Chapter Two

CONCERNING THE EFFECT OF FORCES ACTING ON A FREE POINT.

[p. 39]

DEFINITION 10.

99. ***A force is an action on a free body that either leads to the motion of the body at rest, or changes the motion of that body***. The force of gravity is a force of this kind; for through that force, bodies with obstructions removed and released from rest fall freely downwards, and the descending motion is one of continuous acceleration.

Corollary.

100. Any body left to itself will continue in a state of rest or of uniform motion in a fixed direction. Therefore to what extent it occurs that a free body at rest begins to move, or the uniform motion of a body becomes non-uniform, or the motion changes direction, the cause is to be ascribed to some external force, which we call the force acting on the body, according to whatever state of the body it produces.

[It is almost impossible to give succinctly this definition the exact meaning that I think Euler had in mind, in modern terms, as two different terms are applied to distinguish aspects of forces acting on a mass: at the time of writing the term *vis* (which means force or strength of a force and is a general expression for a force or action, or even an acceleration expressed as the force per unit mass) was applied to the innate or inertial force, the familiar mass × acceleration, or what we might now simply term the resultant of the external forces; and the mass in turn had some external forces applied to it, called by the term *potentia*, (which also means force, but these are usually given by Euler by a formula of some kind). Thus, we might say that the sum of the external forces is to be measured by the size of the inertial force produced on the body itself.]

Scholium 1.

101. Concerning the principles of external forces, in as far as many external forces are applied to a body in equilibrium and they keep the body at rest, that has now been explained in Statics. There the external force has also been defined, as that force denotes all the force that prevails to make the body move. Indeed the motion of the body is not to be considered in Statics, but only these cases are to be investigated in which several forces cancel each other out [p. 40], and the body upon which they act remains in a state of rest. Moreover it is now to be explained by Mechanics, how forces acting on a body, which do not cancel each other out, can produce motion in a body at rest, and indeed for a body in motion, change that motion.

Scholium 2.

102. Whether forces of this kind have their origin in the bodies themselves, or indeed through such forces that do exist in the world, these I will not consider here. For here it is sufficient, in place of the forces that really arise in the world, only to consider the force of gravity, by which all bodies on the earth try to move downwards. Besides truly forces of this nature are evident disturbing the motion of the planets, which unless they were influenced by a certain force, would be progressing in straight lines. Indeed similar forces are understood to arise from magnetic and electric bodies, which surely can attract bodies to a great extent. Some think that all the forces arise from the motion of some subtle matter [*i.e.* followers of Descartes], and others attribute the force to attraction or repulsion between the bodies themselves[*i.e.* followers of Newton]. Moreover, whatever it shall be, certainly we see that forces of this kind can arise from vortices, and from elastic bodies, and we will inquire whether forces can be explained for these phenomena instead. Meanwhile truly we will try to determine the effect of any forces on bodies, so that henceforth, [p. 41] when a fuller understanding of these has been reached and at once elicited, for these to be able to be accomodated.
[Thus, Euler intends to talk only about gravitational forces, for which experimental laws exist, until the other forces of nature have been more fully investigated. Daniel Bernoulli had a lot of influence on Euler's thoughts about mechanics at this time, as Euler actually stayed with him, and both worked at the St. Petersburg Academy; both published papers on the topics considered in this chapter at the time in the early volumes of the St. Petersburg *Commentarii.*]

DEFINITION 11.

103.***The direction of the force is the straight line along which the body is trying to move.*** Thus the direction of the force of gravity is that vertical line, for a heavy body tries to fall along that line.

Scholium 1.

104. In statics, where everything is put in place to remain at rest, all the forces have their directions set up, serving to keep the body at rest all the time. But in mechanics [which we now call dynamics], when the body is always arriving at another place, the direction of the force acting will constantly be changing direction. Indeed for different positions of the body the directions of the forces are either parallel to each other, or converging to a fixed point, or they act in response to some other law, from which so many different forms of the laws governing mechanics arise.

Scholium 2.

105. The comparison and measurement of different forces, likewise from statics, should be recalled. In which some force has been treated *a* that has the ratio to another force *b* as *m* to *n*, when the force *a* is applied *n* times in turn on the point A (Fig. 11) along *AB*, and the force *b* is applied *m* times along the opposite direction *AC*, and the point *A* continues in equilibrium. Then indeed the force *a,* taken *n* times, is equivalent to the force *b,* taken *m* times, and will be related by $na = mb$ or $a : b = m : n$.

Fig. 11.

[p. 42]

Scholium 3.

106. Now in this regard, the measurement of forces differs in mechanics from statics, since in statics all the forces put in place are able to keep the same magnitude, while in mechanics, as with the body arriving in another place, the directions of these are made changeable, thus the magnitude of these is able to be variable following some law.

PROPOSITION 13.

