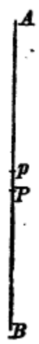
CHAPTER FOUR.
CONCERNING THE MOTION OF FREE POINTS IN A MEDIUM WITH RESISTANCE [p. 188]

PROPOSITION 57.

PROBLEM.

450. *With the time given, in which a body* (Fig. 40) *is projected up from B and has fallen again, in a medium with resistance in the ratio of the square of the speed, and acted on by a constant absolute force g, to determine the height BA to which the body rises, in order that both the initial and the final speed at B after the descent to the same location B are found; and also the ascent time through BA and the descent time through AB.*

SOLUTION.



Let the given time equal t, which is the sum of the times of the ascent and the descent through the line BA, and the exponent of the resistance is equal to k. The altitude sought is put equal to x. The ascent time through BA

$$=2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{x}{k}}-1)}$$

(445) and the descent time from A to B

$$=2\sqrt{\frac{k}{g}}l\left(\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}\right)$$

(427). From which this equation is formed :

$$\frac{t}{2}\sqrt{\frac{g}{k}}=A.\sqrt{(e^{\frac{x}{k}}-1)}+l\left(\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}\right)$$

from which it is possible to find x. Moreover from the height x known, likewise both the times of the ascent through BA and of the descent AB are known. Again from the given height BA = x , the height arising from the speed at B, to which it ascends, equals $gk(e^{\frac{x}{k}}-1)$ (445) and the height generating the speed, to which it descends to B, is equal to $gk(1-e^{-\frac{x}{k}})$ (420). Q.E.I.

Corollary 1.

451. Therefore the ascending speed at B to the speed of descent at the same place is in the ratio $e^{\frac{x}{2k}}$ ad 1. From which it is apparent by how much more is lost from the motion, by the amount the body ascends higher. [p. 189]

Scholium 1.

452. If k is a very large number and with the altitude x not very large, so that it is allowed to have algebraic expressions in place of the times found above, the ascent time is equal to $\frac{2\sqrt{x}}{\sqrt{g}}-\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}$, and the descent time is equal to $\frac{2\sqrt{x}}{\sqrt{g}}+\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}$ (432). Whereby the sum of the given times produces : $t\sqrt{g}=4\sqrt{x}+\frac{x^2\sqrt{x}}{120k^2\sqrt{g}}$, as an approximation.

Scholium 2.

453. Moreover this sum of the times can be defined more accurately by continuing the series expressed for the ascent and descent times. Of course the ascent time can be made

$$=\frac{2\sqrt{x}}{\sqrt{g}}-\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}+\frac{x^3\sqrt{x}}{1344k^3\sqrt{g}}-\frac{x^4\sqrt{x}}{46080k^4\sqrt{g}}-\text{etc.}$$

and the descent time

$$=\frac{2\sqrt{x}}{\sqrt{g}}+\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}-\frac{x^3\sqrt{x}}{1344k^3\sqrt{g}}-\frac{x^4\sqrt{x}}{46080k^4\sqrt{g}}+\text{etc.}$$

On account of which the sum of the times

$$t=\frac{4\sqrt{x}}{\sqrt{g}}+\frac{x^2\sqrt{x}}{120k^2\sqrt{g}}-\frac{x^4\sqrt{x}}{23040k^4\sqrt{g}}+\text{etc.}$$

Where it should be noted, if the time is given in seconds, and k and x are expressed in scruples of Rhenish feet, the above series is to be divided by 250. Thus if the time t is μ seconds, for t must be substituted 250μ.

Corollary 2.

454. From the above equation by inversion, it is possible to extract the series for x. Moreover it becomes :

$$\sqrt{x}=\frac{t\sqrt{g}}{4}-\frac{g^2t^5\sqrt{g}}{2^{15}.15k^2}+\frac{g^4t^9\sqrt{g}}{2^{29}.15k^4}-\text{etc.}$$

and consequently

$$x=\frac{gt^2}{2^4}-\frac{g^3t^6}{2^{16}.15k^2}+\frac{g^5t^{10}\sqrt{g}}{2^{26}.225k^4}-\text{etc.}$$

Corollary 3. [p. 190]

455. The difference between the time of descent and the time of ascent hence will be as an approximation : $\frac{x\sqrt{x}}{3k\sqrt{g}} - \frac{x^3\sqrt{x}}{672k^3\sqrt{g}}$. Hence with the altitude x found, likewise the ascent time and the descent time become known.

Corollary 4.

456. Also the altitude corresponding to the speed in which the body begins the ascent is equal to :
$gx + \frac{gx^2}{2k} + \frac{gx^3}{6k^2} + \frac{gx^4}{24k^3} + \text{etc.}$ and the height corresponding to the speed, with which the body falls, is equal to $gx - \frac{gx^2}{2k} + \frac{gx^3}{6k^2} - \frac{gx^4}{24k^3} + \text{etc.}$

Example.

457. The iron ball fired upwards from a cannon returns to the earth after 34 seconds, and $k = 2250000$ scruples of Rhenish feet and $g = \frac{7499}{7500}$. We will therefore have $t = 8500$ and $\frac{t\sqrt{g}}{4\sqrt{k}} = 1.416572$ and thus $\frac{g^2t^5\sqrt{g}}{2^{15}.15k^2\sqrt{k}} = 0.01188$ and $\frac{g^4t^9\sqrt{g}}{2^{29}.15k^4\sqrt{k}} = 0.0007477$. Therefore $\frac{\sqrt{x}}{\sqrt{k}} = 1.405439$ and $\sqrt{x} = 2108.159$ and the total height x, which the ball reaches in air is equal to 4443 Rhenish feet. Now let δ be the number of seconds, that the descent lasts longer than the ascent ; it will be $\frac{250\delta\sqrt{g}}{\sqrt{k}} = \frac{x\sqrt{x}}{3k\sqrt{k}} - \frac{x^3\sqrt{x}}{672k^3\sqrt{k}}$. Thus indeed, $\sqrt{\frac{x}{k}} = \frac{527}{375}$,from which arises $\frac{x\sqrt{x}}{3k\sqrt{k}} = 0.9913$ and $\frac{x^3\sqrt{x}}{672k^3\sqrt{k}} = 0.01893$. We therefore have $\frac{250\delta\sqrt{g}}{\sqrt{k}} = 0.97237$ and hence δ = 5" 50'''. From which it is apparent that the ascent time is [p. 191] 14" 5''' and the descent time is equal to 19" 55'''. Moreover the altitude generating the speed, with which the body begins the ascent, is found to be 15542 feet, and the altitude corresponding to the speed, with which it starts to fall, is equal to 1969 feet. Concerning which, the comment on page 338 of Book II should be seen.
[Daniel Bernoulli, *Dissertation on the action of fluids on solid bodies and the motion of solids in fluids*. Part four: Concerning the motion of bodies projected up, where the experiments are recalled to the calculation performed by the most distinguished Baron Gunter with canons set in place. *Comment. acad. Petrop*. 2 (1727), 1729, p. 329-342. Pag. 338 it can be read : The time of the whole ascent and descent is 34 sec., with the height to which the cannonball reaches in air with resistance equal to 4550 English feet (an English foot is equivalent to 304.79 mm), the time of the ascent with the air resistance is 14.37sec., and the descent time with air resistance is 19.63 sec., the height to which the cannonball can be projected by the same force in a vacuum is 13694 English feet, the time expended in the motion up and down in a vacuum under the same force is 58 sec.]

Corollary 5.

458. Since the height corresponding to the speed, with which the body is projected up, is equal to

$$gk(e^{\frac{x}{k}}-1)=gx(1+\frac{x}{2k}+\frac{x^2}{6k^2}+\frac{x^3}{24k^3}+\text{etc.})$$

the ascent and descent time likewise accepted, if the body is projected up with this speed in a vacuum acting under the force of gravity only, equals

$$4\sqrt{gx}+\frac{x\sqrt{gx}}{k}+\frac{5x^2\sqrt{gx}}{24k^2}+\frac{x^3\sqrt{gx}}{32k^3}+\text{etc.}$$

Corollary 6.

459. Therefore let the sum of the ascent and descent times in vacuum to the sum of the times in the resisting medium be as

$$g(1+\frac{x}{4k}+\frac{5x^2}{96k^2}+\frac{x^3}{128k^3}+\text{etc.})\text{ to }1+\frac{x^2}{480k^2}-\frac{x^4}{92160k^4}+\text{etc.}$$

If clearly in each case the body is projected with the same speed.

Scholion 3.

460. In the citation in Book II, page 340 the comments concerning the theorem, in which these times in a vacuum and in a medium with the resistance varying as the square ratio of the speed, as we have established here too, are brought together ; and it is asserted that the time in the vacuum is always greater than that in the medium. But indeed from our comparison it is apparent that it can be possible for the time in the vacuum to be less than the time in the medium. For if x were very small, and k truly very large, these times between them are approximately as g to 1. [p. 192] Indeed g in the resisting medium on account of the force of gravity is always a little less than one, and on account of this the time in the vacuum is less than for the time in the medium for these cases. Truly when g differs from unity by a small amount, and x is not very small with respect to k, as in the case of firing the cannonball, the time in the vacuum certainly is greater than in the resisting medium. Then if $x > k$, it is easy to examine the given case too, in which the outcome of that theorem will be different. Moreover the body in the example with the reported speed corresponding to a height of 15542 projected up in a vacuum returns to the earth in 63 seconds, yet when in the air it does not remain longer than 34".

PROPOSITION 58.

PROBLEM.

461. *If a body after some descent* (Fig. 42) *is reflected from O up with the same speed that it acquired in the descent, and again ascends straight up, and these reflections are always repeated when it arrives at O, then the altitudes OA. OB, OC, etc., are sought, which the body in this way successively traverses in a medium with a uniform resistance varying as the square of the speed, and acted on by a constant force g.*

SOLUTION.

With the exponent of the resistance put in place equal to k, as has been done up to the present, let the first height be $AO = a$, and it is the height corresponding to the speed, in which the body starts from rest. The height corresponding to the speed with which the body begins to rise is equal to $gk(e^{\frac{a}{k}}-1)$ (439). [p. 193] Indeed the height corresponding to the speed, which by falling through AO it reaches the point O, is equal to $gk(1-e^{\frac{-a}{k}})$ (420). The following ascent through OB now begins, and OB is made equal to z; and $gk(1-e^{\frac{-a}{k}})=gk(e^{\frac{z}{k}}-1)$. From which is produced :

$$OB=kl(2-e^{\frac{-a}{k}})=z.$$

Indeed the altitude corresponding to the speed, in which the second descent through falling BO, is equal to

$$gk(1-e^{\frac{-z}{k}})=\frac{gk(1-e^{\frac{-a}{k}})}{2-e^{\frac{-a}{k}}},$$

and with this speed the third ascent through OC can begin. Now $OC = z$, and it becomes :

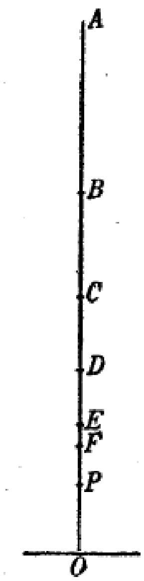


Fig. 42.

$$\frac{gk(1-e^{\frac{-a}{k}})}{2-e^{\frac{-a}{k}}}=gk(e^{\frac{z}{k}}-1), \text{ consequently } z=OC=kl\frac{3-2e^{\frac{-a}{k}}}{2-e^{\frac{-a}{k}}}.$$

Truly the speed, that is acquired in the descent along CO corresponds to the altitude :

$$gk(1-e^{\frac{-z}{k}})=\frac{gk(1-e^{\frac{-a}{k}})}{3-2e^{\frac{-a}{k}}}.$$

With this speed again the ascent along OD begins, as we call the altitude OD anew z : it becomes:

$$gk(e^{\frac{z}{k}}-1)=\frac{gk(1-e^{\frac{-a}{k}})}{3-2e^{\frac{-a}{k}}} \text{ and consequently } z=OD=kl\frac{4-3e^{\frac{-a}{k}}}{3-2e^{\frac{-a}{k}}}.$$

In a similar way the fifth altitude is produced [p. 194]

$OE = kl\frac{5-4e^{\frac{-a}{k}}}{4-3e^{\frac{-a}{k}}}$ and the sixth $OF = kl\frac{6-5e^{\frac{-a}{k}}}{5-4e^{\frac{-a}{k}}}$.

From which it is concluded that the altitude *OP*, the index of which is *n*, is equal to :

$$kl\frac{n-(n-1)e^{\frac{-a}{k}}}{(n-1)-(n-2)e^{\frac{-a}{k}}} = kl\frac{ne^{\frac{a}{k}}-n+1}{(n-1)e^{\frac{a}{k}}-n+2}.$$

Therefore it is clear, how large the altitude of the body reaches after each reflection from the point *O*. Q.E.I.

Corollary 1.

462. Likewise from this solution it is evident that the height corresponding to the speed acquired in falling through the descent *PO* is equal to

$$\frac{gk(1-e^{\frac{-a}{k}})}{n-(n-1)e^{\frac{-a}{k}}} = \frac{gk(e^{\frac{a}{k}}-1)}{ne^{\frac{a}{k}}-n+1}$$

Truly the speed, by which the ascent through *OP* has been assailed, corresponds to the height

$$\frac{gk(1-e^{\frac{-a}{k}})}{n-1-(n-2)e^{\frac{-a}{k}}} = \frac{gk(e^{\frac{a}{k}}-1)}{(n-1)e^{\frac{a}{k}}-n+2}.$$

Corollary 2.

463. Since

$$OA = a = kle^{\frac{a}{k}} \text{ and } OB = kl(2-e^{-\frac{a}{k}}),$$

then it follows that

$$OA + OB = kl(2e^{\frac{a}{k}} - 1).$$

And in the same manner : [p. 195]

$$OA + OB + OC = kl(3e^{\frac{a}{k}} - 2) \text{ and } OA + OB + OC + OD = kl(4e^{\frac{a}{k}} - 3) \text{ etc.}$$

Corollary 3.

464. If *k* is a very large number, in order that $\frac{a}{k}$ almost vanishes, then an approximation is

$$OP = a - \frac{(n-1)a^2}{k} + \frac{(n-1)^2a^3}{k^2} - \frac{(n-1)^3a^4}{k^3} + \text{etc.}$$

Since this series is geometrical, then $OP = \frac{ak}{k+(n-1)a}$ as an approximation.

Corollary 4.

465. If the first height *OA* is indefinitely great, non of the smaller heights are finite. Indeed, there arises :

$OB = k.l2,\quad OC = k.l\frac{3}{2},\quad OD = k.l\frac{4}{3},\quad OP = k.l\frac{n}{n-1}.$

Corollary 5.

466. And if any height is equal to $k.l\,A$, the following height, to which the body after rebounding from the first is able to reach, is equal to $= k.\,l\frac{2A-1}{A}$. Again the third height is equal to $k.\,l\frac{3A-2}{2A-1}$, and similarly the fourth, $k.\,l\frac{4A-3}{3A-2}$ and for that, the index of which is n, is equal to $k.\,l\frac{nA-n+1}{(n-1)A-n+2}$.

Scholium 1.

467. Also negative numbers can be substituted in place of n, and then the preceding altitudes present in the first series can be found. Thus the altitude that follows the first $OA = a$, by putting $n = 0$, is equal to $k.\,l\frac{1}{2-e^{\frac{a}{k}}}$. From which it appears, that if $e^{\frac{a}{k}} = 2$ or $a = k.\,l2$, [p. 196] then the preceding altitude is infinite. But if it is the case that $e^{\frac{a}{k}} > 2$, the preceding altitude on account of the logarithm of a negative quantity is imaginary, that indicates that it is not possible, as so great a real height must be assigned, following the initial taken as a.

Scholium 2.

468. Since after an infinite altitude it is possible for the motion to follow with a finite altitude, which is indeed admirable to consider ; but it is to be considered how a body in a resisting medium falling from an infinite height can acquire such a speed (420), and the reason for this phenomenon can easily be made clear : for the body is only able to rise with this finite speed as far as a certain altitude. Moreover the greatest speed that a body can gain in falling is $\sqrt{gk}$ [i. e. its terminal velocity]. Whereby if the body is initially projected upwards with a speed greater than $\sqrt{gk}$, this speed cannot be generated by a descent of any magnitude ; as also in this case the preceding height calculation of imaginary quantities shows.

PROPOSITION 59.

PROBLEM.

469. *With the resistance of a medium uniform and in proportion to the speed, and with the body acted on by an absolute force pulling downwards, to determine the speed of the body upwards or downwards at any point on a straight line.* [p. 197]

SOLUTION.

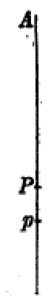


Fig. 39.

At first the body falls along the line *AP* (Fig. 39), and the initial speed of the body at *A* corresponds to the altitude *c*. The absolute force is put equal to *g*, the exponent of the resistance is equal to *k* and *AP* = *x* and the height corresponding to the speed at *P* = *v*. [The original has *x* instead of *v*, which is obviously a misprint.]
With these put in place $dv = gdx - \frac{dx\sqrt{v}}{\sqrt{k}}$; for the force of resistance is equal to $\frac{\sqrt{v}}{\sqrt{k}}$. Hence this becomes : $dx = \frac{dv\sqrt{k}}{g\sqrt{k}-\sqrt{v}}$.
Make
$\sqrt{v} = u$, then $dv = 2udu$ and
$dx = \frac{2udu\sqrt{k}}{g\sqrt{k}-u} = -2du\sqrt{k} + \frac{2gkdu}{g\sqrt{k}-u}$.
By integration this equation gives :
$x = C - 2u\sqrt{k} - 2gkl(g\sqrt{k} - u) = C - 2\sqrt{kv} - 2gkl(g\sqrt{k} - \sqrt{v})$.
For with $x = 0$ it must become v = c, from which $C = 2\sqrt{kc} - 2gkl(g\sqrt{k} - \sqrt{c})$.
Thus we have : $x = 2\sqrt{kc} - 2\sqrt{kv} + 2gkl\frac{(g\sqrt{k}-\sqrt{c})}{(g\sqrt{k}-\sqrt{v})}$,
from which *v* with the aid of logarithms can be deduced.
Which had to be shown for the down motion.

Now for the ascent let the initial speed at *B* (Fig. 40) correspond to the altitude *c* and let *BP* = *x* and the height corresponding to the speed at P is equal to *v*. Since in the ascent with both the absolute force and the force of retardation, it becomes : $dv = -gdx - \frac{dx\sqrt{v}}{\sqrt{k}}$.
Which equation can be deduced from the previous one $dv = dgx - \frac{dx\sqrt{v}}{\sqrt{k}}$ by putting –*g* in place of *g*. On account of which in this manner the required equation can also be deduced from that derived. Therefore with *g* made negative :

$$x = 2\sqrt{kc} - 2\sqrt{kv} - 2gkl\frac{(g\sqrt{k}+\sqrt{c})}{(g\sqrt{k}+\sqrt{v})}.$$

Which had to be shown for the up motion.

Corollary 1.

470. If the initial speed in the descent is zero, then $x = -2\sqrt{kv} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}$. From [p. 198] which equation the speed of the body can be determined dropped from any height.

Corollary 2.

