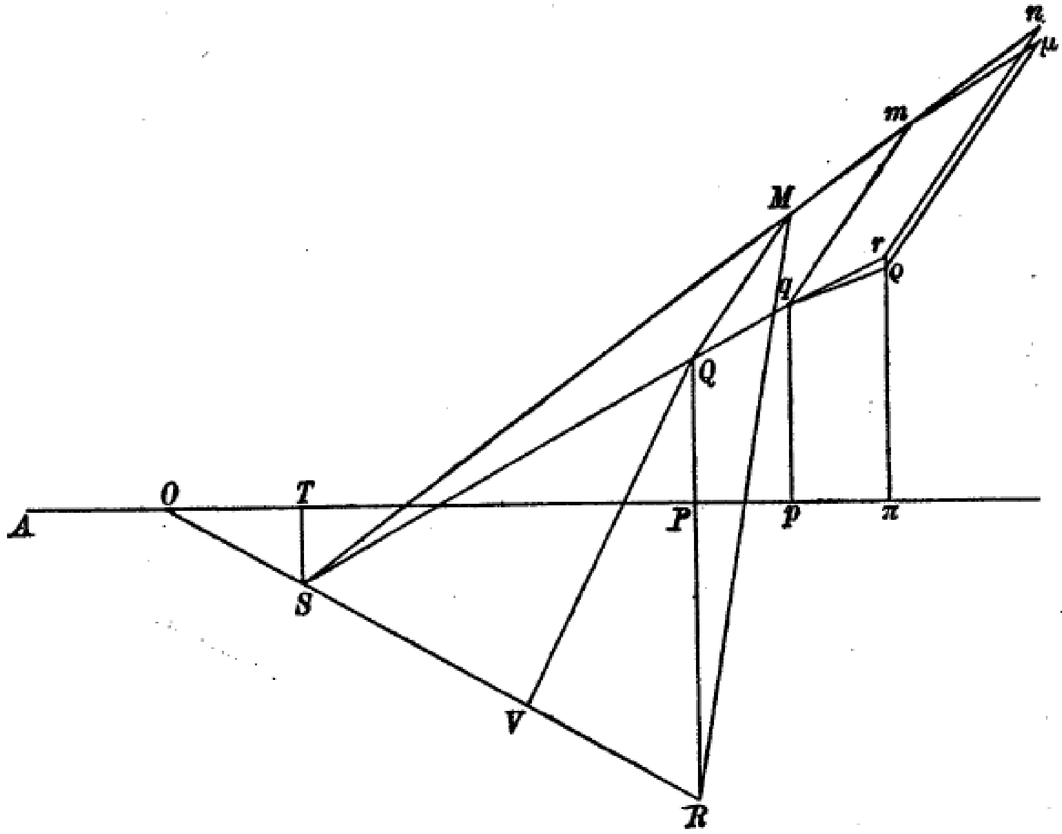


Fig. 76.

producatur, donec ipsi *qQ* productae occurrat in *S*, erit punctum *S* tum in plano $Mm\mu$ tum in plano fixo APQ; transibit ergo per *S* recta, qua haec plana se mutuo intersecant. Manentibus igitur ut ante $AP = x$, $PQ = y$ et $QM = z$ et elementis abscissae *Pp* et $p\pi$ inter se aequalibus, erit $qm - QM : Qq = QM : QS$ hincque

$$QS = \frac{z\sqrt{(dx^2 + dy^2)}}{dz}.$$

Lineae vero *QS* positio cognoscitur ex angulo *PQS*, cuius sinus est

$$= \frac{dx}{\sqrt{(dx^2 + dy^2)}}.$$

Deinde in plano $Mm\mu$ quoque situm est punctum n; hanc ob rem recta per n et μ transiens seu huic parallela per M ducta in eodem extabit plano. Haec autem recta occurret plano APQ in puncto R rectae QP productae, et QR cognoscetur ex hac analogia $rn - \rho\mu : r\rho = QM : QR$; hinc erit

$$QR = \frac{zddy}{ddz} \text{ ideoque } PR = \frac{zddy}{ddz} - y.$$

Est vero $Qq : Pp = QS : PT$ ducta ST perpendiculari in AP. Ex quo oritur

$$PT = \frac{zdx}{dz}.$$

Porro est quoque $Pp : pq - PQ = PT : PQ + ST$ ideoque

$$PQ + ST = \frac{zdy}{dz} \text{ et } ST = \frac{zdy}{dz} - y.$$

Occurrat recta RS producta axi AP in O eritque $PR - ST : PT = PR : PQ,$ ex quo invenitur

$$PO = \frac{zdxddy - ydxddz}{dzddy - dyddz}$$

atque [p. 345]

$$AO = \frac{xdzddy - xdyddz + ydxddz - zdxddy}{dzddy - dyddz}.$$

Porro ex his tangens angulus POR =

$$\frac{PR}{PO} = \frac{dzddy - dyddz}{dxddz};$$

unde positio intersectionis RO plani $Mm\mu$ cum plano fixo APQ innotescit. Inclinatio autem horum planorum mutua invenitur demittendo ex Q in RS perpendiculo QV; tum enim erit anguli inclinationis tangens = $\frac{MQ}{QV}$. Est vero

$$QV = \frac{PO \cdot QR}{RO} = \frac{zdxddy}{\sqrt{(dx^2ddz^2 + (dzddy - dyddz)^2)}}$$

ideoque

$$\frac{MQ}{QV} = \frac{\sqrt{(dx^2ddz^2 + (dzddy - dyddz)^2)}}{dxddy},$$

ex quo angulus inclinationis mutuae planorum $Mm\mu$ et APQ determinatur.
Q.E.I.

Corollarium 1.

813. Si angulus POR semper maneat idem, erit

$$\alpha dxddz + dyddz = dzddy$$

estque huius anguli tangens = α. Aequatio haec integrata dat

$$\alpha dx + dy + \beta dz = 0 \text{ atque } \alpha x + y + \beta z = f.$$

Ex qua aequatione cognoscitur orbitam a corpore descriptam totam fore in eodem plano positam.

Corollarium 2.

814. Si enim fuerit $\alpha x + y + \beta z = f$, erit

$$\alpha dx + dy + \beta dz = 0 \text{ et } ddy + \beta ddz = 0.$$

Hinc fiet

$$AO = x + \frac{ydx + \beta zdx}{-dy - \beta dz},$$

et cum sit $-dy - \beta dz = \alpha dx$, erit

$$AO = x + \frac{y}{\alpha} + \frac{\beta z}{\alpha} = \frac{f}{\alpha}$$

ideoque et AO constans.

