CHAPTER FIVE (*Part d*).

CONCERNING THE CURVILINEAR MOTION OF FREE POINTS ACTED ON BY ABSOLUTE FORCES OF ANY KIND.

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PROPOSITION 92. [p. 316]

PROBLEM.

763. *To find the centripetal forces acting towards two centres C and D* (Fig. 69), *which can be composed, in order that the body moves on a given curve AMB and with the speed given at individual points M.*

SOLUTION.

764. The body is moving from *A* through *M* to *B* and its speed at *M* corresponds to the height *v*; and putting *CM* = *y*, *DM* = *z*. Truly with the tangent *TV* drawn at that point, and with perpendiculars *CT* and *DV* sent from *C* and *D* , called *CT* = *p* and *DV* = *q*. Again the centripetal force, that attracts towards *C* is equal to *P*, and that which pulls towards *D* is equal to *Q*. Therefore the normal force arising from each is equal to

$$\frac{Pp}{y}+\frac{Qq}{z}$$

and the tangential force accelerating the motion of the body is equal to

$$-\frac{P\sqrt{(y^2-p^2)}}{y}-\frac{Q\sqrt{(z^2-q^2)}}{z}$$

for falling tangents as previously. Therefore with the radius of osculation at *M* equal to *r* and with the element of the curve equal to *ds* then [p. 317]

$$\frac{Pp}{y}+\frac{Qq}{z}=\frac{2v}{r}$$

(561) and

$$dv=-\frac{Pds\sqrt{(y^2-p^2)}}{y}-\frac{Qds\sqrt{(z^2-q^2)}}{z}$$

(559). Truly we have :

$$ds=\frac{ydy}{\sqrt{(y^2-p^2)}}=\frac{zdz}{\sqrt{(z^2-q^2)}}$$

and hence $dv=-Pdy-Qdz$.

From these two equations joined together [on eliminating *ds*] we have :

$$P=\frac{2vyzdz+qrydv}{przdz-qrydy}$$

and

$$Q = \frac{2vzydy + przdv}{qrydy - przdz} = \frac{-2vzydy - przdv}{przdz - qrydy}.$$

For indeed:

$$r = \frac{ydy}{dp} = \frac{zdz}{dq} \quad \text{and} \quad zdz = \frac{ydy\sqrt{(z^2 - q^2)}}{\sqrt{(y^2 - p^2)}},$$

hence

$$dq = \frac{dp\sqrt{(z^2 - q^2)}}{\sqrt{(y^2 - p^2)}}.$$

Finally, calling $CD = k$ then

$$k^2 = y^2 + z^2 - 2pq - 2\sqrt{(y^2 - p^2)(z^2 - q^2)};$$

from which P and Q can be determined as you please. Q.E.I.

Corollary 1.

764. If the body should be moving with uniform motion on the curve, thus in order that $v = c$ and $dv = 0$, then we have :

$$P = \frac{2cyzdz}{przdz - qrydy} \quad \text{and} \quad Q = -\frac{2czydy}{przdz - qrydy}.$$

And

$$P : Q = dz : -dy.$$

Corollary 2.

765. If we have $v = \frac{ch^2}{p^2}$ or the speed of the body varies inversely as the perpendicular sent, then $dv = \frac{2ch^2 dp}{p^3}$. With these put in place, then Q = 0; for

$$przdv = -\frac{2ch^2 rzdp}{pp} = -\frac{2ch^2 zydy}{pp} = -2vzydy.$$

And,

$$P = \frac{2ch^2 pyzdz - 2ch^2 qy^2 dy}{p^3 r(pzdz - qydy)} = \frac{2ch^2 y}{p^3 r}$$

(714). For this force alone is acting, in order that the body moves on this curve in this way.

Example.

766. Let the given curve AMB be an ellipse and the centres C and D its foci. The transverse axis AB = A and the latus rectum = L , and from the nature of the ellipse [p. 318]

$$4pp = \frac{ALy}{A-y} \quad \text{et} \quad 4qq = \frac{ALz}{A-z}.$$

But in addition :

$$z = A - y \quad \text{and} \quad r = \frac{4(Ay-yy)^{\frac{3}{2}}}{A\sqrt{AL}} = \frac{4(Az-zz)^{\frac{3}{2}}}{A\sqrt{AL}}.$$

From which there is produced ;

$$P = \frac{Avdy - ydv(A-y)}{2ydy(A-y)} \quad \text{and} \quad Q = \frac{Avdz - zdv(A-z)}{2zdz(A-z)} = \frac{Avdy + ydv(A-y)}{2ydy(A-y)}$$

and thus

$$P + Q = \frac{Av}{yz} \quad \text{et} \quad Q - P = \frac{dv}{dy}.$$

PROPOSITION 93.

PROBLEM.

767. *A body is moving with a given speed on a curve also given AMB* (Fig. 70), *and it is required to find the centripetal force acting towards the centre C together with a force always acting normally to the line AB pulling the body in the direction MP, which two forces have the effect that the body is free to move on this curve with the prescribed speed.*

SOLUTION.

Let the speed of the body present at the point M correspond to the height v and the distance $MC = y$, and the perpendicular $MP = z$. The force pulling the body towards the centre C is put equal to P and the force acting along MP is equal to Q. With the tangent MV at M drawn, the perpendiculars CT and PG are sent to that, which are called p and q. With these put in place, the normal force arising from both these forces is equal to

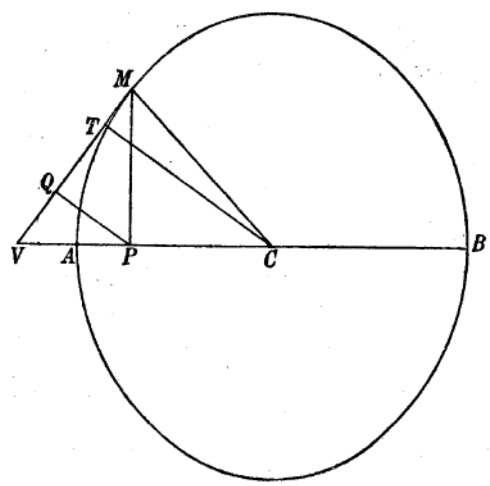

Fig. 70.

$$\frac{Pp}{y} + \frac{Qq}{z}$$

and the tangential force is equal to

$$\frac{P\sqrt{(y^2-p^2)}}{y}+\frac{Q\sqrt{(z^2-q^2)}}{z}.$$

Hence with the radius of curvature at $M = r$ put in place :

$$\frac{Pp}{y}+\frac{Qq}{z}=\frac{2v}{r}$$

(561) and

$$dv=-Pdy-Qdz$$

(559). From these it is thus found that :

$$P=\frac{2vyzdz+qrydv}{przdz-qrydy} \text{ and } Q=\frac{-2vzydy-przdv}{przdz-qrydy}.$$

Moreover, on putting $CP = x$, we have :

$$\sqrt{(dx^2+dz^2)}:dx=z:q, \text{ hence } q=\frac{zdx}{\sqrt{(dx^2+dz^2)}},$$

[p. 319] and with dx made constant, we have :

$$r=\frac{(dx^2+dz^2)^{\frac{3}{2}}}{-dxddz}.$$

Moreover, again we have :

$$y=\sqrt{(x^2+z^2)} \text{ and } p=\frac{zdx-xdz}{\sqrt{(dx^2+dz^2)}}.$$

With these substituted, we have :

$$P=\frac{2vydxdzddz-ydxdv(dx^2+dz^2)}{x(dx^2+dz^2)^2},$$

and

$$Q=\frac{-2vydydxddz+(zdx-xdz)(dx^2+dz^2)dv}{x(dx^2+dz^2)^2}.$$

Q.E.I.

Corollary 1.

768. If the body is required to move uniformly on the curve, thus so that $v = c$ and $dv = 0$, then we have :

$$P=\frac{2cydxdzddz}{x(dx^2+dz^2)^2} \text{ and } Q=\frac{-2cydydxddz}{x(dx^2+dx^2)^2}.$$

Corollary 2.

769. If the curve is a circle, the centre of which is present at C, and the radius is called *a*, then $r = a$, $y = a$, $p = a$ and $q = \frac{z^2}{a}$. Whereby the forces are produced:

$$P = \frac{2v}{a} + \frac{zdv}{adz} \quad \text{and} \quad Q = -\frac{dv}{dz}.$$

Therefore with Q known, then $P = \frac{2v}{a} - \frac{Qz}{a}$. And if $v = c$ and $dv = 0$, then

$$Q = 0 \text{ and } P = \frac{2c}{a}.$$

Scholium.

770. Observe that from this proposition, in which the curve described by the body is given, [conversely] a little of the usefulness follows in the determination of curves which bodies describe, acted on by composite forces. Indeed it is true that it is permitted to progress from this to other propositions, in which the curves described by bodies are not themselves given, but they are generated under the motion of one or more given forces, as in the above propositions in which the motion of the apsides has been treated.

PROPOSITION 94. [p. 320]

PROBLEM.

771. *If the body is moving on the curve AMB* (Fig. 71) *in whatever manner*, *while the curve itself is revolving around the central fixed point C, it is required to find two forces, one of which is always acting towards the fixed point C, and the other is directed normally to the line in the given position PC, which two forces have the effect that the body is free to move in this orbit.*

SOLUTION.

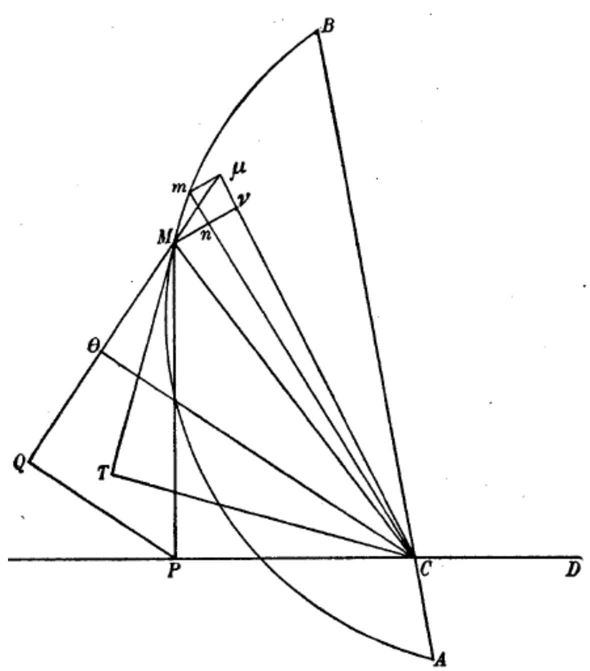

Fig. 71.