471. Since indeed $l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}} = -l(1-\frac{\sqrt{v}}{g\sqrt{k}})$,
the series is obtained :

$$l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}} = \frac{\sqrt{v}}{g\sqrt{k}} + \frac{v}{2g^2k} + \frac{v\sqrt{v}}{3g^3k\sqrt{k}} + \frac{v^2}{4g^4k^2} + \text{etc.}$$

From which on substitution there is given :

$$x = \frac{v}{g} + \frac{2v\sqrt{v}}{3g^2\sqrt{k}} + \frac{v^2}{2g^3k} + \frac{3v^2\sqrt{v}}{5g^4k\sqrt{k}} + \text{etc.}$$

If now k is made a very large number, as an approximation there is obtained :
$v = \frac{v}{g} - \frac{2x\sqrt{gx}}{3\sqrt{k}}$.

Corollary 3.

472. If we make $v = g^2k$, then $x = \infty$ (Fig. 40). From which it appears that the body falling from an infinite height is not able to acquire a velocity greater than $g\sqrt{k}$. And if once the speed satisfies $v = g^2k$, then the body progresses with a constant speed ; then indeed the motion is one of retardation if the speed satisfies $v > g^2k$.

Corollary 4.

473. If the body is again projected from B upwards with a speed $\sqrt{c}$, then the altitude BA can be found from $v = 0$. Moreover the equation is produced :

$$BA = 2\sqrt{kc} - 2gkl(1+\frac{\sqrt{c}}{g\sqrt{k}}) = \frac{c}{g} - \frac{2c\sqrt{c}}{3g^2\sqrt{k}} + \frac{c^2}{2g^3k} + \text{etc.}$$

Indeed the altitude, from which the body by falling can acquire this speed , is given by :

$$-2\sqrt{kc} - 2gkl(1-\frac{\sqrt{c}}{g\sqrt{k}}) = \frac{c}{g} + \frac{2c\sqrt{c}}{3g^2\sqrt{k}} + \frac{c^2}{2g^3k} + \text{etc.}$$

Corollary 5.

474. If the body is again projected with the speed $g\sqrt{k}$, clearly the maximum that it can acquire by falling, then the altitude it can reach is given by : $2gk(2-l\,2)$. [p. 199]

PROPOSITION 60.

PROBLEM.

475. *With the resistance of the medium in the simple ratio of the speed and acted on by a uniform absolute force, to determine the time in which the body traverses some interval, either ascending or descending.*

SOLUTION.

As before with the descent through AP (Fig. 39) in place, with the speed at $A = \sqrt{c}$ and that at $P = \sqrt{v}$, with the exponent of the resistance equal to k and the absolute force equal to g, and with the distance $AP = x$, the element of distance

$$dx = \frac{dv\sqrt{k}}{g\sqrt{k}-\sqrt{v}} = \frac{2udu\sqrt{k}}{g\sqrt{k}-u}$$

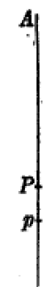


Fig. 39.

on putting u^2 in place of v. Now I say with the time to traverse $AP = t$ that $dt = \frac{du}{u} = \frac{2du\sqrt{k}}{g\sqrt{k}-u}$.

From which is given $t = 2\sqrt{k}\; l\frac{g\sqrt{k}-\sqrt{c}}{g\sqrt{k}-\sqrt{v}}$.

Which had to be shown for the down motion.

Since the ascent through $BP = x$ (Fig. 40), with the initial speed $\sqrt{c}$ given and by putting $-g$ in place of g, the ascent time through BP is given by :

$$t = 2\sqrt{k}l\frac{g\sqrt{k}+\sqrt{c}}{g\sqrt{k}+\sqrt{v}}.$$

Which had to be shown for the up motion.

Indeed from the preceding problem, v is defined from the x. Whereby here the time in which some interval travelled through can become known. Q. E. I.

Corollary 1.

476. If the initial speed, with which the body fell was zero, then the descent time for the interval AP $= 2\sqrt{k}\; l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}$. But v is defined from this equation, [p. 200]

$$x = -2\sqrt{k}v + 2gk\; l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}.$$

Corollary 2.

477. The total altitude BA, to which the body can reach from B ascending, is completed in the time $2\sqrt{k}\, l\frac{g\sqrt{k}+\sqrt{c}}{g\sqrt{k}}$.

Corollary 3.

478. Therefore if the initial speed $\sqrt{c}$ should be infinite, also the time, in which the whole distance BA is traversed, is infinite, clearly equal to $2\sqrt{k}\, l \propto$.

Scholium 1.

479. Therefore this hypothesis for the resistance differs greatly from the previous case, in which the resistance was put in proportion to the square of the speed. For in that case a body projected up with an infinite speed reached the maximum point in a finite time (444). Truly the time $2\sqrt{k}l \propto$ is to be regarded as an infinite amount of the lowest order. From which it seemed to be possible to conclude, if the resistance should be greater than in a simple ratio with the speed, then the time of the total ascent should always be finite, but otherwise if the resistance is in a simple or smaller ratio of the speeds, the time for the whole ascent is infinite, if indeed the initial speed is infinitely great.

Scholium 2.

480. I have thought that these two hypotheses of the resistance should be enlarged upon further than that which have been considered by Newton, and by those who have followed him. Indeed this latter hypothesis, in which we have put the resistance to be in proportion to the speed, is merely a mathematical device and cannot have any use in physics [p. 201]. But since from the beginning, investigators have considered the resistance of fluids arising from their tenacity to be in proportion to the speed of the body, then an enquiry into this kind of motion should be judged more carefully. Yet afterwards, when there has long been an understanding of resistance, this discussion can still be retained. Truly the first, in which the resistance is proportional to the square of the speed, merits to be investigated the most : for indeed with certainty the resistance of particular fluids keeps this ratio. Besides in addition, this second hypothesis has to be investigated by calculation before so many others, since in other hypotheses there is little of merit, yet in this case the calculation is not thwarted. Indeed in nearly all the problems, in which the solutions in the vacuum are not rejected, these can also be resolved according to this hypothesis of resistance. On account of this we will chiefly examine only that kind of resistance sought which can be found by a neat computation, and moreover the rest we mostly ignore.

PROPOSITION 61.

PROBLEM.

481. *The medium offers resistance in a ratio according to some power of the speed and a uniform force is acting, it is necessary to determine the right motion of the body either ascending or descending.* [p. 202]

SOLUTION.

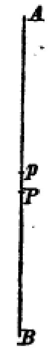


Fig. 40.

We consider (Fig. 40) the first descent and put the speed at $A = \sqrt{c}$, the interval $AP = x$ and the speed at $P = \sqrt{v}$. Let the exponent of the resistance be k and the law of the resistance v^m, and the absolute force equal to g. Therefore with these in place :
$dv = gdx - \frac{v^m dx}{k^m}$ and $dx = \frac{k^m dv}{gk^m - v^m}$. Therefore we have $x = \int \frac{k^m dv}{gk^m - v^m}$,
and with the help of quadrature it is possible to determine v in terms of x. The time in which the distance AP is transversed, is put equal to t, and
$dt = \frac{dx}{\sqrt{v}} = \frac{k^m dv}{gk^m\sqrt{v} - v^m\sqrt{v}}$. And

$$t = \int \frac{k^m dv}{gk^m\sqrt{v} - v^m\sqrt{v}}.$$

Which had to be shown for the motion down.

Now for the ascent the initial speed at B is $\sqrt{c}$, and let $BP = x$ and the speed at P = $\sqrt{v}$ and the time in which BP is traversed is equal to t. With these in place,
$dv = -gdx - \frac{v^m dx}{k^m}$, which equation is extracted from the other by putting $-g$ in place of g. With which accomplished, for the ascent :

$$x = -\int \frac{k^m dv}{gk^m + v^m} \text{ and } t = -\int \frac{k^m dv}{gk^m\sqrt{v} + v^m\sqrt{v}}$$

Which had to be found for the upwards motion.

Corollary 1.

482. If we put $c^m = gk^m$, the body is carried with this uniform motion in the descent. For the absolute force by which the body is accelerated, is always equal to the force of the resistance, by which it is retarded. [p. 203]

Corollary 2.

483. Indeed a body dropped from rest always accelerates unless at some time it acquires a speed corresponding to the altitude $g^{\frac{1}{m}}k$. But this speed is as it were an asymptote, since it has the effect that the body is either made to move faster or to slow down to this speed.

Scholium 1.

484. Since neither of these equations found can be integrated, neither v not t can be defined at x, it is not expedient to linger over these. I will therefore contemplate other mediums in which the resistance is variable, keeping the absolute force constant. Yet I accept a hypothesis of this kind, in which the equation determining dv is made homogeneous and thus it is not dependent upon these difficulties. Then the absolute force is no further uniform, but I put a variable or I consider a centripetal force in its place, which always attracts the body to the some certain fixed point. Since indeed in this first case of another kind of resistance, I only consider it in the case where it is proportional to the square of the speed. Then indeed it will be appropriate with other hypothetical forms of resistance to be introduced, where these are only applied to centripetal forces that permit the integration of the differential equations.

PROPOSITION 62.

PROBLEM.

485. *With a uniform force present, and the exponent of the resistance to be in proportion to the distances from a fixed point C* (Fig. 43)*, and with the law of the resistance in some multiple ratio of the speeds, the speed of the body at some place is required on the line AC, advancing or receding from C.* [p. 204]

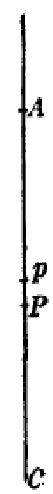


Fig. 43.

SOLUTION.

Let the uniform force acting at C be equal to g, with the height corresponding to the speed at any point P put in place equal to v. AC is put equal to a, which is the maximum height, to which the body reaches, and $CP = x$, the exponent of the resistance is in proportion to x; and this is written as $\lambda^{\frac{1}{m}}x$, and the law of resistance is as v^m . With these quantities put in place the resistive force is equal to $\frac{v^m}{\lambda x^m}$, and for the ascent through CA, in which both the absolute force and the resistive force decelerate the motion, this equation is obtained for the motion : $dv = -gdx - \frac{v^m dx}{\lambda x^m}$. Here we can consider the descent as well as the ascent, and since in the descent the absolute force indeed accelerates, while the resistive force retards, by substituting in the ascent equation, the retarding force in the opposite way, the resisting force is put in place (411), from which this equation arises for the descent :

$dv = -gdx + \frac{v^m dx}{\lambda x^m}$. Which equation can easily be derived from the other by making λ negative, and this on account of the other equation is needed so much in the integration. We take the equation for the descent, which is of this kind :

$$\lambda x^m dv + \lambda g x^m dx = v^m dx,$$

and we put $v = xz$. Therefore it becomes : $dv = xdz + xdx,$ From which this equation arises : $\lambda x^{m+1} dz + \lambda x^m z dx + \lambda g x^m dx = x^m z^m dx.$ [p. 205]
Which divided by $x^{m+1}(z^m - \lambda z - \lambda g)$ it is changed into this : $\frac{\lambda dz}{z^m - \lambda z - \lambda g} = \frac{dx}{x}$,
in which the indeterminate are now separate. Therefore this equation can now be integrated, as by making $x = a$, the speed vanishes; with which done from which equation to be integrated the speed of the descending body can become known at any point. Truly this same equation with λ made negative looks after defining the speed in the ascent through CA. Q.E.I.

Corollary 1.

486. If $m = 1$, or the resistance is in proportion to the square of the speed, the equation becomes :
$\frac{\lambda dz}{(1-\lambda)z - \lambda g} = \frac{dx}{x}$ and $\frac{\lambda}{(1-\lambda)} l\,((1-\lambda)z - \lambda g) = l\,x + C = \frac{\lambda}{(1-\lambda)} l\,\frac{(1-\lambda)v - \lambda g x}{x}$ by substituting $\frac{v}{x}$ in place of z. Moreover since, if $x = a$, it follows that $v = 0$, and the constant $C = \frac{\lambda}{(1-\lambda)} l\,(1-\lambda)g - \lambda a$ and thus $l\,x = l\,a + \frac{\lambda}{(1-\lambda)} l\,\frac{\lambda g x - (1-\lambda)v}{\lambda g x}$. From which is produced:

$$v = \frac{\lambda g x}{(\lambda-1)} \left(\frac{a^{\frac{\lambda-1}{\lambda}} - x^{\frac{\lambda-1}{\lambda}}}{x^{\frac{\lambda-1}{\lambda}}} \right) = \frac{\lambda g}{(\lambda-1)} \left(a^{\frac{\lambda-1}{\lambda}} x^{\frac{1}{\lambda}} - x \right).$$

Corollary 2.

487. If λ is made less than one, this equation must be reduced to the other form ; moreover it becomes : $v = \frac{\lambda g}{(\lambda-1)} \left(\frac{a^{\frac{1-\lambda}{\lambda}} x - x^{\frac{1}{\lambda}}}{a^{\frac{1-\lambda}{\lambda}}} \right)$. [p. 206]

Corollary 3.

488. The case is which $\lambda = 1$ or the exponent of the resistance is equal to the distance itself from the point C, is not present from these formulas, but can be deduced from the differential $\frac{-dz}{g} = \frac{dx}{x}$. Moreover it gives $C - \frac{z}{g} = l\,x$ and hence $v = gx(l\,a - l\,x)$.

Corollary 4.

489. From these in the case $m = 1$ of the body descending the speed at A as at C is equal to 0. For $v = 0$ is produced in these three equations by putting $x = 0$ and by putting $x = a$. Therefore the body falling from A to C loses all the motion and is again at rest for ever at C on account of the resistance of infinite magnitude.

Corollary 5.

490. Therefore while the body traverses the straight line AC, somewhere between A and C the speed is a maximum, which is found from the differential equation by making $dv = 0$. Moreover it then becomes $v = \lambda gx$, from which value put in place of v in the integrated equations gives by substitution :
$\lambda x^{\frac{\lambda-1}{\lambda}} = a^{\frac{\lambda-1}{\lambda}}$ and $x = \frac{a}{\lambda^{\frac{\lambda-1}{\lambda}}}$, if $\lambda > 1$. But if $\lambda < 1$, then $x = \lambda^{\frac{\lambda-1}{\lambda}} a$; and if $\lambda = 1$, then $1 = l\,a - l\,x$ and thus $x = \frac{a}{e}$, with e denoting the number, the logarithm of which is one.

[p. 207] Scholium 1.

491. From these it is gathered for the remaining hypothetical resistance also that it is allowed for the speed of the body to vanish as it approaches C. For the resistive force is $\frac{v^m}{\lambda x^m}$, which therefore becomes infinite if $x = 0$. Whereby if the body has some velocity at C, that force of resistance must be reduced to nothing at once. Truly it has the maximum speed in the descent when $v^m = \lambda g x^m$. From which it is apparent that the maximum speed corresponds to the height $x\sqrt[m]{\lambda g}$. But since x is not known or the place in which the body descends the quickest, also the speed itself cannot be determined, except by the quadrature of the curve, with the help of which the differential equation is constructed.

Corollary 6.

492. For the ascent from C to A, if $m = 1$, the speed of the body at the individual points P is determined from this equation : $v = \frac{\lambda gx}{(\lambda+1)}\left(\frac{a^{\frac{\lambda+1}{\lambda}} x - x^{\frac{\lambda+1}{\lambda}}}{a^{\frac{\lambda+1}{\lambda}}}\right)$,
for which it is necessary to use this equation for the descent, by making λ negative.

Corollary 7.

493. Therefore in the ascent of the body from C the speed is always infinite. For with $x = 0$, since $\frac{\lambda+1}{\lambda}$ is greater than one, the denominator vanishes. [p. 208]

Scholium 2.

494. It is also evident that only the speed at C must be infinite. For unless it is so great, the body is not able to overcome the resistive force at C, but to remain stuck at C for ever.

PROPOSITION 63.

THEOREM.

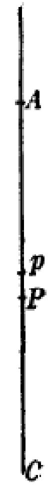


Fig. 43.

495. *With the same quantities put in place as in the previous proposition, if many bodies fall towards C from different heights* (Fig. 43)*, the times at which they arrive there are in the square root ratio of the distances.*

DEMONSTRATION.

In the solution of the preceding problem in finding the speed of the body at P, we found this equation : $\lambda x^m dv + \lambda g x^m dx = v^m dx$ (485). In which equation x and v are put in place with a number of the same dimension everywhere. Therefore with the integral of this equation taken, in order that with $x = a$ making $v = 0$, it has this property, that $x, v,$ and a are represented by numbers with the same dimension. From that it follows that v is equal to a certain function of a and x, in which a and x are everywhere constituted with a number of the same dimension; or v is a function of a and x of dimension one. Whereby in the element of time to pass through CP, which is $\frac{dx}{\sqrt{v}}$, the dimension of dx is that of x, and a is half, and hence on this account the time to pass through CP is equal to a function of a and x agreeing with the dimension of a half. Therefore on putting $x = a$, in which case [p. 209] the time for the whole descent through AC is found, a function only of a will be obtained with the dimension of a half. On account of which the time to pass through AC can be expressed in the form $C\sqrt{a}$, in which C depends on the quantities λ, m and g , and does not depend on the quantity a. Now since a denotes the height AC, it is evident that the times of the descents of many bodies between themselves are in the ratio of the square roots of the heights travelled through. Q.E.D.

Corollary 1.

496. In a like manner it is understood that the times of ascents of many particles from C are in the same ratio of the square roots of the heights to which they rise.

Corollary 2.

497. Therefore in whatever multiple of the speeds ratio the medium resists, as long as the absolute force is constant and the exponent of the resistance are proportional to the distances from C, both the times of the ascents and of the descents keep the ratio of the square root of the height reached.

Scholium 1.

498. Truly is it not permitted to compare either the ascent with the descent nor many ascents or descents between themselves, in which the variables λ*, m* and *g* do not hold the same values. For in the expression $C\sqrt{a}$, in every case the quantity *C*, which is compared, must remain the same. [p. 210]

Scholium 2.

499. In this same proposition we have used the times of the descents towards a fixed point to be compared by a different method than in the above propositions 39 (308) and 46 (354). Moreover in this case it is considered that this method is better than the other, since in that method the speed cannot be determined at *x*. Indeed this alone is seen to be sufficient for us, and a function of this kind of *a* and *x* will be expressed by that, to which *v* shall be made equal. Moreover in the following many outstanding examples of this method occur.

PROPOSITION 64.

PROBLEM.

500. *With a centripetal force to be proportional to some power of the distance from the centre C* (Fig. 43)*, and with the uniform resistance of the medium as the square ratio of the speed, to determine the speed of the body at individual points P on the right line AC, either moving either up or down.*

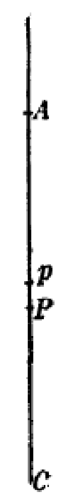


Fig. 43.

SOLUTION.