Corollarium 3.

815. Deinde manente angulo POR constante seu $\alpha x + y + \beta z = f$, [p. 346] erit tangens anguli inclinationis planorum $Mm\mu$ et APQ =

$$\frac{\sqrt{(dx^2 + (-dy - \beta dz)^2)}}{-\beta dx} = \frac{\sqrt{(1 + \alpha^2)}}{-\beta}.$$

Quare et iste angulus erit constans.

Corollarium 4.

816. Neque etiam punctum intersectionis O invariabile poni potest, nisi simul orbita a corpore descripta fiat plana. Nam sit $AO = f$ et ponatur $x - f = t$ et $dx = dt$, erit

$$tdzddy - tdyddz = zdtddy - ydtddz$$

hincque

$$\frac{ddy}{tdy - ydt} = \frac{ddz}{tdz - zdt}.$$

Multiplicetur per t, quo habeatur

$$\frac{tddy}{tdy - ydt} = \frac{tddz}{tdz - zdt}.$$

Quae aequatio ob dt constans est integrabilis; namque erit

Haec divisa per tt et integrata dabit

$$\frac{y}{t} = \frac{\alpha z}{t} + \beta \text{ seu } y = az + \beta x - \beta f.$$

Quam perspicuum est esse ad superficiem planam.

Corollarium 5.

817. At si ponatur tangens anguli inclinationis planorum $Mm\mu$ et APQ constans, huius modi aequatio $\alpha x + y + \beta z = f$ non prodit. Atque aliunde manifestum est orbitam a corpore descriptam tum non necessario esse planam.

Corollarium 6.

818. Quare, ne curva a corpore descripta sit plana, neque punctum O neque angulus POR invariabilia accipi possunt. Haec autem si sint variabilia, nihilio tamen minus angulus inclinationis planorum $Mm\mu$ et APQ constans esse potest. [p. 347]

Corollarium 7.

819. Linea intersectionis RO, quae in astronomia linea nodorum appellatur, si non habeat constantem positionem, convertitur circa punctum S. Nam recta mMS posita est in plano elementorum Mm et praecedentis. Quare intersectio RO et praecedens se in S decussabunt.

Corollarium 8.

820. Punctum igitur hoc S est in eo loco, ubi est

$$AT = \frac{xdz - zdx}{dz} \text{ et } ST = \frac{zdy - ydz}{dz}.$$

Ex quibus positio puncti S cognoscitur.

Corollarium 9.

821. Si ponatur punctum S invariabile, erit

$$xdz - zdx = adz \text{ et } zdy - ydz = bdz,$$

unde reperitur $x - a = \alpha z$ et $y - b = \beta z.$ Hoc igitur casu orbita a corpore descripta non solum est plana, sed etiam linea recta, quia proiectio eius $Qq\rho$ fit recta et propter $y - b = \beta z$ etiam $Mm\mu$.

Scholium.

822. Expositis nunc principiis, quae ad motum corporum in superficiebus non planis pertinent, ipsa tractio ut prior de motu in plano in duas partes potest dividi. In quarum prima docebimus ex datis viribus invenire curvam a corpore descriptam, in altera vero ostendetur, si data fuerit curva, quam corpus describit, quales vires ad hoc requirantur. Hic vel curva ipsa tantum potest esse data [p. 348] vel simul quoque celeritas corporis in singulis locis.

PROPOSITIO 100.

PROBLEMA.

823. *Si corpus sollicitetur a tribus potentiis, quarum directiones Mf, Mg et MQ* (Fig. 77) *sint parallelae tribus coordinatis AP, PQ et QM, determinare motum corporis et orbitam, in qua movebitur.*

SOLUTIO.

Quia *Mf* et *Mg* sunt parallelae ipsis *AP* et *PQ*, erit planum *fMg* parallelum plano *APQ*. In hoc plano ducatur *Mi* parallela elemento *Qq*; erit haec *Mi* quoque posita in plano *Mq*. In elementum *mM* productum demittatur ex *Q* perpendiculum *Qd*; atque ex *f* et *g* in *Mi* perpendicula *fi* et *gk*. Deinde ex *i* et *k* in *Md* cadant perpendicula *ib* et *kc*. Erunt autem *fi* et *gk* perpendiculares in planum *Mq*, quia planum *fMg* est normale ad planum *Mq*. Manentibus nunc ut ante $AP = x$, $PQ = y$ et $QM = z$ sit vis corpus secundum *Mf* trahens = *P*, vis, quae corpus secundum *Mg* trahit, = *Q* et vis secundum *MQ* trahens = *R*. Hae igitur vires, ut earum effectus cognoscantur, resolvi debent in vires tangentialem iuxta Mm agentem, normalem ad *Mm* in plano *Mq* sitam et normalem ad planum *Mq*.

Ob angulum *Mfi* =*Qqp* erit

$$\sqrt{(dx^2+dy^2)} : dy = P : \frac{Pdy}{\sqrt{(dx^2+dy^2)}}$$

exprimitque $\frac{Pdy}{\sqrt{(dx^2+dy^2)}}$ vim ex *P* ortam secundum *if* agentem, et si *P* sola ageret, foret per (802) [p. 349]

$$M = -\frac{Pdy}{\sqrt{(dx^2+dy^2)}}.$$

Deinde vis secundum *Mi* trahens erit =

$$\frac{Pdx}{\sqrt{(dx^2+dy^2)}}.$$

Haec porro resolvitur in vim secundum *bi* trahentem =

$$\frac{Pdxdz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

et vim secundum *Mb* trahentem =

$$\frac{Pdx}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

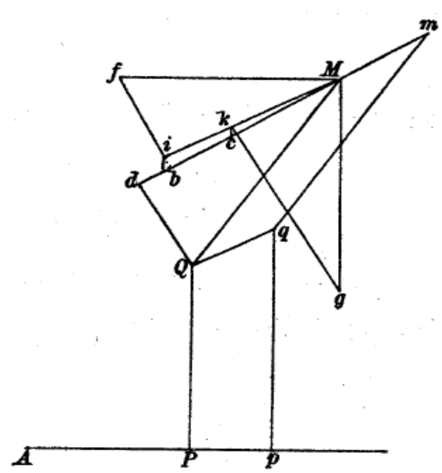


Fig. 77.

