CHAPTER FIVE.
CONCERNING THE CURVILINEAR MOTION OF FREE POINTS ACTED ON BY ABSOLUTE FORCES OF ANY KIND.
[p. 225]

DEFINITION 21.

543. *A body describes the curved line AMB* (Fig. 47) *when acted upon by a force. The tangential force on the body is the component of the force along the direction of the tangent TMt to the curve at the point M.*

Corollary 1.

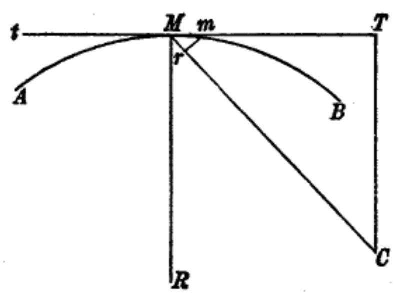


Fig. 47.

544. Therefore the tangential force exerts no other effect on the body while the element *Mm* is traversed, except that the motion of the body is either accelerated or retarded, since clearly the body is pulled either following the direction of *MT* or *Mt*.

Corollary 2.

545. Therefore when a body is moving along a curved line, the tangential force continually changes its direction and exerts its influence at another place.

Scholium.
546. Indeed a tangential force is hardly ever able to arise of its own accord in the nature of things; truly nothing is known of this wider use. For a force acting has a direction that can always be resolved into two parts, one of which is placed along direction of the tangent. [p. 226]

DEFINITION 22.

547. *The normal force is the force acting on the body describing the line AMB* (Fig. 47), the direction of which *MR* is normal to the element of the curve *Mm* or the tangent *MT* .

Corollary 1.

548. Therefore the normal force can neither increase nor decrease the speed of the body, since its direction MR is always at right angles to the direction of motion. (164).

Corollary 2.

549. The effect of this force is agreed upon, as we show in what follows, as only the direction of the body can be changed and affected, because by itself the body progresses along a straight line, and the action of the normal force makes it move along a curve (165).

Scholium 1.

550. If a body moves in the same plane, and also the directions of the forces acting on it are in put in the same plane, then the individual forces can be resolved into two parts, one of which is the normal, and the other the tangent, as is apparent from the principles of statics. Whereby when we have determined the effect of the tangential and normal forces on the body, then likewise also, [p. 227] the effect of any oblique force is also known. Moreover we call all forces acting on the body oblique, which are neither along the normal nor the tangent.

Scholion 2.

551. Hence the first division of this chapter arises. For in the first part we consider these motions that have their paths in the same plane, and likewise all the forces are agreed to be acting in the same plane. Following this, we are to consider motions that follow paths that do not lie in the same plane; for which it is understood that it is not sufficient to resolve the individual forces into two parts, for these are required to be resolved into three parts, on account of the three dimensions in which the body is moving.

PROPOSITION 70.

PROBLEM.

552. *If a body, as it traverses the element Mm* (Fig. 47) *in a plane, is acted on by two forces, the one normal and the other tangential, to determine the effect of each in altering the motion of the body.*

SOLUTION.

Let the speed of the body describing the element *Mm* correspond to the height *v*, the force pulling along the normal *MR* is equal to *N*, and the tangential force pulling along *MT* is equal to *T*, the element *Mm* = *ds* and the radius of osculation at *M* = *r*. [Which we now usually call the radius of curvature.]

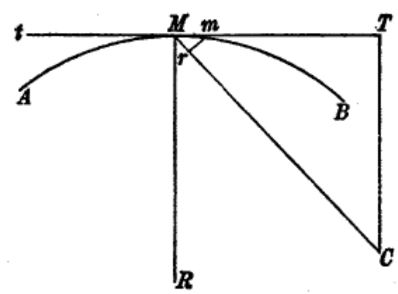


Fig. 47.

To determine the effect of the force *N*, since the direction of this is along the normal at *Mm*, we use the formula (165), which is $npr = Ac^2$. [p. 228]

This becomes $pr = 2Av$ (209). But here N has been put in place of $\frac{p}{A}$ for us ; for we understand by N not only the impression of the force N on the body, but the strength of the acceleration or the absolute force divided by the mass of the body, [which is A] (213). Whereby here in place of $\frac{p}{A}$ must be substituted N, with which done we have :

$$Nr = 2v.$$

Q. E.D. for the first part. [Thus in modern terms, we have the centripetal acceleration at the point M , $a_c = V^2/r$, where V is the tangential speed at this point.]

Then to determine the effect of the tangential force T, I use this rule : $Acdc = npds$ (166), or in place of this, it is modified to give : $Adv = pds$ (209). And on account of the reasons offered, here in place of $\frac{p}{A}$ I substitute T, and it becomes

$$dv = Tds.$$

Q. E.D. for the second part. [We may wish to consider *Tds* as the increase in the kinetic energy of the body supplied by the force *T* acting along the tangent on a unit mass through the increment *ds*, while *dv* is the corresponding change in a hypothetical gravitational potential energy of a unit mass under unit gravitational acceleration. One needs to look at Euler's work in Ch. 2 to see how relations equivalent to these can be found without recourse to work–energy relations.]

Corollary 1.

553. Because the acceleration of gravity is put equal to 1 [here called force, but in the sense force per unit mass], the normal acceleration is to the acceleration of gravity as the height corresponding to the speed to half the radius of osculation [curvature] ; which ratio follows from the equation $Nr = 2v$.

[In modern terms, the centripetal acceleration $a_c = V^2/r = 2gh/r = 2.1.v/r = N$.]

Corollary 2.

554. Therefore with the given acceleration to the normal, and the curve that the body describes, the speed of the body is known at once, for the speed is expressed by $\sqrt{v}$, and $\sqrt{v} = \sqrt{\frac{Nr}{2}}$.

Corollary 3.

555. Truly the increment of the height corresponding to the speed is always equal to the product of the tangential acceleration and the element of distance traversed by the body [p. 229]. Or the element of *v* is to the element of distance described *ds* as the tangential acceleration is to the acceleration of gravity.

PROPOSITION 71.

PROBLEM.

556. *If the body, while it traverses the element Mm* (Fig. 47)*, is acted on by some oblique force in the direction MC, then it is required to determine the effect of this force in changing the motion of the body.*

SOLUTION.

Let this oblique force be in the ratio to the force of gravity, if on the surface of the earth, as *P* to 1, the element *Mm* = *ds* and the speed at *M* corresponds to the height *v*. Since the obliqueness of the force *MC* has been given, the angle *CMT* is given, and on account of this the triangles *CMT* and *Mmt,* which come from the perpendiculars dropped from *C* to *MT* and from *m* to *MC*, are as shown. We can therefore put *Mr* = *dy* and *mr* = *dx*, then $ds^2 = dx^2 + dy^2$, and the ratio between both *ds*, *dx*, and *dy* is given. They are now brought together with these equations (161, 163) that have been presented before (208). For these that we wish to find here are

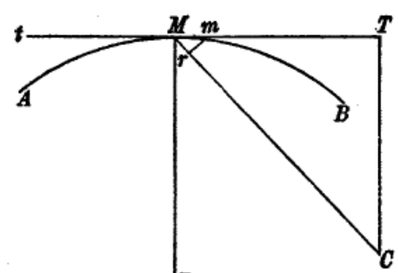


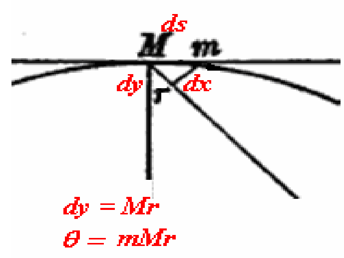
Fig. 47.

completely similar to these previous ones,; only in this respect do they differ, as for us here the ratio is P:1, while there it was $p : A$. On this account, we have :

$$dv = Pdy,$$

and with the radius of osculation MR at M put equal to r this equation is found :

$$Pr\,dx = 2vds$$



(208), I write P in place of $\frac{p}{A}$. Of these equations : $dv = Pdy$ defines the increment of the speed, as the body travels through the element Mm. [p. 230] Truly the other, $Pr\,dx = 2vds$ or $r = \frac{2vds}{Pdx}$ shows the lines of the curvatures at M described by the body. Hence the whole effect of the oblique force on the motion of the body can become known. Q. E. I.

[The component of P along the curve at M is $P\cos\theta = Pdy/ds$: hence $a_t.ds = dv$, where a_t is the tangential acceleration, as in the previous chapters ; the component of P normal to the curve, or the normal acceleration a_n , is given by $a_n = P\sin\theta = Pdx/ds = V^2/r = 2v/r$, where P is taken as the acceleration of a body of unit mass along MC.]

Corollary 1.

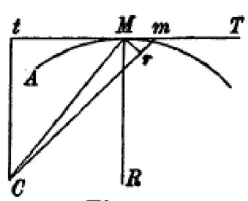


Fig. 48.

557. If the oblique force is put in place at an obtuse angle with the element Mm (Fig. 48), everything remains as before, except that the line element $mr = dy$ must be taken $\sqrt{(1-\lambda^2)} = \mu$. Hence $dv = -Pdy$, and the other equation $Pr\,dx = 2dvds$ remains as before.

Corollary 2.

558. Therefore if the direction of the force MC falls between the normal MR and the element Mm as in Fig. 47, then the motion of the body is one of acceleration. But if CM falls outside each, as in Fig. 48, the motion is one of retardation.

Corollary 3.

559. If the direction of the force MC falls on the tangent MT, then the angle mMr, Mr is made equal to Mm or $dx = 0$ and $dy = ds$. Therefore we have $dv = Pds$ and $r = \propto$, which indicates that the direction of the body is not affected by this tangential force.

Corollary 4.

560. If the direction of the force MC (Fig. 48) on the other part Mt, in which case $Mr = dx = 0$ and $dy = ds$, giving $dv = -Pds$ and $r = \propto$ as before. Therefore only the tangential speed of the body is affected, and the direction of the motion is clearly not changed.(544). [p. 231]

Corollary 5.

561. The direction of the force *MC* falls on the normal MR, where the effect of the normal force is known. Hence in this case $dy = 0$ and $dx = ds$. And consequently $dv = 0$ and $\mathrm{P}r = 2v$ is produced. Hence the normal force does not affect the speed, but only the direction of the motion. (548).

Scholium 1.

562. Therefore both the effect of the normal force and of the tangential force on the motion of the body is known. On which account, when all the forces, as many as act on the body, are put in a plane in the same way in two's, they can be resolved with one part normal and the other tangential, and the effect of any forces on the motion of the body can become known.

Scholium 2.

563. Hence it will be most convenient for motion in the same plane to be subdivided, as in the first place the directions of all the forces acting are parallel to each other, as in our region, the directions of the weights are observed to be parallel to each other. Then we will consider the case, in which the directions of all the forces converge to a single point, to which also the body is always attracted, and which is the case of the centripetal force, such as the singular discoveries made by Newton Part I of the *Princ. Phil. Nat.* Truly in the third place we introduce and investigate any forces acting on a body [p. 232], which the motion shall always be about to follow. Moreover we will turn from these individual cases, as at first we propose direct questions, then indeed also inverse questions, as far as we have done at this stage, that we may be able to resolve. Always finally, as much as is permitted, we will progress from the more simple to the more complicated and difficult cases. [Euler now solves simple projectile motion using his general equations.]

PROPOSITION 72.

PROBLEM.

564. *If there is a constant force the direction of which is normal everywhere to the right line AB* (Fig. 49) *and if a body is projected from A with a given speed along the direction AH, then it is required to find the curve AMDB described by the body, and the motion of the body on this curve.*

SOLUTION.

The force acting is called g, and the speed with which the body is projected from A corresponds to the height c and the cosine of the angle $HAB = \lambda$, with the whole sine taken equal to 1. Now with the body travelling through the element $Mm = ds$, the speed at M corresponds to the height v, and $PM = y$ and likewise the radius of osculation at $M = r$. On account of which $Mr = dx$ and $mr = dy$, and it is apparent that this case can be referred to Fig. 48, where the motion of the body is slowing down. [Note however that the elemental triangle in Fig. 48 has been rotated counterclockwise about M, resulting in x and y being interchanged in the diagram, but not in the equations.]

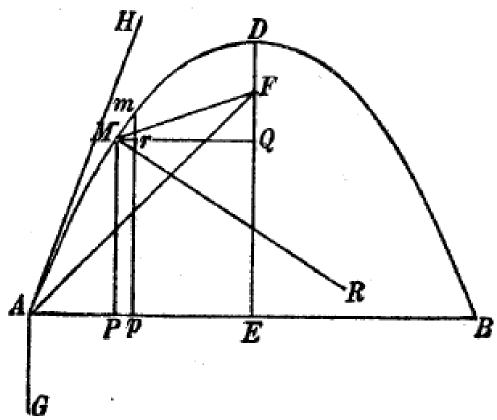
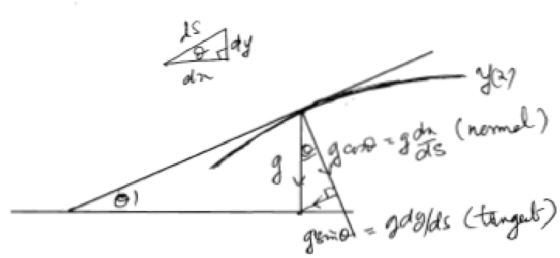


Fig. 49.

Therefore we have : $dv = -gdy$ and $grdx = 2vds$ (557). From the first of these equations there arises : $v = C - gy$, and C must be found from this, that with $y = 0$, it must become $v = c$. Therefore we have : [p. 233]

$$v = c - gy.$$



[Note also that v and c are actually multiplied by 1, which is the assumed acceleration of gravity in Euler's units on earth, in order that the equation has the correct dimensions of L^2/T^2. Thus, g also is one on earth, but it has been made more general in the equation. It is probably a good idea to refresh our memory of the derivation of the radius of curvature for a function $y(x)$ at some regular point. In the extra sketch, the component of the force or acceleration along the normal to the curve shown is $g\cos\theta = gdx/ds$, while the component along the tangent is $g\sin\theta = gdy/ds$. Hence the equation $grdx = 2vds$ can be reduced to the familiar form $V^2/r = gdx/ds$. The radius of curvature is given by $r = ds/d\theta = (ds/dx)(dx/d\theta$; now the element of arc $ds^2 = dx^2 + dy^2$, and $dy/dx = tan\,\theta$, leading to $d^2y/dx^2 = sec^2\theta.d\theta/dx$ and hence $d\theta/dx = (d^2y/dx^2)/(1+tan^2\theta) = ddy/ds^2$. From which it follows that $r = (ds/dx)(ds^2/ddy) = ds^3/dxddy$. The negative sign is applied for a convex curve as we have here.]

Now from the other equation, this value found is put in place of v and the equation is produced :

$$\frac{grdx}{2ds} = c - gy .$$

Truly, $r = \frac{ds^3}{-dxddy}$ with constant dx put in place [i. e. x is an independent variable], with which substituted, the equation becomes : $\frac{-gds^2}{2ddy} = c - gy$ or

$$2cddy = 2gyddy - gdx^2 - gdy^2$$

with $dx^2 + dy^2$ in place of ds^2 . The integral of this equation can be found:

$$\frac{dx}{ds} = \sqrt{\frac{C}{c-gy}} .$$

[This can be shown be assuming $(ds/dx)^2 = p(y)$, differentiating w.r.t. x, and substituting in the original equation $\frac{-gds^2}{2ddy} = c - gy$.]

Moreover with $y = 0$ we designate $C = \lambda^2 c$. We therefore have : $\frac{dx}{ds} = \lambda\sqrt{\frac{c}{c-gy}}$ and hence

$$\frac{\lambda dy\sqrt{c}}{\sqrt{(c(1-\lambda^2)-gy)}} = dx .$$

The integral of which is :

$$C - 2\lambda\sqrt{(c(1-\lambda^2)-gy)} = \frac{gx}{\sqrt{c}} = 2\lambda\sqrt{c(1-\lambda^2)} - 2\lambda\sqrt{(c(1-\lambda^2)-gy)} ,$$

with the constant C determined, so that y vanishes by putting $x = 0$. For the sake of brevity, the sine of the angle HAB or $\sqrt{(1-\lambda^2)} = \mu$, then

$$gx = 2\lambda\mu c - 2\lambda\sqrt{(\mu^2c^2 - gcy)}$$

and hence,

$$y = \frac{\mu x}{\lambda} - \frac{gx^2}{4\lambda^2 c} .$$

Hence also we have :

$$v = c - \frac{\mu gx}{\lambda} + \frac{g^2x^2}{4\lambda^2 c}$$

and

$$ds^2 = dx^2(1+\frac{\mu^2}{\lambda^2}-\frac{\mu gx}{\lambda^3 c}+\frac{g^2x^2}{4\lambda^4c^2}) = \frac{dx^2}{\lambda^2}(1-\frac{\mu gx}{\lambda c}+\frac{g^2x^2}{4\lambda^4c^2}).$$

Consequently the equation arises : $\frac{ds^2}{\mathrm{v}} = \frac{dx^2}{\lambda^2 c}$ and

$$\int\frac{ds}{\sqrt{\mathrm{v}}} = \frac{x}{\lambda\sqrt{c}}$$

is equal to the time, in which the arc *AM* is travelled through. Q.E.I.

Corollary 1.

