CHAPTER FIVE (*Part b*).
CONCERNING THE CURVILINEAR MOTION OF FREE POINTS ACTED ON BY ABSOLUTE FORCES OF ANY KIND.
[p. 261]

PROPOSITION 79.

PROBLEM.

641. *With the attraction from the centre of the forces in the ratio of the distances from C, a body is projected from M with some speed and following some direction MT; it is required to determine the ellipse on which the body is moving.*

SOLUTION.

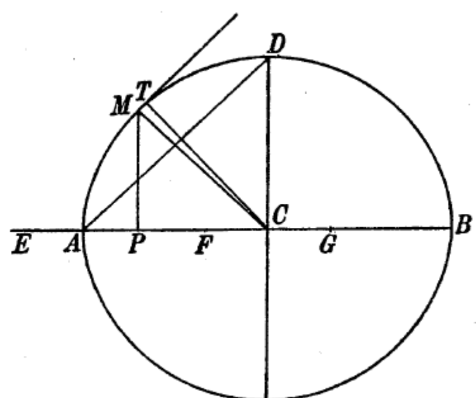
Fig. 56.

The solution of this problem can be understood from this, that the number of those given is not very large for the ellipse to be determined, as will become apparent. The radius $CM = y$, the sine of the angle $CMT = s$, with the total sine equal to 1, and with the height corresponding to the speed at M equal to v, also the distance from the centre C, at which the centripetal force is equal to the force of gravity remains equal to f. These quantities are therefore considered to be given and known; truly unknown and to be found are the axes of the ellipse and the position of these; [p. 263] of these required, put $AC = a$ and $CD = b$, and hence $b = \sqrt{2cf}$ (631).

With these put in place, it is found at once that $2fv = a^2 + b^2 - y^2$ (632), and hence

$$a^2 + b^2 = AD^2 = 2fv + y^2;$$

therefore with the subtangent of AD now known equal to the distance CE determining the speeds. Then also there is found (635) :

$$s = \frac{ab}{\sqrt{(a^2+b^2-y^2)}} = \frac{ab}{y\sqrt{2fv}},$$

and from which there arises :

$$ab = sy\sqrt{2fv}.$$

With these combined together, there arises :

$$(a^2 - b^2)^2 = 4f^2v^2 + 4fvy^2 + y^4 - 8fs^2vy^2 = 4f^2v^2 + y^4 + 4fvy^2(1-2s^2).$$

Truly $1-2s^2$ is the cosine of twice the angle *CMT*, that we call i. Hence we have

$$a^2-b^2=\sqrt{(4f^2v^2+y^4+4fviy^2)}\,.$$

From these equations there comes about :

$$a^2=fv+\tfrac{1}{2}y^2+\sqrt{(f^2v^2+\tfrac{1}{4}y^4+fviy^2)} \text{ and}$$

$$b^2=fv+\tfrac{1}{2}y^2-\sqrt{(f^2v^2+\tfrac{1}{4}y^4+fviy^2)}\,.$$

Truly with the axis found, the position of these can be easily found. For indeed the cosine of the angle *ACM*

$$=\frac{x}{y}=\frac{1}{y}\sqrt{\left(\frac{a^2y^2-a^2b^2}{a^2-b^2}\right)}=\sqrt{\left(\frac{1}{2}+\frac{fiv+\frac{1}{2}y^2}{\sqrt{(4f^2v^2+y^4+4fviy^2)}}\right)}$$

or the cosine of twice the angle *ACM* :

$$\frac{2fiv+y^2}{\sqrt{(4f^2v^2+y^4+4fviy^2)}}\,.$$

and the sine of this angle 2ACM

$$=\frac{2fv\sqrt{(1-i^2)}}{\sqrt{(4f^2v^2+y^4+4fviy^2)}}\,.$$

Q.E.I.

Corollary.

642. Therefore, wherever the body is projected and whatever the speed and the direction, the body moves around the perimeter of an ellipse, the centre of which is the centre of force.

Scholion

643. Up to this stage, a central force proportional to the distance and attracting has been put in place; but in a like manner the curves described are found, [p. 264] if the force from the centre repels the bodies in the same ratio. For everything in the preceding arguments can be adapted, if $-f$ is written in place of f. Truly with this curve made, which before was an ellipse, is now changed into a hyperbola, with the minor axis made imaginary, and the centre of force remaining at the centre of the hyperbola.

PROPOSITION 80.

PROBLEM.

644. *If the centripetal force is inversely proportional to the square of the distance and the body is projected with a given speed from A* (Fig. 57) *with a given speed in a direction normal to the radius AC, it is required to determine the curve AMDBHA, that the body describes, and the motion itself along this curve.*

SOLUTION.

As before with $AC = a$ and with the speed at A corresponding to the height c, f is the distance at which the centripetal force is equal to the force of gravity. Then put $CM = y$, and the perpendicular CT, that is sent to the tangent, is equal to p, and the height corresponding to the speed at M is equal to v. For with these compared with proposition 75 (601), there arises :

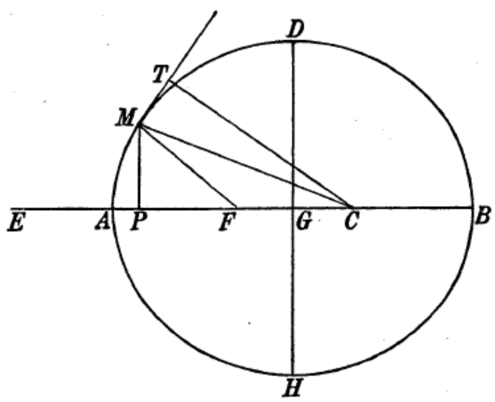
Fig. 57.

$$h = a, \quad P = \frac{f^2}{y^2} \text{ et } Y = \frac{f^2}{a} - \frac{f^2}{y}.$$

The perpendicular MP is dropped from M on AC ; we call $CP = x$, and we put $x = uy$. With this put in place, we have :

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{c}}{y\sqrt{(cy^2 - a^2c - \frac{f^2y^2}{a} + f^2y)}}$$

Put

$$y = \frac{1}{\frac{f^2}{2a^2c} - q},$$

and the equation becomes :

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{dq}{\sqrt{((\frac{2ac-f^2}{2a^2c})^2 - q^2)}}.$$

This formula is brought together with $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ (604), and we find that

$\lambda = 1, \quad A = \frac{2ac - f^2}{2a^2c}$ and $Z = q$.

On account of which the following equation applies :

$$u = \frac{q\sqrt{(A^2 - C^2)} - C\sqrt{(A^2 - q^2)}}{A^2}$$

(609). [p. 265] Truly we have $q = \frac{f^2}{2a^2c} - \frac{1}{y}$, and the constant quantity can be defined from this, since by making $u = 1$ then $y = a$. The equation can be changed a little into this form :

$$C+u\sqrt{(A^2-q^2)}=q\sqrt{(1-u^2)}.$$

In which by making $u=1$ and $q=\frac{f^2-2ac}{2a^2c}$ gives C = 0. Consequently we have :

$$\left(\frac{f^2-2ac}{2a^2c}\right)u^2=q^2,$$

or

$$(f^2-2ac)u=2a^2cq=f^2-\frac{2a^2c}{y}.$$

Since $u=\frac{x}{y}$, this becomes $(f^2-2ac)x=f^2-2a^2c$ and

$$f^4y^2=4a^4c^2+4a^2cx(f^2-2ac)+(f^2-2ac)^2x^2.$$

Calling the line $PM=z$, then

$$f^4z^2=4a^4c^2+4a^2cx(f^2-2ac)-4acf^2x^2+4a^2c^2x^2.$$

On putting $x=t+\frac{2a^2c-af^2}{2f^2-2ac}$, there arises

$$f^4z^2=\frac{a^3cf^4}{f^2-ac}-4act^2(f^2-ac).$$

Which is the equation of an ellipse with the abscissa taken on the transverse axis from the centre G, of which the transverse axis is equal to $\frac{af^2}{f^2-ac}$ and the conjugate axis is equal to $\frac{2a\sqrt{ac}}{\sqrt{(f^2-ac)}}$. And the distance of the centre of force C from the centre of the ellipse G is given by $CG=\frac{2a^2c-af^2}{2f^2-2ac}$, which is equal to half the distance between the focal points. For this reason the centre of force C has been put in one of the focal points of the ellipse.

It is therefore apparent that the body projected from A with a speed $\sqrt{c}$ describes the ellipse *AMDBHA* about the centre C, of which one focus is placed at C.

Thus :

$AC=a,\quad AG=\frac{af^2}{2f^2-2ac}$, $AB=\frac{af^2}{f^2-ac}$, $BC=\frac{a^2c}{f^2-ac}$ and the conjugate axis $DH=\frac{2a\sqrt{ac}}{\sqrt{(f^2-ac)}}$, and the latus rectum of the ellipse is equal to $\frac{4a^2c}{f^2}$.

By making use of the equation $v=\frac{a^2c}{p^2}$ (587), the speed at M can be determined : moreover, $p=\frac{a\sqrt{c}}{\sqrt{(c-Y)}}=\frac{a\sqrt{acy}}{\sqrt{(acy+(a-y)f^2)}}$,

and hence

$$v=\frac{acy+(a-y)f^2}{ay}.$$

Q.E.I. [p. 266]

Corollary 1.

645. The distance determining the speeds *CE* is found equal to $\frac{af^2}{2f^2-2ac}$ (278). From which it is apparent that this distance CE is equal to the transverse axis of the ellipse *AB*.

Corollary 2.

646. Moreover as often as the curve described by the body is an ellipse, so $f^2 > ac$ or $c < \frac{f^2}{a}$. But if $c > \frac{f^2}{a}$, then the transverse axis is made negative and the conjugate axis is an imaginary axis, from which it is evident that the curve has changed into a hyperbola.

Corollary 3.

647. Truly when $c = \frac{f^2}{a}$, the curve holds the middle place between the hyperbola and the ellipse, and it has an infinite transverse axis. Thus the curve then described by the body is a parabola.

Corollary 4.

648. If $c = \frac{f^2}{2a}$, the ellipse becomes a circle; for the transverse and conjugate axis become equal. Truly the centre of the force falls on the centre of the circle.

Corollary 5.

649. The line *FM* is drawn from one focus to *M* on the ellipse, it is given by
$FM = AB - y = \frac{f^2(a-y)+acy}{f^2-ac}$.
From which it will be recognised :

$$v = \frac{(f^2-ac).FM}{a.CM}.$$

Hence the speed at any point *M* is as $\sqrt{\frac{FM}{CM}}$. [p. 267]

Corollary 6.

650. The sine of the angle made by the radius *CM* with the tangent *MT*, is equal to

$$\frac{p}{y} = \frac{a\sqrt{ac}}{\sqrt{(acy^2+(a-y)f^2y)}},$$

with the total sine put equal to 1. But since $FM = \frac{f^2(a-y)+acy}{f^2-ac}$, the sine of the angle

$CMT = \frac{a\sqrt{ac}}{\sqrt{(f^2-ac)FM.CM}}$.

Corollary 7.

651. Because $GC = \frac{2a^2c - af^2}{2f^2 - 2ac}$, the centre of force is in the further focus from the vertex A, as often as $c > \frac{f^2}{a}$. Truly it falls on the closer focus, if $c < \frac{f^2}{2a}$.

Corollary 8.

652. With the major axis *AB* put equal to *E*, with the parameter equal to *L*, then $E = \frac{af^2}{f^2 - ac}$ and $L = \frac{4a^2c}{f^2}$.
Hence it is found that :

$$a = \frac{E \pm \sqrt{(E^2 - EL)}}{2} \text{ and } \sqrt{c} = \frac{f\sqrt{L}}{E \pm \sqrt{(E^2 - EL)}}.$$

Corollary 9.

653. The time, in which the body traverses the whole circumference of the ellipse, is equal to

$$\frac{2.\mathit{Areae\ Ellipticae}}{a\sqrt{c}} = \frac{4.\mathit{Area\ Ellipticae}}{f\sqrt{L}} \text{ (588).}$$

Truly with the ratio of the diameter to the periphery set equal to $1 : \pi$, the area of the ellipse is equal to $\frac{\pi E\sqrt{EL}}{4}$. Therefore the time for one revolution is equal to $\frac{\pi E\sqrt{E}}{f}$.

Corollary 10.

654. Therefore if there are several bodies revolving around the centre of force, attracted in the inverse square of the distances, the periodic times are to each other as the three on two ratio of the transverse axes of the ellipses. [p. 268]

Corollary 11.

655. If the initial speed at A vanishes, the ellipse turns into the right line AC. Therefore the body continually moves on this straight line between A and C, and hence suddenly changes direction again and is returned to A, thus in order that the body never reaches a point beyond the centre C (272). [Assuming elastic collisions at C.]

PROPOSITION 81.

PROBLEM.

656. *With the centripetal force put inversely proportional to the squares of the distances the body is projected from M with any speed and in some nearby direction MT; from which it is required to determine the ellipse MDBHAM, in which the body is moving..*

SOLUTION.

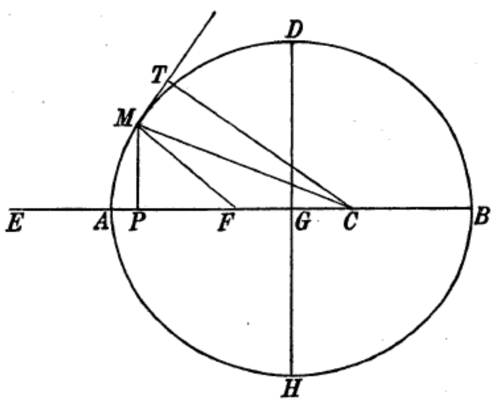

Fig. 57.

