CHAPTER FOUR.
CONCERNING THE MOTION OF FREE POINTS IN A MEDIUM WITH RESISTANCE [p. 153]

DEFINITION 18

367. *The law of the resistance is the force or function of the speed of the body, to which the resistance is in proportion.* Thus if the resistance is proportional to the square of the speed, the law of the resistance is the square of the speed.

Corollary 1.
368. Therefore it is known from the law of the resistance, if many equal points are carried forwards with different speeds, how their motions diminish between themselves. And from the given decrease in the speed of one point the decreases in the speeds of the rest can also be found.

Corollary 2.
369. If therefore from one step in the speed given for the ratio of the resistance to the force of gravity, for all the other steps too the ratio between the resistance and the force of gravity can be found from the law of the resistance. And from this the effect of the resistance on the motion of the body can be found.

[p. 154]

Scholium 1.
370. Certainly the strength of the resistance extends as far as the general idea of force and thus it is homogeneous with the force of gravity, as this will become apparent when the motion of bodies in fluids is treated. Therefore an absolute force can always be assigned to produce the same effect on the body as resistance. Truly this absolute force will always depend on the speed of the body, since on this account, in the expression of this, the speed or the height corresponding to the speed will be present. Therefore in this way the motion of the body in a medium with resistance is reduced to the motion of a body acted on by an absolute force, since the laws will have been set out as in the second chapter above, from which all the questions can be resolved. [Thus, one assumes that the medium is at rest, so that absolute forces are acting.]

Scholium 2.

371. The direction of the force of the resistance in this tract of ours will always be in agreement with the motion of the body and in the opposite direction (117). On account of which the absolute force to be substituted for that will always be one of retardation, with the direction of the motion not changed. And thus it is evident that the expression for the resistive force is made negative as often as the body moves in a contrary direction, and that motion is one of being accelerated. Indeed it is not possible to have this case for a body at rest at a point in a fluid, but yet in the calculation this often occurs for the given motion of the body in the fluid investigated.

[p. 155]

Corollary 3.

372. Therefore the motion of a body in a fluid with resistance, if not acted on by any other force, must be to move in a straight line. For since the direction of the motion is not changed by a resistive force, the motion of the body, that naturally peruses a straight line, must always by necessity be made in the same straight line.

Corollary 4.

373. If in addition an absolute force is acting, the direction of which is always in the same direction as the motion, then the body in the resistive medium also proceeds in a straight line. For neither will this absolute force nor the resistive force change the direction of the motion.

Scholium 3.

374. Therefore in this chapter, in which we have only established for rectilinear motion to be explained, we do not join other absolute forces with the resistive force, except when their directions agrees with the direction of the motion. On account of this, it is possible to consider all the forces that we have used in the preceding chapter with resistance to be included. Moreover before we go forwards to absolute forces, it will be convenient to put the motion of bodies impeded by a single resistive force under scrutiny, from which it will be easier for us to progress from simpler to more complex situations.

[p. 156]

Scholium 4.

375. In the expression of the law of resistance or in that function of the speed, besides the height corresponding to the speed v there can be constant quantities present, but we exclude all variable quantities depending on the motion of the body. Indeed it may happen that the resistance experienced by a wide body with equal speeds, shall be greater or less as that body goes from one place to another; as happens when the fluid in which the body moves is denser in one place and indeed rarer in another, in which case it is necessary to take into account the position in the expression for the resistance. Yet neither is it appropriate to consider the position in the law governing the resistance we wish to express, for in that account of the resistance when the body is put in the same place with various speeds. Truly the distinction that can arise from the variation in the position, we understand to be in the exponent of the resistance, which likewise indicates the intensities of the resistances. [By *intensity* Euler means the force per unit mass or the contribution to the acceleration or deceleration associated with the force acting on the body.]

DEFINITION 19

376. *The exponent of the resistance is the height corresponding to its speed, which if the body has this speed, then the resistance experienced is equal to the force of gravity.* It is clear that in this case the body is slowed down by the resistance to the same extent as it would be slowed in moving up under the force of gravity. [The exponent of the resistance is hence a constant associated with that resistive force, indicating its strength. In certain circumstances where the resistance varies with position, then it is not a constant, as explained below. For uniform resistance it is constant. The resistance is given as a negative acceleration of the form $-\left(\frac{v}{k}\right)^n$, where v is the height the body falls under constant unit acceleration due to gravity, corresponding to the speed $\sqrt{v}$, and n is a whole number power.]

Corollary 1.

377. Therefore if the body in the resisting medium has a motion corresponding to an altitude v and this altitude v is equal to the exponent of the resistance, then [p. 157] the body progresses through the element of distance dx, where $dv = -dx$; since in this case the resistive force is equivalent to the force of gravity, that we always put equal to 1, and the motion is slowed down.

Corollary 2.

378. Therefore with the law and the exponent of the resistance given, the diminution of the motion can be defined. Accordingly from the known exponent, what size of speed the body should have, in order that the resistive force is equal to the force of gravity, and from the given law of the resistance, then the ratio is known, according to which different speeds are lessened by the resistance.

Scholium.

379. The exponent of the resistance is either a constant, or a variable, or depends on the location of the body. The first mentioned happens in a medium or uniform fluid, which has the same resistance acting on bodies everywhere, if they move everywhere with the same speed. A resistive medium of this kind we will call uniform, clearly which is the same to the body in all places. Moreover the exponent of the resistance is variable in a medium or fluid that is not uniform, even if the resistance follows the same law in separate place also. For when the fluid or medium is denser, in which the body moves about, in that too the body experiences the greater resistance, which is equal to the speed of the motion [due to gravity] also. Cleary the resisting speed equal to gravity is greater in a rarer medium, less in a more dense medium. Moreover since dense and rare mediums depend of the location, it is evident [p. 158] that the exponent of the resistance, if it is variable, must depend on the location of the body.

DEFINITION 20

380. *Here resistive mediums are called similar that have the same law of resistance. Truly these mediums are dissimilar which have different laws of resistance.* Thus water and mercury are mediums of the same kind, accordingly as both have fluids resist in the ratio of the square of the speeds.

Corollary.

381. Therefore if the resistances of similar media are different from each other, then the whole difference is consistent with the exponent of the resistance or with the density and rarity. Thus in water the exponent of the resistance is greater than in quicksilver, since this fluid is denser than that.

Scholium.

382. Similar mediums with bodies with equal speeds are able to acquire different resistances, since the densities of these are different from each other, and it can be agreed to measure these densities from the resistances imparted to the bodies with a given speed. Indeed in fluids, as with the motion of the body performed in the fluid, it is taught, that for bodies with equal speeds, the resistances are in proportion to the densities of the fluids. We transfer this property to the resistance of any other mediums whatever the law governing the resistance : since other laws besides the ratio of the square of the speed [p. 159] are purely imaginary and they are accustomed to be used as exercises in analysis only.

PROPOSITION 49.

PROBLEM.

383. *For a body moving along the line AP* (Fig. 37) *in a medium with some kind of resistance, both the law and the exponent of which are known, with a given speed at the point P, to find the decrease in the speed as the element of distance Pp is traversed.*

SOLUTION.


Fig. 37.

With the element $Pp = dx$ let the altitude corresponding to the speed at P be equal to v and the exponent of the resistance be equal to q. Therefore $\sqrt{q}$ denotes the speed, which if the body has at P, then the force of the resistance will equal the strength of gravity, which in turn is equal to 1. On account of which, if $v = q$, then the resistive force is equal to 1 and $dv = - dx$ (376, 377). Moreover let V be that function of the speed $\sqrt{v}$, by which the law of the resistance is expressed, and Q designates a similar function of $\sqrt{q}$, or Q is a quantity of that kind, which is produced if q is substituted in place of v in V. Therefore the resistance, that the body moving with the speed $\sqrt{v}$ experiences, is equal to $\frac{V}{Q}$. Which when the motion is slowed, will be $dv = \frac{-Vdx}{Q}$. Q. E. I. [p. 160]

Corollary 1.

384. Since the quantity V is a function of v and of a constant, and q likewise, and Q is either a constant or some function of x (375), the equation found $dv = \frac{-Vdx}{Q}$ can be freely separated. For we have $\frac{dv}{V} = \frac{-dx}{Q}$, from which by integration or even by construction the whole motion of the body along AP is known.

Corollary 2.

385. Since the force of resistance is equal to $\frac{V}{Q}$, the density of the medium can be found from this. Since indeed the density is measured from the resistance, that the body experiences moving with the given speed, it is necessary to substitute a constant quantity in $\frac{V}{Q}$ in place of v, with which done the resistance is found to vary as $\frac{1}{Q}$. Therefore the density of the medium also will be as $\frac{1}{Q}$, or inversely with Q.

Scholium.

386. Here $\frac{V}{Q}$ denotes not only the force acting, but now the strength of the retarding resistance itself [It is now clear that Euler identifies the strength of a force with the acceleration it produces], on account of which there is no need for the mass of the body to be included in the calculation. Moreover here the mass of the body is constant or we put the masses of many bodies equal to each other. Indeed I have agreed not to deliberate about this here, which will only come to be used in a single case, except to extend the necessity to return to more complicated cases. [p. 161]

PROPOSITION 50.

PROBLEM.

387. *In a medium with uniform resistance, that offers resistance in some power of the speed, to define the speed of the moving body at particular places.*

SOLUTION.

Let the body be moving along the right line AP (Fig. 37), and its speed at the point A corresponds to the altitude c. The distance travelled through $AP = x$, and the height corresponding to the speed at P is equal to v. The exponent of the resistance, which is constant, is called k, and the law of the resistance is v^m, thus in order that the resistance everywhere shall be as the power $2m$ of the exponent of the speed. Therefore in the preceding formula, $dv = -\frac{Vdx}{Q}$ and in this case V is changed into v^m, and Q, since it should be such a function of q or k, as v^m itself is of v, then it is equal to k^m. Therefore we have this equation $dv = -\frac{v^m dx}{k^m}$ or $\frac{dv}{v^m} = -\frac{dx}{k^m}$. The integral of this is : $\frac{v^{1-m}}{1-m} = C - \frac{x}{k^m}$. Indeed the constant C can be determined from this, that by making $x = 0$, v should be changed into c, on which account, $C = \frac{c^{1-m}}{1-m}$. Hence this equation results :

$$v^{1-m} = c^{1-m} - \frac{(1-m)x}{k^m} \text{ or } v = \sqrt[1-m]{\frac{c^{1-m}k^m(1-m)x}{k^m}},$$

if $m < 1$. But if $m > 1$, it becomes :

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m + (m-1)c^{m-1}x}}.$$

Truly in the single case, [p. 162] when $m = 1$, it cannot be found from these formulas, but should be derived from the equation of the differential, which with $m = 1$ is of this kind : $\frac{dv}{v} = -\frac{dx}{k}$, the integral of which is $lv = C - \frac{x}{k}$. Indeed in a similar manner, $C = lc$ and thus $lv = lc - \frac{x}{k}$. With the logarithms reduced to numbers, therefore it is found that : $v = ce^{-\frac{x}{k}}$.

Therefore whatever the value m may have, the speed of the body is known at any point on the line AP. Q. E. I.

Corollary 1.

388. If the resistance of the medium is proportional to the squares of the speeds, then $m = 1$. On account of which in this case, which is thought to be the only kind of resistance found in nature, the case of the singular solution $v = ce^{-\frac{x}{k}}$ prevails. From which it is apparent that the body does not lose the whole speed before x traverses an infinite distance.

Corollary 2.

389. If the medium has a resistance in a ratio greater than twice the square of the speed, then $m > 1$ and

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m + (m-1)c^{m-1}x}}.$$

Moreover it is evident from this equation that the speed does not vanish unless x is made indefinitely large : $x = \infty$.

Corollary 3.

390. Indeed this case differs from the one previously, when $m = 1$, since there, if the initial speed was infinitely great, it would with that great amount everywhere. Truly in

this case, [p. 163] when $m > 1$, by putting $c = \propto$, it comes about that $v = \sqrt[m-1]{\frac{k^m}{(m-1)x}}$.
Therefore v always has a finite magnitude, unless $x = 0$ or $= \propto$.

Corollary 4.

391. Besides in this case, when $m > 1$, the body, before it reaches the point A, always somewhere, such as C (Fig. 38), has a speed of infinite magnitude. In order that the point C can be found, it is necessary for x to become negative,

Fig. 38.

and $k^m + (m-1)c^{m-1}x$ to be put equal to zero. Hence it is found that $AC = \frac{k^m}{(m-1)c^{m-1}}$.

Corollary 5.

392. If the resistance were in a ratio less than the square of the speed, likewise $m < 1$, and it is found that :

$$v = \sqrt[1-m]{\frac{c^{1-m}k^m - (1-m)x}{k^m}}.$$

Therefore the speed of the body vanishes at the point C (Fig. 37), if $AC = \frac{c^{1-m}k^m}{1-m}$.
Consequently, when the body arrives at C, it will perpetually remain at rest there and not progress further.

Corollary 6.

393. If m is put equal to zero, the resistance is constant and acts equally on the body either at rest or moving. Therefore in this case the resistance departs from being a force and is equivalent to the strength of gravity. For since, if $v = k$, the resistance is made equal to the strength of gravity, and also for whatever other speed the body may have, it still experiences the same resistance.[p. 164]

Scholium 1.

