CHAPTER FIVE (*Part c*).
CONCERNING THE CURVILINEAR MOTION OF FREE POINTS ACTED ON BY ABSOLUTE FORCES OF ANY KIND.

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PROPOSITION 86.

PROBLEM.

707. *To find the law of the force continually pulling downwards which can be constructed along the lines MP* (Fig. 62) *parallel to each other, in order that the body moves along a given curve AM, and to determine the speed of the body at individual points M.* [p. 292]

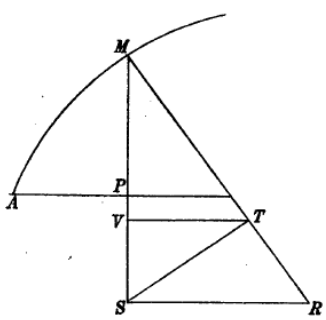
Fig. 62.

SOLUTION.

The normal AP is drawn through the line MP, AP is called x, and PM y. The curve $AM = s$ and the radius os osculation MR at $M = r$, then $r = \frac{dsdy}{ddx}$ with ds taken as constant. Again the speed of the body at M corresponds to the height v, and the body force at M pulling along MP is put as P. From these established, these two equations are obtained : $dv = -Pdy$ and $Pr\,dx = 2vds$ (557), from which, if v is known, then P is itself immediately apparent. Therefore P is eliminated, and v is to be found from this equation : $rdxdv = -2vdsdy$, which with r replaced by its value $\frac{dsdy}{ddx}$, becomes this equation: $-\frac{dv}{v} = \frac{2ddx}{dx}$, which integrated gives: $l\,C - l\,v = l\,\frac{dx^2}{ds^2}$. Moreover, the speed is known at the point A, and that corresponds to the height [$v =$] c, and if the cosine of the angle MAP or the value of $\frac{dx}{ds}$, with the point M falling at A, is equal to λ. Therefore this equation hence arises : $l\,C = l\,c + 2l\,\lambda$ and consequently

$$v = \frac{\lambda^2 cds^2}{dx^2},$$

hence the speed of the body becomes known at the individual points M.

Moreover the force P acting can be found to be :

$$\frac{2\lambda^2 cds^3}{rdx^3} = \frac{2\lambda^2 cds^2 ddx}{dx^3 dy}.$$

Or, if dx taken as constant, then in which case $r = \frac{ds^3}{-dxddy}$, and the force is

$$P = \frac{-2\lambda^2 c\,ddy}{dx^2}.$$

Finally the time, in which the arc AM is traversed, is equal to :

$$\int \frac{ds}{\sqrt{v}} = \frac{x}{\lambda\sqrt{c}}.$$

Q.E.I. [Note that *ds* taken as constant enables us to write $\int \frac{ddx}{dx} = \int \frac{d(dx/ds)}{dx/ds} = log(dx/ds)$. Also recall that $r = -\frac{(1+y'^2)^{3/2}}{y''} = -\frac{ds^3}{(ddy/dx^2)dx^3} = -\frac{ds^3}{ddydx}$.]

Corollary 1.

708. Therefore whatever the force shall be acting, the body always progresses horizontally uniformly on account of the time to travel along *AM* to be proportional to *AP* itself, that has been observed above (579). [p. 293]

Corollary 2.

709. If a perpendicular is sent from *R* to the line *MP* produced *RS,* and from *S* another perpendicular *ST* is sent to *MR* , and finally a third perpendicular *TV* is sent from *T* to *MS*, then $MV = \frac{rdx^3}{ds^3}$. Whereby the force P acting has the ratio to the force of gravity [which equals one] as $2\lambda^2 c$ to *MV*, or *P* is inversely as *MV*.
[For there is a set of nested similar triangles, where *dx/ds* = cosθ = cos (*TSR*). Thus, $MV = MT\cos\theta = MS\cos^2\theta = MR\cos^3\theta$.]

Corollary 3.

710. If the angle at *A* is right, then $\lambda = 0$. In which case the body must ascent straight up. But only if λ is made indefinitely small and *c* infinitely large, in order that $2\lambda^2 c$ has a finite value, is the body able to move by making use of this curve.

Example 1.

711. Let the curve *AM* be a circle, the diameter of which is put on the axis *AP*, and the radius is equal to *a*. Thus $r = a$ and $ds : dx = a : y$. On this account the force P becomes :

$$P = \frac{2\lambda^2 a^2 c}{y^3} \text{ and } v = \frac{\lambda^2 a^2 c}{y^2}$$

Therefore the force pulling downwards on M varies inversely as the cube of the upright MP and the speed varies inversely as this applied line itself [i. e. the *y*-coordinate]. Truly the height generating the speed at the maximum point of the periphery, where it becomes $y = a$, is equal to $\lambda^2 c$.

Example 2.

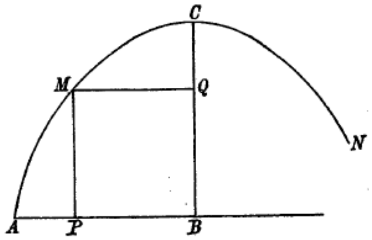

Fig. 63.

712. Let the curve *AMC* (Fig. 63) be a parabola, the axis of which *CB* is vertical and the parameter is equal to *a*. The horizontal line *MQ* is drawn and on putting $CQ = t$ and $MQ = z$, then $z^2 = at$. Truly also $dx = -dz$ and $dy = -dt$ [p. 294] and as before, $\sqrt{(dt^2 + dz^2)} = ds$. On account of which, $v = \frac{\lambda^2 cds^2}{dz^2}$ and $P = \frac{2\lambda^2 cddt}{dz^2}$. Let the speed at the maximum point *C* correspond to the height *b*, and this is given, since $ds = dz$ at *C*, by $\lambda^2 c = b$, and thus $v = \frac{bds^2}{dz^2}$ and $P = \frac{2bddt}{dz^2}$, with *dz* taken as constant. From the equation $z^2 = at$ we have $dt = \frac{2zdz}{a}$ and $ddt = \frac{2dz^2}{a}$ and $ds^2 = dz^2(1 + \frac{4z^2}{a^2})$.
Consequently there is found :

$$v = b\left(\frac{a^2 + 4z^2}{a^2}\right) = b\left(\frac{a + 4t}{a}\right)$$

and $P = \frac{4b}{a}$.
From which it is apparent that the force pulling downwards is constant, which is effective as the body progresses along the parabola. Whereby therefore, if this force is taken as the force of gravity equal to 1, it gives rise to $b = \frac{a}{4}$, equal to the distance of the focus from the vertex. Which are in agreement with what was found above (564 and onwards.)
[This corresponds to our normal presentation of the parabola, $y^2 = 4ax$, where the focus lies on the semi-latus rectum AB, and CB is taken as equal to *a;* if here the vertex to focus distance CB is equal to *b*, while Euler's *a* is equal to the length of the latus rectum, 2AB.]

Example 3.

713. Let *MAN* (Fig. 64) be a hyperbola with centre *C* described having the vertical axis *CP*. The semi-transverse axis $AC = a$ and the semi-conjugate axis $= e$ and $CP = t$, $PM = z$, and as above the height corresponding to the speed that the body has at *A* is equal to *b*, is as we have done for the above parabola : $v = \frac{bds^2}{dz^2}$ and $P = \frac{2bddt}{dz^2}$ with *dz* taken as constant. Truly from the nature of the hyperbola, we have $a^2z^2 = -a^2e^2 + e^2t^2$, from which there arises on differentiation :

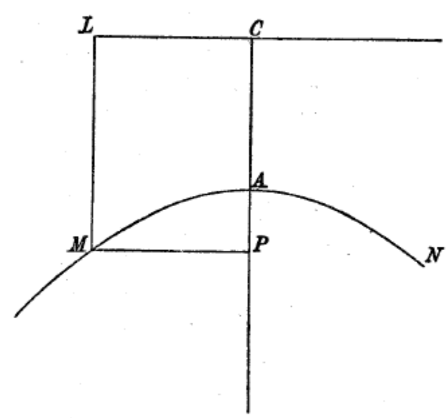

Fig. 64.

$dt = \frac{a^2 zdz}{e^2 t}$ and [on differentiating and substituting these last two equations :]

$$ddt = \frac{a^2 dz^2}{e^2 t} - \frac{a^2 z dz dt}{e^2 t^2} = \frac{a^2 e^2 t^2 dz^2 - a^4 z^2 dz^2}{e^4 t^3} = \frac{a^4 dz^2}{e^2 t^3}.$$

Consequently, the force is given by : $P = \frac{2a^4 b}{e^2 t^3}$, or the force pulling the body downwards everywhere at M is inversely proportional to the cube of the distance ML of the point M from the horizontal LC drawn through the centre C. Again we have :

$$\frac{ds^2}{dz^2} = \frac{e^4 t^2 + a^4 z^2}{e^4 t^2} = \frac{(a^2 + e^2)t^2 - a^4}{e^2 t^2}.$$

And we have besides :

$$v = \frac{b(a^2 + e^2)t^2 - a^4 b}{e^2 t^2}.$$

PROPOSITION 87. [p. 295]

PROBLEM.

714. *For the given curve AMB* (Fig. 65) *together with the centre of attraction C, to find the law of the centripetal force, which must act, in order that the body is free to move along that curve, and to find the speed of the body at any position M.*

SOLUTION.

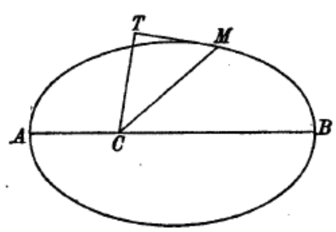

Fig. 65.

Since the curve *AMB* is given together with the point *C*, the equation is sought between the distance *MC* of any point of the curve *M* from the centre *C* and the perpendicular *CT*, that is sent from *C* to the tangent *MT*. Whereby with $CM = y$ and $CT = p$ the equation between p and y can be obtained. Now let the speed of the body at the point *A* correspond to the height c and the perpendicular from *C* sent to the tangent at *A* be equal to h. Truly of these unknown quantities, let the height corresponding to the speed at *M* be equal to v and the centripetal force at *M* is equal to P. With these put in place, we have $v = \frac{ch^2}{p^2}$ (589) and $P = \frac{2ch^2 dp}{p^3 dy}$ (592). Or with the radius of osculation at *M* put equal to r then we have $P = \frac{2ch^2 y}{p^3 r}$ (592). Q.E.I.

Corollary 1.

715. Also the time, in which the body completes some arc AM, is equal to $\frac{2ACM}{h\sqrt{c}}$, or is proportional to the area *ACM* (588).

Corollary 2.

716. Since ch^2 is a constant quantity [i. e. the angular momentum of the unit mass], then the centripetal force at some point M is proportional to this value [p. 296] $\frac{dp}{p^3dy}$ or to this : $\frac{y}{p^3r}$. Therefore the speed $\sqrt{v}$ is proportional to the reciprocal of the perpendicular CT sent to the tangent MT (589).

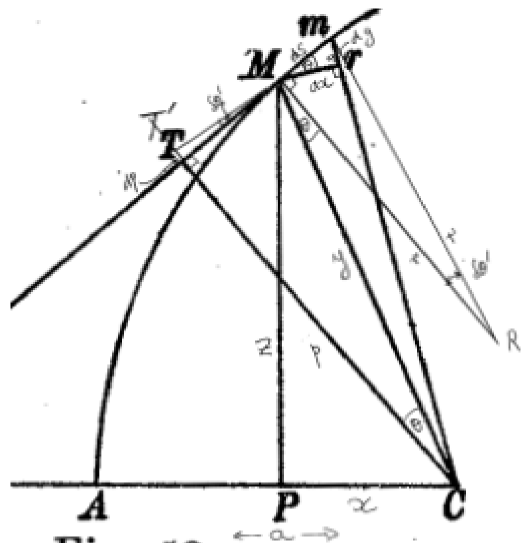

Fig. 52.

[You may wish to remind yourself about r using the sketch supplied here, as in Ch. 5 Part a : note that *Mmr* and MTT' are incremental triangles, and hence the lines TC, MR, and *mR* are treated as parallel. We hence have the similar triangles *Mmr* and *CMT* : giving $dy/ds = MT/y$ and $dy/dx = MT/p$. Likewise, triangles *MTT'* and *RmM* give : $dp/MT = ds/r;$ hence. $r = MT.ds/dp = ydy/dp.$ Note that the first elemental triangle diminishes to a point, keeping its angles intact, while the second diminishes to a finite line segment which stays constant.**]**

Example 1.