Let $MC = y$, the sine of the angle $CMT = s$ and the speed at M correspond to the height v, with the distance from the centre, at which the centripetal force is equal to the force of gravity , to remain equal to f. These quantities are known, from which are sought the transverse axis AB, the position of this and the latus rectum are to be defined. Let the transverse axis be equal to E and the latus rectum equal to L, and the remaining letters a and c keep their meanings as in the preceding proposition. Now from the nature of the ellipse, $FM = E - y$, from which there becomes (649, 652)

$$v = \frac{(f^2 - ac)(E - y)}{ay} = \frac{f^2(E-y)(2E - L \pm 2\sqrt{(E^2 - EL)})}{y(E \pm \sqrt{(E^2 - EL)})^2} = \frac{f^2(E-y)}{Ey}.$$

Hence it is found :

$$E = \frac{f^2 y}{f^2 - vy}.$$

Then the sine of the angle CMT

$$= s = \frac{a\sqrt{ac}}{\sqrt{(f^2 - ac)y(E-y)}}$$

(650). But

$$f^2 - ac = \frac{f^2(E \pm \sqrt{(E^2 - EL)})}{2E} \quad \text{et} \quad a^3c = \frac{1}{8} f^2 L(E \pm \sqrt{(E^2 - EL)})$$

(652) [p. 269] , from $\frac{a\sqrt{ac}}{\sqrt{(f^2 - ac)}} = \frac{1}{2}\sqrt{EL}$ and from which hence

$$s = \frac{\sqrt{EL}}{2\sqrt{(Ey - y^2)}}.$$

Therefore the conjugate axis

$$DH = \sqrt{EL} = 2s\sqrt{(Ey - y^2)} = \frac{2sy\sqrt{vy}}{\sqrt{(f^2 - vy)}}$$

and the latus rectum

$$L = \frac{4s^2vy^2}{f^2}.$$

The cosine of the angle *MCP* sought in determining the position of the transverse axis is given by :

$$\frac{x}{y} = \frac{f^2y - 2a^2c}{y(f^2 - 2ac)} = \frac{2Ey - EL}{2y\sqrt{(E^2 - EL)}} = \frac{ff - 2ssvy}{\sqrt{(f^4 - 4ffssvy + 4s^2v^2y^2)}}.$$

From which the tangent of the angle *MCP*

$$= \frac{2svy\sqrt{(1 - ss)}}{ff - 2ssvy}.$$

Q.E.I.

Corollary 1.

657. If $f^2 > vy$ or $v < \frac{f^2}{y}$, the curve described is always an ellipse. But if $v > \frac{f^2}{y}$, then the curve is a hyperbola; for the transverse axis is negative. Moreover if $v = \frac{f^2}{y}$, then the curve is a parabola.

Corollary 2.

658. Since neither the transverse axis nor the latus rectum can become imaginary in any case, in whatever manner the body may be projected, thus it always moves in a conic section, in which the centre of force is placed in one or the other focus.

Scholium 1.

659. After Kepler had shown that the planets move in ellipses, in which the sun is put in one focus, and the times are in proportion to the areas, which straight lines drawn from the arc to the sun are understood to describe, Newton showed that the force holding the planets in their orbits pulling towards the sun was in the inverse square ratio of the distances of the planets from the sun. [p. 270] This same truth follows from these two propositions; for with the centre of force attracting in the inverse square ratio of the distances, the bodies must be moving in ellipses or hyperbolas, the centre of force of which falls on one or other focus.

Corollary 2.

666. [In the first edition, in place of the number 660, and those which follow, the false numbers 666 and for those following have been written. We have decided to retain the false numbers of the paragraphs. (Note by Paul Stackel, 1912 ed.)]
Following Newton, the force of attraction of the sun is to the force of attraction of the earth at the same distances from the centres of each are in the ratio 227512 to 1. Whereby, when the force of attraction of the earth at a distance of one radius from the centre of the earth on the surface is the gravitational force of 1, a body at a distance of one earth radius from the centre of the sun is attracted to it by a force in turn 227512 greater than that of gravity. From which it is inferred, that if a body were at a distance of 477 earth radii, the force by which it is attracted to the sun, would be equal to the force of gravity on the earth.

Corollary 4.

667. Therefore if the sun is put in place as the centre of force of the attraction in the inverse ratio of the square of the distances, a distance equal to 447 earth radii must be taken for f.

Scholium 2.

668. When these propositions can be adapted to the motion of the planets, the sun is at *C*, and with the planet moving in the ellipse *ADBHA*, the transverse axis of which *AB* is equal to *E*, the interfocal distance $CF = D$ and the speed, [p. 271] that the planet has at the greater apse *A*, corresponds to the height *c*. Hence it becomes :

$$c = \frac{227512(E-D)}{(E+D)E}.$$

Now for Mercury, $E = 15991$ earth radii, that here we constantly use to express distances, $D = 3367$. Whereby $c = 9.278$. For Venus, $E = 29882$ and $D = 206$, hence it is found that $c = 7.509$. For the earth, E = 41312 and D = 743. Hence, c = 5.313. For Mars, $E = 62959$ and $D = 5887$, and thus $c = 2.996$. For Jupiter, $E = 214870$ and D = 22391, hence, c = 0.9615. For Saturn, E = 394042 and D = 22391, hence c = 0.5153. Which are sufficient for the absolute motion of any planet to be determined. [In the first edition, the heights c for Mercury, Venus, the Earth, Mars, Jupiter, and Saturn have been found to have the incorrect values 7.368, 7.598, 5.323, 3.049, 0.9633, 0.5173. Corrected by P. St.]

Scholion 3.

669. Newton, as we have now given the nod to his works, from the known ellipse that the body describes, and with the centre of force placed in one focus, deduced that the centripetal force is inversely proportional to the squares of the distances. Afterwards the inverse of this proposition was considered by the Celebrated Johan Bernoulli and others, as clearly the curve is required, that the body describes around the centre of force attracting in the inverse square ratio of the distances. For they denied that a satisfactory demonstration had been given to the inverse problem by Newton, that besides the conic sections no other curve sought was satisfactory, although prop. XVII. of Book I, *Princ. Philos. Naturalis* seems to show this clearly enough. Therefore with these two propositions, we have shown in full the solution to this question, by which the assertion of Newton is put in place beyond doubt. [p. 272] Although other solutions of this problem can be seen in the Comm. Acad. Paris[1] and Horis subsecivius Francof[2].

[1. *Extrait de la réponse de Monsieur Bernoulli à Monsieur Herman*, datée de Basle le 7 Octobre 1710, Mém. de l' Acad. R. des Sciences de Paris, Année 1710, p.521; *Opera omnia* Tom. I, Lausannae et Genevae 1742, p.470. See also the studies, that P. Varignon published in the volumes of the Comm. Acad. Paris. in the years 1700, 1701, 1710, and the letter that Hermann sent to Bernoulli, 1710, p.519; which is also present in Joh. Bernoulli *Operibus omnis*, Tom. I, p. 469.
2. *Concerning the laws of central forces, by which planets are urged to move in their orbits, and the motions hence arising*, Exercitationum subsecivarum Francofurtensium Tom. I, Sectio 2 (1718), p.181 –194. It is apparent from p.188 that J. Hermann was the author of these studies. P. St.
This seems to be an ongoing dispute, as papers are still written on the topic : see Cohen's *Principia*, p. 136.]

Scholium 4.

670. The time of one revolution of a planet around the sun is $\frac{\pi E\sqrt{E}}{250f}$ seconds (653), if indeed E and f are expressed in the thousandth parts of Rhenish feet. But since is more convenient to express E and f in earth radii, of which one contains 20302353 feet, in place of the fraction $\frac{1}{250}$, that $\frac{\pi E\sqrt{E}}{f}$ is to be multiplied by, it is required to have 569.954. For this reason, the time of a single revolution of the planet is $\frac{569945\pi E\sqrt{E}}{1000f}$ seconds, or with π =3.1415926536 and f = 477 earth radii (567), the periodic time of the planet is equal to $3.754\,E\sqrt{E}$ seconds, with the transverse axis expressed in earth radii.

PROPOSITION 82.

PROBLEM.

671. *If the centripetal force varies inversely as the cube of the distance from the centre, the curve is required that the body describes projected in any manner, and the motion of the body on that curve.*

SOLUTION.

Let C be the centre (Fig. 58) and the body is projected from A with the speed $\sqrt{c}$ and along the direction making an angle with AC, the sine of which is $\frac{h}{a}$, with $AC = a$. [p. 273] The body arrives at M, where $CM = y$ and the perpendicular sent from C to the tangent MT is $CT = p$. Truly the speed at M corresponds to the height v. And the distance from the centre C, at which the centripetal force is equal to the force of gravity, is equal to f. Therefore the centripetal force at is equal to $\frac{f^3}{y^3} = P$; from which there is produced $Y = \frac{f^3}{2a^2} - \frac{f^3}{2y^2}$ (601). Therefore this equation is obtained for the curve sought :

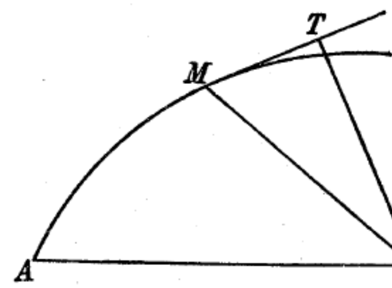

Fig. 58.

$$\frac{f^3}{2a^2} - \frac{f^3}{2y^2} = c - \frac{ch^2}{p^2}.$$

And when $v = \frac{ch^2}{p^2}$ (587), also $v = c - \frac{f^3}{2a^2} + \frac{f^3}{2y^2}$.

We will examine these equations for the following cases.

I. If $c = \frac{f^3}{2a^2}$, then $\frac{1}{y^2} = \frac{h^2}{a^2p^2}$ or

$$p = \frac{hy}{a}.$$

Moreover it is evident from this, that the CMT, the sine of which is $\frac{p}{y}$, is equal to the angle at A. On account of which the curve described in these cases is a logarithmic spiral, the centre of which is the centre of force C itself. Thus moreover the body moves on this spiral, as always $v = \frac{f^3}{2y^2}$, or the speed is inversely proportional to the distance.

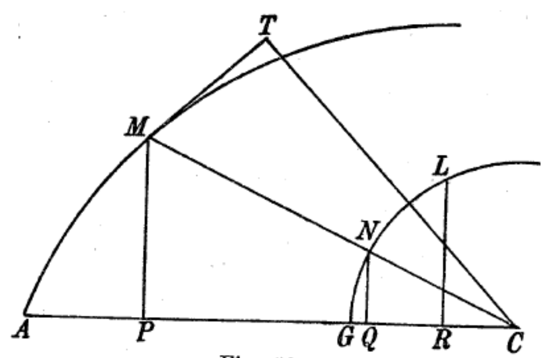

Fig. 59.

If c is not equal to $\frac{f^3}{2a^2}$, the perpendicular MP is sent from M to AC (Fig. 59), and calling $CP = x$ and $x = uy$. Therefore there is had :

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{hdy\sqrt{c}}{y\sqrt{\left(cy^2 - ch^2 - \frac{f^3y^2}{2a^2} + \frac{f^3}{2}\right)}}$$

Truly u is the cosine of the angle ACM, with the whole sine taken as 1. Therefore if the arc of the circle GN is described with radius 1, then $GN = -\int\frac{du}{\sqrt{1-u^2}}$. Moreover calling $GN = t$, and this equation is found :

$$dt = -\frac{hdy\sqrt{c}}{y\sqrt{\left(cy^2 - ch^2 - \frac{f^3y^2}{2a^2} + \frac{f^3}{2}\right)}}.$$

II. [p. 274] If $c = \frac{f^3}{2h^2}$, then $dt = -\frac{hdy\sqrt{c}}{y^2\sqrt{\left(a^2-h^2\right)}}$ and thus $t = C + \frac{ah}{y\sqrt{\left(a^2-h^2\right)}}$

Truly the constant C is equal to $-\frac{h}{y\sqrt{\left(a^2-h^2\right)}}$, which with t vanishing makes $y = a$. But the tangent of the angle at $A = \theta$, and $t = \theta(\frac{a}{y} - 1)$ or $y = \frac{\theta\alpha}{t+\theta}$.

Hence from the given angle ACM the line CM is found, and thus the point M sought on the curve. Moreover this curve is the hyperbolic spiral, that the Cel. Joh. Bernoulli found to his satisfaction for the same cause in Act. Lips. 1713

[*De motu corporum gravium, pendulorum et proiectilium,* Acta erud. 1713, p. 77; *Opera omnia* Tom. I, Lausannae et Genevae 1742, p.548, 552 – 553. P. St.]

If c is neither equal to $\frac{f^3}{2a^2}$ nor $\frac{f^3}{2h^2}$, put $c = \frac{\delta f^3}{2a^2}$; then

$$dt = \frac{-hdy\sqrt{\delta}}{y\sqrt{\left((\delta-1)y^2 + a^2 - \delta h^2\right)}}.$$

Put $y = \frac{1}{q}$, then

$$dt = \frac{hdq\sqrt{\delta}}{\sqrt{\left(\delta - 1 + (a^2 - \delta h^2)q^2\right)}}.$$

Hence two cases of the first kind arise.

III. If $a^2 - \delta h^2$ is a positive number, with the integration depending on logarithms. For

$$t = \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{C}{\sqrt{(\delta - 1 + (a^2-\delta h^2)q^2)} - q\sqrt{(a^2-\delta h^2)}}$$
$$= \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{Cy}{\sqrt{((\delta-1)y^2 + a^2 - \delta h^2)} - \sqrt{(a^2-\delta h^2)}}.$$

And with the constant C determined in this way :

$$t = \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{y\sqrt{\delta(a^2-h^2)} - y\sqrt{(a^2-\delta h^2)}}{a\sqrt{((\delta-1)y^2+a^2-\delta h^2)} - a\sqrt{(a^2-\delta h^2)}}$$

and

$$y = \frac{2ae^{\frac{t\sqrt{(a^2-\delta h^2)}}{h\sqrt{\delta}}}\left(\sqrt{\delta(a^2-h^2)(a^2-\delta h^2)} - a^2 + \delta h^2\right)}{e^{\frac{2t\sqrt{(a^2-\delta h^2)}}{h\sqrt{\delta}}}a^2(\delta-1) - \left(\sqrt{\delta(a^2-h^2)} - \sqrt{(a^2-\delta h^2)}\right)^2}.$$

IV. If $\delta > \frac{a^2}{h^2}$, then $dt = \frac{hdq\sqrt{\delta}}{\sqrt{(\delta-1-(\delta h^2-a^2)q^2)}}$. Truly for the integral of this the part $\frac{h\sqrt{\delta}}{\sqrt{(\delta h^2-a^2)}}$ is taken on the arc of a circle, the sine of which is $q\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$, [p. 275] with the radius equal to 1. And with the addition of a suitable constant, it becomes :

$$\frac{t\sqrt{(\delta h^2-a^2)}}{h\sqrt{\delta}}$$

which is equal to the arc whose sine is $\frac{1}{y}\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$ – the arc whose sine is $\frac{1}{a}\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$
which is equal to the arc whose sine is $\frac{D}{ay}\left(\sqrt{(a^2-D^2)} - \sqrt{(y^2-D^2)}\right)$,
with $\frac{\sqrt{\delta h^2-a^2}}{\delta-1} = D$. Hence the construction of the curve readily flows : for the arc *GL* is taken, (Fig. 59), which has the ratio to GN as $\sqrt{(\delta h^2-a^2)}$ to $h\sqrt{\delta}$. The sine of this arc *LR* is put equal to *R*. Therefore $Ray = D\sqrt{(a^2-D^2)} - D\sqrt{(y^2-D^2)}$ and

$$y = \frac{aDR\sqrt{(a^2-D^2)} - aD^2\sqrt{(1-R^2)}}{a^2R^2-D^2}.$$

Hence both the construction and the equation of the curve sought have been deduced. In these four cases everything pertaining to the problem are contained. Q.E.I.