565. Therefore the body falls back on the horizontal line *AB* at the point *B* with $AB = \frac{4\lambda\mu c}{g}$. Indeed the time, in which the body is turning above AB and the curve ADB is completed, is given by $\frac{4\mu\sqrt{c}}{g}$.

Corollary 2.

566. Moreover $2\lambda\mu$ denotes the sine of the angle, which is twice the sine of the angle *HAB*. Whereby, if the sine of this angle is called χ, [p. 234] then $AB = \frac{2\chi c}{g}$. From which it is apparent that the distance AB to be a maximum, if $\chi = 1$ and thus the angle HAB is half a right angle, if indeed the body is projected with the same speed $\sqrt{c}$.

Corollary 3.

567. It is also understood that the horizontal motion of the body is uniform. For the times, in which any arc is described, are proportional to the corresponding abscissa on the line *AB*.

Corollary 4.

568. If the body is always projected with the same speed $\sqrt{c}$, but at different angles with *AB*, the times in which the motion above *AB* is completed, are to each other as the sines of the angles *HAB* (565).

Corollary 5.

569. The maximum height *DE*, to which the body can reach, is the vertical line through the point *E*, taken from $AE = \frac{2\lambda\mu c}{g}$. From which it is apparent that *AE* is half of *AB*. Truly the maximum height itself *DE* is equal to $\frac{\mu^2 c}{g}$, which hence is in proportion to the square of the sine of the angle *HAB*.

Corollary 6.

570. From the equation $y = \frac{\mu x}{\lambda} - \frac{gx^2}{4\lambda^2 c}$ it is seen that the curve *ADB* is a parabola, the axis of which is the line *DE* and the parameter is equal to $\frac{AE^2}{DE} = \frac{4\lambda^2 c}{g}$. Therefore the parameter is proportional to the square of the cosine of the angle *HAB*. [p. 235]

Corollary 8.

571. Hence the vertex of this parabola is the point D, and the distance DF of the focus F from the vertex D is equal to $\frac{\lambda^2 c}{g}$. Whereby if the line *MF* is drawn, this will be $MF = DQ + DF$ from the nature of the parabola. [For $MF^2 = (y-a)^2 + y^2 = (y+a)^2$.]

Corollary 8.

572. Moreover also $MF = DE - MP + DF$ [$= \frac{c}{g} - (y-a) + a$] $= \frac{c}{g} - y = \frac{c-gy}{g}$. Since truly $v = c - gy$ is the height corresponding to the speed at $M = g.MF$. Therefore it is also apparent that $AF = \frac{c}{g}$, [as $c = g.AF$].

Corollary 9.

573. Therefore it is evident that the body describing this parabola has the same speed at some point *M*, as the same body acted on by the same uniform force falling straight down can acquire from that height, which is equal to the distance of the point M from the focus of the parabola.

Corollary 10.

574. The cosine of the angle FAE is equal to $\frac{AE}{AF} = 2\lambda\mu = \chi$. From which it is evident that the angle *FAE* is equal to the complement of twice the angle *HAE* to the right angle or rather the excess of twice the angle over the right angle. [$\cos FAE = 2\lambda\mu = \sin 2HAB$; FAE = 90 - 2HAB or 2HAB - 90]

Corollary 11.

575. Since the angle *AFD* is the supplement of the angle *AFE*, then the angle *AFD* is twice the angle *HAB*. [p. 236] [AFD = 180 - AFE = 90 + FAE = 2HAB.]

Scholium 1.

576. Therefore the construction of the parabola is easily deduced from these, that describes the projection of the body with a given speed in a given direction. For with *AG* drawn normal to *AB*, and the angle GAF taken equal to twice the angle *HAE*, and in this direction *AF* is taken equal to the height, from which the same body falling in a straight line acquires the speed equal to this, with which it is projected from *A*, from which the focus point *F* of the parabola sought is found (573, 575). Again the normal *DE* through *F* drawn to the line AB is the axis of this parabola, and the vertex D is found by taking $DF = \frac{AF-FE}{2}$. Therefore when the parameter is equal to 4*DF*, the description of the parabola is brought into view.

Scholium 2.

577. If $g = 1$, then we have the case of terrestrial gravity. Concerning which, if there should be no air present, which might impede the motion of the body on account of resistance, all projected bodies move in parabolas. Galileo first elicited this truth, and after him all the writers of mechanics have demonstrated this fact. And indeed it can be arrived at in a much shorter way without differential equations and differentiation, but we have preferred to use a general method, which can be widely extended, rather than apply ourselves too much with the particular solutions, which can only be adapted to that one case. [p. 237]

PROPOSITION 73.

PROBLEM.

578. *If the body at A* (Fig. 50) *with a given speed and projected in a given direction is always attracted to the line AB in some ratio of some function of the distances from this line, to determine the curve ADB, that the body thus acted upon describes and the whole motion on this curve.*

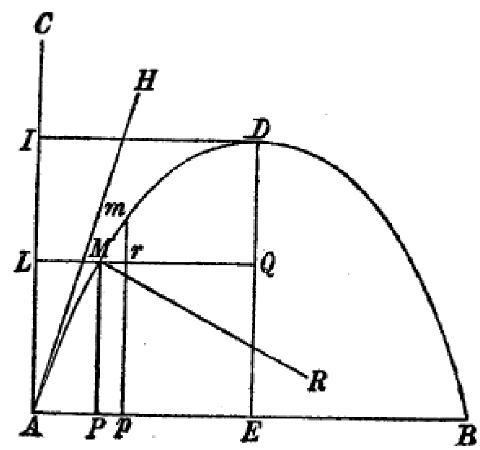


Fig. 50.

SOLUTION.

Let the speed at A = $\sqrt{c}$ and the sine of the angle $HAB = \mu$, with the whole sine taken as 1, and the cosine equal to λ. The speed of the body at *M* corresponds to the height *v* and the element $Mm = ds$, and likewise the abscissa $AP = x$, with the applied line $PM = y$ and the radius of curvature $MR = r$. Moreover while the body is traversing the element *Mm*, it is meanwhile pulled by a given force , which shall be equal to *P*, for a certain function of y, in the direction MP. With these put in place, $dv = -Pdy$, et $Pr\,dx = 2dvds$ (557).

Moreover, $\int Pdy = Y$ and *Y* is such a function of *y*, which is made equal to 0 with *y* put equal to 0. Hence, on this account, $v = c - Y$. Then from the first equation, the value of *P*, i. e. $\frac{-dv}{dy}$, is substituted into the other equation $Pr\,dx = 2dvds$, and there is obtained :

$$\frac{-dv}{v} = \frac{2dyds}{rdx}.$$

It is true, and with *ds* made constant, that with $r = \frac{dyds}{ddx}$, [The reason for introducing the new curvature condition is of course to allow the variables to be separated.] and hence $\frac{-dv}{v}=\frac{2ddx}{dx}$ and by integrating,

$$l\,C-l\,v=l\frac{dx^2}{ds^2}.$$

The constant C is determined from this , that at the point in which $v=c$, put $\frac{dx}{ds}=\lambda$. According to which, $l\,C=l\,c+2l\lambda$ again with the numbers taken

$$\lambda^2cds^2=vdx^2.$$

[p. 238] From this equation the element of the time is at once found : $\frac{ds}{\sqrt{v}}=\frac{dx}{\lambda\sqrt{c}}$.

Moreover the equation of the curve *AMDB* is obtained from the two equations found : $v=c-Y$ et $\lambda^2cds^2=vdx^2$; by eliminating *v* there arises :

$$\lambda^2cds^2=cdx^2-Ydx^2=\lambda^2cdx^2+\lambda^2cdy^2.$$

Bus since $1-\lambda^2=\mu^2$, it gives

$$dx=\frac{\lambda dy\sqrt{c}}{\sqrt{(\mu^2c-Y)}},$$

in which equation, since the variables *x* and *y* can be separated from each other, the curve *ADB* can be constructed. Q.E.I.

Corollary 1.

579. Therefore the times in which any arcs *AM* are described, are in the ratio of the corresponding abscissae *AP*. Indeed the time to traverse *AM* is equal to $\frac{x}{\lambda\sqrt{c}}$.

[Since $\frac{ds}{\sqrt{v}}=\frac{dx}{\lambda\sqrt{c}}$.]

Scholium 1.

580. The motion of the body on the curve *AM* can be considered to be resolved into two other motions, of which the one becomes lines parallel to the line *AB*, and the other follows the perpendiculars to this line *AB*. In that motion the body progresses following the line *AB*, and indeed it either rises or falls with respect to the line *AB*. Now indeed it is evident that the progression of the horizontal motion is not to be changed by a force, the direction of which is always perpendicular to *AB*, and for this reason this motion must always be regular and made with the same speed, which arises from the resolution of the initial motion. Since the direction of the initial motion is along the line *AH*, the speed of this $\sqrt{c}$ to the progressive speed along *AB* is as the whole sine 1 to the cosine of the angle *HAB*, which is λ. [p. 239] Therefore the speed of progression along AB is $\lambda\sqrt{c}$, from which the time, in which the horizontal motion is completed through the distance $AP=x$, comes to equal $\frac{x}{\lambda\sqrt{c}}$, as we have found.

Corollary 2.

581. If the body with the speed $\sqrt{c}$ rises from A perpendicularly along AC and we take $AL = PM = y$, then the speed at L is equal to the speed at M, clearly $\sqrt{v}$. For $dv = -Pdy$ and $v = c - Y$, as we have found for the point M.

Corollary 3.

582. If AC is the total height, to which the body projected up from A with the speed $\sqrt{c}$ is able to reach, then the height CL is that by which the body acquires the same speed by falling, that it had at M.

Corollary 4.

583. The maximum height *DE* is found by making $dy = 0$, in which case $Y = \mu^2 c$, from which equation the value of y extracted must give the height *DE*, and the speed at *D* is of such as size as that acquired by the body falling from the height *CI*.

[We can perhaps also see this from energy conservation: The initial kinetic energy of the body at A is partitioned into a constant amount for the horizontal motion E_h, and an amount that is all transformed into potential energy at D, E_v. Now, $E_h + E_v$ is constant, and is equal to the potential energy at C if the body is projected straight up, in which case the amount of kinetic energy gained in falling is the same as E_h, as the potential energy is the same at D and I in each case.]

Corollary 5.

584. Moreover, we have also found that $v = \frac{\lambda^2 cds^2}{dx^2}$. Whereby, when at the point D, $dx = ds$, and the speed at D = $\lambda\sqrt{c}$, which is equal to the horizontal speed that the body has progressing along *AB*. [p. 240]

Scholium 2.

585. Indeed the size of AB is not apparent from the equation, since it cannot be integrated. Yet it is evident that the parts AE and *EB* are required to be equal, and the branch *DB* is similar and equal to the arc *DA*. For after the body arrives at D, it is accelerated again in a similar way, in which it was retarded before along AD; because the force is the same in these same distances from AB and in this way the motion is again perfectly restored.

Scholium 3.

586. Hence also the inverse problem can be resolved easily, where for a given curve ADB and with the speed of the body at A, the law of the force is sought which must be put in place, in order that the body moves on that curve. For from the equation :

$dx = \frac{\lambda dy\sqrt{c}}{\sqrt{(\mu^2 c - Y)}}$ it is found that $Y = \mu^2 c - \frac{\lambda^2 cdy^2}{dx^2}$ and $dY = Pdy = \frac{-2\lambda^2 cdyddy}{dx^2}$ with dx constant. Since $r = \frac{ds^2}{-dxddy}$ with dx placed constant, then $P = \frac{2\lambda^2 cds^3}{rdx^3}$, which is equal to the force acting on the body at *M,* following the direction *MP*.

PROPOSITION 74.

PROBLEM.

587. *A body is projected at A and everywhere it is pulled* (Fig. 51) *towards the centre of force C by some centripetal force, and it is required to determine the nature of the curve AM, in which the body is moving, and the motion of the body on this curve.* [p. 241]

SOLUTION.

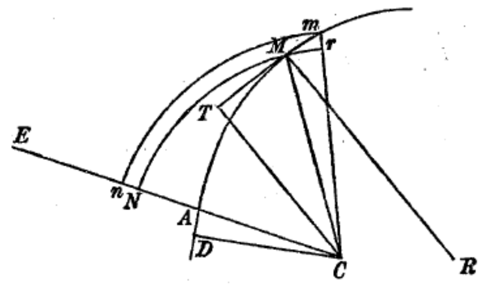


Fig. 51.

The centripetal force acting on the body at *M* towards *C* is put equal to *P*, and the speed of the body at *M* shall correspond to the height *v*. Then the distance *CM* is called *y*, the element of length *Mm* is equal to *ds* and the perpendicular *CT* to the tangent of the curve *MT* is called *p*, also the element *Mr* is *dx* and the radius of curvature *MR* is called *r*. Therefore because of the similar triangles *Mmr* and *CMT* $ds\sqrt{(y^2 - p^2)} = ydy$ and $dx\sqrt{(y^2 - p^2)} = pdy$ [where $MT = \sqrt{(y^2 - p^2)}$]. And the radius of curvature is found $r = \frac{ydy}{dp}$.

[For on drawing the other tangent at *m*, and considering the equality of the small angle between these tangents and the angle *MCm*, $\frac{dp}{MT} = \frac{ds}{r}$ or $r = \frac{ds}{dp}.MT = \frac{ydy}{dp}$. See sketch added to (601)]

With these put in place we have : $dv = -Pdy$ and $Pr\,dx = 2vds$ (557). With these equations collated with the elimination of P, we have : $rdvdx = -2vdsdy$ or $\frac{dv}{v} = \frac{-2dsdy}{rdx}$.

Substituting $\frac{ydy}{dp}$ in place of r and $\frac{y}{p}$ in place of $\frac{ds}{dx}$, there is produced :

$$\frac{dv}{v} = \frac{-2dp}{p}.$$

Which on integrating gives :

$$v = \frac{C}{p^2}.$$

This constant C is defined from the given initial speed at A, which corresponds to the height *c*, and with the direction of projection, that we define thus, in order that with the distance $CA = a$, and the perpendicular *CD* to the tangent at *AD* is equal to *h*. On account of which $C = ch^2$ and

$$v = \frac{ch^2}{p^2}.$$

Hence we have $\frac{ds}{\sqrt{v}} = \frac{pds}{h\sqrt{c}}$, which is equal to the element of the time required to traverse the element *Mm*. Consequently as $pds = 2MCm$ the time [i. e. twice the area of the elemental triangle *MCm*, leading to Kepler's law for equal areas described in equal times in the inverse square law instance; we note also that $v = \frac{ch^2}{p^2}$ is the formula for the conservation of angular momentum of the unit point mass], in which the arc AM is traversed,

$$= \frac{2.ACM}{h\sqrt{c}}.$$

The curve itself can be determined by substituting $\frac{ch^2}{p^2}$ in place of v in the equation $Pr\,dx = 2vds$, and with this done the equation is produced : $Pp^2rdx = 2ch^2ds$. This equation, with the values $\frac{ydy}{dp}$ and $\frac{y}{p}$ substituted in the equation, in place of r and $\frac{ds}{dx}$, is changed into this :

$$Pdy = \frac{2ch^2dp}{p^3}.$$

Which equation, whatever the function of y P shall be, can be constructed on account of the separate variables. Q.E.I. [p. 242]

Corollary 1.

588. Because the time, in which the arc AM is traversed, is $\frac{2.ACM}{h\sqrt{c}}$, the times, in which any arcs can be described, are as the areas taken described by the arc and with the line drawn from the centre C.

Corollary 2.

589. Then since $v = \frac{ch^2}{p^2}$, it follows that $\sqrt{v} = \frac{h\sqrt{c}}{p}$. Therefore the speed of the body at any location of the curve traversed varies inversely as the perpendicular from the centre C sent to the tangent at that point.

Scholium 1.

590. This description of the equality of the areas has been established in Newton's first proposition, from which nearly everything is deduced. Moreover these two properties are the most general and they only require that the direction of the centripetal force is always directed towards the centre of the circle. For whatever the size of the centripetal force, either by a function of the lengths CM, or otherwise expressed, yet each is prevailing equally. For as in these calculations that we have come across, from the calculation the centripetal force P might be removed, but yet in these only the direction is abandoned in the consideration.

Corollary 3.

591. It is necessary that the centripetal force itself should be known for the given curve described by the body, [p. 243] and from that the equation of the curve can be found. For $Pdy = \frac{2ch^2dp}{p^3}$ expresses the nature of the curve, if P is a given quantity.

Corollary 4.

592. Since $\frac{ydy}{dp} = r$, also we can put $P = \frac{2ch^2y}{p^3r}$. This is De Moivre's Theorem,

$P = \frac{2ch^2dp}{p^3dy}$,truly that Keill first contended to have come across.

[Abraham de Moivre (1667–1754) : *Some simple properties of the conic sections deduced from the nature of the foci; with general theorems of centripetal force, by means of which the law of the centripetal force tending to the foci of the sections, the velocities of bodies revolving in them, and the description of the orbits, may be easily determined.* Philosophical Transactions 1717, p. 622; cf. *Miscellanea analytica de seriebus et quadraturis,* London 1730, p.233.
John Keill (1671–1721) : *Of the laws of centripetal force*, Philosophical Transactions 1708, p. 174; " The learned Mr. Halley having showed me a theorem, by which the law of centripetal force can be exhibited in finite quantities, which was communicated to him by Mr. De Moivre, who said that Mr. Is. Newton had before discovered a similar theorem, and as the demonstration of the theorem is very easy, I wish to communicate it to the public, with some other thoughts in the same subject."
Noted by Paul Stackel in the 1912 edition of *Tom I*, which is available from the *Gallica* website, which is used in this translation. You will have to use JSTOR perhaps to access the papers quoted, to which I do not have acces any more.]