717. Let the given curve be an ellipse and the centre of force C placed in the centre of the ellipse. The semi-transverse axis is called a and the semi-conjugate axis b; from the nature of the ellipse it follows that $p = \frac{ab}{\sqrt{(a^2+b^2-y^2)}}$. Therefore we have :

$dp = \frac{abydy}{(a^2+b^2-y^2)^{\frac{3}{2}}}$ and thus $\frac{dp}{p^3dy} = \frac{y}{a^2b^2}$. On this account the centripetal force produced is given by :

$$P = \frac{2ch^2y}{a^2b^2},$$

which is therefore proportional to the distance from the centre of the ellipse.

Example 2.

718. Let the given curve again be an ellipse, but with the centre of force C placed in the other focus. The transverse axis of this is put equal to A and the latus rectum equal to L, and from the nature of the ellipse we find that $4pp = \frac{ALy}{A-y}$. Hence of differentiating, we have $8pdp = \frac{A^2Ldy}{(A-y)^2}$. Truly since $16p^4 = \frac{A^2L^2y^2}{(A-y)^2}$, then $\frac{dp}{2p^3} = \frac{dy}{Ly^2}$ and consequently

$$P = \frac{4ch^2}{Ly^2}.$$

Therefore the centre of force is inversely proportional to the square of the distance of the body from the centre of force C.

Example 3.

719. Let the curve be a logarithmic spiral and the centre of force C is placed at the centre of this; hence $p = ny$ and $\frac{dp}{p^3dy} = \frac{1}{n^2y^3}$, and thus $P = \frac{2ch^2}{n^2y^3}$.

Whereby the centripetal force varies inversely as the cube of the distance of the body from the centre. [p. 297]

PROPOSITION 88.

THEOREM.

720. *The force pulling towards C* (Fig. 66), *that is put in place in order that the body moves along the given curve AM, has the ratio to the force pulling towards another centre of force c, which is put in place in order that the body can move around the same curve with the same periodic time, as the cube of the line cV from c to the tangent TM drawn parallel to the line CM, is to the volume formed from the line cM multiplied by the square of the line CM.*

DEMONSTRATION.

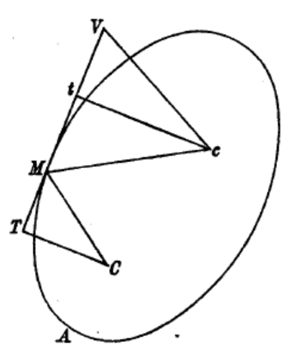

Fig. 66.

Let the speed of the body at a given point A, with the body rotating around the centre of force C, correspond to the height c and the perpendicular sent from C to the tangent at A is equal to h. But when the body is moving around the centre of force c, let the speed at A correspond to the height γ and the perpendicular sent from the centre c to the tangent at A is equal to θ. Moreover since the periodic times around each centre of force are equal, then $h\sqrt{c} = \theta\sqrt{\gamma}$ or $ch^2 = \gamma\theta^2$ (715). From the centre C and from c perpendiculars CT and ct are again sent to the tangent at M, and the radius of osculation at M is equal to r. With these in place, the centripetal force at M pulling towards C, that we call *P*, is equal to $\frac{2ch^2.CM}{r.CT^3}$ and the centripetal force at *M* pulling towards the centre *c*, that we call *Π*, is equal to $\frac{2\gamma\theta^2.cM}{r.ct^3}$ (714). On account of this, since $ch^2 = \gamma\theta^2$ then

$$P:\Pi = \frac{CM}{CT^3} : \frac{cM}{ct^3}.$$

Moreover with the line drawn *cV* parallel to the line *CM*, as triangles *TCM* and *tcV* are similar, we have $CT : ct = CM : cV$. Hence because of this,

$$P:\Pi = \frac{1}{CM^2} : \frac{cM}{cV^3} = cV^3 : cM \cdot CM^2.$$

Q.E.D. [p. 298]

Corollary 1.

721. The speed of the body at the same point *M*, while it is being attracted to the centre of forces *C*, to the speed, while it is being attracted to the other centre *c*, vary inversely as *CT* to *ct* or directly as *cV* to *CM*. This is a consequence of $ch^2 = \gamma\theta^2$.

Corollary 2.

722. If the periodic times are not equal, but are to each other as *T* to *t*, then $T : t = \frac{1}{h\sqrt{c}} : \frac{1}{\theta\sqrt{\gamma}}$ or $ch^2 : \gamma\theta^2 = t^2 : T^2$. Consequently it follows that the ratio of the forces :

$$P:\Pi = t^2 \cdot cV^3 : T^2 \cdot cM \cdot CM^2.$$

Or the forces *P* and *Π* are in the ratio composed from the ratio assigned for equal periods in the theorem, and inversely as the square of the periodic times.

Corollary 3.

723. In this same case, in which the periodic times are unequal, the speed of the body at *M*, with the centre of forces placed at *C*, to the speed at *M*, with the centre of forces placed at *c*, are in the reciprocal ration composed from the ratio of the perpendiculars *CT* and *ct* and in the ratio of the period times *T : t*.

Scholium 1.

724. Newton deduced this Proposition in Book I, prop. VII, coroll. 3 of the *Princ*., and that was used to find the centripetal force acting at some point, from the known force acting at some other centre. Here we show the use of this in the following single example. [p. 299]

Example.

725. Let the given curve be the circle *AMc* (Fig. 67), and one centre of forces is put at the centre of the circle *C*. Therefore the force *P* pulling everywhere towards *C* is constant and is called *g*. From this is sought the force acting towards a centre of forces *c* situated on the periphery and making *Π*, in order that the body is moving

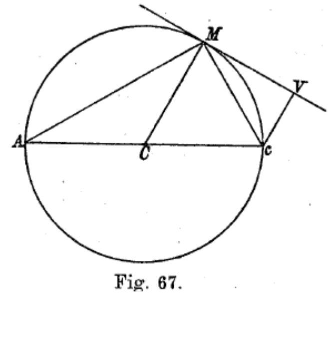

Fig. 67.

around the circle in the same periodic time. Therefore the perpendicular *cV* is sent from *c* to the tangent *MV*, which from the nature of the circle is likewise parallel to the line *CM*. On account of which, $g : \Pi = cV^3 : cM.CM^2$ and thus

$$\Pi = \frac{g \cdot cM \cdot CM^2}{cV^3}.$$

The line AM is drawn, the triangles cVM and cMA are similar, as the angles cMV = cAM and since $cV : cM = cM : cA$. Therefore $cV = \frac{cM^2}{2CM}$, from which the force becomes :

$$\Pi = \frac{8g \cdot CM^5}{cM^5}.$$

Therefore this force *Π* varies inversely as the fifth power of the distance *Mc* of the body from the centre of forces *c*, now as was found above (692).

Corollary 4.

726. Let the speed of the body rotating on the periphery of the circle around the centre *C* correspond to the height *c* and the speed of the body at *M* rotating around the centre of forces c correspond to the height v. Hence :

$$\sqrt{c} : \sqrt{v} = cV : CM = cM^2 : 2CM^2$$

(721) or $v = \frac{4c.CM^4}{cM^4}$.

On account of which the speed of the body rotating around the centre *c* is everywhere reciprocally as the square of the distance of that from *c*. [p. 300] [On account of cons. of ang. mom.]

Corollary 5.

727. For, with the centre of force present at the centre C of the circle, then we have $h = y = p = r =$ radius *CM*, and $P = g = \frac{2c}{CM}$ (592). On this account, [from above] by putting $\Pi = \frac{f^5}{cM^5}$, $g = \frac{f^5}{8CM^5}$ and $c = \frac{f^5}{16CM^5}$. Thus $v = \frac{f^5}{4cM^4}$.

Scholium 2.

728. In these propositions we have put in place the curve that the body describes, which is given completely with the equation for that curve. But there are also the cases, in which the curve itself is not given that describes the motion, but rather the motion itself must first be found by examining certain conditions, so that the law of the centripetal force can then be found. And here these propositions are concerned, and which have been treated everywhere, with the motion of bodies in moving orbits, concerning which therefore we treat in the following proposition. [We note that the fifth power situation with the centre of force lying on the orbit would not be a physically realizable situation.]

PROPOSITION 89.

PROBLEM.

729. *If the orbit* (*A*)(*M*)(*B*) (Fig. 65) *is revolving around some centre of forces C, it is required to define the centripetal force always pulling towards C, which is put in place, in order that the body is moving in this moveable orbit.*

SOLUTION.

While the orbit comes from the situation (*A*)(*M*)(*B*) to the situation *AMB*, the body meanwhile is put to have come from (*A*) to *M*, thus as the body in the orbit meanwhile describes the angle (*A*)*C*(*M*) = *ACM*, but actually the angle (*A*)*CM* = (*A*)C(*M*) + (*A*)*CA* [p. 301] has been described. Initially with the body present at (*A*), the true speed at this point is not that which it has in the orbit, but corresponds to the altitude *c*; and the line *C*(*A*) is perpendicular both to the orbit as well as to the true curve on which the body moves, and which is equal to *a*. Again the speed of the body at *M* , as much as it is moving in the orbit, corresponds to the height *u* and the true speed of the body at M corresponds to the height *v*.

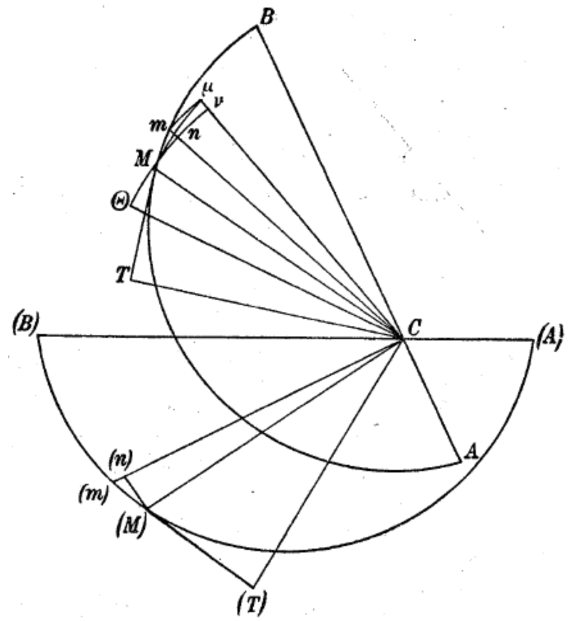

Fig. 68.

But the angular speed in the orbit to the true angular speed around C, while the body is moving at *M*, is as 1 to *w*. Therefore in the orbit considered stationary, the element (*M*)(*m*) is described with the speed $\sqrt{u}$. The distance *C*(*M*) is put equal to *CM* = *y* and the perpendicular $C(T) = CT = p$ is sent from *C* to the tangent to the orbit at (*M*) or *M*, and an equation exists between *p* and *y* on account of the given orbit. Now while the body describes the element *Mm* in the orbit, the orbit itself progresses around *C* with the orbital angular motion through the angle $mC\mu$, and on this account the body is found, not indeed at *m*, but instead at μ, by taking $C\mu = Cm$, and meanwhile the element *M*μ is agreed to have been described by that speed which corresponds to the height *v*. Hence we have :

$$M\mu : Mm = \sqrt{v} : \sqrt{u},$$

and with the small [circular] arc *Mν* described with centre *C* (on account of the given angular motions about *C* in the orbit and in fact in the ratio $1 : w$) for which the ratio is :

$$Mn : M\nu = 1 : w.$$

[Thus, the first proportionality describes the ratio of the displacements of the body in the orbit reference frame to the absolute reference frame in which the orbit itself rotates; the second proportionality considers the corresponding angular speed by taking the corresponding arcs of a circle *Mn* and *Mv,* the first relative to the orbit reference frame and the second relative to the absolute reference frame .]