394. In these corollaries we have explained the first difference of the motions, if m were equal to one, or greater or less than one. Indeed these differences can be understood from these short rules : if $m = 1$, the body neither has an infinite speed or zero speed anywhere in the whole distance. Then if $m > 1$, the body must have an infinite speed somewhere, and indeed the speed never vanishes. Finally if $m < 1$, the body has a zero speed somewhere, and indeed an infinite speed nowhere.

Scholium 2.

395. This diminution of the speed remains the same, in whatever place the body is moving, since the same resistance was shown everywhere. Unless indeed this motion is similar to that which is diminished by a contrary absolute force acting, where it happens that a body in a different place is accelerated as much as it was retarded before. But for the diminished motion in a resistive medium to be restored and again to be equally accelerated, and before it was being diminished, it is necessary for a negative force of resistance to be in place and thus to be changed into a propelling force. Then indeed the equation becomes : $dv = \frac{Vdx}{Q}$, from which it is apparent that the speed is to be increased by the same amount, that before it was diminished. Thus with the resistive force changed into a propelling force, the motion of the body would be made backwards, and it thus reverts from P to A, in order that it recovers the same speed at the individual points of the interval AP that it had everywhere before. [p. 165]

Scholion 3.

396. In the cases, in which $m < 1$ and finally the body has arrived at rest, the same difficulty occurs, of which mention has been made above (316), if we wish the motion to be changed with the force of resistance converted into a force of propulsion. For if the speed of the body is zero when it arrives at C , the propelling force $\frac{v^m}{k^m}$ also vanishes, if indeed m is not a negative number, and on this account the body is never able to leave the place C. Therefore in this case a decreasing motion is not able to be changed into an increasing motion. Indeed the calculation shows the opposite, for if it is said that $CP = y$, the height corresponding to the speed at P, is clearly $v = \sqrt[1-m]{\frac{(1-m)y}{k^m}}$. Moreover from that equation it follows that that is absurd, hence it is apparent, that the exponent $\frac{1}{1-m}$ of y is greater than unity, and thus in the graph of the heights for the corresponding speeds, the line AC is a tangent at C. Indeed as often as this happens, the body is never able to leave the point C, even if the calculation may point out otherwise (319).

PROPOSITION 51.

PROBLEM.

397. *For a body moving in a medium with uniform resistance, which makes the resistance to be in proportion to some power of the speeds, to determine the time in which the body travels through some interval AP* (Fig. 37). [p. 166]

SOLUTION.

As in the previous proposition, with the initial speed at $A = \sqrt{c}$, $AP = x$, with the speed at $P = \sqrt{v}$, with the exponent of the resistance k and with the law equal to v^m , thus as the strength of the resistance shall be as the power of the exponent of the speed $2m$. Which now is :

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m + (m-1)c^{m-1}x}}.$$

(387), if indeed $m > 1$, the speed becomes:

$$\sqrt{v} = \frac{\sqrt{ck^{\frac{m}{m-1}}}}{(k^m + (m-1)c^{m-1}x)^{\frac{1}{2m-2}}}.$$

Therefore the element of time, in which the small interval dx is run through, is

$$\frac{\mathrm{dx}}{\sqrt{v}} = \frac{(k^m + (m-1)c^{m-1}x)^{\frac{1}{2m-2}}}{\sqrt{ck^{\frac{m}{m-1}}}},$$

the integral of which is

$$= C + \frac{2(k^m + (m-1)c^{m-1}x)^{\frac{2m-1}{2m-2}}}{(2m-1)c^{m-1}\sqrt{ck^{\frac{m}{m-1}}}},$$

that expresses the time to traverse the interval AP, only if the constant C is correctly determined, and that can be done, as by making $x = 0$ the whole time vanishes. Therefore it has to be

$$C = \frac{2k^{m\frac{2m^2-m}{2m-2}}}{(2m-1)c^{m-1}\sqrt{ck^{\frac{m}{m-1}}}};$$

on account of which the whole time to travel through the distance AP

$$= \frac{2(k^m + (m-1)c^{m-1}x)^{\frac{2m-1}{2m-2}} - 2k^{\frac{2mm-m}{2m-2}}}{(2m-1)c^{\frac{2m-1}{2}}k^{\frac{m}{2m-2}}}.$$

[p. 167] Which expression also prevails, if $m < 1$. But if $m = 1$, a special operation is necessary, for since $v = ce^{-\frac{x}{k}}$, then $\frac{\mathrm{dx}}{\sqrt{v}} = \frac{ce^{\frac{x}{2k}}dx}{\sqrt{c}}$, the integral of which is $\frac{2ke^{\frac{x}{2k}} - 2k}{\sqrt{c}}$,

which expresses the time in which the interval AP is traversed.

Therefore whatever value m may have, the time for any interval can be determined from these formulas. Q. E. I.

Corollary 1.

398. If therefore the resistance is in proportion to the square of the speed and consequently $m = 1$, the time, in which the body describes an infinite distance, obviously as long as it loses all of its motion, is also infinite.

Corollary 2.

399. If indeed $m > 1$, then also $2m > 1$, and consequently the formula found expressing the time to travel through AP has been correctly set out. Moreover from that the time is apparent, while the moving body loses the whole motion, is infinite, which is easily seen from this, since the distance too is infinite. (389).

Corollary 3.

400. Since indeed in the case of the body, before it arrived at *A* (Fig. 38), had an infinite speed somewhere at *C*, the time also, from which it reaches *A* from *C*, can be known from the product $k^m + (m-1)c^{m-1}x = 0$ (391). From which product the time results to travel through *CA* equal to :

$$\frac{2k^m}{(2m-1)c^{\frac{2m-1}{2}}}.$$

[p. 168]

Corollary 4.

401. If we had $m < 1$, two cases are to be distinguished in turn, for which m is either greater or less than ½. If indeed $m > ½$, there remains the positive number $2m - 1$, and the time in which the interval *AP* is completed, will be

$$\frac{2k^{\frac{m}{2-2m}}}{(2m-1)(c^{1-m}k^m-(1-m)x)^{\frac{2m-1}{2-2m}}} - \frac{2k^m}{(2m-1)c^{\frac{2m-1}{2}}}.$$

Corollary 5.

402. Since the body loses all the speed of the motion, when it reaches C (Fig. 37), with AC proving to be equal to $\frac{c^{1-m}k^m}{1-m}$ (392), the time is infinite, in which this distance AC is traversed, on account of the denominator $(c^{1-m}k^m - (1-m)x)^{\frac{2m-1}{2-2m}}$ vanishing. This therefore arises if m is contained between 1 and ½ .

Corollary 6.

403. But truly if $m < ½$, the time will be, in which any space AP is traversed,

$$= \frac{2c^{\frac{1-2m}{2}}k^m}{1-2m} - \frac{2k^{\frac{m}{2-2m}}(c^{1-m}k^m-(1-m)x)^{\frac{1-2m}{2-2m}}}{(2m-1)}.$$

From which it is apparent that the time, in which the body from A arrives at C, when all of the motion is lost, to be finite and equal to $\frac{2c^{\frac{1-2m}{2}}k^m}{1-2m}$.

Indeed in this case, $c^{1-m}k^m - (1-m)x = 0$ (392). [p. 169]

Scholium 1.

404. Moreover in these formulas, the case is not contained in which $m = ½$, i. e, if the resistance is in proportion to the speed. Therefore this case has to be deduced from the differential formula of the time. For by putting $m = ½$ there arises

$$\frac{dx}{\sqrt{v}} = \frac{2dx\sqrt{k}}{2\sqrt{ck}-x}.$$

Its integral depends on logarithms, and it equals $2\sqrt{k}l\frac{C}{2\sqrt{ck}-x}$. Indeed the constant C must equal $2\sqrt{ck}$, for which the time disappears when x is made equal to 0. Consequently the time, in which the distance *AP* is transversed, will be $2\sqrt{k}l\frac{2\sqrt{ck}}{2\sqrt{ck}-x}$.

Corollary 7.

405. In this case thus, when $m = ½$, since the distance *AC*, in which the body has to traverse in order to lose all its speed, is $2\sqrt{ck}$ (392), and the time in which this distance is traversed is infinite.

Corollary 8.

406. Therefore from all these gathered together, the time is infinite in which the body loses all its speed if $2m$ is either equal to or greater than one; truly otherwise, if $2m$ is less than one, the whole time of the motion is finite.

Scholium 2.

407. If the force of resistance is changed into one of propulsion, in which case the motion becomes backward and is increased in a similar manner, from which before it was being diminished, the times ought to be the same, which have been defined here. For since the speed of the body at the individual points of the interval *AP* is the same, either carried from *A* to *P* by a retarding motion, [p. 170] or in turn from *P* in *A* with one of acceleration, it is not possible to distinguish the time between each. But yet in these cases, in which the whole interval *AC* is completed in a finite time, this rule does not apply, since a body at rest at *C* experiences no force of propulsion (396). Truly we are always able to have confidence in this rule, if bodies are given a finite speed initially.

PROPOSITION 52.

PROBLEM.

408. *A body that is moving in some medium with resistance, is acted on by some absolute force; to determine the increase or decrease of the speed, while it runs through any element Pp* (Fig. 39).

SOLUTION.

Let the speed of the body at *P* correspond to the altitude v and the element to be traversed be $Pp = dx$. Again let the absolute force or rather the accelerating strength of this at $P = p$ and the exponent of the resistance is equal to q. V designates that function of v, to which the resistance is proportional, and Q is such a function of q, as V is of v. From these put in place the body is slowed down, as it is moved through the element *Pp*, by the strength of the resistance $\frac{V}{Q}$ (383); meanwhile likewise it is accelerated by the absolute force [i.e. acceleration or force per unit mass] p. On account of which the body is accelerated in travelling through the element *Pp* by a strength of force equal to $p - \frac{V}{Q}$.

From which it follows that $dv = pdx - \frac{V}{Q}dx$. Q. E. I. [p. 171]

[As expressed earlier, the reader may wish to consider this in modern terms as a work – energy equation : the work done per unit mass against a constant gravitational field of intensity 1 is equal to the work done per unit mass by the applied external force from which is taken the work done against the frictional force, the constant of proportionality of which is 1/Q, also per unit mass. Euler of course could not be completely familiar with this interpretation, although Daniel Bernoulli used similar expressions in his work, corresponding to the modern idea of potential energy. This was highly convenient for Euler, as he could obtain a differential equation in terms of distances only, and bring in the time later via the known speed differential.]

Corollary 1.

409. If therefore it is the case that $p > \frac{V}{Q}$, then the speed of the body travelling through the element *Pp* is increased; but truly for $p < \frac{V}{Q}$, the speed of this is diminished. And if it is the case that $p = \frac{V}{Q}$, then the speed is neither increased or decreased, but remains unchanged in traversing the element *Pp*.

Corollary 2.

410. If the absolute force were contrary to the motion and retarding it, then $dv = -pdx - \frac{V}{Q}dx$. Therefore in this case the body is retarded by each force.

Scholium 1.

411. If the absolute force pulls the body downwards, according to the solution of the problem we have put in place, and the body is moving up, both the absolute force and the resistance force are in the contrary direction. Therefore this equation is then obtained : $dv = -pdx - \frac{V}{Q}dx$. From which it is apparent that the motion of the ascent is not the same as the motion of the descent, since the strength acting in the ascent is not in the negative ratio of the force acting in the descent. When therefore the ascent is the same as the descent and the motion in both cases has the same speed at the same point, it is required that the strength of the resistance in the ascent is changed into a propelling force. With which done we have the equation : $dv = -pdx + \frac{V}{Q}dx$, from which equation it is evident that the ascending body through *Pp* is to be retarded just as much as it was accelerating in the descent. [p. 172]

Scholium 2.

412. The equation found : $dv = pdx - \frac{V}{Q}dx$ with the sum extended, on account of the defective analysis, can neither be separated or constructed; and because of this the speed of the body at *P* cannot be determined. Therefore a much shorter time, in which the distance *AP* is completed must be assigned. Hence it is necessary to abandon this general equation and to examine particular cases of descents and ascents in which the equation can be separated and the speed defined. There are three ways in which the separation of the equations in the unknowns *x* and *v* can be admitted. The first of these is, if *x* does not have more than one dimension. The second, if *v* only maintains a single dimension. The third case occurs, if *x* and *v* likewise everywhere make a number of the same dimension, or if the equation can be reduced to the aforesaid property.

Corollary 3.

413. In the first case therefore we have, if p and q are constants; for then, since Q is a constant, there is produced $dx = \frac{Qdv}{pQ-V}$, in which the indeterminates [or unknowns] are separated from each other in turn. Besides truly also the equation can be separated , if we put $p = \frac{A}{Q}$. Then indeed the equation becomes : $\frac{dv}{A-Q} = \frac{dx}{Q}$, which, since V depends on v and Q on x , it is possible to construct a solution.

Corollary 4.

414. When v has a single dimension, it is required that $V = v$, in which case also $Q = q$, and [p. 173] the equation of the general form will be changed in this case : $dv = pdx - \frac{vdx}{q}$, which allows the separation of the unknowns.

Corollary 5.

415. From which it is apparent, when a homogeneous equation will soon be formed, let $V = v^{\alpha}$ and $q = x^{\beta}$; then $Q = x^{\alpha\beta}$. Again let $p = x^{\gamma}$, and v is computed with δ dimensions, since x has one dimension. With these put in place, that equation is changed into this : $dv = x^{\gamma}dx - \frac{v^{\alpha}dx}{x^{\alpha\beta}}$, in the second case $\gamma + 1$ dimensions and in the third $\alpha\delta + 1 - \alpha\beta$. Therefore we must have $\delta = \gamma + 1$ and $\gamma + 1 = \alpha\gamma + \alpha + 1 - \alpha\beta$ or $\gamma(\alpha - 1) = \alpha(\beta - 1)$. Therefore as often as we have the ratio $\alpha - 1 : \alpha = \beta - 1 : \gamma$, so also the equation can be reduced to the homogeneous form and therefore that speed determined.