Ex P igitur erit

$$N = -\frac{Pdxdz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}} \quad \text{et} \quad T = -\frac{Pdx}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Simili modo vis *Q*, cuius directio est *Mg*, resolvitur in vim secundum *kg* agentem =

$$\frac{Qdx}{\sqrt{(dx^2+dy^2)}}$$

et vim secundum Mk =

$$\frac{Qdy}{\sqrt{(dx^2+dy^2)}}.$$

Haec ulterius resolvitur in vim secundum ok =

$$\frac{Qdydz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

et vim secundum Mc =

$$\frac{Qdy}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Quare, si haec vis sola ageret, haberetur

$$T=-\frac{Qdy}{\sqrt{(dx^2+dy^2+dz^2)}},\ N=-\frac{Qdydz}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}\ \text{ et }\ M=\frac{Qdx}{\sqrt{(dx^2+dy^2)}}.$$

Denique vis R directionem MQ habens resolvitur in vim secundum Md =

$$\frac{Rdz}{\sqrt{(dx^2+dy^2+dz^2)}}$$

et vim secundum dQ =

$$\frac{R\sqrt{(dx^2+dy^2)}}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Et vi R igitur foret

$$T=-\frac{Rdz}{\sqrt{(dx^2+dy^2+dz^2)}}\quad\text{et}\quad N=\frac{R\sqrt{(dx^2+dy^2)}}{\sqrt{(dx^2+dy^2+dz^2)}}.$$

Omnibus igitur hisce tribus viribus P, Q, et R simul agentibus erit vis tangentialis ex omnibus orta

$$T=\frac{-Pdx-Qdy-Rdz}{\sqrt{(dx^2+dy^2+dz^2)}},$$

vis normalis, in plano Mq quae posita est,

$$N=\frac{-Pdxdz-Qdydz+Rdx^2+Rdy^2}{\sqrt{(dx^2+dy^2)(dx^2+dy^2+dz^2)}}$$

atque altera vis normalis

$$M=\frac{-Pdy+Qdx}{\sqrt{(dx^2+dy^2)}}.$$

His valoribus loco T, N et M in aequationibus (809) substitutis prodibunt tres sequentes aequationes

$$dv=-Pdx-Qdy-Rdz,$$

$$\frac{2vdydzddy-2vddz(dx^2+dy^2)}{dx^2+dy^2+dz^2}=-Pdxdz-Qdydz+R(dx^2+dy^2)$$

atque

$$\frac{2vdxddy}{dx^2+dy^2+dz^2} = Pdy - Qdx.$$

[p. 350] Quae aequationes motum corporis determinant. Q.E.I.

Corollarium 1.

824. Duae hae posteriores aequationes dant istam analogiam :

$$dydzddy - ddz(dx^2+dy^2) : dxddy = -Pdxdz - Qdydz + R(dx^2+dy^2) : Pdy - Qdx.$$

Ex qua reperitur

$$ddy : ddz = Pdy - Qdx : Pdz - Rdx.$$

Corollarium 2.

825. Planum ergo $Mm\mu$ (Fig. 76), in quo sunt elementa Mm et $m\mu$, hoc modo definitur. Erit

$$AO = x + \frac{-Pydz+Pzdy-Qzdx+Rydx}{Qdz-Rdy} = \frac{-Pydz+Pzdy+Qxdz-Qzdx+Rydx-Rxdy}{Qdz-Rdy}$$

et

$$\text{tangens anguli } POR = \frac{-Qdz+Rdy}{Pdz-Rdx}$$

atque tangens anguli, quem planum $Mm\mu$ facit cum plano APQ,

$$= \frac{\sqrt{((Pdz-Rdx)^2+(Qdz-Rdy)^2)}}{Pdy-Qdx}$$

Corollarium 3.

826. Si potentia P evanescat, reperitur

$$Q = -\frac{2vddy}{dx^2+dy^2+dz^2} \quad \text{et} \quad R = -\frac{dv}{dz} + \frac{2vdyddy}{dz(dx^2+dy^2+dz^2)}$$

ex duabus aequationibus inventarum. Et ex tertia prodibit

$$\frac{dv}{2v} = \frac{dyddy+dzddz}{dx^2+dy^2+dz^2},$$

cuius integralis est

$$vdx^2 = a(dx^2+dy^2+dz^2) \quad \text{seu} \quad dx\sqrt{v} = \sqrt{a(dx^2+dy^2+dz^2)}.$$

Corollarium 4.

827. Hac igitur hypothesi erit tempus, seu $\int \frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}$,

$$=\int \frac{dx}{\sqrt{a}}=\frac{x}{\sqrt{a}}.$$

Ex quo intelligitur motum corporis progressivum secundum axi *AP* parallas esse uniformem. [p. 351]

Corollarium 5.

828. Eadem porro hypothesi erit

$$Q=-\frac{2addy}{dx^2} \quad \text{et} \quad R=-\frac{2addz}{dx^2}$$

propter $ddy : ddz = Q : R$ (824). Ex quibus aequationibus ipsa curva a corpore descripta determinabitur.

Scholion.

829. Ex resolutione potentiarum facile perspicitur ad tres potentias, quas in hac propositione consideravimus, omnes omnino potentias, quae excogitari possunt, reduci posse. Quare, cum datis his potentiis non difficulter curva a corpore descripta inveniatur, etiam pro quibusque casibus propositis ista propositio maximam habebit utilitatem.

PROPOSITIO 101.

PROBLEMA.

830. *Si corpus perpetuo urgeatur versus axem AP* (Fig. 78) *secundum perpendicula MP a corpore ad axem demissa, determinari oportet motum corporis.*

SOLUTIO.

Ductis coordinatis ut ante $AP = x$, $PQ = y$ et $QM = z$ erit $MP = \sqrt{(y^2+z^2)}$. Sit vis secundum MP agens $= V$ eaque resolvatur in duas secundum MQ et Mg trahentes, ubi Mg est parallela ipsi PQ eique aequalis[p. 352]. Erit igitur vis secundum MQ agens $= \frac{Vz}{\sqrt{(y^2+z^2)}}$.

His cum propositione praecedente comparatis erit $P = 0, Q = \frac{Vy}{\sqrt{(y^2+z^2)}}$ et $R = \frac{Vz}{\sqrt{(x^2+z^2)}}$.

Quare habebitur

$ddy : ddz = Q : R = y : z$ (828) atque

$yddz = zddy$ seu $yddz - zddy = 0,$

cuius aequationis integralis est

$ydz - zdy = bdx.$

Porro erit $\frac{Vy}{\sqrt{(y^2+z^2)}} = \frac{2addy}{dx^2}$.