Scholium 2.

593. These equations are useful in two ways : For in the first place from the given law of the centripetal force, the nature of the curve can be determined that a body projected describes. Then also in turn with the help of these equations, if the curve is given, that the body describes around the centre of force C, it is possible to define the centripetal force bringing about the motion at any place, in order that the body is free to move on this curve.

Corollary 5.

594. Since also $dv = -Pdy$, it is evident, if P is a function of y, the speed everywhere depends on the distance of the body from the centre and at the same distances the body must have the same speed.

Corollary 6.

595. Therefore just as P is a function of y, so also the curve described can thus be compared, as with equal distances from the centre the tangents dropped are always equal to each other, since the speeds vary inversely as the lengths of these perpendiculars (589). [p. 244 : Later of course considered to be an aspect of the conservation of the angular momentum of the rotating body.]

Scholium 3.

596. And always, if P is a function of y, a line EC can be assigned, along which a body acted on by the centripetal force in falling to the individual points N has the same speed, that it has on the curve AM at points M with the points equally distant from the centre. For take $CN = CM = y$ and $Nn = mr = dy$, if the speed at N is equal to the speed at M, clearly corresponding to the height v, it is also the case, as the body traverses through the element Nn, that $dv = -Pdy$. From which it is understood that the speed at n is always equal to the speed at m. And thus the body has a speed at the individual points of the line EC, as great as it has on the curve AM at the same distances from C. If therefore the initial motion starts from E at which the speed is zero, the speed of the body at the individual points on the curve AM is known from the given line EC. Therefore we will use this line EC in later results in defining the speeds of the motion of the body and we will call that the distance defining the speeds.

Corollary 7.

597. Therefore with the speed given at A, clearly corresponding to the height c, from this the whole distance EC can be found. For only the distant point E from C needs to be taken, as the body falling from E is acted upon by this centripetal force and acquires a speed equal to $\sqrt{c}$ at A. [p. 245]

Corollary 8.

598. If the angle $MCm = dw$, then $dw = \frac{dx}{y} = \frac{pdy}{y\sqrt{(y^2-p^2)}}$. Hence there arises

$p = \frac{y^2dw}{\sqrt{(y^2dw^2+dy^2)}}$. Truly the time element, in which the angle MCm is resolved, is equal

to $\frac{pds}{h\sqrt{c}} = \frac{pydy}{h\sqrt{c(y^2-p^2)}}$. Therefore this element of time is equal to $\frac{y^2dw}{h\sqrt{c}}$.

Corollary 9.

599. If we wish to measure the angular speed, or that in which MCm is traversed, by dividing this angle by the time, the angular speed at M is produced equal to $\frac{h\sqrt{c}}{y^2}$.

Therefore the speed of the angle [our angular velocity] varies inversely as the square of the distance of the body from the centre C.

Scholium 4.

600. Moreover since the curve that a body describes acted upon by a given centripetal force, cannot be known otherwise than from the equation between the distance of the body from the centre and the perpendicular dropped on the tangent from the centre, generally to be judged with difficulty, such is the curve found, when we are accustomed to explain the nature of the curves by equations between orthogonal coordinates. Yet nevertheless equations of this kind between the distance from the centre and the perpendicular to the tangent, provided they are either of algebraic or differential forms, in which the variables can be separated from each other, are sufficient for the curves sought to be constructed. But where is possible to investigate thoroughly the nature and order of these curves, [p. 246] we will give here the method in which these equations between the distance and the perpendicular can be reduced to ordinary equations between the coordinates.

PROPOSITION 75.

PROBLEM.

601. *The nature of the curve AM* (Fig. 52), *that a body acted upon by some centripetal force is to be described, in order the equation between the orthogonal coordinates CP and PM referred to fixed axes AC can be defined.*

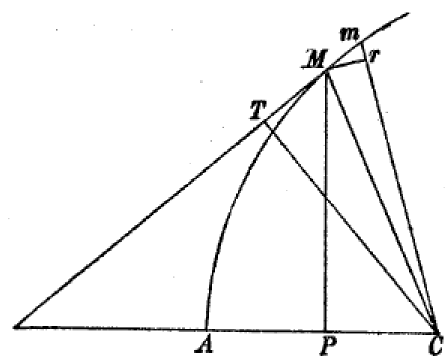


Fig. 52.

SOLUTION.

As before with the initial speed at $A=\sqrt{c}$, $AC = a$ and with the perpendicular from C dropped to the tangent at A equal to h and in addition $CM = y$, $CT = p$ and the centripetal force at M is equal to P, which is a certain function of y, then CP is called equal to x and $PM = z$, then $CM=\sqrt{(x^2+z^2)}=y$ or $z=\sqrt{(y^2-x^2)}$; for we keep y in place of z in the calculation, since in this manner the calculation is easier and shorter, and after handling the whole operation, with z in readiness to be introduced in place of y. With these in place :

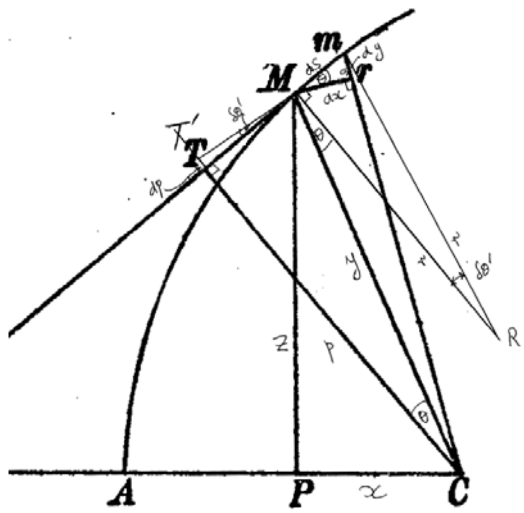
$$Mm=\sqrt{(dx^2+dz^2)}=\frac{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}{\sqrt{(y^2-x^2)}} \text{ and } Mr=\frac{-ydx+xdy.}{\sqrt{(y^2-x^2)}}$$

[This use of this auxillary coordinate y, which is just the radial distance of the body from the centre of force, eases the calculation, where dz^2 is found from $z^2=y^2-x^2$ and $mr = dy$, giving $Mr^2=Mm^2-dy^2,$ leading to Mr.] Therefore we have [see Prop. 74 and following sketch for the similar triangles]:

$$[\cos(\theta)=]\frac{Mr}{Mm}=\frac{p}{y}=\frac{-ydx+xdy.}{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}$$

and consequently

$$p=\frac{yxdy-y^2dx}{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}.$$



But for the curve AM, we have elicited the equation previously (591) : $Pdy=\frac{2ch^2dp}{p^3}$. Moreover we set Y for the integral of Pdy thus assumed, [Y is related to the work done by the centripetal force as the body moves along y, and hence is the potential energy function for unit mass and gravity;] so that it vanishes with $y = a$. With this done, then $Y=C-\frac{ch^2}{p^2}$, where the constant C on account of $p = h$, if $y = a$, must be equal to c. Hence there comes about : $Y=\frac{cp^2-ch^2}{p^2}$, [p. 247] from which there is produced :

$$p=\frac{h\sqrt{c}}{\sqrt{(c-Y)}}.$$

[We note that $h\sqrt{c}$ is the initial angular momentum of unit mass at the point in the orbit where the speed and the distance from C are perpendicular; while Y is the height corresponding to the change in the potenial energy of unit mass in a uniform field of g = 1, and hence $\sqrt{c-Y}$ is the speed elsewhere corresponding to the perpendicular distance p, giving the same angular momentum.]
Which quantity, when Y is a function of y, is also a certain function of y. Hence for this reason, we have this equation :

$$[p=]\ \frac{h\sqrt{c}}{\sqrt{(c-Y)}}=\frac{yxdy-y^2dx}{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}.$$

We put $x=uy$ and $\frac{h\sqrt{c}}{\sqrt{(c-Y)}}=Q$ for the sake of brevity, then $Q=\frac{-y^2dx}{\sqrt{(dy^2-u^2dy^2+y^2du^2)}}$, from which there arises :

$$\frac{du}{\sqrt{(1-u^2)}}=\frac{Qdy}{y\sqrt{(y^2-Q^2)}}=\frac{hdy\sqrt{c}}{y\sqrt{(cy^2-ch^2-y^2Y)}}$$

with $\frac{h\sqrt{c}}{y\sqrt{(c-Y)}}$ restored in place of Q. Therefore which equation, when the variables u and y in that equation are separated [note that u is the cosine of the angle θ' between MC and PC; thus y and θ' are the polar coordinates of M as noted in the next cor.], can always be constructed. Q.E.I.

Corollary 1.

602. Since $u = \frac{x}{y}$, *u* expresses the cosine of the angle *MCA*. And for this reason, this final equation is separated into the distance of the body from the centre and the cosine of the angle *ACM*. Indeed from this equation there emerges immediately the equation between *x* and *z*. [Thus, in this remarkable manner, the equation has been separated into angular and radial components for the general case.]

Corollary 2.

603. Truly the equation cannot be algebraic, unless $\int \frac{hdy\sqrt{c}}{y\sqrt{(cy^2 - ch^2 - y^2Y)}}$ denotes the arc of a circle commensurable with the arc $\int \frac{du}{\sqrt{(1-u^2)}}$. [Note that on putting $u = \cos\theta'$, this integrand becomes –dθ'.]

Corollary 3.

604. Therefore whenever $\frac{hdy\sqrt{c}}{y\sqrt{(cy^2 - ch^2 - y^2Y)}}$ can be reduced to a form of the kind $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ and λ is a rational number, so an algebraic equation for the curve sought can be shown. [p. 248]

Scholium 1.

605. But if $\frac{du}{\sqrt{(1-u^2)}}$ is equal to a quantity of the kind $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$, with the integration carried put with imaginary logarithms, this equation is obtained :

$$\frac{\sqrt{(1-u^2)}+u\sqrt{-1}}{\sqrt{(1-u^2)}-u\sqrt{-1}} = \left(\frac{\sqrt{(A^2-C^2)}-C\sqrt{-1}}{\sqrt{(A^2-C^2)}+C\sqrt{-1}}\right)^{\lambda}\left(\frac{\sqrt{(A^2-Z^2)}+Z\sqrt{-1}}{\sqrt{(A^2-Z^2)}-Z\sqrt{-1}}\right)^{\lambda}.$$

[Note that if $u = \cos\theta$, where we have now taken the angle $MCA = \theta,$ then $\frac{\sqrt{(1-u^2)}+u\sqrt{-1}}{\sqrt{(1-u^2)}-u\sqrt{-1}} = \frac{\sin\theta + i\cos\theta}{\sin\theta - i\cos\theta} = -e^{i.-2\theta};$ similarly, if $Z = A\cos\theta$, then $\left(\frac{\sqrt{(A^2-Z^2)}+Z\sqrt{-1}}{\sqrt{(A^2-Z^2)}-Z\sqrt{-1}}\right)^{\lambda} = \left(\frac{A\sin\theta + Ai\cos\theta}{A\sin\theta - Ai\cos\theta}\right)^{\lambda} = (-1)^{\lambda} e^{i.-2\lambda\theta}$, and similarly, if $C = A\cos\alpha$, then $\left(\frac{\sqrt{(A^2-C^2)}-C\sqrt{-1}}{\sqrt{(A^2-C^2)}+C\sqrt{-1}}\right)^{\lambda} = (-1)^{\lambda} e^{i.2\lambda\alpha}$. Hence, the result of the integration can be written in the convenient form : $e^{i.\theta} = i(Ae^{i.\theta})^{\lambda}.(Ae^{i.-\alpha})^{\lambda} / A^{2\lambda};$ hence, $u = \cos\theta$ gives the expression for *u* subsequently quoted below.

Note also that the angle θ in the original integral can be written as

$2i\theta = i\theta + i\theta = \log e^{i\theta} - \log e^{-i\theta} = \log \frac{\sin\theta + i\cos\vartheta}{\sin\theta - i\cos\vartheta}$.]

Truly here C is a constant quantity to be determined from this equation, since if $CM(y) = CA(a)$, then likewise too, $x = a$ or $u = 1$. Moreover from the above equation, this equation can be constructed :

$$u = \frac{\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\lambda}-\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\lambda}}{2A^{2\lambda}\sqrt{-1}}.$$

Which, whenever λ is a rational number, it is free to be returned from the affects of the imaginary $\sqrt{-1}$ and transformed into an algebraic equation of a certain order. [Various values of λ corrrespond to well-known curves to be investigated; e. g. $\lambda = \frac{1}{2}, 1$ and correspond to ellipses with different centripetal forces.]

Corollary 4.

606. Since $x = uy$, this equation is put in place :

$$x = \frac{\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\lambda}y-\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\lambda}y}{2A^{2\lambda}\sqrt{-1}},$$

which, since Z is a function of y and $y = \sqrt{(x^2 + z^2)}$,can easily be changed into an equation between x and z.

Scholium 2.

607. The above equation can also be transformed into this :

$$Z = \frac{1}{2\sqrt{-1}}\left[\left(\sqrt{(1-u^2)}+u\sqrt{-1}\right)^{\frac{1}{\lambda}}\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)-\left(\sqrt{(1-u^2)}-u\sqrt{-1}\right)^{\frac{1}{\lambda}}\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)\right],$$

[p. 249] which is more convenient, if $\frac{1}{\lambda}$ is a positive whole number.

[Thus, from $e^{i.\theta} = i(Ae^{i.\theta})^{\lambda}.(Ae^{i.-\alpha})^{\lambda}/A^{2\lambda}$, we have $e^{i\theta} = e^{i(\theta+\pi/2)/\lambda}.e^{i\alpha}$; from which $Z = A\cos\theta$ can be found.]

Scholion 3.

608. But if λ is a negative number, equal to $= -\mu$, we have :

$$\frac{\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\mu}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\mu}-\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\mu}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\mu}}{2A^{2\mu}\sqrt{-1}} = u$$

From which it is apparent, if λ is negative, the value of u can only be negative, indeed that which is understood from the differential equation. Truly in a similar manner,

$$Z = \frac{1}{2\sqrt{-1}}\left[\left(\sqrt{(1-u^2)}-u\sqrt{-1}\right)^{\frac{1}{\mu}}\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)-\left(\sqrt{(1-u^2)}+u\sqrt{-1}\right)^{\frac{1}{\mu}}\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)\right].$$

Corollary 5.

609. If $\lambda = 1$, then $u = \frac{Z\sqrt{(A^2-C^2)} - C\sqrt{(A^2-Z^2)}}{A^2}$ and $x = \frac{Zy\sqrt{(A^2-C^2)} - Cy\sqrt{(A^2-Z^2)}}{A^2}$.
If $\lambda = -1$, also u or x has to be taken negative.

Corollary 6.

610. If $\lambda = 2$, then

$$u = \frac{2Z(A^2-2C^2)\sqrt{(A^2-Z^2)} - 2C(A^2-2Z^2)\sqrt{(A^2-C^2)}}{A^4}.$$

But if $\lambda = \frac{1}{2}$, then $Z = C - 2Cu^2 + 2u\sqrt{(A^2 - C^2)(1-u^2)}$. [p. 250]

PROPOSITION 76.

THEOREM.

611. *If the centripetal force is as some function of the distance from the centre C* (Fig. 53), *and a body at A is projected following the direction normal to AC with a speed, of which the corresponding height has a ratio to half AC as the centripetal force at A is to the force of gravity* 1*, then this body moves uniformly on the circumference of the circle AMBA, the centre of which is C.*

DEMONSTRATION.

Indeed the body moves in this circle ; since the distance of this from the centre of the circle does not change ever from the centripetal force acting towards *C*, also for any function of the distance if it is in proportion to the centripetal force. And since it is acting towards the centre *C*, the direction of the force acting is always normal to any small portion of the curve, along which the body is moving. On this account it is always acted on by a normal force, and nowhere by one along the tangent, and hence, and for this reason this speed always remains the same (561), and thus the body describes the periphery with a uniform motion. Then since the centripetal or normal force is always the same, put this equal to *g* and likewise the constant speed corresponds to the height *c* and the radius *AC* = *a*, which quantity *a* is everywhere shown to be the radius of curvature. Therefore with these in place *ag* = 2*c* (561)[$dv = 0$ and P$r = 2v$]. From which this ratio arises : As the height to the speed of the body, with which it is initially projected from A, there corresponds *c* to half the distance AC, $\frac{1}{2}a$, thus as the centripetal force *g* to the force of gravity 1. Q.E.D. [p. 251]

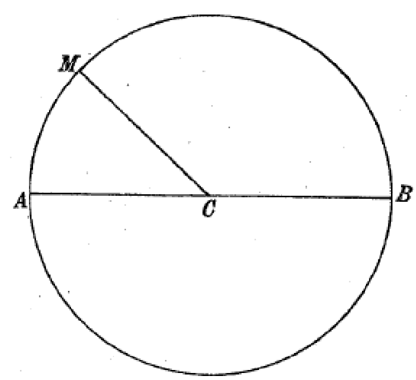


Fig. 53.