Scholium 3.

416. In place of *V, q,* and *p* no other functions are permitted to be formed except the powers of *v* and of *x* themselves. For, since a power of *x* cannot be present in *V* and neither *q* nor *p* can enter *v,* and above the number of the dimensions of *x* and *v* themselves everywhere must be the same or be able to be reduced to the same, in place of these quantities by necessity the powers ought to be assumed. On this account, I have put $V = v^{\alpha}$, $q = x^{\beta}$ and $p = x^{\gamma}$ and I have extracted the above ratio $\alpha - 1 : \alpha = \beta - 1 : \gamma$. I have neglected certain coefficients, which can safely be added in this reduction, for with these the homogeneity cannot be disturbed. Thus I can put $q = Bx^{\beta}$ and $p = Cx^{\gamma}$, with the same ratio kept. [p. 174] For *x* truly not only the distance traversed *AP* can be substituted, but increased by some other constant, provided the differential of this is *dx* or some multiple of this. But it is not necessary to add the coefficient $V = v^{\alpha}$, since by V only, the ratio of the resistance is indicated.

Corollary 6.

417. If the resisting medium is uniform medium and thus $\beta = 0$, then $\alpha - 1 : \alpha = -1 : \gamma$. Hence $\gamma = \frac{\alpha}{1-\alpha}$. Whereby if the law of the resistance is v^{α} , then the absolute force must be equal to $Bx^{\frac{\alpha}{1-\alpha}}$, where the equation defining the speed can be reduced to the homogeneous case.

Scholium 4.

418. Thus we are about to explore these rectilinear motions in resistive mediums, as first we are to put in place an absolute constant force, and then we progress to several centripetal forces. And with these explained we contemplate inverse questions, as we did in the preceding chapter, and from the given properties of the motion so we elicit the absolute force and then the resistive force. [p. 175]

PROPOSITION 53.

PROBLEM.

419. *With a uniform absolute force and medium resistance put in place, to determine the speed of descent of the body at individual points, if the resistance is proportional to the square of the speed.* [p. 175]

SOLUTION.

Keeping as before $AP = x$ (Fig. 39), with the speed at $P = \sqrt{v}$, let the uniform force be equal to g, and the exponent of the resistance equal to k. Since the resistance law is v, the strength of the resistance is equal to $\frac{v}{k}$. From which it follows that

$dv = gdx - \frac{vdx}{k}$ or $dx = \frac{kdv}{gk-v}$; the integral of which is $x = kl\frac{C}{gk-v}$. Let the initial speed at A corresponding to the height c, the constant $C = gk - c$ and $x = kl\frac{gk-c}{gk-v}$. In place of the logarithms numbers are taken : $e^{\frac{x}{k}} = \frac{gk-c}{gk-v}$; from which the equation arises :

$e^{\frac{x}{k}}v = c + gk(e^{\frac{x}{k}}v - 1)$ or $v = e^{-\frac{x}{k}}c + gk(1 - e^{-\frac{x}{k}}) = e^{-\frac{x}{k}}(c - gk) + gk.$ Q.E.I.

Corollary 1.

420. If the motion of the body starts from rest at A, then $c = 0$. In this case we have

$v = gk(1 - e^{-\frac{x}{k}})$, which expression, when a greater x is taken, is also made larger, yet a certain limit can never be exceeded. For by taking $x = \infty$ then it is found that $v = gk$.

Therefore $\sqrt{gk}$ is the asymptotic speed of descent of the body, that it had not acquired before, as it was required to fall an infinite distance.

Corollary 2.

421. If the initial speed $\sqrt{c}$ is equal to this asymptotic value $\sqrt{gk}$, the motion of the descending body is uniform; indeed it shall be $v = gk = c$. It is apparent also from this equation $dv = gdx - \frac{vdx}{k}$. [p. 176] For if once $v = gk$, the increments in the speed is always zero.

Corollary 3.

422. If it is the case that $c < gk$, the falling body will accelerate, as long as it has not yet fallen far enough to acquire the speed $\sqrt{gk}$ unless it has traversed an infinite distance. For if $c < gk$, the quantity $e^{-\frac{x}{k}}(c - gk) + gk$ is always negative, and on account of this v is always less than gk.

Corollary 4.

423. If the initial speed $\sqrt{c}$ is greater than $\sqrt{gk}$, then $e^{-\frac{x}{k}}(c - gk)$ is a positive quantity, and thus v is always greater than gk. Indeed by traversing an infinite distance it becomes $v = gk$. From which it is evident that in this case the body falls with a retarding motion. e.

Scholion 1.

424. It must be understood in the equation $v = e^{-\frac{x}{k}}(c - gk) + gk$ also in the case, in which the body falls in a vacuum only under the action of an absolute force. And indeed here the resistance is put to vanish or the exponent k is made infinite ; then indeed the acceleration associated with the resistance $\frac{v}{k}$ vanishes. But it is seen that there is a difficulty with the determination, as to what the value for the altitude v should be for $k = \infty$. In order that this can be found it is most useful to change $e^{-\frac{x}{k}}$ into an equivalent series :
$1 - \frac{x}{k} + \frac{x^2}{2k^2} - \frac{x^3}{6k^3} + \text{etc.}$, [p. 177] of which, if k is ∞, in its place it suffices to put $1 - \frac{x}{k}$. With this value put in place of $e^{-\frac{x}{k}}$ we have $v = c - \frac{cx}{k} + gx = c + gx$ on account of the vanishing of the term $\frac{cx}{k}$. Which equation agrees with that, as we found above (239) ; for that everywhere is $\frac{g}{A}$, for us here it is only g. Since g is not only the absolute force, but shows the acceleration of this force.

[Thus, if the force acts on unit mass, then the variables for the force and the acceleration have the same magnitude. As we have mentioned previously, the Latin *vi* can be a general term for force, strength, or power, and can also be used to express acceleration as the force on a unit mass. The use of the word *potentia* is usually more explicit, and requires the force to be defined by some formula. This is contrary to accepted wisdom, which regards the words *vi* and *potentia* to relate to absolute and relative forces, at least as defined by D. Speiser in his analysis of the works of Daniel Bernoulli. However, this does not seem to be the case for Euler's works, who seems occasionally to use the words as synonyms, and at other times as we have said here. In the final analysis you the reader must make up your own mind as to what Euler is saying; any comments made by me are my own views at the time, which can change as more of the work is examined.]

Scholium 2.

425. If k does not indeed have an infinite value, but yet is extremely large, thus as it happens, when extremely heavy bodies are held and released in a fluid, the use of the above series is outstanding, with as many as the three first terms taken of the above series $1-\frac{x}{k}+\frac{x^2}{2k^2}-\frac{x^3}{6k^3}$ in place of $e^{-\frac{x}{k}}$: indeed the error is negligible. Therefore in this case, if the body is released from rest, thus as $c = 0$. then the height $v = gx-\frac{gx^2}{2k}$. From which equation an approximate value of v is extracted. But if again we do not wish to ignore anything, then the height is given by : $v = gx-\frac{gx^2}{2k}+\frac{gx^3}{6k^2}-\frac{gx^4}{24k^3}+\frac{gx^5}{120k^4}-\text{etc.},$ from which infinite series the value of v is expressed.

PROPOSITION 54.

PROBLEM.

426. *To determine the time, in which a body in a medium with a uniform resistance present in proportion to the square of the speed, descends through a given interval AP, acted on by a uniform absolute force* (Fig. 39) . [p. 178]

SOLUTION.

As above by placing $AP = x$, with the speed at $A = \sqrt{c}$, and the speed at $P = \sqrt{v}$, and with the exponent of the resistance k, the corresponding distance is given by :

$v = gk+e^{-\frac{x}{k}}(c-gk)$ (419). From which the element of the time is :

$$\frac{dx}{\sqrt{v}} = \frac{dx}{\sqrt{(gk+e^{-\frac{x}{k}}(c-gk))}}.$$

In order to integrate this expression I put $e^{-\frac{x}{k}} = z$ and $c - gk = b$, and for the sake of brevity; this becomes $\frac{dx}{\sqrt{v}} = \frac{-kdz}{z\sqrt{(gk+bz)}}$. Again by making $gk + bz = r^2$, then $z = \frac{r^2 - gk}{b}$ and

$$\frac{dx}{\sqrt{v}} = \frac{-2kdr}{r^2 - gk} = \frac{dr\sqrt{\frac{k}{g}}}{r+\sqrt{gk}} - \frac{dr\sqrt{\frac{k}{g}}}{r-\sqrt{gk}}.$$

On account of which :

$$\int \frac{dx}{\sqrt{v}} = C + \sqrt{\frac{k}{g}}\, l\frac{r+\sqrt{gk}}{r-\sqrt{gk}}.$$

In place of r, its value is restored: $\sqrt{gk + e^{-\frac{x}{k}}(c - gk)}$, and the time is obtained to travel through the interval AP :

$$= C + \sqrt{\frac{k}{g}}\, l\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}-\sqrt{gk}}.$$

Because the time vanishes on making $x = 0$, the constant must be

$C = -\sqrt{\frac{k}{g}}\, l\frac{\sqrt{c}+\sqrt{gk}}{\sqrt{c}-\sqrt{gk}}$. From these the time to descend through AP is put together

$$= \sqrt{\frac{k}{g}}\, l\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}-\sqrt{gk}} - \sqrt{\frac{k}{g}}\, l\frac{\sqrt{c}+\sqrt{gk}}{\sqrt{c}-\sqrt{gk}}.$$

Which expression can in turn be simplified in order to give :

$$\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}\, l\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{c}+\sqrt{gk}},$$

and to which the descent time through AP is equal. Q.E.I. [p. 179]

Corollary 1.

427. If the initial speed is equal to 0 and thus $c = 0$, then the time in which the body descends through AP, $\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}\, l(\sqrt{(1-e^{-\frac{x}{k}})}+1) = \frac{x}{\sqrt{gk}} - 2\sqrt{\frac{k}{g}}\, l\frac{1-\sqrt{(1-e^{-\frac{x}{k}})}}{e^{-\frac{x}{k}}}$

$$= 2\sqrt{\frac{k}{g}}\, l\frac{1}{\sqrt{e^{\frac{x}{k}}}-\sqrt{(e^{\frac{x}{k}}-1)}} = 2\sqrt{\frac{k}{g}}\, l\sqrt{e^{\frac{x}{k}}} + \sqrt{(e^{\frac{x}{k}}-1)}.$$

Scholium 1.

428. Also the formula for the general time to travel through AP with the initial speed $\sqrt{c}$ can be changed with $2\sqrt{\frac{k}{g}}le^{\frac{x}{2k}}$ in place of $\frac{x}{\sqrt{gk}}$ into this :

$$2\sqrt{\frac{k}{g}}l\frac{\sqrt{(e^{\frac{x}{k}}gk+c-gk)}+\sqrt{e^{\frac{x}{k}}gk}}{\sqrt{c}+\sqrt{gk}}.$$

From which this, as in the manner we found for the case $c = 0$, freely follows; giving indeed :

$$2\sqrt{\frac{k}{g}}l\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}.$$

Corollary 2.

429. If we put $c = gk$, in which case the motion in descend is uniform (421), the time of descent through the interval *AP* is produced in this final form:

$$2\sqrt{\frac{k}{g}}l\sqrt{e^{\frac{x}{k}}}=\frac{x}{\sqrt{gk}},$$

as is also found from the nature of the uniform motion too. For the time must be expressed from the distance x travelled through divided by the speed divided by the speed $\sqrt{gk}$.

Scholium 2.

430. Moreover these times can be obtained in minutes and seconds, if the expressions found are divided by [p. 180] 250 and the lines c, k, x are shown in scruples of Rhenish feet. (222)

Scholium 3.

431. With the initial speed put as $\sqrt{c} = 0$, if the time is given, in which the interval *AP* is traversed in the descent, the distance *AP* can also be found. For if the time t is given in seconds and both k and x are given in scruples of Rhenish feet, then the time is :

$$t=\frac{1}{125}\sqrt{\frac{k}{g}}l\sqrt{e^{\frac{x}{k}}}+\sqrt{(e^{\frac{x}{k}}-1)}.$$

Scholion 4.

432. If k is an exceedingly large quantity and yet the time is to be found as closely as possible for the distance AP, with the initial speed being $\sqrt{c} = 0$, I assume this formula :

$$\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}\, l(\sqrt{(1-e^{-\frac{x}{k}})} + 1).$$

In which, since $\sqrt{(1-e^{-\frac{x}{k}})}$ nearly disappears, if k is very large, the time becomes :

$$l(1+\sqrt{(1-e^{-\frac{x}{k}})}) = \sqrt{(1-e^{-\frac{x}{k}})} - \frac{1-e^{-\frac{x}{k}}}{2} + \frac{(1-e^{-\frac{x}{k}})^{\frac{3}{2}}}{3} - \frac{(1-e^{-\frac{x}{k}})^2}{4} + \frac{(1-e^{-\frac{x}{k}})^{\frac{5}{2}}}{5} \quad \text{[p. 181]}$$

Indeed it is the closest :

$$\sqrt{(1-e^{-\frac{x}{k}})} = \frac{\sqrt{x}}{\sqrt{k}} - \frac{x\sqrt{x}}{4k\sqrt{k}} + \frac{5x^2\sqrt{x}}{96k^2\sqrt{k}}.$$

From which it arises :

$$l(1+\sqrt{(1-e^{-\frac{x}{k}})}) = \frac{\sqrt{x}}{\sqrt{k}} - \frac{x}{2k} + \frac{x\sqrt{x}}{12k\sqrt{k}} + \frac{x^2\sqrt{x}}{480k^2\sqrt{k}}.$$

On account of which the descent time to pass through the interval AP

$$= \frac{2\sqrt{x}}{\sqrt{g}} + \frac{x\sqrt{x}}{6k\sqrt{g}} + \frac{x^2\sqrt{x}}{240k^2\sqrt{g}}.$$

Corollary 3.