Corollary 1.

672. The distance determining the speed is found to equal $\frac{af\sqrt{f}}{\sqrt{(f^3-2a^2c)}}$. Whereby if $c = \frac{f^3}{2a^2}$, in which case the body is moving in a logarithmic spiral, and the distance determining the speeds is infinite.

Corollary 2.

673. Because, if the body is moving along a logarithmic spiral, then $p=\frac{hy}{a}$ and $v=\frac{f^3}{2y^2}$, then the time in which the arc AM is completed, is equal to $\frac{a^3-ay^2}{f\sqrt{2f(a^2-h^2)}}$.

Corollary 3.

674. If this time is put as T and the cosine of the angle $CMT = i$, then
$y=\sqrt{(a^2-fiT\sqrt{2f})}$.
From which equation, after some given time, the distance of the body from the centre can be found. [p. 276]

Corollary 4.

675. Therefore the body arrives at the centre C in the time $T=\frac{a^2}{fi\sqrt{2f}}$, while meanwhile making an infinite number of revolutions around C.

Corollary 5.

676. If T is taken greater than $\frac{a^2}{fi\sqrt{2f}}$, then y becomes imaginary. From which it follows, after the body has arrived at *C*, to be found nowhere greater, but as if annihilated.

Corollary 6.

677. If $\delta=\frac{a^2}{h^2}$ or $c=\frac{f^3}{2h^2}$, the body moves in a hyperbolic spiral and even after performing an infinite number of revolutions around C, it reaches the centre C in some finite time. For the time in which the arc *AM* is traversed is equal to $\frac{2\theta\alpha(a-y)}{f\sqrt{2f}}$, and thus the time, in which the body reaches C, is equal to $\frac{2\theta a^2}{f\sqrt{2f}}$.

Corollary 7.

678. If $\delta<\frac{a^2}{h^2}$, or $c<\frac{f^3}{2h^2}$, which is the third case, the body is also moving along spiral lines, and at last after an infinite number of circuits it arrives at the centre. This is apparent from the equation; for by making $t=\infty$ at last $y=0$.

Scholium 1.

679. When $c=\frac{f^3}{2h^2}$, the hyperbolic spiral that the body describes has this property, [p. 277] as by putting $t=-\theta$ makes $y=\infty$. Therefore that radius drawn from the centre C is seen to be the asymptote of the curve as it approached from infinity. But the curve continually recedes from that, but yet the distance never goes beyond the given interval θa. Therefore the right line parallel to this radius for each θa distance is the true asymptote. Besides it is a property of this conspicuous curve, that equal arcs can be cut from the curve by an infinite number of concentric circles, from which more can be elucidated about the nature and properties of the curve. Likewise too, the curves have this property which are described if $\delta < \frac{a^2}{h^2}$, likewise truly $\delta > 1$. For $y = \infty$, if t is taken, given by :

$$t = -\frac{h\sqrt{\delta}}{\sqrt{(a^2 - \delta h^2)}}\, l \frac{a\sqrt{(\delta - 1)}}{\sqrt{\delta(a^2 - h^2)} - \sqrt{(a^2 - \delta h^2)}}.$$

From which it is evident, if it is the case that $\delta < 1$, that the place in which $y = \infty$ is imaginary. But if $\delta = 1$, then $t = -\infty$; for the curve that arises in this case is the logarithmic spiral. Moreover all these curves are thus of comparable shapes, as they are all concave everywhere towards C; and indeed never from the nature of the motion are they able to have a point of flexion or of turning back.

Corollary 8.

680. If $\delta > \frac{a^2}{h^2}$, or $c > \frac{f^3}{2h^2}$, the curve described by the body is no longer that of a spiral, but an algebraic curve, if indeed these can also be referred to among the algebraic equations, as there are irrational exponents contained in their equations.

Scholium 2.

681. Therefore in order that these can be found, it is necessary to put $h = a$ (640). With this accomplished this equation is put in place : [p. 278]

$$dt = \frac{adq\sqrt{\frac{\delta}{\delta - 1}}}{\sqrt{(1 - a^2q^2)}}.$$

Or by putting $aq = r$ and dt replaced by the value $\frac{-du}{\sqrt{(1-u^2)}}$, we have

$$\frac{du}{\sqrt{(1 - u^2)}} = \frac{-dr\sqrt{\frac{\delta}{\delta - 1}}}{\sqrt{(1 - r^2)}}.$$

By comparing this form with $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ (605) it follows that $Z = -r, A = 1, \lambda = \sqrt{\frac{\delta}{\delta - 1}}$.

We therefore have besides this equation: $u =$

$$\frac{(\sqrt{(1-C^2)}-C\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}(\sqrt{(1-r^2)}-r\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}-(\sqrt{(1-C^2)}+C\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}(\sqrt{(1-r^2)}+r\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}}{2\sqrt{-1}}$$

or

$$2r\sqrt{-1} = \left(\sqrt{(1-u^2)} - u\sqrt{-1}\right)^{\sqrt{\frac{\delta-1}{\delta}}}\left(\sqrt{(1-C^2)} - C\sqrt{-1}\right)$$
$$-\left(\sqrt{(1-u^2)} + u\sqrt{-1}\right)^{\sqrt{\frac{\delta-1}{\delta}}}\left(\sqrt{(1-C^2)} + C\sqrt{-1}\right)$$

(605 et 607). Truly the constant C can be determined from this, since by making $u = 1$, y becomes equal to a, or $q = \frac{1}{a}$ or $r = 1$. Therefore with C determined, the equation becomes :

$$2r = \left(u + \sqrt{(u^2-1)}\right)^{\sqrt{\frac{\delta-1}{\delta}}} + \left(u - \sqrt{(u^2-1)}\right)^{\sqrt{\frac{\delta-1}{\delta}}} = \frac{2a}{y}$$

or

$$u = \frac{1}{2y^{\sqrt{\frac{\delta}{\delta-1}}}}\left(\left(a + \sqrt{(a^2-y^2)}\right)^{\sqrt{\frac{\delta}{\delta-1}}} + \left(a - \sqrt{(a^2-y^2)}\right)^{\sqrt{\frac{\delta}{\delta-1}}}\right) = \frac{x}{y}.$$

Which is the equation of the curve sought.

Example 1.

682. Let $\sqrt{\frac{\delta}{\delta-1}} = 2$ or $\delta = \frac{4}{3}$ and $c = \frac{2f^3}{3a^2}$. Then

$$2xy = \left(a + \sqrt{(a^2-y^2)}\right)^2 + \left(a - \sqrt{(a^2-y^2)}\right)^2 = 4a^2 - 2y^2.$$

With the line $PM = z$ put in place, the equation becomes

$$x\sqrt{(x^2+z^2)} = 2a^2 - x^2 - z^2$$

and by taking the squares :

$$0 = 4a^4 - 4a^2x^2 - 4a^2z^2 + x^2z^2 + z^4$$

or

$$x = \frac{2a^2 - z^2}{\sqrt{(4a^2 - z^2)}}.$$

[p. 279] This curve therefore progresses according to the shape of a parabola, but has the asymptote parallel to the line *AC* and situated apart from that in the interval $2a$, that it never reaches.

Example 2.

683. Let $\sqrt{\frac{\delta}{\delta-1}} = 3$ or $\delta = \frac{9}{8}$ and $c = \frac{9f^3}{16a^2}$. Therefore there arises :

$$2xy^2 = \left(a + \sqrt{(a^2-y^2)}\right)^3 + \left(a - \sqrt{(a^2-y^2)}\right)^3 = 8a^3 - 6ay^2.$$

And with the applied line $PM = z$ drawn, we have :

$$x^3 = 4a^3 - xz^2 - 3ax^2 - 3az^2.$$

This curve of the third order belongs to example 41 in the enumeration of cubic curves made by Newton. [*Enumeratio linearum tertii ordinis*, Londini 1704; J. NEWTONI *Opuscula mathematica etc.* Lausannae et Genevae 1744, Tom. I, p. 259. P. St.]

Scholium 3.

684. Innumerable other algebraic curves can be found, which are described by a body projected according to this hypothesis, if it is considered that $\delta = \frac{m^2}{m^2-1}$, with m denoting some rational number greater than one, least δ becomes negative. Moreover these three hypotheses handled, in which the centripetal force has been in proportion in the first case to the distance, in the second case to the inverse square of the distance, and in the third case to the inverse cube of the distance, are the only ones that lead either to algebraic curves or to curves depending on circular or hyperbolic quadrature; if some power of y is substituted in place of P. But nevertheless with other powers of y put in place of P the equation for the curve described is not able to be free from irrationality and on this account neither algebraic nor curves depending on circular or hyperbolic quadrature are possible; yet there are special cases given, in which the curve sought is algebraic. [p. 280] In as much as a straight line and circle are able to satisfy all the hypothesis, thus also as above whenever these other algebraic curves are found. Moreover, it is required to show how these too can be found, as we indicate in the following proposition.

PROPOSITION 83.

PROBLEM.

685. *With a centripetal force present as some power of the distances, to find the special cases in which a body can be projected in certain ways in order that is moves along a line algebraically.*

SOLUTION.

With everything kept as up until now, this equation for the curve sought has been found : $\frac{du}{\sqrt{(1-u^2)}} = \frac{hdy\sqrt{c}}{y\sqrt{(cy^2-ch^2-y^2Y)}}$ (601). Moreover, in our case the force is given by : $P = \left(\frac{y}{f}\right)^n$, with f denoting the distance at which the centripetal force is equal to the force of gravity. Hence we have $Y = \int Pdy = \frac{y^{n+1}-a^{n+1}}{(n+1)f^n}$. Further, because we require an algebraic curve, we put $h = a$ (640). With these quantities substituted, there is :
[p. 281]

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{cf^n(n+1)}}{y\sqrt{((n+1)cf^n(y^2-a^2)-y^2(y^{n+1}-a^{n+1}))}}.$$

The [starting] speeds are determined by the distance k, then we have $c = \frac{k^{n+1}-a^{n+1}}{(n+1)f^n}$.

Therefore with this equation introduced in place of c :

$$\frac{du}{\sqrt{(1-u^2)}}=\frac{ady\sqrt{(k^{n+1}-a^{n+1})}}{y\sqrt{(a^{n+3}-a^2k^{n+1}+k^{n+1}y^2-y^{n+3})}}.$$

Hence it is apparent that if $k = a$, by making $c = 0$, to become u = const. = 1 and likewise $x = y$, which is the case in which the body has descended to the centre. If k is infinite, then c also is infinite, as long as $n + 1$ is a positive number. Therefore in this case as well, the body must be progressing along a straight line, since a finite force acting on a body with an infinite speed is not strong enough to change the direction of the motion.
[Thus, positive values of n lead to straight line motions to the centre.]
Therefore otherwise we put $n + 1$ to be a negative number $= -m$ or $n = -m - 1$; and the equation becomes :

$$\frac{du}{\sqrt{(1-u^2)}}=\frac{y^{\frac{m-4}{2}}dy\sqrt{(k^m-a^m)}}{\sqrt{(a^{m-2}k^m+a^my^{m-2}-k^my^{m-2}-a^{m-2}y^m)}}$$

With k made indefinitely large [i. e. the place where $h = a$], this becomes :

$$\frac{du}{\sqrt{(1-u^2)}}=\frac{y^{\frac{m-4}{2}}dy}{\sqrt{(a^{m-2}-y^{m-2})}}$$

with the remainder of the terms vanishing ; and on putting $y^{m-2}=q^2$, then $y=q^{\frac{2}{m-2}}$.
With this substitution made, there arises the following equation :

$$\frac{du}{\sqrt{(1-u^2)}}=\frac{2dq}{(m-2)\sqrt{(a^{m-2}-q^2)}}.$$

When this formula is compared with the general case $\frac{\lambda dZ}{\sqrt{(A^2-Z^2)}}$ (604), then

$\lambda=\frac{m}{m-2}$, $A=a^{\frac{m-2}{2}}$ and $Z=q=y^{\frac{m-2}{2}}$.
Therefore, this formula arises for the algebraic equation sought (607) :

$$\begin{aligned}2y^{\frac{m-2}{2}}\sqrt{-1}=&\left(\sqrt{(1-u^2)}+u\sqrt{-1}\right)^{\frac{m-2}{2}}\left(\sqrt{(a^{m-2}-C^2)}+C\sqrt{-1}\right)\\&-\left(\sqrt{(1-u^2)}-u\sqrt{-1}\right)^{\frac{m-2}{2}}\left(\sqrt{(a^{m-2}-C^2)}-C\sqrt{-1}\right)\end{aligned}$$

[p. 282] The constant C is determined from this equation, since by making $u = 1$ then y must become equal to a. [The initial normality condition.]
On account of this, we have :

$$2y^{\frac{m-2}{2}}=a^{\frac{m-2}{2}}\left(\left(u-\sqrt{(u^2-1)}\right)^{\frac{m-2}{2}}+\left(u+\sqrt{(u^2-1)}\right)^{\frac{m-2}{2}}\right).$$

Therefore, provided m is a positive rational number, an algebraic curve is found that the body projected normally describes, with the speeds arising determined from an infinite [initial] distance. Q.E.I.

Corollary 1.

686. By taking squares we have :

$$4y^{m-2} - 2a^{m-2} = a^{m-2}\left((u+\sqrt{(u^2-1)})^{m-2} + (u-\sqrt{(u^2-1)})^{m-2}\right)$$

or since $n = -m - 1$, this equation

$$4y^{-n-3} - 2a^{-n-3} = a^{-n-3}\left((u+\sqrt{(u^2-1)})^{-n-3} + (u-\sqrt{(u^2-1)})^{-n-3}\right).$$

Corollary 2.

687. It is required that m is a positive number greater than zero. For if m should be equal to 0 or a negative number, the speed of the body becomes infinite and the curve therefore becomes a straight line.

Corollary 3.

688. If $n = -2$, we have this equation: $\frac{4a}{y} - 2 = 2u = \frac{2x}{y}$ or $2a = x + y$. Which with the vertical coordinate PM, or $\sqrt{(y^2 - x^2)}, = z$ becomes equal to this equation: $z^2 = 4a^2 - 4ax$ for a parabola with the centre of force at the position of the focus, as we have thus found. (647). [p. 283]

Corollary 4.

689. If $n = -3$ or $m = 2$, which is the case handled in the preceding proposition, then the equation becomes : $2y^{\frac{m-2}{2}} = 2a^{\frac{m-2}{2}}$ or $y = a$. Therefore the curve described by the body is a circle, at the centre of which lies the centre of force.