Let the speed of the body present at M, in which it traverses the element of the curve itself Mm, correspond to the height v, and the angular speed of the body in the orbit to the true angular speed of the body be in the ratio about C as 1 to w, or the angular speed of the body in orbit to the angular speed of the orbit itself, while the body is at M, is as 1 to $w -$ 1. [If $\omega_a, \omega_o,$ and $\omega_{b/o}$ are the absolute, orbital, and relative to the orbital, angular speeds of the body then $\omega_a = \omega_o + \omega_{b/o}$; Euler sets $\frac{\omega_{b/o}}{\omega_a} = \frac{1}{\omega}$, then $\frac{\omega_{b/o}}{\omega_a - \omega_{b/o}} = \frac{\omega_{b/o}}{\omega_o} = \frac{1}{\omega - 1}$] The radius CM is put equal to y and the perpendicular CT from C sent to the tangent of the orbit at M is equal to p, while the tangent itself MT is equal to q, thus in order that $q = \sqrt{(y^2 - p^2)}$. From M the perpendicular MP which is called z, is sent to the position of the given line DP, and CP is called x, thus in order that $x = \sqrt{(y^2 - z^2)}$. Now while the element Mm is traversed, meanwhile the orbital angle is put equal to $mC\mu$ of the circumference; as from the composite motion of the body it arrives at μ, with $C\mu = Cm$, Mμ is an element of the true curve in which the body is moving, in which produced the perpendicular $C\Theta$ is sent from C. With centre C the little arc $Mn\nu$ is described, [p. 321] where we have : $mn = \mu\nu = dy$ et $Mn : M\nu = 1 : w$. Truly, we have

$$Mm = \frac{ydy}{q},$$

hence

$$Mv = \frac{wpdy}{q},$$

from which is found :

$$M\mu = \frac{dy}{q}\sqrt{(w^2p^2+q^2)}$$

and again, [as CΘM and $M\mu\nu$ are similar triangles :]

$$C\Theta = \frac{wpy}{\sqrt{(w^2p^2+q^2)}}$$

and

$$M\Theta = \frac{qy}{\sqrt{(w^2p^2+q^2)}}.$$

The ratio is made $Mm : M\mu = \sqrt{v : \frac{\sqrt{v(w^2p^2+q^2)}}{y}}$, the square of which $\frac{v(w^2p^2+q^2)}{y^2}$ shows the height corresponding the true speed of the body ; therefore the increment of this is equal to

$$\frac{dv(w^2p^2+q^2)}{y^2} + \frac{2(w^2-1)vpdp}{y^2} - \frac{2(w^2-1)vp^2dy}{y^3} + \frac{2vp^2wdw}{y^2}.$$

Moreover, the radius of osculation of the true curve, in which the body falls, is equal to :

$$\frac{ydy}{d.C\Theta} = \frac{ydy(w^2p^2+q^2)^{\frac{3}{2}}}{w(w^2-1)p^3dy+wy^3dp+py^3dw-p^3ydw}$$
$$= \frac{ydy(w^2p^2+q^2)^{\frac{3}{2}}}{w(w^2-1)p^3dy+wy^3dp+pq^2ydw}.$$

With the whole sine put as 1, the sine of the angle *CMP* is equal to $\frac{x}{y}$ and the cosine is equal to $\frac{z}{y}$. But the sine of the angle $CM\Theta$ is equal to $\frac{wp}{\sqrt{(w^2p^2+q^2)}}$ and the cosine of this is equal to $\frac{q}{\sqrt{(w^2p^2+q^2)}}$. From which the sine of the angle $PM\Theta$ is equal to $\frac{qpz-qz}{y\sqrt{(w^2p^2+q^2)}}$ and the cosine of this is equal to $\frac{wpx+qz}{y\sqrt{(w^2p^2+q^2)}}$.

By sending the perpendicular *PQ* from *P* to the tangent $M\Theta$, then

$$PQ = \frac{wpz^2-qxz}{y\sqrt{(w^2p^2+q^2)}} \quad \text{and} \quad MQ = \frac{wpxz+qz^2}{y\sqrt{(w^2p^2+q^2)}}.$$

And with the body traversing the element $M\nu$ the increment of the line PM is

$$dz = \frac{wpxdy+qzdy}{qy}.$$

From which equation the relation between *w* and *x* becomes known and likewise the position of the apsidal line *AB* with respect to the line *CP* can be found.

Now put the force acting on the body towards *MC* equal to *P* and the force pulling along *MP* equal to *Q*, [p. 322] from which there arises the tangential force retarding the motion of the body, which is equal to :

$$\frac{Pq}{\sqrt{(w^2p^2+q^2)}}+\frac{Qwpx+Qqz}{y\sqrt{(w^2p^2+q^2)}},$$

which therefore taken by $\frac{dy}{q}\sqrt{(w^2p^2+q^2)}$ can be put equal to :

$$-\frac{dv(w^2p^2+q^2)}{y^2}-\frac{2(w^2-1)vpdp}{y^2}+\frac{2(w^2-1)vp^2dy}{y^3}-\frac{2vp^2wdw}{y^2},$$

thus in order to give :

$$Pdy+\frac{Qwpxdy}{qy}+\frac{Qzdy}{y}$$

$$=-\frac{dv(w^2p^2+q^2)}{y^2}-\frac{2(w^2-1)vpdp}{y^2}+\frac{2(w^2-1)vp^2dy}{y^3}-\frac{2vp^2wdw}{y^2}.$$

Moreover, the normal force arising from both is equal to :

$$\frac{Pwp}{\sqrt{(w^2p^2+q^2)}}+\frac{Qwpz-Qqx}{y\sqrt{(w^2p^2+q^2)}},$$

which must be equal to :

$$\frac{2w(w^2-1)vp^3dy+2wvy^3dp+2vpq^2ydw}{y^3dy\sqrt{(w^2p^2+q^2)}}$$

(561), whereby we have :

$$Pwpydy+Qwpzdy-Qqxdy=\frac{2w(w^2-1)vp^3dy}{y^2}+2wvydp+\frac{2vpq^2dw}{y}.$$

From which equations solved for *P* and *Q*, we have :

$$Q=-\frac{wpqdv}{yxdy}-\frac{2wvqdp}{yxdy}-\frac{2vpqdw}{yxdy}$$

and

$$P=\frac{qdv(wpz-qx)}{y^2xdy}+\frac{2(w^2-1)vp^2}{y^3}+\frac{2vdp(wqz+px)}{y^2xdy}+\frac{2vpqzdw}{y^2xdy}.$$

Truly the angle, that the apsis line *AB* makes with the line *CP*, is equal to

$$\int\frac{(w-1)pdy}{qy},$$

thus the position of this is known for any time. Q.E.I.

Corollary 1.

772. Since $dz = \frac{wpxdy+qzdy}{qy}$, putting $z = ty$ then there comes about

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{wpdy}{qy}.$$

From which equation, if *w* is given in terms of *y*, in which also on account of the given curve *AMB*, *p* and *q* are expressed, *t* can be found and likewise also *z* and *x*.

Corollary 2.

773. If the speed of the body in the orbit $\sqrt{v}$ varies inversely as the perpendicular *CT* sent from *C* to the tangent or $v = \frac{a^2c}{p^2}$, then we can write :

$$P = \frac{2a^2cdp}{p^3dy} + \frac{2a^2c(w^2-1)}{y^3} + \frac{2a^2cqzdw}{py^2xdy} \text{ and } Q = -\frac{2a^2cqdw}{pyxdy}$$

Corollary 3. [p. 323]

774. If in this case *w* is constant, the force *Q* vanishes and there remains alone

$$P = \frac{2a^2cdp}{p^3dy} + \frac{2a^2c(w^2-1)}{y^3}$$

pulling towards the centre *C*, which effects the motion, as the body progresses in the orbit *AMB* moving around *C*, evidently as has been found above (734)

Corollary 4.

775. If *v* is not $\frac{a^2c}{p^2}$, but *w* is constant, thus as the angular motion of the orbit is proportional to the angular motion of the body in the orbit truly as $w-1$ to 1, then we have

$$Q = \frac{-wpqdv - 2wvqdp}{yxdy} \text{ and } P = \frac{qdv(wpz-qx)}{y^2xdy} + \frac{2(w^2-1)vp^2}{y^3} + \frac{2vdp(wqz+px)}{y^2xdy}.$$

Example.

776. On putting $v = \frac{a^2c}{p^2}$ the curve AMB is an ellipse having C in either focus. Therefore if the transverse axis of this ellipse is called A and the latus rectum L, then
Whereby we have

$$4pp = \frac{ALy}{A-y} \quad \text{or} \quad p = \frac{\surd ALy}{2\surd(A-y)} \text{ and } q = \frac{\surd(4Ay^2 - 4y^3 - ALy)}{2\surd(A-y)} \quad \text{and} \quad \frac{dp}{p^3} = \frac{2dy}{Ly^2}.$$

$$Q = -\frac{2a^2cdw\surd(4Ay - 4y^2 - AL)}{yxdy\surd AL}$$

and

$$P = \frac{4a^2c}{Ly^2} + \frac{2a^2c(w^2-1)}{y^3} + \frac{2a^2czdw\surd(4Ay - 4y^2 - AL)}{y^2xdy\surd AL}.$$

And

$$\frac{dt}{\surd(1-tt)} = \frac{wdy\surd AL}{y\surd(4Ay - 4y^2 - AL)}.$$

Scholion 1.

777. These formulae for the curve of the ellipse can be made simpler in various ways, if the curve in which the body is moving is approximately circular. And in this case it is of some use in the theoretical motion of the moon to be defined. [p. 324] For the earth is put at rest at *C* and the sun on the line *CP* perpendicular at C is considered equally as being at rest ; with which put in place and with these forces compared both with the forces of the sun and the earth, the synodal motion of the moon is elicited for some position of the apsidal line and likewise the motion of the apsidal line, which only differs slightly from the true motion of the moon.

Scholium 2.

778. This proposition certainly appears of greater extent than the above (729), in which all the force was directed towards the centre of rotation of the orbit; indeed the former is included with the force Q vanishing. Yet the quadrature cannot account perfectly for the motion of the moon because the proportional force in P varies inversely with the cube of the distance *MC* (773). Because of this we offer other orbits besides the gyrations in the middle, and which appear wider and agree more with the physics questions. Of this kind are the motions of orbits by which curves are always in orbits parallel to themselves, which on contemplation deserve to be preferred from others, since the forces acting are both easier to find and the formulas are simpler to understand. Moreover there is an outstanding need for this, and the following theorem is presented.

PROPOSITION 95.

THEOREM.

779. *The body is moving along the curve AM* (Fig. 72), *by some force acting around the point C, and in addition* [p. 325] *both the body and the point C are acted on by a force in the same direction; there is the relative motion of the body M with respect to the point C or the motion of the body M such as is seen from C, and likewise, if this new force is not to be added..*

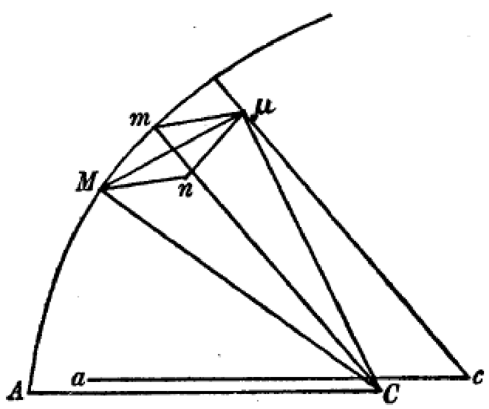

Fig. 72.

DEMONSTRATION.

With the point *C* at rest in the element of time *dt* the body arrives at the point *m* from *M* in *m*. Therefore in this small finite time the body *M* is apart from *C* by the interval *mC* and makes the angle *mCA* with a certain fixed line *AC* expressed in the region. Now with the body put present at *M* , both the body M and the point *C* are to be acted on by a force towards the same place, thus in order that the point *C* can be advanced through *Cc* in the small time *dt*. Therefore in the same short time *dt* the body *M*, if at rest, by this force is carried through the distance *Mn* parallel and itself equal to *Cc*. But since the body *M* now has the motion in place, in which element of time *dt* it traverses the distance *Mm*, then with both motions connected together it describes the diagonal *Mμ* to complete the parallelogram *Mmμn*. Wherefore with the occurrence of this new force, the body *M* in the short finite time *dt* is at a distance from the point *C*, that meanwhile is translated to *c* by the interval *μc,* and with the line *ac* drawn parallel to *AC* in the fixed direction, it makes the angle *μca*. But since *mCcμ* is a parallelogram *μc* = *mC* and the angle *mCA* is equal to the angle *μca.* Consequently the force is acting equally at each point *M* and *C* equally and acting along the same direction, it does not change the motion of *M* relative to the point *C*. Q.E.D. [p. 326]

Corollary 1.

780. Therefore whatever the force acting at the point *C*, if the same likewise acts on the body *M* along the same direction, then the relative motion of the body *M* with respect to the point *C* does not change.

Corollary 2.

781. Therefore a force of this kind can be put into effect acting equally on *M* and *C*, in order that the body *M* is moving in the moving orbit *AM* in some manner. Moreover the orbit by its own motion is able to follow the motion of the point *C*, as its position always remains parallel to the position of that point.

Corollary 3.

782. It is also seen from the demonstration of the proposition, if an equal speed is impressed on both the point C and the body M along the same direction, then the relative motion is not going to be disturbed.

Corollary 4.

783. With such motion impressed on the point C, it continues to move indefinitely in some direction without the continuation of any force, and the body M follows moving uniformly around the point C, that can be equally progressing or at rest. For this situation is only maintained, provided the same direction and speed of the body C are added to the body *M*.