433. When the resistance also vanishes, and thus k becomes equal to $\propto$, the descent time for the body to pass through AP by the absolute force g acting alone. Truly the descent time from this last formula, on account of all the terms vanishing except the first, is found to equal $\frac{2\sqrt{x}}{\sqrt{g}}$, as now has been found above (218), by ignoring the number m and with g put in place of $\frac{g}{A}$, as we have established here.

PROPOSITION 55.

PROBLEM.

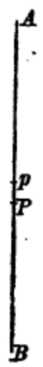
Fig. 40.

434. *If a body in a medium with uniform resistance, which resists in the square ratio, is projected up again from B with a given speed, and is acted upon by a given uniform force g,* (Fig. 40)*, it is required to determine the speed of the body at individual places.*

SOLUTION.

Let the speed at $B = \sqrt{c}$ and the speed at $P = \sqrt{v}$, put $BP = x$ and the exponent of the resistance equal to k. [p. 182] Since now the motion is retarded by the absolute force g and by the resistance $\frac{v}{k}$, then $dv = -gdx - \frac{vdx}{k}$. Hence $dx = \frac{-kdv}{gk+v}$ and $x = kl\frac{C}{gk+v}$, where it is necessary that $C = gk + c,$ when $v = c$ by making $x = 0$. On account of which we have $x = kl\frac{gk+c}{gk+v}$; from which

$$v = e^{-\frac{x}{k}}(c + gk) - gk,$$

from which the speed can be found at any point P. Q.E.I.

Corollary 1.

435. In this way the body can reach as far as A on being projected up, and let the speed at $A = 0$. On which account the total height BA can be found by setting $v = 0$, in which case it becomes :

$e^{-\frac{x}{k}} = \frac{gk}{c+gk}$ or $x = kl\frac{c+gk}{gk}$, which quantity is equal to the height BA.

Corollary 2.

436. The resistance can vanish or by making $k = \infty$, in order that the motion in a vacuum is given by : $e^{-\frac{x}{k}} = 1 - \frac{x}{k}$. Therefore in this case, $v = c - gx.$ And the same equation is found, if the body is considered to be acted on by the absolute force g alone.

Corollary 3.

437. If the resisting medium is very rare, so that thus the number k indicates a very large number, in place of $e^{-\frac{x}{k}}$, $1 - \frac{x}{k} + \frac{x^2}{2k^2} - \frac{x^3}{6k^3}$ is taken . From which arises

$v = c - \frac{cx}{k} + \frac{cx^2}{2k^2} - gx + \frac{gx^2}{2k} - \frac{gx^3}{6k^2}$ as an approximation.

Moreover when the value of this quantity is to be shown as an approximation of *v*, [p. 183] it is not only necessary that the number k is very large, but the height *x* is much smaller than *k*, in which case $e^{-\frac{x}{k}}$ is not very different from 1.

Scholium 1.

438. Now we turn our attention to the case where the descent of the body through AB is not similar to the ascent, if the medium in each case can be resisting. Yet the descent along AB can be conceived by thinking about the case which is similar in a straight forwards way to the ascent, thus as the body in the ascent so in the descent at the point *P* has the same speed and the medium is one that propels [rather than retards; a purely hypothetical case obviously for mechanical systems, though energy can be removed and then restored in electrical systems if we are talking about electric currents]. For since in the ascent it is necessary that both the forces oppose the motion, as in the descent, each force is set up following the motion, from which the motion is completely backwards.

Corollary 4.

439. Therefore with the height AP = y and with the speed in this descent equal to $\sqrt{v}$, then $dv = gdy + \frac{vdy}{k}$. From which by integration is produced $v = gk(e^{\frac{y}{k}} - 1)$.

Scholium 2.

440. This equation agrees with the previous, as we elicited on contemplating the ascent. For it is :

$y = AB\text{-}x = kl\frac{c+gk}{gk} - x$ and thus $e^{\frac{x}{k}} = \frac{c+gk}{gk}.e^{-\frac{x}{k}}$.

Therefore from this there arises : $v = e^{-\frac{x}{k}}(c + gk) - gk$, as [p. 184] we found above in the solution of the problem. Therefore it is apparent in this descent that the body has the same speeds at the individual points that it had at the same point in the ascent. Therefore likewise it is also necessary that the time of descent can be found to be considered in this method, as came about from the ascent.

PROPOSITION 56.

PROBLEM.

441. *To determine the time of ascent through BP* (Fig. 40) *of a body in a medium with resistance in the ratio of the square of the speed from B to be projected up with a given speed, and meanwhile to be acted on by an absolute force g pulling downwards.*

SOLUTION.

With the speed at B equal to $\sqrt{c}$ and at P equal to $\sqrt{v}$, then put $BP = x$ and with the exponent of the resistance equal to k, then the distance $v = e^{-\frac{x}{k}}(c + gk) - gk$ (434). From which the element of the time arises :

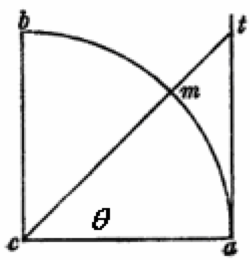

Fig. 41.

$$\frac{dx}{\sqrt{v}} = \frac{dx}{\sqrt{(e^{-\frac{x}{k}}(c+gk)-gk)}}.$$

In order that this can be integrated, I put as above : $e^{-\frac{x}{k}} = z$ and $c + gk = b$, for the sake of brevity, from which completed we have : $\frac{dx}{\sqrt{v}} = \frac{-kdz}{z\sqrt{(bz-gk)}}$. Let $bz - gk = r^2$, then $z = \frac{r^2+gk}{b}$ and $\frac{dx}{\sqrt{v}} = \frac{-2kdr}{r^2+gk}$, the integration of which depends on the quadrature of the circle. Therefore to this end, it is necessary to construct the quadrant of a circle abc (Fig. 41), the radius of which ac is equal to 1, the tangent line is taken to that radius : $at = \frac{r}{\sqrt{gk}}$, and the arc length $am = \int\frac{dr\sqrt{gk}}{r^2+gk}$, which can be designated in this manner $A\frac{r}{\sqrt{gk}}$.

[For if $t = \tan\theta$, then $d\theta = \frac{dt}{1+t^2}$, from which the result follows by substitution; θ has been added to the diagram][p. 185] Clearly an expression of the kind $A.t$ denotes the arc of the circle for us, the tangent of which is t, with the radius being 1. Hence because of this :

$$\int\frac{-2kdr}{r^2+gk} = -2\sqrt{\frac{k}{g}}\int\frac{dr\sqrt{gk}}{r^2+gk} = C - 2\sqrt{\frac{k}{g}}A.\frac{r}{\sqrt{gk}}.$$

Moreover since $r = \sqrt{(e^{-\frac{x}{k}}(c+gk)-gk)}$, then

$$\int\frac{dx}{\sqrt{v}} = C - 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}.$$

Which quantity since it must vanish by making $x = 0$, gives $C = 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}}$.

Therefore the time of the ascent through the interval *BP* is found :

$$= 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}} - 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}$$

$$= 2\sqrt{\frac{k}{g}}\left(A.\sqrt{\frac{c}{gk}} - A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}\right)$$

$$= 2\sqrt{\frac{k}{g}}A.\frac{\sqrt{cgk}-\sqrt{(gke^{-\frac{x}{k}}(c+gk)-g^2k^2)}}{gk+\sqrt{(ce^{-\frac{x}{k}}(c+gk)-cgk)}}.$$

Q.E.I.

Corollary 1.

442. Since the total height AB, to which the body is able to reach, is obtained by making $x = kl\frac{c+gk}{gk}$, in which case $e^{-\frac{x}{k}} = \frac{gk}{c+gk}$, and the total time for the ascent is equal to $2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}}$

Corollary 2.

443. Whereby if $c = gk$, the total time to ascend through $BA = 2\sqrt{\frac{k}{g}}A.1$.

But the arc, of which the tangent is equal to the radius, is the 8th part of the periphery. With the quarter part of the periphery $amb = \pi$, the ascent time through $BA = = \pi\sqrt{\frac{k}{g}}$. [p. 186; note that π was not yet established in its present meaning, as the ratio of the circumference to the diameter of a circle; in which case Euler's formula needs to be divided by 2.]

Corollary 3.

444. From this it is also understood, that if the body is projected up with an infinite speed from B, the ascent time is to be no less finite ; indeed it becomes equal to $2\sqrt{\frac{k}{g}}A.\propto$; which when the quarter part of the periphery is equal to π, then the total ascent time is $2\pi\sqrt{\frac{k}{g}}$.

Corollary 4.

445. If in place of the initial speed $\sqrt{c}$ the total height is given $BA = a$, to which the body reaches, since it is $a = kl\frac{c+gk}{gk}$ and therefore $c = gk(e^{\frac{a}{k}} - 1)$, the total time is found for the ascent through *BA,* $= 2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{a}{k}} - 1)}$.

Scholium 1.

446. if the ascent in the medium is changed into a descent, as we have assumed above (438), and by calling $AP = y$, the time of descent through AP

$$= 2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{y}{k}} - 1)},$$

with y substituted in place of a in the above formula. Here indeed a denotes the height traversed in the ascent, this indeed is the height y from integration, to which the body with the speed that it has at P is able to reach.

Scholium 2.

447. The fundamental equation for a descent of this kind to be considered is $dv = gdy + \frac{vdy}{k}$ (439), which from the equation for a true descent is $dv = gdy - \frac{vdy}{k}$ (419) which can be formed with y in place of x and $-k$ in place of k. Whereby also the approximate expression for the time truly [p. 187] for the interval AP from that, as we found above (432) for the true descent, can be adapted to this imaginary descent the inverse of the ascent, also by putting y in place of x and $-k$ in place of k. And thus in this way the time can be found for the descent through the interval AP, equal to $\frac{2\sqrt{y}}{\sqrt{g}} - \frac{y\sqrt{y}}{6k\sqrt{g}} + \frac{y^2\sqrt{y}}{240k^2\sqrt{g}}$, as an approximation, provided $\frac{y}{k}$ is a number smaller than one.

Corollary 5.

448. If the resistance of the medium completely disappears, as happens if $k = \infty$, then the total time of the ascent through $BA = \frac{2\sqrt{a}}{\sqrt{g}}$. Which expression comes about from above with a in place of y : $\frac{2\sqrt{y}}{\sqrt{g}} - \frac{y\sqrt{y}}{6k\sqrt{g}} + \frac{y^2\sqrt{y}}{240k^2\sqrt{g}}$; indeed all the terms except the first vanish.

Corollary 6.

449. Hence also from the given time of the ascent t from the integration the height traversed a can be found. For since the time is given by $t = 2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{a}{k}} - 1)}$, the tangent of the arc $\frac{t}{2}\sqrt{\frac{g}{k}} = \sqrt{(e^{\frac{a}{k}} - 1)}$. That tangent is called T, it satisfies the equation $T^2 + 1 = e^{\frac{a}{k}}$ and $a = kl(T^2 + 1)$.

CAPUT QUARTUM

DE MOTU RECTILINEO PUNCTI LIBERI IN MEDIO RESISTENTE

[p. 153]

DEFINITIO 18

367. *Lex resistaniae est potestas seu functio celeritatis corporis, cui ipsa resistentia est proportionalis.* Sic si resistentia est celeritatis quadrato proportionalis, lex resistenctiae est celeritatis quadratum.

Corollarium 1.

368. Cognoscitur igitur ex lege resistentiae, si plura puncta aequalia diversis ferantur celeritatibus, quomodo se habeant motus diminutiones inter se. Atque dato celeritatis decremento unius puncti reliquorum quoque celeritatis decrementa inveniuntur.

Corollarium 2.

369. Si ergo pro uno celeritatis gradu datur ratio resistentiae ad vim gravitatis, pro omnibus aliis quoque gradibus ratio inter resistentiam et vim gravitatis ex lege resistentiae innotescet. Atque ex hoc effectus resistentiae in corpus motum invenietur.

[p. 154]

Scholion 1.

370. Pertinet utique vis resistentiae ad potentias atque ideo cum vi gravitatis est homogenea, quemadmodum, cum de motu corporum in fluidis tractabitur, apparebit. Semper igitur potentia absoluta poterit assignari eundem in corpore effectum, quem resistentia, prodens. Haec vero potentia absoluta pendebit a celeritate corporis, quam ob rem in eius expressione celeritas inerit seu altitudo celeritati debita. Hoc igitur modo corporis motus in medio resistente reducetur ad corporis motum a potentiss absolutis sollicitatum, cuius cum supra in capite secundo leges sint expositae, ex iis omnes quaestiones poterunt resolvi.

Scholion 2.