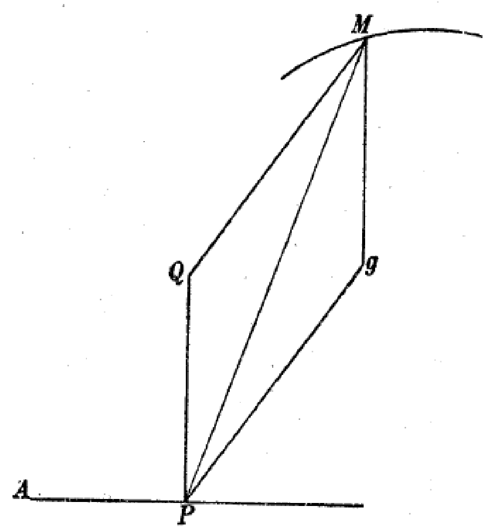


Fig. 78.

Quae aequationes coniunctae determinant curvam a corpore descriptam.

Corporis autem celeritas dabitur per aequationem

$$vdx^2 = a(dx^2+dy^2+dz^2)$$

seu ipsa celeritas erit $= \frac{\sqrt{a(dx^2+dy^2+dz^2)}}{dx}$.

Q.E.I.

Corollarium 1.

831. Ponatur $dx = pdy$, erit ob dx constans

$$0 = pddy + dpdy \quad \text{seu} \quad ddy = -\frac{dpdy}{p}.$$

His substitutis ad cognoscendam curvam descriptam habebuntur istae aequationes

$$ydz - zdy = bpdy \quad \text{et} \quad \frac{Vy}{\sqrt{(y^2+z^2)}} = \frac{2adp}{p^3 dy}.$$

Corollarium 2.

832. Si porro ponatur $z = qy$, istae aequationes transibunt in

$$y^2 dq = bpdy \quad \text{et} \quad \frac{V}{\sqrt{(1+qq)}} = \frac{2adp}{p^3 dy}.$$

Quae etiam tres continent variables.

Scholion 1.

833. Ad haec clarius exponenda maxime convenit exempla adhibere. Quamobrem aliquot afferemus, in quibus vis V a distantia MP pendere ponitur, eamque potestatibus aliquibus ipsius MP proportionalem ponemus, quo iste motus cum [p. 353] motu in plano a vi centripeta distantiarum potestati cuidam proportionali orto comparari possit. Inter hos enim casus magna est similitudo, cum, quod est in plano centrum virium, hoc loco est quasi axis virium, ad quem corpus perpetuo attrahitur. Atque si initio corpus ita proiiciatur, ut non habeat motum progressivum secundum axem AP, motus eius fiet in plano PQM, et corpus attrahetur perpetuo ad punctum P, centrum virium.

Exemplum 1.

834. Sit vis V distantiae MP directe proportionalis ponaturque $V = \frac{\sqrt{(y^2+z^2)}}{f}$. Erit ergo $\frac{y}{f} = \frac{2adp}{p^3 dy}$ (831) et integrando

$$\frac{y^2}{2f} = c - \frac{a}{p^2} \quad \text{seu} \quad \frac{1}{pp} = \frac{2cf - yy}{2af} \quad \text{et} \quad p = \frac{\sqrt{2af}}{\sqrt{(2cf-y^2)}}.$$

Cum autem sit (832)

$$dq = \frac{bpdy}{y^2}, \quad \text{erit} \quad \frac{bdy\sqrt{2af}}{y^2\sqrt{(2cf-y^2)}} = dq.$$

Cuius integralis est

$$q = \alpha - \frac{\beta\sqrt{(2cf-y^2)}}{y} \quad \text{denotante} \quad \beta = \frac{b\sqrt{2af}}{2cf}.$$

Quare habebitur

$$z = \alpha y - \beta\sqrt{(2cf - y^2)},$$

quae aequatio exprimit proiectionem curvae descriptae in plano ad axem AP normali, quam igitur perspicitur esse ellipsin, cuius centrum in axe AP est positum. Deinde cum sit $dx = pdy$, erit $dx = \frac{dy\sqrt{daf}}{\sqrt{(2cf-y^2)}}$, quae aequatio exprimit proiectionem curvae quaesitae in plano APQ. Haec itaque est linea sinuum Leibnitiana, cum abscissa x sit ut arcus, cuius sinus est applicata y.

Exemplum 2.

835. Sit fuerit vis V reciproce ut quadratum distantiae MP seu

$$V=\frac{ff}{y^2+z^2}=\frac{ff}{yy(1+q^2)}$$ ob $z=qy$. [p. 354]

Quamobrem habebitur

$$\frac{ff}{y^2(1+q^2)^{\frac{3}{2}}}=\frac{2adp}{p^3dy}.$$

Quia autem est $dy=\frac{y^2dq}{bp}$ (832), erit

$$\frac{f^2dq}{(1+q^2)^{\frac{3}{2}}}=\frac{2abdp}{p^2};$$

cuius integralis est

$$\frac{ffq}{\sqrt{(1+qq)}}=C-\frac{2ab}{p}=C-\frac{2ab^2dy}{y^2dq}$$

loco p ipsius valore $\frac{y^2dq}{bdy}$ substituto. Hinc fit

$$\frac{f^2qdq}{\sqrt{(1+q^2)}}=Cdq-\frac{2ab^2dy}{y^2}$$

et integrando

$$f^2\sqrt{(1+q^2)}=Cq+\frac{2ab^2}{y}+D.$$

Quare, cum sit $q=\frac{z}{y}$, prodibit

$$f^2\sqrt{(y^2+z^2)}=Cz+Dy+2ab^2.$$

Quae est aequatio pro proiectione curvae descriptae in plano ad axem AP normali; quam igitur colliti potest esse ad sectionem conicam, cuius alteruter focus sit in axe AP positus.

Scholion 2.

836. Ex his intelligitur proiectiones curvaru descriptarum in plano ad axem AP normali congruere cum curvis, quas corpora in hoc plano mota describerent ab eadem vi solliciatat. Neque autem hoc mirum est; nam motus, quem hoc loco consideramus, reduci potest ad motum in plano ad axem AP normali factum a corpore ad axem perpetuo attracto, dummodo huic plano motus uniformis secundum axem AP impressus concipiatur. Namque iste motus progressivus, quia fit uniformiter in directum, motum corporis in plano turbare nequit. Haec igitur convenientia iam deduci potuisset ex (827), ubi, si vis P evanescit, motus corporis [p. 355] secundum axem progressivus aequabilis est ostensus. Quamobrem quoties vis P in nihilum abit, tum semper motus quaesitus ad motum in plano factum potest reduci. Hoc scilicet fiet, si plano ad axem AP normali tantus motus retro secundum PA imprimatur, quantum habere inventum est secundum AP progressivum (827).

Scholion 3.