Let the body at *P* have a speed corresponding to the height *v*. *CP* is called *x*, [*AP* in the original and in *Opera Omnia*, but clearly a typographical error,] and the centripetal force shall be as x^n , and that distance at which the centripetal force is equal to gravity is equal to *f*. Then the exponent of the resistance is put equal to *k*. With these put in place, the absolute force acting on the body at *P* is equal to $\frac{x^n}{f^n}$, and the force of resistance at this place is $\frac{v}{k}$, with the force of gravity set equal to 1. [p. 211] Now the body is descending to *C* and it has, as it moves through the element *pP*, the centripetal accelerating force and the resistive retarding force acting. Moreover since here when the body in the inverse motion goes from *P* to *p* , it is necessary, with increasing *x*, that the inverses of these forces are considered to be put in place or, since it gives the same result, *dx* is made negative, since in the descent the distance *PC* = *x* is being made less. Hence there arises : $dv = -\frac{x^n}{f^n}dx + \frac{vdx}{k}$. Moreover in

the ascent of the body through *Pp* each force is retarding, and thus we have :
$dv = -\frac{x^n}{f^n}dx - \frac{vdx}{k}$. From which it is evident that the one equation can arise from the other by making *k* negative. On account of this it is only necessary for one equation to be integrated. We take that for the ascent : $dv = -\frac{x^n}{f^n}dx - \frac{vdx}{k}$ or $dv + \frac{vdx}{k} = -\frac{x^n}{f^n}dx$,

and this is to be multiplied by $e^{\frac{x}{k}}$, in order that it gives $e^{\frac{x}{k}}(dv + \frac{vdx}{k}) = -\frac{e^{\frac{x}{k}}x^n}{f^n}dx$, and the integral of this is [An early use of an Euler *integrating factor*] :

$e^{\frac{x}{k}}v = -\int \frac{e^{\frac{x}{k}}x^n}{f^n}dx$. Hence it becomes : $v = -e^{-\frac{x}{k}}\int \frac{e^{\frac{x}{k}}x^n}{f^n}dx$.

Therefore for the descent : $v = -e^{\frac{x}{k}}\int \frac{e^{-\frac{x}{k}}x^n}{f^n}dx$.

In each integration a constant quantity must be added, to be determined from that, since the speed of the motion of the body is given everywhere, [p. 212] : for otherwise the speed of the body cannot be determined. Q.E.I.

Corollary 1.

501. It is therefore evident, if n is a positive whole number, that these formulas can be integrated. Indeed the integral is :

$$\int e^{\frac{x}{k}}x^n dx = ke^{\frac{x}{k}}x^n - nk^2e^{\frac{x}{k}}x^{n-1} + n(n-1)k^3e^{\frac{x}{k}}x^{n-2} - n(n-1)(n-2)k^4e^{\frac{x}{k}}x^{n-3} + \text{etc.} + \text{C}.$$

Which series is not infinite, as often as *n* is a positive whole number.

Corollary 2.

502. Let the speed at *C* be given and let it correspond to the height *c*, the series for *v* is :

$$v = e^{-\frac{x}{k}}c - \frac{kx^n}{f^n} + \frac{nk^2x^{n-1}}{f^n} - \frac{n(n-1)k^3x^{n-2}}{f^n} + \text{etc.} \pm n(n-1)(n-2)...2.1.\frac{k^{n+1}e^{-\frac{x}{k}}}{f^n};$$

the upper changeable sign of which prevails if $n + 1$ is an odd number, and the lower if $n + 1$ is an even number.

Moreover for the descent the corresponding height is :

$$v = e^{\frac{x}{k}}c + \frac{kx^n}{f^n} + \frac{nk^2x^{n-1}}{f^n} + \frac{n(n-1)k^3x^{n-2}}{f^n} + \text{etc.} - n(n-1)(n-2)...2.1.\frac{k^{n+1}e^{\frac{x}{k}}}{f^n},$$ with the constant put in place of *C*. [p. 213]

Corollary 3.

503. With the integral of $\frac{e^{\frac{x}{k}}x^n dx}{f^n}$ thus taken, in order that it is equal to zero by making $x = 0$, it is put equal to *X*. And hence $v = e^{-\frac{x}{k}}(c - X)$, since by making $x = 0$ it must become $v = c$. Indeed this equation takes care of the ascent, but it is adapted for the descent by making k negative.

Corollary 4.

504. With X known it is apparent that the height CA, to which the body either can ascend, or, from which dropped, it acquires a speed equal to $\sqrt{c}$. For, from the equation $X = c,$ the root x gives the height CA.

Corollary 5.

505. From the differential equation for the descent, $dv = -\frac{x^n}{f^n}dx + \frac{vdx}{k}$, it is apparent that the body has a maximum speed somewhere, before it reaches C, which will be there, where $v = \frac{kx^n}{f^n}$, if indeed n is not a negative number.

Corollary 6.

506. With the height given $CA = a$, and if c is sought from this, it is necessary to have that quantity, in which X results with a put in place of x. Let that value be A, for which the ascent $v = e^{-\frac{x}{k}}(A - X)$ and for the descent [p. 214] $v = e^{\frac{x}{k}}(A - X)$. For by making $x = a$, v should vanish. Now by making $x = 0$, in which case also X is made zero (503), then $v = c = A$.

Corollary 7.

507. It is evident from these , how the time can be found in which the interval CP is traversed. Clearly for the ascent, the time for $CP = \int \frac{e^{\frac{x}{2k}}dx}{\sqrt{(A-X)}}$, and for the descent, the time for $PC = \int \frac{dx}{e^{\frac{x}{2k}}\sqrt{(A-X)}}$.

EXAMPLE.

508. Let the centripetal force be as the distance from the centre C, in which case $n = 1$ (502); hence for the ascent :

$$v = e^{-\frac{x}{k}}c - \frac{kx}{f} + \frac{k^2}{f} - \frac{k^2 e^{-\frac{x}{k}}}{f}.$$

Truly for the descent :

$$v = e^{\frac{x}{k}}c + \frac{kx}{f} + \frac{k^2}{f} - \frac{k^2 e^{\frac{x}{k}}}{f}$$

(502). In the descent, the maximum speed is where $v = \frac{kx}{f}$ (505), from which equation with this put in place we have : $fe^{\frac{x}{k}} + k^2 = k^2 e^{\frac{x}{k}}$. Hence this will be

$$e^{\frac{x}{k}} = \frac{k^2}{k^2 - cf} \text{ and } x = k\,l\frac{k^2}{k^2 - cf}.$$

Therefore this distance is infinite if $cf = k^2$, and everything is truly imaginary if $cf > k^2$. Again let the speed of the body at A be zero, and with $AC = a$, the height a

corresponding to the initial speed for the ascent at C is given by [from the last two formulas in the present Prop. 64] :

$$\frac{e^{\frac{a}{k}}ka}{f}-\frac{e^{\frac{a}{k}}k^2}{f}+\frac{k^2}{f}.$$

[p. 215] In the descent, the height corresponding to the final speed is given by :

$$-\frac{e^{-\frac{a}{k}}ka}{f}-\frac{e^{-\frac{a}{k}}k^2}{f}+\frac{k^2}{f}.$$

From which it is apparent, if a becomes infinite, the height corresponding to the final speed at C is equal to $\frac{k^2}{f}$.

PROPOSITION 65.

PROBLEM.

509. *For some centripetal force acting towards C, and with the resistance to the motion following the squares of the velocities changed in some way, the motion of the body is to be determined on the line either accelerating towards, or decelerating from C.*


Fig. 43.

SOLUTION.

Let the body be at P and put $CP = x$, and the speed at $P = \sqrt{v}$. Then the centripetal force at $P = p$, with the acceleration of gravity put as 1, and with the exponent of the resistance equal to q, which letters p and q denote some functions of x. Hence the force of resistance is equal to $\frac{v}{q}$. On account of this we have for the ascent : $dv=-pdx-\frac{vdx}{q}$. For the descent indeed there is this equation : $dv=-pdx+\frac{vdx}{q}$. Of which one can be changed into the other by making q negative. Therefore we will consider only the one suited for the ascent, which adopts this form :

$$dv+\frac{vdx}{q}=-pdx.$$

This is multiplied by $e^{\int\frac{dx}{q}}$, so that it becomes integrable. But the equation of this integral is : [p. 216] $e^{\int\frac{dx}{q}}v=-\int e^{\int\frac{dx}{q}}pdx$, hence :

$$v=-e^{-\int\frac{dx}{q}}\int e^{\int\frac{dx}{q}}pdx.$$

Let the speed of the body at A be zero, with $AC = a$, and X written in place of the integral of $e^{\int\frac{dx}{q}}pdx$ itself is thus taken, so that it vanishes when x made equal to zero. Then with a put in place of x [in the limit of the integral; note that Euler does not distinguish between the variable in the integration, and the upper value of this variable], X becomes A, and hence : $-\int e^{\int\frac{dx}{q}}pdx=A-X$ and $v=e^{-\int\frac{dx}{q}}(A-X)$.

Therefore the time, in which the distance *CP* is completed, is given by :

$$\int \frac{e^{-\frac{1}{2}\int \frac{dx}{q}} dx}{\sqrt{(A-X)}}.$$

Moreover for the descent, with $A - X$ written in a similar way $-\int e^{-\int \frac{dx}{q}} pdx$, the height corresponding to the speed at P is given by :

$$v = e^{\int \frac{dx}{q}} (A - X),$$

and the time, in which the distance PC is completed , is given by :

$$\int \frac{dx}{e^{-\frac{1}{2}\int \frac{dx}{q}} \sqrt{(A-X)}}.$$

Q.E.I.

Corollary 1.

510. The speed at the lower point *C* is found by making $x = 0$, in which case X vanishes and we put $\int \frac{dx}{q}$ to disappear. Therefore there arises for the ascent as for the descent $v = A$. Moreover it is to be noted that *A* does not have the same value in each case, but different values. For it is formed from *X*, which in the ascent is equal to $\int e^{\int \frac{dx}{q}} pdx$, and for the descent truly equal to $\int e^{-\int \frac{dx}{q}} pdx$.

Corollary 2.

511. In the descent the body has the maximum speed when $v = pq$, as indeed it makes $dv = 0$. [p. 217] Hence the place, in which the speed is a maximum, is determined from the equation $pq = e^{\int \frac{dx}{q}} (A - X)$.

Scholium.

512. In the hypothesis for a force in a uniform medium, the body finally falls through an infinite distance to acquire its maximum speed and if, in the beginning it should suddenly be moved to that speed, then that speed is always kept. Here indeed, where *p* and *q* are variable quantities, the body can fall for a finite time to acquire the maximum speed and is not required to keep that speed, once acquired, unless *pq* is always a constant quantity, or the density of the medium is proportional to the centripetal force (385).

PROPOSITION 66.

PROBLEM.

513. *With the law of the centripetal force pulling towards the centre C given* (Fig.44*), and with the medium resisting in the ratio of the square of the speed, if the speed of the body is given that the body acquires on being dropped from any height, to determine the density or the exponent of the resistance at individual places.*

SOLUTION.

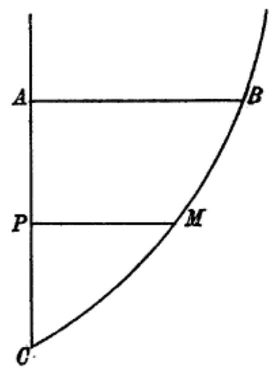


Fig. 44.

With some distance $CP = x$, and the centripetal force at $P = p$, let CMB be the curve associated with this force, in order that any coordinate [called 'applied line' in the original] of this curve AB is equal to the height corresponding to the speed, that the body dropped from A acquires at C, and which curve is therefore given. Truly the exponent of the resistance sought is equal to q at P. Again, if the distance AC, [p. 218] from which the body is dropped, is equal to a, then AB is a certain function of a, that we put as corresponding to the value L. Indeed PM is the coordinate of the same curve corresponding to the value R, and R is the same function of x as L is of a. Moreover from the preceding proposition, it is apparent from the distance of the body dropped $AC = a$, that the height corresponding to the speed acquired at C is equal to A (510). On account of which $L = A$ and also $R = X$; for also X is such a function of x, as A is of a ; R therefore must be such a function of x, so that it vanishes when $x = 0$. Since indeed :

$$X = \int e^{-\int \frac{dx}{q}} pdx, \text{ hence } R = \int e^{-\int \frac{dx}{q}} pdx \text{ and } l\,\frac{pdx}{dR} = \int \frac{dx}{q}.$$

From which, with dx put constant, it is found that

$q = \frac{pdxdR}{dpdR - pddR}$. Q.E.I.

[For, from $R = \int e^{-\int \frac{dx}{q}} pdx$, written with limits to the integral in the modified form :

$R = \int_0^x e^{-\int \frac{dt}{q}} pdt$, we have

$\frac{dR}{dx} = e^{-\int \frac{dt}{q}} p$, and hence, $\frac{1}{p}\frac{dR}{dx} = e^{-\int \frac{dt}{q}}$, and $\int \frac{dt}{q} = \ln(p\frac{dx}{dR}) = \ln p + \ln \frac{dx}{dR}$. On again expressing the integral with limits, and differentiating w.r.t. R, with dx kept constant [in the limit of the integral], then we have :

$\frac{dx}{q} = \frac{dp}{pdR} - \frac{ddR}{dR} = \frac{dpdR - pdR.ddR}{pdR}$, or $q = \frac{dxpdR}{dpdR - pdR.ddR}$.]

Corollary 1.

514. If *CMB* is a straight line and hence $R = ax$, then $ddR = 0$ and $q = \frac{pdx}{dp}$. If in addition, $p = \beta x^n$, then $q = \frac{x}{n}$, or the density of the medium is reciprocally proportional to the distance from the centre.

Corollary 2.

515. If $n = 0$, or the centripetal force is the same everywhere, then $q = \infty$, and thus the density of the medium is zero and thus the resistance vanishes. This is the case of bodies descending in a vacuum acted on by absolute uniform forces.

Corollary 3.

516. If n is a negative number, then q also has a negative value.
From which it is to be understood that the resistance is to be changed into a propelling force. [p. 219]

Scholium 1.

517. From these the question is easily adapted to resolve ascent, if clearly the height is given, to which the body reaches projected with some speed from *C*. Indeed with the speed set equal to $\sqrt{R}$, in which the distance x is completed, q only needs to be changed into its negative, with which done we have : $q = \frac{pdxdR}{pddR - dpdR}$.

Scholium 2.

518. Each equation defining q for the ascent as well as for the descent is thus compared, so that the same value of q is found, whatever multiples of p and R are put in place. Yet it is not permitted to conclude from these, if q and p are to be determined, that R can have a certain variable value; for by necessity the corresponding value of R must be determined. Moreover when that assumed value of R does not produce some multiple of q, the centripetal force p according to this is either to be returned or modified. [Thus, the functions found must be consistent.] Moreover when the magnitude of the centripetal forces in individual places is given, then the problem is over–determined, if indeed the speeds corresponding to the distances are given as you please : for only the ratio of these need be proposed. Truly the ratio is consistent with this difficulty, that we have found q from the equation $R = \int e^{-\int \frac{dx}{q}} pdx$ by differentiating twice. [p. 220] For the differential equation appears broader and to embrace more than the integral.

Scholium 3.

519. It follows naturally from the solution of the problem, how, if the density of the medium is given at individual places, or the quantity q determined, then it is required instead to find the centripetal force p, with everything given remaining as before. Indeed from this equation : $l\frac{pdx}{dR} = \int \frac{dx}{q}$ it can be deduced that $p = \frac{dR}{dx} e^{\int \frac{dx}{q}}$, the value of which has been determined, since we have put the selected value of $\int \frac{dx}{q}$ in place, which vanishes when $x = 0$ (510).

PROPOSITION 67.

PROBLEM.

520. *With the medium resistance in the square ratio of the speed, and with the density of the medium given or the exponent of this at individual places given, to determine the centripetal force which can be acting , in order that the body released from any height towards the centre C* (Fig.43)*, still always takes the same time to arrive there.*

SOLUTION.

The body descends from some point A, and $AC = a$. The unknown quantity $CP = x$, and the height corresponding to the speed at P is equal to v, the exponent of the resistance at $P = q$, and the centripetal force at the same place is equal to p, which is to be found. Therefore the equation $v = e^{\int \frac{dx}{q}}(A - X)$ is established, and the time, in which the interval PC is completed, [p. 221]

$$= \int \frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}.$$

(509) . X is taken everywhere to be defined thus by $X = \int e^{-\int \frac{dx}{q}} pdx$, so that it vanishes when $x = 0$, and A arises from X by putting $x = a$. Therefore the time to traverse AC is had, if in the integral of $\frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}$ we put $x = a$ or $X = A$. Truly the resulting expression is to be prepared so that neither a nor A is present : which is the case, if $\int \frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}$ should be a function of a and x or A and X of zero dimension. [Recall that X = $e^{-\int \frac{dx}{q}} pdx$, and A similarly defined, with suitable limits, as above; Euler had established this dimensional approach in one of his early papers on isochronous curves.] On account of which the

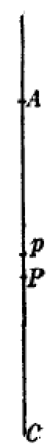


Fig. 43.

differential by necessity is a function of this kind. Therefore put : $\frac{dx}{e^{\int\frac{dx}{2q}}}=\frac{dX}{P}$; we have for the differential of the time : $\frac{dX}{P\sqrt{(A-X)}}$, in which A and X maintain the dimension of a half ; hence P, which has no given dimension, must also have the dimension of a half. But neither a nor A can be present in P; for the magnitude of this must depend only on the point P, and not on the point A. On this account, $P=\frac{\sqrt{X}}{b}$ and the element of the time is $\frac{bdX}{\sqrt{(AX-XX)}}$, which has the required property. Therefore we have : $\frac{dx}{e^{\int\frac{dx}{2q}}}=\frac{bdX}{\sqrt{X}}$, and with the integration completed : $2b\sqrt{X}=\int\frac{dx}{e^{\int\frac{dx}{2q}}}$, which must thus be accepted for the integral, as it vanishes by making $x = 0$. [p. 222] Moreover since $X=\int e^{-\int\frac{dx}{q}}pdx$, we have $4b^2\int e^{-\int\frac{dx}{2q}}pdx=\left(\int\frac{dx}{e^{\int\frac{dx}{2q}}}\right)^2$ and hence by differentiation we have finally :

$$p=\frac{e^{\int\frac{dx}{2q}}}{2b^2}=\int\frac{dx}{e^{\int\frac{dx}{2q}}}.$$

Q.E.I.

Corollary 1.

521. Since the element of the time is $\frac{bdX}{\sqrt{(AX-XX)}}$, the time in which the interval PC is completed, is equal to the arc of a circle, of which the versed sine is X, with the diameter present A, multiplied by $\frac{2b}{A}$. And with the ratio of the periphery of the circle to the diameter put as $\pi:1$, the time for the whole descent through $AC=\pi b$, which is a constant not depending on a.

Corollary 2.

522. Since

$$\int\frac{dx}{e^{\int\frac{dx}{2q}}}=2b\sqrt{X}\text{ and }e^{\int\frac{dx}{2q}}=\frac{dx\sqrt{X}}{bdX}\text{, then }p=\frac{Xdx}{b^2dX}\text{. And also }X=\frac{1}{4b^2}\left(\int\frac{dx}{e^{\int\frac{dx}{2q}}}\right)^2.$$

Corollary 3.

523. Let the medium have a uniform resistance and thus $q = k$; then $e^{\int \frac{dx}{2q}} = e^{\frac{x}{2k}}$ and $\int \frac{dx}{e^{\int \frac{dx}{2q}}} = 2k(1-e^{\frac{-x}{2k}})$. From which is produced : $p = \frac{k}{b^2}(e^{\frac{x}{2k}} - 1)$ [p. 223]. Therefore the centripetal force is equal to zero at C.

Corollary 4.

524. If q is a constant equal to k; then $2b\sqrt{X} = 2k(1-e^{\frac{-x}{2k}})$ and $X = \frac{k^2}{b^2}(1-e^{\frac{-x}{2k}})^2$.