Corollary 5.

784. Therefore the body is free to describe the same curve about the centre of force *C* that is progressing uniformly [p. 327], that it describes about *C* at rest, but to M must be added as well as much speed as has been accepted by the centre of force.

Corollary 6.

785. Moreover, as the body left to itself is unable by itself to progress along the curved line, nor to move unequally along a straight line; thus the body moving around the moving centre of force, either in performing the required curved motion or changes in the direction, cannot describe the same curve that is does for the resting position, but it is always urged on by the above force, to the extent required to maintain the motion along its own path.

PROPOSITION 96.

PROBLEMA.

786. *If the body M* (Fig. 71) *is revolving around the centre of force L at rest in the curve BM, to determine the force that is effective in order that the body in the same orbit following the same curve AL is moving in a direction always parallel to itself.*

SOLUTION.

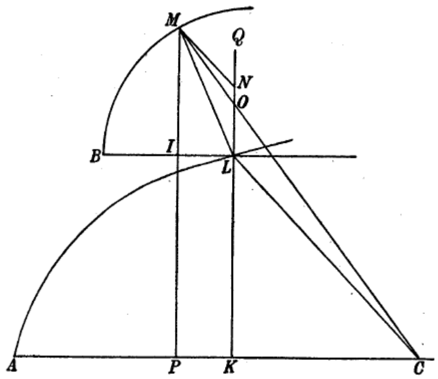

Fig. 73.

Since the body M is freely moving in the orbit BM about the centre of force L at rest, then, with the distance $LM = y$ and with the perpendicular sent from L to the tangent at M equal to p, the height corresponding to the speed at $M = \frac{a^2c}{p^2}$ (589) and the centripetal force acting towards L is equal to $\frac{2a^2cdp}{p^3dy}$ (592). Now the centre L is put to be moving on the curve AL attracted to the centre of force C, and $CL = c$ and the perpendicular sent from C to the tangent [p. 328] at L is equal to w. With which put in place the height corresponding to the speed at $L = \frac{b^2e}{w^2}$ and the force attracting the point L to C is equal to $\frac{2b^2edw}{w^3ds}$ (592). Moreover the speed impressed on the first point L along the tangent is equal to $\frac{b\sqrt{e}}{w}$ and it is also the same for the body M following the direction parallel to this tangent; and it is evident, if no additional force acts on the point L, but only this impressed motion is conserved, then the motion of the body M gone through is thus obtained following the motion of the point L, moving freely on the same curve BM, in order that the axis BL always remains parallel to the motion of L (783). But when the body M is moving in the same way around the advancing point L on the curve AL, it is required, as such a force is itself always acting, that it is of such a size as is necessary to keep the point L on this curve AL (781). Therefore, with MN drawn parallel to LC, the body M besides the force by which it is urged towards L, must also be acted on by a force equal to $\frac{2b^2edw}{w^3ds}$ along the direction MN. Whereby this two-fold force has the effect of solving the problem of the motion. For which moreover it is apparent, since the body M is acted on by the force with respect to the point C and to the fixed line AC parallel to BL, that the forces acting along MN and ML on the body M are to be resolved in two others, of which one has the direction MC, and the other MP, which line MP has been drawn perpendicular to AC. To this outstanding line *MP* is drawn the parallel line *KLN* cutting the line *MC* at *O* and the distance *LI* = *x*, *MI* = *z*, *CK* = *r*, *KL* = *t* and *CP* = *X*, *PM* = *Z* and *CM* = *Y*. On this account,

$$y = \sqrt{(x^2 + z^2)}, \quad s = \sqrt{(r^2 + t^2)} \quad \text{and} \quad Y = \sqrt{(X^2 + Z^2)}.$$

And again,

$$p = \frac{xdz - zdx}{\sqrt{(dx^2 + dz^2)}} \quad \text{and} \quad w = \frac{rdt - tdr}{\sqrt{(dr^2 + dt^2)}}.$$

[p. 329] Besides indeed $X = r + x$ and $Z = t + z$. Whereby on account of the given curves *AL* and *BM* since *z, y*, and *p* in *x*, and likewise *t, s*, and *w* in *r* are given, all these quantities in *X, Z*, and Y are able to be shown. Truly the equation between X and *Z* can be deduced from that, because the increment of the time through *BM* is equal to the increment through AL. Hence we therefore have :

$$\frac{p\sqrt{(dx^2 + dz^2)}}{a\sqrt{c}} = \frac{w\sqrt{(dr^2 + dt^2)}}{b\sqrt{e}},$$

and from the integrals taken, the area *ACL* $\pm$ some constant area is to the area *BLM* as $b\sqrt{e}$ to $a\sqrt{c}$. Therefore the force pulling along *ML* is resolved into two forces pulling along *MO* and *LO* or *MI*, and in a similar way the force along *MN* is resolved into two forces pulling along *MO* and *NO* , of which the latter acts on the opposite direction to *MI* on the body *M* ; to which resolved forces put in place it is necessary to know the angles. For indeed the angle *MLO* is equal to the angle *LMI* and therefore the sine of this is equal to $\frac{x}{y}$ and the cosine $\frac{z}{y}$, by taking 1 for the total sine. Similarly the angles *MON* and *CMP* are equal, therefore the sine of this is $\frac{X}{Y}$ and the cosine $\frac{Z}{Y}$. Therefore the sine of the angle *LMO*, which is the difference of these, is equal to $\frac{Xz - Zx}{Yy}$. And finally the angles *MNQ* and *CLK* are equal, whereby the sine is equal to $\frac{r}{s}$ and the cosine $\frac{t}{s}$. Consequently as NMO = NMO = MNQ – MON the sine of this angle is $\frac{Zr - Xt}{Ys}$. Also, $Xx - Zx = Zr - Xt$ as $X = r + x$ and $Z = t + z$.

From these the ratio of *MN:MO* or $\frac{X}{Y} : \frac{r}{s}$ can be made and thus the ratio of the force pulling along *MN* , $\frac{2b^2edw}{w^3ds}$ to the force along *MC*, which hence is equal to

$$\frac{2b^2eYrdw}{Xsw^3ds}.$$

And the ratio *MN:NO* or $\frac{X}{Y} : \frac{Zr - Xt}{Ys}$, thus the force along *MN* , $\frac{2b^2edw}{w^3ds}$, to the force along *ON*, [p. 330] which is equal to

$$\frac{2b^2edw(Zr - Xt)}{Xsw^3ds}.$$

In a like manner the ratio $ML : MO = \frac{X}{Y} : \frac{x}{y}$, and thus the ratio of the force along *ML*, $\frac{2a^2cdp}{w^3dy}$, to the force along *MC*, which is hence equal to

$$\frac{2a^2cYxdp}{Xyp^3dy}.$$

And $ML : MO = \frac{X}{Y} : \frac{Zr-Xt}{Yy}$, and thus the ratio of the force along *ML*, $\frac{2a^2cdp}{p^3dy}$, to the force along *OL* or *MP*, which is hence equal to $\frac{2a^2cdp(Zr-Xt)}{Xyp^3dy}$.

From these collected together the body *M* is acted on by a force pulling in the direction of *MC* equal to:

$$\frac{Y}{X}\left(\frac{2b^2erdw}{sw^3ds}+\frac{2a^2cxdp}{yp^3dy}\right)$$

and by a force along *MP* :

$$\frac{(Zr-Xt)}{X}\left(\frac{2a^2cdp}{yp^3dy}-\frac{2b^2edw}{sw^3ds}\right).$$

Which expressions can all be shown in terms of *X, Z* and *Y* and besides an equation can be assigned between these quantities which relate to the true curve described by the body *M*. Q.E.I.

Corollary 1.

787. If the curve *AL* is the periphery of a circle, the centre of which is *C* and the radius $AC = b$, then $s = w = b$ and $r^2 + t^2 = b^2$. Therefore in this case the force pulling along *MC* is equal to :

$$\frac{Y}{X}\left(\frac{2er}{b^2}+\frac{2a^2cxdp}{yp^3dy}\right)=\frac{2eY}{b^2}-\frac{2Yx}{X}\left(\frac{e}{b^2}-\frac{a^2cdp}{yp^3dy}\right)$$

and the force along *MP* is equal to :

$$\frac{(Zr-X\sqrt{(b^2-r^2)})}{X}\left(\frac{2a^2cdp}{yp^3dy}-\frac{2e}{b^2}\right)$$

And *e* is the height corresponds to the speed of the point *L*.

Corollary 2.

788. Let the curve *BM* be an ellipse having the centre at *L*, and *BL*= *a* is the transverse semi-axis of this curve, thus in order that the height corresponding to the speed at *B* is equal to *c*, and indeed the other axis is set equal to *h*. And the equations arise :

$$a^2z^2+h^2x^2=a^2h^2 \quad\text{and}\quad \frac{2a^2cdp}{p^3dy}=\frac{2cy}{h^2}.$$

Whereby the force along MC becomes equal to :

$$\frac{Y}{X}\left(\frac{2b^2 erdw}{sw^3 ds}+\frac{2cx}{h^2}\right)=\frac{2b^2 eYdw}{sw^3 ds}-\frac{2Yx}{X}\left(\frac{b^2 edw}{sw^3 ds}-\frac{c}{h^2}\right).$$

[p. 331] And the force along *MP* is equal to :

$$\frac{(Zr-Xt)}{X}\left(\frac{2c}{h^2}-\frac{2b^2 edw}{sw^3 ds}\right).$$

Corollary 3.

789. And if the curve *AL* is the circle as in Cor.1 and the curve *BM* the ellipse as in Cor. 2, then the force along *MC* is equal to :

$$\frac{Y}{X}\left(\frac{2er}{b^2}+\frac{2cx}{h^2}\right)=\frac{2eY}{b^2}-\frac{2Yx}{X}\left(\frac{e}{b^2}-\frac{c}{h^2}\right)$$

and the force along *MP* is equal to :

$$\frac{(Zr-Xt)}{X}\left(\frac{2c}{h^2}-\frac{2e}{b^2}\right),$$

where *r* and *t* and *x* are to be determined in terms of *X, Z*, and *Y* from these equations :

$$a^2z^2+h^2x^2=a^2h^2,\quad r^2+t^2=b^2,\quad r+x=X \text{ and } t+z=Z.$$

Moreover the equation emerging rises to the fourth power.

Corollary 4.

790. If the ellipse *BM* is put infinitely small or even extremely small with respect to the circle *AL*, yet thus, as the periodic time of the ellipse is a finite quantity, then

$$x=\frac{a^2X(Y^2-b^2)\pm ahZ\sqrt{(4a^2X^2+4h^2Z^2-(Y^2-b^2)^2)}}{2a^2X^2+2h^2Z^2}$$

and

$$z=\frac{h^2Z(Y^2-b^2)\mp ahX\sqrt{(4a^2X^2+4h^2Z^2-(Y^2-b^2)^2)}}{2a^2X^2+2h^2Z^2}.$$

Corollary 5.

791. If the curve *BM* is also a circle having centre *L*, we can place $h=a$ and thus

$$x^2+z^2=a^2,\ r^2+t^2=b^2,\ r=X-x \text{ and } t=Z-z,$$

from which on putting $b^2-a^2=f^2$ it is found that

$$2x=\frac{(Y^2-f^2)X\pm Z\sqrt{(4a^2Y^2-(Y^2-f^2)^2)}}{Y^2}$$

and

$$2z=\frac{(Y^2-f^2)Z\mp X\sqrt{(4a^2Y^2-(Y^2-f^2)^2)}}{Y^2}.$$

And,

$$2Zr - 2Xt = \mp \sqrt{\left(4a^2Y^2 - (Y^2 - f^2)^2\right)}.$$

Scholium 1. [p. 332]

792. Finally we can add this corollary, as it is apparent that such forces are required for a body to be turning in an epicycle about a centre of force, as the adherents of Ptolemy considered the planets to move.

Corollary 6.