837. Interim tamen hoc maxime attendi meretur, quod tam facile in exemplis propositis aequationes inter coordinatas orthogonales pro curvarum proiectionibus in plano ad axem AP normali atque adeo pro ipsis curvis a corpore descriptis, si motus progressivus evanescat, invenerimus. In huius capitis enim priore parte, qua motus in plano a vi centripeta generatos consideravimus, multo maiore opus fuit labore et comparatione arcuum circularium, ut ad aequationes consuetas pro curvis descriptis pervenerimus. Maior igitur generalitas, quae saepissime inventionem quaesiti difficiliorem reddit, hoc loco non solum non est impedimento, sed etiam facillime id determinat, quod in particulariori sensu difficile erat inventu.

Corollarium 3.

838. In casu huius propositionis planum elementorum *Mm* et $m\mu$ (Fig. 76) facile determinatur. Nam ob $ddy : ddz = y : z$ erit $PO = 0$ et $AO = x$ incidetque *O* in *P*. [p. 356] Porro tangens anguli $PQR = \frac{ydz - zdy}{zdx} = \frac{b}{z}$ ob $ydz - zdy = bdx$. Cotangens igitur anguli *POR* est ut *QM*. Denique tangens anguli inclinationis plani $Mm\mu$ ad planum *APQ* est = $\frac{\sqrt{(z^2+b^2)}}{y}$.

Scholion 4.

839. Cum casus, quo vis *P* evanescit, ad motum in plano possit reduci, reduci quoque poterit ad motum in plano, si vel *Q* vel *R* desit. Nam si axis capiatur in recta ad *AP* normali in plano *APQ*, vis *Q* directionem habebit axi parallelam et reliquae *P* et *R* tractabuntur ut ante *Q* et *R*. At si axis sumatur normalis ad *AP* et ad planum *APQ*, vis *R* locum vis *P* axi parallelae occupabit. Scilicet quemadmodum coordinatae *x, y* et *z* respectu situs inter se possint commutari, similiter etiam de viribus *P, Q* et *R* est iudicandum.

PROPOSITIO 102.

PROBLEMA.

840. *Si corpus in singulis punctis M* (Fig. 79) *duplici vi sollicitetur, una, cuius directio est MA, et altera, cuius directio est MQ normalis ex M in planum APQ demissa, oportet determinari motum corporis M et orbitam eius.*

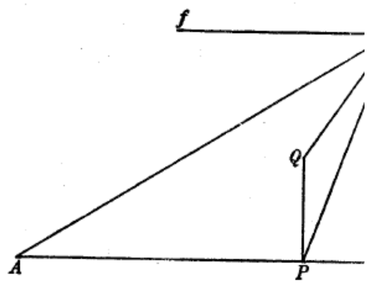
SOLUTIO.

Ducta *MP*, quae sit normalis in *AP*, vis *MA* resolvatur in vires secundum *Mf* ipsi *AP* parallelam et secundum *MP* agentes. Haec vero vis iuxta *MP* resolvitur [p. 357] in vires secundum *MQ* et *Mg* agentes. Positis igitur ut ante $AP = x$, $PQ = y$ et $QM = z$ et vi secundum *MA* trahents = V et vi secundum $MQ = W$ atque resolutione virium instituta et comparatione facta cum prop. 100 (823) reperietur

$$P = \frac{Vx}{\sqrt{(x^2+y^2+z^2)}}, \quad Q = \frac{Vy}{\sqrt{(x^2+y^2+z^2)}}$$

et

$$R = W + \frac{Vz}{\sqrt{(x^2+y^2+z^2)}}.$$



Atque ex aequationibus eiusdem propositionis prodibit

$$V = \frac{2v\,dx\,ddy\sqrt{(x^2+y^2+z^2)}}{(xdy-ydx)(dx^2+dy^2+dz^2)} \quad \text{et} \quad W = \frac{2v\,ddy(xdz-zdx)-2v\,ddz(xdy-ydx)}{(xdy-ydx)(dx^2+dy^2+dz^2)}.$$

Ex his invenitur

$$\frac{dv}{2v} = \frac{dyddy+dzddz}{dx^2+dy^2+dz^2} - \frac{xddy}{xdy-ydx}$$

hincque integrando

$$l\sqrt{v} = l\frac{\sqrt{(dx^2+dy^2+dz^2)}}{xdy-ydx} + la\sqrt{a}$$

seu

$$\sqrt{v} = \frac{a\sqrt{a}(dx^2+dy^2+dz^2)}{xdy-ydx} \quad \text{sive} \quad v = \frac{a^3(dx^2+dy^2+dz^2)}{(xdy-ydx)^2}.$$

Hic valor loco v substitutus dat has aequationes

$$V = \frac{2a^3dx\,ddy\sqrt{(x^2+y^2+z^2)}}{(xdy-ydx)^3} \quad \text{et} \quad W = \frac{2a^3ddy(xdz-zdx)-2a^3ddz(xdy-ydx)}{(xdy-ydx)^3}.$$

Ex quibus curva descripta determinatur. Q.E.I.

Corollarium 1.

841. Tempus, quo corpus in M usque pervenit, est

$$\int \frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{v}}.$$

Cum autem sit

$$\sqrt{v} = \frac{a\sqrt{a}(dx^2+dy^2+dz^2)}{xdy-ydx},$$

erit illud tempus

$$\frac{\int xdy - \int ydx}{a\sqrt{a}},$$

quod per quadraturam proiectionis curvae descriptae in plano *APQ* cognoscitur.

Corollarium 2.

842. Si ponatur $y = px$ et $z = qx$, prodibunt sequentes aequationes [p. 358]

$$V = \frac{2a^3dx(xddp+2dxdp)\sqrt{(1+p^2+q^2)}}{x^5dp^3} \quad \text{et} \quad W = \frac{2a^3dqddp-2a^3dpddq}{x^3dp^3}.$$

Quae ad curvam cognoscendam inserviunt.

Exemplum.