371. Directio vis resistentiae in hac tractatione nobis semper erit congruens cum directione motus corporis (117) et contraria. Quamobrem potentia absoluta ei substituenda motum semper retardabit, directione motus non mutata. Perspicuum itaque est vim resistentiae, quoties eius expressio prodit negativa, habituram directionem contrariam motumque corporis esse acceleraturam. Hic quidem casus in fluidis quiescentibus locum habere nequit, sed tamen in calculo, cum ex dato corporis motu resistentia investigabitur, saepe occurret.

[p. 155]

Corollarium 3.

372. Corpus igitur in medio resistente motum, si a nulla alia potentia sollicitetur, in linea recta moveri debeat. Quia enim a vi resistente directio motus non mutatur, eius motus, quem a natura in linea recta prosequitur, perpetuo in eadem recta fiat necesse est.

Corollarium 4.

373. Si praeteria accedit potentia absoluta, cuius directio perpetuo cum directione motus congruit, corpus quoque in medio resistente in linea recta progredietur. Neque enim potentia haec absoluta neque vis resistentiae directionem motus immutabit.

Scholion 3.

374. In hoc igitur capite, in quo motus tantum rectilineos exponere constituimus, alias potentias absolutas cum vi resistentiae non coniungemus, nisi quarum directio cum motus directione convenit. Hanc ob rem omnes potentias, quas in capite praecedente adhibuimus, etiam in hoc capite cum vi resistentiae coniunctas considerare licebit. Antequam autem potentias absolutas inducemus, convenit motum corporum a sola resisentiae vi impeditum examini subiicere, quo facilius a simplicioribus ad magis composita progrediamur.

[p. 156]

Scholion 4.

375. In legis resistentiae expressione seu illa celeritatis functione praeter altitudinem celeritati debitam v inesse possunt quantitates constantes, set excludimus omnes quantitates variabiles a loco corporis pendentes. Fieri quidem potest, ut resistentia, quam corpus aequali celeritate latum patitur, maior minorve sit, prout corpus in alium atque alium locum perveniat; quemadmodum evenit, quando fluidum, in quo corpus movetur, in alio loco est densius, in alis vero rarius, quo casu in resistentiae expressione loci rationem haberi oportet. Neque tamen in lege resistentiae locum respici convenit, nam per eam resistentiae rationem, quando corpus in eodem loco variis celeritatibus moveri ponitur, exprimere volumus. Discremen vero, quod ex loci varietate oriri potest, in exponente resistentiae comprehendemus, quo simul resistentiae intensitas indicatur.

DEFINITIO 19

376. *Exponens resistentiae est altitudo debita celeritati ei, quam si corpus habet, resistentiam patitur aequalem vi gravitatis.* Hac scilicet celeritate motum corpus tantum a resistantia retardatur, quantum a vi gravitatis sursum proiectum.

Corollarium 1.

377. Si igitur corpus in medio resistente motum celeritatem habeat altitudini v debitam atque haec altitudo v sit ipsi exponenti resistentiae aequalis, erit, [p. 157] dum corpus per spatiolum dx progreditur, $dv = -dx$; quia potentia resistentiae aequivalens hoc casu aequalis vi gravitas, quam semper ponimus = 1, et motum retardat.

Corollarium 2.

378. Datis ergo lege et exponente resistentiae motus diminutio potest difiniri. Namque ex exponente intelligitur, quantam corpus habere deberet celeritatem, ut resistentiae vis aequalis esset gravitati, ex ex lege resistentiae cognoscitur ratio, secundum quam diversae celeritates a resistentia deminuuntur.

Scholion.

379. Exponens resistentiae est vel constans vel variabilis seu a loco, in quo est corpus, pendens. Illud accidit in medio seu fluido uniformi, quod corporibus ubique eandem resistentiam infert, si quidem eadem ubique moveantur celeritate. Huiusmodi medium resistens appellabimus uniforme, quippe quod in omnibus locis sui est simile. Exponens autem resistentiae variabilis est in medio seu fluido difformi, etiamsi in quoque loco seorsim resistentia eandem teneat legem. Nam quo densius est fluidum seu medium, in quo corpus versatur, eo quoque maiorem patitur corpus resistentiam aequali etiam motum celeritate. Maior scilicet erit celeritas resistentiam gravitati aequalem patiens in fluido rariore, minor vero in densiore. Quia autem densitas [p. 158] et raritas medii a loco pendet, per spicuum est resistentiae exponentem, si est variabilis, a loco corporis pendere debere.

DEFINITIO 20

380. *Media resistentia similia hic vocantur, quae eandem habent resistentiae legem. Dissimilia vero, quae resistentiae lege differunt.* Sic aqua et mercurius sunt hiusmodi media similia, siquidem ambo haec fluida resistunt, uti videntur, in duplicata celeritatum ratione.

Corollarium.

381. Si ergo media resistentia similia inter se differunt, tota differentia consistit in exponente resistestiae seu densitate et raretate. Sic in aqua exponens resistentiae est maior quam in argento vivo, quia hoc est fluidum densius illo.

Scholion.

382. Media similia cum corporibus aequaliter celeribus diversas facere queant resistentias, prout eorum densitates inter se differunt, has ipsas densitates ex resistentia, quam corpori data celeritate moto inferunt, metiri convenit. In fluidis enim, ut, cum de motu corporum in fluido agitur, docetur, resistentiae aequalibus celeritatibus sunt densitatibus fluidorum proportionales. Hancque proprietatem ad alia media quamcunque resistentiae legem tenentia transferimus : quia aliae resistentiae [p. 159] leges praeter duplicatam celeritatum rationem mere sunt imaginariae et ad analysin tantum exercendam adhiberi solent.

PROPOSITIO 49.

PROBLEMA.

383. *Corporis in recta AP (Fig. 37) moti in medio quocunque resistente, cuius et lex et exponens resistentiae sunt cognita, data celeritate in puncto P, invenire celeritatis decrementum, dum spatii elementum Pp percurrit.*

SOLUTIO.

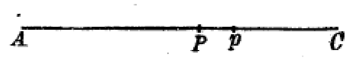

Fig. 37.

Posito elemento $Pp = dx$ sit altitudo celeritati in P debita $= v$ et exponens resistentiae $= q$. Denotat ergo $\sqrt{q}$ celeritatem, quam si corpus in P haberet, vis resistentiae aequalis foret vi gravitatis = 1. Quamobrem, si esset $v = q$, tum haberetur vis resistentiae = 1 atque $dv = - dx$ (376, 377). Sit autem V ea celeritatis $\sqrt{v}$ functio, qua resistentiae lex exprimitur, atque designet Q similem functionem ipsius $\sqrt{q}$, seu Q est huiusmodi quantitas, quae prodit, si in V loco v substituitur q. Resistentia ergo, quam patitur corpus celeritate $\sqrt{q}$ motum, quae est = 1, se habetit ad resistentiam corporis celeritate $\sqrt{v}$ latum patitur, $= \frac{V}{Q}$. Quae cum motum retardet, erit $dv = \frac{-Vdx}{Q}$. Q. E. I. [p. 160]

Corollarium 1.

384. Quia quantitas V est functio ipsius v et constantium atque q ideoque et Q vel est constans vel functio quaedam ipsius x (375), aequatio inventa $dv = \frac{-Vdx}{Q}$ sponte separatur. Habetur enim $\frac{dv}{V} = \frac{-dx}{Q}$, ex qua integrata vel saltem constructa totus corporis motus per AP cognoscitur.

Corollarium 2.

385. Cum vis resistantiae sit $= \frac{V}{Q}$, poterit ex hac medii densitas cognosci. Quia enim densitatem metimur ex resistentia, quam corpus data celeritate motum patitur, oportebit in $\frac{V}{Q}$ loco v substiture quantitatem constantem, quo facto habebitur resistentia ut $\frac{1}{Q}$. Densitas igitur medii quoque erit ut $\frac{1}{Q}$ seu reciproce ut Q.

Scholion.

386. Denotat hic $\frac{V}{Q}$ non tantum potentiam sollicitantem, set iam ipsam vim retardatricem resisitentiae, et hanc ob rem non opus est massam corporis in calculum inducere. Ceterum hic corporis massam constantem seu plurium corporum massas inter se aequales ponimus. Non enim consultum esse iudico hanc tractationem, quae non nisi in unico casu in usum venire potest, praeter necessitatem extendere et magis complicatam reddere. [p. 161]

PROPOSITIO 50.

PROBLEMA.

387. *In medio resistenti uniformi, quod resistit in ratione quacunque multiplicata celeritatum, definire corporis moti celeritatem in singulis locis.*

SOLUTIO.

Moveatur corpus in recta *AP* (Fig. 37), sitque celeritas eius in puncto *A* debita altitudini *c*. Ponatur spatium percursum $AP = x$ et altitudo celeritati in *P* debita = *v*. Exponens resistentiae, qui est constans, vocetur = *k*, et lex resistentiae sit v^m, ita ut resistentia ubique sit ut celeritatis potestas exponentis $2m$. In praecedente igitur formula $dv = -\frac{Vdx}{Q}$ abit hoc casu *V* in v^m, et *Q*, quia talis esse debet functio *q* seu *k*, qualis est v^m ipsius v, erit = k^m. Habemus ergo hanc aequationem $dv = -\frac{v^m dx}{k^m}$ seu $\frac{dv}{v^m} = -\frac{dx}{k^m}$. Cuius integralis est $\frac{v^{1-m}}{1-m} = C - \frac{x}{k^m}$. Constans vero *C* ex hoc determinabitur, quod facto $x = 0$ transmutari debeat *v* in *c*, quamodrem erit $C = \frac{c^{1-m}}{1-m}$. Hinc itaque resultabit aequatio ista

$$v^{1-m} = c^{1-m} - \frac{(1-m)x}{k^m} \text{ seu } v = \sqrt[1-m]{\frac{c^{1-m}k^m(1-m)x}{k^m}},$$

si est $m < 1$. At se erit $m > 1$, habebitur

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m + (m-1)c^{m-1}x}}.$$

Unicus vero casus, [p. 162]
quo est $m = 1$, in his formulis non comprehenditur, sed derivari debet ex aequatione differentiali, quae facto $m = 1$ erit huiusmodi $\frac{dv}{v} = -\frac{dx}{k}$,cuiis integralis est $lv = C - \frac{x}{k}$. Simili vero modo erit $C = lc$ ideoque $lv = lc - \frac{x}{k}$. Logarithmis ad numeros reductis habebitur ergo $v = ce^{-\frac{x}{k}}$.

Quemcunque igitur *m* habeat valorem, corporis celeritas in quovis loco rectae *AP* innotescit. Q. E. I.

Corollarium 1.

388. Si resistentia medii est quadratis celeritatum proportionalis, erit $m = 1$. Quamobrem pro hoc casu, qui solus in rerum natura existere putatur, valet singularis solutionis casus $v = ce^{-\frac{x}{k}}$. Ex quo apparet corpus celeritatem ante non amittere totam, quam spatium infinitum *x* percurrit.

Corollarium 2.

389. Si medium in maiore quam duplicata celeritatum ratione resistit, erit $m > 1$ atque

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m + (m-1)c^{m-1}x}}.$$

Perspicitur autem ex hac aequatione celeritatem non evanescere, nisi ponatur $x = \infty$.

Corollarium 3.

390. Hoc vero differt iste casus a priore, quo erat $m = 1$, quod in illo, si fuerit celeritas initialis infinite magna, prodeat ubique ea tanta. Hoc vero casu, [p. 163] quo est $m > 1$, si ponatur $c = \infty$, provenit $v = \sqrt[m-1]{\frac{k^m}{(m-1)x}}$. Erit ergo v semper finitae magnitudinis, nisi sit $x = 0$ vel $= \infty$.

Corollarium 4.

391. Praetera hoc casu, quo $m > 1$, corpus, antequam pervenit in punctum A, semper alicubi, puta in C (Fig. 38), habuit celeritatem infinite magnam. Ad hoc punctum C inveniendum, fieri debet x negativum et $k^m + (m-1)c^{m-1}x$ poni aequale nihilo. Unde invenitur $AC = \frac{k^m}{(m-1)c^{m-1}}$.

Fig. 38.

Corollarium 5.

392. Si resistentia fit in minore quam duplicata ratione celeritatum ideoque est $m < 1$, erit $v = \sqrt[1-m]{\frac{c^{1-m}k^m - (1-m)x}{k^m}}$. Celeritas corporis ergo evanescit in puncto C (Fig. 37), si accipiatur $AC = \frac{c^{1-m}k^m}{1-m}$. Consequenter, cum corpus in C pervenerit, ibi perpetuo quiescet neque ultra progredietur.

Corollarium 6.

393. Si ponatur $m = 0$, erit resistantia constans et aequaliter aget in corpus sive quiescens sive motum. Abit ergo hoc casu resistentia in potentiam absolutam et aequalem vi gravitatis. Nam quoniam, si est $v = k$, resistantia aequalis ponitur vi gravitatis, etiam quacunque alia celeritate corpus moveatur, tantundem resistentiae patietur. [p. 164]

Scholion 1.

394. Exposuimus in his corollaries primarias motuum differentias, si fuerit vel $m = 1$ vel maior vel minor unitate. Hae vero differentiae brevibus hisce canonibus comprehendi possunt : si est $m = 1$, corpus per totum spatium nusquam neque celeritatem infinitam neque nullam habebit. Deinde si $m > 1$, corpus alicubi habeat celeritatem infinitam necesse est, evanescentem vero nusquam. Denique si $m < 1$, corpus alicubi celeritatem habebit nullam, infinitam vero nusquam.

Scholion 2.