793. From the hypothesis of Cor. 3 it is understood, that if $\frac{e}{b^2} = \frac{c}{h^2}$, or the speed of the body *L* to the speed of the body *M* at *B* as the diameter of the circle *AL* is to the conjugate axis of the ellipse *BM*, then the true curve described by the body *M* is an ellipse with the centre at *C*, when the force acting towards *C* becomes equal to $\frac{2eY}{b^2}$ and with the force along *MP* vanishing. The semi-major axis of this ellipse is $b + a$, and the minor truly $b + h$.

Scholium 2.

794. I have therefore especially reported on this Proposition, because in an appendix to the new edition of Newton's Principia in English, the most distinguished Machin asserts [John Machin (1680-1751), *The mathematical principles of natural philosophy translated into English*, London 1729; to which book the Appendix has been added : *The laws of the motion of the moon*. P. St.] that the motion of the moon can be considered as in an ellipse, the transverse axis of which shall be in the ratio of 2 : 1 to the conjugate axis made around the centre of the ellipse, while meanwhile with that ellipse itself moving parallel to the periphery of a circle, on which it progresses freely, as I have explained in Cor. 3. For my part, I do not deny that this motion is extremely similar to the motion that the moon can show, but I would doubt very much that it was an exact ratio. Moreover in the following proposition I have decided to determine, what needs to be indicated to determine the motion of the moon, [p. 333]. Even if indeed this proposition pertains to astronomy, yet it is assumed that the method reported here can be used to examine fundamental questions of this kind that are to be resolved.
[At this time, the laws governing angular momentum and the conservation of energy were not yet fully understood; in the following proposition the inverse square law solution is taken for the earth in a circular orbit, and the moon is assumed to rotate about the earth in an elliptical orbit according to the same law. The moon is given, along with the sun, the negative of the earth's centripetal acceleration, resulting in the moon's orbit being transported parallel to itself according to a person on the earth, Thus, the problem is reduced to one of kinematics.]

PROPOSITION 97.

PROBLEM.

795. *With the sun at rest at S* (Fig. 74) *and with the earth T moving around it uniformly in the circle TD while the moon L is attracted to the earth T as to the sun S in the inverse square of the distances; with which put in place it is required to determine the motion of the moon, such as can be seen from the earth T.*

SOLUTION.

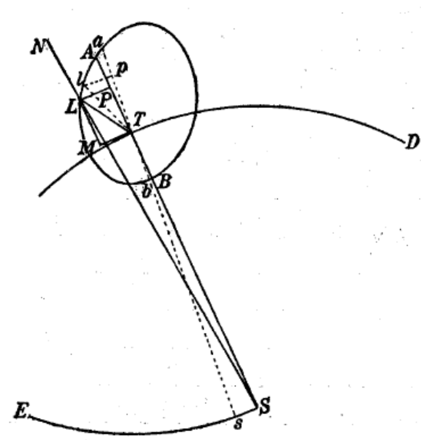

Fig. 74.

The distance of the earth from the sun *ST* is put equal to *a* and the force, and the force which attracts the earth to the sun is equal to $\frac{f}{a^2}$. The distance of the moon from the earth is equal to *y* and the distance of the moon from the sun LS is equal to *z*. The force, by which the moon is attracted to the earth, is equal to $\frac{h}{y^2}$; and indeed the force, by which the moon is attracted to the sun along LS, is equal to $\frac{f}{z^2}$. [Paul Stackel's note : In the formulas $\frac{f}{a^2}$ and $\frac{f}{z^2}$, the letter *f* does not have the same value. On this account the solution to the problem would have to be modified. This translator's note : Euler always deals with accelerations, or the forces per unit mass; thus, I am inclined to believe he has the accelerations of the bodies in mind here, rather than the actual forces, in which case the formulas are the same and so are correct. Euler talks about forces or strengths of forces where we would use the word acceleration.] Therefore from these forces acting, such lunar motion produced is to be investigated. But since it is agreed that such motion of the moon is to be viewed from the earth, then the earth is considered at rest ; when it is done, while the motion for the whole system is viewed relative to the earth, then equal and likewise opposite accelerations must be applied to that which the earth receives from the sun [and the moon], with the moon and the sun known to be carried round in the opposite direction to their true motion. [Euler is concerned with a kinematic problem involving relative accelerations; the basic physics has been attended to already in the setting up of the orbits. There is no question of an earth-centred dynamics problem being solved. He tries to fit a solution to this vexing problem.][p. 334] Moreover the speed of the earth in the orbit *TD* corresponds to the height $\frac{f}{2a}$, as can be gathered from the force $\frac{f}{a^2}$ by which the earth is drawn towards the sun. Therefore such a speed must be impressed both on the sun and the moon along the direction normal to *TS*. Besides, since the earth is drawn to the sun by the force $\frac{f}{a^2}$, it is necessary that the effect of this force is the destruction of the original force, and if the sun is always attracted to the earth by such a force, then the moon truly is acted on by the same force along the line *LN* parallel to *ST* itself [meaning that they have a common acceleration which is the opposite of the centripetal acceleration of the earth towards the sun]. With this done the sun describes a circle *SE* around the earth *T* at rest with the same speed, which before the earth was carried around the sun. Truly the moon besides the forces pulling along LT and LS above is urged by a force equal to $\frac{f}{a^2}$ in the direction LN. With LM drawn parallel to TS also, the force acting along LS $\frac{f}{z^2}$ is resolved into these two forces, of which one is in the direction LT, and the other along LM. Hence by considering the triangle LTS the force arises acting along LT equal to $\frac{fy}{z^3}$ and the force pulling along LM equal to $\frac{af}{z^3}$. Whereby with these forces combined, the moon is pulled in the direction LT by a force equal to :

$$\frac{h}{y^2}+\frac{fy}{z^3}$$

and in the direction LM by a force equal to :

$$\frac{f}{a^2}-\frac{af}{z^3}=\frac{f(z^3-a^3)}{a^2z^3},$$

from which forces the motion of the moon must be determined. Moreover it is to be noted that the direction *LM* is not constant but variable, clearly always parallel to the radius *TS*, which on account of the motion of the sun is carried along the periphery *SE*. Therefore with *ST* produced in *A*, in order that *AB* is the line of the conjunctions, and from *L* by sending the perpendicular *LP* to the line *AB*, *TP* is equal and parallel to *LM*. [p. 335] In the small interval of time *dt* the moon travels from *L* to *l*, and moreover the sun from *S* to *s*; and therefore meanwhile the line of conjunctions is carried to *ab* and the moon at *l* is acted on in part by a force along *lT*, and in part by a force pulling along the line parallel to *Tp*, clearly with the perpendicular *lp* sent from *l* to *Ta*. Moreover from these forces resolved, the normal and tangential force can be found, of which either gives the speed of the moon. Moreover these two equations can be solved to eliminate the speed, and present the equation of the curve *ABL*, in which the moon can be determined to move. Q.E.I.

Scholium 1.

796. The equations which hence are deduced for the motion of the moon, become so complex that from them neither the orbit of the moon nor the position of the apsides of this motion can be exactly determined. Moreover truly from the same calculation by neglecting very small quantities in a certain way approximate conclusions for the use of astronomy can be drawn, as the great Newton did in Book III of the *Phil. Princ*. Moreover even if this inconvenient calculation does not work, yet from this proposition without a great deal of rigor, the motion of the moon may soon be demonstrated. For we have put the sun forwards again as being at rest, which in a short while disagrees with the truth; then we consider the earth moving in a circle, and the orbit of the moon placed in the same plane with the earth, which likewise they have otherwise. Yet meanwhile it is certain, if the solution of this proposition can evolve and from that a table constructed, then it would be of the most use in astronomy. [p. 336]

Corollary 1.

797. Since the distance of the moon from the earth is very small with respect to the distance of the earth from the sun, it is possible to put $z = a$ without sensible error and it is almost possible to put in that case the force acting along *LM* to vanish and the only force drawing the moon to the earth is equal to :

$$\frac{h}{y^2}+\frac{fy}{a^3}.$$

Corollary 2.

798. When the orbit of the moon does not differ much from a circle, it is possible in that case to consider the form of the moving ellipse, as we have done in Prop. 91 (747). Whereby a knowledge of the motion of the apsides is obtained from Coroll. 3 of this Proposition (750).

$$P=\frac{a^3hy+fy^4}{a^3},$$

and the moon arrives at the apogee from the perigee from the absolute angular motion around the earth by turning through an angle equal to :

$$180\sqrt{\frac{a^3h+fy^3}{a^3h+4fy^3}}\quad \text{degrees},$$

where *y*, which does not change much, can be considered as a constant.

Scholium 2.

799. Therefore the line of the apsides of the moon's motion is continually regressing, since $\frac{a^3h+fy^3}{a^3h+4fy^3}$ is less than one, which is contrary to observation. Truly the reason for this error is that we have considered the quantity *z* as being constant. For although [the sun-moon distance] *z* is neither much increased or decreased in the ratio of this to itself, yet the increments and decrements with respect to the increments of *y* are small enough to be disregarded. [p. 337] Whereby, when the differential of *P* must be taken, in that equation we have wrongly considered *z* as constant and put *a* in its place. Moreover, since *z* cannot be given by *y*, the motion of the apsides cannot be determined in this way. Meanwhile nevertheless this is gathered, if it is the case that $ady > ydz$, then the line of apsides is as in the preceding, but if $ady < ydz$, as a consequence it is to be moving forwards, if indeed we stop thinking about a force acting along *LM* .

Corollary 3.

800. With $LM = TP = x$ then we have approximately $z = a + x$, where x very small with respect to a. Therefore by ignoring x before a then the force, which pulls the moon to the earth is equal to $\frac{h}{y^2} + \frac{fy}{a^3}$ and the force which is pulling along LM is equal to $\frac{3fx}{a^3}$. This therefore vanishes when the moon is at right angles in its orbit to the earth, and is a maximum when the moon is in conjugation.

Scholium 3.

801. Moreover since this is not the place to pursue considerations of the motion of the moon in more detail, which are clearly pertinent to the theoretical astronomy, we will proceed to piece together what remains from our present arrangement. For the principles are sufficiently well understood to the extent that tables of the movements can be constructed for any case of interest and the respective motions can be found from the tables. Moreover, the motions of free bodies which are not made in the same plane remain to be included in this chapter [p. 338]. Indeed from the preceding it is evident that for a singe centripetal force present, the motion of the body always takes place in the same plane as the body was initially projected; and if there are several centres of force placed in the same plane in which the body is projected, then the curve described by the body is likewise completely in the same plane. What follows must therefore refer to the situation when a body is acted on by several forces, the directions of which lie in different planes; or also when the direction, along which the body is initially projected, is not in situated in that plane in which the directions of the forces are placed. Therefore in these cases, the motion of the body must be considered as if a certain curve is described on a certain convex or concave surface. Moreover the nature of the surface is expressed by an equation involving three variables, and from the nature of the line drawn on that surface that same equation is solved with another equation either in these three variables as well, or only two. For from these, the projection of the curved line onto a given plane can become known, and from the projection and the surface, likewise the curve described by the body placed on the surface is known. Again as in the co-planar case, the forces can be reduced to two, along the normal and the tangent, thus in this current business, the forces can be reduced to three (551), and what effect they exert on the body, we are about to find out. [p. 339]

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PROPOSITIO 92. [p. 316]

PROBLEMA.

763. *Invenire vires centripetas tendentes ad duo virium centra C et D* (Fig. 69), *quae faciant, ut corpus in data curva AMB et data in singulis punctis M celeritate moveatur.*

SOLUTIO.