843. Si vis *V* fuerit distantiae *MA* proportionalis et vis *W* perpendiculo *MQ*, ponatur

$$V = \frac{\sqrt{(x^2+y^2+z^2)}}{f} \quad \text{et} \quad W = \frac{z}{g},$$

ut itaque sit

$$V = \frac{x\sqrt{(1+p^2+q^2)}}{f} \quad \text{et} \quad W = \frac{qx}{g}.$$

Quare per aequationes praecedentes corollarii erit

$$x^6dp^3 = 2a^3fdx(xddp+2dxdp) \quad \text{et} \quad x^4qdp^3 = 2a^3gdqddp - 2a^3gdpddq.$$

Illius aequationis integralis est $x^2 = C - \frac{2a^3fdx^2}{x^4dp^2}$, ex qua oritur

$$dp = \frac{adx\sqrt{2af}}{x^2\sqrt{(c^2-x^2)}},$$

huius integralis

$$p = \frac{y}{x} = C - \frac{a\sqrt{2af(c^2-x^2)}}{c^2x} \quad \text{seu} \quad y = nx - \frac{a\sqrt{2af(c^2-x^2)}}{c^2}$$

pro aequatione proiectionis curvae descriptae in plano *APQ*, quae igitur est ellipsis, cuius centrum in *A* est positum. Ex invento ipsius *dp* valore erit porro

$$ddp = -\frac{adx^2(2c^2-3x^2)\sqrt{2af}}{x^3(c^2-x^2)^{\frac{3}{2}}};$$

quibus loco *dp* et *ddp* valoribus in altera aequatione substitutis orietur

$$afqxdx^2 = -2accgdxdq + 3agx^2dxdq - ac^2gxddq + agx^3ddq.$$

Ponatur $q = e^{\int rdx}$, prodibit

$$fxdx = -c^2gxdr + gx^3dr - 2ccgrdx + 3gx^2rdx - c^2gxr^2dx + gx^3r^2dx.$$

Fiat $r = \frac{u}{x^2\sqrt{(c^2-x^2)}}$, provenietque

$$fxdx = -\frac{gdu\sqrt{(cc-x^2)}}{x} - \frac{gu^2dx}{x^3} \quad \text{seu} \quad du + \frac{u^2dx}{x^2\sqrt{(c^2-x^2)}} + \frac{fx^2dx}{g\sqrt{(c^2-x^2)}} = 0.$$

Quae positi $t = \frac{\sqrt{(c^2-x^2)}}{x}$ seu $x = \frac{c}{\sqrt{(1+tt)}}$ transibit in hanc :

$$du - \frac{u^2dt}{cc} = \frac{ccfdt}{g(1+tt)^2};$$

cuius integrale postea exhibebimus.

Ad planum cognoscendum, in quo elementa *Mm* et $m\mu$ (Fig. 76) sunt sita, ob [p. 359]

$$ddy : ddz = gxdy - gydx : gxdz - hzdx$$

posito *h* loco *f* + *g* reperietur

$$AO = \frac{fz(xdy - ydx)}{hzdy - gydz},$$

$$\text{tang. ang. } POR = \frac{hzdy - gydz}{gxdz - hzdx}$$

atque tangens anguli plani $Mm\mu$ cum plano *APQ* =

$$\frac{\sqrt{((hzdy - gydz)^2 + (gxdz - hzdx)^2)}}{g(xdy - ydx)}.$$

Denique tempus, quo corpus in M pervenit, cum sit $= \frac{\int xxdp}{a\sqrt{a}}$, erit =

$$\int \frac{dx\sqrt{2f}}{\sqrt{(c^2-x^2)}}.$$

Erit itaque hoc tempus proportionale angulo, cuius sinus est abscissa *x*, existente sinu toto = *c*, seu cuius sinus est $\frac{x}{c}$, si sinus totus capiatur = 1. Ex quo perspicitur motum corporis in proiectione in plano *APQ* facta angularem circa A esse uniformem et tempus unius revolutionis esse ut $\sqrt{f}$.

Corollarium 3.

844. Proiecto curvae descriptae in plano *APQ* fit circulus, si est $n = 0$ et $-a\sqrt{2af} = cc$; cuius centrum est in *A* et radius = *c*. Erit igitur $y = \sqrt{(c^2 - x^2)}$. Atque z dabitur ex hac aequatione $\frac{gddz}{dx} = \frac{gxdz - hzdx}{cc - xx}$, quae aeque late patet ac casus exempli praecedentis, etiamsi hic casus particularis sit consideratus.

Corollarium 4.

845. Ad inveniendum ipsius z valorem ex aequatione $\frac{gddz}{dx} = \frac{gxdz - hzdx}{cc - xx}$ pono $z = e^{\int rdx}$. Quo facto prodibit aequatio differentialis primi gradus haec :

$$gdr + gr^2dx = \frac{grxdx - hdx}{cc - xx}.$$

Fiat $r = \frac{u}{\sqrt{(c^2 - x^2)}}$ prodibitque

$$gdu + \frac{gu^2dx}{\sqrt{(c^2 - x^2)}} + \frac{hdx}{\sqrt{(c^2 - x^2)}} = 0.$$

[p. 360] Posito $\frac{h}{g}$ seu $\frac{f+g}{g} = m^2$ orietur aequatio ista :

$$\frac{du}{u^2 + m^2} + \frac{dx}{\sqrt{(c^2 - x^2)}} = 0,$$

in qua indeterminatae iam sunt a se invicem separatae.

Corollarium 5.

846. Est vero

$$\int \frac{dx}{\sqrt{(c^2 - x^2)}} = +\sqrt{-1}\, l\left(\sqrt{(cc - xx)} - x\sqrt{-1}\right)$$

et

$$\int \frac{du}{u^2 + m^2} = \frac{1}{2m\sqrt{-1}}\, l\frac{u - m\sqrt{-1}}{u + m\sqrt{-1}}.$$

Addita igitur constante et sumtis numeris habebitur

$$\left(\frac{\sqrt{(cc - xx)} - x\sqrt{-1}}{b}\right)^{2m} = \frac{u - m\sqrt{-1}}{u + m\sqrt{-1}}$$

hincque

$$u = \frac{b^{2m} + \left(\sqrt{(c^2 - x^2)} - x\sqrt{-1}\right)^{2m}}{b^{2m} - \left(\sqrt{(c^2 - x^2)} - x\sqrt{-1}\right)^{2m}} \cdot m\sqrt{-1}.$$

Corollarium 6.