395. Haec celeritatis diminutio permanet eadem, in quamcunque plaga, corpus moveatur, quia resistentiam ubique eandem ostendit. Neque enim hic motus similis est ei, qui a potentia absoluta contra urgente diminuitur, quo fit, ut corpus in contrariam plagam motum tantumdem acceleretur, quantum ante erat retardatum. Sed ad motum in medio resistente diminutum restituendum atque rursus pariter accelerandum, ac ante diminuebatur, oportet vim resistantiae negativam statui atque ita in vim propellentem transmutari. Tum enim fiet $dv = \frac{Vdx}{Q}$, ex quo apparet celeritatem tantundem augeri, quantum ante minuebatur. Vi ergo resistente in propellentem transmutata motus corporis fiet retrogradus, atque ex *P* in *A* revertetur ita, ut in singulis punctis spatii *AP* easdem recuperet celerites, quas ante ibidem habebat. [p. 165]

Scholion 3.

396. In casibus, quibus $m < 1$ corpusque tandem ad quietem pervenit, occurrit eadem difficultas, cuius supra (316) mentio est facta, si motum transmutanda vi resistents in propellentem velimus convertere. Nam si corporis, cum in C pervenerit, celeritas est nulla, vis propellens $\frac{v^m}{k^m}$ quoque evanescit, si quidem m non est numerum negativus, et hanc ob rem nunquam ex loco *C* corpus poterit depelli. Hoc igitur casu motus diminutionis non poterit in motum augmentationis converti. Calculus quidem contrarium ostendit, nam si dicatur $CP = y$, erit altitudo debita celeritati in *P*, nempe *v*, $= \sqrt[1-m]{\frac{(1-m)y}{k^m}}$. Quod autem ex hac aequatione ipsa absurdum sequatur, hinc apparet, quod $\frac{1}{1-m}$ expons ipsius y est unitate maior, ideoque scala altitudinum, celeritatibus debitarum rectam *AC* in *C* tangat. Quoties enim hoc evenit, corpus ex puncto *C* nunquam egredi potest, etiamsi calculus secus commonstret (319).

PROPOSITIO 51.

PROBLEMA.

397. *Moto corpore in medio resistenti uniformi, quod resistentiam facit potestati cuicunque celeritatum proportionalem, determinare tempus, quo corpus spatium quodcunque AP percurrit* (Fig. 37). [p. 166]

SOLUTIO.

Positis, ut in praecedente problemate, celeritate initiali in $A = \sqrt{c}$, $AP = x$, celeritate in $P = \sqrt{v}$, exponente resistentiae = k et lege = v^m, ita ut vis resistentiae sit ut celeritatis potestas exponentis $2m$. Quia iam est

$$v = \frac{ck^{\frac{m}{m-1}}}{\sqrt[m-1]{k^m+(m-1)c^{m-1}x}}.$$

(387), si quidem $m > 1$, erit

$$\sqrt{v} = \frac{\sqrt{ck^{\frac{m}{m-1}}}}{(k^m+(m-1)c^{m-1}x)^{\frac{1}{2m-2}}}.$$

Elementum ergo temporis, quo spatiolum dx percurritur, est

$$\frac{dx}{\sqrt{v}} = \frac{(k^m+(m-1)c^{m-1}x)^{\frac{1}{2m-2}}}{\sqrt{ck^{\frac{m}{m-1}}}},$$

cuius integralis est

$$= C + \frac{2(k^m+(m-1)c^{m-1}x)^{\frac{2m-1}{2m-2}}}{(2m-1)c^{m-1}\sqrt{ck^{\frac{m}{m-1}}}},$$

quod exprimit tempus per spatium AP, si modo constans C recte determinatur, id quod fit efficiendo, ut facto $x = 0$ totum tempus evanescat. Debebit itaque esse

$$C = \frac{2k^{m\frac{2m^2-m}{2m-2}}}{(2m-1)c^{m-1}\sqrt{ck^{\frac{m}{m-1}}}};$$

quamobrem totum tempus per spatium AP

$$= \frac{2(k^m+(m-1)c^{m-1}x)^{\frac{2m-1}{2m-2}} - 2k^{\frac{2mm-m}{2m-2}}}{(2m-1)c^{\frac{2m-1}{2}}k^{\frac{m}{2m-2}}}.$$

[p. 167] Quae expressio quoque valet, si $m < 1$. At si est $m = 1$, peculiari operatione opus est, nam quia est $v = ce^{-\frac{x}{k}}$, erit $\frac{dx}{\sqrt{v}} = \frac{ce^{\frac{x}{2k}}dx}{\sqrt{c}}$, cuius integrale est $\frac{2ke^{\frac{x}{2k}} - 2k}{\sqrt{c}}$,

quod exprimit tempus, quo spatium AP percurritur.

Quemcunque ergo valorem habeat *m*, tempus ex his formulis per spatium quodcunque determinatur. Q. E. I.

Corollarium 1.

398. Si ergo resistentia quadratis celeritatum est proportionalis et consequenter $m = 1$, tempus, quo corpus spatium infinitum describit, quoad scilicet totum suum motum amittit, erit quoque infinitum.

Corollarium 2.

399. Si vero est $m > 1$, erit quoque $2m > 1$, et consequenter formula exprimens tempus per AP inventa recte est disposita. Ex ea autem apparet tempus, donec corpus totum motum amittat, fore infinitum, id quod facile ex hoc perspicitur, quod spatium quoque sit infinitum (389).

Corollarium 3.

400. Quia vero casu corpus, antequam pervenit in *A* (Fig. 38), alicubi in *C* habuit celeritatem infinitam, tempus etiam, quo ex *C* in *A* pertingit, innotescet facto $k^m + (m-1)c^{m-1}x = 0$ (391). Quo facto resultat tempus per *CA*

$$= = \frac{2k^m}{(2m-1)c^{\frac{2m-1}{2}}}.$$

[p. 168]

Corollarium 4.

401. Si fuerit $m < 1$, duo casus sunt a se invecem distinguendi, quibus *m* vel maius est quam ½ vel minus. Si enim est $m > ½$, manet $2m - 1$ numerus affirmativus, et tempus, quo spatium *AP* absolvitur, erit

$$\frac{2k^{\frac{m}{2-2m}}}{(2m-1)(c^{1-m}k^m-(1-m)x)^{\frac{2m-1}{2-2m}}} - \frac{2k^m}{(2m-1)c^{\frac{2m-1}{2}}}.$$

Corollarium 5.

402. Quia corpus hac hypothesi motum omnem celeratem amittit, cum in C (Fig. 37) pervenit, existente AC = $\frac{c^{1-m}k^m}{1-m}$ (392), erit tempus, quo spatium hoc AC percurrit, ob denominatorem $(c^{1-m}k^m-(1-m)x)^{\frac{2m-1}{2-2m}}$ evanescentem, infinitum. Hoc igitur evenit, si m intra limites 1 et ½ continetur.

Corollarium 6.

403. Sin vero fuerit m < ½, erit tempus, quo spatium quodcunque AP percurritur,

$$= \frac{2c^{\frac{1-2m}{2}}k^m}{1-2m} - \frac{2k^{\frac{m}{2-2m}}(c^{1-m}k^m-(1-m)x)^{\frac{1-2m}{2-2m}}}{(2m-1)}.$$

Ex quo apparet tempus, quo corpus ex A in C pervenit, ubi totum suum motum amittit, esse finitum et $= \frac{2c^{\frac{1-2m}{2}}k^{m}}{1-2m}$.
Fit enim hoc casu $c^{1-m}k^{m}-(1-m)x=0$ (392). [p. 169]

Scholion 1.

404. In his autem formulis non continetur casus, quo m = ½, i. e. si resistentia est celeritatibus proportionalis. Hic igitur casus ex formula differentiali temporis est deducendus. Posito autem m = ½ prodit

$$\frac{dx}{\sqrt{v}}=\frac{2dx\sqrt{k}}{2\sqrt{ck}-x}.$$

Cuius intergrale a logarithmis pendet atque est $=2\sqrt{k}l\frac{C}{2\sqrt{ck}-x}$. Constans vero C debet esse $2\sqrt{ck}$, quo tempus evanescat facto x = 0. Consequenter tempus, quo spatium AP percurritur, erit $2\sqrt{k}l\frac{2\sqrt{ck}}{2\sqrt{ck}-x}$.

Corollarium 7.

405. In hoc itaque casu, quo m = ½, quia spatium AC, quo corpus percurrendo totum motum perdit, est $2\sqrt{ck}$ (392), tempus, quo hoc spatium percurritur, est infinitum.

Corollarium 8.

406. Ex his igitur omnibus colligitur tempus, quo corpus totum suum motum amittit, esse infinitum, si fuerit 2m vel aequalis unitati, vel ea maior; contra vero, si est 2m vel aequalis unitate numerus minor, tempus totius motus esse finitum.

Scholion 2.

407. Si vis resistentiae transmutatur in propellentem, quo casu motus fit retrogradus et simili modo augetur, quo ante minuebatur, tempora eadem esse debebunt, quae hic sunt difinita. Nam quia corporis spatium AP percurrentis in singulis locis eadem est velocitas, sive ex A in P motu retardato, [p. 170] sive vicissim ex P in A accelerato feratur, inter utrumque tempus discrimen esse non potest. Attamen iis casibus, quibus spatium totum AC tempore finito absolvitur, haec regula non valet, quia corpus in C quiescens nullam vim propellentem sentire potest (396). Semper vero huic regulae confidere possumus, si corpori finita celeritas initialis tribuatur.

PROPOSITIO 52.

PROBLEMA.

408. *Sollicitetur corpus, quod movetur in medio quocunque resistente, a potentia quacunque absoluta; determinare celeritatis incrementum vel decrementum, dum quodvis elementum Pp percurrit* (Fig. 39).

SOLUTIO.

Sit corporis celeritas in *P* debita altitudini *v* et elementum percurrendum $Pp = dx$. Sit porro potentia absoluta seu potius eius vis accelerans in $P = p$ atque exponens resistentiae $= q$. Designet *V* eam ipsius *v* functionem, cui resistentia proportionalis est, sitque *Q* talis functio ipsius *q,* qualis *V* est ipsius *v*. His positis retardabitur corpus, dum per elementum *Pp* movetur, vi resistentiae $\frac{V}{Q}$ (383); interea vero simul acceleratur potentia absoluta *p*. Quamobrem corpus elementum *Pp* percurrens accelerabitur a vi $p - \frac{V}{Q}$. Ex quo igitur erit $dv = pdx - \frac{V}{Q}dx$.

Fig. 39.

Q. E. I. [p. 171]

Corollarium 1.

409. Si igitur est $p > \frac{V}{Q}$,celeritas corporis elementum *Pp* percurrentis augebitur; sin vero $p < \frac{V}{Q}$, eius celeritas diminuetur. Atque si fuerit $p = \frac{V}{Q}$, celeritas neque augebitur neque minuetur, sed immutata manebit per elementum *Pp*.

Corollarium 2.

410. Si potentia absoluta fuerit motui contraria eumque retardet, erit $dv = -pdx - \frac{V}{Q}dx$. Hoc igitur casu corpus ab utraque vi retardabitur.

Scholion 1.

411. Si potentia absoluta corpus deorsum trahat, ut in solutione problematis posuimus, atque corpus sursum moveatur, habebit et potentiam absolutam et vim resistentiae contrariam. Tum igitur habebitur ista aequatio $dv = -pdx - \frac{V}{Q}dx$. Ex quo apparet motum ascendentem non similem fore descendenti, quia vis sollicitans in ascensu non est negativa ratione vis sollicitantis in descensu. Quo igitur ascensus similis sit descensui atque corpus in utroque motu in iisdem locis eandem habeat celeritatem, oportet vim resistentiae in ascensu transmutari in propellentem. Quo facto habebitur $dv = -pdx + \frac{V}{Q}dx$, ex qua aequatione perspicitur corpus ascendens per *Pp* tandundem retardari, quantum ante in descensu accelerabatur. [p. 172]

Scholion 2.

412. Aequatio inventa $dv = pdx - \frac{V}{Q}dx$ hac summa extensione ob defectum analyseos neque separari neque construi potest; et hanc ob rem celeritas corporis in P non potest determinari. Multo minus igitur tempus, quo spatium AP absolvitur, poterit assignari. Relinqui ergo oportet hanc generalem aequationem atque descendi ad casus particulares, quibus aequatio potest separari ac celeritas definire. Triplici vero modo aequatio ista separationem indeterminatarum x et v admittit. Quorum primus est, si x plus una dimensione non habet. Secundus, si v unicam tantum obtinet dimensionem. Tertius casus habebitur, si x et v simul ubique eundem dimensionum numerum constituunt, vel si aequatio ad aliam hac proprietate praeditam reduci poterit.

Corollarium 3.

413. Primus casus ergo habetur, si et p et q fuerint constantes; tum enim, quia et Q constans erit, prodit $dx = \frac{Qdv}{pQ-V}$, in qua indeterminatae sunt a se invicem separatae. Praeterea vero etiam aequatio separari poterit, si fuerit $p = \frac{A}{Q}$. Tum enim erit $\frac{dv}{A-Q} = \frac{dx}{Q}$, quae, quia V ab b et Q ab x pendent, construi potest.

Corollarium 4.

414. Quo v unicam habeat dimensionem, opertet sit $V = v$, quo casu quoque erit $Q = q$, et [p. 173] aequatio generalis abibit in hanc $dv = pdx - \frac{vdx}{q}$, quae indeterminatarum separationem admittit.

Corollarium 5.