764. Moveatur corpus ab *A* per *M* ad *B* et sit eius celeritas in *M* debita altitudini *v*; ponatur $CM = y$ et $DM = z$. Ducta vero tangente *TV* in eamque ex *C* et *D* demissis perpendiculis *CT* et *DV* dicantur $CT = p$ et $DV = q$. Vis centipeta porro, quae ad *C* tendit, sit $= P$, et ea, quae ad *D* trahit, sit $= Q$. Vis igitur normalis ex utraque orta erit =

$$\frac{Pp}{y}+\frac{Qq}{z}$$

et vis tangentialis accelerans motum corporis erit =

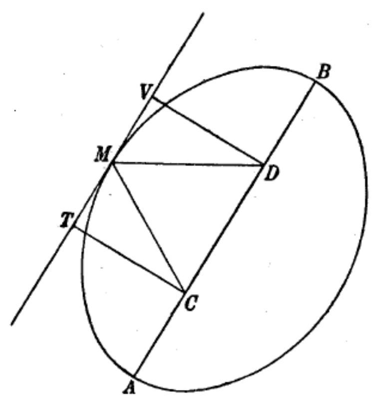

Fig. 69.

$$-\frac{P\sqrt{(y^2-p^2)}}{y}-\frac{Q\sqrt{(z^2-q^2)}}{z}$$

cadentibus tangentibus in antecedentia. Posito ergo radio osculi in $M = r$ et elemento curvae = *ds* erit [p. 317]

$$\frac{Pp}{y}+\frac{Qq}{z}=\frac{2v}{r}$$

(561) et

$$dv=-\frac{Pds\sqrt{(y^2-p^2)}}{y}-\frac{Qds\sqrt{(z^2-q^2)}}{z}$$

(559). Est vero

$$ds=\frac{ydy}{\sqrt{(y^2-p^2)}}=\frac{zdz}{\sqrt{(z^2-q^2)}}$$

ideoque $dv = -Pdy - Qdz$.
Ex his duabus aequationibus coniunctis habebitur
$$P = \frac{2vyzdz + qrydv}{przdz - qrydy}$$
et
$$Q = \frac{2vzydy + przdv}{qrydy - przdz} = \frac{-2vzydy - przdv}{przdz - qrydy}.$$
Est vero
$$r = \frac{ydy}{dp} = \frac{zdz}{dq} \quad \text{atque} \quad zdz = \frac{ydy\sqrt{(z^2 - q^2)}}{\sqrt{(y^2 - p^2)}},$$
ergo
$$dq = \frac{dp\sqrt{(z^2 - q^2)}}{\sqrt{(y^2 - p^2)}}.$$
Tandem dicta $CD = k$ erit
$$k^2 = y^2 + z^2 - 2pq - 2\sqrt{(y^2 - p^2)(z^2 - q^2)};$$
ex quibus P et Q, prout libuerit, determinari possunt. Q.E.I.

Corollarium 1.

764. Si corpus in curva motu aequabili debeat moveri, ita ut sit $v = c$ et $dv = 0$, erit
$$P = \frac{2cyzdz}{przdz - qrydy} \quad \text{et} \quad Q = -\frac{2czydy}{przdz - qrydy}.$$
Atque
$$P : Q = dz : -dy.$$

Corollarium 2.

765. Si fuerit $v = \frac{ch^2}{p^2}$ seu celeritas corporis reciproce ut perpendiculum ex centro in tangentem demissum, erit $dv = \frac{2ch^2dp}{p^3}$. His substitutis fit Q = 0; erit enim
$$przdv = -\frac{2ch^2rzdp}{pp} = -\frac{2ch^2zydy}{pp} = -2vzydy.$$
Atque
$$P = \frac{2ch^2pyzdz - 2ch^2qy^2dy}{p^3r(pzdz - qydy)} = \frac{2ch^2y}{p^3r}$$
(714). Haec enim vis sola efficiet, ut corpus hoc modo in ista curva moveatur

Exemplum.

766. Sit curva data AMB ellipsis et centra C et D eius foci. Ponatur eius axis transversus AB = A et latus rectum = L eritque ex natura ellipsis [p. 318]

$$4pp = \frac{ALy}{A-y} \quad \text{et} \quad 4qq = \frac{ALz}{A-z}.$$

At praeterea erit

$$z = A - y \quad \text{et} \quad r = \frac{4(Ay-yy)^{\frac{3}{2}}}{A\sqrt{AL}} = \frac{4(Az-zz)^{\frac{3}{2}}}{A\sqrt{AL}}.$$

Ex quo prodibit

$$P = \frac{Avdy - ydv(A-y)}{2ydy(A-y)} \quad \text{et} \quad Q = \frac{Avdz - zdv(A-z)}{2zdz(A-z)} = \frac{Avdy + ydv(A-y)}{2ydy(A-y)}$$

ideoque

$$P + Q = \frac{Av}{yz} \quad \text{et} \quad Q - P = \frac{dv}{dy}.$$

PROPOSITIO 93.

PROBLEMA.

767. *Moveatur corpus data celeritate in curva etiam data AMB* (Fig. 70), *et oportet inveniri vim centripetam ad centrum C tendentem una cum vi perpetuo ad rectam AB normaliter in directione MP corpus trahente, quae duae vires efficiant, ut corpus in hac curvae cum praescripta celeritate libere moveatur.*

SOLUTIO.

Sit corporis in puncto *M* existentis celeritas debita altitudini *v* et distantia *MC* = *y*, perpendicularis vero *MP* = *z*. Ponatur vis corpus ad centrum *C* trahens = *P* et vis secundum *MP* trahens = *Q*. Ducta tangente *MV* in *M* demittuntur in eam perpendicula *CT* et *PG*, quae dicantur *p* et *q*. His factis erit vis normalis ex utraque orta

$$= \frac{Pp}{y} + \frac{Qq}{z}$$

et vis tangentialis

$$= \frac{P\sqrt{(y^2-p^2)}}{y} + \frac{Q\sqrt{(z^2-q^2)}}{z}.$$

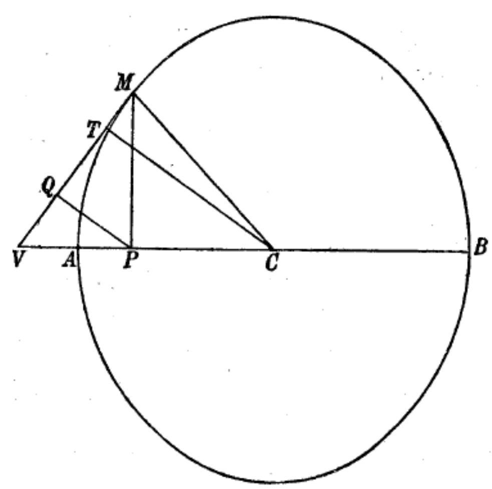

Fig. 70.

Posito ergo radio osculi in $M = r$ erit

$$\frac{Pp}{y} + \frac{Qq}{z} = \frac{2v}{r}$$

(561) et

$$dv = -Pdy - Qdz$$

(559). Ex his itaque reperietur

$$P = \frac{2vyzdz + qrydv}{przdz - qrydy} \quad \text{et} \quad Q = \frac{-2vzydy - przdv}{przdz - qrydy}.$$

Ponatur autem $CP = x$, erit

$$\sqrt{(dx^2 + dz^2)} : dx = z : q, \quad \text{unde} \quad q = \frac{zdx}{\sqrt{(dx^2 + dz^2)}},$$

[p. 319] et posito dx constante est

$$r = \frac{(dx^2 + dz^2)^{\frac{3}{2}}}{-dxddz}.$$

Erit autem porro

$$y = \sqrt{(x^2 + z^2)} \quad \text{et} \quad p = \frac{zdx - xdz}{\sqrt{(dx^2 + dz^2)}}.$$

His substitutis prodibit

$$P = \frac{2vydxdzddz - ydxdv(dx^2 + dz^2)}{x(dx^2 + dz^2)^2},$$

et

$$Q = \frac{-2vydydxddz + (zdx - xdz)(dx^2 + dz^2)dv}{x(dx^2 + dz^2)^2}.$$

Q.E.I.

Corollarium 1.

768. Si corpus aequabiliter in curva moveri debeat, ita ut sit $v = c$ et $dv = 0$, erit

$$P = \frac{2cydxdzddz}{x(dx^2 + dz^2)^2} \quad \text{et} \quad Q = \frac{-2cydydxddz}{x(dx^2 + dx^2)^2}.$$

Corollarium 2.

769. Si curva sit circulus, cuius centrum in C existat, et radius dicatur = a, erit $r = a$, $y = a$, $p = a$ et $q = \frac{z^2}{a}$. Quare prodibit

$$P = \frac{2v}{a} + \frac{zdv}{adz} \quad \text{et} \quad Q = -\frac{dv}{dz}.$$

Ergo cognita Q erit $P = \frac{2v}{a} - \frac{Qz}{a}$. Atque si est $v = c$ et $dv = 0$, erit

$$Q = 0 \text{ et } P = \frac{2c}{a}.$$

Scholion.

770. Ex hac propositione in se specta, qua ipsa curva a corpore descripta datur, parum utilitatis consequitur ad curvas, quas corpora a compositis viribus sollicitata describunt, determinandas. Ab hac vero ad alias propositiones progredi licet, in quibus curvae a corporibus descriptae non ipsae dantur, sed generantur ex motu unius pluriumve datarum, quemadmodum in superioribus propositionibus, in quibus de motu apsidum tractavimus, est factum.

PROPOSITIO 94. [p. 320]

PROBLEMA.

771. *Si moveatur corpus utcunque in curva AMB* (Fig. 71), *ipsa vero curva interea revolvatur circa punctum fixum C, inveniri oportet duas vires, quarum altera perpetuo ad punctum fixum C, altera normaliter ad rectum positione datam PC set directa, quae duae vires efficiant, ut corpus in hac orbita mobili libere moveatur .*

SOLUTIO.

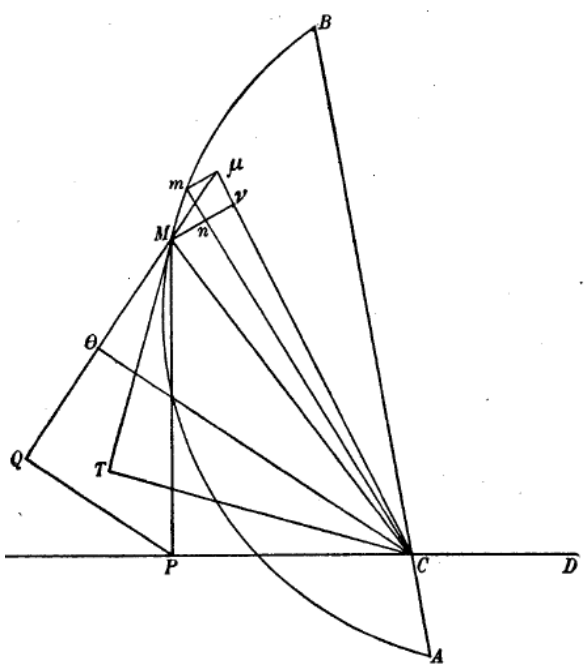

Fig. 71.