847. Cum vero sit $lz = \int rdx$ et $r = \frac{u}{\sqrt{(c^2-x^2)}}$, erit

$$lz = \int \frac{mdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}.$$

Ponatur $\int \frac{dx}{\sqrt{(c^2-x^2)}} = s,$ erit

$$s = \sqrt{-1}\, l \frac{\sqrt{(c^2-x^2)} - x\sqrt{-1}}{b} \qquad \text{et} \qquad e^{\frac{s}{\sqrt{-1}}} b = \sqrt{(c^2-x^2)} - x\sqrt{-1}$$

atque

$$lz = \int \frac{mds\left(1 + e^{\frac{2ms}{\sqrt{-1}}}\right)\sqrt{-1}}{1 - e^{\frac{2ms}{\sqrt{-1}}}},$$

seu posito $e^{\frac{2ms}{\sqrt{-1}}} = t,$ ut sit $ds = \frac{dt\sqrt{-1}}{2mt}$, erit

$$lz = \int \frac{-dt(1+t)}{2t(1-t)} = l\frac{(1-t)k}{\sqrt{t}},$$

unde fit

$$z = \frac{\left(1 - e^{\frac{2ms}{\sqrt{-1}}}\right)k}{e^{\frac{ms}{\sqrt{-1}}}} = \frac{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)k}{b^m\left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^m}.$$

Corollarium 7. [p. 361]

848. Invento nunc valore ipsius z erit

$$dz = \frac{mzdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}$$

atque

$$\frac{gxdz - hzdx}{z}$$
$$= \frac{mgdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)x\sqrt{-1} - m^2gdx\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{(c^2-x^2)}}.$$

Porroque posito $y = \sqrt{(c^2-x^2)}$ erit

$$\frac{hzdy - gydz}{z} = \frac{m^2gxdx}{\sqrt{(c^2-x^2)}} - \frac{mgdx\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}}.$$

Corollarium 8.

849. Ex his denique invenietur

$$AO = \frac{fcc\left(b^{2m} - \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right)}{m^2gx\left(b^{2m} - \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right) + mg\left(b^{2m} + \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right)\sqrt{-1}\,(c^2-x^2)}$$

et tang. $PQR =$

$$\frac{mx\left(b^{2m} - \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right) + \left(b^{2m} + \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right)\sqrt{-1}\,(c^2-x^2)}{m\left(b^{2m} - \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right)\sqrt{(c^2-x^2)} - x\left(b^{2m} + \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m}\right)\sqrt{-1}}.$$

Simili modo ex his invenitur angulus inclinationis plani, in quo corpus movetur, ad planum *APQ*. [p. 362]

Scholion.

850. Inventorum valorum ipsius z et orbitae inclinationis applicatio fit admodum difficilis ob quantitates imaginarias invicem permixtas. Hanc obrem iis longius immorari noluimus ad intersectiones curvae a corpore descriptae cum plano *APQ* determinandas. Quia autem magni est momenti haec quaestio in astronomia ad motum nodorum inveniendum, sequens huic negetio destinata est propositio, in quo loco, ubi corpus motu suo in planum *APQ* perveniat, investigabimus. Corpus enim partim supra hoc planum motum suum absolvit, partim infra; et sive supra sit sive infra, corpus perpetuo trahitur altera vi W ad hoc planum in ratione directa distantiae ab hoc plano. Punctum vero in plano *APQ*, per quod corpus ex superiore parte in inferiorem transit, vocatur nodus descendens, punctum vero, per quod in superiora revertitur, nodus ascendens.

PROPOSITIO 103.

PROBLEMA.

851. *Si corpus attrahatur perpetuo partim ad punctum fixum A* (Fig. 79) *in ratio distantiarum ab eodem, partim normaliter ad planum APQ in ratione quoque distantiarum ab hoc plano, determinare nodes seu puncta, in quibus corpus in hoc planum pervenit, et praeterea etiam puncta,* [p. 363] *in quibus corpus ab hoc plano maxime distat.*

SOLUTIO.

Manentibus ut ante tribus coordinatis x, y et z atque vi, qua corpus ad A trahitur, $= \frac{\sqrt{(x^2+y^2+z^2)}}{f}$ ac vi, qua corpus ad planum *APQ* trahitur, $\frac{z}{g}$ positoque $\frac{f+g}{g} = m^2$ manifestum est corpus in ipsum planum *APQ* incidere, ubir erit $z = 0$. Fit autem $z = 0$, quoties est

$$b^{2m} - \left(\sqrt{(c^2-x^2)} - x\sqrt{-1}\right)^{2m} = 0$$

(847). Hoc evenit, quoties est $u = \infty$ (846). Cum igitur sit (845)

$$\frac{du}{u^2+m^2}+\frac{dx}{\sqrt{(c^2-x^2)}}=0,$$

centro *A* describatur circulus *BQC* (Fig. 80) radio *AB = c* corpusque iuxta plagam *BQC* moveatur et loco illius aequationis sumatur haec aequivalens

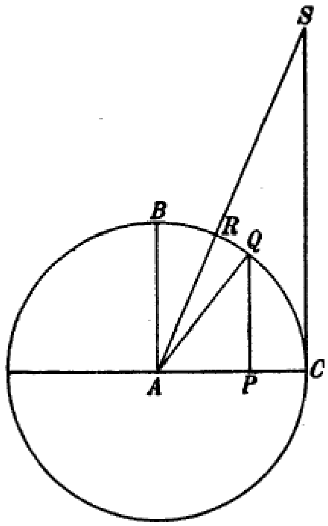


Fig. 80.

$$\frac{\frac{c^3}{m^2}du}{\frac{c^2u^2}{m^2}+c^2}+\frac{cdx}{\sqrt{(c^2-x^2)}}=0 \quad \text{seu} \quad \frac{\frac{1}{m}\cdot cc\cdot\frac{cdu}{m}}{\frac{c^2u^2}{m^2}+c^2}=-\frac{cdx}{\sqrt{(c^2-x^2)}}.$$

Cuius integralis est

$$\frac{1}{m}\int\frac{cc\cdot\frac{cdu}{m}}{\frac{c^2u^2}{m^2}+c^2}=C-\int\frac{cdx}{\sqrt{(c^2-x^2)}},$$

in qua

$$\int\frac{cc\cdot\frac{cdu}{m}}{\frac{ccuu}{m^2}+c^2}$$

exprimit arcum, cuius tangens est $\frac{cu}{m}$, et

$$\int\frac{cdx}{\sqrt{(c^2-x^2)}}$$

arcum *BQ* cuius sines est $AP = x$. Sit *C* arcus BQC seu quadrans circuli et tangens $CS=\frac{cu}{m}$, cui respondeat arcus *CR*. Transibit ergo illa aequatio in hanc $\frac{1}{m}CQR = CQ$ atque $CR = m.CQ.$ Ex hoc perspicuum est fore $u=\infty$, [p. 364] quoties angulus *CAR* fuerit rectus vel aequalis tribus aut quinque aut 7 etc. Sequendo itaque corporis motum erit *u* infinitum, si arcus *CR* successive fiat aequalis sequentibus graduum numeris :

$90, -90, -270, -450, -630$ etc.