415. Quo appareat, quando aequatio homogenea sit futura, sit $V = v^{\alpha}$ et $q = x^{\beta}$; erit $Q = x^{\alpha\beta}$. Sit porro $p = x^{\gamma}$, et computetur v constituere δ dimensiones, dum x unam adimplet. His positis aequatio illa abibit in hanc $dv = x^{\gamma}dx - \frac{v^{\alpha}dx}{x^{\alpha\beta}}$, in secundo $\gamma + 1$ dimensiones et in tertio $\alpha\delta + 1 - \alpha\beta.$ Debet igitur esse $\delta = \gamma + 1$ atque $\gamma + 1 = \alpha\gamma + \alpha + 1 - \alpha\beta$ seu $\gamma(\alpha - 1) = \alpha(\beta - 1)$. Quoties igitur fuerit $\alpha - 1 : \alpha = \beta - 1 : \gamma$, toties aequatio ad homogeneitatem potest reduci adeoque celeritas ipsa determinari.

Scholion 3.

416. Loco V, q, et p alias functiones non assumere licet nisi potestates ipsarum v et x. Nam, quia in V non inesse potest x atque in q et p non ingreditur v ac insuper numerus dimensionum ipsarum x et v ubique vel debet esse idem vel ad eundem reducibilis, loco harum quantitatum necessario potestates debent assumi. Hanc ob rem posui $V = v^{\alpha}$, $q = x^{\beta}$ et $p = x^{\gamma}$ atque superiorem analogiam $\alpha - 1 : \alpha = \beta - 1 : \gamma$ elicui. Neglexi quidem coefficientes, qui salva hac reductione possunt adiici, nam homogeneitas hisce perturbari non potest. Itaque potest poni $q = Bx^{\beta}$ et $p = Cx^{\gamma}$, manente eadem analogia. [p. 174] Pro x vero non solum spatium percursum AP potest substitui, sed illud ipsum quacunque constante auctum, dummodo eius differentiale sit dx vel huius aliquod multiplum. Ad $V = v^{\alpha}$ coefficientem addere non est opus, quia per V ratio tantum resistentiae indicatur.

Corollarium 6.

417. Si medium resistens est uniforme ideoque $\beta = 0$, erit $\alpha - 1 : \alpha = -1 : \gamma$. Unde fit $\gamma = \frac{\alpha}{1-\alpha}$. Quare si fuerit lex resistentiae v^{α}, potentia absoluta debebit esse $= Bx^{\frac{\alpha}{1-\alpha}}$, quo aequatio celeritatem definiens ad homogeneitatem possit reduci.

Scholion 4.

418. Hos motus rectilineos in medio resistente ita sumus pertractaturi, ut primo potentiam absolutam constantem ponamus tumque ad quasvis vires centripetas progrediamur. Hisque expositis quaestiones inversas contemplabimur, quamadmodum in praecedente capite fecimus, atque ex datis proprietatibus motus cum potentiam absolutam, tum resistentiae vim eruemus. [p. 175]

PROPOSITIO 53.

PROBLEMA.

419. *Posita potentia absoluta et medio resistente uniforme determinare corporis descendentis celeritatem in singulis locis, si resistentia fuerit quadratis celeritatum proportionalis.* [p. 175]

SOLUTIO.

Manentibus ut hactenus $AP = x$ (Fig. 39), celeritate in $P = \sqrt{v}$, sit potentia uniformis $= g$, exponens resistentiae $= k$. Quiae lex resistentia est v, erit vis resistentiae $= \frac{v}{k}$. Ex quibus prodit $dv = gdx - \frac{vdx}{k}$ seu $dx = \frac{kdv}{gk-v}$; cuius integralis est $x = kl\frac{C}{gk-v}$. Sit celeritas initialis in A debita altutudini c, debebit esse $C = gk - c$ atque $x = kl\frac{gk-c}{gk-v}$.

Logarithmorum loco sumantur numeri : $e^{\frac{x}{k}} = \frac{gk-c}{gk-v}$; ex qua aequatione provenit

$$e^{\frac{x}{k}}v = c + gk(e^{\frac{x}{k}}v - 1) \text{ seu } v = e^{-\frac{x}{k}}c + gk(1 - e^{-\frac{x}{k}}) = e^{-\frac{x}{k}}(c - gk) + gk.$$ Q.E.I.

Corollarium 1.

420. Si corpus in A motum ex quiete inchoet, erit $c = 0$. Hoc igitur in casu habebitur $v = gk(1 - e^{-\frac{x}{k}})$, quae expressio, quo maior accipitur x, magis quoque augetur, certum tamen terminum nunquam potest transgredi. Nam sumto $x = \infty$ habebitur $v = gk$. Est ergo $\sqrt{gk}$ asymtotos celeratum corporis descendentis, quam ante non acquirit, quam ex spatio infinito fuerit delapsum.

Corollarium 2.

421. Si celeritas initialis $\sqrt{c}$ fuerit huic asymptoto $\sqrt{gk}$ aequalis, motus corporis descendentis erit uniformis; fit enim $v = gk = c$. Apparet hoc etiam ex aequatione differentiali $dv = gdx - \frac{vdx}{k}$. [p. 176] Nam si semel fuerit $v = gk$, incrementum celeritatis erit perpetuo evanescens.

Corollarium 3.

422. Si fuert $c < gk$, corpus descendens movebitur motu accelerato, nunquam tamen celeritatem $\sqrt{gk}$ acquiret nisi spatio infinito percurso. Si enim est $c < gk$, quantitas $e^{-\frac{x}{k}}(c - gk) + gk$ semper est negativa, et hanc ob rem v perpetuo erit minor quam gk.

Corollarium 4.

423. Si celeritas initialis $\sqrt{c}$ fuerit maior quam $\sqrt{gk}$, erit $e^{-\frac{x}{k}}(c - gk)$ quantitas positiva ideoque v ubique maior quam gk. Percurso vero spatio infinito fiet $v = gk$. Ex quo perspicitur corpus hoc casus motu retardato descendere.

Scholion 1.

424. Comprehendi debet in hac aequatione $v = e^{-\frac{x}{k}}(c - gk) + gk$ etiam casus, quo corpus in vacuo a sola potentia absoluta sollicitatum descentit. Hicque enim resistentia ponatur evanescens seu exponens k infinitus ; tum enim resistentiae vis $\frac{v}{k}$ evanescit. Difficule autem videtur determinatu, quem valorem habitura sit altitudo v facto $k = \infty$. Ad hunc vero inveniendum plurimum conducit $e^{-\frac{x}{k}}$ in seriem aequavalentem $1 - \frac{x}{k} + \frac{x^2}{2k^2} - \frac{x^3}{6k^3} + \text{etc.}$ transmutare, [p. 177] cuius, si k est ∞, sufficit loco accipere $1 - \frac{x}{k}$. Quo valore loco $e^{-\frac{x}{k}}$ substituto habebitur $v = c - \frac{cx}{k} + gx = c + gx$ ob evanescentem terminum $\frac{cx}{k}$. Quae aequatio convenit cum ea, quam supra (239) invenimus ; nam quod ibi est $\frac{g}{A}$, hic nobis est tantum g. Quoniam g non solum potentiam absolutam,sed vim eius accelerantem exhibet.

Scholion 2.

425. Si k non quidem habet valorem infinitum, set tamen perquam ingentem, prout accidit, quando corpora vehementer gravia in fluido tenui delabuntur, magnum praestabit utilitatem superior series, sumendis tantum tribus terminis primis $1-\frac{x}{k}+\frac{x^2}{2k^2}-\frac{x^3}{6k^3}$ loco $e^{-\frac{x}{k}}$: error enim erit insensibilis. Hoc igitur casu, si corpus ex quiete delabatur, ut sic c = 0. erit $v=gx-\frac{gx^2}{2k}$. Ex qua aequatione vero proximus ipsius v valor eruitur. At si prorsus nihil negligere velimus, erit $v=gx-\frac{gx^2}{2k}+\frac{gx^3}{6k^2}-\frac{gx^4}{24k^3}+\frac{gx^5}{120k^4}-\text{etc.},$ qua series infinita verus valor ipsius v exprimitur.

PROPOSITIO 54.

PROBLEMA.

426. *Determinare tempus, quo corpus in medio resistente uniforme, existente resistentiia celeritatem quadratis proportionali, a potentia absoluta uniformi sollicitatum per spatium AP descendit* (Fig. 39) . [p. 178]

SOLUTIO.

Positis ut supra $AP = x$, celeritate in $A = \sqrt{c}$, celeritate in $P = \sqrt{v}$, exponente resistentiae = k, erit $v=gk+e^{-\frac{x}{k}}(c-gk)$ (419). Ex quo elementum temporis

$$\frac{dx}{\sqrt{v}}=\frac{dx}{\sqrt{(gk+e^{-\frac{x}{k}}(c-gk))}}.$$

Ad quod integrandum pono $e^{-\frac{x}{k}}=z$ et $c-gk=b$, brevitatis causa; eritque $\frac{dx}{\sqrt{v}}=\frac{-kdz}{z\sqrt{(gk+bz)}}$. Fiat porro $gk+bz=r^2$, erit $z=\frac{r^2-gk}{b}$ et

$$\frac{dx}{\sqrt{v}}=\frac{-2kdr}{r^2-gk}=\frac{dr\sqrt{\frac{k}{g}}}{r+\sqrt{gk}}-\frac{dr\sqrt{\frac{k}{g}}}{r-\sqrt{gk}}.$$

Quocirca erit

$$\int\frac{dx}{\sqrt{v}}=C+\sqrt{\frac{k}{g}}\,l\frac{r+\sqrt{gk}}{r-\sqrt{gk}}.$$

Resistuatur loco r suus valor $\sqrt{gk + e^{-\frac{x}{k}}(c - gk)}$, et habebitur tempus descencus per spatium AP

$$= C + \sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}-\sqrt{gk}}.$$

Quod quo evanescat facto $x = 0$, debebit esse

$C = -\sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{c}+\sqrt{gk}}{\sqrt{c}-\sqrt{gk}}$. Ex his conficitur tempus descensus per AP

$$= \sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}-\sqrt{gk}} - \sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{c}+\sqrt{gk}}{\sqrt{c}-\sqrt{gk}}.$$

Quae expressio simplicior reddi potest ita, ut prodeat

$$\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{e^{-\frac{x}{k}}(c-gk)+gk}+\sqrt{gk}}{\sqrt{c}+\sqrt{gk}},$$

cui itaque tempus descensus per AP aequale est. Q.E.I. [p. 179]

Corollarium 1.

427. Si celeritas initialis fuerit = 0 ideoque $c = 0$, erit tempus, quo corpus per altitudinem AP descendit, $\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}\,l(\sqrt{(1-e^{-\frac{x}{k}})}+1) = \frac{x}{\sqrt{gk}} - 2\sqrt{\frac{k}{g}}\,l\,\frac{1-\sqrt{(1-e^{-\frac{x}{k}})}}{e^{-\frac{x}{k}}}$

$$= 2\sqrt{\frac{k}{g}}\,l\,\frac{1}{\sqrt{e^{\frac{x}{k}}}-\sqrt{(e^{\frac{x}{k}}-1)}} = 2\sqrt{\frac{k}{g}}\,l\sqrt{e^{\frac{x}{k}}} + \sqrt{(e^{\frac{x}{k}}-1)}.$$

Scholion 1.

428. Formula etiam generalis temporis per AP cum celeritate initiali $\sqrt{c}$ transmutatur posito $2\sqrt{\frac{k}{g}}\,le^{\frac{x}{2k}}$ loco $\frac{x}{\sqrt{gk}}$ in hanc

$$2\sqrt{\frac{k}{g}}\,l\,\frac{\sqrt{(e^{\frac{x}{k}}gk+c-gk)}+\sqrt{e^{\frac{x}{k}}gk}}{\sqrt{c}+\sqrt{gk}}.$$

Ex qua ea, quam modo pro casu $c = 0$ invenimus, sponte sequitur; prodit enim

$$2\sqrt{\frac{k}{g}}\,l\sqrt{e^{\frac{x}{k}}} + \sqrt{(e^{\frac{x}{k}}-1)}.$$

Corollarium 2.

429. Si ponatur $c = gk$, quo casu motus descensu est aequabilis (421), prodibit tempus descensus per spatium *AP* ex hac ultima forma

$$2\sqrt{\frac{k}{g}}l\sqrt{e^{\frac{x}{k}}} = \frac{x}{\sqrt{gk}} ,$$

quemadmodum ex natura motus aequabilis quoque reperitur. Tempus enim debet exprimi spatio per curso x diviso per celeritatem $\sqrt{gk}$.

Scholion 2.

430. Tempora haec autem habebuntur in minutis secundis, si inventae expressiones per [p. 180] 250 dividantur et lineae *c, k, x* in scrupulis pedis Rhenani exhibeantur (222)

Scholion 3.

431. Posita celeritate initiali $\sqrt{c} = 0$, si detur tempus, quo spatium *AP* descendendo percurritur, poterit ipsum spatium *AP* determinari. Nam sit tempus *t* minutorum secundorum denturque *k* et *x* in scrupuli pedis Rhenani, erit

$$t = \frac{1}{125}\sqrt{\frac{k}{g}}l\sqrt{e^{\frac{x}{k}}} + \sqrt{(e^{\frac{x}{k}} - 1)} .$$

Scholion 4.