Corollary 1.

612. Therefore when the body has once described an arc of the circle, the centre of which is itself the centre of force C, then it rotates for ever on the periphery of that circle. If indeed the centripetal force only depends on the distances from the centre, thus as with equal distances, the centripetal force is always equal.

Corollary 2.

613. With the ratio of the diameter to the periphery put as 1 : π, the periphery of the circle in which the body is moving is equal to $2\pi a$. Then since the speed, with which the body is moving, is equal to $\sqrt{c} = \sqrt{\frac{ag}{2}}$, the time of one period along the whole periphery is equal to $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$.

Corollary 3.

614. If therefore many bodies are moving in different circles, the time of the revolutions are in the square root ratio composed as directly from the radii of the circles and inversely as the centripetal forces.

Corollary 4.

615. If the body is projected from A perpendicularly to AC, but with a speed either greater or less than $\sqrt{\frac{ag}{2}}$, then the body describes the arc of a circle, the radius of which is either greater or less than AC. [p. 252]

Corollary 5.

616. Therefore in this case, when the body begins to move along the arc of the circle, the centre of which is not at C, suddenly it approaches more towards, or recedes from, the centre of the force. And this sudden motion arises from the action of another centripetal force, unless perhaps the centripetal force is the same everywhere.

Scholium.

617. Moreover the body is projected with some speed from A (Fig. 54), but the direction of this is a perpendicular crossing the line AC through the centre of force C, and the curve BMAND has this property, in order that the potions AMB and AND of this, placed on this side and the other side of AC are similar and equal to each other, and AC is the axis and diameter of this curve. For since, as we have now agreed, the centripetal force is proportional to a certain function of the distances from the centre, the body either above or below AC is acted on by an equal force at the same distance from C, and for this reason this body must move towards A in the same way along DNA, as it moves away from A along AMB, and also it has the same speed at the homologous points M and N.

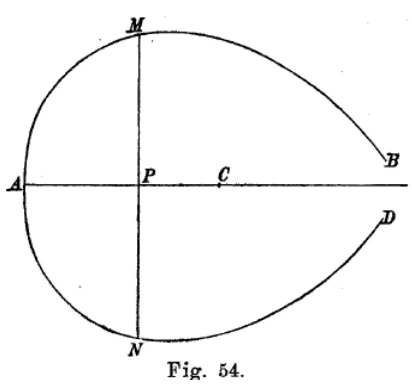


Fig. 54.

Corollary 6.

618. Therefore every line drawn from the centre C, which is normal to the curve, is likewise a diameter of the curve ; thus in order that the parts of the curve on placed on this or that side of this line are equal and similar to each other.[p. 253]

PROPOSITION 77.

THEOREM.

619. *If more bodies are moving around the centre of force C* (Fig. 55) *and describe the similar curves AM and am about C, the speeds at the similar points M and m are in the square root ratio composed from the ratios of the homologous sides and of the centripetal forces at the homologous points M and m.*

DEMONSTRATION.

Because $AC : aC = MC : mC$, also the radius of osculation at M is in the same ratio to the radius of osculation at m, as also is the perpendicular CT to Ct. From proposition 74 (587) it is truly apparent that $Pr\,dx = 2vds$, [i. e. the centripetal force equation normal to the curve with radius of curvature r] or, with $\frac{y}{p}$ put in place of $\frac{ds}{dx}$, $Ppr = 2vy$. Hence therefore the speed is given by :

$$\sqrt{v} = \sqrt{\frac{Ppr}{2y}}.$$

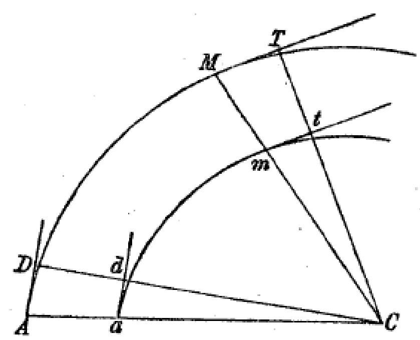


Fig. 55.

From which this ratio is produced : the speed at M is to the speed at m in the square root ratio composed from the direct ratios of the centripetal forces at M and m and the perpendiculars CT to cT, and in the inverse ratio of the distances MC to mC. Since truly CT is to cT as MC is to mC and the radius of osculation at M is to the radius of osculation at m as MC is to mC, then the speed at M to the speed at m is in the square root ratio composed from the ratios of the centripetal forces at M and m and of the homologous sides MC to mC. Q.E.D.

Corollary 1.

620. If therefore we call $AC = A$, $aC = a$, $CD = H$ and $Cd = h$, and likewise the speed at $A = \sqrt{C}$ and at $a = \sqrt{c}$, the angular speed at $M = \frac{H\sqrt{C}}{MC^2}$ [p. 254] and the angular speed at $m = \frac{h\sqrt{C}}{mC^2}$ (599). But since $H : h = MC : mC = A : a,$ the angular speeds at M and m are as $\frac{\sqrt{C}}{A}$ to $\frac{\sqrt{c}}{a}$, i. e. in a constant ratio.

Corollary 2.

621. Therefore the times, in which the equal angles *ACM* and *aCm* or the homologous distances *AM* and *am* are completed, are reciprocally as the angular speeds at *M* and *m*, i. e. directly as the homologous sides and reciprocally as the speeds at the homologous points.

Scholium 1.

622. Indeed the speeds at the homologous points maintain the same ratio everywhere. For the speed at $M = \frac{H\sqrt{C}}{CT}$ and the speed at $m = \frac{h\sqrt{c}}{Ct}$ (589). On account of which, since $H : h = CT : Ct$, the speed at *M* to the speed at *m* is in the ratio $\sqrt{C}$ to $\sqrt{c}$, i. e. as the speed at *A* to the speed at *a*.

Corollary 3.

623. Moreover from this proposition it is evident that the speed at *A*, $\sqrt{C}$, to the speed at *a*, $\sqrt{c}$, are in the square root ratio composed from the ratios of the centripetal forces at *A* and *a* and of the homologous sides *A* and *a*. Whereby if the said centripetal force at $A = G$ and the centripetal force at $a = g$, then we have the ratio :

$$\sqrt{C} : \sqrt{c} = \sqrt{AG} : \sqrt{ag}.$$ [p. 255]

Corollary 4.

624. Consequently the time to pass through AM is to the time to pass through *am* as $\sqrt{\frac{A}{G}} : \sqrt{\frac{a}{g}}$, i. e. in ratio composed with the square root from the direct proportions of the homologous sides and inversely as the centripetal forces at the points *A* and *a*. Therefore this ratio is constant, and the whole times of the revolutions must keep the same values between each other.

Scholium 2.

625. Also in this case, in which several similar figures are described around the centre *C*, the centripetal forces in the homologous points must maintain the same ratio. For since the force $P = \frac{2ch^2y}{p^3r}$ (592), the centripetal force at *M* to the centripetal force at *m* is directly as the square of the speed at *A* to the square of the speed at *a* and inversely as *AC* to *aC*, which is a constant ratio. On account of which, when more similar figures are to be described around the centre *C*, it is required that a centripetal force of this kind of the distances can be expressed by a function, which presents the centripetal forces in the same ratio at the homologous positions. Clearly with the centripetal force *P* put at the distance *y* and *Q* at the distance *my*, *P* to *Q* must have a constant ratio, in which *y* is not present. For unless this is done, then it cannot be made to happen, that more similar figures can be described around the centre *C*.

Corollary 5.

626. Moreover it is not possible to obtain this result, unless *P* is a certain power of *y*, such as $\frac{y^n}{f^n}$. [p. 256] For in this case, $Q=\frac{m^n y^n}{f^n}$, and the ratio P : Q is 1 : m^n, which is constant.

Corollary 6.

627. Therefore unless the centripetal force is in proportion to a certain power of the distance from the centre *C*, then indeed it is not possible to happen, that more similar figures can be described around the centre *C*. And only with these cases do they have the properties in place, that we have elicited from this proposition.

Corollary 7.

628. Moreover, if $P=\frac{y^n}{f^n}$, then $G=\frac{A^n}{f^n}$ and $g=\frac{a^n}{f^n}$. Therefore the speeds at the homologous places maintain the ratio $A^{\frac{n+1}{2}}$ to $a^{\frac{n+1}{2}}$.

Corollary 8.

629. And the times, in which similar arcs *AM* and *am* are completed, are in the ratio $A^{\frac{1-n}{2}}$ to $a^{\frac{1-n}{2}}$ or in the ratio of a multiple of the homologous sides, the exponent of which is $\frac{1-n}{2}$.

Scholium 3.

630. In the preceding and in this proposition all the theorems are in place, which Huygens put as an appendix concerning centrifugal forces to his tract *de Horologio*. [p. 257] And that part of the appendix that I have placed here, is the part usually deduced from these propositions. [The reader can view a translation of these at the end of the *Horologium* on this website.]

PROPOSITION 78.

PROBLEM.

631. *If the centre of force C attracts directly in the ratio of the distances* (Fig. 56) *and a body is projected from A following the direction normal to a given radius AC, to determine the curve AMDBH that the body describes, and the speed of the body at particular points.*

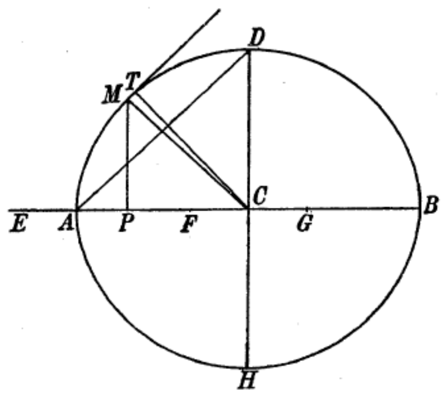


Fig. 56.

SOLUTION.

The distance *CA* is put equal to *a*, and the perpendicular dropped from *C* on the direction of the motion is equal to *a*. Truly the speed at *A* corresponds to the height *c*. On arriving at *M*, the distance $CM = y$; and the perpendicular *CT* from *C* sent to the tangent at *M* is put equal to *p*, and the speed at *M* corresponds to the height *v*. Again the distance at which the centripetal force is equal to the force of gravity is *f*; and the centripetal force at $M = \frac{y}{f}$, with the force of gravity equal to 1. When these are compared with Prop.75 (601), $P = \frac{y}{f}$ and $Y = \frac{y^2 - a^2}{2f}$. On account of which we have : $p = \frac{a\sqrt{2cf}}{\sqrt{a^2 + 2cf - y^2}}$.

For truly $v = \frac{a^2 c}{p^2}$ (587), from which it follows that $v = \frac{a^2 + 2cf - y^2}{2f}$. Moreover, as the element of the curve is equal to $\frac{ydy}{\sqrt{y^2 - p^2}}$, the element of time is equal to :

$$\frac{ydy\sqrt{2f}}{\sqrt{a^2y^2 + 2cfy^2 - y^4 - 2a^2cf}}.$$

For the perpendicular *MP* dropped from *M* at *CA* is called $CP = x$, and $x = uy$. [p. 258] With these in place,

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{2cf}}{y\sqrt{2cfy^2 - 2a^2cf - y^4 + a^2y^2}}$$

(601). Put $y = \frac{1}{\sqrt{(q + \frac{2cf + a^2}{4a^2cf})}}$, and there comes about

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{-\frac{1}{2}dq}{\sqrt{((\frac{a^2 - 2cf}{4a^2cf})^2 - q^2}}.$$

From which, collated with the formula $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ (604) there is produced :

$$\lambda = \tfrac{1}{2}, Z = -q \text{ and } A = \tfrac{a^2-2cf}{4a^2cf}.$$

Moreover from this equation there arises ;

$$Z = -q = C - 2Cu^2 + 2u\sqrt{(A^2 - C^2)(1-u^2)} = \tfrac{a^2+2cf}{4a^2cf} - \tfrac{1}{y^2}.$$

(610). The constant amount C can be determined from this equation, since by making $u = 1$ it comes about that $y = a$. Hence therefore, $C = \tfrac{2cf-a^2}{4a^2cf}$, thus $\sqrt{(A^2 - C^2)} = 0$.

With these in place we have :

$$\tfrac{1}{2cf} - \tfrac{1}{y^2} = \tfrac{(a^2-2cf)u^2}{4a^2cf} = \tfrac{(a^2-2cf)x^2}{4a^2cfy^2}$$

on account of $u = \tfrac{x}{y}$. Therefore this equation results :

$$a^2y^2 - a^2 2cf = (a^2 - 2cf)x^2.$$

The applied line MP is put equal to z, and $y^2 = x^2 + z^2$. Hence the following equation is produced between the orthogonal coordinates for the curve sought:

$$a^2z^2 + 2cfx^2 = 2a^2cf.$$

This is the equation for an ellipse, the centre of which is situated at C, and $AB = 2a$ is the length of one axis, and indeed the length of the other DH = $2\sqrt{2cf}$. Q.E.I.

Corollary 1.

632. The height v corresponding to the speed at M is equal to $\tfrac{a^2+2cf-y^2}{2f}$. Moreover since $a^2 + 2cf = AC^2 + CD^2 = AD^2$, then

$$v = \tfrac{AD^2-CM^2}{2f}.$$

Corollary 2.

633. In a similar manner the perpendicular CT dropped on the tangent MT is given by : $p = \tfrac{AC.CD}{\sqrt{(AD^2-CM^2)}}$ and the tangent itself : [p. 259]

$$MT = \frac{\sqrt{(AD^2.CM^2 - CM^4 - AC^2.CD^2)}}{\sqrt{(AD^2-CM^2)}} = \frac{(AD^2-CD^2).CP.PM}{AC.CD\sqrt{(AD^2-CM^2)}}$$

if indeed $AC > CD$.

Corollary 3.

634. In this case, when $AC > CD$, AB is the transverse axis of the ellipse, and on which the foci F and G are placed. But $CF = CH = \sqrt{(AC^2 - CD^2)}$, and hence

$$MT = \frac{CF^2.CP.PM}{AC.CD\sqrt{(AD^2 - CM^2)}}.$$

Corollary 4.

635. Therefore the sine of the angle TMC, which the direction of the body at M makes with the radius MC , is equal to :

$$\frac{AC.CD}{CM\sqrt{(AD^2 - CM^2)}}$$

and the cosine is equal to:

$$\frac{CF^2.CP.PM}{CM.AC.CD\sqrt{(AD^2 - CM^2)}}.$$

Scholion 1.

636. The distance determining the speeds CE (596, 597) is always equal to the sub-tangent AD. For by putting $CE = k$ the body falls from E towards C drawn by the centripetal force; the body must have, when it arrives at A, a speed corresponding to the height c. On account of which, $k = \sqrt{(a^2 + 2cf)}$ (275). Moreover since

$$AC = a \text{ and } CD = \sqrt{2}cf, \text{ then } CE = \sqrt{(AC^2 + CD^2)} = AD.$$

Corollary 5.

637. The height corresponding to the speed, which the body moving along the line EC can acquire, when it comes to C , is given by :

$$\frac{k^2}{2f} = \frac{a^2 + 2cf}{2f} = \frac{AD^2}{2f}.$$

Corollary 6.

638. The time, in which the arc AM is absolved, is equal to $\frac{2.ACM}{a\sqrt{c}}$ (588) as $h = a$ in this case. [p. 260] On this account the time of the whole revolution around the perimeter of the ellipse $ADBHA$ is equal to $\frac{2.\text{Area of Ellipse}}{a\sqrt{c}}$. With the ratio of the diameter to the periphery put in place $1 : \pi$, the elliptic distance is equal to $\pi a\sqrt{2cf}$. Consequently the time of one revolution is $2\pi\sqrt{2f}$.

Corollary 7.

639. Therefore if more bodies are rotating in ellipses around the same centre of force attracting in the direct ratio of the distances, the times for the whole revolutions are equal to each other.

Scholium 2.

640. When the initial direction of the body at the point *A* is not put normal to the radius *AC*, the calculation does not present an ellipse for the curve described by the body, but another curve of the fourth order, which yet cannot be considered to be satisfactory. The reason for this disagreement between the calculation and the truth depends on this, that the expression of the sine of the angle, which is put in place by the curve with the radius, taken in *y* and *u* always avoids being equal to 1, with *y* put equal to *a* and $u = 1$, even if the following hypothesis must produce another quantity. For the sine of the angle, which the curve makes with the radius, is given by :

$$\frac{ydu}{\sqrt{dy^2 - u^2dy^2 + y^2du^2}},$$

which expression with *u* made equal to 1 evidently departs from unity, when yet it should give $\frac{h}{a}$. On account of this it is evident, unless *h* is put equal to *a*, a calculation set up in this way can never agree with the truth, if indeed the equation between *u* and *y* must be found. Therefore this rule must always be stretched, [p. 261] as often as the curve to be investigated follows the teaching of Prop.75 (601). Moreover the method handled in this proposition has only been found to be convenient for algebraic curves ; and indeed it is required to use another method, if the curves are transcendental. But all algebraic curves enjoy the use of this property, so that in these from any point it is possible to draw a perpendicular.