Truly, with the tangent put in place : $MT = \sqrt{(y^2 - p^2)} = q$, $Mm = \frac{ydy}{q}$ and $Mn = \frac{pdy}{q}$

[In a small displacement along the orbital curve, C*M* corresponds to *y* and *Cm* corresponds to *y + dy*; hence *mn* is equal to *dy* in the elemental triangle *mMn*. The curvature remains the same on *Mm*, and the centre of curvature lies along the normal to the curve and the tangent at M, i. e. on a line parallel to CT (but not passing through *C* unless the orbit is a circular one); hence, the angle TCM is a measure of the angle between the direction of MC or *y* and the direction of the normal to the curve at M, which stays constant along Mm. Hence the angle *Mmn* is the complement of this angle, and hence the angle *mMn* is equal to the angle *TCM*. It follows that the triangles *Mnm* and MTC are similar; hence the ratio *dy/Mm = TM/MC = q/y* follows. Meanwhile, the arc *Mn* is traced out by the same elemental angle *MCn* or *MCm*, for which the constant radius is *y*, and the arc length *Mn* is to *dy* (the cotangent of the angle *mMn* or *MCT*) as *p* is to *q* as above. We have met this result several times already, the last occurrence in (716) above.]

On which account we have : $M\mu = \frac{ydy\sqrt{v}}{q\sqrt{u}}$ and $M\nu = \frac{wpdy}{q}$

[The multiples of the distance and angle gone through in the same time on the true curve. Hence, $\nu\mu^2 + M\nu^2 = M\mu^2$ or $1 + \frac{M\nu^2}{dy^2} = \frac{M\mu^2}{dy^2}$, leading to the next result.]

From which as $\mu\nu = mn = dy$ there is produced $1 + \frac{w^2 p^2}{q^2} = \frac{vy^2}{uq^2}$ or

$$vy^2 = uq^2 + w^2 up^2.$$

Because *Mμ* is the element [p. 302] of the true curve that the body describes, the perpendicular $C\Theta$ is sent from *C* to this element produced, and the ratio becomes [on using arguments similar to the above for the equality of the angles] :
$M\mu : M\nu = CM : C\Theta$, thus we have :

$$C\Theta = \frac{wp\sqrt{u}}{\sqrt{v}}.$$

Truly from this perpendicular, the true speed of the body is known; for $v = \frac{a^2cv}{w^2up^2}$ (589)

[Recall that since for any curve, $v = \frac{ch^2}{p^2}$, and that the speed $\sqrt{v} = \frac{h\sqrt{c}}{p}$; here $p = C\Theta$, and thus $v = \frac{a^2cv}{w^2up^2}$ and *v* cancels out.]

$$u = \frac{a^2c}{w^2p} \text{ and } v = \frac{a^2c(q^2 + w^2p^2)}{w^2p^2y^2}.$$

With these values of *u* and *v* put in place, then

$$C\Theta = \frac{wpy}{\sqrt{(q^2 + w^2p^2)}},$$

which for the sake of brevity we call π. Moreover from this known value π the centripetal force P can itself be found, which comes about, in order that the body can move in this given orbit, and that the orbit in turn can be moved in this way. For it is given by $P=\frac{2a^2cd\pi}{\pi^3dy}$ (592) [De Moivre's Theorem]. But since $\pi^2=\frac{w^2p^2y^2}{q^2+w^2p^2}$, then [as $q^2+p^2=y^2$:]

$$\frac{1}{\pi^2}=\frac{q^2}{w^2p^2y^2}+\frac{1}{y^2}=\frac{1}{y^2}-\frac{1}{w^2y^2}+\frac{1}{w^2p^2},$$

and thus

$$-\frac{2d\pi}{\pi^3}=-\frac{2dy}{y^3}+\frac{2dw}{w^3y^2}+\frac{2dy}{w^2y^3}-\frac{2dw}{w^3p^2}-\frac{2dp}{w^2p^3}$$

or

$$\frac{d\pi}{\pi^3}=\frac{dy(w^2-1)}{w^2y^3}+\frac{q^2dw}{w^3y^2p^2}+\frac{dp}{w^2p^3}.$$

Consequently we have :

$$P=\frac{2a^2cdp}{w^2p^3dy}+\frac{2a^2c(w^2-1)}{w^2y^3}+\frac{2a^2cq^2dw}{w^3y^2p^2dy}.$$

Q.E.I.

Corollary 1.

730. With the radius put equal to 1 then $\frac{m\mu}{CM}=\frac{(w-1)pdy}{qy}$ is the element of the angle $(A)CA$, that the orbit has completed, while the body travels through the arc $(A)(M)$. For this reason the angle $(A)CA=\int\frac{(w-1)pdy}{qy}$. And $(w-1):1$ is as the angular speed of the orbital to the angular speed of the body, while it is at M in its orbit.

Corollary 2.

731. The speed of the body in the orbit, which is as $\sqrt{u}$, is inversely proportional to wp itself. Therefore, unless w is constant, this cannot happen, as the body in the stationary orbit may be moved by this force in this way, attracted to the centre C. [p. 303] [One would need to add another force as a cause for w to change.]

Corollary 3.

732. Therefore with w constant, i. e. with the ratio of the angular motion of the body to the angular motion in the orbit always the same, also the speed of the body in the orbit $\sqrt{u}$ is inversely proportional to the perpendicular $C(T)$ to the tangent. And the centripetal force attracting and being effective towards C, because the body is moving in a stationary orbit, is equal to $\frac{2a^2cdp}{w^2p^3dy}$. For the speed with respect to the orbit, that the body has at (A), corresponds to the height γ, is $\sqrt{\gamma}:\sqrt{c}=1:w$ (by hypothesis) and $c=w^2\gamma$, from which the centripetal force acting towards C, as the body in the stationary orbit is moved , is equal to $\frac{2a^2\gamma\,dp}{p^3dy}$, as indeed has been found from the above treatment (591).

Corollary 4.

733. Therefore the angle (*A*)*CA* in this hypothesis, where *w* is put constant, which is completed by the orbit, while the body traverses the arc (*A*)(*M*), is equal to

$$(w-1)\int\frac{pdy}{qy} = (w-1) \text{ ang. } (A)C(M).$$

Hence for one complete revolution of the body in the orbit, the orbit itself rotates about C by the angle $(w-1).360$ degrees.

Corollary 5.

734. Moreover the force, which it effects, as the body moving in this orbit proportionately to the angular motion in the orbit itself, is [p. 304]

$$\frac{2a^2cdp}{w^2p^3dy} + \frac{2a^2c(w^2-1)}{w^2y^3}$$

or is equal to

$$\frac{2a^2\gamma dp}{p^3dy} + \frac{2a^2\gamma(w^2-1)}{y^3}.$$

Whereby the difference between the centripetal force for the stationary orbit and the force for the moving orbit is the latter part which is inversely proportional to the cube of the distance of the body from the centre of the forces C.

Corollary 6.

735. If we put $w = 1$, then $w-1=0$ and then there is no motion of the orbit, in which case also the centripetal force is equal to $\frac{2a^2\gamma\, dp}{p^3dy}$ with the other term vanishing. Likewise it comes about, if $w=-1$ or $w-1=-2$, in which case the orbit in the preceding moves with twice the speed that the body itself moves in the orbit. [From the factor $w-1$ in (734)] But the true curve, that in this motion is described by the body, does not differ from the orbit, except that it is in the opposite sense.

Corollary 7.

736. If $w > 1$, then as a consequence the orbit is moving; and the greater this motion becomes, the greater also becomes the centripetal force. But if $w < 1$, with the orbit pulling in the opposite direction, and the centripetal force is less, as w^2-1 is negative.

Corollary 8.

737. If $w = 0$, then this makes $c = 0$, and the body is moving along a straight line, since in this case the angular motion of the orbit is equal and opposite to the angular motion of the body in its orbit.

Corollary 9.

738. If w is negative, surely equal to $-n$, then the body moves on the same curve as if [p. 305] $w = +n$, only with this distinction, that the body proceeds in the opposite sense. And on this account the centripetal force retains the same value, whether w takes positive or negative values. Likewise the same result holds generally, if w is a variable quantity.

Example.

739. Let the curve (A)(M)(B) be an ellipse and the centre of forces C of this one of the focal points. The latus rectum of this is put equal to L and the transverse axis (A)(B) = A; the distance a [from the focus as origin] is given by :

$$a = \frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - AL)}$$

and

$$4pp = \frac{ALy}{A-y}.$$

[Note that the use of the eccentricity e had not yet been developed for conic sections, and conics were specified by their width and the length of the latus rectum or focal chord. In modern terminology, with the semi-latus rectum $l = a(1-e^2)$ and $b^2 = a^2(1-e^2)$, and the pedal equation of an ellipse is given by $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, see Lockwood p. 21. Hence, $p^2 = \frac{b^2 r}{2a-r} = \frac{a^2(1-e^2)r}{2a-r} = \frac{alr}{2a-r} = \frac{(A/2)(L/2)y}{A-y}$, in Euler's notation. The first result is easily found from the 'constant length of string principle' used to draw an ellipse, applied to the rt. triangle formed by the semi-latus rectum $FP = L/2$, the interfocal distance FF', and the distance $F'P$, where $FP + F'P = A$.]
If besides, w is constant; the force operating in order that the body travels in a rotating ellipse is equal to: $\frac{4a^2\gamma}{Ly^2} + \frac{2a^2\gamma(w^2-1)}{y^3}$ (734). [For, on differentiating the $4p^2$ equation above, we have $\frac{dp}{dy} = \frac{2p^3}{Ly^2}$.] Truly the angle ($A$)$CA$, that the orbit completes, while the body travels through the arc (A)(M), is equal to $(w-1)(A)C(M)$ (733). The equation for the curve itself that the body describes, of which the element is $M\mu$, is found by finding the equation between $CM = y$ and $C\Theta = \pi$. Moreover, we have

$$pp = \frac{ALy}{4A - 4y}$$

and

$$qq = \frac{4Ay^2 - 4y^3 - ALy}{4A - 4y},$$

which values substituted in the equation $\pi = \frac{wpy}{\sqrt{(qq + w^2p^2)}}$ give this [pedal] equation for the curve itself described:

$$\pi\pi = \frac{w^2 ALy^2}{4Ay - 4y^2 + (w^2 - 1)AL}.$$

Scholium 1.

740. The curves themselves, which bodies describe acted on by centripetal forces of this kind, otherwise are most difficult to be recognised and their form from this consideration does not mean that in any way they can be determined. [p. 306] Therefore investigations of centripetal forces of this kind have the maximum use for curves generated by some given force, from which in turn, from the given centripetal forces, properties of the curves themselves can become known. For so complex expressions of the forces acting there occur in the motions of heavenly bodies, so that none of the orbits of these can be determined, except perhaps these forces can be understood in some such case, for which the orbits can be determined after the centripetal force has been found.

Scholium 2.

741. If a body is taken to be moving in a moving orbit of this kind, the motion of this body and the distance at some time from the centre C can be determined. And just as often as the body in the orbit arrives at the points(A) and (B), so the distance from C is a minimum or a maximum. Whereby when the rotary motion of the line (A)(B), which is called the line of the apses, is given, it is possible to define when the distance from the centre C is a maximum or a minimum. Newton has explored this problem in the *Principia*, Book I, in the whole of Section IX, and that theory applies the motion of the apses in the determination of the moon's orbit. But this examination is applied with less accuracy to the moon, since the lunar force is not acting at any fixed point, as we have put here, but is always exerted from some variable point. Therefore we will give the work, in order that, after we have explained relevant matters here, we will offer other more suitable propositions, which can be transferred to the motion of the moon. [p. 307]

PROPOSITION 90.

PROBLEM.

742. *For the known curve, that a body describes acting under some central force V, to determine the curve, that the body describes acted on the centripetal force* $V + \frac{C}{y^3}$, *with y denoting the distance MC of the body from the centre of forces C.*

[See also L. Euler Commentationem 232 (the Enestrom index) : *De motu corporum coelestium a viribus quibuscunque perturbato*, Novi comment. acam. se. Petrop. (1752/53); Leonardi Euleri *Opera omnia*, series II, vol. 21. P. St.]

SOLUTION.