432. Si *k* fuerit quantitas vehementer magna et tempus tantum quam proxime desideretur per *AP*, existente celeritate initiali $\sqrt{c} = 0$, assumo hanc formulam

$$\frac{x}{\sqrt{gk}} + 2\sqrt{\frac{k}{g}}l(\sqrt{(1 - e^{-\frac{x}{k}})} + 1) .$$

In qua, quia $\sqrt{(1 - e^{-\frac{x}{k}})}$ fere evanescit, si *k* est valde magnum, erit

$$l(1 + \sqrt{(1 - e^{-\frac{x}{k}})}) = \sqrt{(1 - e^{-\frac{x}{k}})} - \frac{1-e^{-\frac{x}{k}}}{2} + \frac{(1-e^{-\frac{x}{k}})^{\frac{3}{2}}}{3} - \frac{(1-e^{-\frac{x}{k}})^2}{4} + \frac{(1-e^{-\frac{x}{k}})^{\frac{5}{2}}}{5}$$ [p. 181]

Est vero etiam quam proxime

$$\sqrt{(1 - e^{-\frac{x}{k}})} = \frac{\sqrt{x}}{\sqrt{k}} - \frac{x\sqrt{x}}{4k\sqrt{k}} + \frac{5x^2\sqrt{x}}{96k^2\sqrt{k}} .$$

Ex quo proveniet

$$l(1 + \sqrt{(1 - e^{-\frac{x}{k}})}) = \frac{\sqrt{x}}{\sqrt{k}} - \frac{x}{2k} + \frac{x\sqrt{x}}{12k\sqrt{k}} + \frac{x^2\sqrt{x}}{480k^2\sqrt{k}} .$$

Quamobrem habebitur tempus descensus per spatium *AP*

$$= \frac{2\sqrt{x}}{\sqrt{g}} + \frac{x\sqrt{x}}{6k\sqrt{g}} + \frac{x^2\sqrt{x}}{240k^2\sqrt{g}} .$$

Corollarium 3.

433. Quando itaque resistentia evanescit ideoque fiat $k=\infty$, prodibit descensus corporis per AP a sola potentia absoluta g sollicitati. Huius vero descensus tempus ex hac postrema formula, ob evanescentes omnes terminos praeter primum, invenitur $=\frac{2\sqrt{x}}{\sqrt{g}}$, quemadmodum iam supra (218) est inventum, neglecto numero *m* et *g* posito loco $\frac{g}{A}$, ut hic instituimus.

PROPOSITIO 55.

PROBLEMA.

434. *Si corpus in medio resistente uniformi, quod resistit in duplicata ratione, ex B* (Fig. 40) *data celeritate sursum proiiciatur atque sollicitetur potentia uniformi g, determiari oporteat celeritatem corporis in singulis locis .*

SOLUTIO.

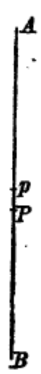
Fig. 40.

Sit celeritas in puncto $B=\sqrt{c}$ et celeritas in $P=\sqrt{v}$, ponatur $BP=x$ et resistentiae exponens $=k$. [p. 182] Quia nunc motus et a potentia absoluta *g* et a resistentia $\frac{v}{k}$ retardatur, erit $dv=-gdx-\frac{vdx}{k}$. Hinc fit $dx=\frac{-kdv}{gk+v}$ et $x=kl\frac{C}{gk+v}$, ubi debet esse $C=gk+c,$ quo fiat $v=c$ facto $x=0$. Quamobrem habebimus $x=kl\frac{gk+c}{gk+v};$ ex qua erit

$$v=e^{-\frac{x}{k}}(c+gk)-gk,$$

quae determinat celeritatem in quovis loco *P*. Q.E.I.

Corollarium 1.

435. Pertingat corpus hoc modo sursum proiectum ad A usque , et sit celeritas in $A=0$. Quocirca altitudo tota *BA* reperietur faciendo $v=0$, quo casu fit

$e^{-\frac{x}{k}}=\frac{gk}{c+gk}$ seu $x=kl\frac{c+gk}{gk},$ cui quantitati aequalis est altitudo *BA*.

Corollarium 2.

436. Evanescat resistentia seu fiat $k=\infty$, ut motus fiat in vacuo, erit $e^{-\frac{x}{k}}=1-\frac{x}{k}$. Hoc igitur casu erit $v=c-gx.$ Eademque aequatio reperitur, si corpus a potentia sola absoluta *g* sollicitam consideretur.

Corollarium 3.

437. Si medium resistens fuerit valde rarum, ita ut k numerum vehementer magnum significet, poterit loco $e^{-\frac{x}{k}}$ accipi $1-\frac{x}{k}+\frac{x^2}{2k^2}-\frac{x^3}{6k^3}$. Ex quo erit

$$v=c-\frac{cx}{k}+\frac{cx^2}{2k^2}-gx+\frac{gx^2}{2k}-\frac{gx^3}{6k^2}$$ quam proxime.

Quo autem haec quantitas valorem vero proximum ipsius v exhibeat, [p. 183] non solum opus, ut k sit numerus valde magnus, set insuper requiretur, ut altitudo x multo sit minor quam k, quo $e^{-\frac{x}{k}}$ non multum ab unitate differat.

Scholion 1.

438. Iam animadvertimus descensum corporis per AB non similem esse ascensui, si medium in utroque casu resistens ponatur. Potest tamen descensus per AB cogitatione concipi, qui prorsus similis sit ascensui, ita ut corpus tam in ascensu quam descensu in puncto P eandem habeat celeritatem et medium propellens. Nam quia in ascensu ambae motui erant contrariae, necesse est, ut in descensu utraque secunda constituatur, quo motus fiat perfecte retrogradus.

Corollarium 4.

439. Posita igitur altitudine AP = y et celeritate in hoc descensu $=\sqrt{v}$, erit

$dv=gdy+\frac{vdy}{k}$. Ex qua integrata prodit $v=gk(e^{-\frac{x}{k}}-1)$.

Scholion 2.

440. Congruit haec aequatio cum priore, quam ascensum contemplantes eruimus. Est enim

$$y=AB\text{-}x=kl\frac{c+gk}{gk}-x \text{ ideoque } e^{\frac{x}{k}}=\frac{c+gk}{gk}.e^{-\frac{x}{k}}.$$

Ex hoc igitur prodit $v=e^{-\frac{x}{k}}(c+gk)-gk$, quemadmodum [p. 184] supra invenimus in solutione problematis. Apparet igitur in hoc descensu corpus in singulis punctis P easdem esse habiturum celeritates, quas habuit ibidem in ascensu. Idem ergo etiam necesse est reperiatur tempus descensus hoc modo considerati, ac ex ascensu provenit.

PROPOSITIO 56.

PROBLEMA.

441. *Determinare tempus ascensus perBP* (Fig. 40) *corporis in medio resistente in duplicata ratione celeritatum ex B data celeritate sursum proiecti et interim sollicitati a potentia absoluta g deorsum tendente.*

SOLUTIO.

Positis celeritate in $B = \sqrt{c}$ eaque in $P = \sqrt{v}$, deinde $BP = x$ et resistentiae exponente resistantiae $= k$, erit $v = e^{-\frac{x}{k}}(c + gk) - gk$ (434). Ex quo oritur elementum temporis

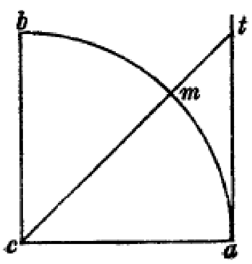
Fig. 41.

$$\frac{dx}{\sqrt{v}} = \frac{dx}{\sqrt{(e^{-\frac{x}{k}}(c+gk)-gk)}}.$$

Ad quod integrandum pono ut supra

$e^{-\frac{x}{k}} = z$ et $c + gk = b$, brev. gr., quo facto habebitur

$\frac{dx}{\sqrt{v}} = \frac{-kdz}{z\sqrt{(bz-gk)}}$. Sit $bz - gk = r^2$, erit $z = \frac{r^2+gk}{b}$ atque

$\frac{dx}{\sqrt{v}} = \frac{-2kdr}{r^2+gk}$, cuius integratio a quadratura circuli pendet. Ad hoc igitur construendum constituatur quadrans circuli *abc* (Fig. 41), cuius radius *ac* sit = 1, sumatur in eo tangens $at = \frac{r}{\sqrt{gk}}$, eritque arcus $am = \int \frac{dr\sqrt{gk}}{r^2+gk}$, qui signetur hoc modo $A\frac{r}{\sqrt{gk}}$ [p. 185]. Huiusmodi scilicet expressio *A.t* nobis denotet arcum circuli, cuius tangens est *t*, existente radio = 1. Hanc ob rem erit

$$\int \frac{-2kdr}{r^2+gk} = -2\sqrt{\frac{k}{g}}\int \frac{dr\sqrt{gk}}{r^2+gk} = C - 2\sqrt{\frac{k}{g}}A.\frac{r}{\sqrt{gk}}.$$

Quia autem est $r = \sqrt{(e^{-\frac{x}{k}}(c+gk) - gk)}$, erit

$$\int \frac{dx}{\sqrt{v}} = C - 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}.$$

Quae quantitas cum debeat evanescere facto $x = 0$, dabit $C = 2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}}$.

Tempus igitur ascens per spatium *BP* invenitur

$$=2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}}-2\sqrt{\frac{k}{g}}A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}$$

$$=2\sqrt{\frac{k}{g}}\left(A.\sqrt{\frac{c}{gk}}-A.\sqrt{\frac{(e^{-\frac{x}{k}}(c+gk)-gk)}{gk}}\right)$$

$$=2\sqrt{\frac{k}{g}}A.\frac{\sqrt{cgk}-\sqrt{(gke^{-\frac{x}{k}}(c+gk)-g^2k^2)}}{gk+\sqrt{(ce^{-\frac{x}{k}}(c+gk)-cgk)}}.$$

Q.E.I.

Corollarium 1.

442. Quia tota altitudo AB, ad quam corpus pertingere potest, habetur faciendo $x=kl\frac{c+gk}{gk}$, quo casu fit $e^{-\frac{x}{k}}=\frac{gk}{c+gk}$, invenietur tempus totius ascensus $=2\sqrt{\frac{k}{g}}A.\sqrt{\frac{c}{gk}}$

Corollarium 2.

443. Quare si fuerit $c = gk$, erit tempus totius ascensus per BA $=2\sqrt{\frac{k}{g}}A.1$.

At arcus, cuius tangens aequatur rado, est peripheriae pars octava. Posita itaque quarta peripheriae parte $amb=\pi$, erit tempus ascensus per $BA = =\pi\sqrt{\frac{k}{g}}$. [p. 186]

Corollarium 3.

444. Ex hoc quoque intelligitur, si celeritate infinita corpus ex B sursum proiiciatur, tempus ascensus totius nihilo minus fore finitum; fit enim $=2\sqrt{\frac{k}{g}}A.\propto$; qui arcus cum sit quarta peripheriae pars = π, erit tempus totius ascensus $=2\pi\sqrt{\frac{k}{g}}$.

Corollarium 4.

445. Si loco celeritatis initialis $\sqrt{c}$ detur tota altitudo $BA = a$, ad quam corpus pertingit, quia est $a=kl\frac{c+gk}{gk}$ et propterea $c=gk(e^{\frac{a}{k}}-1)$, reperiatur tempus totius ascensus per $BA=2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{a}{k}}-1)}$.

Scholion 1.

446. Si transmutetur ascensus in descensum medio accelerante, ut supra assumsimus (438), et vocetur $AP = y$, erit tempus descensus per AP

$$=2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{y}{k}}-1)},$$

substituto y loco a in superiore formula. Ibi enim a denotabat altitudinem in ascensu percursam, hic vero est y altitudo integra, ad quam corpus celeritate, quam in P habet, pertingere potest.

Scholion 2.

447. Aequatio fundamentalis pro descensu hoc modo considerata est $dv = gdy + \frac{vdy}{k}$ (439), quae ex aequatione fundamentali pro vero descensu $dv = gdy - \frac{vdy}{k}$ (419) potest formari ponendo y loco x et $-k$ loco k. Quare etiam expressio temporis vero [p. 187] proxima per spatium AP ex ea, quam supra invenimus (432) pro vero descensu, accommodari poterit ad hunc descensum imaginarium ascensus inversum, ponendo quoque y loco x et $-k$ loco k. Hoc itaque modo invenitur tempus descensus per spatium AP
$= \frac{2\sqrt{y}}{\sqrt{g}} - \frac{y\sqrt{y}}{6k\sqrt{g}} + \frac{y^2\sqrt{y}}{240k^2\sqrt{g}}$, quam proxime, dummodo $\frac{y}{k}$ fuerit numerus unitate minor.

Corollarium 5.

448. Si medium resistantia prorsus evanescat, ut fiat $k = \infty$, erit tempus totius ascensus per spatium $BA = \frac{2\sqrt{a}}{\sqrt{g}}$. Quae expressio provenit ex superiore $\frac{2\sqrt{y}}{\sqrt{g}} - \frac{y\sqrt{y}}{6k\sqrt{g}} + \frac{y^2\sqrt{y}}{240k^2\sqrt{g}}$ posito a loco y; omnes enim termini praeter primum evanescunt.

Corollarium 6.

449. Poterit hinc etiam ex dato tempore integri ascensus t reperiri altitudo percursa a. Nam quia est $t = 2\sqrt{\frac{k}{g}}A.\sqrt{(e^{\frac{a}{k}} - 1)}$, erit tangens arcus $\frac{t}{2}\sqrt{\frac{g}{k}} = \sqrt{(e^{\frac{a}{k}} - 1)}$. Vocetur illa tangens T, erit $T^2 + 1 = e^{\frac{a}{k}}$ et $a = kl(T^2 + 1)$.