Scholium 1.

690. If n is an odd number, then m is even and likewise $\frac{m-2}{2}$ is an integer. Therefore for these cases, the formula found for the solution of the problem is convenient to use. But if n is an even number, the formula (686) produced by taking the squares should be used. Indeed in each case a rational equation between x and y is immediately arrived at, as becomes apparent in the following examples.

Example 1.

691. The centre attracts in the inverse ratio of the fourth power of the distances, and the body is projected normally with the initial speeds arising determined at infinity ; the curve that the body describes is the following algebraic one. By putting $n = -4$, then

$4y-2a=2au=\frac{2ax}{y}$ or $2y^2=a(y+x)$. Truly with the coordinate put in place, $z=\sqrt{(y^2-x^2)}$ it becomes :

$$2x^2+2z^2=ax+a\sqrt{(x^2+z^2)},$$

which is the equation for a fourth order curve.

Example 2.

692. Let the centripetal force vary inversely as the fifth power of the distances, or $n=-5$, and $m=4$. [p. 284] Therefore we have $2y=2au=\frac{2ax}{y}$ and

$$yy=ax=x^2+z^2,$$

which is the equation for a circle with the centre of force put on the periphery of this circle. Newton discusses this case in the *Princ.* Book I. Prop. VII. [Note that here a is the diameter of the circle.]

Example 3.

693. Let $n=-7$, or $m=6$, then $2y^2=4a^2u^2-2a^2$ or $y^4=2a^2x^2-a^2y^2$. With the coordinate $PM=z$ in place, this equation arises :

$$(x^2+z^2)^2=a^2x^2-a^2z^2$$

for a curve of order four, in which the centre of the force also lies on the periphery of the curve.

Scholium 2.

694. With regard to the figures that bodies describe under the action of given forces, it is not worth the effort to add more here, as in Physics and Astronomy, the hypotheses of centripetal forces other than those in proportion to the inverse square of the distances have no use. Yet in Astronomy, when a body must be considered to be acted on by several forces of this kind, of which one exercises a maximum influence on the body over the others, these extra forces, as the problem demands, do not have to be introduced into the calculation, as they only augment or diminish a little, that by which even the approximate motion of the body is known. Therefore in these cases the curve the body describes does not disagree much with an ellipse. For this reason, Astronomies usually consider the curve to be in the form of an ellipse, however one which is not fixed, but with movement, so that thus they consider the body revolving in an ellipse about a focus which in turn is moving [p. 285]. From which the motions of the orbits of the planets arise, where the lines of the apses continually move to another place. We ourselves, when we proceed closer to the truth, besides the mobility of the axis of the ellipse, will also consider the form of this motion as a variable. Therefore we rotate with this ellipse, so that we can determine any element of the curve that the body describes, which is part of the ellipse having the focus at the centre of force, from which position the nature of the ellipse can become known. Moreover all these ellipses have their focus in turn in the position of centre of force, by which the body is continually drawn.

PROPOSITION 84.

PROBLEM.

695. *If the centripetal force does not disagree much with the ratio of the inverse square with the distances, to determine the motion of the ellipse, and the continual change of the form of the ellipse, and the change of the motion of the body associated with this change of the ellipse.*

SOLUTION.

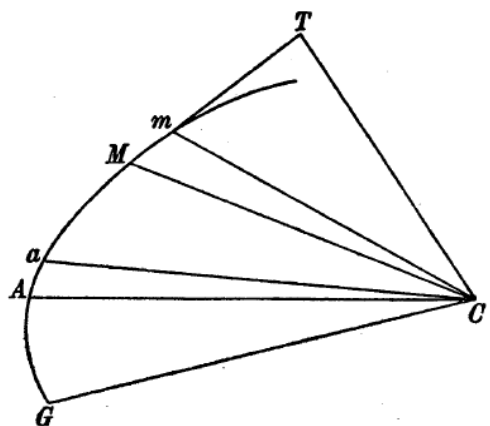

Fig. 60.

Let C be the centre of force (Fig. 60), and the body at M has a speed along the direction MT corresponding to the height v. The centripetal force acting at M is equal to P, the distance $CM = y$ and the sine of the angle $CMT = s$. Now it is evident, as the law holds, that P, can still always be put equal to this form: $\frac{f^2}{y^2}$, although f [p. 286] is not a constant quantity as before, but denotes a variable. Therefore we have $f^2 = Py^2$. Since moreover, while the body traverses the element Mm, f can remain constant, and the ellipse can be determined, the focus of which is C and Mm an element, in which the body moves, if f remains the same constant. Therefore with AC put as the transverse axis of this ellipse, the tangent of the angle $MCA = \frac{2svy\sqrt{(1-ss)}}{ff-2ssvy}$ (656), which therefore is equal to $\frac{2sv\sqrt{(1-ss)}}{Py-2s^2v}$. But the latus rectum is equal to $\frac{4s^2v}{P}$ and with the transverse axis $\frac{Py^2}{Py-v}$ (cit.) From the centre C the perpendicular CT is dropped to the tangent MT, which is put equal to p; and $s = \frac{p}{y}$. And where the perpendicular to the tangent is equal to h, that becomes with G in position $CG = a$, the speed of the body corresponding to the height c, from which we have $v = \frac{ch^2}{p^2}$ (589). Truly the tangent $MT = \sqrt{(y^2 - p^2)}$ for brevity is put equal to t. With these in place, the tangent of the angle is equal to $\frac{2ch^2t}{Py^3p-2ch^2p}$ and the latus rectum, truly the transverse chord, is equal to $\frac{Py^2p^2}{Pyp^2-ch^2}$. Truly besides, from the nature of the attraction, we have $Pdy = \frac{2ch^2dp}{p^3}$ (587).

From which equation p is to be determined in terms of y and thus the whole ellipse from only y, c and h. Now CG is the fixed line, with which the angle that it makes with the line CM, from the nature of the curve can be defined. Therefore from this angle, if the angle MCA is taken away, the inclination of the apse line AC in the fixed line CG is found. Moreover the motion of the body from the known speed, $v=\frac{ch^2}{p^2}$, can easily become known. Q.E.I. [p. 287]

Scholium 1.

696. The ellipse determined in this way deserves to be called the osculating curve of the ellipse from the likeness of the circles of osculation, by which the curvatures of lines are measured. Truly this consideration is not purely geometrical, for by finding the osculating ellipse, besides the nature of the curve, it is also necessary to know the speed of the body and the centripetal force.

Corollary 1.

697. If t vanishes, the line of the apses AC of the axis of the osculating ellipse falls on the radius CM on account of the angle MCA vanishing.

Corollary 2.

698. If we put $P=\frac{2ch^2}{y^3}$, the angle MCA is made right. Therefore in this case the centripetal force varies inversely as the cube of the distance. Whereby with $P=\frac{f^3}{y^3}$, then $c=\frac{f^3}{2h^2}$. Moreover the curve, which is then described by the body, is a hyperbolic spiral (679). Therefore for this line the apsidal line is always normal with the radius MC. But the latus rectum of the osculating ellipse is $2y=2MC$.

Corollary 3.

699. If we put $P=\frac{f^n}{y^n}$, then we have

$$\int Pdy=-\frac{f^n}{(n-1)y^{n-1}}=C-\frac{ch^2}{p^2}.$$

Since truly with $y=a$, we must have $p=h$, then $C=c-\frac{f^n}{(n-1)a^{n-1}}$. [p. 288] Hence the equation is found :

$$p^2=\frac{(n-1)ch^2a^{n-1}y^{n-1}}{(n-1)ca^{n-1}y^{n-1}+f^n(a^{n-1}-y^{n-1})}.$$

Hence the latus rectum of the oscillating ellipse is $\frac{4ch^2y^{n-2}}{f^n}$. The latus transversum and the angle MCA are also determined from these in terms of y.

Corollary 4.

700. If $P = \frac{ch^2}{yp^2}$, then the osculating ellipse is always a parabola. Moreover by substituting $p^2 = \frac{h^2 y}{a}$ in the equation for P in $Pdy = \frac{2ch^2 dp}{p^3}$ And consequently $P = \frac{ac}{yy}$. Moreover in this case the curve described is this parabola on account of $f^2 = ac$ (647)

Scholium 2.

701. The theory of osculating ellipses is not to be confused with the motion of bodies in moving orbits, concerning which Newton and others after him have worked on. [*Princ. phil. nat*., Book I. Section IX : De motu corporum in orbibus mobilibus, deque motu apsidum. P. St.] For here we have determined that some part of an ellipse is some portion of the curve described by a body. But, when the talk is about moving orbits (729), the centripetal force bringing about the motion is being investigated, as the body rotating about a given centre of force is moved.

PROPOSITION 85.

THEOREM.

702. *If a body projected in some way is attracted to several centres of force A, B, C* (Fig. 61), *of which the individual forces are proportional to the distances of the body from these,* [p. 289] *then the body moves in the same manner, as if it is attracted equally by the common centre of gravity O of the points A, B, C in the simple ratio of the distances.*

DEMONSTRATION.

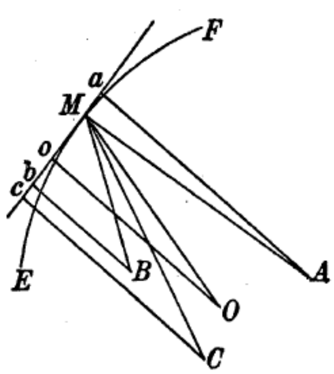

Fig. 61.

With the forces of the centres put at A, B, and C, and with the forces which they exercise at unit distance put as α, β, γ respectively, let *ac* be the direction of the motion that the body has at M, and thus the tangent of the curve EMF described at M. But O is the centre of gravity of the bodies α, β, γ put at the points A, B, C, and at O it is understood that the force varies directly with the distance, which at unit distance attracts with a force equal to $\alpha + \beta + \gamma$. With these in place, a body at M is attracted to A by the force $AM.\alpha$, at B by the force $BM.\beta$ and at C by the force $CM.\gamma$. Moreover with these forces acting together, it is requires to show that a force equal to $OM.(\alpha + \beta + \gamma)$ is attracting the body towards O. In order that this can be shown, perpendiculars *Aa, Bb, Cc* and *Oo* are sent from the points *A, B, C*, and *O* to the tangent *ac*. In this manner any force can be resolved into normal and tangential components, and the sum of the normals arising from the attractions towards A, B,C is equal to :

$$\alpha.Aa+\beta.Bb+\gamma.Cc,$$

and the sum of the tangents is equal to :

$$-\alpha.Ma+\beta.Mb+\gamma.Mc.$$

But since O is the centre of gravity of the bodies α, β, γ situated at A, B, C, it has been proven from statics that :

$$\alpha \cdot Aa+\beta \cdot Bb+\gamma \cdot Cc=(\alpha+\beta+\gamma) \cdot Oo$$

and

$$-\alpha \cdot Ma+\beta \cdot Mb+\gamma \cdot Mc=(\alpha+\beta+\gamma) \cdot Mo.$$

From which is evident for the forces acting together that $\alpha.AM, \beta.BM, \gamma.CM$ is equivalent to the force $(\alpha+\beta+\gamma).OM$. Q.E.D. [p. 290]

Corollary 4.

703. Therefore the body according to this hypothesis describes an ellipse, the centre of which is placed in the centre of gravity itself O (631). For all the forces have the same effect, that a single force placed at O and attracting in the direct ratio of the distances.

Corollary 5.

704. It is also evident, however many there are centres of this kind of force attracting in the ratio of the distances, the body still always moves in an ellipse, and as if it is entirely attracted by a single force at the common centre of gravity.

Scholium 1.

705. The demonstration again succeeds in an equal manner, if some number of centres of force are not put in the same plane, as is evident from the principles of statics. Hence it is understood that the body is nevertheless moving in the same place, even if the centres of force are scattered in the most diverse of planes.

Scholium 2.

706. If the centres of forces are attracting in some other ratio besides the simple ratio of the distances, a reduction of this kind to a single central position of the forces cannot be had in a straightforward manner, and I can hardly calculate the motion of the body, nor indeed can hardly anything be determined about the motion. [p. 291] Therefore in these cases it is necessary to flee to approximations, which are set up in different ways according to the various conditions. And on this account, Newton was unable to determine the motion of the moon, which arises from two attractions, but truly this is by far the most outstanding nearest attempt. Moreover, concerning this it is necessary to give this problem the most singular consideration, and the inverse method has to be called upon, where the body is receding from a known curve that it describes, under the influence of attracting forces. On this account, we will explain in the following what aids can be put in place, when we are to investigate the force acting as the unknown in the inverse order. Therefore as we are progressing through this discussion, which can be established in two ways. Firstly, for besides the curve described is taken as known we take the direction of the forces acting at individual points, and from these quantities the forces acting, and the motion of the body itself is found. In the other way, by considering the curve and the motion of the body on that curve is taken as given, from which it is required to extract the force acting.

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[p. 262]

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PROPOSITIO 79.

PROBLEMA.

641. *Attrahente centro virium in ratione distantiarum proiiciatur corpus in M celeritate quacunque et secundum quamcunque directionem MT; determinari oportet ellipsin, in qua corpus movebitur.*

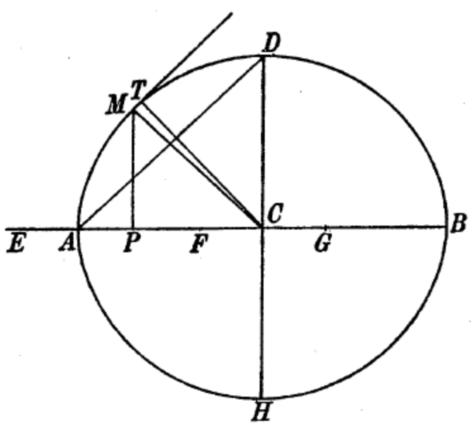

Fig. 56.

SOLUTIO.

Solutionem huius problematis esse possibilem ex hoc intelligi potest, quod numerus datorum non sit nimis magnus ad ellipsin determinandam, prout apparebit. Ponatur radius $CM = y$, sinus anguli $CMT = s$, posito sinu toto $= 1$, et altitudo celeritati in M debita $= v$, distantia praeterea a centro C, in qua vis centripeta aequalis est vi gravitatis, maneat $= f$. Hae igitur quantitates tanquam datae et cognitae erunt considerandae; incognitae vero et inveniendae sunt ellipsis axes eorumque positio; [p. 263] horum ponatur $AC = a$ et $CD = b$, eritque $b = \sqrt{2cf}$ (631).