With the centripetal force $V + \frac{C}{y^3}$ acting, the speed of the body, which is projected at (A) along a direction normal to the direction of the radius C(A), corresponding to the height c, and C(A) is put equal to a. Moreover with the force V acting, (A)(M)(B) is the orbit, in which the body moves projected at (A) along the same direction, but with a speed corresponding to the height γ. Now from the preceding proposition it has been shown that a force of the form $V+\frac{C}{y^3}$ is to be acting, in order that the body in the same orbit (A)(M)(B), but moving around the centre C in a given ratio to the angular motion of the orbit. Therefore the ratio becomes $w-1$ to 1, as the orbital angular motion to the angular motion of the body in that orbit, while it is at M, and also it is given that $c=w^2\gamma$ (732) and the perpendicular from C sent to the tangent of the orbit at M is called $CT=p$. Hence making the centripetal force equal to $\frac{2a^2\gamma dp}{p^3dy}$, in order that the body moves in the stationary orbit (A)(M)(B), and thus

$$V=\frac{2a^2\gamma dp}{p^3dy},$$

as therefore we are able to construct the equation on account of the given curve (A)(M)(B) (by hypothesis). Moreover the force acting V, [p. 308] in order that the body in the same orbit can move in the manner described, is given by :

$$\frac{2a^2\gamma dp}{p^3dy}+\frac{2a^2\gamma(w^2-1)}{y^3}$$

(734). On account of which we have :

$$V+\frac{C}{y^3}=\frac{2a^2\gamma dp}{p^3dy}+\frac{2a^2\gamma(w^2-1)}{y^3}.$$

From which there is produced :

$$C=2a^2\gamma(w^2-1)=\frac{2a^2c(w^2-1)}{w^2}$$

and

$$w^2=\frac{2a^2c}{2a^2c-C}.$$

Hence:

$$(w-1):1=\frac{\sqrt{2a^2c}-\sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}:1$$

and, [as $c=w^2\gamma$],

$$\gamma=\frac{2a^2c-C}{2a^2}.$$

Therefore the curve, that the body at (A) describes with the speed of projection corresponding to the height $\frac{2a^2c-C}{2a^2}$ acted on by the force V, the force $V+\frac{C}{y^3}$ acting on the body, in order that the body at (A) projected with the speed $\sqrt{c}$ is moving in the same mobile orbit thus, in order that the angular motion of the orbit to the angular motion of the body in this orbit, shall be as $\frac{\sqrt{2a^2c}-\sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}$ to 1. Moreover the motion of the body in that orbit is the same, as that which it has in a stationary orbit acted on only by the force V and projected with a speed at (A) corresponding to the height $\frac{2a^2c-C}{2a^2}$, which motion by hypothesis is known. Q.E.I.

Corollary 1.

743. Therefore while the body reaches (*B*) from (*A*) in the orbit, or rotates around the centre *C* by an angle of 180, meanwhile the orbit itself has turned about *C* through an angle of

$$\frac{\sqrt{2a^2c}-\sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}\cdot 180 \text{ deg.}$$

Corollary 2.

744. Therefore if the line (*A*)(*B*) is the apse line, the point (*A*) is the closer point, and (*B*) truly the greater of the apses, as they are called in Astronomy; therefore the body arrives at the larger from the smaller apse in an absolute motion around C of [p. 309]

$$\frac{180}{\sqrt{\left(1-\frac{C}{2a^2c}\right)}}$$

degrees.

Corollary 3.

745. The time, in which the body arrives at M from (*A*) in the moving orbit, is equal to the time, in which in the stationary orbit it reaches (*M*) from (*A*). Therefore the angle (*A*)*CM* has to the angle (*A*)C (*M*) the ratio *w* to 1, i. e. as

$$\frac{1}{\sqrt{\left(1-\frac{C}{2a^2c}\right)}} \text{ to } 1.$$

Example.

746. Let the force *V* be inversely proportional to the square of the distance from the centre or $V=\frac{ff}{yy}$, the curve (*A*)(*M*)(*B*) is an ellipse, in the focus of which is put the centre of the forces *C*. Let the transverse axis of this ellipse be (*A*)(*B*) = A and the latus rectum = *L*, then [as above] $a=\frac{1}{2}A-\frac{1}{2}\sqrt{(A^2-AL)}=(A)C$ and $(B)C=\frac{1}{2}A+\frac{1}{2}\sqrt{(A^2-AL)}$. On account of $4pp=\frac{ALy}{A-y}$ then we have :

$$\frac{2a^2\gamma dp}{p^3dy}=\frac{4a^2\gamma}{Ly^2}=\frac{ff}{yy}.$$

Hence there becomes :

$4a^2\gamma=Lff=4a^2c-2C$, thus $\frac{4a^2c-2C}{ff}$ and $c=\frac{Lff+2C}{4a^2}$.

Which is the height corresponding to the speed of the body at (*A*) for the orbit to be moving under the central force $\frac{ff}{y^2}+\frac{C}{y^3}$. Truly the orbital angular motion is to the angular motion of the body in the orbit as :

$$\frac{\sqrt{(\frac{1}{2}Lff+C)}-\sqrt{\frac{1}{2}Lff}}{\sqrt{\frac{1}{2}Lff}}$$

to 1. And the body arrives at the further apse from the nearer one, after the angular motion has completed an angle of :

$$\frac{180\sqrt{(\frac{1}{2}Lff+C)}}{\sqrt{\frac{1}{2}Lff}}=180\sqrt{\left(1+\frac{2C}{Lff}\right)}$$

degrees. [p. 310]

PROPOSITION 91.

PROBLEM.

747. *If the figure that the body describes on being acted on by some central force does differ much from a circle, to determine the motion of the apses.*

SOLUTION.

The motion of this is to be compared with the motion of the body in a moving ellipse with a small eccentricity, of which one or other focus is placed at the centre of the forces. Therefore in this stationary orbit the body is moving acted on by a centripetal force inversely proportional to the square of the distance. Truly the body is moving in the same orbit if the centripetal force is equal to $\frac{ffy+C}{y^3}$ (746). With the preceding denominators kept in place, and putting $y=a+z;$ where z is extremely small with respect to a, since the curve described by the body is put as nearly circular. Whereby the former centripetal force is equal to $\frac{aff+C+ffz}{y^3}$, and as $2a$ is approximately equal to the latus rectum L. Now the centripetal force acting is put equal to $\frac{P}{y^3}$, in which P is some function of y. The value $a+z$ is put in place of y in P, and P can be changed into $E+Fz$ by rejecting terms in which z has a dimension greater than one, on account of z being so small. Therefore this formula has to be compared with $aff+C+ffz$:
$F=ff$ or $f=\sqrt{F}$ and $aF+C=E$ or $C=E-aF$. With these substituted, the body acted on by this centripetal force $\frac{P}{y^3}$ comes from the smaller to the larger apse, with the angle $180\sqrt{\left(1+\frac{2C}{Lff}\right)}$ degrees [p. 311] for the absolute angular motion (746). Or with $2a$ put in place of L and F in place of ff and $C=E-aF$, this angle becomes $180\sqrt{\frac{E}{aF}}$ degrees. If indeed the orbit does not disagree much from being circular. Q.E.I.

Corollary 1.

748. Truly the apsidal line (*A*)(*B*), while the body rotates through an angle of 360 degrees about C, is moved by the orbital angular motion through an angle $\frac{\sqrt{E}-\sqrt{aF}}{\sqrt{E}}.360$ degrees. For the angular motion of the orbital is put proportional to the angular motion of the body on account of the centripetal force $\frac{ff}{y^2}+\frac{C}{y^3}$ (734).

Corollary 2.

749. Since *E* is such a function of *a*, just as *P* is of *y*, *Fz* is the increment of *E* from the increase of *a* by the element *z*. Whereby on putting $z = da$ then $Fda = dE$, and likewise the angle, by which the body arrives at the larger apse from the smaller one, is equal to $180\sqrt{\frac{Eda}{adE}}$ degrees.

Corollary 3.

750. Since *E* is such a function of *a*, as *P* is of *y*, *y* can be put in $\frac{Eda}{adE}$ in place of a and *P* in place of *E*. On account of which by the presence of the centripetal force $\frac{P}{y^3}$, the body moves from the small apse to the large apse in an absolute angle of $180\sqrt{\frac{pdy}{ydp}}$ degrees. And if *y* remains in this expression, a can be written in place of *y*, clearly with a small discrepancy. [p. 312]

Corollary 4.

751. If we set $\frac{da}{a} > \frac{dE}{E}$ or $\frac{dy}{y} > \frac{dP}{P}$, then the ellipse by its own motion expresses the true motion of the body by moving forwards [as $\frac{Eda}{adE} > 1$, etc]. For if $\frac{dy}{y} < \frac{dP}{P}$, then the line of the apses moves backwards. But if $\frac{dy}{y} = \frac{dP}{P}$ or $P = \alpha y$, in which case the centripetal force is inversely proportional to the square of the distance, and the apse line remains at rest, or the body, after the angular motion has completed 180 degrees, has gone in turn from the nearer apse to the further apse.

Corollary 5.

752. Moreover for a given angle, in which the body goes from one apse to the other and back again, which is 360μ degrees, we have $\mu^2 = \frac{Pdy}{ydP}$ and likewise $P^{\mu\mu} = \alpha y$ or $P = (\alpha y)^{\frac{1}{\mu\mu}}$.

Therefore the centripetal force $\frac{P}{y^3}$ which makes the apsidal line have such a motion, is $y^{\frac{1-3\mu^2}{\mu^2}}$. [*P* is thus the function of *y* required in the proposition].

Corollary 6.

753. If it happens, that $\frac{Pdy}{ydP}$ or dP is made negative, then the motion of the apses is imaginary. From which it is understood that the body is never able to proceed from one apse to the other, but continually either recedes further from the centre or approaches the centre, or evidently it is in a closed orbit that does not change.

Corollary 7.

754. If the centripetal force is proportional to the power of the distance y^n, then $P = y^{n+3}$. Whereby $\frac{Pdy}{ydP} = \frac{1}{n+3}$, [p. 313] and the body goes from the closer to the further apse while going through the absolute angle about *C* of $\frac{180}{\sqrt{(n+3)}}$ degrees; and to go from the greater or smaller apse to return to the same by going through an angle of $\frac{360}{\sqrt{(n+3)}}$ degrees.

Corollary 8.

755. If $\sqrt{(n+3)}$ is a rational number and *m* the smallest whole number, from which $\frac{m}{\sqrt{(n+3)}}$ makes some whole number, then after completing $\frac{m}{\sqrt{(n+3)}}$ revolutions about the centre C the body has fallen on the same point and the curve described by the body has completed as many whole turns before it is restored and closed. But if $n + 3$ is not a perfect square, then the curve can never be restored to its starting point, but the body indefinitely travels around the centre *C*, and neither does it at any time revert to the same path.

Example 1.

756. If the centre of forces attracts in the inverse cube of the distances, then $n + 3 = 0$. Hence in this hypothesis, the body cannot travel from one apse to the other except by completing an infinite number of revolutions. And if the centripetal force decreases in a greater ratio than the third power of the distances, the curve clearly does not have two apses, but will either go to infinity or terminate as a logarithmic spiral towards the centre. [p. 314]

Example 2.

757. If the centripetal force is inversely proportional to the square of the distance, then $n + 3 = 1$. Whereby while the body by the absolute angular motion completes an angle of 180 degrees it travels from one apse to the other, and the curve after any revolution returns to its initial condition. For the body it travelling in an ellipse, in either focus of which is placed the centre of forces, and the transverse axis of this ellipse is the line of the apses.

Example 3.

758. If the centripetal force varies inversely as the distance, then $n + 3 = 2$. Therefore the body arrives at one apse from the other after the orbit has turned through an angle of $\frac{180}{\sqrt{2}}$ or 127 degrees and 17'. Truly the orbit never returns on itself on account of the irrational $\sqrt{2}$.

Example 4.

759. If the centripetal force is constant at all distances, then $n = 0$. In this case the body goes from one apse to the other while the angular motion carries the orbit through an angle of $\frac{180}{\sqrt{3}}$ degrees, i. e. 103 degrees and 55' approximately.

Example 5.

760. If the centripetal force varies directly as the distance of the body from the centre, in which case the body is agreed to be moving in an ellipse, in the centre of which the centre of force has been placed. (631). Therefore with the apses standing apart by an angle of 90 degrees. The same can truly be deduced from this rule. for on account of $n = 1$ the angle is $\frac{180}{\sqrt{n+3}} = 90$. [p. 315]

Scholium 1.

761. Therefore as often as the body is projected around the centre of force with such a velocity, so that it almost revolves in a circle, with the help of this proposition the true curve that the body describes can be determined, which is not possible by considering the centripetal force alone. For which it is evident on greater contemplation that there are more uses of this kind for these orbits, by determining other more difficult orbits from these that can easily be defined. Newton has set out the same proposition in Sect. IX , Prop. 45.