Tunc autem arcus CQ tenebit gradus :

$\frac{90}{m}, -\frac{90}{m}, -\frac{270}{m}, -\frac{450}{m}, -\frac{630}{m}$ etc. atque arcus *BQ* :

$90-\frac{90}{m}, 90+\frac{90}{m}, 90+\frac{270}{m}, 90+\frac{450}{m}, 90+\frac{630}{m}$ etc.

Quare si corpus alicubi fuerit in nodo, ad alios nodos perveniet successive absolutis motu angulari circa *A* angulis graduum

$\frac{180}{m}, \frac{360}{m}, \frac{540}{m}, \frac{720}{m}$ etc.

Duo igitur nodi proximi a se invicem distabunt angulo $\frac{180}{m}$ graduum. Atque nodus ascendens vel descendens distabit a sequente nodo eiusdem nominis angulo $\frac{360}{m}$ graduum, i. e, angulo $\frac{360.\sqrt{g}}{\sqrt{(g+f)}}$ graduum. Q. E. Prius.

Maxime deinde corpus a plano *APQ* distabit, ubi erit $dz = 0$; id quod evenit, quoties

$$b^{2m} + (\sqrt{(c^2 - x^2)} - x\sqrt{-1})^{2m} = 0$$

(848). His autem in casibus fit $CR = 0$. Manente ergo priore constructione evadet $u = 0$, quoties fit $CR = 0$ vel $- 180$ vel $- 360$ etc. gradibus. Tum autem arcus CQ continebit gradus :

$0, -\frac{180}{m}, -\frac{360}{m}, -\frac{540}{m}$ etc.

ideoque arcus BQ habebit gradus :

$90, 90+\frac{180}{m}, 90+\frac{360}{m}, 90+\frac{540}{m}$ etc.

Maxima igitur corporis a plano APQ distantia aequaliter distat a nodis utrinque proximis. Q. E. Posterius. [p. 365]

Corollarium 1.

852. Nodi igitur in idem tandem recident punctum, si fuerit m numerus rationalis seu $\frac{f+g}{g}$ numerus quadratus. At si $\frac{f+g}{g}$ non fuerit numerus quadratus, corpus nunquam in eodem puncto in planum APQ incidet, in quo ante aliquando inciderit.

Corollarium 2.

853. Si g est numerus affirmativus seu si corpus perpetuo ad planum attrahatur vi positiva, erit m numerus unitate maior. Intervallum ergo inter duos nodos proximos minus erit quam angulus 180 graduum. Quare nodi retrogrediuntur et nodus sequens ab oppositione praecedentis nodi distabit angulo $\frac{180(m-1)}{m}$ graduum.

Corollarium 3.

854. Si corpus perpetuo a plano APQ repellatur, fit g numerus negativus atque $m=\sqrt{\frac{g-f}{g}}$. Quare si $g > f$, est m numerus realis, sed unitate minor; tum igitur nodi in consequentia progrediuntur eo celerius, quo minus f ab g distabit. Atque si $f = g$, tum corpus e nodo egressum nunquam in superficiem APQ revertetur. At si $f > g$, corpus perpetuo ab hoc plano discedet. [p. 366]

Corollarium 4.

855. Quia corpus a plano APQ maxime distat, quando sit

$$(\sqrt{(c^2 - x^2)} - x\sqrt{-1})^{2m} = -b^{2m},$$

ipsa maxima distantia habebitur, si in valore ipsius z invento (847) haec substitutio. Invenietur autem haec maxima distantia $=\frac{2k}{\sqrt{-1}}$, quae igitur ubique est eadem.

Corollarium 5.

856. Si hic circulus *BQC* (Fig. 80) fuerit proiectio orbitae a corpore descriptae in plano *APQ*, erit anguli inclinationis plani orbitae ad planum *APQ* tangens $= \frac{2m^2k}{c\sqrt{-1}}$ in locis, ubi corpus maxime ab hoc plano distat, seu $= \frac{2m^2k}{c}$ posito k loco $= \frac{k}{\sqrt{-1}}$, cum constans k talem debeat habere valorem ad imaginaria evitanda.

Corollarium 6.

857. Eadem manente hypothesi in locis, ubi corpus in planum *APQ* incidit, tangens anguli inclinationis erit $= \frac{dz\sqrt{(c^2-k^2)}}{cdx}$ existente

$$(\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m} = b^{2m}.$$

Fiet itaque ista tangens =

$$\frac{mz\left(b^{2m} + (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)\sqrt{-1}}{\left(b^{2m} - (\sqrt{(c^2-x^2)} - x\sqrt{-1})^{2m}\right)c}$$

et loco z suo posito valore (847)

$$= \frac{2mk\sqrt{-1}}{c},$$

[p. 367] seu idoneo loco k valore substituto erit illa tangens $= \frac{2mk}{c}$.

Corollarium 7.

858. Tangens igitur inclinationis orbitae a corpore descriptae ad planum *APQ*, si corpus a plano *APQ* maxime distat, est ad eandem tangentem, si corpus in hoc planum incidit, ut m ad 1. Fiet ergo ut distantia duorum nodorum ad 180 gradus, ita inclinatio orbitae, si corpus a nodis maxime distat, ad inclinationem orbitae a corpore descriptae, si corpus in ipsius versatur nodis.

Scholion.

859. Haec quidem propositio parum utilitatis habere videtur in astronomia, eo quod vim, qua corpus ad punctum fixum A trahitur, distantia proportionalem faciamus, in corporibus vero coelestibus vis reciproce quadratis distantiarum proportionalis locum habeat. Usus tamen eius eximius est, si orbitae corporum non multum a circulis differant; nam orbita in circulum abeunte nihil interest, quomodocunque vis centripeta a distantiis pendeat. Quamobrem, cum orbitae planetarum non multum a circulus discrepent, haec propositio [p. 368] bono cum successu ad eos motus potest accommodari. Hocque tum maxime est faciendum, ut linea f inveniatur, quae se habeat ad corporis a centro distantiam ut vis gravitatis ad vim centripetam. Altera vis, quae corpus ad planum datum trahit, quodammodo distantiae proportionalis effici potest ; id tamen si non accidat, litera g ut variabilis debet considerari, ex quo vero proxime motus nodorum poterit colligi inter omnes ipsius g valores quasi medium elegendo. In motu lunae nodorum motus maxime attendi meretur, quippe qui iuxta nostram determinationem fit in antecedentia. Observatur autem nodi ab oppositione praecedentis nodi distantia fere 43', ita ut sit

$$\frac{180(m-1)}{m} = \frac{43}{60} \quad \text{atque} \quad m = 1 + \frac{43}{10757} = 1\frac{1}{250}.$$

Ex quo vis lunam ad planum eclipticae perpetuo trahens a posteriore potest cognosi.