His positis habebitur statim $2fv = a^2 + b^2 - y^2$ (632), unde fit

$$a^2 + b^2 = AD^2 = 2fv + y^2;$$

cogniscitur ergo iam subtensa AD huicque aequalis distantia CE celeritates determinans. Deinde quoque habetur (635)

$$s = \frac{ab}{\sqrt{(a^2+b^2-y^2)}} = \frac{ab}{y\sqrt{2fv}},$$

et qui oritur

$$ab = sy\sqrt{2fv}\ .$$

His coniunctis oritur

$$(a^2 - b^2)^2 = 4f^2v^2 + 4fvy^2 + y^4 - 8fs^2vy^2 = 4f^2v^2 + y^4 + 4fvy^2(1 - 2s^2).$$

Est vero $1 - 2s^2$ cosinus dupli anguli *CMT*, quem vocemus *i*. Erit ergo

$$a^2 - b^2 = \sqrt{(4f^2v^2 + y^4 + 4fviy^2)}\ .$$

Ex his aequationibus provenit

$$a^2 = fv + \tfrac{1}{2}y^2 + \sqrt{(f^2v^2 + \tfrac{1}{4}y^4 + fviy^2)} \text{ et}$$

$$b^2 = fv + \tfrac{1}{2}y^2 - \sqrt{(f^2v^2 + \tfrac{1}{4}y^4 + fviy^2)}\ .$$

Inventis vero axibus posito eorum facile habebitur. Est enim consinus angulum ACM

$$= \frac{x}{y} = \frac{1}{y}\sqrt{\left(\frac{a^2y^2 - a^2b^2}{a^2 - b^2}\right)} = \sqrt{\left(\frac{1}{2} + \frac{fiv + \frac{1}{2}y^2}{\sqrt{(4f^2v^2 + y^4 + 4fviy^2)}}\right)}$$

seu cosinus dupli anguli *ACM*

$$\frac{2fiv + y^2}{\sqrt{(4f^2v^2 + y^4 + 4fviy^2)}}\ .$$

atque sinus huius anguli 2ACM

$$= \frac{2fv\sqrt{(1 - i^2)}}{\sqrt{(4f^2v^2 + y^4 + 4fviy^2)}}\ .$$

Q.E.I.

Corollarium.

642. Ubicunque igitur corpus proiiciatur et quacunque celeritate atque secundum quamvis directionem, corpuus circa centrum virium in perimetro ellipsis movebitur, cuius centrum est in ipso centro virium.

Scholion

643. Hactenus vis centralis distantiis proportionalis attrahens est posita; simili autem modo reperientur curvae descriptae, [p. 264] si vis corpora a centro in eadem ratione repellat. Praecedentia enim omnia ad hunc casum accommodantur, si modo $-f$ loco f scribatur. Hoc vero facto curva, quae ante erat ellipsis, transmutatur in hyperbolam, axe minore imaginario facto, centrumque virium manet in centro hyperbolae.

PROPOSITIO 80.

PROBLEMA.

644. *Si vis centripeta fuerit quadratis distantiarum reciproce proportionalis corpusque in A* (Fig. 57) *data celeritate proiiciatur in directione ad radium AC normali, oportet determinari curvam AMDBHA, quam corpus describet, et motum ipsum per illam.* .

SOLUTIO.

Positis ut ante $AC = a$ et celeritate in A debita altitudii c, sit f distantia, in qua vis centripeta aequalis est gravitati. Ponatur tum $CM = y$, perpendiculum, quod in tangentem demittitur, $CT = p$ et altitudo celeritati in M debita = v. His cum propositione 75 (601) comparatis erit

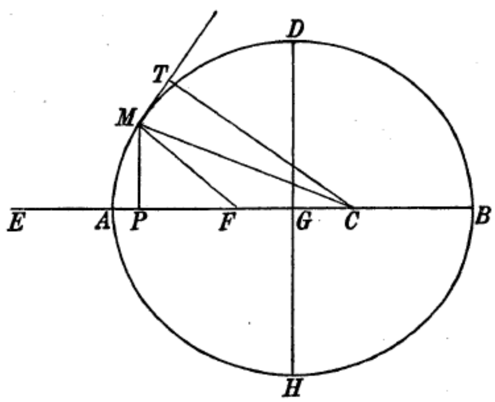

Fig. 57.

$$h = a, \quad P = \frac{f^2}{y^2} \text{ et } Y = \frac{f^2}{a} - \frac{f^2}{y}.$$

Demittatur ex M in AC perpendiculum MP et vocetur $CP = x$ ponaturque $x = uy$. Quo facto erit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{c}}{y\sqrt{(cy^2 - a^2c - \frac{f^2y^2}{a} + f^2y)}}$$

Ponatur

$$y = \frac{1}{\frac{f^2}{2a^2c} - q},$$

eritque

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{dq}{\sqrt{((\frac{2ac-f^2}{2a^2c})^2 - q^2)}}.$$

Conferatur haec formula cum $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ (604), eritque

$\lambda = 1, \quad A = \frac{2ac - f^2}{2a^2c}$ et $Z = q$.

Quamobrem habebitur sequens aequatio

$$u = \frac{q\sqrt{(A^2 - C^2)} - C\sqrt{(A^2 - q^2)}}{A^2}$$

(609). [p. 265] Est vero $q = \frac{f^2}{2a^2c} - \frac{1}{y}$, et constans quantitas C ex eo debet definiri, quod facto $u = 1$ fiat $y = a$. Aequatio vero illa parum mutata in hanc transit

$$C + u\sqrt{(A^2 - q^2)} = q\sqrt{(1 - u^2)}.$$

In qua facto $u = 1$ et $q = \frac{f^2 - 2ac}{2a^2c}$ prodit C = 0. Consequenter habebitur

$$\left(\frac{f^2-2ac}{2a^2c}\right)u^2 = q^2,$$

seu

$$(f^2-2ac)u = 2a^2cq = f^2 - \frac{2a^2c}{y}.$$

Quia vero est $u = \frac{x}{y}$, erit $(f^2-2ac)x = f^2 - 2a^2c$ atque

$$f^4y^2 = 4a^4c^2 + 4a^2cx(f^2-2ac) + (f^2-2ac)^2x^2.$$

Vocetur applicata $PM = z$, erit

$$f^4z^2 = 4a^4c^2 + 4a^2cx(f^2-2ac) - 4acf^2x^2 + 4a^2c^2x^2.$$

Ponatur $x = t + \frac{2a^2c-af^2}{2f^2-2ac}$, quo facto habebitur

$$f^4z^2 = \frac{a^3cf^4}{f^2-ac} - 4act^2(f^2-ac).$$

Quae est aequatio ad ellipsin abscissis in axe transverso a centro G sumtis, cuius axis transversus est $= \frac{af^2}{f^2-ac}$ et coniugates $= \frac{2a\sqrt{ac}}{\sqrt{(f^2-ac)}}$. Centrique virium C a centro ellipsis G distantia $CG = \frac{2a^2c-af^2}{2f^2-2ac}$, quae aequalis est semidistantiae focorum. Hanc ob rem centrum virium C in alterutro foco ellipsis est positum.

Apparet igitur corpus in A celeritate $\sqrt{c}$ proiectum circa centrum C descripturum esse ellipsin *AMDBHA*, cuius focus alter in C est situs.
Eritque

$AC = a,\quad AG = \frac{af^2}{2f^2-2ac},\quad AB = \frac{af^2}{f^2-ac},$ BC $= \frac{a^2c}{f^2-ac}$ axisque coniungatus $DH = \frac{2a\sqrt{ac}}{\sqrt{(f^2-ac)}}$, latus vero rectum ellipsis est $= \frac{4a^2c}{f^2}$.

Ad celeritatem in M determinandam usu erit aequatio $v = \frac{a^2c}{p^2}$ (587). Est autem

$$p = \frac{a\sqrt{c}}{\sqrt{(c-Y)}} = \frac{a\sqrt{acy}}{\sqrt{(acy+(a-y)f^2)}},$$

ergo

$$v = \frac{acy+(a-y)f^2}{ay}.$$

Q.E.I. [p. 266]

Corollarium 1.

645. Distantia celeritas determinans *CE* invenitur $= \frac{af^2}{2f^2-2ac}$ (278). Ex quo apparet distantiam hanc CE aequalem esse axi transverso ellipsis *AB*.

Corollarium 2.

646. Toties autem curva a corpore descripta est re vera ellipsis, quoties est $f^2 > ac$ seu $c < \frac{f^2}{a}$. At si est $c > \frac{f^2}{a}$, tum axis transversus fit negativus et coniugatus imaginarius, id quod indicio est curvam his casibus abire in hyperbolam.

Corollarium 3.

647. Quando vero est $c = \frac{f^2}{a}$, curva medium tenebit inter hyperbolam et ellipsin axemque transversum habebit infinitum. Erit itaque tum curva a corpore descripta parabola.

Corollarium 4.

648. Si est $c = \frac{f^2}{2a}$, ellipsis abit in circulum; aequales enim evadunt axes, transversus et coniugatus. Centrum vero virium in ipsum centrum circuli incidet.

Corollarium 5.

649. Ducatur in ellipsi ex altero foco recta *FM* ad *M*, erit $FM = AB - y = \frac{f^2(a-y)+acy}{f^2-ac}$. Ex quo cognoscitur fore

$$v = \frac{(f^2-ac).FM}{a.CM}.$$

Est ergo celeritas in puncto quovis *M* ut $\sqrt{\frac{FM}{CM}}$. [p. 267]

Corollarium 6.

650. Sinus anguli, quem radius *CM* cum tangente *MT* constituit, est

$$= \frac{p}{y} = \frac{a\sqrt{ac}}{\sqrt{(acy^2+(a-y)f^2y)}},$$

posito sinu toto = 1. Sed quia est $FM = \frac{f^2(a-y)+acy}{f^2-ac}$, erit sinus anguli *CMT*

$$= \frac{a\sqrt{ac}}{\sqrt{(f^2-ac)FM.CM}}.$$

Corollarium 7.

651. Quia est $GC = \frac{2a^2c - af^2}{2f^2 - 2ac}$, centrum virium erit in remotiore foco a vertice A, quoties est $c > \frac{f^2}{a}$. In propiorem vero incidet, si $c < \frac{f^2}{2a}$.

Corollarium 8.

652. Posito axe maiore $AB = E$, parametro = L, erit $E = \frac{af^2}{f^2 - ac}$ et $L = \frac{4a^2c}{f^2}$.
Hinc reperitur

$$a = \frac{E \pm \sqrt{(E^2 - EL)}}{2} \text{ atque } \sqrt{c} = \frac{f\sqrt{L}}{E \pm \sqrt{(E^2 - EL)}}.$$

Corollarium 9.

653. Tempus, quo corpus integram ellipsis circumferentiam percurrit, est

$$= \frac{2.\textit{Areae Ellipticae}}{a\sqrt{c}} = \frac{4.\textit{Area Ellipticae}}{f\sqrt{L}}$$

(588). Posita vero ratione diametri ad peripheriam $1 : \pi$, est area ellipticae $= \frac{\pi E\sqrt{EL}}{4}$.
Tempus ergo unius revolutionis est = $\frac{\pi E\sqrt{E}}{f}$.

Corollarium 10.

654. Si ergo plura corpora circa centrum virium C revolvantur attracta in reciproca duplicata ratione distantiarum, erunt tempora periodica inter se in sesquiplicata ratione axium transversorum ellipsium. [p. 268]

Corollarium 11.

655. Si celeritas initialis in A evanescit, abit ellipsis in lineam rectam AC. In hac igitur recta corpus perpetuo movebitur ab A ad C hincque subito rursus ad A revertetur, ita ut numquam ultra centrum C pertingat (272).

PROPOSITIO 81.

PROBLEMA.

656. *Posita vi centripeta distantiarum quadratis reciproce proportionali proiiciatur corpus ex M celeritate quacunque et iuxta directionem quamvis MT; ex quo oporteat determinari ellipsin MDBHAM, in qua corpus movebitur.*

SOLUTIO.

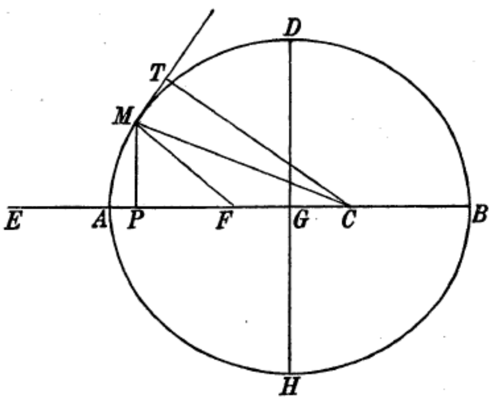

Fig. 57.

Sit $MC = y$, sinus anguli $CMT = s$ et celeritas in M debita altitudini v, distantia a centro, in qua vis centripeta aequalis est vi gravitatis, maneat $= f$. Haecque sunt quantitaties cognitae, ex quibus quaesitae, axis transversus AB, eius positio et latus rectum debent definiri. Sit axis transversus $= E$ et latus rectum $= L$, reliquaeque litterae a, c servent suas significationes ut in praecedente propositione. Erit iam ex natura ellipsis $FM = E - y$, ex quo fiet (649, 652)

$$v = \frac{(f^2 - ac)(E - y)}{ay} = \frac{f^2(E-y)(2E - L \pm 2\sqrt{(E^2 - EL)})}{y(E \pm \sqrt{(E^2 - EL)})^2} = \frac{f^2(E-y)}{Ey}.$$

Hinc reperitur

$$E = \frac{f^2 y}{f^2 - vy}.$$

Deinde est sinus anguli CMT

$$= s = \frac{a\sqrt{ac}}{\sqrt{(f^2 - ac)y(E-y)}}$$

(650). At est

$$f^2 - ac = \frac{f^2(E \pm \sqrt{(E^2 - EL)})}{2E} \quad \text{et} \quad a^3 c = \frac{1}{8} f^2 L(E \pm \sqrt{(E^2 - EL)})$$

(652) [p. 269] , ex $\frac{a\sqrt{ac}}{\sqrt{(f^2 - ac)}} = \frac{1}{2}\sqrt{EL}$ quo fit

hincque

$$s = \frac{\sqrt{EL}}{2\sqrt{(Ey - y^2)}}.$$

Erit igitur axis coniugatus

$$DH = \sqrt{EL} = 2s\sqrt{(Ey - y^2)} = \frac{2sy\sqrt{vy}}{\sqrt{(f^2 - vy)}}$$

et latus rectum

$$L = \frac{4s^2vy^2}{f^2}.$$

Ad positionem axis transversi inveniendam quaeratur cosinus anguli *MCP* seu

$$\frac{x}{y} = \frac{f^2y - 2a^2c}{y(f^2 - 2ac)} = \frac{2Ey - EL}{2y\sqrt{(E^2 - EL)}} = \frac{ff - 2ssvy}{\sqrt{(f^4 - 4ffssvy + 4s^2v^2y^2)}}.$$

Ex quo erit tangens ang. *MCP*

$$= \frac{2svy\sqrt{(1 - ss)}}{ff - 2ssvy}.$$

Q.E.I.