Scholium 2.

762. Now we have shown above that a body acting under a hypothetical centripetal force varying inversely as the cube of the distance falling to the centre, arrives in a finite time and does not to escape from that point, but as it were, to be annihilated suddenly (675 and 676). The same also prevails if the body falls to the centre along a straight line. And in a similar manner, if the centripetal force decreases in a ratio greater than the inverse cube of the distance, the body at once arrives at the centre, where it vanishes and neither progresses further than the centre nor returns. For whatever you please, it would be absurd for the curve that the body describes, projected with a certain velocity, to have two apses (756). Moreover as often as the centripetal force decreases in a ratio less than the cube, as in the simple ratio of the distance or greater than that, the body recedes, after it arrives at the centre, [p. 316] along the same line by which it approached; for this is evident for the inverse square of the distance (655) and for the simple ratio, from which it is apparent (271) that the body cannot progress beyond the centre. But if $n + 1 > 0$, the body falling towards the centre along a straight line has a finite speed, by which beyond the centre it can progress along the same straight line, as long as it was losing speed (273). Therefore in this way we have satisfied the above desire (272), in which it was necessary to define the linear motion of the falling body, when it arrived at the centre.

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PROPOSITIO 86.

PROBLEMA.

707. *Invenire legem vis perpetuo deorsum secundum rectas MP* (Fig. 62) *inter se parallelas tendentis, quae faciat, ut corpus in data curva AM moveatur, atque determinare corporis in singulus locis M celeritatem.* [p. 292]

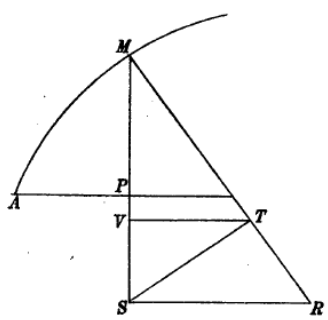

Fig. 62.

SOLUTIO.

Per rectas *MP* ducatur normalis *AP*, et vocetur *AP* x et *PM* y. Ponatur curva $AM = s$ et radius osculi *MR* in $M = r$, erit $r = \frac{dsdy}{ddx}$ sumto ds pro constante. Porro debita sit corporis in *M* celeritas altitudini v, et vis corpus in *M* trahens secundum *MP* ponatur *P*. Ex his habebuntur duae istae aequationes $dv = -Pdy$ et $Pr\,dx = 2vds$ (557), ex quibus, si cognita esset v, statim apparet quantitas ipsius *P*. Eliminatur igitur *P* ad v inveniendum ex hac aequatione $rdxdv = -2vdsdy$, quae posito loco r eius valore $\frac{dsdy}{ddx}$ abit in hanc $-\frac{dv}{v} = \frac{2ddx}{dx}$, quae integrata dat $l\,C - l\,v = l\,\frac{dx^2}{ds^2}$. Cognita autem sit celeritas in puncto *A*, eaque debeatur altitudini c, atque sit consinus anguli *MAP* seu valor ipsius $\frac{dx}{ds}$ incidente puncto *M* in $A = \lambda$. Hinc ergo erit $l\,C = l\,c + 2l\,\lambda$ et consequenter

$$v = \frac{\lambda^2 cds^2}{dx^2},$$

unde corporis in singulis locis *M* celeritas innotescit.

Vis autem sollicitans *P* reperietur

$$= \frac{2\lambda^2 cds^3}{rdx^3} = \frac{2\lambda^2 cds^2 ddx}{dx^3 dy}.$$

Sive sumto dx pro constante, quo casu est $r = \frac{ds^3}{-dxddy}$, erit

$$P = \frac{-2\lambda^2 c\,ddy}{dx^2}.$$

Tempus denique, quo arcus AM percurritur, erit

$$= \int \frac{ds}{\sqrt{v}} = \frac{x}{\lambda\sqrt{c}}.$$

Q.E.I.

Corollarium 1.

708. Quaecunque ergo sit vis sollicitans, corpus perpetuo aequabiliter secundum horizontem progreditur ob tempus per *AM* proportionale ipsi *AP*, uti iam supra est observatum (579). [p. 293]

Corollarium 2.

709. Si ex *R* in rectam *MP* productam demittatur perpendicularis *RS* et ex *S* in *MR* quoque perpendicularum *ST* atque denique tertium perpendiculum *TV* ex *T* in *MS*, erit $MV = \frac{rdx^3}{ds^3}$. Quare vis sollicitans P se habebit ad vim gravitatis ut $2\lambda^2 c$ ad *MV*, seu *P* est reciproce ut *MV*.

Corollarium 3.

710. Si angulus ad *A* est rectus, fiet λ = 0. Quo casu corpus directe sursum ascendere debebit. At si tantum sit λ infinite parvum atque *c* infinite magnum, ita ut $2\lambda^2 c$ finitum habeat valorem, corpus in huiusmodi curva utique moveri poterit.

Exemplum 1.

711. Sit curva *AM* circulus, cuius diameter posita sit in axe *AP*, et radius = *a*. Erit itaque *r* = a et $ds : dx = a : y.$ Hanc ob rem fiet

$$P = \frac{2\lambda^2 a^2 c}{y^3} \text{ et } v = \frac{\lambda^2 a^2 c}{y^2}$$

Vis ergo corpus in M deorum trahens est reciproce ut cubis applicatae MP et celeritas reciproce ut haec ipsa applicata. Altitudo vero generans celeritatem in summo peripheriae puncto, ubi fit $y = a$, est = $\lambda^2 c$.

Exemplum 2.

712. Sit curva *AMC* (Fig. 63) parabola, cuius axis *CB* est verticalis et parameter = *a*. Ducatur horizontalis *MQ* et ponatur $CQ = t$ et $MQ = z$, erit $z^2 = at.$ Praeterea vero erit $dx = -dz$ et $dy = -dt$ [p. 294] et ut ante $\sqrt{(dt^2 + dz^2)} = ds$. Quocirca fiet $v = \frac{\lambda^2 cds^2}{dz^2}$ and $P = \frac{2\lambda^2 cddt}{dz^2}$. Debita sit celeritas in puncto summo C altitudini b, eritque,

ob $ds = dz$ in C, $\lambda^2 c = b$ ideoque $v = \frac{bds^2}{dz^2}$ et $P = \frac{2bddt}{dz^2}$, sumto dz pro constante. Ex aequatione vero $z^2 = at$ fit $dt = \frac{2zdz}{a}$ et $ddt = \frac{2dz^2}{a}$ atque $ds^2 = dz^2(1 + \frac{4z^2}{a^2})$.
Consequenter invenitur

$$v = b\left(\frac{a^2 + 4z^2}{a^2}\right) = b\left(\frac{a + 4t}{a}\right)$$

et $P = \frac{4b}{a}$.
Ex quo apparet potentiam deorsum tendentem, quae efficit, ut corpus in hac parabola progrediatur, esse constantem. Quare igitur, si aequalis sit gravitati = 1, fiet $b = \frac{a}{4} =$ distantiae foci a vertice. Quae conveniunt cum supra inventis (564 et sqq.)

Exemplum 3.

713. Sit curva *MAN* (Fig. 64) hyperbola centro *C* descripta habens axem *CP* verticalem. Ponatur semiaxis transversus *AC* = *a* et semiaxis coniugatus = *e* atque *CP* = *t* ac *PM* = *z*, sitque insuper altitudo celeritati, quam corpus in *A* habet, debita = *b*, erit ut supra pro parabola fecimus,

$$v = \frac{bds^2}{dz^2} \quad \text{et} \quad P = \frac{2bddt}{dz^2}$$

sumto dz constante. Est vero ex natura hyperbola $a^2z^2 = -a^2e^2 + e^2t^2$, ex qua fit $dt = \frac{a^2zdz}{e^2t}$ et

$$ddt = \frac{a^2dz^2}{e^2t} - \frac{a^2zdzdt}{e^2t^2} = \frac{a^2e^2t^2dz^2 - a^4z^2dz^2}{e^4t^3} = \frac{a^4dz^2}{e^2t^3}.$$

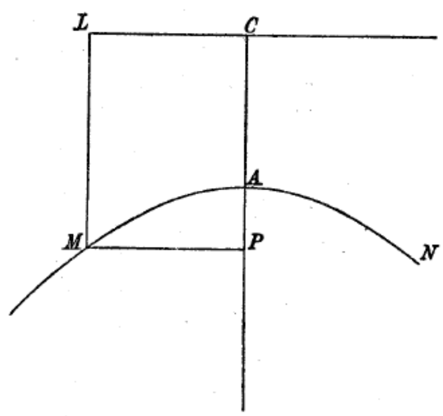

Fig. 64.

Consequenter erit
$P = \frac{2a^4b}{e^2t^3}$,
seu potentia corpus deorsum trahens ubique in *M* proportionalis est reciproce cubo distantiae *ML* puncti *M* ab horizontali *LC* per centrum *C* ducta. Porro erit

$$\frac{ds^2}{dz^2} = \frac{e^4t^2 + a^4z^2}{e^4t^2} = \frac{(a^2 + e^2)t^2 - a^4}{e^2t^2}.$$

Atque praeterea habebitur

$$v = \frac{b(a^2 + e^2)t^2 - a^4b}{e^2t^2}.$$

[p. 295]
PROPOSITIO 87.

PROBLEMA.

714. *Data curva AMB* (Fig. 65) *una cum centro virium C invenire legem vis centripetae, quae faciat, ut corpus in hac curva libere moveatur, ut et celeritatem corporis in loco quavis M.*

SOLUTIO.

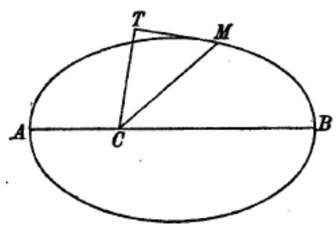

Fig. 65.

Quia curva *AMB* una cum puncto *C* est dat, quaeratur aequatio inter distantiam *MC* cuiusque curvae puncti *M* a centro *C* et perpendiculum *CT*, quod ex *C* in tangentem *MT* demittitur. Quare posita $CM = y$ et $CT = p$ habebitur inter *p* et *y* aequatio. Iam sit corporis in dato loco *A* celeritas debita altitudini *c* atque perpendiculum ex *C* in tangentem in *A* demissum = *h*. Eorum vero, quae sunt incognita, vocetur altitudo debita celeritati in $M = v$ et vis centripeta in $M = P$. His positis erit $v = \frac{ch^2}{p^2}$ (589) atque $P = \frac{2ch^2 dp}{p^3 dy}$ (592). Vel posito radio osculi in $M = r$ erit $P = \frac{2ch^2 y}{p^3 r}$ (592). Q.E.I.

Corollarium 1.

715. Tempus etiam, quo corpus quemvis arcum AM absolvit, erit $= \frac{2ACM}{h\sqrt{c}}$ seu erit proportionale areae *ACM* (588).

Corollarium 2.

716. Quia ch^2 est quantitas constans, erit vis centripeta in puncto quovis M proportionalis [p. 296] huic valori $\frac{dp}{p^3 dy}$ seu huic $\frac{y}{p^3 r}$. Celeritas vero $\sqrt{v}$ proportionalis est reciproce perpendiculo *CT* in tangentem *MT* demisso (589).

Exemplum 1.

717. Sit curva data ellipsis et centrum virium *C* in ipso eius centro positum. Vocetur eius semiaxis transversus *a* et semiaxis coniugatus *b*; erit ex natura ellipsis $p = \frac{ab}{\sqrt{(a^2+b^2-y^2)}}$.

Habebitur ergo $dp = \frac{abydy}{(a^2+b^2-y^2)^{\frac{3}{2}}}$ ideoque $\frac{dp}{p^3 dy} = \frac{y}{a^2 b^2}$. Quocirca prodibit vis centripeta

$$P = \frac{2ch^2 y}{a^2 b^2},$$

quae igitur proportionalis est distantiae corporis a centro.

Exemplum 2.