[In the *Opera Omnia* there is a note added by Paul Stackel :
'This proposition seized upon by Euler without demonstration is easily shown to be false by considering Neil's parabola ' . However , this curve is not a closed orbit, as it is a function of the form $y = ax^{\frac{3}{2}}$, and it appears that Euler is concerned here with closed or negative energy orbits, for which there must be turning points in a physical sense, at which the radius is perpendicualar to the curve, so that the body can return to its original position. There are of course open or positive energy hyperbolic and parabolic orbits where this condition is satisfied, but which do not have a finite period. Comets, for example, may have bound orbits and return periodically, or be free on hyperbolic orbits and never return. Other motions such as the two dimensional mass on a massless spring are performing bound orbits. I have not had time to check the results in this last proposition; if someone feels they would like to do so, I would be glad to hear from them regarding their conclusions.]

Concerning which, as often as the body revolves on an algebraic curve around the centre of curvature, always one of more points can be assigned, at which the radius is perpendicular to the curve. Therefore the body should be put to begin moving at points of this kind, and the calculation is always in agreement with the truth. Moreover from such a solution the speed of the body is easily found at any other place, and hence by the inverse method from the given speed of the body at a point, at which the radius is not normal to the curve, the speed may be found at a place, where the radius falls normal to the curve. Moreover how this should be effected, is evident from the following proposition.

Truly from the properties of algebraic curves recalled above, where from any given point a perpendicular can be dropped on these, but not all transcendental curves agree with this. For in the logarithmic spiral no radius can be drawn from the centre normal to the curve, but always with at some constant angle that has been found.

When finally the ratio can be realised, whereby

$$\frac{ydu}{\sqrt{dy^2 - u^2dy^2 + y^2du^2}}$$

u is always made equal to 1, can be changed into one, since still any other angle too, besides a right angle, can be produced from that formula, u departing from one should be considered 1, [p. 262] and the element du is to vanish before dy, unless the tangent at A is normal to the radius AC. Hence on this account by putting $u = 1$ the element $dy^2(1-u^2)$ is not to be neglected with the ratio y^2du^2 , since each vanishes, as ydu in the numerator. By which it happens, that the sine of any arbitrary angle from that formula by making $u = 1$ can be expressed. But since this warning in the calculation cannot be observed, except the case $h = a$ the calculation cannot show the true curve anywhere.

CAPUT QUINTUM

DE MOTU CURVILINEO PUNCTI LIBERI A QUIBUSCUNQUE PONTENTIIS ABSOLUTIS SOLLICITATI

[p. 225]

DEFINITIO 21.

543. *Vis tangentialis est potentia corpus, quod lineam curvam AMB* (Fig. 47) *describit,sollicitans, cuius directio incidit in tangentem TMt puncti M, in quo corpus, cum sollicitatur, versatur.*

Corollarium 1.

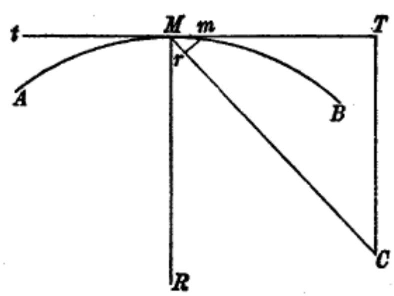


Fig. 47.

544. Vis igitur tangentialis in corpus, dum elementum *Mm* percurrit, alium effectum non exerit, nisi quod motum eius vel acceleret vel retardet, prout scilicet corpus trahit vel secundum directionem *MT* vel *Mt*.

Corollarium 2.

545. Cum igitur corpus in linea curva movetur, vis tangentialis directionem suam perpetuo mutat et secundum aliam plagam effectum suum exerit.

Scholion.

546. Vis quidem tangentialis per se in rerum natura vix unquam oriri potest ; nihil vero eius usus latissime patet. Quamcunque enim potentia sollicitans habeat directionem, ea [p. 226] semper in duas alias potest resolvi, quarum alterius directio in tangente sit sita.

DEFINITIO 22.

547. *Vis normalis est potentia corpus lineam AMB* (Fig. 47) describens sollicitans, cuius directio *MR* est normalis in curvae elementum *Mm* seu tangentem *MT*.

Corollarium 1.

548. Vis igitur normalis corporis celeritatem neque auget neque minuit, quia eius directio MR neque in sequentia neque in antecedentia vergit (164).

Corollarium 2.

549. In hoc vero eius effectus constitit, ut in sequentibus ostendetur, ut corporis tantum directum immutet et efficiat, ut corpus, quod per se in recta esset progressurum, in linea curva promoveatur. (165).

Scholion 1.

550. Si corpus in eodem plano moveatur in eoque etiam positae sint potentiarum sollicitantium directiones, singulae potentiae resolvi possunt in binas, quarum altera sit normalis, altera tangentialis, quamadmodum ex principiis staticis apparet. Quare cum virium tangentialis et normalis effectus in corpus determinaverimus, simul quoque [p. 227] cuiusque potentiae obliquae effectus innotescet. Vocamus autem potentiam seu vim obliquam omnes potentias corpus sollicitantes, quae neque normales sint neque tangentiales.

Scholion 2.

551. Hinc oritur primaria huius capitis divisio. Primo enim eos motus considerabimus, quorum semitae sunt in eodem plano atque simul omnes potentiae sollictantes in eodem ipso plano constitutae. Deinceps autem de iis motibus explicabimus, quorum semitae non sunt positae in eodem plano; ad quos cognoscendos singulas potentias in binas resolvere non sufficit, sed eas in ternas resolvi oportet, propter tres spatii, in quo corpus movetur, dimensiones.

PROPOSITIO 70.

PROBLEMA.

552. *Si corpus, dum elementum Mm* (Fig. 47) *percurrit, sollicitatur a duabus potentiis, altera normali, altera tangentiali, determinare utriusque effectum in motu corporis alterando.*

SOLUTIO.

Sit corporis elementum *Mm* describentis celeritas debita altitudini v et vis normalis secundum *MR* trahens = N et vis tangentialis secundum *MT* trahens = T, ponaturque elementum $Mm = ds$ et radius osculi in $M = \tau$.

Ad effectum vis N determinandum, quia eius directio est normalis in *Mm*, utemur formula [p. 228] (165), quae erat $npr = Ac^2$. Haec vero transmutata est in hanc $pr = 2Av$ (209). Quod autem hic nobis est N, id in citatis locis erat $\frac{p}{A}$; intelligimus enim per N non impressionem solum potentiae N in corpus, sed ipsam vim acceleratricem seu potentiam absolute consideratam divisam per massam corporis (213). Quare loco $\frac{p}{A}$ hic substituti debet N, quo facto habebitur.

$$Nr = 2v .$$

Q. E. Alteram.

Deinde ad effectum vis tangentialis T determinandum adhibeo canonem hunc $Acdc = npds$ (166) seu huius loco mutatum $Adv = pds$ (209). Atque ob allegatas rationes hic loco $\frac{p}{A}$ substituo T, erit

$$dv = Tds .$$

Q. E. Alteram.

Corollarium 1.

553. Quia vis gravitatis ponitur 1, erit vis normalis ad vim gravitatis ut altitudo celeritati debita ad dimidium radii osculi curvae; quae analogia sequitur ex aequatione $Nr = 2v$.

Corollarium 2.

554. Ex data igitur vi normali et curva, quam corpus describit,statim innotescit corporis celeritas. Nam expressa celeritate per $\sqrt{v}$ erit $\sqrt{v} = \sqrt{\frac{Nr}{2}}$.

Corollarium 3.

555. Incrementum vero altitudinis celeritatem exponentis semper aequale est facto ex vi tangentiali [p. 229] et spatiolo a corpore percurso. Seu hoc elementum ipsius v est ad spatiolum descriptum ds ut vis tangentialis ad vim gravitatis.

PROPOSITIO 71.

PROBLEMA.

556. *Si corpus, dum elementum Mm* (Fig. 47) *percurrit, sollicitatur a potentia quacunque obliqua directionis MC, oportet determinare effectum huius potentiae in corporis motu alterando.*

SOLUTIO.

Sit potentia haec obliqua ad corporis gravitatem, si in superficie terrae esset positum, ut P ad 1, elementum $Mm = ds$ et celeritas in M debita altitudini v. Quia vero obliquitas potentiae MC data esse ponitur, angulus CMT datus erit, et propterea triangula CMT, Mmt, quae oriuntur ex C in MT et ex m in MC perpendiculis demissis, erunt specie data. Ponamus igitur $Mr = dy$ et $mr = dx$, erit $ds^2 = dx^2 + dy^2$, et ratio inter ds et dx et dy data erit. Conferantur iam cum his, quae supra (161, 163) nec non postea (208) tradita sunt. Ea enim prorsus similia sunt hisce, de quibus hic quaerimus; hoc tantum differunt, quod hic nobis sit P:1, quod ibi erat $p : A$. Hanc ob rem habebimus

$$dv = Pdy ,$$

posito radio osculi MR in $M = r$ habebitur haec aequatio

$$Pr\,dx = 2vds$$

(208), scripto P loco $\frac{p}{A}$. Harum aequationum illa $dv = Pdy$ definit celeritatis incrementum, dum corpus elementum Mm percurrit. [p. 230] Altero vero $Pr\,dx = 2vds$ seu $r = \frac{2vds}{Pdx}$ exhibet lineae a corpore descriptae curvaturam in M. Innotescit ergo totius potentiae oblique effectus in corpus motum. Q. E. I.

Corollarium 1.

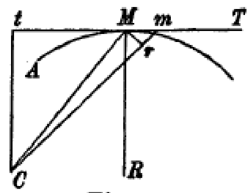


Fig. 48.

557. Si potentia obliqua constituit angulum obtusum cum elementum Mm (Fig. 48), omnia manent ut ante, nisi quod lineola $mr = dy$ accipi debeat negativa. Erit ergo $dv = -Pdy$, et altera aequatio $Pr\,dx = 2dvds$ ut ante.

Corollarium 2.

558. Si ergo potentiae directio MC intra normalem MR et elementum Mm cadit, ut in Fig. 47, motus corporis accelleratur. At si CM extra utramque cadit, ut in Fig. 48, motus retardatur.

Corollarium 3.

559. Si directio potentiae MC incidit in tangentem MT, evanescit angulus mMr, et fit $Mr = Mm$ seu $dx = 0$ et $dy = ds$. Habebitur itaque $dv = Pds$ et $r = \infty$, id quod indicat corporis directionem ab hac potentia tangentiali non affici.

Corollarium 4.

560. Si directio potentiae MC (Fig. 48) incidit in alteram partem Mt, quo casu fit $Mr = dx = 0$ et $dy = ds$, erit $dv = -Pds$ et $r = \infty$ ut ante. Vis igitur tangentialis celeritatem tantum afficit, directionem vero prorsus non immutat. (544). [p. 231]

Corollarium 5.

561. Incidat potentiae MC in normalem MR, quo effectum vis normalis cognoscamus. Hoc ergo casu erit $dy = 0$ et $dx = ds$. Et consequenter prodit $dv = 0$ et $Pr = 2v$. Vis ergo normalis celeritatem non afficit, sed tantum motus directionem (548).

Scholion 1.

562. Hinc igitur tam vis normalis quam vis tangentalis effectus in corpus motum cognoscuntur. Quamobrem cum omnes potentiae, quotquot corpus sollicitant, modo sint in eodem plano positae cum motus directione, in binas, alteram normalem, alteram tangentialem, possint resolvi, etiam quaruncunque potentiarum effectus in corpus motum cognoscitur.

Scholion 2.

563. Hanc ergo priorem partem de motu in eodem plano facto ita subdivididi maxime convenit, ut primo directiones potentiae sollicitantis sint inter se parallelae, quemadmodum in nostris regionibus vis gravitatis directiones sibi parallelae observantur. Deinde contemplabimur casus, quibus potentiae directiones omnes in uno puncto convergunt, quo etiam corpora perpetuo attrahuntur, et qui est casus virium centripetarum, qua de re tam egregiae inventiones a Neutono sunt factae in *Princ. Phil. Nat.* Parte I. Tertio vero potentias [p. 232] quascunque corpus sollicitantes inducemus et investigabimus, qui motus sit perpetuo secuturus. Ita autem in his singulis versabimur, ut primo quaestiones directas proponamus, tum vero etiam quaestiones quasque inversas, ut hactenus fecimus, simus resoluturi. Perpetuo denique, quantum licet, a casibus simplicioribus ad magis compositos et difficiliores progrediemur.

PROPOSITIO 72.

PROBLEMA.

564. *Si fuerit potentia constans eiusque directio ubique normalis ad rectam AB* (Fig. 49) *atque ex A data celeritate secundum directionem AH proiiciatur corpus, determinari oportet curvam AMDB, quam corpus describet, motumque corporis in hac curva.*

SOLUTIO.

Vocetur potentia sollicitans g, et sit celeritas, qua corpus in A proiicitur, debita altitudini c atque cosinus anguli $HAB = \lambda$, posito sinu toto = 1. Iam percurrente corpore elementum $Mm = ds$ sit celeritas in M debita altitudini v, et $PM = y$ itemque radius osculi in $M = r$. Propterea erit $Mr = dx$ et $mr = dy$, ex quo apparet hunc casum referri ad Fig. 48, quo corporis motus retardabatur.

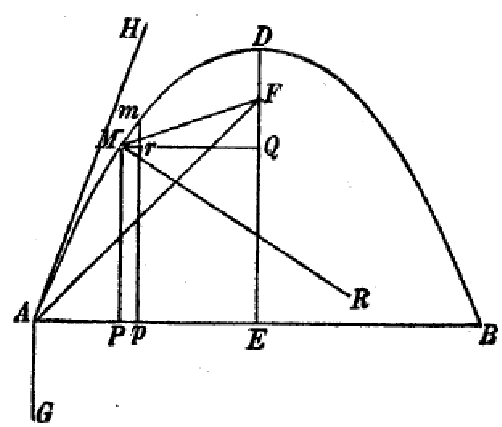


Fig. 49.

Habebimus ergo $dv = -gdy$ et $grdx = 2vds$ (557). Ex altera harum vero aequationum illa oritur $v = C - gy$, in qua constans C ex hoc debet determinari, quod posito $y = 0$ fieri debeat $v = c$. Habetur ergo [p. 233]

$$v = c - gy.$$

In altera nunc aequatione loco v substituatur hic valor inventus, et prodibit

$$\frac{grdx}{2ds} = c - gy.$$

Est vero $r = \frac{ds^3}{-2dxddy}$ posito dx constante, quo substituto erit $\frac{-gds^2}{2ddy} = c - gy$ seu

$$2cddy = 2gyddy - gdx^2 - gdy^2$$

posito $dx^2 + dy^2$ loco ds^2. Huius aequationis integralis reperitur

$$\frac{dx}{ds} = \sqrt{\frac{C}{c-gy}}.$$

Posito autem $y = 0$ designat $C = \lambda^2 c$. Habebimus igitur $\frac{dx}{ds} = \lambda\sqrt{\frac{c}{c-gy}}$ et hinc

$$\frac{\lambda dy\sqrt{c}}{\sqrt{(c(1-\lambda^2)-gy)}} = dx.$$

Cuis integralis est

$$C - 2\lambda\sqrt{(c(1-\lambda^2)-gy)} = \frac{gx}{\sqrt{c}} = 2\lambda\sqrt{c(1-\lambda^2)} - 2\lambda\sqrt{(c(1-\lambda^2)-gy)},$$

ita determinate constante *C*, ut evanescat *y* posito $x = 0$. Ponatur brevitatis gratia sinus anguli *HAB* seu $\sqrt{(1-\lambda^2)} = \mu$, erit $gx = 2\lambda\mu c - 2\lambda\sqrt{(\mu^2c^2 - gcy)}$
atque hinc

$$y = \frac{\mu x}{\lambda} - \frac{gx^2}{4\lambda^2 c}.$$

Erit ergo etiam

$$v = c - \frac{\mu gx}{\lambda} + \frac{g^2x^2}{4\lambda^2 c}$$

et

$$ds^2 = dx^2\left(1 + \frac{\mu^2}{\lambda^2} - \frac{\mu gx}{\lambda^3 c} + \frac{g^2x^2}{4\lambda^4c^2}\right) = \frac{dx^2}{\lambda^2}\left(1 - \frac{\mu gx}{\lambda c} + \frac{g^2x^2}{4\lambda^4c^2}\right).$$

Consequenter erit $\frac{ds^2}{v} = \frac{dx^2}{\lambda^2 c}$ et

$$\int\frac{ds}{\sqrt{v}} = \frac{x}{\lambda\sqrt{c}}$$

= tempori, quo arcus *AM* percurritur. Q.E.I.

Corollarium 1.

565. Recidet ergo corpus in puncto *B* in lineam horizontalem *AB* sumpta $AB = \frac{4\lambda\mu c}{g}$.

Tempus vero, quo corpus supra AB versatur et curvam ADB absolvit, est = $\frac{4\mu\sqrt{c}}{g}$.

Corollarium 2.

566. Denotat autem $2\lambda\mu$ sinum anguli, qui est duplus anguli *HAB*. Quare, si huius dupli anguli sinus vocetur χ, [p. 234] erit $AB = \frac{2\chi c}{g}$. Ex quo apparet distantiam AB fore maximam, si fuerit $\chi = 1$ ideoque angulus HAB semirectus, si quidem corpus eadem celeritate $\sqrt{c}$ proiiciatur.

Corollarium 3.