Since indeed X becomes A by putting $x = a$, then $A = \frac{k^2}{b^2}(1-e^{\frac{-a}{2k}})^2$ and

$$v = A = \frac{k^2}{b^2}e^{\frac{a}{k}}\left((1-e^{\frac{-a}{2k}})^2 - (1-e^{\frac{-x}{2k}})^2\right).$$

Corollary 5.

525. Therefore the speed at the lowest height C corresponds to the height

$$A = \frac{k^2}{b^2}\left(1-e^{\frac{-a}{2k}}\right)^2.$$

Corollary 6.

526. The body has its maximum speed when $v = pk$ (511). Therefore :

$$e^{\frac{-x}{2k}}\left(1-e^{\frac{-x}{2k}}\right) = \left(1-e^{\frac{-a}{2k}}\right)^2 - \left(1-e^{\frac{-x}{2k}}\right)^2.$$

From which it is found that : $e^{\frac{-x}{2k}} = 2e^{\frac{-a}{2k}} - e^{\frac{-a}{k}}$, and hence

$$x = 2a - 2k\, l\left(2e^{\frac{a}{2k}} - 1\right).$$

Scholium.

527. If q and k can be taken negative, the law of the centripetal force can be found which has the effect that all the ascents from C can be completed in equal times.[p. 224] Indeed this always has the descent location transformed into an ascent by making the resistive force negative. In which all the ascents become isochronous, and the centripetal force is given by : $p = \frac{e^{-\int \frac{dx}{2q}}}{2b^2} \int e^{\int \frac{dx}{2q}} dx$. In the case of a uniform medium

$p = \frac{k}{b^2}\left(1 - e^{\frac{-x}{2k}}\right)$.

PROPOSITION 68.

PROBLEM.

528. *If the centripetal force is in proportion to the distance from the centre C* (Fig.45*) and the medium has a resistance in the simple ratio of the speed, it is required to determine the motion of the body as it approaches towards and as it recedes from the centre C.*
[The case of simple harmonic motion, with resistance proportional to the speed.]

SOLUTION.

Let the distance, at which the centripetal force is equal to the force of gravity, be equal to f, and the exponent of the resistance is equal to k. Now the body approaches towards the centre C along the straight line AC to the centre C, and the height corresponding to the speed of the body at C is equal to c. And then with this speed the body recedes from C along the straight line CB. First we consider the approach, and we put $CP = x$ and the height corresponding to the speed at P is equal to v. With these in place the centripetal force at $P = \frac{x}{f}$ and the force of the resistance $= \frac{\sqrt{v}}{\sqrt{k}}$, from which this equation arises : $dv = -\frac{xdx}{f} + \frac{dx\sqrt{v}}{\sqrt{k}}$. In which this equation can be made homogeneous, by putting $\sqrt{v} = u$ and $\sqrt{k} = h$; therefore it becomes : $dv = 2udu$ and $2udu = -\frac{xdx}{f} + \frac{udx}{h}$. [p. 225]

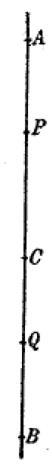


Fig. 45.

By making $u = rx$, then $2r^2xdx + 2rx^2dr = -\frac{xdx}{f} + \frac{rxdx}{h}$, from which arises :

$$\frac{dx}{x} = \frac{2fhrdr}{fr - h - 2fhr^2}$$

Which on integrating, with a constant requiring to be added, and with v and k restored, it changes into this :

$$\frac{v}{c} - \frac{x\sqrt{v}}{2c\sqrt{k}} + \frac{x^2}{2fc} = \left(\frac{4\sqrt{fkv} - x\sqrt{f} + x\sqrt{(f-8k)}}{4\sqrt{fkv} - x\sqrt{f} - x\sqrt{(f-8k)}}\right)^{\sqrt{\frac{f}{f-8k}}}$$

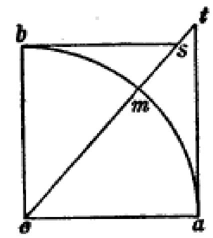


Fig. 46.

Moreover if $8k > f$, the differential equation should be constructed with the help of the quadrature of the circle. [See (441)] Clearly putting $h = \frac{1}{4\alpha}$ and $f = \frac{1}{4\beta}$, and this differential equation is obtained :

$$0 = \frac{dx}{x} + \frac{rdr}{r^2 - 2\alpha r + \beta} = \frac{dx}{x} + \frac{rdr - \alpha dr}{r^2 - 2\alpha r + \beta} + \frac{\alpha dr}{r^2 - 2\alpha r + \beta}.$$

The integral of which is :

$$C = l\sqrt{u^2 - 2\alpha ux + \beta x^2} + \int \frac{\alpha dr}{r^2 - 2\alpha r + \beta}.$$

For $\int \frac{\alpha dr}{r^2 - 2\alpha r + \beta} = \frac{\alpha}{\sqrt{(\beta - \alpha^2)}}.am$ (Fig. 46), and the tangent of the arc am, $at = \frac{r - \alpha}{\sqrt{(\beta - \alpha^2)}}$, with the radius present $ac = 1$. Therefore with $\frac{u}{x}$ in place of r, it becomes $at = \frac{u - \alpha x}{x\sqrt{(\beta - \alpha^2)}}$.

To determine the constant C , put $x = 0$ and $u = \sqrt{c}$, with which done in place of $\frac{\alpha}{\sqrt{(\beta - \alpha^2)}}.am$ there is obtained : $\frac{\alpha}{\sqrt{(\beta - \alpha^2)}}.amb$. Hence, the equation becomes :

$$l\sqrt{c} + \frac{\alpha}{\sqrt{(\beta - \alpha^2)}}.amb = l\sqrt{u^2 - 2\alpha ux + \beta x^2} + \frac{\alpha}{\sqrt{(\beta - \alpha^2)}}.am.$$

Hence, $bm = \frac{\sqrt{(\beta - \alpha^2)}}{\alpha} l\sqrt{\frac{u^2 - 2\alpha ux + \beta x^2}{c}}$, and the arc bm, the tangent of which is bs, is equal to $\frac{x\sqrt{(\beta - \alpha^2)}}{u - \alpha x}$.

For the case of receding from the centre C if as before we put $CQ = x$ and the speed at $Q = \sqrt{v}$, with $h, f, k, \alpha, \beta, r,$ and u keeping the same values as before, we have :

$$0 = \frac{dx}{x} + \frac{rdr}{r^2 + 2\alpha r + \beta}.$$

From which there is obtained :

$bm = -\frac{\sqrt{(\beta - \alpha^2)}}{\alpha} l\sqrt{\frac{u^2 + 2\alpha ux + \beta x^2}{c}}$, and the tangent of the arc bm , $bs = \frac{x\sqrt{(\beta - \alpha^2)}}{u + \alpha x}$.

If $\alpha^2 > \beta$ or $f > 8k$, the integration can be shown algebraically ; it becomes :

$$\frac{v}{c} + \frac{x\sqrt{v}}{2c\sqrt{k}} + \frac{x^2}{2fc} = \left(\frac{4\sqrt{fkv} + x\sqrt{f} - x\sqrt{(f - 8k)}}{4\sqrt{fkv} + x\sqrt{f} + x\sqrt{(f - 8k)}}\right)^{\sqrt{\frac{f}{f - 8k}}}. \text{[p. 226]}$$

Moreover there remains the case in which $\alpha^2 = \beta$ or $f = 8k$, which needs to be handled separately. Moreover this equation is found for the approach :

$$l\frac{4\sqrt{kv} - x}{4\sqrt{kv}} = \frac{x}{4\sqrt{kv} - x}.$$

And for receding :

$l\,\frac{4\sqrt{kv}+x}{4\sqrt{kv}} = \frac{-x}{4\sqrt{kv}+x}$. Q.E.I.

Corollary 1.

529. In the case, in which $f = 8k$, $4\sqrt{kv} > x$ for the approach always, otherwise $\frac{x}{4\sqrt{kv}-x}$ is equal to an imaginary quantity. Whereby, unless $x = 0$, it is not possible for $v = 0$, and likewise the speed at C by necessity must be equal to 0. On account of which if that is put as a finite $\sqrt{c}$, the beginning of the descent will be imaginary.

Corollary 2.

530. Moreover in the case $f = 8k$ from the known equation of the recession, for with $v = 0$ it is found that $l\,\frac{x}{4\sqrt{kc}} = -1$, and hence $x = BC = \frac{4\sqrt{kc}}{e}$ with e denoting the number, the logarithm of which is one. Hence the distance BC is proportional to the speed at C.

Corollary 3.

531. Therefore because, when the resistance is of such a size that $8\,k = f$, the body approaching C loses all its speed, and with a greater ratio, if $8k < f$ or the resistance becomes greater, then the body loses all its speed on approaching C. [p. 227]

Corollary 4.

532. Whereby, if either $8k = f$ or $8k < f$, after arriving at C the body is in perpetual rest, and in these cases no receding motion is possible. But if the resistance is less or $8k > f$, then the approaching body can have a finite speed at C, which then recedes from C, and will move in an oscillatory motion.

Corollary 5.

533. Moreover if $8k > f$, this equation is obtained for the accession : the tangent of the arc is $\frac{x\sqrt{(\beta-\alpha^2)}}{u-\alpha x}$

$$= \frac{\sqrt{(\beta-\alpha^2)}}{\alpha}\, l \sqrt{\frac{u^2-2\alpha ux+\beta x^2}{c}}\,.$$

Hence the beginning of the approaching motion is found by putting $u = 0$; moreover the tangent of this arc is $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$

$$= \frac{\sqrt{(8k-f)}}{\sqrt{f}}\, l\, \frac{\sqrt{2fc}}{x}\,.$$

For the receding motion similarly the tangent of the arc is $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$

$$= \frac{\sqrt{(8k-f)}}{\sqrt{f}}\, l\, \frac{\sqrt{2fc}}{x}\,.$$

Scholium 1.

534. Hence it considered to follow that the distance BC is always equal to the distance AC, as these two equations are similar to each other. But since, if $8k < f$, absolutely nothing is given for the recession, and it cannot happen, as if $8k$ is only a little amount greater than f, the interval of the recession becomes equal to the interval of the accession. This difficulty is removed, if we consider a large number of arcs corresponding to the same tangent $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$, [p. 228] of which one is taken for the accession, and another must be taken for the recession. Putting $\frac{\sqrt{(8k-f)}}{\sqrt{f}} = \tau$, and the smallest arc corresponding to the tangent τ is γ and the semi–circumference of the circle is π : the tangent τ of all these arcs $\gamma, \pi+\gamma, 2\pi+\gamma, 3\pi+\gamma,$ etc. but not of these $-\pi+\gamma, -2\pi+\gamma,$ etc. Now for the recession BC the arc γ must be taken, then $\frac{\gamma}{\tau} = l\frac{\sqrt{2fc}}{BC}$ and $BC = e^{\frac{-\gamma}{\tau}}\sqrt{2fc}$. And for the accession the arc $-\pi+\gamma$ must be taken, and it gives $AC = e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$. The remaining arcs give the points, in which the body oscillating about C has speeds successively equal to zero. Since indeed in the first oscillation, the approach distance is equal to $e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$, the accession interval of the second oscillation is equal to the recession interval of the first oscillation and thus $e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$. In the third oscillation, the access interval is equal to $e^{\frac{-\pi-\gamma}{\tau}}\sqrt{2fc}$. And in the oscillation, which is indicated by the number n, the access interval is $e^{\frac{-(n-2)\pi-\gamma}{\tau}}\sqrt{2fc}$. And from this ratio, any of the access intervals as well as the recession intervals can be determined.

Corollary 6.

535. Therefore when the oscillations of the body are completed about the centre C, they constitute a geometric progression of accession intervals, of which the denominator is $e^{\frac{-\gamma}{\tau}}$ [p. 229]. And in a like manner the recession intervals constitute a progression and also the whole distance traversed by the individual oscillations.

Scholium 2.

536. Because the differential equation for the descent : $2udu = -\frac{xdx}{f} + \frac{udx}{h}$, and the equation $2udu = -\frac{xdx}{f} - \frac{udx}{h}$ for the ascent is homogeneous, in each case u is equal to a function of x and a of one dimension, with a denoting the maximum elongation AC or BC from C. On account of which in the expression for the time $\int \frac{dx}{u}$ there is no dimension present of a and x, and likewise all the times for the ascents as for the descents are equal to each other. For indeed the integral of $\frac{dx}{u}$ is a function of a and x of zero dimensions, and this expression with $x = a$ is equal to a constant quantity. In a similar manner the times of all the descents as far as the point with the maximum speed are equal to each other. For the distance of the points in which the body has the maximum speed, is proportional to a or to the maximum elongation from the centre C (528). [p. 230] [Thus we have the first exposition of damped S.H.M., with over–, under–, and critically–damped motions described, and with the isochronous nature of the motion discussed. As Euler has indicated earlier, he does not go into the nature of the force, be it electrical, magnetic, or mechanical in origin; instead, everything is related to a uniform gravitational force.]

PROPOSITION 69.

THEOREM.

537. *If the centripetal force is as the n^{th} power of the distance from the centre C* (Fig.43*), and the medium resists in the ratio of the 2m multiple of the speed, truly the resistance is proportional to the* $\frac{mn+m-n}{m}$ *power of the distance from the centre C, and the times of all the descents or ascents in the whole distance described are in the ratio of the* $\frac{1-n}{2}$ *power. .*

DEMONSTRATION.

Let AC be the whole interval described either in ascent or descent, and it equal a , and any part of this $CP = x$ and the speed of the body at P is equal to $\sqrt{v}$. The distance f is put in place, in which the centripetal force is equal to the force of gravity. From these put in place the centripetal force at $P = \frac{x^n}{f^n}$, and, with the exponent taken for the

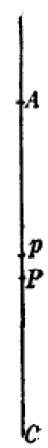


Fig. 43.

resistance $\lambda^{\frac{1}{m}} x^{\frac{mn+m-n}{m}}$, the force of the resistance is $\frac{v^m}{\lambda x^{mn+m-n}}$. Hence for the descent we have this equation :

$$dv = \frac{-x^n dx}{f^n} + \frac{v^m}{\lambda x^{mn+m-n}},$$

and for the ascent :

$$dv = \frac{-x^n dx}{f^n} - \frac{v^m}{\lambda x^{mn+m-n}}$$

Which equations in short agree between themselves, except that λ in one has a negative value. Now put $v = u^{n+1}$, and we have

$$(n+1)u^n du = \frac{-x^n dx}{f^n} \mp \frac{u^{mn+m} dx}{\lambda x^{mn+m-n}},$$

in which equation *u* and *x* put in place a number of the same dimension everywhere. Indeed this equation thus must be integrated, in order that by making $x = a$, *u* vanishes. As on this account the equation of the integral is thus prepared, in order that *a , x,* and *u* everywhere [p. 231] constitute a number of the same dimension. From that therefore it is found that *u* is equal to a function of *a* and *x* of the one dimension *n*. Consequently *v* is equal to a function of *a* and *x* of dimension *n* + 1. On account of which the time, in which the interval *PC* is traversed, clearly $\int \frac{dx}{\sqrt{v}}$, is a function of *a* and *x*, which has the dimensions $\frac{1-n}{2}$. Therefore the whole time either for the ascent or the descent is equal to $Aa^{\frac{1-n}{2}}$, where *A* is a constant quantity made from *f* and λ, which remain unchanged. It is therefore evident that all of the ascent as well as descent times in the description of the whole interval are in the ratio $\frac{1-n}{2}$ to each other. Q.E.D.

Corollary 1.

538. If the resistance of the medium is constant, and thus $mn + m - n = 0$, then $n = \frac{m}{1-m}$ or the centripetal force shall be as the distance raised to the power $\frac{m}{1-m}$. Truly the times for the ascents and descents in the intervals traversed are in the ratio of the power $\frac{1-2m}{2-2m}$.

Corollary 2.

539. If $n = 1$ or the centripetal force is in proportion to the distance from the centre C, then all the ascent and descent times are equal to each other. [p. 232]
Truly in this case, since the law of the resistance is proportional to the speed raised to the power 2*m*, the exponent of the resistance is as the distance from the centre *C* raised to the power $\frac{2m-1}{m}$.

Corollary 3.

540. From this it is apparent, that we have found from the preceding proposition (536), if the resistance is proportional to the speeds, and on account of this, $m = \frac{1}{2}$ and with a uniform medium, all the ascent times as well as the descent times are equal to each other.

Corollary 4.

541. If the centripetal force is constant or $n = 0$, the ascent or descent times are in the square root ratio of the intervals traversed. Truly the exponent of the resistance is proportional to the distance from the centre C. This is now the same case that we have presented above. (495).

Scholium.

542. And with these we conclude this chapter concerning the rectilinear motion of points with resistance ; and we move on in a like manner to the division made, to consider the curvilinear motions of bodies in a vacuum acted upon by some kind of absolute forces.

CAPUT QUARTUM

DE MOTU RECTILINEO PUNCTI LIBERI IN MEDIO RESISTENTE

[p. 188]

PROPOSITIO 57.

PROBLEMA.

450. *Dato tempore, quo corpus ex B* (Fig. 40) *sursum proiectum iterum in B decidit in medio resistence in duplicata celeritatum ratione et sollicitante potentia absoluta uniformi g, determinare altitudinem BA, ad quam corpus pervenit, ut et celeritatem initialem in B finalem post descensum in eodem loco B; nec non tempus ascensus per BA et tempus descensus per AB.*

SOLUTIO.

Sit datum tempus = t, quod est summa temporum ascensus et descensus per rectam BA, et exponens resistentiae = k. Ponatur altitudo BA quaesita = x. Erit tempus ascensus per BA

$$=2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{x}{k}}-1)}$$

(445) atque tempus descensus sequentis ex A in B

$$=2\sqrt{\frac{k}{g}}l\left(\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}\right)$$

(427). Ex quibus conflatur ista aequatio

$$\frac{t}{2}\sqrt{\frac{g}{k}}=A.\sqrt{(e^{\frac{x}{k}}-1)}+l\left(\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}\right)$$

ex qua inveniri potest x. Cognita autem altitudine x dabitur simul et tempus ascensus per BA et tempus descensus per AB. Porro data altitudine BA = x erit altitudo generans celeritatem in B, qua ascendit, = $gk(e^{\frac{x}{k}}-1)$ (445) et altitudo generans celeritatem, qua decidit in B, $gk(1-e^{-\frac{x}{k}})$ (420). Q.E.I.

Corollarium 1.

451. Erit ergo celeritas ascendens in B ad celeritatem descendentem ibidem ut $e^{\frac{x}{2k}}$ ad 1. Ex quo apparet tanto magis de motu amitti, quanto corpus altius ascendat. [p. 189]

Scholion 1.

452. Si fuerit k numerus valde magnus neque altitudo x admodum magna, ut loco temporum supra inventas expressiones algebraicas adhibere liceat, erit tempus ascensus = $\frac{2\sqrt{x}}{\sqrt{g}}-\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}$ atque tempus descensus $=\frac{2\sqrt{x}}{\sqrt{g}}+\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}$ (432). Quarum temporum summa quia data est, habebitur $t\sqrt{g}=4\sqrt{x}+\frac{x^2\sqrt{x}}{120k^2\sqrt{g}}$ quam proxime.