718. Sit curva data iterum ellipsis, at centrum virium C in eius alterutro foco positum. Ponatur eius axis transversus = A et latus rectum = L, eritque ex natura ellipsis $4pp = \frac{ALy}{A-y}$. Differentiando ergo fit $8pdp = \frac{A^2Ldy}{(A-y)^2}$. Quia vero est $16p^4 = \frac{A^2L^2y^2}{(A-y)^2}$, erit $\frac{dp}{2p^3} = \frac{dy}{Ly^2}$ et consequenter

$$P = \frac{4ch^2}{Ly^2}.$$

Vis igitur centripeta reciproce erit proportionalis quadrato distantiae corporis a centro virium C.

Exemplum 3.

719. Sit curva spiralis logarithmica et centrum virium C in eius centro positum; erit $p = ny$ et $\frac{dp}{p^3dy} = \frac{1}{n^2y^3}$, ideoque $P = \frac{2ch^2}{n^2y^3}$.

Quare vis centripeta erit reciproce ut cubus distantiae corporis a centro. [p. 297]

PROPOSITIO 88.

THEOREMA.

720. *Vis tendens ad centrum C* (Fig. 66), *quae facit, ut corpus in data curva AM moveatur, se habet ad vim tendentem ad aliud centriud c, quae facit, ut corpus in eadem curva et eodem tempore periodico moveatur, ut cubis rectae cV ex c ad tangentem TM parallele rectae CM ductae ad solidum ex recta cM in quadratu rectae CM.*

DEMONSTRATIO.

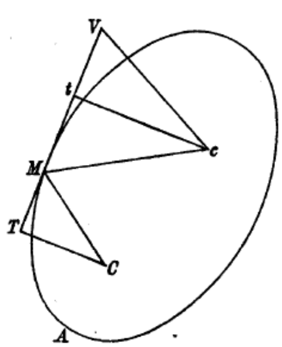

Fig. 66.

Sit corporis celeritas in dato puncto A, cum corpus circa centrum virium C revolvitur, debita altitudini c et perpendiculum ex C in tangentem in A demissum = h. At cum corpus circa centrum virium c movetur, sit celeritas in A debita altitudini γ et perpendiculum ex centro c in tangentem in A demissum = θ. Quia autem tempora periodica circa utrumque virium centrum sunt aequalia, erit $h\sqrt{c} = \theta\sqrt{\gamma}$ seu $ch^2 = \gamma\theta^2$ (715). Ex centro C et c porro in tangentem in M demittantur perpendicula CT et ct, sitque radius osculi in M = r. His positis erit vis centripeta in M ad centrum C tendens, quam vocemus P, $= \frac{2ch^2.CM}{r.CT^3}$ atque vis centripeta in M ad centrum c tendens, quam vocemus Π, $= \frac{2\gamma\theta^2.cM}{r.ct^3}$ (714). Quamobrem ob $ch^2 = \gamma\theta^2$ erit

$$P : \Pi = \frac{CM}{CT^3} : \frac{cM}{ct^3}.$$

Ducta autem cV parallela rectae CM erit, ob triangula TCM et tcV similia, $CT : ct = CM : cV$. Hance ob rem erit

$$P : \Pi = \frac{1}{CM^2} : \frac{cM}{cV^3} = cV^3 : cM \cdot CM^2.$$

Q.E.D. [p. 298]

Corollarium 1.

721. In eodem puncto *M* erit celeritas corporis, dum ad virium centrum *C* attrahitur, ad celeritatem, dum ad alterum centrum *c* attrahitur, reciproce ut *CT* ad *ct* sive directe ut *cV* ad *CM*. Sequitur hoc ex eo, quod est $ch^2 = \gamma\theta^2$.

Corollarium 2.

722. Si tempora periodica non sint aequalia, sed sint inter se ut *T* ad *t*, erit $T : t = \frac{1}{h\sqrt{c}} : \frac{1}{\theta\sqrt{\gamma}}$ seu $ch^2 : \gamma\theta^2 = t^2 : T^2$. Consequenter erit

$$P : \Pi = t^2 \cdot cV^3 : T^2 \cdot cM \cdot CM^2.$$

Seu vires *P* et *Π* erunt in ratione composita ex ratione in theoremate assignata et inversa duplicata temporum periodicorum.

Corollarium 3.

723. Eodem hoc casu, quo tempora periodica sunt inaequalia, erit celeritas in *M*, centro virium in *C* posito, ad celeritatem in *M*, centro virium in *c* posito, in ratione reciproca composita ex ratione perpendiculorum *CT* et *ct* et ratione temporum periodicorum *T : t*.

Scholion 1.

724. Propositionem hanc Neutonus deduxit ex Lib. I. prop. VII in coroll. 3 *Princ*. eaque utitur ad inveniendam vim centripetam tendentem ad punctum quodcunque ex cognita vi ad aliud quodpiam centrum trahente. Nos hic usum eius in unico exemplo sequente ostendemus. [p. 299]

Exemplum.

725. Sit curva data circulus *AMc* (Fig. 67), alterumque centrum virium positum sit in ipso circuli centro *C*. Vis igitur centripeta *P* ad *C* tendens ubique erit constans dicaturque *g*. Ex hac quaeratur vis ad centrum virium c in peripheria situm tendens *Π* faciensque, ut corpus eodem tempore periodico in circulo moveatur. Demittatur ergo ex *c* in tangentem *MV* perpendiculum *cV*, quod ex natura circuli simul parallelum erit rectae *CM*. Quamobrem erit $g : \Pi = cV^3 : cM.CM^2$ ideoque

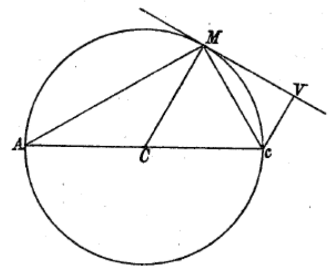

Fig. 67.

$$\Pi = \frac{g \cdot cM \cdot CM^2}{cV^3}.$$

Ducatur recta AM, erunt triangula cVM, cMA ob ang.cMV = cAM similia et propterea $cV : cM = cM : cA$. Habetur ergo $cV = \frac{cM^2}{2CM}$, ex quo fit

$$\Pi = \frac{8g \cdot CM^5}{cM^5}.$$

Est igitur haec vis *Π* reciproce ut potestas quinta distantiae *Mc* corporis a centro virium *c*, ut iam supra (692) est inventum.

Corollarium 4.

726. Sit celeritas corporis in peripheria circuli circa centrum *C* revolventis debita altitudini *c* et celeritas corporis in *M* circa centrum virium *c* revolventis debita altitudini *v*. Eritque

$$\sqrt{c} : \sqrt{v} = cV : CM = cM^2 : 2\,CM^2$$

(721) seu $v = \frac{4c.CM^4}{cM^4}$.

Quamobrem celeritas corporis circa centrum *c* revolventis erit ubique reciproce ut quadratum distantiae eius ab *c*. [p. 300]

Corollarium 5.

727. Quia, centro virium in centro circuli C existente, est $h = y = p = r =$ radio *CM*, erit $P = g = \frac{2c}{CM}$ (592). Hanc ob rem fiet, posito $\Pi = \frac{f^5}{cM^5}$, $g = \frac{f^5}{8CM^5}$ et $c = \frac{f^5}{16CM^5}$. Ideoque $v = \frac{f^5}{4cM^4}$.

Scholion 2.

728. In his propositionibus posuimus curvam, quam corpus describit, absolute esse datam et aequationem pro ea haberi. Sed dantur etiam casus, quibus curva ipsa, quam corpus describit, non datur, sed ex certis conditionibus ad ipsum motum spectantibus ante debet inveniri, quam lex vis centripetae potest determinari. Hucque pertinent, quae passim tradita sunt de motu corporum in orbibus mobilibus, qua de re igitur in sequenti propositione tractabimus.

PROPOSITIO 89.

PROBLEMA.

729. *Si orbita* (*A*)(*M*)(*B*) (Fig. 65) *utcunque revolvatur circa centrum virium C, oportet definiri vim centripetam perpetuo ad C tendentem, quae faciat, ut corpus in hac orbita mobili moveatur.*

SOLUTIO.

Dum orbita ex situ (*A*)(*M*)(*B*) in situm *AMB* pervenit, ponatur corpus interea ex (*A*) in *M* pervenisse, ita ut corpus interea in orbita angulum (*A*)*C*(*M*) = *ACM*, revera autem angulum (*A*)*CM* = (*A*)C(*M*) + (*A*)*CA* [p. 301] descripserit. Existente initio corpore in (*A*) sit eius celeritas vera, non ea, quam in orbita habet, debita altitudini *c*, et recta *C*(*A*), quae tam in orbitam quam in veram curvam, in qua corpus movetur, sit perpendicularis, = *a*. Porro sit celeritas corporis in *M* , quatenus in orbita movetur, debita altitudini u et vera corporis celeritas in M debita altitudine *v*. At celeritas angularis in orbita sit ad veram celeritatem angularem circa C, dum corpus in *M* versatur, ut 1 ad *w*. In

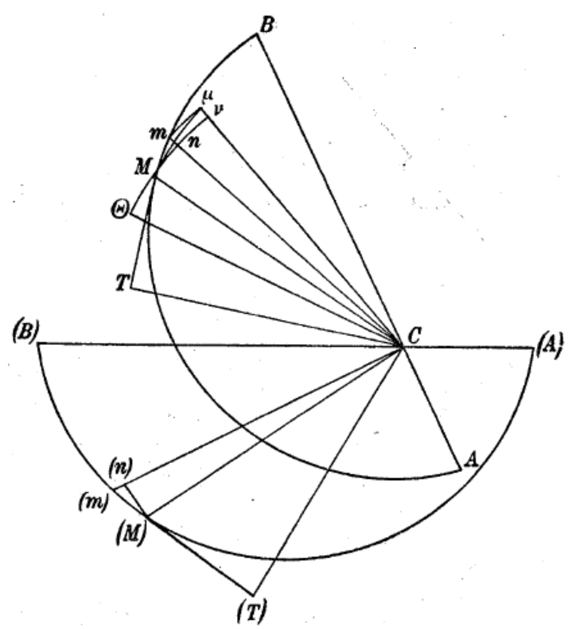

Fig. 68.

orbita igitur tanquam immobili spectata elementum (*M*)(*m*) celeritate $\sqrt{u}$ describetur. Ponatur distantia *C*(*M*) = $CM - y$ et perpendiculum in tangentem orbitae in (*M*) vel *M* ex *C* demissum $C(T) = CT = p$, habebiturque ob orbitam datam aequatio inter *p* et *y*. Iam dum corpus in orbita elementum *Mm* describit, progrediatur ipsa orbita motu angulari circa *C* per angulum $= mC\mu$, et hanc ob rem corpus reipsa non in *m*, sed in μ reperietur, sumto $C\mu = Cm$, atque idcirco interea elementum $M\mu$ descripsisse censendum est, id quod fecit celeritate debita altitudini *v*. Erit itaque

$$M\mu : Mm = \sqrt{v} : \sqrt{u},$$

atque centro *C* descripto arculo *Mv* (ob datam motuum angularium circa *C* in orbita et revera rationem $1 : w$) erit

$$Mn : M\nu = 1 : w.$$

Est vero, posita tangente $MT = \sqrt{(y^2 - p^2)} = q$,

$$Mm = \frac{ydy}{q} \quad \text{et} \quad Mn = \frac{pdy}{q}.$$

Quocirca habebitur

$$M\mu = \frac{ydy\sqrt{v}}{q\sqrt{u}} \quad \text{et} \quad M\nu = \frac{wpdy}{q}.$$

Ex quo ob $\mu\nu = mn = dy$ prodibit $1 + \frac{w^2p^2}{q^2} = \frac{v^2y^2}{uq^2}$ seu

$$vy^2 = uq^2 + w^2up^2.$$

Quia $M\mu$ est elementum [p. 302] verae curvae, quam corpus describit, demittatur in hoc productum ex C perpendiculum $C\Theta$, eritque $M\mu : M\nu = CM : C\Theta$, unde fit

$$C\Theta = \frac{wp\sqrt{u}}{\sqrt{v}}.$$

Ex hoc vero perpendiculo cognoscitur vera corporis celeritas ; erit enim $v = \frac{a^2cv}{w^2up^2}$ (589) ideoque

$$u = \frac{a^2c}{w^2p^2} \quad \text{et} \quad v = \frac{a^2c(q^2 + w^2p^2)}{w^2p^2y^2}.$$

His loco u et v positis valoribus erit

$$C\Theta = \frac{wpy}{\sqrt{(q^2 + w^2p^2)}},$$

quam brevitatis gratia vocemus π. Ex hac autem π cognita innotescit ipsa vis centripeta P, quae facit, ut corpus in hac data orbita hocque modo mobili moveatur. Namque erit $P = \frac{2a^2cd\pi}{\pi^3dy}$ (592). At ob $\pi^2 = \frac{w^2p^2y^2}{q^2+w^2p^2}$, erit

$$\frac{1}{\pi^2} = \frac{q^2}{w^2p^2y^2} + \frac{1}{y^2} = \frac{1}{y^2} - \frac{1}{w^2y^2} + \frac{1}{w^2p^2},$$

ideoque

$$-\frac{2d\pi}{\pi^3} = -\frac{2dy}{y^3} + \frac{2dw}{w^3y^2} + \frac{2dy}{w^2y^3} - \frac{2dw}{w^3p^2} - \frac{2dp}{w^2p^3}$$

seu

$$\frac{d\pi}{\pi^3} = \frac{dy(w^2-1)}{w^2y^3} + \frac{q^2dw}{w^3y^2p^2} + \frac{dp}{w^2p^3}.$$

Consequenter habebitur

$$P = \frac{2a^2cdp}{w^2p^3dy} + \frac{2a^2c(w^2-1)}{w^2y^3} + \frac{2a^2cq^2dw}{w^3y^2p^2dy}.$$

Q.E.I.