Sit corporis in *M* existentis celeritas, qua in ipsa curva elementum *Mm* percurrit, debita altitudini *v*, atque celeritas angularis corporis in orbita ad veram celeritatem angularem corporis circa *C* ut 1 ad *w*, seu celeritas angularis corporis in orbita ad celeritatem angularem ipsius orbitae, dum corpus est in *M*, ut 1 ad $w-1$. Ponatur radius $CM = y$ et perpendiculum *CT* ex *C* in tangentem orbitae in *M* demissum = *p*, ipsa vero tangens $MT = q$, ita ut sit $q=\sqrt{(y^2-p^2)}$. Ex *M* in rectam positione datam *DP* demittatur perpendiculum *MP*, quod dicatur *z*, et *CP* *x*, ita ut sit $x=\sqrt{(y^2-z^2)}$. Iam dum corpus elementum Mm percurrit, ponatur orbita interea angulo $= mC\mu$ circumferri; quamobrem motu composito corporis in μ perveniet, sumto $C\mu = Cm$, eritque Mμ elementum verae curvae, in qua corpus movetur, in quod productum demittatur ex *C* perpendiculum $C\Theta$. Centro *C* describatur arculus $Mn\nu$, [p. 321] erit $mn=\mu\nu=dy$ et $Mn : M\nu = 1 : w$. Est vero

$$Mm=\frac{ydy}{q},$$

unde

$$M\nu=\frac{wpdy}{q},$$

ex quo habebitur

$$M\mu=\frac{dy}{q}\sqrt{(w^2p^2+q^2)}$$

atque porro

$$C\Theta = \frac{wpy}{\sqrt{(w^2p^2+q^2)}}$$

et

$$M\Theta = \frac{qy}{\sqrt{(w^2p^2+q^2)}}.$$

Fiet $Mm : M\mu = \sqrt{v : \frac{\sqrt{v(w^2p^2+q^2)}}{y}}$, cuius quadratum $\frac{v(w^2p^2+q^2)}{y^2}$ exhibet altitudinem debitam verae corporis celeritati; huius igitur incrementum est =

$$\frac{dv(w^2p^2+q^2)}{y^2} + \frac{2(w^2-1)vpdp}{y^2} - \frac{2(w^2-1)vp^2dy}{y^3} + \frac{2vp^2wdw}{y^2}.$$

Radius osculi autem verae curvae, in qua corpus incedit, est =

$$\frac{ydy}{d.C\Theta} = \frac{ydy(w^2p^2+q^2)^{\frac{3}{2}}}{w(w^2-1)p^3dy + wy^3dp + py^3dw - p^3ydw}$$

$$= \frac{ydy(w^2p^2+q^2)^{\frac{3}{2}}}{w(w^2-1)p^3dy + wy^3dp + pq^2ydw}.$$

Posito sinu toto = 1, erit sinus anguli $CMP = \frac{x}{y}$ et cosinus = $\frac{z}{y}$. At anguli $CM\Theta$ sinus erit $= \frac{wp}{\sqrt{(w^2p^2+q^2)}}$ eiusque cosinins $= \frac{q}{\sqrt{(w^2p^2+q^2)}}$. Ex quibus reperitur anguli $PM\Theta$ sinus $= \frac{qpz-qz}{y\sqrt{(w^2p^2+q^2)}}$ eiusque cosinus $= \frac{wpx+qz}{y\sqrt{(w^2p^2+q^2)}}$.

Demisso ex P in tangentem $M\Theta$ perpendiculo PQ erit

$$PQ = \frac{wpz^2 - qxz}{y\sqrt{(w^2p^2+q^2)}} \quad \text{et} \quad MQ = \frac{wpxz + qz^2}{y\sqrt{(w^2p^2+q^2)}}.$$

Atque percurrente corpore elementum $M\nu$ erit linea PM incrementum

$$dz = \frac{wpxdy + qzdy}{qy}.$$

Ex qua aequatione relatio inter w et x innotescit et simul posito lineae absidum AB respectu rectae CP inveniri potest.

Iam ponatur vis corpus sollicitans versus $MC = P$ et vis secundum MP trahens = Q, [p. 322] ex quibus oritur vis tangentialis motum corporis retardans =

$$\frac{Pq}{\sqrt{(w^2p^2+q^2)}} + \frac{Qwpx + Qqz}{y\sqrt{(w^2p^2+q^2)}},$$

quae ergo ducta in $\frac{dy}{q}\sqrt{(w^2p^2+q^2)}$ aequalis poni debet =

$$-\frac{dv(w^2p^2+q^2)}{y^2} - \frac{2(w^2-1)vpdp}{y^2} + \frac{2(w^2-1)vp^2dy}{y^3} - \frac{2vp^2wdw}{y^2},$$

ita ut prodeat

$$Pdy + \frac{Qwpxdy}{qy} + \frac{Qzdy}{y}$$

$$= -\frac{dv(w^2p^2+q^2)}{y^2} - \frac{2(w^2-1)vpdp}{y^2} + \frac{2(w^2-1)vp^2dy}{y^3} - \frac{2vp^2wdw}{y^2}.$$

Vis autem normalis ex ultraque orta est

$$\frac{Pwp}{\sqrt{(w^2p^2+q^2)}} + \frac{Qwpz - Qqx}{y\sqrt{(w^2p^2+q^2)}},$$

quae aequalis esse debet =

$$\frac{2w(w^2-1)vp^3dy + 2wvy^3dp + 2vpq^2ydw}{y^3dy\sqrt{(w^2p^2+q^2)}}$$

(561), quare habebitur

$$Pwpydy + Qwpzdy - Qqxdy = \frac{2w(w^2-1)vp^3dy}{y^2} + 2wvydp + \frac{2vpq^2dw}{y}.$$

Ex quibus aequationibus coniunctis obtinetur

$$Q = -\frac{wpqdv}{yxdy} - \frac{2wvqdp}{yxdy} - \frac{2vpqdw}{yxdy}$$

atque

$$P = \frac{qdv(wpz-qx)}{y^2xdy} + \frac{2(w^2-1)vp^2}{y^3} + \frac{2vdp(wqz+px)}{y^2xdy} + \frac{2vpqzdw}{y^2xdy}.$$

Angulus vero, quem linea absidum *AB* facit cum recta *CP*, erit =

$$\int \frac{(w-1)pdy}{qy},$$

unde eius positio quovis tempore innotescit. Q.E.I.

Corollarium 1.

772. Quia est $dz = \frac{wpxdy+qzdy}{qy}$, ponatur $z = ty$ eritque

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{wpdy}{qy}.$$

Ex qua aequatione, si detur *w* in *y*, in qua etiam ob curvam *AMB* datam *p* et *q* exprimuntur, invenietur *t* idioque etiam *z* et *x*.

Corollarium 2.

773. Si fuerit celeritas corporis in orbita $\sqrt{v}$ reciproce ut perpendiculum *CT* ex *C* in tangentem demissum seu $v = \frac{a^2c}{p^2}$, erit

$$P = \frac{2a^2cdp}{p^3dy} + \frac{2a^2c(w^2-1)}{y^3} + \frac{2a^2cqzdw}{py^2xdy} \quad \text{et} \quad Q = -\frac{2a^2cqdw}{pyxdy}$$

Corollarium 3. [p. 323]

774. Si hoc casu est *w* constans, evanescit vis *Q* et sola remanet

$$P = \frac{2a^2cdp}{p^3dy} + \frac{2a^2c(w^2-1)}{y^3}$$

ad centrum *C* tendens, quae efficiet, ut corpus in orbita *AMB* circa *C* mobili progrediatur, prorsus ut supra inventum est (734)

Corollarium 4.

775. Si *v* non est $= \frac{a^2c}{p^2}$, sed w constans, ita ut motus angularis orbitae proportionalis sit motui angulari corporis in orbita, nempe ut $w-1$ ad 1, erit

$$Q = \frac{-wpqdv - 2wvqdp}{yxdy} \quad \text{et} \quad P = \frac{qdv(wpz-qx)}{y^2xdy} + \frac{2(w^2-1)vp^2}{y^3} + \frac{2vdp(wqz+px)}{y^2xdy}.$$

Exemplum.

776. Posito $v = \frac{a^2c}{p^2}$ sit curva AMB ellipsis alterutrum focum in C habens. Cuius igitur axis transversus si vocetur A et latus rectum L, erit

$$4pp = \frac{ALy}{A-y} \quad \text{seu} \quad p = \frac{\sqrt{ALy}}{2\sqrt{(A-y)}} \quad \text{et} \quad q = \frac{\sqrt{(4Ay^2 - 4y^3 - ALy)}}{2\sqrt{(A-y)}} \quad \text{atque} \quad \frac{dp}{p^3} = \frac{2dy}{Ly^2}.$$

Quare habebitur

$$Q = -\frac{2a^2cdw\sqrt{(4Ay - 4y^2 - AL)}}{yxdy\sqrt{AL}}$$

et

$$P = \frac{4a^2c}{Ly^2} + \frac{2a^2c(w^2-1)}{y^3} + \frac{2a^2czdw\sqrt{(4Ay - 4y^2 - AL)}}{y^2xdy\sqrt{AL}}.$$

Atque

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{wdy\sqrt{AL}}{y\sqrt{(4Ay - 4y^2 - AL)}}.$$

Scholion 1.

777. Hae formulae existente curva ellipsi variis modis simpliciores effici possunt, si curva, in qua corpus movetur, ad circulum proxime accedat. Atque hic casus tum non parum habebit utilitatis in motu lunae theoretice definiendo. [p. 324] Terra enim ut in *C* quiescens ponatur et sol in recta *CP* in C perpendiculari parite tanquam quiescens consideretur; quo facto et his viribus cum viribus solis et terrae comparatis elicietur motus lunae synodicus pro quavis lineae absidum positione et simul ipsius lineae absidum motus, qui a vero lunae motu quam minime differet.

Scholion 2.

778. Multo latius quidem patet ista propositio quam superior (729), in qua vis omnis ad centrum rotationis orbitae erat directa; haec enim illam in se complectitur evanescente vi Q. Neque tamen perfecte ad motum lunae explicandum quadrat propter vim reciproce cubo distantiae MC proportionalem in vi P (773). Hanc ob rem alios orbitae motus praeter gyratorium in medium proferemus, que et latius pateant et magis cum quaestionibus physicis congruant. Huiusmodi sunt motus orbitarum per quasque curvas manente orbita sibi semper parallela, quae contemplatio aliis ideo anteferri meretur, quod vires sollicitantes et facile invenire et simplicioribus formulis possint comprehendi. Ad hoc autem praestandum opus est sequens theorema praemitti.

PROPOSITIO 95.

THEOREMA.

779. *Moveatur corpus in curva AM* (Fig. 72), *a vi quacunque sollicitantum circa punctum C, atque insuper* [p. 325] *et corpus et punctum C ab aequali vi et in eadem directione sollicitentur; erit motus relativus corporis M respectu puncti C seu motus corporis M, qualis ex C spectatur, idem, ac si haec nova vis non accessisset.*

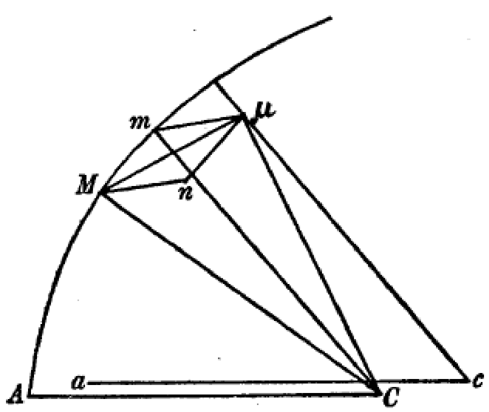

Fig. 72.

DEMONSTRATIO.

Puncto *C* quiescente perveniat puncto temporis dt corpus ex *M* in *m*. Hoc igitur tempusculo finito corpus *M* distabit a *C* intervallo *mC* et cum plaga quadam fixa recta *AC* expressa constituet angulum *mCA*. Ponatur iam corpore existente in *M* et corpus *M* et punctum *C* ab aequali et in eandem plagam tendente vi urgeri, ita ut punctum *C* ab hac vi tempusculo *dt* promoveatur per *Cc*. Eodem igitur tempusculo *dt* corpus *M*, si quiesceret, ab hac vi transferretur per *Mn* parallelam et aequalem ipsi *Cc*. At quia corpus *M* iam habet motum insitum, quo tempusculo *dt* elementum *Mm* percurrit, utroque motu coniuncto describet diagonalem *Mμ* completo parallelogrammo *Mmμn*. Quocirca accidente hac nova vi corpus *M* finito tempusculo *dt* distabit a puncto *C*, quod interea in *c* est translatum, intervallo *μc* et ducta *ac* parallera ipsi *AC* cum plaga fixa constituet angulum *μca*. At est ob *mCcμ* parallelogrammum *μc* = *mC* et ang. *mCA* = ang. *μca.* Consequenter vis utrumque punctum *M* et *C* aequaliter et secundum eandem directionem sollicitans non immutat motum relativum corporis *M* respectu punctum *C*. Q.E.D. [p. 326]

Corollarium 1.

780. Quaecunque igitur vis punctum *C* sollicitat, si eadem simul corpus *M* secundem eandem plagam urgeat, motus relativus corporis *M* respectu puncti *C* non mutabitur.

Corollarium 2.

781. Huiusmodi ergo vi *M* et *C* aequaliter sollicitante effici potest, ut corpus *M* in orbita *AM* quomodocunque mobili moveatur. Orbita autem ipsa motu suo ita sequetur puncti *C* motum, ut eius positio sibi semper maneat parallela.