Scholion 2.

453. Accuratius autem definietur haec temporum summa magis continuandis seriebus tempora ascensus et descensus exprimentibus. Fit scilicet tempus ascensus

$$=\frac{2\sqrt{x}}{\sqrt{g}}-\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}+\frac{x^3\sqrt{x}}{1344k^3\sqrt{g}}-\frac{x^4\sqrt{x}}{46080k^4\sqrt{g}}-\text{etc.}$$

et tempus descensus

$$=\frac{2\sqrt{x}}{\sqrt{g}}+\frac{x\sqrt{x}}{6k\sqrt{g}}+\frac{x^2\sqrt{x}}{240k^2\sqrt{g}}-\frac{x^3\sqrt{x}}{1344k^3\sqrt{g}}-\frac{x^4\sqrt{x}}{46080k^4\sqrt{g}}+\text{etc.}$$

Quamobrem horum temporum summa

$$t=\frac{4\sqrt{x}}{\sqrt{g}}+\frac{x^2\sqrt{x}}{120k^2\sqrt{g}}-\frac{x^4\sqrt{x}}{23040k^4\sqrt{g}}+\text{etc.}$$

Ubi notandum, si tempus detur in minutis secundis et k et x in scruplis pedis Rhenani exprimantur, superiorem seriem per 250 esse dividendam. Ita si tempus t sit μ minotorum secondorum, debebit pro t substituti 250μ.

Corollarium 2.

454. Ex superiore aequatione poterit serie invertenda elici x per seriem. Fiet autem

$$\sqrt{x}=\frac{t\sqrt{g}}{4}-\frac{g^2t^5\sqrt{g}}{2^{15}.15k^2}+\frac{g^4t^9\sqrt{g}}{2^{29}.15k^4}-\text{etc.}$$

et consequenter

$$x=\frac{gt^2}{2^4}-\frac{g^3t^6}{2^{16}.15k^2}+\frac{g^5t^{10}\sqrt{g}}{2^{26}.225k^4}-\text{etc.}$$

Corollarium 3. [p. 190]

455. Differentia inter tempus descensus et tempus ascensus erit ergo $\frac{x\sqrt{x}}{3k\sqrt{g}}-\frac{x^3\sqrt{x}}{672k^3\sqrt{g}}$ quam proxime. Inventa ergo altitudine x simul et tempus ascensus et tempus descensus innotescunt.

Corollarium 4.

456. In serie erit etiam altitudo debita celeritati, qua corpus ascensum inchoat, = $gx+\frac{gx^2}{2k}+\frac{gx^3}{6k^2}+\frac{gx^4}{24k^3}+\text{etc.}$ et altitudo debeta celeritati, qua corpus delabitur, $=gx-\frac{gx^2}{2k}+\frac{gx^3}{6k^2}-\frac{gx^4}{24k^3}+\text{etc.}$

Exemplum.

457. Globus ferreus ex tormento bellico sursum explosus recidebat in terram post 34 minuta secunda, eratque k = 2250000 scrupulorum pedis Rhenani et $g=\frac{7499}{7500}$.
Habebimus ergo t = 8500 et $\frac{t\sqrt{g}}{4\sqrt{k}}=1{,}416572$ adeoque $\frac{g^2t^5\sqrt{g}}{2^{15}.15k^2\sqrt{k}}=0{,}01188$ atque $\frac{g^4t^9\sqrt{g}}{2^{29}.15k^4\sqrt{k}}=0{,}0007477.$ Fiet igitur $\frac{\sqrt{x}}{\sqrt{k}}=1.405439$ et $\sqrt{x}=2108.159$ et tota altitudo x , ad quam globus in aere pervenit, = 4443 ped. Rhen. Sit nunc δ numerus minutorum secundorum, quibus descensus longius durat quam ascensus; erit $\frac{250\delta\sqrt{g}}{\sqrt{k}}=\frac{x\sqrt{x}}{3k\sqrt{k}}-\frac{x^3\sqrt{x}}{672k^3\sqrt{k}}$. Est vero $\sqrt{\frac{x}{k}}=\frac{527}{375}$, ex quo prodit $\frac{x\sqrt{x}}{3k\sqrt{k}}=0.9913$ et $\frac{x^3\sqrt{x}}{672k^3\sqrt{k}}=0.01893.$ Habebimus ergo $\frac{250\delta\sqrt{g}}{\sqrt{k}}=0.97237$ et hinc δ = 5'' 50'''. Ex quo apparet tempus ascensus fuisse [p. 191] 14'' 5''' et tempus descensus = 19'' 55'''. Altitudo autem generans celeritatem, qua corpus ascensum inchoavit, reperitur 15542 ped. et altitudo debita celeritati, qua delabitur, = 1969 ped. Videatur de his Comment. Tom. II. pag.338.
[Daniel Bernoulli, *Dissertatio de actione fluidorum in corpora solida et motu solidorum in fluidis*. Pars quarta: de motu corporum sursum proiectorum, ubi ad calculum revocantur experimenta ab Excellentissimo Domino Gunter cum tormentis instituta. *Comment. acad. Petrop*. 2 (1727), 1729, p. 329-342. Pag. 338 legitur : Tempus totum ascensus et descensus 34 min.sec., altitudo, ad quam globus pervenit in aere resistente, 4550 ped. Angl. (Ped. Angl. continentur 304.79 mm), tempus ascensus in aere resistente 14.37 min.sec., tempus descensus in aere resistente 19.63 min. sec., altitudo, in quam globus eadem vi proiectus in vacuo ascendere potest, 13694 ped. Angl., tempus, quod eadem vi proiectus in vacuo in ascensum et descensum impendit, 58 min. sec.]

Corollarium 5.

458. Quia altitudo debita celeritati, qua corpus sursum proiicitur,
$$=gk(e^{\frac{x}{k}}-1)=gx(1+\frac{x}{2k}+\frac{x^2}{6k^2}+\frac{x^3}{24k^3}+\text{etc.})$$
erit tempus ascensus et descensus simul sumtum, si corpus hac celeritate in vacuo sursum proiiceretur sola sollicitante vi gravitatis,

$$= 4\sqrt{gx} + \frac{x\sqrt{gx}}{k} + \frac{5x^2\sqrt{gx}}{24k^2} + \frac{x^3\sqrt{gx}}{32k^3} + \text{etc.}$$

Corollarium 6.

459. Erit ergo summa temporum ascensus et descensus in vacuo ad temporum summam in medio resistente ut

$$g(1 + \frac{x}{4k} + \frac{5x^2}{96k^2} + \frac{x^3}{128k^3} + \text{etc.})\ \text{ad}\ 1 + \frac{x^2}{480k^2} - \frac{x^4}{92160k^4} + \text{etc.}$$

Si scilicet in utroque casu corpus eadem celeritate proiiciatur.

Scholion 3.

460. In citato Tomo II. Commentar. pag. 340 est Theorema, quo haec tempora in vacuo et medio resistente in duplicata celeritatum ratione, ut hic quoque statuimus, inter se conferuntur; atque asseritur tempus in vacuo semper esse maius tempore in pleno. At vero ex nostra comparatione apparet fieri posse, ut tempus in vacuo etiam minus sit tempore in pleno. Nam si fuerit x valde parvum, k vero vehementer magnum, ea tempora inter se erunt proxime ut g at 1. [p. 192] Est vero g in medio resistente ob vim gravitatis densitate medii minutam semper minor unitate, et hanc ob rem tempus in vacuo minus erit his casibus tempore in pleno. Quando vero g quam minime ab unitate differt et x non est admodum parvum respectu k, prout in eiaculatione globorum ex tormentis evenit,tempus in vacuo utique perpetuo erit maius quam in medio resistente. Deinde si fuerit $x > k$, facile perspicitur dari quoque casus, quibus eventus illi Theoremati futuris sit contrarius. In exemplo autem allato corpus celeritate altitudini 15542 ped. debita in vacuo sursum proiectum recidet in terram post 63 minuta secunda, cum tamen in aere non diutius quam 34" moretur.

PROPOSITIO 58.

PROBLEMA.

461. *Si corpus post quemvis descensum ex O* (Fig. 42) *reflectatur eaque celeritate, quam in descensu est adeptum, iterum recta ascendat, atque hae reflexiones perpetuo, cum in O pervenerit, repetantur, quaerendae sunt altitudines OA. OB, OC, etc., quas corpus hoc modo successive percurrit in medio resistent uniformi iuxta quadrata celeritatum et sollicitatum a potentia uniformi g.*

SOLUTIO.

Posito, ut hactenus est factum, exponente resistentiae = k, sit altitudo primo $AO = a$, eritque celeritati, qua corpus ascensum inchoavit, altitudo debito. Erit tempus ascensus per BA $= gk\sqrt{(e^{\frac{a}{k}}-1)}$ (439). [p. 193] Altitudo vero debita celeritati, qua per AO delapsum punctum O attingit, est $= gk(1-e^{\frac{-a}{k}})$ (420). Hac celeritate iam secundum ascensum per OB incipiat, sitque OB = z; erit $gk(1-e^{\frac{-a}{k}}) = gk(e^{\frac{z}{k}}-1)$. Ex quo prodit

$$OB = kl(2-e^{\frac{-a}{k}}) = z.$$

Altitudo vero debita celeritati, qua in hoc secundo descensu per BO delabitur, erit

$$= gk(1-e^{\frac{-z}{k}}) = \frac{gk(1-e^{\frac{-a}{k}})}{2-e^{\frac{-a}{k}}},$$

hacque celeritate tertium ascensum per OC incipiet. Sit nunc OC = z, erit

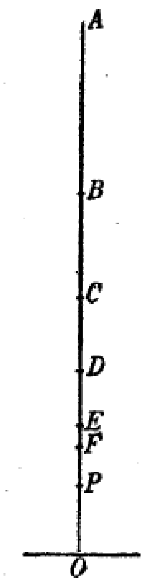


Fig. 42.

$$\frac{gk(1-e^{\frac{-a}{k}})}{2-e^{\frac{-a}{k}}} = gk(e^{\frac{z}{k}}-1), \text{ consequenter } z = OC = kl\frac{3-2e^{\frac{-a}{k}}}{2-e^{\frac{-a}{k}}}.$$

Celeritas vero, quam in descensu per CO acquirit, debita erit altitudini

$$gk(1-e^{\frac{-z}{k}}) = \frac{gk(1-e^{\frac{-a}{k}})}{3-2e^{\frac{-a}{k}}}.$$

Hac porro celeritate quartum ascensum per OD incipit, quam altitudinem OD denuo vocemus z : erit

$$gk(e^{\frac{z}{k}}-1) = \frac{gk(1-e^{\frac{-a}{k}})}{3-2e^{\frac{-a}{k}}} \text{ et consequenter } z = OD = kl\frac{4-3e^{\frac{-a}{k}}}{3-2e^{\frac{-a}{k}}}.$$

Simili modo prodibit [p. 194]

altitudo quinta $OE = kl\frac{5-4e^{\frac{-a}{k}}}{4-3e^{\frac{-a}{k}}}$ et sexta $OF = kl\frac{6-5e^{\frac{-a}{k}}}{5-4e^{\frac{-a}{k}}}$.

Ex quibus concluditur ea altitudo *OP*, cuius index est *n*, fore

$$= kl\frac{n-(n-1)e^{\frac{-a}{k}}}{(n-1)-(n-2)e^{\frac{-a}{k}}} = kl\frac{ne^{\frac{a}{k}}-n+1}{(n-1)e^{\frac{a}{k}}-n+2}.$$

Manifestum igitur est, ad quantam altitudinem corpus in quaque reflexione ex puncto *O* perveniat. Q.E.I.

Corollarium 1.

462. Ex hac solutione simul perspicitur celeritati, qua in hoc descensu per *PO* delabitur, debitam altitudinem fore

$$= \frac{gk(1-e^{\frac{-a}{k}})}{n-(n-1)e^{\frac{-a}{k}}} = \frac{gk(e^{\frac{a}{k}}-1)}{ne^{\frac{a}{k}}-n+1}$$

Celeritas vero, qua ascensum per *OP* est adorsum, debita est altitudini

$$\frac{gk(1-e^{\frac{-a}{k}})}{n-1-(n-2)e^{\frac{-a}{k}}} = \frac{gk(e^{\frac{a}{k}}-1)}{(n-1)e^{\frac{a}{k}}-n+2}.$$

Corollarium 2.

463. Quia

$$OA = a = kle^{\frac{a}{k}} \text{ et } OB = kl(2-e^{-\frac{a}{k}}),$$

erit

$$OA + OB = kl(2e^{\frac{a}{k}} - 1).$$

Eodemque modo [p. 195]

$$OA + OB + OC = kl(3e^{\frac{a}{k}} - 2) \text{ et } OA + OB + OC + OD = kl(4e^{\frac{a}{k}} - 3) \text{ etc.}$$

Corollarium 3.

464. Si *k* est numerus valde magnus, ut $\frac{a}{k}$ fere evanescat, erit quam proxime

$$OP = a - \frac{(n-1)a^2}{k} + \frac{(n-1)^2a^3}{k^2} - \frac{(n-1)^3a^4}{k^3} + \text{etc.}$$

Quae series cum sit geomettica, erit $OP = \frac{ak}{k+(n-1)a}$ quam proxime.

Corollarium 4.

465. Si altitudo prima *OA* fuerit infinite magna, reliquae nihilo minus erunt finitae. Prodibit enim

$$OB = k\,l2,\quad OC = k\,l\frac{3}{2},\quad OD = k\,l\frac{4}{3},\quad OP = k\,l\frac{n}{n-1}.$$

Corollarium 5.

466. Et si altitudo quaecunque fuerit = $kl\,A$, erit altitudo sequens, ad quam corpus post repercussionem e prima pertingere valet, $= k\,l\frac{2A-1}{A}$. Porro altitudo tertia erit $= k\,l\frac{3A-2}{2A-1}$ et similiter quarta $= k\,l\frac{4A-3}{3A-2}$ et ea, cuius index est n, erit $= k\,l\frac{nA-n+1}{(n-1)A-n+2}$.

Scholion 1.

467. Possunt etiam loco n numeri negative substitui, tumque invenientur altitidines praecedentes in quarum serie prima existit. Sic altitudo, quam sequitur prima $OA = a$, posito $n = 0$, erit $= kl\frac{1}{2-e^{\frac{a}{k}}}$. Ex quo apparet, si fuerit $e^{\frac{a}{k}} = 2$ seu $a = k\,l2$, [p. 196] altitudinem praecedentem fuisse infinitam. At si fuerit $e^{\frac{a}{k}} > 2$, altitudo praecedens ob logarithmum quantitatis negativae erit imaginaria, id quod indicat fieri non posse, ut altitudo tanta possit assignari, cuius sequens sit haec assumta a.

Scholion 2.

468. Quod post altitudinem infinitam sequi possit altitudo finita, admirabile quidem videtur; sed consideranti, quod corpus in medio resistente ex infinita altitudine delapsum acquirat tantum celeritatem (420), ratio huius phaenomeni facile patebit. Hac enim finita celeritate ad finitam tantum altitudinem reascendere poterit. Maxima autem celeritas, quam corpus in descensu potest adipisci, est $\sqrt{gk}$. Quare si corpus initio sursum proiiciatur celeritate maiore quam $\sqrt{gk}$, haec celeritas ex nullo quamvis magno descensu antecedente generari potuit; quemadmodum etiam hoc casu calculus altitudinem praecedentem exhibet imaginariae quantitatis.

PROPOSITIO 59.

PROBLEMA.

469. *Resistente medio uniformi in celeritatum ratione simplici et sollicitante potentia absoluta uniformi deorsum tendente, determinare corporis recta vel ascendentis vel descenditis celeritatem in quovis puncto.* [p. 197]

SOLUTIO.

Descendat primo corpus in recta *AP* (Fig. 39), sitque celeritas eius initialis in *A* debita altitudini c. Ponatur potentia absoluta = g, exponens resistentiae = k atque $AP = x$ et altitudo debita celeritati in $P = x$. His positis erit $dv = gdx - \frac{dx\sqrt{v}}{\sqrt{k}}$; est enim vis resistentiae $= \frac{\sqrt{v}}{\sqrt{k}}$. Hinc fit $dx = \frac{dv\sqrt{k}}{g\sqrt{k} - \sqrt{v}}$.

$\sqrt{v} = u,$ erit $dv = 2udu$ atque
Fiat $dx = \frac{2udu\sqrt{k}}{g\sqrt{k}-u} = -2du\sqrt{k} + \frac{2gxdu}{g\sqrt{k}-u}$.
Integrata hac aequatione prodit
$x = C - 2u\sqrt{k} - 2gkl(g\sqrt{k} - u) = C - 2\sqrt{kv} - 2gkl(g\sqrt{k} - \sqrt{v})$.
Posito vero $x = 0$ fieri debet v = c, ex quo fit $C = 2\sqrt{kc} - 2gkl(g\sqrt{k} - \sqrt{c})$.
Habebimus itaque $x = 2\sqrt{kc} - 2\sqrt{kv} + 2gkl\frac{(g\sqrt{k}-\sqrt{c})}{(g\sqrt{k}-\sqrt{v})}$,
ex qua v ope logarithmicae potest deduci. Q. E. Altrum.

Iam pro ascensu sit celeritas initialis in *B* (Fig. 40) altitudini *c* debita et $BP = x$ et altitudo celeritati in P debita = *v*. Quia in ascensu tam potentia absoluta quam resistentiae vis retardant, erit $dv = -gdx - \frac{dx\sqrt{v}}{\sqrt{k}}$. Quae aequatio directe ex priore $dv = dgx - \frac{dx\sqrt{v}}{\sqrt{k}}$ deducitur ponendo $-g$ loco g. Quamobrem etiam hoc modo requisitam aequationem integralem ex illa derivare licet. Fit igitur g negativo.

$x = 2\sqrt{kc} - 2\sqrt{kv} - 2gkl\frac{(g\sqrt{k}+\sqrt{c})}{(g\sqrt{k}+\sqrt{v})}$. Q. E. Altrum.

Corollarium 1.

470. Si celeritas initialis in descensu fuerit = 0, erit $x = -2\sqrt{kv} + 2gkl\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}$. Ex qua [p. 198] aequatione determinantur celeritas corporis ex quacunque altitudine delapsi.

Corollarium 2.

471. Quia vero est $l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}} = -l(1 - \frac{\sqrt{v}}{g\sqrt{k}})$,
habebitur in serie

$$l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}} = \frac{\sqrt{v}}{g\sqrt{k}} + \frac{v}{2g^2k} + \frac{v\sqrt{v}}{3g^3k\sqrt{k}} + \frac{v^2}{4g^4k^2} + \text{etc.}$$

Qua substituta prodibit

$$x = \frac{v}{g} + \frac{2v\sqrt{v}}{3g^2\sqrt{k}} + \frac{v^2}{2g^3k} + \frac{3v^2\sqrt{v}}{5g^4k\sqrt{k}} + \text{etc.}$$

Si nunc fuerit k numerus valde magnus, erit $v = \frac{v}{g} - \frac{2x\sqrt{gx}}{3\sqrt{k}}$ quam proxime.

Corollarium 3.