567. Intelligitur etiam motum corporis horizontalem esse aequabilem. Tempora enim, quibus quilibet arcus describitur, sunt respondentibus abscissis in recta *AB* proportionalia.

Corollarium 4.

568. Si corpus perpetui eadem celeritate $\sqrt{c}$ proiiciatur, sed sub diversis angulis cum *AB*, erunt tempora, quibus supra *AB* versabitur, inter se ut sinus angulorum *HAB* (565) .

Corollarium 5.

569. Maxima altitudo *DE*, ad quam corpus pertintet, erit applicata in puncto *E*, sumto $AE = \frac{2\lambda\mu c}{g}$. Ex quo apparet esse *AE* subduplam ipsius *AB*. Ipsa vero maxima altitudo *DE* erit = $\frac{\mu^2 c}{g}$, quae proinde quadrato sinus anguli *HAB* est proportionalis.

Corollarium 6.

570. Ex aequatione $y = \frac{\mu x}{\lambda} - \frac{gx^2}{4\lambda^2 c}$ perspicitur curvam *ADB* esse parabolam, cuius axis sit recta sit recta *DE* et parameter = $\frac{AE^2}{DE} = \frac{4\lambda^2 c}{g}$. Parameter ergo proportionalis est quadrato cosinus anguli *HAB*. [p. 235]

Corollarium 7.

571. Huius ergo parabolae vertex erit punctum D, et distantia DF foci F a vertice D est $= \frac{\lambda^2 c}{g}$. Quare si ducatur recta MF, erit haec $MF = DQ + DF$ ex natura parabolae.

Corollarium 8.

572. Porro autem erit $MF = DE - MP + DF = \frac{c}{g} - y = \frac{c-gy}{g}$. Quia vero est $v = c - gy$, erit altitudo debita celeritati in $M = g.MF.$ Patet ergo etiam fore $AF = \frac{c}{g}$.

Corollarium 9.

573. Perspicuum igitur est corpus hanc parabolam describens in loco quovis *M* tantam habere celeritatem, quantam corpus idem ab eadem potentia uniformi *g* sollicitatum recta descendens acquire potest ex altitudine, quae aequalis sit distantiae puncti *M* a foco *F* parabolae

Corollarium 10.

574. Cosinus anguli FAE est = $\frac{AE}{AF} = 2\lambda\mu = \chi$. Ex quo manifestum est angulum *FAE* esse complementum dupli anguli *HAE* ad rectum seu potius excessum huius dupli anguli super angulum rectum.

Corollarium 11.

575. Quia angulus *AFD* est deinceps positus anguli *AFE* hicque complementum anguli *FAE*, erit angulus *AFD* duplus anguli *HAB*. [p. 236]

Scholion 1.

576. Facile ergo ex his deducitur constructio parabolae, quam corpus data celeritate et in data directione proiectum describet. Ducta enim *AG* normali ad *AB* capiatur angulus GAF = duplo angulo *HAE*, et in hac directione sumatur *AF* aequalis altitudini, ex qua idem corpus recta descendens acquirit celeritatem aequalem ei, qua ex *A* proiicitur, quo facto erit punctum *F* focus parabolae quaesitae (573, 575). Normalis porro *DE* per *F* in rectam *AB* ducta erit axis huius parabolae, atque vertex *D* reperitur sumendo $DF = \frac{AF-FE}{2}$. Cum igitur parameter aequalis sit 4*DF*, in promtu erit parabolae descriptio.

Scholion 2.

577. Si est $g = 1$, habemus casum gravitatis terrestris. Quocirca, si nullus adesset aer, qui ob resistentiam corpora mota impediret, omna corpora proiecta moverentur in parabolis. Hanc veritatem primus elicuit Galilaeus, et post eum omnes scriptores mechanici demonstravere. Plerique quidem multo breviore modo et sine differentio–differentialibus ad eam pervenerunt, sed nos hic maluimus uti methodo universali, quae latissime pateret, quam nimis particulari, quae ad hunc solum casum esset accommodata. [p. 237]

PROPOSITION 73.

PROBLEMA.

578. *Si corpus in A* (Fig. 50) *data celeritate et in data directione proiectum perpetuo attrahatur ad rectam AB in ratione quarumcunque functionum distantiarum ab hac recta, determinare curvam ADB, quam corpus ita sollicitatum describet et motum totum in hac curva.*

SOLUTIO.

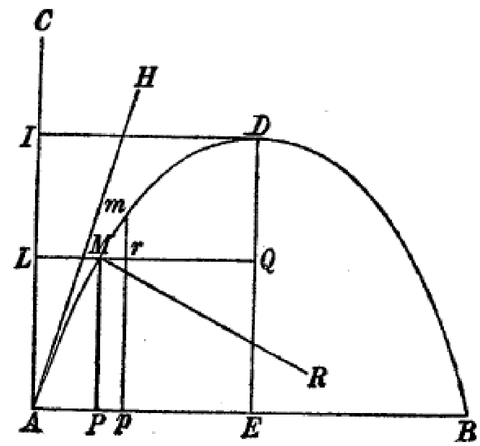


Fig. 50.

Sit celeritas in A = $\sqrt{c}$ et sinus anguli $HAB = \mu$, posito sinu toto = 1, et cosinus = λ. Ponatur celeritas corporis in *M* debita altitudini *v* et elementum $Mm = ds$, itemque abscissa *AP* = *x*, applicata *PM* = *y* et radius osculi *MR* = *r*. Dum autem corpus elementum *Mm* percurrit, trahitur interea a potentia data, quae sit = *P*, functioni cuidam ipsius *y*, in directione MP. His positis erit $dv = -Pdy$, et $Pr\,dx = 2dvds$ (557).

Sit autem $\int Pdy = Y$ atque *Y* talis functio ipsius *y*, quae fiat = 0 posito *y* = 0. Hanc ob rem erit $v = c - Y$. Deinde ex priore aequatione valor ipsius *P*, i. e. $\frac{-dv}{dy}$, substituatur in altera $Pr\,dx = 2dvds$**,** et habebitur

$$\frac{-dv}{v} = \frac{2dyds}{rdx}.$$

Est vero, posito *ds* constante, $r = \frac{dyds}{ddx}$, unde fit $\frac{-dv}{v} = \frac{2ddx}{dx}$ et integrando

$$l\,C - l\,v = l\frac{dx^2}{ds^2}.$$

Constans C determinabitur ex hoc , quod in puncto, in quo fit $v = c$, fiat $\frac{dx}{ds} = \lambda$.
Quocirca erit $l\,C = l\,c + 2l\lambda$ porro sumendis numeris

$$\lambda^2 cds^2 = vdx^2.$$

[p. 238] Ex hac aequatione statim habetur elementum temporis $\frac{ds}{\sqrt{v}} = \frac{dx}{\lambda\sqrt{c}}$.

Pro ipsa autem curva *AMDB* habebitur aequatio ex duabus aequationibus inventis $v = c - Y$ et $\lambda^2 cds^2 = vdx^2$; eliminando *v* orietur enim

$$\lambda^2 cds^2 = cdx^2 - Ydx^2 = \lambda^2 cdx^2 + \lambda^2 cdy^2.$$

Sed quia $1 - \lambda^2 = \mu^2$, prodibit

$$dx = \frac{\lambda dy\sqrt{c}}{\sqrt{(\mu^2 c - Y)}},$$

in qua aequatione, cum sint indeterminatae x et y a se invicem separatae, construi poterit curva quaesita *ADB*. Q.E.I.

Corollarium 1.

579. Tempora igitur, quibus arcus quicunque *AM* describuntur, sunt ut abscissae respondentes *AP*. Est enim tempus per $AM = \frac{x}{\lambda\sqrt{c}}$.

Scholion 1.

580. Motus corporis in curva *AM* in duos alios cogitatione potest resolvi, quorum alter fiat secundum parallelas rectae *AB*, alter secundum perpendiculares in hanc *AB*. Illo motu corpus progreditur secundum rectam *AB*, isto vero vel ascendit vel descendit respectu huius rectae *AB*. Iam vero perspicuum est a potentia, cuius directio perpetuo est in perpendicularis ad *AB*, motum progressivum horizontalem non immutari, et hanc ob rem hic motus perpetuo esse debebit aequabilis et ea celeritate factus, quae oritur ex simili motus resolutione initialis. Cum vero motus initialis directio sit recta *AH*, erit eius celeritas $\sqrt{c}$ ad celeritatem progressivam secundum *AB* ut sinus totus 1 ad cosinum anguli *HAB*, [p. 239] qui est λ. Celeritas ergo progressiva secundum AB erit $\lambda\sqrt{c}$, ex qua tempus, quo motus horizontalis perficitur per spatium $AP = x$, provenit $= \frac{x}{\lambda\sqrt{c}}$, ut invenimus.

Corollarium 2.

581. Si corpus celeritate $\sqrt{c}$ ex A perpendiculariter ascendat in AC et sumatur $AL = PM = y$, erit celeritas in L aequalis celeritati in M, nempe $= \sqrt{v}$. Est enim $dv = -Pdy$ atque $v = c - Y$, quemadmodum pro puncto M invenimus.

Corollarium 3.

582. Si fuerit AC tota altitudo, ad quam corpus ex A celeritate $\sqrt{c}$ sursum proiectum pertingere potest, erit CL altitudo, ex quo corpus cadendo eandem acquirit celeritatem, quam habet in M.

Corollarium 4.

583. Maxima altitudo *DE* reperitur faciendo $dy = 0$, quo casu erit $Y = \mu^2 c$, ex qua aequatione valor ipsius y erutus dabit altitudinem *DE*, et celeritas in *D* tanta erit, quantam corpus ex altitudine *CI* delapsum acquirit.

Corollarium 5.

584. Invenimus autem quoque $v = \frac{\lambda^2 cds^2}{dx^2}$. Quare, cum in puncto *D* sit $dx = ds$, erit celeritas in D = $\lambda\sqrt{c}$, quae est aequalis ipsi celeritati horizontali, qua corpus secundum *AB* progreditur. [p. 240]

Scholion 2.

585. Amplitudo quidem *AB* ex aequatione, quia non potest integrari, non apparet. Nihilo tamen minus perspicuum est partes AE et *EB* opportere esse aequales et ramum *DB* similem et aequalem ipsi arcui *DA*. Namque postquam corpus in D pervenit, simili modo rursus accelerabitur, quo ante per AD erat retardatum; quia potentia in iisdem ab AB distantiis est eadem hocque modo motus iterum perfecte restituitur.

Scholion 3.

586. Facile hinc etiam problema reciprocum, quo ex data curva ADB et celeritate corporis in A quaeritur potentiarum lex, quae faciat, ut corpus in hac curva moveatur, resolvi poterit. Ex aequatione enim $dx = \frac{\lambda dy\sqrt{c}}{\sqrt{(\mu^2 c - Y)}}$ habetur $Y = \mu^2 c - \frac{\lambda^2 cdy^2}{dx^2}$ atque $dY = Pdy = \frac{-2\lambda^2 cdyddy}{dx^2}$ posito *dx* constante. Quia est vero $r = \frac{ds^2}{-dxddy}$ eodem *dx* posito constante, erit $P = \frac{2\lambda^2 cds^3}{rdx^3}$ = potentiae sollicatanti corpus in *M* secundum directum *MP*.

PROPOSITIO 74.

PROBLEMA.

587. *Trahatur corpus in A* (Fig. 51) *proiectum ubique ad virium centrum C a vi centripeta quacunque, oporteatque determinari naturam curvae AM, in qua corpus movebitur, et motum corporis in hac curva.* [p. 241]

SOLUTIO.

Ponatur vis centripeta corpus in *M* ad *C* sollicitans = *P*, et celeritas corporis in *M* debita sit altitudine *v*. Vocentur deinde distantia *CM y*, elementum *Mm ds* et perpendiculum *CT* in tangentem curvae *MT p*, nec non elementum *Mr dx* et radius osculi *MR r*. Erit ergo ob triangula similia *Mmr* et *CMT*

$$ds\sqrt{(y^2 - p^2)} = ydy \text{ et } dx\sqrt{(y^2 - p^2)} = pdy.$$

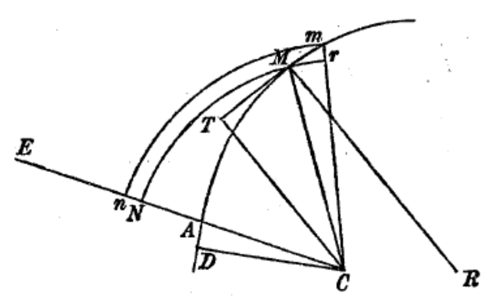


Fig. 51.

Atque radius osculi r reperitur $=\frac{ydy}{dp}$. His positis habebimus $dv=-Pdy$ et $Pr\,dx=2vds$ (557). His aequationibus coniunctis eliminato P erit $rdvdx=-2vdsdy$ seu $\frac{dv}{v}=\frac{-2dsdy}{rdx}$.

Substituantur $\frac{ydy}{dp}$ loco r et $\frac{y}{p}$ loco $\frac{ds}{dx}$, prodibitque

$$\frac{dv}{v}=\frac{-2dp}{p}.$$

Quae integrata dat

$$v=\frac{C}{p^2}.$$

Constans haec C definietur ex data celeritate initiali in A, quae sit debita altitudini c, et directione proiectionis, quam ita definiemus, ut posita distantia $CA=a$ perpendiculum CD in tangentem AD sit = h. Propterea erit $C=ch^2$ et

$$v=\frac{ch^2}{p^2}.$$

Hinc erit $\frac{ds}{\sqrt{v}}=\frac{pds}{h\sqrt{c}}$ = elemento temporis per Mm. Consequenter ob $pds=2MCm$ erit tempus, quo arcus AM percurritur,

$$=\frac{2.ACM}{h\sqrt{c}}.$$

Ipsa curva vero determinabitur substituendo $\frac{ch^2}{p^2}$ loco v in aequatione $Pr\,dx=2vds$, quo facto prodibit $Pp^2rdx=2ch^2ds$. Haec aequatio, loco r et $\frac{ds}{dx}$ substitutis valoribus $\frac{ydy}{dp}$ et $\frac{y}{p}$, transmutabitur in hanc

$$Pdy=\frac{2ch^2dp}{p^3}.$$

Quae aequatio, quaecunque P sit functio ipsius y, ob indeterminatas separatas poterit construi. Q.E.I. [p. 242]

Corollarium 1.

588. Quia tempus, quo arcus AM percurritur, est $\frac{2.ACM}{h\sqrt{c}}$, erunt tempora, quibus arcus quicunque describuntur, ut areae comprehensae arcu descripto et rectis ad centrum C ductis.

Corollarium 2.

589. Deinde cum sit $v=\frac{ch^2}{p^2}$, erit $\sqrt{v}=\frac{h\sqrt{c}}{p}$. Celeritas igitur corporis in quocunque loco curvae percursae est reciproce ut perpendiculum ex centro C in tangentem in illo puncto demissum.

Scholion 1.

590. Haec arearum aequabilis descripto constituit apud Neutonum primam propositionem, ex qua sequentia fere omnia deducit. Sunt autem hae duae proprietates maxime generales et id tantum requirunt, ut vis centripetae directio sit perpetuo versus centrum. Quaecunque enim vis centripeta quantitate, sive ipsarum CM functione, sive secus exprimatur, aeque tamen utraque valet. Nam cum in eas incidissemus, ex calculo vis centripeta P exterminabatur, eiusque tantum directio in consideratione relinquebatur.

Corollarium 3.

591. Ad curvam, quam corpus describit, cognoscendam ipsam vim centripetam dari oportet, [p. 243] ex hacque data aequatio pro curva habebitur. Est enim $Pdy = \frac{2ch^2dp}{p^3}$, quae exprimit curvae naturam, si P fuerit quantitas data.

Corollarium 4.

592. Quia est $\frac{ydy}{dp} = r$, erit etiam $P = \frac{2ch^2y}{p^3r}$. Atque hoc est Theorema Moivreanum, illud vero $P = \frac{2ch^2dp}{p^3dy}$ Keillius primus se invenisse contendit.

[Abraham de Moivre (1667–1754) : *Some simple properties of the concic sections deduced from the nature of the foci; with general theoremsof centripetal force, by means of which the law of the centripetal force tending to the foci of the sections, the velocities of bodies revolving in them, and the description of the orbits, may be easily determined.* Philosophical Transactions 1717, p. 622; cf. *Miscellanea analytica de seriebus et quadraturis,* London 1730, p.233.
John Keill (1671–1721) : *Of the laws of centripetal force*, Philosophical Transactions 1708, p. 174; " The learned Mr. Halley having showed me a theorem, by which the law of centripetal force can be exhibited in finite quantities, which was communicated to him by Mr. De Moivre, who said that Mr. Is. Newton had before discovered a similar theorem, and as the demonstration of the theorem is very easy, I wish to communicate it to the public,with some other thoughts in the same subject."
Noted by Paul Stackel in the 1912 edition of *Tom I*, which is available from the *Gallica* website.]

Scholion 2.

593. Hae aequationes duplicem habent utilitatem : Primo enim ex data vi centripeta poterit natura curvae, quam corpus proiectum describit, determinari. Deinde etiam vicissim ope harum aequationum, si fuerit data curva, quam corpus circa virium centrum C describit, definiri potest in quovis loco vis centripeta efficiens, ut corpus in hac curva libere moveatur.

Corollarium 5.