Corollarium 3.

782. Perspicitur etiam ex demonstratione propositionis, si et puncto C et corpori M aequalis celeritas imprimatur secundum eandem plagam, motum relativum non perturbatum iri.

Corollarium 4.

783. Atque cum talis motus puncto C impressus perpetuo duret aequabilis in directum sine ulla vis continuatione, sequitur corpus M circa punctum C aequabiliter in directum progrediens aeque moveri possi ac circa quiescens. Hoc enim obtinebitur, si modo corpori M aequalis celeritas in eandem plagam directa adiiciatur.

Corollarium 5.

784. Corpus ergo circa centrum virium uniformiter in directum progrediens eandem curvam [p. 327] libere describere poterit, quam circa quiescens describeret, modo ei superaddatur tanta celeritas, quantam accepit centrum virium.

Corollarium 6.

785. Quemadmodum autem corpus sibi ipsum relictum non potest in linea curva progredi neque inaequabiliter in recta, ita corpus circa centrum virium vel in curva motum vel difformiter in directum non potest libere circa id eandem curvam describere quam circa quiescens, sed perpetuo tanta vi insuper urgeri debet, quanta ad centrum in sua via retinendum requiritur.

PROPOSITIO 96.

PROBLEMA.

786. *Si corpus M* (Fig. 71), *circa centrum virium L quiescens revolvatur in curva BM, determinare vim, quae efficit, ut corpus in eadem orbita secundum curvam AL sibi ipsi semper parallela mota ingrediatur.*

SOLUTIO.

Quia corpus M in orbita BM libere movetur circa centrum virium quiescens L, erit, posita distantia $LM = y$ et perpendiculo ex L in tangentem in M demisso $= p$, altitudino debita celeritati in $M = \frac{a^2c}{p^2}$ (589) et vis centrapeta ad L tendens $= \frac{2a^2cdp}{p^3dy}$ (592). Iam ponatur centrum L in curva AL moveri attractum ad centrum virium C, sitque $CL = c$ et perpendiculum ex C in tangentem [p. 328] in L demissum $= w$. Quo posito erit altitudo debita celeritati in $L = \frac{b^2e}{w^2}$ et vis punctum L ad

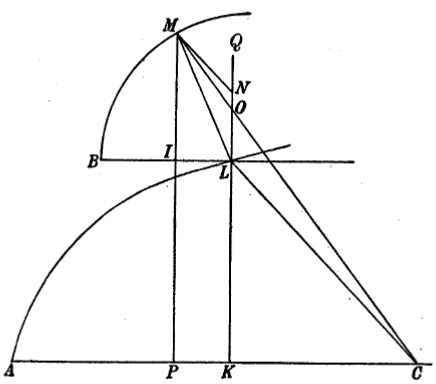

Fig. 73.

C trahens $= \frac{2b^2edw}{w^3ds}$ (592). Ponatur autem primo puncto L celeritas secundum tangentem imprimi $= \frac{b\sqrt{e}}{w}$ eademque etiam corpori M secundum directionem huic tangenti parallelam; atque perspicium erit, si nulla insuper vis punctum L sollicet, sed tantum hunc motum impressum conservet, corpus M libere in eadem curva BM motum puncti L ita sequente, ut axis BL sibi semper maneat parallelus, motum iri (783). At quo corpus M eodem modo circa punctum L in curva AL incedens moveatur, oporteat, ut ipsi perpetuo tanta vis imprimatur, quanta requiritur ad punctum L in hac curva AL retinendum (781). Ducta igitur MN parallela ipsi LC corpus M praeter vim, qua ad L urgetur, sollicitari debebit vi $= \frac{2b^2edw}{w^3ds}$ secundum directionem MN. Quare haec duplex vis efficiet, ut problemati satisfiat. Quo autem appareat, quanam vi corpus M respectu puncti C et rectae fixae AC ipsi BL parallelae sollicitari opporteat, resolvantur vires corpus M secundum MN et ML trahentes in duas alias quarum altera habeat directionem MC, altera MP, quae linea MP ad AC perpendiculariter est ducta. Ad hoc praestandum ipsi MP ducatur parallel KLN rectam MC in O secans et vocetur $LI = x$, $MI = z$, $CK = r$, $KL = t$ atque $CP = X$, $PM = Z$ et $CM = Y$. Quocirca erit

$$y = \sqrt{(x^2 + z^2)}, \quad s = \sqrt{(r^2 + t^2)} \quad \text{et} \quad Y = \sqrt{(X^2 + Z^2)}.$$

Atque porro

$$p = \frac{xdz - zdx}{\sqrt{(dx^2 + dz^2)}} \quad \text{et} \quad w = \frac{rdt - tdr}{\sqrt{(dr^2 + dt^2)}}.$$

[p. 329] Praeterea vero erit $X = r + x$ et $Z = t + z$. Quare cum ob curvas *AL* et *BM* datas, *z, y*, et *p* in *x*, itemque *t, s*, et *w* in *r* dentur, poterunt hae omnes quantitates in *X, Z*, et Y exhibiri. Aequatio vero inter X et *Z* ex eo deducetur, quod temporis incrementum per *BM* aequale esse debeat temporis incremento per AL. Hinc ergo erit

$$\frac{p\sqrt{(dx^2 + dz^2)}}{a\sqrt{c}} = \frac{w\sqrt{(dr^2 + dt^2)}}{b\sqrt{e}},$$

et integrabilis sumtis erit area *ACL* $\pm$ constante quadam area ad aream *BLM* ut $b\sqrt{e}$ ad $a\sqrt{c}$. Vis igitur secundum *ML* trahens resolvatur in binas secundum *MO* et *LO* seu *MI* trahentes et simili modo vis secundum *MN* in binas secundum *MO* et *NO* trahentes, quarum posterior corpus *M* secundum directionem ipsi *MI* contrariam sollicitabit; ad quas resolutiones instituendas angulos nosse oportet. Est vero *MLO = LMI* eiusque igitur sinus $= \frac{x}{y}$ et cosinus $= \frac{z}{y}$, sumto 1 pro sinu toto. Similiter est *MON = CMP*, huius igitur sinus $= \frac{X}{Y}$ et cosinus $= \frac{Z}{Y}$. Anguli ergo *LMO*, qui horum est differentia, sinus erit $= \frac{Xz - Zx}{Yy}$.

Denique est *MNQ = CLK*, quare eius sinus est $= \frac{r}{s}$ et cosinus $= \frac{t}{s}$. Consequenter ob NMO = NMO = MNQ – MON erit sinus $= \frac{Zr - Xt}{Ys}$. Est vero
$Xx - Zx = Zr - Xt$ ob $X = r + x$ et $Z = t + z$.

Ex his fiet *MN:MO* seu $\frac{X}{Y} : \frac{r}{s}$ ita vis secundum *MN* tendens $\frac{2b^2 edw}{w^3 ds}$ ad vim secundum *MC*, quae ergo erit =

$$\frac{2b^2 eYrdw}{Xsw^3 ds}.$$

Atque *MN:NO* seu $\frac{X}{Y} : \frac{Zr - Xt}{Ys}$ ita vis secundum *MN* $\frac{2b^2 edw}{w^3 ds}$ ad vim secundum *ON*, [p. 330] quae ergo est =

$$\frac{2b^2 edw(Zr - Xt)}{Xsw^3 ds}.$$

Simili modo erit $ML : MO = \frac{X}{Y} : \frac{x}{y}$ ita vis secundum *ML* $\frac{2a^2 cdp}{w^3 dy}$ ad vim secundum *MC*, quae ergo erit =

$$\frac{2a^2 cYxdp}{Xyp^3 dy}.$$

Atque $ML : MO = \frac{X}{Y} : \frac{Zr - Xt}{Yy}$ ita vis secundum *ML* $\frac{2a^2 cdp}{p^3 dy}$ ad vim secundum *OL* seu *MP*, quae ergo est = $\frac{2a^2 cdp(Zr - Xt)}{Xyp^3 dy}$.

Ex quibus colligitur corpus *M* trahi debere a vi tendente secundum *MC* =

$$\frac{Y}{X}\left(\frac{2b^2erdw}{sw^3ds}+\frac{2a^2cxdp}{yp^3dy}\right)$$

atque a vi secundum *MP* =

$$\frac{(Zr-Xt)}{X}\left(\frac{2a^2cdp}{yp^3dy}-\frac{2b^2edw}{sw^3ds}\right).$$

Quae expressiones omnes in *X, Z* et *Y* exhiberi poterunt et praeterea inter has quantitates, quae veram curvam a corpore *M* descriptam pertinent, aequatio assignari. Q.E.I.

Corollarium 1.

787. Si curva *AL* sit peripheria circuli, cuius centrum in *C* et radius *AC* = *b*, erit $s = w = b$ et $r^2+t^2=b^2$. Hoc ergo casu vis secundum *MC* trahens erit =

$$\frac{Y}{X}\left(\frac{2er}{b^2}+\frac{2a^2cxdp}{yp^3dy}\right)=\frac{2eY}{b^2}-\frac{2Yx}{X}\left(\frac{e}{b^2}-\frac{a^2cdp}{yp^3dy}\right)$$

et vis secundum *MP* =

$$\frac{\left(Zr-X\sqrt{(b^2-r^2)}\right)}{X}\left(\frac{2a^2cdp}{yp^3dy}-\frac{2e}{b^2}\right)$$

Atque *e* erit altitudo debita celeritati, quam habet punctum *L*.

Corollarium 2.

788. Sit curva *BM* ellipsis centrum habens in *L*, et *BL* eius semiaxis transversus = *a*, ita ut altitudino debita celeritati in *B* sit = *c*, alter vero semiaxis fit = *h*. Eritque

$$a^2z^2+h^2x^2=a^2h^2 \quad \text{et} \quad \frac{2a^2cdp}{p^3dy}=\frac{2cy}{h^2}.$$

Quare vis secundum MC fit =

$$\frac{Y}{X}\left(\frac{2b^2erdw}{sw^3ds}+\frac{2cx}{h^2}\right)=\frac{2b^2eYdw}{sw^3ds}-\frac{2Yx}{X}\left(\frac{b^2edw}{sw^3ds}-\frac{c}{h^2}\right).$$

[p. 331] Atque vis secundum *MP* =

$$\frac{(Zr-Xt)}{X}\left(\frac{2c}{h^2}-\frac{2b^2edw}{sw^3ds}\right).$$

Corollarium 3.

789. Si et curva *AL* fuerit circulus ut coroll. 1 et curva *BM* ellipsis ut coroll. 2, erit vis secundum *MC* =

$$\frac{Y}{X}\left(\frac{2er}{b^2}+\frac{2cx}{h^2}\right)=\frac{2eY}{b^2}-\frac{2Yx}{X}\left(\frac{e}{b^2}-\frac{c}{h^2}\right)$$

et vis secundum *MP* =

$$\frac{(Zr-Xt)}{X}\left(\frac{2c}{h^2}-\frac{2e}{b^2}\right),$$

ubi *r* et *t* et *x* in *X, Z*, et *Y* poterunt determinari ex his aequationibus

$$a^2z^2+h^2x^2=a^2h^2,\quad r^2+t^2=b^2,\quad r+x=X\quad \text{et}\quad t+z=Z.$$

Aequatio autem emergens ad quatuor dimensiones ascendit.

Corollarium 4.

790. Si ellipsis *BM* ponatur infinite parva seu saltem perquam exigua respectu circuli *AL*, ita tamen, ut tempus periodicum ellipsis sit finite magnum, erit

$$x=\frac{a^2X(Y^2-b^2)\pm ahZ\sqrt{(4a^2X^2+4h^2Z^2-(Y^2-b^2)^2)}}{2a^2X^2+2h^2Z^2}$$

et

$$z=\frac{h^2Z(Y^2-b^2)\mp ahX\sqrt{(4a^2X^2+4h^2Z^2-(Y^2-b^2)^2)}}{2a^2X^2+2h^2Z^2}.$$

Corollarium 5.