THEOREM.

107. *When a point is acted on by many forces, the same motion comes about from these, as if the point is acted on by a single force equivalent to all of these forces.*

DEMONSTRATION.

Let the point A (Fig. 12) be acted on by the forces *AB, AC, AD,* and *AE,* to which the force *AM* is equivalent. The equal and opposite position to this, *AN,* is taken, and as it is known from statics, it will cancel the action of the forces *AB, AC, AD,* and *AE*. Therefore [harking back to (104),] in the first place only the force *AN* is impressed on the point A and the motion is along *AN*, the magnitude of the forces *AB, AC, AD,* and *AE* acting together impress on the point *A* the force along the line *AM* in the central direction. Truly the force *AM* alone, since it is equal to the force *AN*, is of such a size that it moves the point A along AM, to the same extent as the force AN moves the point towards AN. Whereby the force *AM* alone impresses on the point *A* a motion along *AM*, as much as the forces *AB, AC, AD, AE* acting together along the same direction [p. 43] AM. In each case therefore, the effect is the same. Q. E. D.

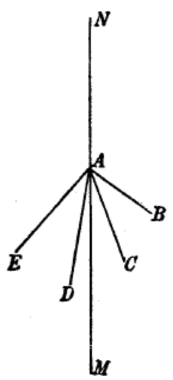


Fig. 12.

Corollary 1.

108. If a point is therefore influenced by many forces, that can be considered to be influenced as it were by a single force, which is equivalent to all these forces.

Corollary 2.

109. And in turn in place of a single force acting on a point, there can be considered to be many forces acting, to which that is equivalent ; that which, as has been shown from statics, can be made in a limitless number of ways.

Scholium.

110. Because truly, as the body first has moved from its position, the forces acting on it change their directions and magnitudes or they are put in place to move with the body, there will be some other equivalent force for any moment. Hence on account of this circumstance, for any time, the equivalent force of all the forces acting on the point ought to be investigated, and thus not the long term effect of the same force is to be put in place [i. e. the time average force,] but rather that from an infinitely small element of time.

DEFINITION 12.

111. ***The absolute force is the force that acts equally on a body either at rest or moving.*** An absolute force of this kind is the force of gravity. which acts equally downwards on a body which is either at rest or moving. [p. 44]

Corollary.

112. If therefore the absolute effect of a force acting on a body at rest is known, then the effect of the force on the body is also known for any kind of motion.

DEFINITION 13.

113. ***A force is relative, which acts in one way on a body at rest, in another on a body in motion.*** A force of this kind is the force acting on a body dragged away by a river ; where indeed the faster the body moves in the river, the smaller the force shall be : and that therefore the force vanishes when the speed of the body is the same as that of the river.

Corollary 1.

114. If therefore the speed of the body is given together with the law of the relative force, it is possible to find the strength, the magnitude of the force is exerted on the body. And hence from this as the absolute force has to be considered, as long as the body has the same speed, the effect of this from the action of the absolute action can be determined. For the strength of the relative force is to be determined from the given motion of the body.

Corollary 2.

115. Therefore these relative and absolute forces in turn are different from each other; for the magnitude and direction of the absolute force acting on the body may only depend on the location of the body ; while truly the magnitude and direction of the relative force acting on the body depends in addition on the speed of the body. [p. 45]

Scholium 1.

116. Relative forces are chiefly to be considered in the relative motion of bodies in fluids; for the action of these forces on bodies depends on their relative speed; for when that is greater, it is apparent that the force of the fluid acting on the body is also greater. Moreover as well as the other causes of motion in fluids, which require a greater understanding of fluids, there are two which are easier to handle; the one when the fluid is at rest, and the other when the fluid is moving uniformly in a given direction. Truly it is always possible for the one to be substituted for the other, with the relative motion always reduced to the absolute ; likewise, clearly for the state of a fluid considered to be at rest the proper forces will remain [i. e. those of a relative nature have vanished]. Therefore in what follows, concerning the relative forces which may be proposed, those properties which also pertain to fluids at rest will be the chief concern. For truly the action of fluids on the motion of bodies consists wholly in the diminution of their speed and on this account it is called the resistance, which is also greater when bodies move faster, and which always disappears when the bodies are at rest. On this account we can put the true motions in place of the relative motion in what follows, and these are only affected by the absolute forces, and can be placed in a vacuum [in general].

Scholium 2.