594. Quia est etiam $dv = -Pdy$, perspicuum est, si *P* fuerit functio ipsius *y*, celeritatem ubique a distantia corporis a centro pendere et in iisdem distantiis corpus eandem habere debere celeritatem.

Corollarium 6.

595. Quoties igitur *P* est functio ipsius *y*, toties etiam curva descripta ita erit comparata, ut in aequalibus a centro distantiis perpendicula ex centro in tangentes demissa sint inter aequalia, quia celeritates sunt reciproce ut haec perpendicula (589). [p. 244]

Scholion 3.

596. Atque semper, si *P* fuerit functio ipsius *y*, poterit assignari recta *EC*, in qua corpus a vi centripeta sollicitatum descendens in singulis punctis *N* eandem habebit celeritatem, quam habet in curva *AM* motum in punctis *M* aequaliter a centro distantibus. Sumto enim $CN = CM = y$ et $Nn = mr = dy$, si fuerit celeritas in *N* aequalis celeritati in *M*, scilicet debita altitudini *v*, erit etiam, dum corpus per elementum *Nn* moveatur, $dv = -Pdy$. Ex quo intelligitur celeritatem in *n* aequalem fore celeritati in *m*. Atque ita in singulis punctis rectae *EC* corpus tantam habebit celeritatem, quantam habet in curva *AM* in iisdem a centro *C* distantiis. Si ergo fuerit *E* motus initium, in quo celeritas est = 0, ex data linea *EC* corporis in curva *AM* moti in singulis punctis innotescet celeritas. Hac igitur recta EC in posterum utemur ad celeritates corporis in curva moti definiendas eamque vocabimus distantiam celeritates determinantem.

Corollarium 7.

597. Cum igitur data sit celeritas in *A*, nempe debita altitudini *c*, ex hoc tota distantia *EC* reperiatur. Tantum enim punctum *E* distans a *C* accipi debet, ut corpus ex *E* hac vi centripeta sollicitatum descendens in A acquirat celeritatem = $\sqrt{c}$. [p. 245]

Corollarium 8.

598. Si vocatur angulus $MCm = dw$, erit $dw = \frac{dx}{y} = \frac{pdy}{y\sqrt{(y^2-p^2)}}$. Unde prodit

$p = \frac{y^2dw}{\sqrt{(y^2dw^2+dy^2)}}$. Tempusculum vero, quo angulus *MCm* absolvitur, est =

$\frac{pds}{h\sqrt{c}} = \frac{pydy}{h\sqrt{c(y^2-p^2)}}$. Hoc igitur tempusculum erit = $\frac{y^2dw}{h\sqrt{c}}$.

Corollarium 9.

599. Si celeritatem angularem seu eam, qua angulus *MCm* percurritur, metiri velimus ipso hoc angulo per tempus diviso, prodibit celeritas angularis in M = $\frac{h\sqrt{c}}{y^2}$. Celeritas igitur angularis est reciproce ut quadratum distantiae corporis a centro C.

Scholion 4.

600. Quia autem curva, quam corpus a data vi centripeta sollicitatum describit, aliter non cognoscitur nisi per aequationem inter corporis distantiam a centro et perpendicularum in tangentem ex centro demissum, difficile plerumque est iudicare, qualis sit curva inventa, cum curvarum naturas consueverimus aequationibus inter coordinatas orthogonales exponere. Interim tamen huiusmodi aequationes inter distantiam a centro et perpendiculaum in tangentem, se modo sint vel algebraicae vel differentiales, in quibus indeterminatae sint a se invicem separatae, sufficiunt ad curvas quaesits construendas. Sed quo natura et ordo curvarum harum penitus possit investigari, [p. 246] dabimus hic methodum aequationes inter distantiam et perpendiculum ad consuetas aequationes inter coordinatas reducendi.

PROPOSITIO 75.

PROBLEMA.

601. *Naturam curvae AM* (Fig. 52), *quam corpus a quacunque vi centripeta sollicitatum describit, definire aequatione inter coordinatas orthogonales CP et PM ad axem fixum AC relatus.*

SOLUTIO.

Positis ut ante celerate initiali in $A = \sqrt{c}$, $AC = a$ et perpendiculo ex C in tangentem in A demisso = h atque praeterae $CM = y$, $CT = p$ et vi centripeta in $M = P$, quae sit functio quaedam ipsius y, vocetur deinde $CP = x$ et $PM = z$, erit $CM = \sqrt{(x^2+z^2)} = y$ seu $z = \sqrt{(y^2-x^2)}$; retinebimus enim loco z in calculo y, quia hoc modo calculus fit facilior et brevior atque post peractam totam operationem in promptu est z loco y introducere. His positis erit

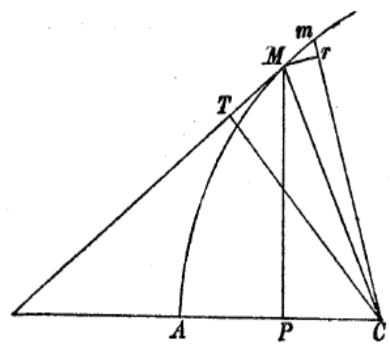


Fig. 52.

$$Mm = \sqrt{(dx^2+dz^2)} = \frac{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}{\sqrt{(y^2-x^2)}} \quad \text{et } Mr = \frac{-ydx+xdy.}{\sqrt{(y^2-x^2)}}$$

Fiet igitur

$$\frac{Mr}{Mm} = \frac{p}{y} = \frac{-ydx+xdy.}{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}$$

et consequenter

$$p = \frac{yxdy-y^2dx}{\sqrt{(y^2dy^2-2yxdydx+y^2dx^2)}}.$$

Sed pro curva AM hanc ante (591) elicuimus aequationem $Pdy = \frac{2ch^2dp}{p^3}$. Ponamus autem Y integrale ipsius Pdy ita assumtum, ut evanescat posito $y = a$. Quo facto erit

$Y = C - \frac{ch^2}{p^2}$, ubi constans C ob $p = h$, si est $y = a$, debet esse = c. Erit ergo $Y = \frac{cp^2 - ch^2}{p^2}$, [p. 247] ex qua prodit

$$p = \frac{h\sqrt{c}}{\sqrt{(c-Y)}}.$$

Quae quantitas, cum sit Y functio ipsius y, est quoque functio quaedam ipsius y. Hanc ob rem habebimus hanc aequationem

$$\frac{h\sqrt{c}}{\sqrt{(c-Y)}} = \frac{yxdy - y^2dx}{\sqrt{(y^2dy^2 - 2yxdydx + y^2dx^2)}}.$$

Ponamus $x = uy$ et $\frac{h\sqrt{c}}{\sqrt{(c-Y)}} = Q$ brevitatis gratia, erit $Q = \frac{-y^2dx}{\sqrt{(dy^2 - u^2dy^2 + y^2du^2)}}$,

ex qua oritur

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{Qdy}{y\sqrt{(y^2 - Q^2)}} = \frac{hdy\sqrt{c}}{y\sqrt{(cy^2 - ch^2 - y^2Y)}}$$

restituto $\frac{h\sqrt{c}}{y\sqrt{(c-Y)}}$ loco Q. Quae igitur aequatio, cum in ea indeterminatae u et y sint separatae, semper poterit construi. Q.E.I.

Corollarium 1.

602. Quia est $u = \frac{x}{y}$, exprimet u cosinum anguli MCA. Et hanc ob rem ultima aequatio separata erit inter distantiam corporis a centro et anguli ACM cosinum. Ex hac vero aequatione sponte oritur aequatio inter x et z.

Corollarium 2.

603. Aequatio vero nunquam esse poterit algebraica, nisi denotet $\int \frac{hdy\sqrt{c}}{y\sqrt{(cy^2 - ch^2 - y^2Y)}}$ arcum circuli commensurabilem cum arcu $\int \frac{du}{\sqrt{(1-u^2)}}$.

Corollarium 3.

604. Quoties ergo $\frac{hdy\sqrt{c}}{y\sqrt{(cy^2 - ch^2 - y^2Y)}}$ reduci poterit ad formam huiusmodi $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ et sit λ numerus rationalis, aequatio algebraica pro curva quaesita poterit exhiberi. [p. 248]

Scholion 1.

605. Sin autem $\frac{du}{\sqrt{(1-u^2)}}$ aequabitur huiusmodi quantitati $\frac{\lambda dZ}{\sqrt{(A^2-Z^2)}}$, habebitur integratione per logarithmos imaginarios peracta haec aequatio

$$\frac{\sqrt{(1-u^2)}+u\sqrt{-1}}{\sqrt{(1-u^2)}-u\sqrt{-1}}=\left(\frac{\sqrt{(A^2-C^2)}-C\sqrt{-1}}{\sqrt{(A^2-C^2)}+C\sqrt{-1}}\right)^{\lambda}\left(\frac{\sqrt{(A^2-Z^2)}+Z\sqrt{-1}}{\sqrt{(A^2-Z^2)}-Z\sqrt{-1}}\right)^{\lambda}.$$

Est vero hic C constans quantitas ex eo determinanda, quod, si fit $CM(y)=CA(a)$, simul quoque fieri debeat $x=a$ seu $u=1$. Ex illa autem aequatione conficitur ista

$$u=\frac{\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\lambda}-\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\lambda}}{2A^{2\lambda}\sqrt{-1}}.$$

Quae, quoties λ est numerus rationalis, ab imaginariis $\sqrt{-1}$ affectis libera redditur et in algebraicam certi ordinis transmutatur.

Corollarium 4.

606. Quia est $x=uy$, habebitur ista aequatio

$$x=\frac{\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\lambda}y-\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\lambda}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\lambda}y}{2A^{2\lambda}\sqrt{-1}},$$

quae, cum sit *Z* functio ipsius *y* et $y=\sqrt{(x^2+z^2)}$, facile in aequationem inter *x* et *z* mutatur.

Scholion 2.

607. Superior aequatio etiam in hanc potest transmutari

$$Z=\frac{1}{2\sqrt{-1}}\left[\left(\sqrt{(1-u^2)}+u\sqrt{-1}\right)^{\frac{1}{\lambda}}\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)-\left(\sqrt{(1-u^2)}-u\sqrt{-1}\right)^{\frac{1}{\lambda}}\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)\right],$$

[p. 249] quae commodior est, si $\frac{1}{\lambda}$ fuerit numerus integer affirmativus.

Scholion 3.

608. At si λ fuerit numerus negativum $=-\mu$, habebitur

$$\frac{\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)^{\mu}\left(\sqrt{(A^2-Z^2)}-Z\sqrt{-1}\right)^{\mu}-\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)^{\mu}\left(\sqrt{(A^2-Z^2)}+Z\sqrt{-1}\right)^{\mu}}{2A^{2\mu}\sqrt{-1}}=u$$

Ex quo apparet, si λ fuerit negatimum, ipsius *u* valorem tantum fieri negativum, id quod quidem ex aequatione differentiali intelligitur. Simili vero modo erit etiam

$$Z=\frac{1}{2\sqrt{-1}}\left[\left(\sqrt{(1-u^2)}-u\sqrt{-1}\right)^{\frac{1}{\mu}}\left(\sqrt{(A^2-C^2)}+C\sqrt{-1}\right)-\left(\sqrt{(1-u^2)}+u\sqrt{-1}\right)^{\frac{1}{\mu}}\left(\sqrt{(A^2-C^2)}-C\sqrt{-1}\right)\right].$$

Corollarium 5.

609. Si fuerit $\lambda=1$, erit $u=\frac{Z\sqrt{(A^2-C^2)}-C\sqrt{(A^2-Z^2)}}{A^2}$ et $x=\frac{Zy\sqrt{(A^2-C^2)}-Cy\sqrt{(A^2-Z^2)}}{A^2}$.
Si fuerit $\lambda=-1$, etiam u vel x sumi debet negativum.

Corollarium 6.

610. Si fuerit $\lambda=2$, erit

$$u=\frac{2Z(A^2-2C^2)\sqrt{(A^2-Z^2)}-2C(A^2-2Z^2)\sqrt{(A^2-C^2)}}{A^4}.$$

At si fuerit $\lambda=\frac{1}{2}$, erit $Z=C-2Cu^2+2u\sqrt{(1-u^2)(A^2-C^2)}$. [p. 250]

PROPOSITIO 76.

THEOREMA.

611. *Si fuerit vis centripeta ut functio quaecunque distantiarum a centro C* (Fig. 53), *et corpus in A proiiciatur secundum directionem normalem in AC celeritate, cuius altitudo debita se habeat ad dimidiam AC ut vis centripeta in A ad vim gravitas 1, hoc corpus in peripheria circuli AMBA, cuius centrum est C, movebitur aequaliter.*

DEMONSTRATIO.

Moveatur enim corpus in hoc circulo ; quia eius distantia a centro non variatur perpetuo ab eadem vi centripeta versus *C* sollicitabitur, cuicunque functioni distantiarum etiam si sit proportionalis vis centripeta. Atque quia ad centrum *C* sollicitatur, erit directio potentiae sollicitantis normalis semper in curvae portiunculam, in qua corpus movetur. Quodcirca corpus sollicitabitur perpetuo a vi normali, nunquam a tangentiali, et hanc ob rem eius celeritas semper manebit eadem (561), ac ideo corpus motu aequabili peripheriam describet. Deinde quia vis centripeta seu normalis ubique est eadem, ponatur ea = *g* et celeritas itidem constans debita altitudini *c* atque radius *AC* = *a*, quae quantitas *a* ubique exhibet curvae radium osculi. His igitur positis erit $ag=2c$ (561). Ex quo haec oritur analogia : Ut altitudo celeritati corporis, qua initio in A proiicitur, debita *c* ad dimidian distantiam AC, $\frac{1}{2}a$, ita vis centripeta *g* ad vim gravitatis 1. Q.E.D. [p. 251]

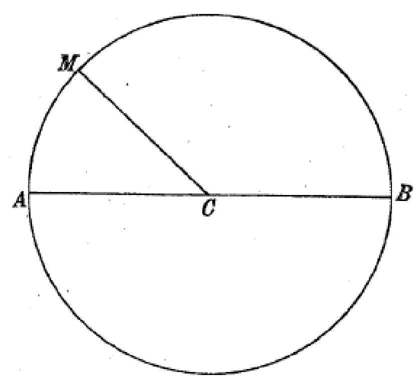


Fig. 53.

Corollarium 1.

612. Quando ergo corpus semel arcum circuli describit,cuius centrum est in ipso centro virium C, tum perpetuo in ea circuli peripheria revolvetur. Si quidem vis centripeta a solis distantiis a centro pendeat, ita ut in aequalibus distantiis vis centripeta sit aequalis.

Corollarium 2.

613. Posita ratione diametri ad peripheriam 1 : π erit peripheria circuli, in quo corpus movetur, = $2\pi a$. Quia deinde celeritas, qua corpus movetur, est $=\sqrt{c}=\sqrt{\frac{ag}{2}}$, erit tempus unius periodi per totam peripheriam = $\frac{2\pi\sqrt{2a}}{\sqrt{g}}$.

Corollarium 3.

614. Si ergo plura corpores in diversis circulis moveantur, erunt tempora revolutionum in subduplicata ratione composita ex directa radiorum circulorum et inversa virium centripetarum.

Corollarium 4.

615. Si corpus ex A perpendiculariter ad AC proiiaciatur, set celeritate vel maiore vel minore quam $\sqrt{\frac{ag}{2}}$, corpus arculum circuli describet, cuius radius erit vel maior vel minor quam AC. [p. 252]

Corollarium 5.

616. Hoc ergo casu, quo corpus incipit in arcu circulari moveri, cuius centrum non est in C, statim ad centrum vel magis accedet vel magis ab eo recedet. Et hanc ob rem statim ab alia sollicitabitur vi centripeta, nisi forte vis centripeta ubique est eadem.

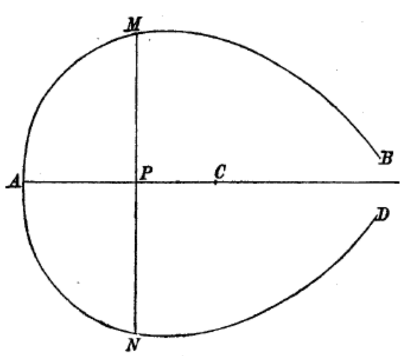


Fig. 54.

Scholion.

617. Quacunque autem corpus in A (Fig. 54) celeritate proiiciatur, modo sit eius directio in rectam AC per centrum virium C transeuntem perpendicularis, curva BMAND hanc habebit proproprietatem, ut eius portiones AMB, AND, cis et ultra rectam AC positae sint inter se similes et aequales, atque AC axis et diameter huius curvae. Nam quia, ut iam innuimus, vis centripeta functioni cuidam distantiarum a centro est proportionalis, corpus sive supra sive infra AC in aequalibus a C distantiis aequaliter sollicitatur et hanc ob rem eodem modo per DNA ad A accedere debet, quo ab A per AMB recedit, atque in punctis homologis M et N eandem quoque habebit celeritatem.

Corollarium 6.

618. Omnis igitur recta ex centro C ducta, quae in curvam est normalis, erit simul curvae diameter ; ita ut curvae partes cis et ultra hanc rectam positae sint inter se similes et aequales. [p. 253]

PROPOSITIO 77.

THEOREMA.