Corollarium 1.

657. Si est $f^2 > vy$ vel $v < \frac{f^2}{y}$, curva descripta semper erit ellipsis. At si $v > \frac{f^2}{y}$, curva erit hyperbola; fit enim axis transversus negativus. Sin autem $v = \frac{f^2}{y}$, curva erit parabola.

Corollarium 2.

658. Quia neque axis transversus neque latus rectum ullo casu fieri potest imaginarium, quomodocunque corpus proiiciatur, ideo semper in sectione conica movebitur, in cuius alterutro foco positum est centrum.

Scholion 1.

659. Postquam Keplerus ostendisset planetas in ellipsibus moveri, in quarum alterutro foco sol esset positus, atque tempora esse proportionalia areis, quas rectae ad solem ductae cum arcu descripto comprehenderent, Neutonus demonstravit vim planetas in orbitis continentem ad solem tendere [p. 270] atque esse quadratis distantiarum planetarum a sole reciproce proportionalem. Haec eadem veritas ex hisce duabus proportionibus consequitur; nam centro virium in ratione inversa duplicata distantiarum attrahente corpora in ellipsibus vel hyperbolis moveri debebunt, quarum alteruter focus in centrum virium incidit.

Corollarium 2.

666. [In editione principe loco numerorum 660 et qui sequuntur falso numeri 666 et qui sequuntur scripti sunt. Falsos paragraphorum numeros retinendos esse putavimus.] Secundum Neutonum vis attrahens solis est ad vim attrahentem terrae in aequalibus ab utriusque centris distantiis ut 227512 ad 1. Quare, cum vis attrahens terrae in distanta semidiametri suae a centro seu in superficie aequalis sit vi gravitatis 1, corpus a centro solis semidiametrum terrestrem distans ad id trahetur vi 227512 vicibus maiore quam gravitate. Ex quo concluditur, si corpus a centro solis distet 477 semidiametros terrae, vim, qua ad solem trahetur, fore aequalem vi gravitatis.

Corollarium 4.

667. Si igitur sol locum centri virium in reciproca duplicata ratione distantiarum attrahentis sustineat, pro f accipi debebit distantia 477 semidiametris telluris aequalis.

Scholion 2.

668. Quo hae propositiones ad motum planetarum possint accommodari, sit sol in C, et planeta moveatur in ellipsi $ADBHA$, cuius axis transversus AB sit = E, distantia focorum $CF = D$ et celeritas, [p. 271] quam planeta habet in summa abside A, debita sit altitudini c. Eritque

$$c = \frac{227512(E-D)}{(E+D)E}.$$

Iam pro Mercurio est E = 15991 semid. telluris, quam mensuram hic constanter ad distantias expremendas adhibebimus, D = 3367. Quare erit c = 9.278. Pro Venere est E = 29882 et D = 206, unde reperitur c = 7.509. Pro tellure est E = 41312 et D = 743. Unde c = 5.313. Pro Marte est E = 62959 et D = 5887, unde sit c = 2.996. Pro Iove est E = 214870 et D = 22391, unde c = 0.9615. Pro Saturno est E = 394042 et D = 22391, unde c = 0.5153. Quae sufficiunt ad cuiusque planetae motum absolutum determinandum.
[In editione principe altitudines c pro Mercurio, Venere, Tellure, Marte, Iove, Saturno repertae falsos valores 7.368, 7.598, 5.323, 3.049, 0.9633, 0.5173 habent. Correxit P. St.]

Scholion 3.

669. Neutonus, prout iam innuimus, ex ellipsi, quam corpus describit, cognita et centro virium in alterutro foco posito deduxit vim centripetam esse quadratis distantiarum reciproce proportionalem. Postmodum autem a Cel. Joh. Bernoulli aliisque huius propositionis inversa fuit agitata, ut scilicet quaeratur curva, quam corpus circa centrum virium in ratione reciproca duplicata distantiarum attrahens describet. Negabant enim a Neutono satis esse demonstratum praeter sectiones conicas nullam aliam curvam quaesito satisfacere, quamvis prop. XVII. lib. I. *Princ. Philos. Naturalis* hoc satis clare evincere videatur. Huius igitur quaestionis his duabus propositionibus plenam exhibuimus solutionem, qua Neutoni assertio extra dubium ponitur. [p. 272] Alias huius problematis solutiones videre licet in Comm. Acad. Paris et Horis subsecivius Francof.

[*Extrait de la réponse de Monsieur Bernoulli à Monsieur Herman*, datée de Basle le 7 Octobre 1710, Mém. de l' Acad. R. des Sciences de Paris, Année 1710, p.521; *Opera omnia* Tom. I, Lausannae et Genevae 1742, p.470. Vide etiam commentationes, quas P. Varignon in tomis Comm. Acad. Paris. annorum 1700, 1701, 1710 publicavit, et epistolam, quam Hermannus ad Bernoullium misit, 1710, p.519; que etiam exstat in Joh. Bernoulli *Operibus omnis*, Tom. I, p. 469.
De legibus virium centralium, quibus planetae in obitis suius urgentur, et motibus inde oriundis, Exercitationum subsecivarum Francofurtensium Tom. I, Sectio 2 (1718), p.181 –194. Cuius commentationis autorem J. Hermannum fuisse apparet ex p.188. P. St.]

Scholion 4.

670. Tempus unius revolutionis planetae circa solem est $\frac{\pi E\sqrt{E}}{250f}$ minutorum secondorum (653), si quidem E et f in partibus millesemis pedis Rhenani exprimantur. Quia autem E et f commodius in semidiameteris telluris exhibentur, quarum una continet 20302353 pedes, loco fractionis $\frac{1}{250}$, per quam $\frac{\pi E\sqrt{E}}{f}$ multiplicandum est, adhiberi oportet 569.954. Hanc ob rem tempus unius revolutionis planetae erit $\frac{569945\pi E\sqrt{E}}{1000f}$ minutorum secundorum, seu ob π =3.1415926536 et f = 477 semidiam. terrae (567), erit tempus periodicum planetae = $3.754\,E\sqrt{E}$ minutorum secundorum, expresso axe traverso in semidiametris terrae.

PROPOSITIO 82.

PROBLEMA.

671. *Si vis centripeta fuerit reciproce ut cubus distantiae a centro, requiritur curva, quam corpus utcunque proiectum describet, corporisque motus in ea.*

SOLUTIO.

Sit centrum C (Fig. 58) et proiiciatur corpus ex A celeritate $\sqrt{c}$ et secundum directionem cum AC faciendem angulum, cuius sinus sit $\frac{h}{a}$, posita $AC = a$. [p. 273] Perveneril corpus in M, ubi sit $CM = y$ et in tangentem MT ex C dimissum perpendiculum $CT = p$. Celeritas vero in M debita sit altitudini v. Atque distantia a centro C, in qua vis centripeta aequalis est gravitati, sit $= f$. Hinc ergo erit vis centripeta in $M = \frac{f^3}{y^3} = P$; ex quo prodit $Y = \frac{f^3}{2a^2} - \frac{f^3}{2y^2}$ (601). Habebitur igitur pro curva quaesita ista aequatio

$$\frac{f^3}{2a^2} - \frac{f^3}{2y^2} = c - \frac{ch^2}{p^2}.$$

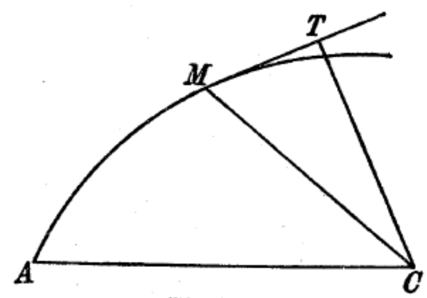

Fig. 58.

Atque cum sit $v = \frac{ch^2}{p^2}$ (587), erit etiam $v = c - \frac{f^3}{2a^2} + \frac{f^3}{2y^2}$.

Has aequationes per sequentes casus examinabimus.

I. Si fuerit $c = \frac{f^3}{2a^2}$, erit $\frac{1}{y^2} = \frac{h^2}{a^2p^2}$ seu $p = \frac{hy}{a}$.

Perspicitur autem ex hoc angulum CMT, cuius sinus est $\frac{p}{y}$, aequalem fore angulo ad A. Quamobrem curva his casibus descripta erit spiralis logarithmica, cuius centrum est in ipso centro virium C. Ita autem corpus movebitur in hac spirali, ut semper sit $v = \frac{f^3}{2y^2}$ seu celeritas distantiae a centro reciproce proportionalis.

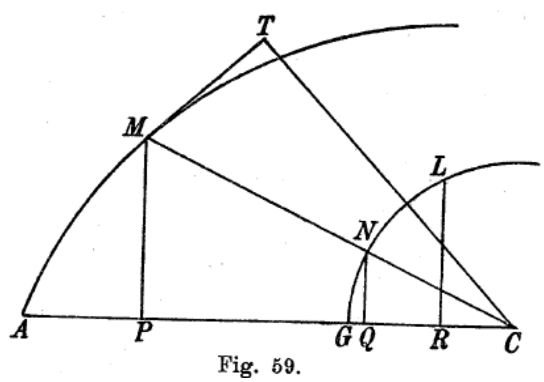

Fig. 59.

Si non fuerit $v = \frac{f^3}{2y^2}$, demittatur ex *M* in *AC* perpendicularis *MP* (Fig. 59), et vocetur *CP* = *x* atque *x* = *uy*. Habebitur igitur

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{hdy\sqrt{c}}{y\sqrt{\left(cy^2 - ch^2 - \frac{f^3y^2}{2a^2} + \frac{f^3}{2}\right)}}$$

Est vero *u* cosinus anguli *ACM*, sinu toto existente = 1. Radio igitur 1 si describatur arcus circuli *GN*, erit $GN = -\int \frac{du}{\sqrt{1-u^2}}$. Vocetur autem *GN* = *t*, atque habebitur ista aequatio

$$dt = -\frac{hdy\sqrt{c}}{y\sqrt{\left(cy^2 - ch^2 - \frac{f^3y^2}{2a^2} + \frac{f^3}{2}\right)}}.$$

II. [p. 274] Sit $c = \frac{f^3}{2h^2}$, erit $dt = -\frac{hdy\sqrt{c}}{y^2\sqrt{\left(a^2-h^2\right)}}$ ideoque $t = C + \frac{ah}{y\sqrt{\left(a^2-h^2\right)}}$

Constans vero *C* erit $= -\frac{h}{y\sqrt{\left(a^2-h^2\right)}}$, quia evanescente *t* fit *y* = *a*. Sit autem tangens anguli ad $A = \theta$, eritque $t = \vartheta(\frac{a}{y} - 1)$ seu $y = \frac{\theta\alpha}{t+\theta}$.

Unde ex dato angulo *ACM* reperitur recta *CM* ideoque punctum *M* in curva quaesita. Haec curva autem est spiralis hyperbolica, quam Cel. Joh. Bernoulli pro eadem causa satisfacientem invenit in Act. Lips. 1713

[*De motu corporum gravium, pendulorum et proiectilium,* Acta erud. 1713, p. 77; *Opera omnia* Tom. I, Lausannae et Gevevae 1742, p.548, 552 – 553. P. St.]

Si *c* neque $= \frac{f^3}{2a^2}$ neque $= \frac{f^3}{2h^2}$, ponatur $c = \frac{\delta f^3}{2a^2}$; erit

$$dt = \frac{-hdy\sqrt{\delta}}{y\sqrt{\left((\delta-1)y^2 + a^2 - \delta h^2\right)}}.$$

Ponatur $y = \frac{1}{q}$, erit

$$dt = \frac{hdq\sqrt{\delta}}{\sqrt{\left(\delta - 1 + (a^2 - \delta h^2)q^2\right)}}.$$

Hinc duo oriuntur casus primarii.

III. Si $a^2 - \delta h^2$ sit numerus affirmativus, pendebit integratio a logarithmis. Erit enim

$$\begin{aligned} t &= \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{C}{\sqrt{(\delta - 1 + (a^2-\delta h^2)q^2)} - q\sqrt{(a^2-\delta h^2)}} \\ &= \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{Cy}{\sqrt{((\delta-1)y^2 + a^2 - \delta h^2)} - \sqrt{(a^2-\delta h^2)}}. \end{aligned}$$

Atque determinata debeto modo constante *C* erit

$$t = \frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{y\sqrt{\delta(a^2-h^2)} - y\sqrt{(a^2-\delta h^2)}}{a\sqrt{((\delta-1)y^2+a^2-\delta h^2)} - a\sqrt{(a^2-\delta h^2)}}$$

atque

$$y = \frac{2ae^{\frac{t\sqrt{(a^2-\delta h^2)}}{h\sqrt{\delta}}}\left(\sqrt{\delta(a^2-h^2)(a^2-\delta h^2)} - a^2+\delta h^2\right)}{e^{\frac{2t\sqrt{(a^2-\delta h^2)}}{h\sqrt{\delta}}}a^2(\delta-1) - \left(\sqrt{\delta(a^2-h^2)} - \sqrt{(a^2-\delta h^2)}\right)^2}.$$

IV. Si $\delta > \frac{a^2}{h^2}$, erit $dt = \frac{hdq\sqrt{\delta}}{\sqrt{(\delta-1-(\delta h^2-a^2)q^2)}}$. Integrale vero huius membri est $\frac{h\sqrt{\delta}}{\sqrt{(\delta h^2-a^2)}}$ ductum in arcum circuli, cuius sinus est $q\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$, [p. 275] posito radio = 1. Atque addita idonea constante erit

$$\frac{t\sqrt{(\delta h^2-a^2)}}{h\sqrt{\delta}}$$

= Arcui cuius sinus est $\frac{1}{y}\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$ – Arcui cuius sinus est $\frac{1}{a}\sqrt{\frac{\delta h^2-a^2}{\delta-1}}$
= Arcui cuius sinus est $\frac{D}{ay}\left(\sqrt{(a^2-D^2)} - \sqrt{(y^2-D^2)}\right)$,

posito $\frac{\sqrt{\delta h^2-a^2}}{\delta-1} = D.$ Hinc constructio curvae facilis fluit : sumatur enim arcus GL (Fig. 59), qui ad GN habeat rationem ut $\sqrt{(\delta h^2-a^2)}$ ad $h\sqrt{\delta}$. Huius arcus sinus LR ponatur = R. Erit ergo $Ray = D\sqrt{(a^2-D^2)} - D\sqrt{(y^2-D^2)}$ atque

$$y = \frac{aDR\sqrt{(a^2-D^2)} - aD^2\sqrt{(1-R^2)}}{a^2R^2-D^2}.$$

Unde et constructio et aequatio curvae quaesitae deducitur. In his autem quatuor casibus omnia ad problema pertinentia continentur. Q.E.I.