472. Si fit $v = g^2 k$, prodit $x = \infty$ (Fig. 40). Ex quo apparet corpus ex infinita altitudine delapsum maiorem acquirere non posse celeritatem quam $g\sqrt{k}$. Et si fuerit semel $v = g^2 k$, corpus motu aequabli esse progressurum; tum vero motum eius retardari, si sit $v > g^2 k$.

Corollarium 4.

473. Si corpus ex *B* sursum proiiciatur celeritate $\sqrt{c}$, altitudo *BA* reperietur facto $v = 0$. Prodibit autem

$$BA = 2\sqrt{kc} - 2gkl(1 + \frac{\sqrt{c}}{g\sqrt{k}}) = \frac{c}{g} - \frac{2c\sqrt{c}}{3g^2\sqrt{k}} + \frac{c^2}{2g^3 k} + \text{etc.}$$

Altitudino vero, ex qua corpus descendendo hanc celeritatem acquirere potest, erit

$$= -2\sqrt{kc} - 2gkl(1 - \frac{\sqrt{c}}{g\sqrt{k}}) = \frac{c}{g} + \frac{2c\sqrt{c}}{3g^2\sqrt{k}} + \frac{c^2}{2g^3 k} + \text{etc.}$$

Corollarium 5.

474. Si corpus sursum proiiciatur celeritate $g\sqrt{k}$, maxima scilicet, quam descendu acquirere potest, erit altitudo, ad quam pertingit, $= 2gk(2 - l2)$.

[p. 199]

PROPOSITIO 60.

PROBLEMA.

475. *Resistente medio uniformi in ratione simplici celeritatum et sollicitante potentia absoluta uniformi, determinare tempus, quo corpus vel ascendens vel descendens spatium quodvis percurrit.*

SOLUTIO.

Positis ut ante descensu per *AP* (Fig. 39), celeritate in $A = \sqrt{c}$ et ea in $P = \sqrt{v}$, exponente resistentiae = k et potentia absoluta = g spatioque $AP = x$, erit

$$dx = \frac{dv\sqrt{k}}{g\sqrt{k} - \sqrt{v}} = \frac{2udu\sqrt{k}}{g\sqrt{k} - u}$$

posito u^2 posito v. Iam dicto tempore per $AP = t$ erit $dt = \frac{du}{u} = \frac{2du\sqrt{k}}{g\sqrt{k} - u}$.

Ex qua prodit $t = 2\sqrt{k}l\frac{g\sqrt{k}-\sqrt{c}}{g\sqrt{k}-\sqrt{v}}$. Q.E.Alterum.

Quia ascensus per $BP = x$ (Fig. 40), cum celeritate initiali $\sqrt{c}$ prodit ponendo $-g$ loco g, erit tempus ascensus per BP

$$t = 2\sqrt{k}l\frac{g\sqrt{k}+\sqrt{c}}{g\sqrt{k}+\sqrt{v}}.$$ Q.E.Alterum.

Problemate vero praecedente definitur v ex dato x. Quare et hic tempus, quo spatium quodvis percurritur, poterit cognosci. Q. E. I.

Corollarium 1.

476. Si celeritas initialis, qua corpus descentit, fuerit nulla, erit tempus descensus per spatium AP $= 2\sqrt{k}l\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}$. At v definitur ex hac aequatione

$$x = -2\sqrt{k}v + 2gkl\frac{g\sqrt{k}}{g\sqrt{k}-\sqrt{v}}.$$

[p. 200]

Corollarium 2.

477. Tota altitudo BA, ad quam corpus ex B ascendens pervenire potest, absolvetur tempore $= 2\sqrt{k}l\frac{g\sqrt{k}+\sqrt{c}}{g\sqrt{k}}$.

Corollarium 3.

478. Si ergo celeritas initialis $\sqrt{c}$ fuerit infinita, erit etiam tempus, quo tota altitudo BA percurritur, infinitum, nempe $= 2\sqrt{k}l\propto$.

Scholion 1.

479. Hoc igitur vehementer differt haec resistentiae hypothesis a priore, quae quadratis celeritatum posita erat proportionalis. Nam illo casu corpus infinita celeritate sursum proiectum ad summum punctum pertingit tempore finito (444). Hoc vero notandum est tempus $2\sqrt{k}l\propto$ esse numerum infinitum infimi ordinis. Ex quo concludi posse videtur, si resistentia fuerit in maiore quam simplici ratione celeritatum, tempus ascensus totius semper esse finitum, sin autem resistentia sit in simplici vel minore celeritatum ratione, tempus ascensus totius esse infinitum, si quidem celeritas initialis est infinite magna.

Scholion 2.

480. Has duas resistentiae hypotheses ideo fusius pertractandas esse censui, quod eae a Neutono aliisque, qui eum secuti sunt, praecipue sint consideratae. Haec quidem posterior hypothesis, qua resistentiam celeritatibus proportionalem posuimus, [p. 201] mere mathematica est neque ullum in physicis habere potest usum. Sed quia initio putarunt resistentiam fluidorum a tenacitate oriundam celeritatibus esse proportionalem, diligentius in huiusmodi motus inquirendum esse existimaverunt. Postmodum tamen, cum resistentiam tenacitatis longe aliter se habere intellexissent, hanc tractationem nihilo minus retinuerunt. Prior vero, qua resistentia quadratis celeritatum proportionalis est, maxime explorari meretur : certum enim est praecipuam fluidorum resistentiam hanc tenere rationem. Praeterea etiam haec hypothesis in calculo prae reliquis tantam habet praerogativam, ut, quod in aliis hypothesibus minime potest praestari, in hac tamen sola calculus non refragetur. Omnia enim fere problemata, quae in vacuo solutionem non respuunt, in hac resistentiae hypothesi resolvi possunt. Hanc ob rem in sequentibus istam resistentiam potissimum examinabimus, reliquas autem, nisi concinno computo quaesitum invenire potest, plerumque negligemus.

PROPOSITIO 61.

PROBLEMA.

481. *Resistat medium uniformi in ratione quacunque multiplicata celeritatum sitque potentia sollicitans uniformis, determinare oportet motum corporis recta vel ascendentis vel descendentis.* [p. 202]

SOLUTIO.

Consideramus (Fig. 40) primo descensum et ponamus celeritatem in $A = \sqrt{c}$, spatium $AP = x$ et celeritatem in $P = \sqrt{v}$. Sit resistentiae exponens = k et lex resistentiae = v^m et potentia absoluta = g. His igitur positis erit $dv = gdx - \frac{v^m dx}{k^m}$ et $dx = \frac{k^m dv}{gk^m - v^m}$. Habemus

ergo $x = \int \frac{k^m dv}{gk^m - v^m}$,

ex qua ope quadraturarum v in x determinare potetit. Ponatur tempus, quo spatium AP percurritur, = t, erit $dt = \frac{dx}{\sqrt{v}} = \frac{k^m dv}{gk^m\sqrt{v} - v^m\sqrt{v}}$. Atque

$$t = \int \frac{k^m dv}{gk^m\sqrt{v} - v^m\sqrt{v}}.$$

Q.E. Alterum.

Iam pro ascensu maneat $\sqrt{c}$ celeritas initialis in B, sitque $BP = x$ et celeritas in P = $\sqrt{v}$ atque tempus, quo spatium BP percurritur, = t. His positis erit $dv = -gdx - \frac{v^m dx}{k^m}$, quae aequatio ex illa elicitur ponendo $-g$ loco g. Quo facto erit pro ascensu

$$x = -\int \frac{k^m dv}{gk^m + v^m} \text{ et } t = -\int \frac{k^m dv}{gk^m\sqrt{v} + v^m\sqrt{v}}$$

Q.E. Alterum inveniendorum.

Corollarium 1.

482. Si fuerit $c^m = gk^m$, corpus hac celeritate descensens motu aequabili feretur. Nam perpetuo potentiae absolutae, qua corpus acceleratur, aequalis erit vis resistentiae, qua retardatur. [p. 203]

Corollarium 2.

483. Corpus vero ex quiete delapsum perpetuo accelerabitur neque tamen unquam celeritatem acquiret altitudini $g^{\frac{1}{m}}k$ debitam. Sed haec celeritas est quasi asymtotos, quam, sive celerius sive tardius corpus moveatur, affectat.

Scholion 1.

484. Quia hae inventae aequationes neque integrari possunt neque v vel t in x definiri, diutius iis immorari non expedit. Ad alia igitur progredior atque medium resistens variabile contemplabor manente potentia absoluta uniformi. Huiusmodi tamen accipiam hypothesin, qua aequatio dv determinans fiat homogenea ideoque his difficultatibus non sit obnoxia. Deinde potentia absolutam non amplius uniformem, sed variabilem pono seu eius loco vim centripetam considero, qua corpus perpetuo ad certum aliquod punctum fixum attrahitur. Cum hac quidem primum aliam resistentiam non coniungam, nisi quae quadratis celeritatum est proportionalis. Deinde vero cum aliis resistentiae hypothesibus eas tantum vires centripetas coniungi convenit, quae integrationem aequationis differentialis admittunt.

PROPOSITIO 62.

PROBLEMA.

485. *Potentia absoluta existente uniformi ex exponente resistentiae distantiis a puncto C* (Fig. 43) *proportionali atque lege resistentiae celeritatum ratione quacunque multiplicata, requiritur corpus in recta AC ad C vel accedintis vel ab eo recedentis celeritas in quavis loco.* [p. 204]

SOLUTIO.

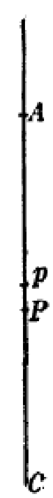


Fig. 43.

Potentia uniformis ad C urgens sit = g, altitudo debita celeritati loco quocunque $P = v$. Ponatur AC, quae est maxima altitudo, ad quam corpus pertingit, = a et $CP = x$, erit exponens resistentiae ut x; sit is $\lambda^{\frac{1}{m}}x$, et lex resistentiae sit v^m. His positis erit vis resistentiae = $\frac{v^m}{\lambda x^m}$, et pro ascensu per CA, quo et potentia absoluta et vis resistentiae retardant, habebitur ista aequatio $dv = -gdx - \frac{v^m dx}{\lambda x^m}$. Descensum hic quoque tanquam ascensum consideremus, et quia in descensu vero potentia accelerans, resistentia vero retardans est, in hoc ascensu substituto contrario modo potentia retardans et resistentia accelerans poni debet (411), ex quo oritur pro descensu haec aequatio $dv = -gdx + \frac{v^m dx}{\lambda x^m}$. Quae aequatio ex illa derivatur faciendo λ negativum, et hanc ob rem alteram tantum aequationem integrari opus est. Sumamus aequationem pro descensu, quae erit huiusmodi

$$\lambda x^m dv + \lambda g x^m dx = v^m dx,$$

et ponamus $v = xz$. Erit ergo $dv = xdz + xdx,$ ex quo prodibit ista aequatio

$\lambda x^{m+1} dz + \lambda x^m z dx + \lambda g x^m dx = x^m z^m dx.$ [p. 205]

Quae divisa per $x^{m+1}(z^m - \lambda z - \lambda g)$ abit in hanc $\frac{\lambda dz}{z^m - \lambda z - \lambda g} = \frac{dx}{x}$,

in qua indeterminatae iam sunt separatae. Haec igitur aequatio ita integretur, ut facto $x = a$ celeritas evanescat; quo facto ex aequatione integrali celeritas corporis descendentis in quovis loco innotescet. Eadem vero ipsa aequatio facto λ negativo inserviet ad celeritates in ascensu per CA definiendas. Q.E.I.

Corollarium 1.

486. Si fuerit $m = 1$ seu resistentia quadratis celeritatum proportionalis, erit

$\frac{\lambda dz}{(1-\lambda)z - \lambda g} = \frac{dx}{x}$ atque $\frac{\lambda}{(1-\lambda)} l((1-\lambda)z - \lambda g) = lx + C = \frac{\lambda}{(1-\lambda)} l \frac{(1-\lambda)v - \lambda g x}{x}$ substituto $\frac{v}{x}$ loco z. Quia autem, si $x = a$, debet esse $v = 0$, erit

$C = \frac{\lambda}{(1-\lambda)} l(1-\lambda)g - \lambda a$ ideoque $lx = la + \frac{\lambda}{(1-\lambda)} l \frac{\lambda g x - (1-\lambda)v}{\lambda g x}$. Ex qua prodit

$$v = \frac{\lambda g x}{(\lambda-1)} \left(\frac{a^{\frac{\lambda-1}{\lambda}} - x^{\frac{\lambda-1}{\lambda}}}{x^{\frac{\lambda-1}{\lambda}}} \right) = \frac{\lambda g}{(\lambda-1)} \left(a^{\frac{\lambda-1}{\lambda}} x^{\frac{1}{\lambda}} - x \right).$$

Corollarium 2.

487. Si fuerit λ unitate minor, haec aequatio ad aliam formam redigi debet; prodibit autem

$$v = \frac{\lambda g}{(\lambda-1)} \left(\frac{a^{\frac{1-\lambda}{\lambda}} x - x^{\frac{1}{\lambda}}}{a^{\frac{1-\lambda}{\lambda}}} \right).$$

[p. 206]

Corollarium 3.

488. Casus, quo λ = 1 seu exponens resistentiae ipsi distantia a puncto C est aequlais, in his formulis non continetur, sed ex differerentiali $\frac{-dz}{g} = \frac{dx}{x}$ est deducendas. Prodibit autem $C - \frac{z}{g} = l\,x$ hincque $v = gx(l\,a - l\,x)$.

Corollarium 4.

489. Ex his intelligitur casu $m = 1$ corporis descendentis celeritatem tam in A quam in C fore = 0. Fit enim $v = 0$ in tribus hisce aequationibus, tam posito $x = 0$ quam $x = a$. Corpus igitur ex A in C delapsum omnem motum amittet atque in C perpetuo quiescet ob resistentiam in eo loco infinite magnam.

Corollarium 5.

490. Dum igitur corpus rectam AC percurrit, alicubi inter A et C habebit celeritatem maximam, quae invenitur ex aequatione differentiali faciendo $dv = 0$. Fiet autem tum $v = \lambda gx$, quo valore loco v in integratis aequationibus substituto prodibit $\lambda x^{\frac{\lambda-1}{\lambda}} = a^{\frac{\lambda-1}{\lambda}}$ atque $x = \frac{a}{\lambda^{\frac{\lambda-1}{\lambda}}}$, si λ > 1. Sin autem λ < 1, erit $x = \lambda^{\frac{\lambda-1}{\lambda}} a$. At si λ = 1, erit $1 = la - lx$ ideoque $x = \frac{a}{e}$, denotante e numerum, cuius logarithmus est unitas. [p. 207]

Scholion 1.

491. Ex his colligere licet etiam in reliquis resistentiae hypothesibus celeritatem corporis, cum ad C pervenerit, esse evanituram. Vis enim resistentiae est $\frac{v^m}{\lambda x^m}$, quae ergo fit infinita, si $x = 0$. Quare si corpus in C quandam haberet velocitatem, ea a vi resistentiae infinita statim in nihilum redigi deberet. Maximam vero in descensu celeritatem habebit, qando est $v^m = \lambda gx^m$. Ex quo apparet maximam celetatem esse debitam altitudini $x\sqrt[m]{\lambda g}$. Sed quia x ignoratur seu locus, quo corpus celerrime descendit, etiam ipsa celeritas non potest determinari, nisi per quadraturas curvarum, quarum ope aequatio differerentialis construitur.

Corollarium 6.

492. Pro ascensu ex C in A, si $m = 1$, celeritates corporis in singulis locis P determinabuntur ex hac aequatione $v = \frac{\lambda gx}{(\lambda+1)}\left(\frac{a^{\frac{\lambda+1}{\lambda}} x - x^{\frac{\lambda+1}{\lambda}}}{a^{\frac{\lambda+1}{\lambda}}}\right)$,
quae ex illis pro descensu formatur, facto λ negativo, uti oportet.

Corollarium 7.

493. In ascensu ergo corporis celeritas in C semper est infinita. Facto enim $x = 0$, quia $\frac{\lambda}{(\lambda+1)}$ est unitate maius, denominator evanescit. [p. 208]

Scholion 2.

494. Perspicium etiam est ex sola contemplatione celeritatem in C esse debere infinitam. Nam nisi tanta esset, corpus vim resistentiae in C infinitam superare non posset, sed perpetuo in C haerere deberet.

PROPOSITIO 63.

THEOREMA.

495. *Iisdem positis, quae in praecedente propositione, si plura corpora ex diversis distantiis ad punctum C* (Fig. 43) *accedent, erunt tempora, quibus eo perveniunt, in duplicata ratione distantiarum.*

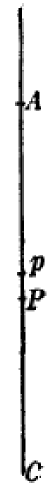


Fig. 43.

DEMONSTRATIO.

In solutione praecedentis problematis ad celeritatem in P determinandam obtinuimus hanc aequationem $\lambda x^m dv + \lambda g x^m dx = v^m dx$ (485). In qua aequatione x et v ubique dimensionum numerum constituunt. Eius igitur integralis ita accepta, ut posito $x = a$ fiat $v = 0$, habebit hanc proprietatem, ut x, v, et a ubique eundem dimensionum numerum constituant. Ex ea ergo prodibit v aequalis functioni cuidam ipsarum a et x, in qua a et x unicam ubique dimensionem constituunt; seu v erit functio ex a et x constans unius dimensionis. Quare in elemento temporis per CP, quod est $\frac{dx}{\sqrt{v}}$, erit ipsarum x, dx et a dimidia demendio, et hanc ob rem tempus per CP aequalibitur functioni ex a et x constanti dimidiae dimensionis. Posito ergo $x = a$, quo [p. 209] casu totum tempus descensus per AC invenitur, habibitur functio ipsius solius a dimidiae dimensionis. Quamobrem tempus per AC exprimetur huiusmodi expressione $C\sqrt{a}$, in qua C ex quantitibus λ, m et g constat, non vero pendet ab a. Quia iam a denotat altitudinem AC, perspicuum est plurium descensuum tempora esse inter se in subduplicata ratione altitudinum percursarum. Q.E.D.

Corollarium 1.

496. Simili modo intelligitur plurium ascensuum ex C tempora tenere etiam rationem altitudinum, ad quas pervenitur, subduplicatam.

Corollarium 2.

497. In quacunque igitur multiplicata celeritatum ratione medium resistat, dummodo potentia absoluta est constans et resistentiae exponens distantiis a C proportionalis, tempora vel ascensuum vel descensuum rationem tenent subduplicatam altitudinum.

Scholion 1.

498. Neque vero ascensus cum descensus comparare licet neque plures ascensus vel descensus inter se, in quibus litterae λ, m et g non eosdem tenent valores. Nam in expressione $C\sqrt{a}$ quantitas C in omnibus casibus, qui inter se comparantur, eadem esse debet. [p. 210]

Scholion 2.

499. In hac propositione eadem usi sumus methodo varia descensuum ad punctum fixum tempora comparandi quam supra in propositionibus 39 (308) et 46 (354). Hoc autem casu eo magis huius methodi praestantia cernitur, quia nequidem celeritatem in x determinare licebat. Hoc enim solum nobis, perspicere sufficiebat, cuiusmodi functio ipsarum a et x futura sit ea expressio, cui v esset aequalis. In sequentibus autem plura specimina egregia huius methodi occurrent.

PROPOSITIO 64.

PROBLEMA.