Corollarium 1.

730. Posito radio = 1 est $\frac{m\mu}{CM} = \frac{(w-1)pdy}{qy}$ elementum anguli (A)CA, quem orbita confecit, dum corpus arcum (A)(M) percurrit. Hanc ob rem erit ang. (A)$CA$$= \int \frac{(w-1)pdy}{qy}$. Estque $(w-1):1$ ut celeritas angularis orbitae ad celeritatem angularem corporis, dum est in M, ipsa orbita.

Corollarium 2.

731. Cereritas corporis in orbita, quae est ut $\sqrt{u}$, reciproce proportionalis est ipsi wp. Ergo, nisi w sit constans, fieri non potest, ut corpus hoc modo in orbita quiescente moveatur attractum ad centrum C. [p. 303]

Corollarium 3.

732. Posito igitur w constante, i. e. ratione motus angularis corporis ad motum angularem orbitae perpetuo eadem, erit etiam celeritas corporis in orbita $\sqrt{u}$ reciproce proportionalis perpendiculo $C(T)$ in tangentem. Atque vis centripeta ad C tendens atque efficiens, ut corpus hac ratione in orbita quiescente moveatur, erit $= \frac{2a^2cdp}{w^2p^3dy}$. Sit enim celeritas respectu orbitae, quam corpus in (A) habet, debita altitudini γ, erit $\sqrt{\gamma} : \sqrt{c} = 1 : w$ (p. hyp.) atque $c = w^2\gamma$, ex quo vis centripeta ad C tendens faciensque, ut corpus in orbita quiescente moveatur, erit $= \frac{2a^2\gamma dp}{p^3dy}$, ut etiam ex supra traditis invenitur (591).

Corollarium 4.

733. Angulus igitur $(A)CA$ in hac hypothesi, qua w ponitur constans, qui ab orbita absolvitur, dum corpus arcum $(A)(M)$ percurrit, erit

$$= (w-1)\int\frac{pdy}{qy} = (w-1) \text{ ang. } (A)C(M).$$

Ergo una tota corporis in orbita revolutione ipsa orbita circa C gyrabitur angulo $(w-1)360$ graduum.

Corollarium 5.

734. Vis autem, quae efficit, ut corpus in hac orbita mobili proportionaliter motui anguli in ipsa orbita moveatur, erit [p. 304]

$$= \frac{2a^2cdp}{w^2p^3dy} + \frac{2a^2c(w^2-1)}{w^2y^3} \quad \text{seu} \quad = \frac{2a^2\gamma dp}{p^3dy} + \frac{2a^2\gamma(w^2-1)}{y^3}.$$

Quare differentia inter vim centripetam pro orbita immobili et vim pro orbita mobili reciproce proportionalis est cubo distantiae corporis a centro virium C.

Corollarium 6.

735. Si sit $w = 1$, erit $w-1=0$ motusque orbitae nullus, quo casu etiam vis centripeta fit $= \frac{2a^2\gamma dp}{p^3dy}$ evanescente altero termino. Idem evenit, si $w=-1$ seu $w-1=-2$, quo casu orbita in antecedentia movetur duplo velocius, quam ipsum corpus in orbita ingreditur. At vera curva, quae hoc motu a corpore describitur, non differt ab orbita, nisi quod sit inversa.

Corollarium 7.

736. Si $w > 1$, orbita in consequentia movetur; qui motus quo sit maior, eo maior etiam erit vis centripeta. At si $w < 1$, orbita in antecedentia tendit, et vis centripeta fit minor, ob $w^2 - 1$ negativum.

Corollarium 8.

737. Si $w = 0$, fit etiam $c = 0$, corpusque in recta linea movebitur, quia motus angularis orbitae hoc casu aequalis fit et contrarius motui angulari corporis in orbita.

Corollarium 9.

738. Si *w* est numerus negativus, nempe $= -n$, corpus in eadem movebitur curva, [p. 305] ac si esset $w = +n$, hoc tantum discrimine, quod corpus in contrarius plagas progrediatur. Et hanc ob rem vis centripeta eundem retinet valorem, sive *w* affirmative sive negative accipiatur. Idem etiam universaliter, si *w* est quantitas variabilis, obtinet.

Exemplum.

739. Sit curva (*A*)(*M*)(*B*) ellipsis et centrum virium *C* eius alteruter focus. Ponatur eius latus rectum = *L* et axis transversus (*A*)(*B*) = A; erit

$$a = \frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - AL)} \quad \text{et} \quad 4pp = \frac{ALy}{A-y}.$$

Sit praeterea *w* constans; erit vis, quae facit, ut corpus in hac ellipsi mobili moveatur, = $\frac{4a^2\gamma}{Ly^2} + \frac{2a^2\gamma(w^2-1)}{y^3}$ (734). Angulus vero (*A*)*CA*, quem orbita absolvit, dum corpus in ea arcum (*A*)(*M*) percurrit,erit $=(w-1)(A)C(M)$ (733). Aequatio vero pro ipsa curva, quam corpus describit, cuius elementum est $M\mu$, habebitur invenienda aequatione inter $CM = y$ et $C\Theta = \pi$. Est autem

$$pp = \frac{ALy}{4A - 4y} \quad \text{et} \quad qq = \frac{4Ay^2 - 4y^3 - ALy}{4A - 4y},$$

qui valores in aequatione $\pi = \frac{wpy}{\sqrt{(qq + w^2p^2)}}$ substituti dabunt aequationem pro ipsa curva descripta hanc

$$\pi\pi = \frac{w^2ALy^2}{4Ay - 4y^2 + (w^2 - 1)AL}.$$

Scholion 1.

740. Curvae ipsae, quas corpora a huiusmodi viribus centripetis sollicitata describunt, difficillime alias cognoscerentur earumque forma hac consideratione non adhibita nequaquam posset determinari. [p. 306]Maximam igitur habent utilitatem huiusmodi virium centripetarum investigationes pro curvis ex datis utcunque generatis, quo reciproce ex viribus centripetis datis ipsae curvae earumque proprietates innotescant. Occurrunt enim in motibus corporum coelestium tam complexae virium ea sollicitantium expressiones, ut omnino eorum orbitae determinari nequaeant, nisi forte illae vires comprehendantur in tali quodam casu, de quo a posteriori vis centripeta est inventa.

Scholion 2.

741. Si corpus in huiusmodi orbita mobili moveri deprehenditur, motus eius et distantia quovis tempore a centro *C* poterit determinari. Atque quoties corpus in orbita in puncta (*A*) et (*B*) pervenit, tum in minima vel maxima a *C* erit distantia. Quare cum motus gyratorius lineae (*A*)(*B*), quae linea absidum vocatur, sit datus, definire poterit, quando corporis a centro C distantia sit maxima vel minima. Neutonus hanc rem pertractavit in Prin. Libro I. tota Sectione IX, eaque theoria utitur ad motum lineae absidum orbitae lunaris determinandum. Sed minus accurate haec consideratio ad lunam accommodari potest, cum vis lunam sollicitans non ad punctum quoddam fixum C, ut hic posuimus, sed perpetuo variabile tendat. Operam igitur dabimus, ut, postquam reliqua huc pertinentia explicaverimus, alias propositiones magins idoneas afferamus, quae ad motum lunae transferri queant. [p. 307]

PROPOSITIO 90.

PROBLEMA.

742. *Cognita curve, quam corpus vi quacunque centripeta V sollicitatum describit, determinare curvam, quam corpus a vi centripeta* $V+\frac{C}{y^3}$, *denotante y distantium MC corporis a centro virium C, sollicitatum describet.*

[Vide etiam L. Euleri Commentationem 232 (indices Enestroemiani) : *De motu corporum coelestium a viribus quibuscunque perturbato*, Novi comment. acam. se. Petrop. (1752/53); Leonardi Euleri *Opera omnia*, series II, vol. 21. P. St.]

SOLUTIO.

Agente vi centripeta $V+\frac{C}{y^3}$ sit corporis celeritas, qua in (A) secundum directionem ad radium C(A) normalem proiicitur, debita altitudini *c* et ponatur C(A) = *a*. Urgente autem vi V sit (*A*)(*M*)(*B*) orbita, in qua corpus movebitur proiectum in (*A*) secundum eandem directionem, sed celeritate debita altitudini γ. Iam ex praecedente propositione manifestum est vi $V+\frac{C}{y^3}$ efficii, ut corpus in eadem orbita (*A*)(*M*)(*B*), sed circa centrum C in data ratione ad motum angularem in ipsa orbita mobili moveatur. Sit igitur $w-1$ ad 1 ut motus angularis orbitae ad motum angularum corporis in ipsa orbita, dum est in M, et sit etiam $c=w^2\gamma$ (732) atque vocetur perpendiculum ex C in tangentem orbitae in M demissum CT = p. Hinc est vim centripetam facientem, ut corpus in orbita immobili (*A*)(*M*)(*B*) moveatur, fore = $\frac{2a^2\gamma dp}{p^3 dy}$ ideoque

$$V=\frac{2a^2\gamma dp}{p^3 dy},$$

quam aequationem ergo ob datam curvam (*A*)(*M*)(*B*) construibilem ponimus (p. hyp.). Vis autem efficiens, [p. 308] ut corpus in eadem orbita descripto modo mobili moveatur, erit

$$= \frac{2a^2\gamma dp}{p^3 dy} + \frac{2a^2\gamma(w^2-1)}{y^3}$$

(734). Quamobrem habebitur

$$V + \frac{C}{y^3} = \frac{2a^2\gamma dp}{p^3 dy} + \frac{2a^2\gamma(w^2-1)}{y^3}.$$

Ex quo prodit

$$C = 2a^2\gamma(w^2-1) = \frac{2a^2c(w^2-1)}{w^2}$$

atque

$$w^2 = \frac{2a^2c}{2a^2c - C}.$$

Erit ergo

$$(w-1):1 = \frac{\sqrt{2a^2c} - \sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}:1$$

atque

$$\gamma = \frac{2a^2c - C}{2a^2}.$$

Inventa igitur curva, quam corpus in (*A*) celeritate altitudinis $\frac{2a^2c-C}{2a^2}$ proiectum describit sollicitatum a vi *V*, vis $V+\frac{C}{y^3}$ efficiet, ut corpus in (*A*) celeritate $\sqrt{c}$ proiectum moveatur in eadem orbita mobili ita, ut sit motus angularis orbitae ad motum angularem corporis in hac orbita, quaemadmodum est $\frac{\sqrt{2a^2c}-\sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}$ ad 1. Motus autem corporis in ipsa orbita idem erit, quem habet in orbita immobili a vi tantum *V* sollicitatum et in (*A*) celeritate debita altitudini $\frac{2a^2c-C}{2a^2}$ proiectum, qui motus per hypothesin est cognitus. Q.E.I.

Corollarium 1.