Corollarium 1.

672. Distantia celeritates determinans reperitur = $\frac{af\sqrt{f}}{\sqrt{(f^3-2a^2c)}}$. Quare si $c = \frac{f^3}{2a^2}$, quo casu corpus in spirali logarithmica movetur, distantia celeritates determinans est infinita.

Corollarium 2.

673. Quia, si corpus in spirali logarithmica movetur, est $p = \frac{hy}{a}$ et $v = \frac{f^3}{2y^2}$, erit tempus, quo arcus AM absolvitur, = $\frac{a^3-ay^2}{f\sqrt{2f(a^2-h^2)}}$.

Corollarium 3.

674. Si hoc tempus ponatur T et cosinus anguli $CMT = i$, erit $y = \sqrt{(a^2 - fiT\sqrt{2f})}$. Ex qua aequatione post quodvis tempus datum reperitur corporis a centro C distantia. [p. 276]

Corollarium 4.

675. Corpus igitur in ipsum centrum C perveniet tempore $T = \frac{a^2}{fi\sqrt{2f}}$, dum interim revolutiones infinitas circa C perfecerit.

Corollarium 5.

676. Si T capiatur maius quam $\frac{a^2}{fi\sqrt{2f}}$, fit y imaginaria. Ex quo sequitur, postquam corpus in C pervenerit, nusquam amplius reperiri, sed quasi annihilari.

Corollarium 6.

677. Si sit $\delta = \frac{a^2}{h^2}$ seu $c = \frac{f^3}{2h^2}$, corpus movebitur in spirali hyperbolica atque etiam post infinitas circa C peractas revolutiones in ipsum centrum C perveniet tempore quoque finito. Est enim tempus, quo arcus AM percurritur, $= \frac{2\theta\alpha(a-y)}{f\sqrt{2f}}$, ideoque tempus, quo corpus in C pervenit, $= \frac{2\theta a^2}{f\sqrt{2f}}$.

Corollarium 7.

678. Si $\delta < \frac{a^2}{h^2}$, seu $c < \frac{f^3}{2h^2}$, qui erat casus tertius, corpus in lineis spiralibus quoque movebitur atque tandem post infinitos percursos gyros in centrum C perveniet. Apparet hoc ex aequatione; nam facto $t = \infty$ demum fit $y = 0$.

Scholion 1.

679. Quando est $c = \frac{f^3}{2h^2}$, spiralis hyperbolica, quam corpus describit, hanc habet proprietatem, [p. 277] ut posito $t = -\theta$ fiat $y = \infty$. Iste igitur radius ex centro C ductus videtur esse curvae ex infinito accedentis asymtotos. At potius curva ab eo perpetuo recedit neque tamen ultra datum intervallum θa. Recta ergo ipsi huic radio parallela ab eoque θa distans erit vera asymtotos. Huius praeterea curvae insignis est proprietas, quod ex infinitis circulis concentricis arcus abscindat aequales, ex qua natura et forma curvae magis elucet. Similem quoque proprietatem habent curvae, quae describuntur, si $\delta < \frac{a^2}{h^2}$, simul vero $\delta > 1$. Fit enim $y = \infty$, si capiatur

$$t = -\frac{h\sqrt{\delta}}{\sqrt{(a^2-\delta h^2)}}\, l\, \frac{a\sqrt{(\delta-1)}}{\sqrt{\delta(a^2-h^2)} - \sqrt{(a^2-\delta h^2)}}$$

Ex quo perspicitur, si sit $\delta < 1$, locum, quo fit $y = \infty$, fore imaginarium. At si $\delta = 1$, erit $t = -\infty$; curva enim hoc casu orta est spiralis logarithmica. Ceterum omnes hae curvae ita sunt comparatae, ut sint ubique versus C concave; neque enim ex motus natura usquam habere possunt flexus vel reversionis.

Corollarium 8.

680. Si $\delta > \frac{a^2}{h^2}$, seu $c > \frac{f^3}{2h^2}$, curva a corpore descripta non erit amplius spiralis, sed algebraica, si quidem inter algebraicas etiam eae referantur, in quarum aequationibus continentur exponentes irrationales.

Scholion 2.

681. Ad has igitur inveniendas oportet poni $h = a$ (640). Quo facto haec habebitur aequatio [p. 278]

$$dt = \frac{a\,dq\sqrt{\frac{\delta}{\delta-1}}}{\sqrt{(1-a^2q^2)}}.$$

Seu posito $aq = r$ et loco dt valore $\frac{-du}{\sqrt{(1-u^2)}}$ erit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{-dr\sqrt{\frac{\delta}{\delta-1}}}{\sqrt{(1-r^2)}}.$$

Hac forma comparat cum $\frac{\lambda dZ}{\sqrt{(A^2-Z^2)}}$ (605) erit $Z = -r, A = 1, \lambda = \sqrt{\frac{\delta}{\delta-1}}$. Habebitur properea ista aequatio $u =$

$$\frac{(\sqrt{(1-C^2)}-C\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}(\sqrt{(1-r^2)}-r\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}-(\sqrt{(1-C^2)}+C\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}(\sqrt{(1-r^2)}+r\sqrt{-1})^{\sqrt{\frac{\delta}{\delta-1}}}}{2\sqrt{-1}}$$

seu

$$\begin{aligned} 2r\sqrt{-1} = {} & \left(\sqrt{(1-u^2)} - u\sqrt{-1}\right)^{\sqrt{\frac{\delta-1}{\delta}}}\left(\sqrt{(1-C^2)} - C\sqrt{-1}\right) \\ & -\left(\sqrt{(1-u^2)} + u\sqrt{-1}\right)^{\sqrt{\frac{\delta-1}{\delta}}}\left(\sqrt{(1-C^2)} + C\sqrt{-1}\right) \end{aligned}$$

(605 et 607). Constans vero C ex hoc determinabitur, quod facto $u = 1$ fieri debeat $y = a$ seu $q = \frac{1}{a}$ seu $r = 1$. Determinata igitur C erit

$$2r = \left(u + \sqrt{(u^2-1)}\right)^{\sqrt{\frac{\delta-1}{\delta}}} + \left(u - \sqrt{(u^2-1)}\right)^{\sqrt{\frac{\delta-1}{\delta}}} = \frac{2a}{y}$$

seu

$$u = \frac{1}{2y^{\sqrt{\frac{\delta}{\delta-1}}}}\left(\left(a + \sqrt{(a^2-y^2)}\right)^{\sqrt{\frac{\delta}{\delta-1}}} + \left(a - \sqrt{(a^2-y^2)}\right)^{\sqrt{\frac{\delta}{\delta-1}}}\right) = \frac{x}{y}.$$

Quae est aequatio pro curva quaesita.

Exemplum 1.

682. Sit $\sqrt{\frac{\delta}{\delta-1}}=2$ seu $\delta=\frac{4}{3}$ atque $c=\frac{2f^3}{3a^2}$. Erit

$$2xy=\left(a+\sqrt{(a^2-y^2)}\right)^2+\left(a-\sqrt{(a^2-y^2)}\right)^2=4a^2-2y^2.$$

Ponatur applicata $PM=z$, fiet

$$x\sqrt{(x^2+z^2)}=2a^2-x^2-z^2$$

et sumendis quadratis

$$0=4a^4-4a^2x^2-4a^2z^2+x^2z^2+z^4$$

sive

$$x=\frac{2a^2-z^2}{\sqrt{(4a^2-z^2)}}.$$

[p. 279] Haec curva igitur ad instar parabolae progreditur, sed habet asymptoton rectae *AC* parallelam et ab ea intervallo $2a$ dissitam, quam nunquam attingit.

Exemplum 2.

683. Sit $\sqrt{\frac{\delta}{\delta-1}}=3$ seu $\delta=\frac{9}{8}$ atque $c=\frac{9f^3}{16a^2}$. Oritur ergo

$$2xy^2=\left(a+\sqrt{(a^2-y^2)}\right)^3+\left(a-\sqrt{(a^2-y^2)}\right)^3=8a^3-6ay^2.$$

Atque inducta applicata $PM=z$ habebitur

$$x^3=4a^3-xz^2-3ax^2-3az^2.$$

Haec curva ordinis tertii pertinet ad speciem 41 in enumeratione a Neutono facta. [*Enumeratio linearum tertii ordinis*, Londini 1704; J. NEWTONI *Opuscula mathematica etc.* Lausannae et Genevae 1744, Tom. I, p. 259. P. St.]

Scholion 3.

684. Innumerabiles aliae inveniri possunt curvae algebraicae, quae a corpore proiecto in hac hypothesi describuntur, si ponatur $\delta=\frac{m^2}{m^2-1}$, denotate m numerum quemcunque rationalem unitate maiorem, ne fiat δ negativum. Hae autem tres hypothesis pertractatae, quibus vis centripeta primo distantiis, secundo reciproce quadratis distantiarum et tertio reciproce cubis distantiarum est posita proportionalis, sunt solae, quae deducunt ad curvas vel algebraicas vel a circuli aut hyperbolae quadraturis pendentes; si quidem loco *P* potentia ipsius y substituatur. Quanquam autem in aliis potestatibus ipsius y loco *P* substitutis aequatio pro curva descripta non potest ab irrationalitate liberari et hanc ob rem neque algebraica neque a quadratura circuli vel hyperbolae pendens esse potest, tamen dantur casus speciales, quibus curva quaesita fit algebraica. [p. 280] Namque quemadmodum linea recta et circulus in omnibus hypothesibus satisfacere possunt, ita etiam quandoque aliae curvae algebraicae reperiuntur. Has autem quomodo invenire oporteat, in sequenti propositione declarabimus.

PROPOSITIO 83.

PROBLEMA.

685. *Existente vi centripeta ut potestas distantiarum quaecunque, invenire casus speciales, quibus corpus certo quodam modo proiectum in linea algebraica moveatur.*

SOLUTIO.

Manentibus omnibus ut hactenus, pro curva quaesita inventa est haec aequatio $\frac{du}{\sqrt{(1-u^2)}} = \frac{hdy\sqrt{c}}{y\sqrt{(cy^2-ch^2-y^2Y)}}$ (601). Nostro autem case est $P = \left(\frac{y}{f}\right)^n$, denotante f distantiam, in qua vis centripeta gravitate aequatur. Hinc erit $Y = \int Pdy = \frac{y^{n+1}-a^{n+1}}{(n+1)f^n}$.
Deinde, quia curvas algebraicas requiremus, ponimus $h = a$ (640). His substitutis erit [p. 281]

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{cf^n(n+1)}}{y\sqrt{((n+1)cf^n(y^2-a^2)-y^2(y^{n+1}-a^{n+1}))}}.$$

Sit distantia celeritates determinans k, erit $c = \frac{k^{n+1}-a^{n+1}}{(n+1)f^n}$. Hac igitur loco c introducta erit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{(k^{n+1}-a^{n+1})}}{y\sqrt{(a^{n+3}-a^2k^{n+1}+k^{n+1}y^2-y^{n+3})}}.$$

Patet hinc, si $k = a$, quo casu fit $c = 0$, fore u = const. = 1 ideoque $x = y$, qui est casus, quo corpus recta descendit ad centrum. Si k est infinita, erit c quoque infinita, quoties $n + 1$ est numerus affirmativus. Hoc igitur casu corpus etiam in recta progredi debebit, quia vis finita corporis infinita celeritate moti directionem non valet mutare. Ponamus ergo $n + 1$ esse numerum negativum $= -m$ seu $n = -m - 1$; fiet

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{y^{\frac{m-4}{2}}dy\sqrt{(k^m-a^m)}}{\sqrt{(a^{m-2}k^m+a^my^{m-2}-k^my^{m-2}-a^{m-2}y^m)}}$$

Fiat k infinita magna, erit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{y^{\frac{m-4}{2}}dy}{\sqrt{(a^{m-2}-y^{m-2})}}$$

evanescentibus reliquis terminis ; et ponatur $y^{m-2} = q^2$, erit $y = q^{\frac{2}{m-2}}$. Hacque facta substitutione oritur sequens aequatio

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{2dq}{(m-2)\sqrt{(a^{m-2}-q^2)}}.$$

Qua formula comparata cum universali $\frac{\lambda dZ}{\sqrt{(A^2-Z^2)}}$ (604), erit $\lambda = \frac{m}{m-2}$, $A = a^{\frac{m-2}{2}}$ et $Z = q = y^{\frac{m-2}{2}}$.
Emerget igitur ista aequatio algebraica pro curva quaesita

$$2y^{\frac{m-2}{2}}\sqrt{-1} = \left(\sqrt{(1-u^2)} + u\sqrt{-1}\right)^{\frac{m-2}{2}}\left(\sqrt{(a^{m-2}-C^2)} + C\sqrt{-1}\right)$$
$$-\left(\sqrt{(1-u^2)} - u\sqrt{-1}\right)^{\frac{m-2}{2}}\left(\sqrt{(a^{m-2}-C^2)} - C\sqrt{-1}\right)$$

(607). [p. 282] Constans C ex hoc determinatur, quod facto $u = 1$ fieri debeat $y = a$. Hanc ob rem erit

$$2y^{\frac{m-2}{2}} = a^{\frac{m-2}{2}}\left(\left(u-\sqrt{(u^2-1)}\right)^{\frac{m-2}{2}} + \left(u+\sqrt{(u^2-1)}\right)^{\frac{m-2}{2}}\right).$$

Quoties igitur m est numerus rationalis affirmativus, invenitur curva algebraica, quam corpus normaliter proiectum existente distantia celeritates determinante infinita describet. Q.E.I.

Corollarium 1.

686. Sumendis quadratis obtinebitur

$$4y^{m-2} - 2a^{m-2} = a^{m-2}\left((u+\sqrt{(u^2-1)})^{m-2} + (u-\sqrt{(u^2-1)})^{m-2}\right)$$

seu ob $n = -m-1$ haec aequatio

$$4y^{-n-3} - 2a^{-n-3} = a^{-n-3}\left((u+\sqrt{(u^2-1)})^{-n-3} + (u-\sqrt{(u^2-1)})^{-n-3}\right).$$

Corollarium 2.

687. Oportet esse m numerum affirmativum nihilo maiorem. Nam si esset m vel = 0 vel numerus negativus, celeritas corporis foret infinita et curva propterea linea recta.

Corollarium 3.

688. Si sit $n = -2$, havebitur ista aequatio $\frac{4a}{y} - 2 = 2u = \frac{2x}{y}$ seu $2a = x + y$. Quae posita applicata PM, $\sqrt{(y^2 - x^2)}, = z$ abit in hanc aequationem $z^2 = 4a^2 - 4ax$ pro parabola centro virium in foco posito, ut iam invenimus (647). [p. 283]

Corollarium 4.