619. *Si plura corpora circa centrum virium C* (Fig. 55) *moveantur atque*, *describant curvas AM, am circa C similes, erunt celeritates in punctis similibus M et m in subduplicitata ratione composita ex rationibus laterum homologorum et virium centripetarum in locis homologis M et m.*

DEMONSTRATIO.

Quia est $AC : aC = MC : mC$, erit etiam radius osculi in M ad radium osculi in m in eadem ratione, nec non etiam perpendiculum CT ad Ct. Ex propositione 74 (587) vero apparet esse

$Pr\,dx = 2vds$, seu, loco $\frac{ds}{dx}$ posito $\frac{y}{p}$. $Ppr = 2vy$.

Hinc ergo erit celeritas

$$\sqrt{v} = \sqrt{\frac{Ppr}{2y}} .$$

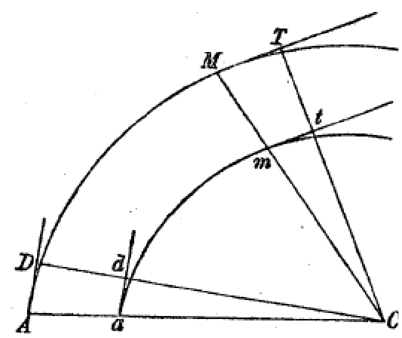


Fig. 55.

Ex quo prodit haec analogia : celeritas in M est ad celeritatem in m in ratione subduplicata composita ex directis virium centripetarum in M et m atque perpendiculorum CT ad cT, atque ratio inverse distantiarum MC ad mC. Quia vero est CT ad cT ut MC ad mC et radius osculi in M ad radium osculi in m ut MC ad mC, erit celeritas in M ad celeritatem in m in ratione subduplicata composita ex rationibus virium centripetarum in M et m et laterum homologorum MC ad mC. Q.E.D.

Corollarium 1.

620. Si ergo dicatur $AC = A$, $aC = a$, $CD = H$ et $Cd= h$, itemque celeritas in $A = \sqrt{C}$ et in $a = \sqrt{c}$, erit celeritas angularis in $M = \frac{H\sqrt{C}}{MC^2}$ [p. 254] et celeritas angularis in $m = \frac{h\sqrt{C}}{mC^2}$ (599). Quia autem est $H : h = MC : mC = A : a,$ erunt celeritates angulares in M et m ut $\frac{\sqrt{C}}{A}$ ad $\frac{\sqrt{c}}{a}$, i. e. in ratione constante.

Corollarium 2.

621. Tempora igitur, quibus aequales anguli *ACM*, *aCm* seu spatio homologia *AM* et *am* absolventur, sunt reciproce ut celeritates angulares in *M* et *m*, i. e. directe ut latera homologa et reciproca ut celeritates in punctis homologis.

Scholion 1.

622. Quin etiam celeritates in punctis homologis ubique eandem tenent rationem. Est enim celeritates in $M = \frac{H\sqrt{C}}{CT}$ et celeritates in $m = \frac{h\sqrt{c}}{Ct}$ (589). Quamobrem, cum sit $H : h = CT : Ct$, erit celeritas in *M* ad celeritatem in *m* ut $\sqrt{C}$ ad $\sqrt{c}$, i. e. ut celeritates in *A* ad celeritatem in *a*.

Corollarium 3.

623. Ex ipsa autem propositione perspicitur esse celeritatem in *A*, $\sqrt{C}$, ad celeritatem in *a*, $\sqrt{c}$, in ratione subduplicata composita ex rationibus virium centripetarum in *A* et *a* et laterum homologorum *A* et *a*. Quare si dicatur vis centripeta in $A = G$ et vis centripeta in $a = g$, erit

$$\sqrt{C} : \sqrt{c} = \sqrt{AG} : \sqrt{ag}.$$ [p. 255]

Corollarium 4.

624. Consequenter tempus per *AM* erit ad tempus per *am* ut $\sqrt{\frac{A}{G}} : \sqrt{\frac{a}{g}}$, i. e. in ratione subduplicata composita ex directa laterum homologorum et inversa virium centripetarum in punctis *A* et *a*. Quae ergo ratio est constans, eandemque tenere debent inter se integra revolutionum tempora.

Scholion 2.

625. Hoc etiam casu, quo plures figures similes circa centrum *C* describuntur, vires centripetae in punctis homologis eandem ubique tenere debent rationem. Quia enim est $P = \frac{2ch^2y}{p^3r}$ (592), erit vis centripeta in *M* ad vim centripetam in *m* directe ut quadratum celeritatis in *A* ad quadratum celeritatis in *a* et reciproce ut *AC* ad *aC*, quae est ratio constans. Quamobrem, quo possint plures figurae similes circa centrum *C* describi, oportet, ut vis centripeta huiusmodi distantiarum functione exprimatur, quae in locis homologis vires centripetas eandem rationem tenentes praebeat. Scilicet posita *P* vi centripeta in distantia *y* et *Q* in distantia *my*, debebit *P* ad *Q* habere rationem constantem, in qua non insit *y*. Hoc enim nisi fuerit, fieri non potest, ut circa centrum *C* plures figurae similes describantur.

Corollarium 5.

626. Hoc autem obtineri non potest, nisi sit P potestas quaedam ipsius y, ut $\frac{y^n}{f^n}$. [p. 256] Hoc enim casu fit $Q=\frac{m^n y^n}{f^n}$, et ratio P : Q erit 1 : m^n, quae est constans.

Corollarium 6.

627. Nisi ergo vis centripeta cuidam dignitati distantiarum a centro *C* sit proportionalis, ne fieri quidem potest, ut plures figurae similes circa centrum *C* describantur. Atque in his solis casibus locum habebunt proprietates, quas ex hac propositione eruimus.

Corollarium 7.

628. Si autem fuerit $P=\frac{y^n}{f^n}$, erit $G=\frac{A^n}{f^n}$ et $g=\frac{a^n}{f^n}$. Celeritates ergo in locis homologis tenebunt rationem $A^{\frac{n+1}{2}}$ ad $a^{\frac{n+1}{2}}$.

Corollarium 8.

629. Atque tempora, quibus arcus similes *AM* et *am* absolvuntur, erunt in ratione $A^{\frac{1-n}{2}}$ ad $a^{\frac{1-n}{2}}$ seu in ratione multiplicata laterum homologorum, cuius exponens est $\frac{1-n}{2}$.

Scholion 3.

630. In praecedente et hac propositione continentur omnia theoremata, quae Huygenius de viribus centrifugis in tractatu suo de Horologio oscillatorio annexit. [p. 257] Eaque partim re ipsa hic apposui, partim eventissime ex ipsis propositionibus deduci possunt.

PROPOSITIO 78.

PROBLEMA.

631. *Si centrum virium C attrahat in ratione distantiarum directa* (Fig. 56) *et corpus ex A secundum directionem ad radium AC normalem data cum celeritate proiiciatur, determinare curvam AMDBH, quam corpus describet, corporisque celeritatem in singulis locis.*

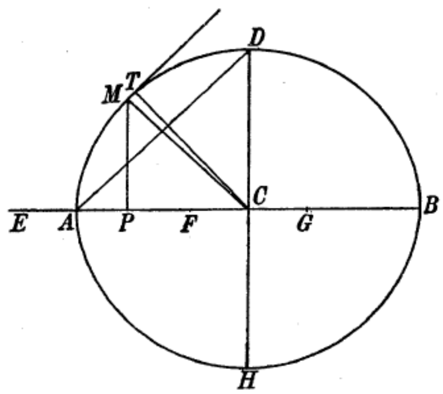


Fig. 56.

SOLUTIO.

Ponatur distantia $CA = a$, erit perpendiculum, quod ex centro C in motus directionem demittitur, $= a$. Celeritas vero in A debita sit altitudini c. Pervenerit corpus in M, sitque $CM = y$; et perpendiculum ex C in tangentem in M demissum CT ponatur $= p$, atque celeritas in M debita sit altitudini v. Sit porro distantia, in quia vis centripeta aequalis est gravitati, f; erit vis centripeta in $M = \frac{y}{f}$, posita vi gravitatis = 1.

His cum prop. 75 (601) comparitis erit $P = \frac{y}{r}$ et $Y = \frac{y^2 - a^2}{2f}$. Quamobrem habebitur $p = \frac{a\sqrt{2cf}}{\sqrt{a^2 + 2cf - y^2}}$.

Est vero $v = \frac{a^2 c}{p^2}$ (587), ex quo erit $v = \frac{a^2 + 2cf - y^2}{2f}$. Ob elementum curvae autem $= \frac{ydy}{\sqrt{y^2 - p^2}}$ erit elementum temporis $\frac{ydy\sqrt{2f}}{\sqrt{a^2 y^2 + 2cfy^2 - y^4 - 2a^2 cf}}$.

Demisso ex M in CA perpendiculo MP vocetur $CP = x$, ponaturque $x = uy$. [p. 258] His positis erit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{ady\sqrt{2cf}}{y\sqrt{2cfy^2 - 2a^2 cf - y^4 + a^2 y^2}}$$

(601). Fiat $y = \frac{1}{y\sqrt{(q + \frac{2cf + a^2}{4a^2 cf})}}$,

et prodibit

$$\frac{du}{\sqrt{(1-u^2)}} = \frac{-\frac{1}{2}dq}{\sqrt{((\frac{a^2 - 2cf}{4a^2 cf})^2 - q^2}}.$$

Quo collato cum formula $\frac{\lambda dZ}{\sqrt{(A^2 - Z^2)}}$ (604) erit

$$\lambda = \frac{1}{2}, Z = -q \text{ et } A = \frac{a^2 - 2cf}{4a^2 cf}.$$

Ex his autem invenitur

$$Z = -q = C\text{-}2Cu^2 + 2u\sqrt{(A^2 - C^2)(1-u^2)} = \frac{a^2+2cf}{4a^2cf} - \frac{1}{y^2}.$$

(610). Constans quantitas C ex hoc determinabitur, quod facto $u = 1$ fieri debeat $y = a$. Hinc ergo erit $C = \frac{2cf-a^2}{4a^2cf}$, unde fit $\sqrt{(A^2 - C^2)} = 0$. His substitutis habebitur

$$\frac{1}{2cf} - \frac{1}{y^2} = \frac{(a^2-2cf)u^2}{4a^2cf} = \frac{(a^2-2cf)x^2}{4a^2cfy^2}$$

ob $u = \frac{x}{y}$. Aequatio igitur resultabit ista

$$a^2y^2 - a^2 2cf = (a^2 - 2cf)x^2.$$

Ponatur applicata $MP = z$, erit $y^2 = x^2 + z^2$. Hinc sequens prodibit pro curva quaesita aequatio inter coordinatas orthogonales $a^2z^2 + 2cfx^2 = 2a^2cf$.
Haec aequatio est ad ellipsin, cuius centrum in C est situm,et $AB = 2a$ est alter eius axis, alter vero DH = $2\sqrt{2cf}$. Q.E.I.

Corollarium 1.

632. Attitudo celerati in M debita v est = $\frac{a^2+2cf-y^2}{2f}$. Quia autem est $a^2 + 2cf = AC^2 + CD^2 = AD^2$, erit

$$v = \frac{AD^2-CM^2}{2f}.$$

Corollarium 2.

633. Simili modo perpendiculum CT in tangentem MT demissum erit $p = \frac{AC.CD}{\sqrt{(AD^2-CM^2)}}$ et ipsa tangens [p. 259]

$$MT = \frac{\sqrt{(AD^2.CM^2-CM^4-AC^2.CD^2)}}{\sqrt{(AD^2-CM^2)}} = \frac{(AD^2-CD^2).CP.PM}{AC.CD\sqrt{(AD^2-CM^2)}}$$

si quidem est $AC > CD$.

Corollarium 3.

634. Hoc autem casu, quo $AC > CD$, erit AB ellipsis axis transversus, in eoque foci F et G siti. Erit autem $CF = CH = \sqrt{(AC^2 - CD^2)}$. Adeoque

$$MT = \frac{CF^2.CP.PM}{AC.CD\sqrt{(AD^2-CM^2)}}.$$

Corollarium 4.

635. Anguli ergo *TMC,* quem corporis in *M* directio cum radio *MC* constituit, sinus est,

$$= \frac{AC.CD}{CM\sqrt{(AD^2 - CM^2)}}$$

et cosinus

$$= \frac{CF^2.CP.PM}{CM.AC.CD\sqrt{(AD^2 - CM^2)}}.$$

Scholion 1.

636. Distantia celeritas deteminans *CE* (596, 597) aequalis est semper subtensae *AD*. Nam posita *CE* = *k* descendat corpus ex *E* versus *C* a vi centripeta tractum; habere debebit corpus, cum in *A* pervenerit, celeritatem altitutudini *c* debitam. Quamobrem erit $k = \sqrt{(a^2 + 2cf)}$ (275). Quia autem est

$$AC = a \text{ et } CD = \sqrt{2}cf, \text{ erit } CE = \sqrt{(AC^2 + CD^2)} = AD.$$

Corollarium 5.

637. Altitudo debita celeritati, quam corpus in recta *EC* motum acquiret, cum in *C* pervenerit, erit

$$\frac{k^2}{2f} = \frac{a^2 + 2cf}{2f} = \frac{AD^2}{2f}.$$

Corollarium 6.

638. Tempus, quo arcus AM absolvitur, est $= \frac{2.ACM}{a\sqrt{c}}$ (588) ob $h = a$ hoc casu. [p. 260] Quamobrem tempus totius revolutionis per ellipsis perimetrum *ADBHA* erit $= \frac{2.\text{Areae Ellipticae}}{a\sqrt{c}}$. Est vero posita ratione diametri ad peripheriam 1 : π spatium ellipticum $= \pi a\sqrt{2cf}$. Consequenter tempus unius revolutionis erit $2\pi\sqrt{2f}$.

Corollarium 7.

639. Si ergo plura corpora circa idem centrum virium attrahens in ratione distantiarum revolvantur in ellipsibus, erunt singulorum tempora integrarum revolutionum inter se aequlia.

Scholion 2.

640. Quando corporis directio initialis in puncto *A* non ponitur normalis ad radium *AC*, calculus non praebet ellipsin pro curva a corpora descripta, set aliam curvam ordinis quarti, quae tamen nullo modo satisfacere potest. Causa huius discrepentiae calculi a veritate in hoc consistit, quod expressio sinus anguli, quem curva cum radio constituit, in *y* et *u* sumta semper evadet = 1, posito *y* = *a* et *u* = 1, etiamsi secundum hypothesin alia prodire debeat quantitas. Sinus enim anguli, quem curva cum radio comprehendit, est

$$\frac{ydu}{\sqrt{dy^2 - u^2dy^2 + y^2du^2}},$$

quae expressio facto $u = 1$ evidenter abit in unitatem, cum tamen prodire debeat $\frac{h}{a}$. Quamobrem perspicuum est, nisi ponatur $h = a$, calculum hoc modo institutum nunquam cum veritate conspirare posse, si quidem aequatio inter u et y quaeri debeat. Haec ergo regula perpetuo est tendenda, [p. 261] quoties curva descripta secundum praecepta propositionis 75 (601) investigabitur. Methodus autem hac propositione tradita ad curvas algebraicas tantum inveniendas est accommodata; alia enim methodo, si curvae sint transcendentes, uti oportet. At omnes curvae algebraicae hac gaudent proprietate, ut in eas ex puncto quocunque duci possit perpendicularis. [Hanc propositionem ab Eulero sine demonstratione usurpatam falsam esse ex consideratione parabolae Neilianae facile perspicitur. Notari per Paul Stackel in *Opera Omnia*] Quocirca, quoties corpus in curva algebraica circa centrum virium revolvitur, semper in ea vel unum vel plura poterunt assignari puncta, in quibus radius ad curvam sit perpendicularis. In huiusmodi igitur punctis corpus motum inchoare ponendum est, atque calculus semper veritati erit consentaneus. Ex tali autem solutione facile invenietur corporis in quovis alio loco celeritas, atque hinc methodo inversa data corporis celeritate in loco, in quo radius non est ad curvam normalis, reperietur celeritas in loco, ubi radius in curvae normalem incidit. Quomodo autem hoc effici debeat, ex propositione sequente perspicietur.

Supra memorata curvarum algebraicarum proprietas vero, qua ex quocunque puncto dato in eas perpendicularis potest demitti, non in omnes curvas transcendentes competit. In spirali enim logarithmica nullum radium ex cento ad curvam ductum esse normalem, sed perpetuo cum ea constantem constituere angulum cuique est notum.

Quo tandem ratio intelligatur, quare

$$= \frac{ydu}{\sqrt{dy^2 - u^2dy^2 + y^2du^2}}$$

semper facto $u = 1$ mutetur in unitatem, cum tamen quivis alius quoque angulus praeter rectum ex ea produci deberet, animadvertendum est abeunte u in 1 [p. 262] elementum du evanescere prae dy, nisi sit tangens in A ad radium AC normalis. Hancque ob causam posito u = 1 elementum $dy^2(1-u^2)$ non est negligendum ratione y^2du^2, cum utrumque evanescat, ut in numerator ydu. Quo fit, ut cuiusvis arbitrarii anguli sinus illa formula facto $u = 1$ possit exprimi. Sed quoniam haec cautela in calculo non potest observari, praeter casus $h = a$ calculus nunquam veram curvam exhibet.