500. *Existente vi centipeta cuicunque potestati distantiarum a centro C* (Fig. 43) *proportionali medioque uniformi resistente in duplicata celeritatum ratione, determinare corporis in recta AC moti sive sursum sive deorsum in singulis locis P celeritatem.*

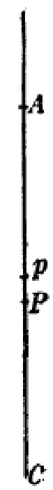


Fig. 43.

SOLUTIO.

Sit corpus in P habeatque celeritatem altitudini v debitam. Vocetur AP x, et sit vis centripeta ut x^n atque ea distantia, in qua vis centripeta aequalis est gravitati, $= f$. Deinde ponatur exponens resistentiae k. His praemissis erit vis absoluta, qua corpus in P sollicitatur, $= \frac{x^n}{f^n}$ et vis resistentiae in hoc loco $\frac{v}{k}$, existente vi gravitatis = 1. [p. 211]

Descendat iam corpus ad C et habebit, dum per elementum pP movetur, vim centripetam accelerentem atque vim resistentiae retardentem. Quia hic autem corpus inverso ex P in p pervenire ponimus crescente x, contrarius harum virium actiones statui oportet, seu, quod eodem redit, dx negativum est ponendum, quia descensu distantia $PC = x$ minuitur. Prodibit ergo $dv = -\frac{x^n}{f^n}dx + \frac{vdx}{k}$. In vero autem corporis ascensu per Pp utraque vis erit retardans, ideoque habebitur $dv = -\frac{x^n}{f^n}dx - \frac{vdx}{k}$.

Ex quo perspicitur alteram aequationem ex altera oriri faciendo k negativum. Hanc ob rem alterutram tantum aequation integrari opus est. Summus eam pro ascensu

$$dv = -\frac{x^n}{f^n}dx - \frac{vdx}{k} \text{ seu } dv + \frac{vdx}{k} = -\frac{x^n}{f^n}dx,$$

haecque multiplicetur per $e^{\frac{x}{k}}$, ut prodeat $e^{\frac{x}{k}}(dv + \frac{vdx}{k}) = -\frac{e^{\frac{x}{k}}x^n}{f^n}dx$, cuius integralis est

$e^{\frac{x}{k}}v = -\int \frac{e^{\frac{x}{k}}x^n}{f^n}dx$. Erit ergo $v = -e^{-\frac{x}{k}}\int \frac{e^{\frac{x}{k}}x^n}{f^n}dx$.

Pro descensu igitur erit $v=-e^{\frac{x}{k}}\int\frac{e^{-\frac{x}{k}}x^n}{f^n}dx$.

In utraque vero integratione quantitas constans adiicenda ex eo determinari debet, [p. 212] quod corporis moti alicubi celeritas sit data : alioquin enim motus non esset determinatus. Q.E.I.

Corollarium 1.

501. Perspicitur igitur, si *n* fuerit numerus integer affirmativus, has formulas fore integrabiles. Est enim

$$\int e^{\frac{x}{k}}x^n dx=ke^{\frac{x}{k}}x^n-nk^2e^{\frac{x}{k}}x^{n-1}+n(n-1)k^3e^{\frac{x}{k}}x^{n-2}-n(n-1)(n-2)k^4e^{\frac{x}{k}}x^{n-3}+\text{etc.}+\text{C}.$$

Quae series non sit infinita, quoties *n* est numerus integer affirmativus.

Corollarium 2.

502. Sit celeritas in *C* data et debita altitudini *c*, erit pro ascensu

$$v=e^{-\frac{x}{k}}c-\frac{kx^n}{f^n}+\frac{nk^2x^{n-1}}{f^n}-\frac{n(n-1)k^3x^{n-2}}{f^n}+\text{etc.}\pm n(n-1)(n-2)...2.1.\frac{k^{n+1}e^{-\frac{x}{k}}}{f^n};$$

quorum signorum ambiguorum superius valet, si $n+1$ fuerit numerus impar, inferius vero, si $n+1$ fuerit numerus par.

Pro descensu autem erit

$$v=e^{-\frac{x}{k}}c+\frac{kx^n}{f^n}+\frac{nk^2x^{n-1}}{f^n}+\frac{n(n-1)k^3x^{n-2}}{f^n}+\text{etc.}-n(n-1)(n-2)...2.1.\frac{k^{n+1}e^{\frac{x}{k}}}{f^n},$$ loco C debita substituta constante. [p. 213]

Corollarium 3.

503. Integrale ipsius $\frac{e^{\frac{x}{k}}x^n dx}{f^n}$ ita acceptum, ut fiat = 0 facto $x=0$, ponatur $=X$. Eritque

$v=e^{-\frac{x}{k}}(c-X)$, quia facto $x=0$ fieri debet $v=c$. Inservit quidem haec aequatio ascensui, sed facto *k* negativo ad descensum accommodatur.

Corollarium 4.

504. Cognito *X* apparebit altitudo *CA*, ad quam corpus vel ascendere potest, vel ex qua delapsum celeritatem acquirit $=\sqrt{c}$. Ex hac enim aequatione $X=c$ radix *x* dabit altitudinem *CA*.

Corollarium 5.

505. Ex differentiali aequatione pro descensu $dv=-\frac{x^n}{f^n}dx+\frac{vdx}{k}$ apparet alicubi corpus habiturum esse celeritatem maximam, antequam ad C pertingit, quae ibi erit, ubi est $v=\frac{kx^n}{f^n}$, si quidem *n* non est numerus negativum.

Corollarium 6.

506. Detur altitudo $CA = a$, ex hacque si quaeratur c, oportet eam habere quantitatem, quae resultat in X posito a loco x. Sit ea = A, erit pro ascensu $v = e^{-\frac{x}{k}}(A - X)$ et pro [p. 214] descensu $v = e^{\frac{x}{k}}(A - X)$. Facto enim $x = a$ debet evanescere v. Iam facto $x = 0$, quo casu etiam fit $X = 0$ (503), erit $v = c = A$.

Corollarium 7.

507. Manifestum est ex hisce, quomodo tempus, quo spatium CP percurritur, inveniendum sit. Scilicet pro ascensu erit tempus per $CP = \int \frac{e^{\frac{x}{2k}}dx}{\sqrt{(A-X)}}$ atque pro descensu tempus per PC erit = $\int \frac{dx}{e^{\frac{x}{2k}}\sqrt{(A-X)}}$.

EXEMPLUM.

508. Sit vis centripeta pro descensu ut distantia a centro C, quo casu fit $n = 1$; erit ergo pro ascensu

$$v = e^{-\frac{x}{k}}c - \frac{kx}{f} + \frac{k^2}{f} - \frac{k^2e^{-\frac{x}{k}}}{f}.$$

Pro descensu vero

$$v = e^{\frac{x}{k}}c + \frac{kx}{f} + \frac{k^2}{f} - \frac{k^2e^{\frac{x}{k}}}{f}$$

(502). In descensu maxima celeritas erit, ubi est $v = \frac{kx}{f}$ (505), qua aequatione cum illa coniuncta habebitur $fe^{\frac{x}{k}} + k^2 = k^2e^{\frac{x}{k}}$. Erit ergo

$$e^{\frac{x}{k}} = \frac{k^2}{k^2-cf} \text{ et } x = k\, l\frac{k^2}{k^2-cf}.$$

Haec igitur distantia fit infinita, si $cf = k^2$, omnio vero imaginaria, si $cf > k^2$. Sit porro in A corporis celeritas = 0, positoque $AC = a$ erit pro ascensu celeritati initiali in C altitudo debita

$$= \frac{e^{\frac{a}{k}}ka}{f} - \frac{e^{\frac{a}{k}}k^2}{f} + \frac{k^2}{f}.$$

[p. 215] In descensu vero erit celeritati finali in C altitudo debita

$$= -\frac{e^{-\frac{a}{k}}ka}{f} - \frac{e^{-\frac{a}{k}}k^2}{f} + \frac{k^2}{f}.$$

Ex qua apparet, si fuerit a infinitum, fore celerati ultimae in C altutudinem debitam $= \frac{k^2}{f}$.

PROPOSITIO 65.

PROBLEMA.

509. *Existente vi centipeta cuicunque ad C quacunque et motio resistente secundum quadrata celeritatum utcunque difformi, determinatur motum corporis recta vel accelerentis vel recedentis a C.*

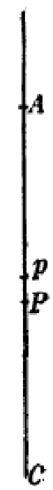


Fig. 43.

SOLUTIO.

Sit corpus in P et ponatur $CP = x$, et celeritas in $P = \sqrt{v}$. Deinde sit vis centipeta in $P = p$, posita vi gravitas = 1, et exponens resistentia = q, quae litterae p et q denotant functiones quascunque ipsius x. Erit ergo vis resistentiae = $\frac{v}{q}$. Hanc ob rem habebitur pro ascensu

$dv = -pdx - \frac{vdx}{q}$. Pro descensu vero haec $dv = -pdx + \frac{vdx}{q}$. Quarum altera in alteram transmutatur facto q negativo. Considerimus igitur alterutram tantum ascensui accommodatam, quae induit hanc formam

$$dv + \frac{vdx}{q} = -pdx.$$

Multiplicetur haec per $e^{\int \frac{dx}{q}}$, ut fiat integrabilis. Erit autem aequatio integralis [p. 216] $e^{\int \frac{dx}{q}} v = -\int e^{\int \frac{dx}{q}} pdx$, ergo

$$v = -e^{-\int \frac{dx}{q}} \int e^{\int \frac{dx}{q}} pdx.$$

Sit corporis in A, positis $AC = a$, celeritas nulla, et scribatur X loco integralis ipsius $e^{\int \frac{dx}{q}} pdx$ ita accepti, ut evanescat facto $x = 0$. Deinde loco x posito a abeat X in A, erit $-\int e^{\int \frac{dx}{q}} pdx = A - X$ atque $v = e^{-\int \frac{dx}{q}} (A - X)$.

Tempus igitur, quo spatium CP absolvitur, est

$$\int \frac{e^{-\frac{1}{2}\int \frac{dx}{q}} dx}{\sqrt{(A-X)}}.$$

Pro descensu erit autem, scripto $A - X$ simili modo $-\int e^{-\int \frac{dx}{q}} pdx$, altitudo celeritati in P debita

$$v = e^{\int \frac{dx}{q}} (A - X),$$

et tempus, quo spatium PC absolvitur, erit

$$\int \frac{dx}{e^{-\frac{1}{2}\int \frac{dx}{q}} \sqrt{(A-X)}}.$$

Q.E.I.

Corollarium 1.

510. Celeritas in puncto infimo C reperiatur facto $x = 0$, quo casu et X evanescat et $\int \frac{dx}{q}$ evanescere ponamus. Prodibit igitur tam pro ascensu quam pro descensu $v = A$. Notandum autem est A in utroque casu non eundem habere valorem, sed diversum. Formatur enim ex X, quod pro ascensu est $= \int e^{\int \frac{dx}{q}} pdx$, pro descensu vero $= \int e^{-\int \frac{dx}{q}} pdx$.

Corollarium 2.

511. In descensu maxima habebit corpus celeritatem, quando est $v = pq$, tum enim fit $dv = 0$. [p. 217] Locus ergo, in quo celeritas est maxima, determinabitur ex ista aequatione $pq = e^{\int \frac{dx}{q}} (A - X)$.

Scholion.

512. In hypothesi tam potentiae quam medii uniformis corpus delapsum spatio demum infinito percurso acquirebat maximam suam celeritatem et , si principio ea statim promovebatur, eam perpetuo retinebat. Hic vero, ubi p et q sunt quantitaties variables, corpus ex quiete delapsum finito tempore maximam celeritatem acquirere potest neque, si eam semel habuit, retinere debet, nisi sit pq perpetuo quantitas constans seu medii densitas vi centripetae proportionalis (385).

PROPOSITIO 66.

PROBLEMA.

513. *Data lege vis centripetae ad centrum C* (Fig.44*) trahentis et medio resistente celeritatum ratione, se dentur celeritates corporis, quas ex quibuscunque altitudinibus delapsum in C acquirit, determinare densitatem seu resistentiae exponentem in singulis locis.*

SOLUTIO.

Posita quacunque distantia $CP = x$, et vis centipeta in $P = p$, sit curva CMB huius indolis, ut eius applicata quaevis AB sit aequalis altitudini debitae celeritati, quam corpus ex A delapsum in C acquirit, quae curva igitur data erit. Exponens vero resistentiae, qui quaeritur, sit in $P = q$. Sit porro distantia AC, [p. 218] ex qua corpus delabitur, $= a$, erit AB certa quaedam functio ipsius a, quam ponamus L. Eiusdem vero curvae applicata PM sit R, eritque R talis functio ipsius x, qualis L est ipsius a. Ex praecedente autem propositione apparet corporis ex distantia $AC = a$ delapsi altitudinem celeritati in C acquisitae debitam fore $= A$ (510). Quamobrem

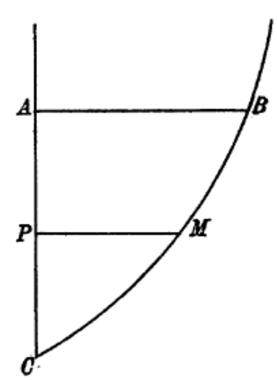


Fig. 44.

erit $L = A$ atque etiam $R = X$; est enim quoque X talis functio ipsius x, qualis A est ipsius a; R igitur talis esse debet functio ipsius x, ut evanescat facto $x = 0$. Quoniam vero est

$$X = \int e^{-\int \frac{dx}{q}} pdx, \text{ erit } R = \int e^{-\int \frac{dx}{q}} pdx \text{ et } l\,\frac{pdx}{dR} = \int \frac{dx}{q}.$$

Ex quo, posito dx constante, elicitur

$q = \frac{pdxdR}{dpdR - pddR}$. Q.E.I.

Corollarium 1.

514. Si fuerit CMB linea recta adeoque $R = ax$, erit $ddR = 0$ et $q = \frac{pdx}{dp}$. Si fit praeterea $p = \beta x^n$, erit $q = \frac{x}{n}$, seu medii densitas erit distantiis a centro reciproce proportionalis.

Corollarium 2.

515. Si fuerit $n = 0$ seu vis centripeta ubique eadem, erit $q = \infty$, ideoque medii densitas nulla et ipsa resistentia evanescens. Hicque est casus corporis in vacuo descendentis a potentia absoluta uniformi sollicitati.

Corollarium 3.

516. Si fuerit n numerus negativus, habebit q quoque valorem negativum.
Ex quo cognoscitur resistantiam transmutandam esse in vim propellentem. [p. 219]

Scholion 1.

517. Ex hisce facile quoque resolvitur eadem quaestio ad ascensum accommodata, si nimirum detur altitudo, ad quam corpus ex C quacunque celeritate proiectum pertingit. Posita enim celeritate = $\sqrt{R}$, qua spatium x absolvitur, q tantummodo in sui negativum transmutari debebit, quo facto habebitur $q = \frac{pdxdR}{pddR - dpdR}$.

Scholion 2.

518. Utraque aequatio definiens q tam pro ascensu quam pro descensu ita est comparata, ut idem valor ipsius q inveniatur, quaecunque multipla loco p et R accipiantur. Neque tamen ex his concludere licet, si determinatae sint q et p, R vagum quendam habere posse valorem; sed necessario debebit esse determinatus. Quo autem ille ipse valor ipsius R, qui est assumtus, prodeat, non vero eius quoddam multiplum, vis centripeta p ad hoc est vel remittenda vel intendenda. Quando autem vis centripetae in singulis locis quantitas ipsa datur, problema erit plus quam determinatum, si quidem celeritates quibusvis distantiiis respondentes dentur : sed duntaxat earum ratio proposita esse debet. Ratio vero difficultatis in hoc consistit, quod invenimus q ex aequatione $R = \int e^{-\int \frac{dx}{q}} pdx$ bis differentia. [p. 220] Differenta enim aequatio latius patet plusque in se complectitur quam integralis.

Scholion 3.

519. Ex solutione problematis sponte sequitur, quomodo, si data fuerit medii densitas in singulis locis seu quantitas q, inveniri oporteat vim centripetam p, reliquis iisdem quibus ante manentibus datis. Ex hac enim aequatione $l\frac{pdx}{dR} = \int \frac{dx}{q}$ dedecetur $p = \frac{dR}{dx} e^{\int \frac{dx}{q}}$, qui valor est determatus, quia $\int \frac{dx}{q}$ ita sumtum ponimus, ut evanescat facto $x = 0$ (510).

PROPOSITIO 67.

PROBLEMA.

520. *Resistente medio in duplicata celeritatum ratione dataque eius densitate seu exponentiae in singulis locis, determinare vim centripetam, quae faciat, ut corpus, ex quacunque altitudine ad centrum C* (Fig.43*) delabatur, perpetuo tamen eodem tempore eo perveniat .*

SOLUTIO.

Descendat corpus ex puncta quocunque A, sitque $AC = a$. Vocetur indeterminata $CP = x$, et ponatur altitudo celeritati in P debita = v, exponens resistentiae in $P = q$ et vis centripeta ibidem = p, quae est invenienda. Habebitur igitur $v = e^{\int \frac{dx}{q}}(A - X)$ et tempus, quo spatium PC absolvitur, [p. 221]

$$= \int \frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}.$$

(509) . Ubi est $X = \int e^{-\int \frac{dx}{q}} pdx$ ita acceptum, ut evanescat posito $x = 0$, et A oritur ex X posito $x = a$. Totum igitur tempus per AC habebitur, si in integrali ipsius $\int \frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}$ ponatur $x = a$ vel $X = A$. Expressio vero resultans ita esse debebit comparata, ut in ea omnino non insit a vel A : quod obtinetur, si $\int \frac{dx}{e^{\int \frac{dx}{2q}} \sqrt{(A-X)}}$ fuerit functio ipsarum a et x vel A et X nullius dimensions. Quamobrem et differentialis huiusmodi sit functio necesse est. Ponatur igitur $\frac{dx}{e^{\int \frac{dx}{2q}}} = \frac{dX}{P}$; habebimus pro differentiali temporis $\frac{dX}{P\sqrt{(A-X)}}$, in quo A et X dimensionem obtinent dimidiam; P ergo, quo nulla adsit dimensio, quoque dimediam dimidiam dimensionem haberer debebit. Sed in P non inesse potest a vel A; eius enim quantitas a solo puncto P pendere debet, non a puncto A. Hanc ob rem erit $P = \frac{\sqrt{X}}{b}$ et elementum temporis $= \frac{bdX}{\sqrt{(AX - XX)}}$, quod requisitam habet proprietatem. Erit igitur

$\frac{dx}{e^{\int\frac{dx}{2q}}}=\frac{bdX}{\sqrt{X}}$, et integratione peracta $2b\sqrt{X}=\int\frac{dx}{e^{\int\frac{dx}{2q}}}$, quod integrale ita esse debet sumtum, ut evanescat facto $x = 0$. [p. 222] Quia autem est $X=\int e^{-\int\frac{dx}{q}}pdx$, habebitur

$$4b^2\int e^{-\int\frac{dx}{2q}}pdx=\left(\int\frac{dx}{e^{\int\frac{dx}{2q}}}\right)^2$$ et hinc differentiando tandem $$p=\frac{e^{\int\frac{dx}{2q}}}{2b^2}=\int\frac{dx}{e^{\int\frac{dx}{2q}}}.$$

Q.E.I.

Corollarium 1.