791. Si curva *BM* fuerit quoque circulus in *L* centrum habens, fiet $h=a$ ideoque

$$x^2+z^2=a^2,\ r^2+t^2=b^2,\ r=X-x\ \text{ et }\ t=Z-z,$$

ex quibus positio $b^2-a^2=f^2$ reperiatur

$$2x=\frac{(Y^2-f^2)X\pm Z\sqrt{(4a^2Y^2-(Y^2-f^2)^2)}}{Y^2}$$

et

$$2z=\frac{(Y^2-f^2)Z\mp X\sqrt{(4a^2Y^2-(Y^2-f^2)^2)}}{Y^2}.$$

Atque

$$2Zr-2Xt=\mp\sqrt{(4a^2Y^2-(Y^2-f^2)^2)}.$$

Scholion 1. [p. 332]

792. Extremum hoc corollarium adiecimus, ut appareat, quales vires requirantur ad corpus in epicyclo circa centrum virium circumagendum, quemadmodum Ptolemaici planetas moveri existimaverunt.

Corollarium 6.

793. Ex hypothesi coroll. 3 intelligitur, si fuerit $\frac{e}{b^2} = \frac{c}{h^2}$, seu celeritas corporis *L* ad celeritatem corporis *M* in *B* ut diameter circuli *AL* ad axem coniugatum ellipsis *BM*, tum veram curvam a corpore *M* descriptam fore ellipsin centrum habentem in *C*, cum vis sollicitans ad *C* tendens sit $\frac{2eY}{b^2}$ evanescente vi secundum *MP*. Huius ellipsis semiaxis maior erit $b + a$, minor vero $b + h$.

Scholion 2.

794. Propositionem hanc ideo praecipue attuli, quod in appendice novae Principiorum Neutoni editionis anglicae Cl. Machin assererat [John Machin (1680-1751), *The mathematical principles of natural philisophy translated into english*, London 1729; cui libro Appendix addita est : *The laws of the motion of the moon*. P. St.] lunae motum considerari posse tanquam in ellipsi, cuius axis transversus sit ad coniugatum ut 2 : 1, circa centrum factum, dum interea ipsa ellipsis motu sibi semper parallelo secundum peripheriam circuli libere progrediatur, quemadmodum in coroll. 3 explicui. Equidem non nego hac ratione motum perquam conformem motui lunae posse exhiberi, sed, an exacte congruat, vehementer dubito. Sequente autem propositione determinare statui, [p. 333] quid ad lunae motum indicandum requiratur. Etiamsi vero ista proposito ad astronomiam pertineat, tamen eam, ut genuina huiusmodi quaestiones resolvendi methodus perspiciatur, hic afferre e re visum est.

PROPOSITIO 97.

PROBLEMA.

795. *Quiescente sole in S* (Fig. 74) *et terra T circa eum in circulo TD uniformiter mota attrahatur luna L tum ad terram T tum ad solem S in reciproce distantiarum duplicata; quibus positis determinari oporteat motum lunae,qualis ex terrae T spectatur.*

SOLUTIO.

Ponatur distantia terrae a sole $ST = a$ et vis, qua terra ad solem trahitur, $= \frac{f}{a^2}$. Distantia lunae a terrae sit $= y$ et distantia lunae a sole LS sit $= z$. Vis, qua luna ad terram trahitur, sit $= \frac{h}{y^2}$; vis vero, qua luna ad solem trahitur secundum LS, erit $= \frac{f}{z^2}$.

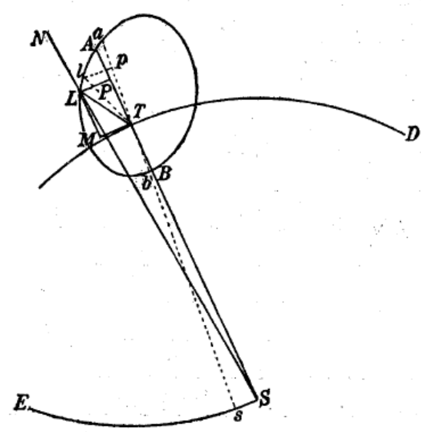

Fig. 74.

[In formulis $\frac{f}{a^2}$ et $\frac{f}{z^2}$ littera f non eundem valorem repraesentat. Quamobrem problematis solutio modificanda est. P. St.]

Ab his igitur viribus lunam sollicitantibus qualis motus producatur, est investigandum. At quia lunae motus, qualis a spectore in terra constituto observatur, definiri debet, terra tanquam quiescens est consideranda; id quod fit, dum toti systemati motus ei, quem terra habet, aequalis et contrarius imprimitur simulque sollicitationes, quas terra a sole recipit, contrario modo in lunam et solem cogitatione transferuntur. [p. 334] Celeritas autem terrae in circulo *TD* debita est altitudini $\frac{f}{2a}$, uti ex vi $\frac{f}{a^2}$, qua terra ad solem urgetur, colligi potest. Tanta igitur celeritas et soli et lunae secundum directionem ad *TS* normalem imprimi debet. Praeterea, quia terra ad solem trahitur vi $\frac{f}{a^2}$, oportet ad huius vis effectum destruendum res ita concipi, ac si sol perpetuo tanta vi ad terram traheretur, luna vero eadem vi secundum *LN* ipsi *ST* parallam. Hoc facto sol describet circa terram in *T* quiescentem circulum *SE* eadem celeritate, qua ante terra circa solem ferebatur. Luna vero praeter vires secundum *LT* et *LS* tendentes insuper urgibitur versus *LN* vi $= \frac{f}{a^2}$.

Ducta *LM* parallela quoque ipsi *TS* resolvatur vis secundum *LS* agens $\frac{fy}{z^3}$ in has duas, quarum alterius directio sit LT, alterius LM. Ex consideratione ergo triangulo LTS orietur vis secundum *LT* agens $= \frac{fy}{z^3}$ et vis secundum LM trahens $= \frac{af}{z^3}$. Quare omnibus coniunctis luna trahetur versus *LT* vi =

$$\frac{h}{y^2}+\frac{fy}{z^3}$$

atque versus *LM* vi =

$$\frac{f}{a^2}-\frac{af}{z^3}=\frac{f(z^3-a^3)}{a^2z^3},$$

ex quibus viribus motus lunae debet determinari. Notandum autem directionem *LM* non esse constantem sed variabilem, quippe perpetuo parallelam radio *TS*, qui ob motum solis secundum peripheriam *SE* circumfertur. Producta igitur *ST* in *A*, ut *AB* sit linea syzygiarum, et ex *L* in eam demisso perpendiculo LP erit *TP* aequalis et parallela ipsi *LM*. [p. 335] Perveniat tempusculo dt luna ex *L* in *l*, sol autem ex *S* in *s*; transferetur ergo interia linea syzygiarum in *ab* et luna in *l* sollicitabitur partim a vi secundum *lT*, partim a vi secundum parallelam ipsi *Tp* trahente, demisso scilicet ex *l* in Ta perpendiculo *lp*. Ex his autem viribus resolvendis reperiuntur vis normalis et tangentialis, quarum utraque celeritatem lunae dabit. Hae autem aequationes coniunctae eliminata celeritate praebebunt aequationem pro curva *ABL*, in qua luna moveri cernitur. Q.E.I.

Scholion 1.

796. Aequationes, quae hinc ad motum lunae deducuntur, tam fiunt complexae, ut ex iis neque lunae neque orbita neque positio lineae absidum eiusque motus exacte possint determinari. Vero autem proxime ex eodem calculo neglegendis quantitatibus vehementer exiguis quadammodo conclusiones in usum astronomiae possunt elici, quemadmodum fecit Summus Neutonus in *Phil. Princ*. Libr. III. Etiamsi autem hoc incommodo calculus non laboraret, tamen ista proposito non summo rigore motum lunae esset exhibitura. Posuimus enim solem prorsuns quiescere, quod a vero parumper discrepat; deinde terrram in circulo motam consideramus et orbitam lunae in ipso terrae plano positam, quae itidem re ipsa secus se habent. Interim tamen certum est, si huius propositionis solutio posset evolvi ex eaque tabula confici, [p. 336] hoc in astronomia maximam habiturum esse utilitatem.

Corollarium 1.

797. Quia lunae a terrae distantia est admodum parvum respectu distantiae terrae a sole, sine sensibili errore fere poterit poni $z = a$, quo casu vis secundum *LM* agens evanescit et luna tantum ad terram trahetur vi =

$$\frac{h}{y^2}+\frac{fy}{a^3}.$$

Corollarium 2.

798. Cum orbita lunae non multum differat a circulo, poterit ea instar ellipsis mobilis considerari, ut fecimus prop. 91 (747). Quare ad motum absidum cognoscendum erit ex illius prop. coroll. 3 (750)

$$P=\frac{a^3hy+fy^4}{a^3},$$

atque luna a perigaeo ad apogaeum perveniet absoluto motu angulari circa terram angulo

$$=180\sqrt{\frac{a^3h+fy^3}{a^3h+4fy^3}}\text{ graduum,}$$

ubi y, quia non multum variatur, tanquam constans est considerandum.

Scholion 2.

799. Regrederetur ergo perpetuo linea absidum motus lunaris, quia $\frac{a^3h+fy^3}{a^3h+4fy^3}$ minor est unitate, id quod est contra observationes. Ratio vero huius erroris est, quod *z* ut constantem quantitatem consideravimus. Nam etsi *z* non multum neque augeatur neque minuatur ratione sui ipsius, tamen eius incrementa et decrementa respectu incrementorum ipsius *y* minime negligi possunt. [p. 337] Quare, cum ipsius *P* differentiale sit accipiendum, in eo perperam z tanquam constantem sumus contemplati eiusque loco a posuimus. Quia autem *z* non potest dari per *y*, motus absidum non potest hoc modo determinari. Interim tamen hoc colligitur, si fuerit $ady > ydz$, lineam absidum in antecedentia, at si $ady < ydz$, in consequentia promoveri, si quidem cogitatem abstrahamus a vi secundum *LM* agente.

Corollarium 3.

800. Posito $LM = TP = x$ erit proxime $z = a + x$, ubi x est admodum parvum respectu *a*. Neglecto ergo *x* prae *a* erit vis, qua luna ad terram trahitur, $= \frac{h}{y^2} + \frac{fy}{a^3}$ et vis, qua secundum LM trahitur, $= \frac{3fx}{a^3}$. Haec igitur evanescit, quando luna est in quadraturis, maxima vero est, quando luna est in syzygiis.

Scholion 3.

801. Cum autem non sit huius loci haec ad motum lunae spectantia fusius persequi, quippe quae ad astronomiam theoreticam pertinent, ad reliqua instituto nostro accommodata progrediemur. Sufficere enim possunt ista ad intelligendum, quomodo canones motuum trahiti ad quosvis casus motusque respectivos inveniendos in usum verti queant. Quae autem in hoc capite restant, motum corporum liberum, qui non fit in eadem plano, [p. 338] complectuntur. Ex praecedentibus quidem manifestum est unica existente vi centripeta motum corporis semper fieri in eodem plano, quomodocunque etiam corpus initio fuerit proiectum; et si plura sint centra virium in eodem plano sita in eodemque plano corporis fiat proiectio, curva a corpora descripta similite tota in eodem plano erit posita. Ad sequentia igitur referri debet, quando corpus in pluribus viribus, quarum directiones in diversis planis existunt, sollicitatur, vel etiam quando directio, secundum quam corpus initio proiicitur, non in eo, in quo sunt virium directiones, sita est plano. His igitur in casibus motus corporis ita debet considerari, quasi fieret in superficie quadam convexa seu concava in eaque lineam quandam describeret. Natura autem superficiei exprimitur aequatione tres indeterminantas involvente, et lineae in ea superficie ductae natura continetur eadem illa aequatione coniuncta cum alia aequatione verl tres quoque illas indeterminatas complectente vel duas tantum. Ex his enim deduci poterit curvae lineae proiecto in dato plano et ex proiectione et superficie simul innotescit ipsa curva a corpora descripta et in superficie posita. Quemadmodum porro in plano quaevis vires ad duas, normalem et tangentialem, possunt reduci, ita in hoc negotio virium reductio ad tres fieri debet (551), quae, quales in corpus exerant effectus, primum sumus investigaturi. [p. 339]