689. Si sit $n=-3$ seu $m = 2$, qui est casus in praecedente propositione tractatus, erit $2y^{\frac{m-2}{2}}=2a^{\frac{m-2}{2}}$ seu $y = a$. Curva ergo, quam corpus describet, erit circulus, in cuius centrum ipsum centrum virium incidit.

Scholion 1.

690. Si n est numerus impar, erit m par ideoque $\frac{m-2}{2}$ integer. His igitur casibus formula in solutione problematis inventa uti conveniet. At si n fuerit par, formula (686) quadratis sumtis producta debet adhiberi. Statim enim utroque casu ad aequationem rationem inter x et y pervenitur, ut in sequentibus exemplis apparebit.

Exemplum 1.

691. Attrahat centrum in ratione reciproca quadruplicata distantiarum, et corpus normaliter proiiciatur existente distantia celeritates determinante infinita; curva, quam corpus describit, erit algebraica sequens. Ob sit $n=-4$, erit $4y-2a=2au=\frac{2ax}{y}$ seu $2y^2=a(y+x)$. Introducta vero applicata $z=\sqrt{(y^2-x^2)}$ erit

$$2x^2+2z^2=ax+a\sqrt{(x^2+z^2)},$$

quae aequatio est pro curva ordinis quarti.

Exemplum 2.

692. Sit vis centripeta reciproce in quintuplicata distantiarum, ratione seu $n=-5$, et m = 4. [p. 284] Habebitur ergo $2y=2au=\frac{2ax}{y}$ atque

$$yy=ax=x^2+z^2,$$

quae est aequatio ad circulum centro virium in eius peripheria posito. Casum hunc habet Neutonus in *Princ.* Lib. Prop. VII.

Exemplum 3.

693. Sit $n=-7$, seu $m = 6$, erit $2y^2=4a^2u^2-2a^2$ seu $y^4=2a^2x^2-a^2y^2$. Posita applicata $PM = z$ prodibit ista aequatio

$$(x^2+z^2)^2=a^2x^2-a^2z^2$$

pro linea ordinis quarti, in qua centrum virium quoque peripheriam incidit.

Scholion 2.

694. De figuris, quas corpora a datis viribus centripetis sollicitata describunt, hic plura addere operae non esset pretium, cum in Physica et Astronomia aliae virium centripetarum hypotheses, nisi quae quadratis distantiarum reciproce est proportionnalis, usum habeant nullum. Quando tamen in Astronomia corpus a pluribus huiusmodi viribus sollicitatum considerari debet, quarum una prae reliquis maximam vim exercet, haec, prout res postulat, ne reliquas in computum ducere opus sit, aliquantum vel augetur vel diminuitur, quo saltem quam fieri potest proxime motus illius corporis cognoscatur. His igitur casibus curva, quam corpus describit, non multum discrepabit ab ellipsi. Hanc ob rem Astrononomi hanc curvam ad instar ellipsis contemplari solent, quae autem non est fixa, sed mobilis, ita ut corpus in ellipse circa focum revolvente moveri concipiant [p. 285]. Ex hocque oritur mobilitas orbitarum planetarum, qua lineae absidum perpetuo in alium situm transferuntur. Nos vero, quo propius ad veritatem accedamus, praeter mobilitatem axis ellipsis etiam speciem eius tanquam variabilem considerabimus. Ita igitur hac in re versabimur, ut de quolibet elemento curvae, quam corpus describit, determinemus, cuiusnam ellipsis focum in centro virium habentis sit portio, ex quo tam positio quam species istius ellipsium innotescet. Omnes autem hae ellipses alterutrum focum in ipso centro virium positum habebunt, quia ad id punctum corpus perpetuo retrahitur.

PROPOSITIO 84.

PROBLEMA.

695. *Si vi centripeta non multum discrepat a ratione reciproca duplicata distantiarum, determinare motum ellipsis eiusque continuam speciei mutationem atque corporis motum in ista mutabili ellipsi.*

SOLUTIO.

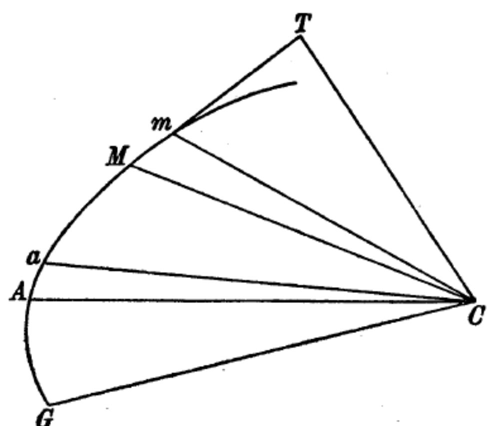

Fig. 60.

Sit centrum virium C (Fig. 60), habeatque corpus in M secundum directionem MT celeritatem altitudini v debitam. Ponatur vis centripeta in M agens = P, distantia $CM = y$ et sinus anguli $CMT = s$. Iam perspicuum est, quamcunque legem teneat P, semper tamen aequari posse huic formae $\frac{f^2}{y^2}$, dummodo f [p. 286] non quantitatem constantem ut hactenus, sed variabilem denotet. Erit ergo $f^2 = Py^2$. Quia autem, dum corpus elementum Mm percurrit, f constans manet, poterit determinari ellipsis, cuius focus est C et elementum Mm, in qua corpus, si f perpetuo maneret constans, moveretur. Sit igitur AC huius ellipsis axis transversi positio, erit tangens anguli $MCA = \frac{2svy\sqrt{(1-ss)}}{ff-2ssvy}$ (656), quae ergo erit $= \frac{2sv\sqrt{(1-ss)}}{Py-2s^2v}$. At latus rectum erit $= \frac{4s^2v}{P}$ et axis et axis transversu $\frac{Py^2}{Py-v}$ (cit.) Ex centro C demittatur perpendiculum CT in tangentem MT, quod ponatur $= p$; erit $s = \frac{p}{y}$. Atque ubi perpendicularum in tangentem est = h, quod fiat in G posita $CG = a$, sit corporis celeritas debita altitudini c, ex quo erit $v = \frac{ch^2}{p^2}$ (589). Tangens vero $MT = \sqrt{(y^2 - p^2)}$ ponatur brevitatis gratia = t. His positis erit tang. ang. $= \frac{2ch^2t}{Py^3p - 2ch^2p}$ et latus rectum, transversum vero = $\frac{Py^2p^2}{Pyp^2 - ch^2}$. Praeterqea vero est ex natura attractionis $Pdy = \frac{2ch^2dp}{p^3}$ (587). Ex qua aequatione p in y poterit determinari adeoque tota ellipsis ex sola y et c et h. Sit iam CG linea fixa, cum qua angulus, quem recta CM constituit, ex natura curvae potest definiri. Ab hoc igitur angulo, si auferatur angulus MCA, habebitur inclinatio linea absidium AC in rectam fixam CG. Motus autem corporis ex cognita celeritate, $v = \frac{ch^2}{p^2}$, facile innotescet. Q.E.I. [p. 287]

Scholion 1.

696. Ellipsis hoc modo determinata merito potest vocari ellipsis curvaqm osculans ad similitudinem circulorum osculatorum, quibus curvidines linearum mensurantur. Haec vero consideratio non est pure geometrica, sed ad hanc ellipsin osculantem inveniendam praeter curvae naturam nosse oportet celeritatem corporis et vim centripetam.

Corollarium 1.

697. Si t evanescit, linea absidum AC seu positio axis ellipsis osculantis incidit in radium CM ob evanescentem angulum MCA.

Corollarium 2.

698. Si fuerit $P = \frac{2ch^2}{y^3}$, fiet angulus MCA rectus. Hoc igitur casu erit vis centripeta ut cubus distantia reciproce. Quare posito $P = \frac{f^3}{y^3}$, erit $c = \frac{f^3}{2h^2}$. Curva autem, quae tum a corpore describatur, est spiralis hyperbolica (679). Pro hac igitur curva linea absidum perpetuo est normalis cum radio MC. At latus rectum ellipsis osculantis erit $= 2y = 2MC$.

Corollarium 3.

699. Si fuerit $P = \frac{f^n}{y^n}$, erit

$$\int Pdy = -\frac{f^n}{(n-1)y^{n-1}} = C - \frac{ch^2}{p^2}.$$

Quia vero facto $y = a$ fieri debet $p = h$, erit $C = c - \frac{f^n}{(n-1)a^{n-1}}$. [p. 288] Hinc invenietur

$$p^2 = \frac{(n-1)ch^2a^{n-1}y^{n-1}}{(n-1)ca^{n-1}y^{n-1} + f^n(a^{n-1} - y^{n-1})}.$$

Latus rectum ergo ellipsis osculantis erit $\frac{4ch^2y^{n-2}}{fn}$. Latus transversum et ang. *MCA* ex his etiam in *y* determinabuntur.

Corollarium 4.

700. Si fuerit $P = \frac{ch^2}{yp^2}$, ellipsis osculans semper erit parabola. Fiet autem hoc valore loco *P* in aequatione $Pdy = \frac{2ch^2dp}{p^3}$ substituto $p^2 = \frac{h^2y}{a}$. Et consequenter $P = \frac{ac}{yy}$. Hoc autem casu curva descripta haec ipsa est parabola ob $f^2 = ac$ (647)

Scholion 2.

701. Non confundenda est haec ellipsium osculantium doctrina cum motu corporum in orbibus mobilibus, de quo Neutonus aliique post eum tractaverunt. [*Princ. phil. nat.*, Lib. I Sectio IX : De motu corporum in orbibus mobilibus, deque motu apsidum. P. St.] Hoc enim loco determinavimus, cuiusnam elliipsis portio sit quodvis elementum curvae a corpore descriptae. At, quando de orbibus mobilibus sermo erit (729), investigabitur vis centripeta efficiens, ut corpus in data curva circa centrum virium revolvente moveatur.

PROPOSITIO 85.

THEOREMA.

702. *Si corpus utcunque proiectum attrahatur ad quotcunque centra virium A, B, C* (Fig. 61), *quarum singulorum vires sint distantiis ab ipsis proportionales,* [p. 289] *corpus eodem modo movebitur, ac si ad punctorum A, B, C commune centrum gravitatis O attraheretur pariter in ratione distantiarum simplici.*

DEMONSTRATIO.

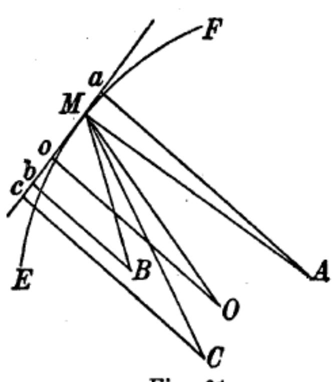

Fig. 61.

Positis centrorum A, B, C viribus, quas in distantia 1 exercent, α, β, γ respective, sit ac directio motus, quam corpus in M habet, ideoque tangens curvae EMF descriptae in M. At sit O centrum gravitatis corporum α, β, γ in punctis A, B, C positorum, et in O concipiatur vis distantiis directe proportionalis, quae in distantia 1 attrahat vi = $\alpha + \beta + \gamma$. His positis corpus in M attrahatur ad A vi $AM.\alpha$, ad B vi $BM.\beta$ et ad C vi $CM.\gamma$. His autem viribus simul agentibus demonstrandum est aequalem esse vim $OM.(\alpha + \beta + \gamma)$ corpus ad O trahentem. Ad hoc ostendendum ex punctis

A, B, C, et *O* in tangentem *ac* demittantur perpendiculara *Aa, Bb, Cc* et *Oo*. Hoc modo quaelibet vis in normalem et tangentialem resolvetur, eritque summa normalium ex viribus ad A, B,C tendentibus ortarum

$$\alpha.Aa+\beta.Bb+\gamma.Cc,$$

summa vero tangentialium erit

$$-\alpha.Ma+\beta.Mb+\gamma.Mc.$$

At quia O est centrum gravitatis corporum α, β, γ in A, B, C sitorum, liquet ex Staticis fore

$$\alpha\cdot Aa+\beta\cdot Bb+\gamma\cdot Cc=(\alpha+\beta+\gamma)\cdot Oo$$

et

$$-\alpha\cdot Ma+\beta\cdot Mb+\gamma\cdot Mc=(\alpha+\beta+\gamma)\cdot Mo.$$

Ex quo perspicitur viribus $\alpha.AM, \beta.BM, \gamma.CM$ coniunctis aequivalere vim $(\alpha+\beta+\gamma).OM$. Q.E.D. [p. 290]

Corollarium 4.

703. Corpus igitur in hac hypothese ellipsin describet, cuius centrum in ipso centro gravitatis O est positum (631). Omnes enim vires idem efficiunt, quod unica in O posita et attrahens in ratione directa distantiarum.

Corollarium 5.

704. Manifestum quoque est, quotcunque etiam sint huiusmodi virium centra in ratione distantiarum attrahentia, semper tamen corpus in ellipsi motum iri, omnino ac si ad unicum virium centrum in eorum communi centro gravitatis positum attraheretur.

Scholion 1.

705. Demonstratio porro pari quoque modo succedit, si illa quotcunque virium centra non sint in eodem plano posita, quamadmodum ex staticis principiis cuique perspectum esse potest. Hincque intelligitur corpus nihilominus perpetuo in eodem plane moveri debere, etiamsi centra virium in diversissimis planis sint dispersa.

Scholion 2.

706. Si centra virium in alia ratione quacunque praeter simplicem distantiarum ad se trahant, huiusmodi reductio ad unicum virium centrum locum prorsus non habet , atque motus corporis calculo vix, re ipsa autem ne vix quidem potest determinari. [p. 291] His igitur in casibus ad approximationes erit confugiendum, quae pro variis conditionibus diversimode sunt instituendae. Atque hanc ob rem Neutonus verum lunae motum, qui ex duplici attractione oritur, determinare non suscepit, sed vero tantum proxime hoc praestare conatus est. Ad hoc autem opus est peculiaribus considerationibus, atque inversa methodus est adhibenda, qua ex curva, quam corpus describit, cognita ad vires attrahentes receditur. Quamobrem, quae subsidia ad hoc institutum afferri possunt, ea in sequentibus, cum inverso ordine vim solicetantem tanquam incognitam sumus investigaturi, explicabimus. Ad hanc igitur tractationem progrediemur, quae duplici modo instituti potest. Primo enim praeter curvam descriptam ut cognita sumetur potentiae sollicitantis directio in singulis locis, ex hisque quantitas potentiae sollicitantis et ipse corporis motus determinabitur. In altera contemplatione curva et corporis motus in ea pro datis accipientur, ex quibus potentiam sollicitantem erui oportebit.