521. Quia elementum temporis est $\frac{bdX}{\sqrt{(AX-XX)}}$, erit tempus, quo spatium *PC* absolvitur, = arcui circuli, cuius sinus versus est *X*, existente diametro = *A*, ducto in $\frac{2b}{A}$. Et posita ratione peripheriae ad diametrum $\pi:1$, erit tempus totius descensus per $AC=\pi b$, quod est constans neque ab *a* pendens.

Corollarium 2.

522. Quia est $\int\frac{dx}{e^{\int\frac{dx}{2q}}}=2b\sqrt{X}$ et $e^{\int\frac{dx}{2q}}=\frac{dx\sqrt{X}}{bdX}$, erit $p=\frac{Xdx}{b^2dX}$. Estque $X=\frac{1}{4b^2}\left(\int\frac{dx}{e^{\int\frac{dx}{2q}}}\right)^2$.

Corollarium 3.

523. Sit medium resistens uniforme et ideo $q = k$; erit $e^{\int\frac{dx}{2q}}=e^{\frac{x}{2k}}$ et $\int\frac{dx}{e^{\int\frac{dx}{2q}}}=2k(1-e^{\frac{-x}{2k}})$.

Ex quo prodibit $p=\frac{k}{b^2}(e^{\frac{x}{2k}}-1)$ [p. 223]. Vis igitur centripeta in C erit = 0.

Corollarium 4.

524. Si *q* est constans et = *k*; erit $2b\sqrt{X}=2k(1-e^{\frac{-x}{2k}})$ et $X=\frac{k^2}{b^2}(1-e^{\frac{-x}{2k}})^2$.

Quia vero *X* abit in *A* posito $x = a$, erit $A=\frac{k^2}{b^2}(1-e^{\frac{-a}{2k}})^2$ atque

$$v=A=\frac{k^2}{b^2}e^{\frac{a}{k}}\left((1-e^{\frac{-a}{2k}})^2-(1-e^{\frac{-x}{2k}})^2\right).$$

Corollarium 5.

525. In infimo igitur loco C altitudo celeritati debita erit $=A=\frac{k^2}{b^2}\left(1-e^{\frac{-a}{2k}}\right)^2$.

Corollarium 6.

526. Maximam habebit corpus celeritatem, ubi est $v = pk$ (511). Erit ergo

$$e^{\frac{-x}{2k}}\left(1-e^{\frac{-x}{2k}}\right)=\left(1-e^{\frac{-a}{2k}}\right)^2-\left(1-e^{\frac{-x}{2k}}\right)^2.$$

Ex quo reperitur $e^{\frac{-x}{2k}} = 2e^{\frac{-a}{2k}} - e^{\frac{-a}{k}}$ hincque

$$x = 2a - 2k\, l\left(2e^{\frac{a}{2k}} - 1\right).$$

Scholion.

527. Si q et k accipiantur negative, invenitur lex vis centripetae, quae efficit, ut omnes ascensus ex C facti absolvantur aequalibus temporibus. [p. 224]
Hoc enim semper locum habet descensum in ascensum transmutari vi resistentiae negativa facta. Quo igitur omnes ascensus fiant isochroni, erit

$$p = \frac{e^{-\int\frac{dx}{2q}}}{2b^2}\int e^{\int\frac{dx}{2q}}dx.$$

In casuque medii uniformis erit $p = \frac{k}{b^2}\left(1-e^{\frac{-x}{2k}}\right)$.

PROPOSITIO 68.

PROBLEMA.

528. *Si vis centripeta sit distantiis a centro C* (Fig.45)*proportionalis et medium uniforme resistat in simplice celeritatum ratione, oportet determinari motum corporis tam recta accedintis ad centum C quam recedentis ab eo.*

SOLUTIO.

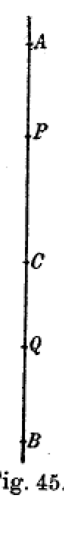


Fig. 45.

Sit distantia, in qua vis centripeta aequalis est vi gravitatis, $= f$ et exponens resistentiae $= k$. Iam accedat corpus in recta AC ad centrum C, et ponatur altitudo celeritati, quam C habebit, debita $= c$. Hacque celeritate tum ultra C in recta CB recedat a C. Consideremus primo accessum, et ponamus $CP = x$ et celeritati in P altitudinem debitam $= v$. His positis erit vis centripeta in $P = \frac{x}{f}$ et vis resistentiae $= \frac{\sqrt{v}}{\sqrt{k}}$; ex quibus oritur ista aequatio $dv = -\frac{xdx}{f} + \frac{dx\sqrt{v}}{\sqrt{k}}$. Quo haec aequatio fiat homogenea, ponatur $\sqrt{v} = u$ et $\sqrt{k} = h$; erit ergo $dv = 2udu$ et $2udu = -\frac{xdx}{f} + \frac{udx}{h}$. [p. 225]

Fiat $u = rx$, erit $2r^2xdx + 2rx^2dr = -\frac{xdx}{f} + \frac{rxdx}{h}$, ex qua oritur

$$\frac{dx}{x} = \frac{2fhrdr}{fr-h-2fhr^2}$$

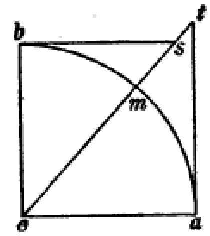


Fig. 46.

Quae integrata cum debita adiecta constante et restitutis v et k, abit in hanc

$$\frac{v}{c}-\frac{x\sqrt{v}}{2c\sqrt{k}}+\frac{x^2}{2fc}=\left(\frac{4\sqrt{fkv}-x\sqrt{f}+x\sqrt{(f-8k)}}{4\sqrt{fkv}-x\sqrt{f}-x\sqrt{(f-8k)}}\right)^{\sqrt{\frac{f}{f-8k}}}$$

Si autem $8k > f$, aequatio differentialis ope circuli quadraturae debet construi. Ponatur scilicet $h=\frac{1}{4\alpha}$ et $f=\frac{1}{4\beta}$, et habebitur ista aequatio differentialis

$$0=\frac{dx}{x}+\frac{rdr}{r^2-2\alpha r+\beta}=\frac{dx}{x}+\frac{rdr-\alpha dr}{r^2-2\alpha r+\beta}+\frac{\alpha dr}{r^2-2\alpha r+\beta}.$$

Cuius integralis est

$$C=l\sqrt{u^2-2\alpha ux+\beta x^2}+\int\frac{\alpha dr}{r^2-2\alpha r+\beta}.$$

Est vero $\int\frac{\alpha dr}{r^2-2\alpha r+\beta}=\frac{\alpha}{\sqrt{(\beta-\alpha^2)}}.am$ (Fig. 46), arcusque am tangens $at=\frac{r-\alpha}{\sqrt{(\beta-\alpha^2)}}$, existente radio ac = 1. Posito ergo $\frac{u}{x}$ loco r, erit $at=\frac{u-\alpha x}{x\sqrt{(\beta-\alpha^2)}}$. Ad constantem C determinandam ponatur x = 0 et $u=\sqrt{c}$, quo facto loco $\frac{\alpha}{\sqrt{(\beta-\alpha^2)}}.am$ habebitur $\frac{\alpha}{\sqrt{(\beta-\alpha^2)}}.amb$. Erit ergo $l\sqrt{c}+\frac{\alpha}{\sqrt{(\beta-\alpha^2)}}.amb=l\sqrt{u^2-2\alpha ux+\beta x^2}+\frac{\alpha}{\sqrt{(\beta-\alpha^2)}}.am$.

Unde fit $bm=\frac{\sqrt{(\beta-\alpha^2)}}{\alpha}l\sqrt{\frac{u^2-2\alpha ux+\beta x^2}{c}}$,

estque bm arcus, cuius tangens bs est = $\frac{x\sqrt{(\beta-\alpha^2)}}{u-\alpha x}$.

Pro recessu a centro C si ponatur ut ante $CQ = x$ et celeritas in $Q=\sqrt{v}$, retinentibus $h, f, k, \alpha, \beta, r$, et u eosdem quos ante valores, habebitur $0=\frac{dx}{x}+\frac{rdr}{r^2+2\alpha r+\beta}$.

Ex qua obtinebitur

$bm=-\frac{\sqrt{(\beta-\alpha^2)}}{\alpha}l\sqrt{\frac{u^2+2\alpha ux+\beta x^2}{c}}$, estque arcus bm tangens $bs=\frac{x\sqrt{(\beta-\alpha^2)}}{u+\alpha x}$.

Si fuerit $\alpha^2>\beta$ seu $f>8k$, poterit integratio algebraice exhiberi; erit

$$\frac{v}{c}+\frac{x\sqrt{v}}{2c\sqrt{k}}+\frac{x^2}{2fc}=\left(\frac{4\sqrt{fkv}+x\sqrt{f}-x\sqrt{(f-8k)}}{4\sqrt{fkv}+x\sqrt{f}+x\sqrt{(f-8k)}}\right)^{\sqrt{\frac{f}{f-8k}}}$$.[p. 226]

Restat autem casus quo $\alpha^2 = \beta$ seu $f = 8k$, qui seorsim pertractari debet. Invenitur autem pro accessu haec aequatio $l\frac{4\sqrt{kv}-x}{4\sqrt{kv}} = \frac{x}{4\sqrt{kv}-x}$. Atque pro recessu ista $l\frac{4\sqrt{kv}+x}{4\sqrt{kv}} = \frac{-x}{4\sqrt{kv}+x}$.
Q.E.I.

Corollarium 1.

529. In casu ergo, quo $f = 8k$, semper pro accessu esse debet $4\sqrt{kv} > x$, alioquin $\frac{x}{4\sqrt{kv}-x}$ aequaretur quantitati imaginariae. Quare, nisi x = 0, non poterit esse v = 0, atque ideo celeritas in C necessario debet esse = 0. Quamobrem si ea ponitur finita $\sqrt{c}$, initium descensus erit imaginarium.

Corollarium 2.

530. In autem casu $f = 8k$ recessus ex aequatione cognoscitur; facto enim v= 0 invenitur $l\frac{x}{4\sqrt{kc}} = -1$, hincque $x = BC = \frac{4\sqrt{kc}}{e}$ denotante e numerum, cuius logarithmus est unitas. Ergo distantia BC est proportionalis celeriti in C.

Corollarium 3.

531. Quia igitur, quando resistentia tanta est, ut sit $8f = k$, corpus in accessu ad C omnem amittit celeritatem, multo maiore ratione, si $8k < f$ seu resistentia adhuc maior fuerit, corpus ad C accedens omnem celeritatem amittet. [p. 227]

Corollarium 4.

532. Quare, si vel $8k = f$ vel $8k < f$, corpus post accessum ad C in C perpetuo quiescet, atque his casibus nullus recessus sequi poterit. At si resistancia fuerit minor seu $8k > f$, tum corpus accedens in C finitam celeritatem habere poterit, qua deinde a C recedet, atque motu oscillatorio movebitur.

Corollarium 5.

533. Sin autem $8k > f$, pro accessu haec habetur aequatio : arcus cuius tangens est $\frac{x\sqrt{(\beta-\alpha^2)}}{u-\alpha x}$

$$= \frac{\sqrt{(\beta-\alpha^2)}}{\alpha} l \sqrt{\frac{u^2-2\alpha ux+\beta x^2}{c}}.$$

Unde initium accessus A invenitur ponendo $u = 0$; prodit autem arcus cuius tangens est $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$

$$= \frac{\sqrt{(8k-f)}}{\sqrt{f}} l \frac{\sqrt{2fc}}{x}.$$

Pro recessu vero similiter invenitur arcus cuius tangens est $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$

$$= \frac{\sqrt{(8k-f)}}{\sqrt{f}}\, l\, \frac{\sqrt{2fc}}{x}.$$

Scholion 1.

534. Hinc sequi videtur distantiam *BC* semper aequalem esse distantiae AC, quia hae duae aequationes inter se congruunt. At cum,si $8k < f$, nullus omnino detur recessus, fieri non potest, ut, si $8k$ aliquantulum tantum maius fuerit quam f, spatium recessus aequale fiat spatio accessus. Difficultas haec tollitur, si attendamus innumerabiles arcus eidem tangenti $\frac{\sqrt{(8k-f)}}{\sqrt{f}}$ respondere, [p. 228] quorum alius pro accessu, alius pro recessu accipi debet. Ponatur $\frac{\sqrt{(8k-f)}}{\sqrt{f}} = \tau$, et minimus arcus tangenti τ respondens sit γ et semipheripheria circuli π : erit τ tangens omnium horum arcuum $\gamma, \pi+\gamma, 2\pi+\gamma, 3\pi+\gamma,$ etc. nec non horum $-\pi+\gamma, -2\pi+\gamma,$ etc. Pro recessu nunc *BC* sumi debet arcus γ, erit $\frac{\gamma}{\tau} = l\frac{\sqrt{2fc}}{BC}$ atque $BC = e^{\frac{-\gamma}{\tau}}\sqrt{2fc}$. Atque pro accessu sumi debet arcus $-\pi+\gamma$, fitque $AC = e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$. Reliqui arcus dant puncta, in quibus corpus circa C oscillando successive habet celeritatem = 0. Cum igitur in prima oscillatione sit spatium accessus $= e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$, erit spatium accessus secundae oscillationis aequale spatio recessus in prima oscillatione atque ideo $e^{\frac{\pi-\gamma}{\tau}}\sqrt{2fc}$. In tertia oscillatione erit spatium accessus = $e^{\frac{-\pi-\gamma}{\tau}}\sqrt{2fc}$. Atque in oscillatione, quae numero n indicatur, est spatium accessus $e^{\frac{-(n-2)\pi-\gamma}{\tau}}\sqrt{2fc}$. Hacque ratione cuiuscunque oscillationis tam spatium accessus quam spatium recessus poterit determinari.

Corollarium 6.

535. Quando igitur corpus oscillationes absolvit circa centrum C, constituent spatia accessus progressionem geometricam, [p. 229] cuius denominator est $e^{\frac{-\gamma}{\tau}}$. Similemque progressionem constituunt spatia recessus atque etiam integra spatia singulis oscillationibus percursa.

Scholion 2.

536. Quia aequatio differentialis $2udu = -\frac{xdx}{f} + \frac{udx}{h}$ pro descensu, et aequatio $2udu = -\frac{xdx}{f} - \frac{udx}{h}$ pro ascensu est homogenea, erit in utroque casu u = functioni ipsarum x et a unius dimensionis, denotante a maximam a centro C elongationem *AC* aut *BC*.

Quamobrem in temporis expressione $\int \frac{dx}{u}$ nulla inerit dimensio ipsarum *a* et *x*, et ideo omnia tempora tam ascensuum quam descensuum erunt inter se aequalia. Integrale enim ipsius $\frac{dx}{u}$ erit functio ipsarum *a* et *x* nullius dimensionis, haecque expressio posito $x = a$ erit aequalis quantitati constanti. Simili modo erunt omnium descensuum tempora usque ad punctum maxima celeritatis inter se aequalis. Distantia enim puncti in quo corpus maximum habet celeritatem, proportionalis est ipsa a seu maximae elongationi a centro *C* (528). [p. 230]

PROPOSITIO 69.

THEOREMA.

537. *Si fuerit vis centripeta ut potestas distantiae a centro C* (Fig.43*) cuius expones ist n, et medium resistat in ratione 2m –multiplicata celeritatum, exponens vero resistentiae sit proportionalis distantiarum a centro C potestati exponentis* $\frac{mn+m-n}{m}$*, erunt plurium descensuum vel ascensuum tempora in spatiurum totorum descriptorum ratione* $\frac{1-n}{2}$ *— multiplicata.*

DEMONSTRATIO.

Sit *AC* spatium totum vel ascensu vel descensu descriptum = *a* eiusque portio quaecunque $CP = x$ et celeritas corporis in $P = \sqrt{v}$. Ponatur distantia *f*, in qua vis centripeta aequalis est vi gravitatis. His positis erit vis centripeta in $P = \frac{x^n}{f^n}$, et, sumto pro resistentiae exponente $\lambda^{\frac{1}{m}} x^{\frac{mn+m-n}{m}}$, erit vis resistentiae $\frac{v^m}{\lambda x^{mn+m-n}}$. Hinc pro descensu habebitur ista aequatio

$$dv = \frac{-x^n dx}{f^n} + \frac{v^m}{\lambda x^{mn+m-n}},$$

pro ascensu vero

$$dv = \frac{-x^n dx}{f^n} - \frac{v^m}{\lambda x^{mn+m-n}}$$

Quae aequationes inter se prorsus conveniunt, nisi quod λ in altera negativum habeat Ponatur nunc $v = u^{n+1}$, et habebitur

$$(n+1)u^n du = \frac{-x^n dx}{f^n} \mp \frac{u^{mn+m} dx}{\lambda x^{mn+m-n}},$$

in qua aequatione *u* et *x* eundem ubique dimensionum numerum constituunt. Haec vero aequatio ita debet integrari, ut facto $x = a$ evanescat *u*. Quam ob rem aequatio integralis ita erit comparato, ut *a , x,* et *u* ubique [p. 231] eundem constituant dimensionum numerum. Ex ea igitur reperietur *u* aequalis functioni ipsarum *a* et *x* unius dimensionis. Consequenter aequabitur *v* functioni ipsarum *a* et *x* dimensionum $n + 1$. Quocirca tempus, quo spatium PC percurritur, nempe $\int \frac{dx}{\sqrt{v}}$, erit functio ipsarum a et x, quae habebit $\frac{1-n}{2}$ dimensions. Totum ergo tempus vel ascensus vel descensus erit $= Aa^{\frac{1-n}{2}}$, ubi A est quantitas constans ex literas f et λ, quae immutantae manent. Perspicuum igitur est omnes tam ascensus quam descensus esse inter se in totorum spatiorum descriptorum ratione $\frac{1-n}{2}$—multiplicata. Q.E.D.

Corollarium 1.

538. Si medium resistens sit uniforme, ideoque $mn + m - n = 0$, erit $n = \frac{m}{1-m}$ seu vis centripeta ut distantia elevata ad $\frac{m}{1-m}$. Tempora vero vel ascensuum vel descensuum erunt in spatiorum percursorum ratione $\frac{1-2m}{2-2m}$—multiplicata.

Corollarium 2.

539. Si fuerit $n = 1$ seu vis centripeta distantiis a centro C proportionalis, erunt omnia tempora tam ascensuum quam descensuum inter se aequalia. [p. 232]
Hoc vero casu cum resistentiae lex sit celeritatum potestas exponentis $2m$, erit resistantiae exponens ut distantia a centro C elevata ad $\frac{2m-1}{m}$.

Corollarium 3.

540. Ex hoc patet, quod ex praecedente propositione invenimus (536), si resistentia sit celeritatibus proportionalis et hanc ob rem $m = \frac{1}{2}$ et medium uniforme, omnia tempora tam ascensuum quam descensuum fore inter se aequalia.

Corollarium 4.

541. Si vis centripeta fuerit constans seu $n = 0$, erunt tempora vel ascensuum vel descensuum in subduplicata spatiorum percursorum ratione. Exponens vero resistentiae fit distantiis a centro C proportionalis. Eundem hunc casum iam exposuimus supra (495).

Scholion.

542. Hisce concludimus hoc Caput de motu puncti rectilineo in medio resistente; atque iuxta divisionem factam progredimur ad motus curvilineos in vacuo corporum a quibuscunque potentiis absolutis sollicitatorum.