743. Dum igitur corpus in orbita ex (*A*) ad (*B*) pervenit seu circa centrum *C* angulo 180 graduum revolvitur, ipsa orbita interea angulo

$$\frac{\sqrt{2a^2c}-\sqrt{(2a^2c-C)}}{\sqrt{(2a^2c-C)}}\cdot 180$$

graduum circa C gyrabitur.

Corollarium 2.

744. Si igitur recta (*A*)(*B*) est linea absidium, erit punctum (*A*) ima, punctum (*B*) vero summa absis, uti in Astronomia vocantur; corpus igitur ab abside ima ad summa perveniet absoluto motu angulari circa C graduum [p. 309]

$$\frac{180}{\sqrt{\left(1-\frac{C}{2a^2c}\right)}}.$$

Corollarium 3.

745. Tempus, quo corpus in orbita mobili ex (*A*) in *M* pervenit, aequatur tempori, quo in quiescente ex (*A*) in (*M*) pertingit. Angulus vero (*A*)*CM* se habet ad angulum (*A*)C (*M*) ut *w* ad 1, i. e. ut

$$\frac{1}{\sqrt{\left(1-\frac{C}{2a^2c}\right)}}$$

ad 1.

Exemplum.

746. Sit vis *V* reciproce proportionalis quadrato distantiae a centre seu $V=\frac{ff}{yy}$, erit curva (*A*)(*M*)(*B*) ellipsis, in cuius foco positum est centrum virium *C*. Sit eius axis transversus (*A*)(*B*) = A et latus rectum = *L*, erit

$a=\frac{1}{2}A-\frac{1}{2}\sqrt{(A^2-AL)}=(A)C$ et $(B)C=\frac{1}{2}A+\frac{1}{2}\sqrt{(A^2-AL)}$. Ob $4pp=\frac{ALy}{A-y}$ erit

$$\frac{2a^2\gamma dp}{p^3dy}=\frac{4a^2\gamma}{Ly^2}=\frac{ff}{yy}.$$

Hinc fit

$4a^2\gamma=Lff=4a^2c-2C$, unde $\frac{4a^2c-2C}{ff}$ atque $c=\frac{Lff+2C}{4a^2}$.

Quae est altitudo debita celeritati corporis in (*A*) pro orbita mobili ex vi centripta $\frac{ff}{y^2}+\frac{C}{y^3}$. Motus vero angularis orbitae erit ad motum angularem corporis in orbita ut

$$\frac{\sqrt{(\frac{1}{2}Lff+C)}-\sqrt{\frac{1}{2}Lff}}{\sqrt{\frac{1}{2}Lff}}$$

ad 1. Atque corpus abside ima ad summam perveniet, postquam motu angulari angulum

$$\frac{180\sqrt{(\frac{1}{2}Lff+C)}}{\sqrt{\frac{1}{2}Lff}}=180\sqrt{\left(1+\frac{2C}{Lff}\right)}$$

graduum absolverit. [p. 310]

PROPOSITIO 91.

PROBLEMA.

747. *Si figura, quam corpus a quacunque vi centripeta sollicitatum describit, non multum differt a circulo, determinare motum absidum.*

SOLUTIO.

Comparandus est huiusmodi motus cum motu corporis in ellipsi mobili parum excentrica, cuius focus alteruter positus sit in centro virium. In hac igitur orbita quiescente corpus movebitur sollicitantum a vi centripeta quadratis distantiarum reciproce proportionali. In eadem vero orbita mobili corpus movebitur, si fuerit vis centripeta = $\frac{ffy+C}{y^3}$ (746). Manentibus praecendentibus denominationibus ponatur $y = a + z;$ erit z respectu ipsius a valde parvum, quia curva a corpore descripta circulo proxima ponitur. Quare vis centripeta illa erit $\frac{aff+C+ffz}{y^3}$, et $2a$ quam proxime erit aequalis ipsi lateri recto L. Iam ponatur vis centripeta corpus sollicitans $=\frac{P}{y^3}$, in qua P sit functio quacunque ipsius y. Ponatur in P loco y eius valor $a + z$ abeatque P reiectis terminis, in quibus z plus una habet dimensionem, ob z tam parvum in $E + Fz$. Hac ergo formula cum $aff + C + ffz$ comparata habebitur
$F = ff$ seu $f = \sqrt{F}$ et $aF + C = E$ seu $C = E - aF$. His substitutis corpus ab hac vi centripeta $\frac{P}{y^3}$ sollicitatum ab abside ima ad summam perveniet [p. 311] absoluto motu angulari angulo $180\sqrt{\left(1+\frac{2C}{Lff}\right)}$ graduum (746). Seu posita $2a$ loco L et F loco ff atque $C = E - aF$. Seu posito $2a$ loco L et F loco ff atque $E - aF$ loco C erit iste angulus graduum $180\sqrt{\frac{E}{aF}}$. Si quidem orbita non multum a circulari discrepat. Q.E.I.

Corollarium 1.

748. Linea vero absidum $(A)(B)$, dum corpus circa C revolvitur angulo 360 graduum, movibitur motu angulari per angulum $\frac{\sqrt{E}-\sqrt{aF}}{\sqrt{E}}.360$ graduum. Motus enim angularis orbitae proportionalis ponitur motui angulari corporis ob vim centripetam = $\frac{ff}{y^2}+\frac{C}{y^3}$ (734).

Corollarium 2.

749. Quia E est functio talis ipsius a, qualis P est ipsius y, erit Fz incrementum ipsius E crescente a elemento z. Quare posito $z = da$ erit $Fda = dE$, ideoque angulus, quo corpus ab abside ima ad summam pervenit, erit $=180\sqrt{\frac{Eda}{adE}}$ graduum.

Corollarium 3.

750. Quia E talis est functio ipsius a, qualis P ipsius y, poterit in $\frac{Eda}{adE}$ poni y loco a et P loco E. Quamobrem existenta vi centripeta $\frac{P}{y^3}$ corpus ab abside ima ad summam perveniet absoluto angulo $180\sqrt{\frac{pdy}{ydp}}$ graduum. Atque in hac expressione si restet y, poterit eius loco a scribi, quippe parum ab y descrepans. [p. 312]

Corollarium 4.

751. Si fuerit $\frac{da}{a} > \frac{dE}{E}$ seu $\frac{dy}{y} > \frac{dP}{P}$, ellipsis motu suo verum corporis motum exprimens movebitur in consequentia. Sin vero $\frac{dy}{y} < \frac{dP}{P}$, linea absidum in antecedentia movebitur. At si $\frac{dy}{y} = \frac{dP}{P}$ seu $P = \alpha y$, quo casu vis centripeta reciproce proportionalis est quadratis distantiarum, linea absidum quiescet seu corpus, postquam motu angulari angulum 180 graduum absolverit, ab abside ima ad summam et vicissim pertingit.

Corollarium 5.

752. Dato autem angulo, quo corpus ab abside altera ad alteram pervenit, qui sit 360μ graduum, erit $\mu^2 = \frac{Pdy}{ydP}$ ideoque $P^{\mu\mu} = \alpha y$ seu $P = (\alpha y)^{\frac{1}{\mu\mu}}$.

Vis ergo centripeta, quae facit, ut lineae absidum tantus sit motus, erit $y^{\frac{1-3\mu^2}{\mu^2}}$.

Corollarium 6.

753. Si accidit, ut $\frac{Pdy}{ydP}$ seu dP fiat negativum, motus absidum erit imaginarius. Ex quo cognoscitur corpus nunquam ad absidem alteram pervenire posse ab altera progressum, sed perpetuo vel magis recessurum a centro vel ad id accessurum sive in orbita clausa prorsus non moveri.

Corollarium 7.

754. Si vis centripeta proportionalis sit distantiarum potestati y^n, erit $P = y^{n+3}$. Quare fiet $\frac{Pdy}{ydP} = \frac{1}{n+3}$, [p. 313] atque corpus ab abside ima ad summam perveniet absoluto angulo circa C graduum $\frac{180}{\sqrt{(n+3)}}$; a summa vero vel ima abside ad eadem revertetur absoluto angulo $\frac{360}{\sqrt{(n+3)}}$ graduum.

Corollarium 8.

755. Si ergo fuerit $\sqrt{(n+3)}$ numerus rationalis et m minimus numerus integer, quo fiat $\frac{m}{\sqrt{(n+3)}}$ quoque numerus integer, tum corpus post $\frac{m}{\sqrt{(n+3)}}$ revolutiones circa centrum C peractas in idem punctum incidet totidemque curva a corpore descripta absolvet spiras, antequam in se ipsam redeat atque claudatur. At si $n + 3$ non est quadratum, curva nunquam in se redibit, sed infinitas circa centrum C habebit spiras neque unquam corpus in eandem viam revertetur.

Exemplum 1.

756. Attrahat centrum virium in ratione reciproca triplicata distantiarum; erit $n + 3 = 0$. Hac ergo hypothesi corpus ab altera abside egressum ad alteram nisi infinitis revolutionibus peractis non perveniet. Atque si vis centripeta in maiore ratione quam triplicata distantiarum decrescat, curva prorsus non habebit duas absides, sed vel in infinitum abibit vel in ipso centro ut spiralis logarithmica terminabitur. [p. 314]

Exemplum 2.

757. Si vis centripeta est quadratis distantiarum reciproce proportionalis erit $n + 3 = 1$. Quare tum corpus motu angulari absolutis 180 gradibus ab altera abside ad alteram perveniet et curva post quamvis revolutionem in se ipsam redibit. Corpus enim in ellipse, in cuius alterutro foco centrum virium est positum, movebitur eiusque axis transversus est ipsa linea absidum.

Exemplum 3.

758. Si vis centripeta est distantiis reciproce proportionalis, est $n + 3 = 2$. Corpus igitur ab abside ima ad summam perveniet absoluto angulo $\frac{180}{\sqrt{2}}$ graduum seu 127 gr. 17'. Orbita vero ob $\sqrt{2}$ irrationale nusquam in se redibit.

Exemplum 4.

759. Si vis centripeta est constans in omni distantia, est $n = 0$. Hoc casu corpus ab altera abside egressum ad alteram perveniet motu angulari percurso angulo $\frac{180}{\sqrt{3}}$ graduum , i. e. 103 gr. 55' quam proxime.

Exemplum 5.

760. Si vis centripeta directe ut corporis a centro distantia, quo casu corpus in ellipsi moveri constat, in cuius centro centrum virium est positum (631). Absis igitur ima ad summa distabit angulo 90 graduum. Idem vero ex hac regula deducitaur; nam ob n = 1 erit $\frac{180}{\sqrt{n+3}} = 90$. [p. 315]

Scholion 1.

761. Quoties igitur corpus circa centrum virium tanta velocitate proiicitur, ut fere in circulo deberet revolvi, ope huius propositionis vera curva, quam corpus describet, potest determinari, id quod ex solo vis centripetae consideratione fieri non potest. Ex quibus eo magis huiusmodi contemplationum usus perspicitur, cum res, quae alias determinatu essent difficillimae, ex iis facile definiantur. Neutonus eandem hanc propositionem exposuit Sect. IX prop. 45.

Scholion 2.

762. Iam supra ostendimus corpus in hypothesi vis centripetae cubo distantiae reciproce proportionalis ad centrum descendens ad id tempore finito pervenire neque deinde ex eo egredi, sed quasi subito annihilari (675 et 676). Idem etiam valet, si corpus recta ad centrum descendat. Atque simuli modo, si vis centripeta in maiore quam triplicata distantiarum ratione decrescat, corpus, statim ac in centrum pervenerit, ibi evanescet neque ultra centrum progredietur neque revertetur. Utrumvis enim eveniat, curva, quam corpus velocitate quadam proiectum decrescit, haberet duas absides, quod esset absurdum (756). Quoties autem vis centripeta in minore quam triplicata ratione descescit, ut in simplici distantiarum ratione vel ea maiore, corpus, postquam in centrum pervernerit, [p. 316] in eadem recta, qua ad centrum accessit, recedet; perspicitur hoc enim ex ratione reciproca duplicata (655) et simplici, de qua patet (271) corpus non ultra centrum posse progredi. At si $n + 1 > 0$, corpus recta ad centrum descendens finitam habebit celeritatem, qua ultra centrum in eadem recta progredietur, quoad motum amiserit (273). Hoc ergo modo satisfecimus desiderato superiori (272), quo motum corporis recta descendentis, cum in centrum pervenisset, definiri oportebat.