117. Indeed the motion in mediums with resistance, if we wish the greatest order to be followed in the following chapters, must be referred to the last chapter [of vol. 1], in which the motion of the fluid is to be determined, since also it is not now agreed upon by which law the fluids resist the motions of bodies. [p. 46] Truly, since this matter is usually considered from many points of view, in order that the nature of fluids can be examined in a straight forward manner, these have been revoked and instead a purely mathematical hypothesis to be used : I have decided to retain this method as many elegant problems are passed over, which otherwise are not to be found in discussions on fluids. However I will apply this method only to the motion of points in fluids, as the calculations associated with bodies of finite magnitudes become insurmountable. Moreover when the shape of a body taken as a finite number of points is considered, there is a convenient outcome from this, which is that the direction of the resistive force is in agreement with the direction of the motion, since indeed that arises from a fluid at rest. Moreover, on account of this we agree that the motion of points in relative motion in fluids are always to be considered with relative forces in the same direction as the point itself, and that always as we will consider the motion to be decreasing.

PROPOSITION 14.

PROBLEM.

118. *For the given effect of an absolute force on a point at rest, to find the effect of the same absolute force on the same point in some kind of motion.*

SOLUTION.

Let the point be placed at A (Fig. 13), from where it can be moved with speed *c* following the direction *AB*, and indeed the direction of the force acting on it is in the direction *AC*. Some element of time is taken *dt*, and in this short length of time the point *A* is pulled forwards, if it were at rest at *A*, through the small distance *AC*, [p. 47] that may be called *dz*, so that after the time *dt* the point is no longer at *A*, but at *C*. This motion of the point along *AC* is the effect of the force acting on the point at rest. The effect of the same force, which is put as absolute, should be the same on a moving point as on one at rest (111). Now the direction of the point is taken so that it travels the distance *AB* along *AB*, since it travels the distance $AB = cdt$ (30), with its speed *c* acting for the short time *dt*, if it is not influenced by any force. Truly with the force acting after the short element of time *dt*, the point will no longer be found at *B*, but to be elsewhere at *D*, thus so that the effect, which is to be measured by the deviation from the point *B*, which is the distance *BD*, which is equal to the effect of the same force on the resting point (111), i. e. *AC*. Hence $BD = AC$. Besides indeed *BD* is parallel to *AC* itself, since *BD* has been the effect of the force, and thus should be acting in the same direction, which does not change during the indefinitely short time *dt*. On account of this the point *A* having the speed *c* along the direction *AB* and influenced by the absolute force, for the elapse of the short time, is to be found not from *B* but from *D*, with *BD* equal and parallel to *AC* itself. Truly the distances traversed in the infinitely short time can be considered to be straight lines; on account of which the distance *AD* traversed in the very short time *dt* is agreed upon. Q. E. I.

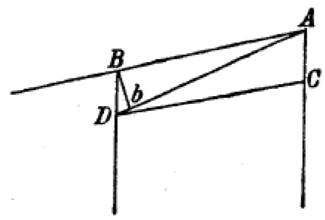


Fig. 13.

Corollary 1.

119. Since the motions in infinitely short distance traversed can be uniform, the speed with which the element AD (33) is traversed, is equal to $\frac{AD}{dt}$ (30) [p. 48]

Corollary 2.

120. The speed along AD is put equal to $c + dc$, which preceding was *c* (35), will be given by $c + dc = \frac{AD}{dt}$; but before $AB = cdt$, from which $c = \frac{AB}{dt}$. Hence there is produced: $dc = \frac{AD-AB}{dt}$. Therefore on cutting off the portion $Ab = AB$ from *AD*, there is left the equation $dc = \frac{Db}{dt}$.

Scholium 1.

121. Moreover, it is to be understood that *AC* or *BD* is infinitely smaller than *AB*, since *AB* is the distance traveled with a finite speed in the time *dt*, but the absolute small distance *AC* is traversed in the same time element with an infinitely small speed; indeed it is not possible to infer a finite speed for a body at rest in an infinitely small increment of time.
[Euler's way of saying that the distance AB is a first order increment, while *AC* or *BD* is a second or higher order increment.]

Corollary 3.

122. Hence on account of this, the angle *BAD* is indefinitely small, and with the points *B* and *b* joined, the line increment *Bb* is perpendicular to *AD*. The sine of the angle *BAC*, which is surely to be given, is called *k*, with the total sine taken as 1, then the sine of the angle *BDb* is *k* also, and since $BD = AC = dz$, then $Db = dz\sqrt{(1-kk)}$ and $Bb = kdz$.

Corollary 4.

123. Therefore, the increment of the speed *dc*, that we found before to be equal to $\frac{Db}{dt}$, will be $\frac{dz\sqrt{(1-kk)}}{dt}$. Indeed it is understood that the distance *dz* is infinitely smaller than *dt*; for *dz* is infinitely small with respect to *AB*, i. e. *cdt*, and likewise with respect to *dt*, since *c* is put of finite magnitude.
[p.49; thus, we see that even at this time, 1733, the problems involved with an inadequate notation for orders of increments had not been resolved, and much was left to the intuitive powers of the practitioner. It does appear in general that Euler considers his infinitesimal elements set out in the diagrams as *dx, dy*, etc, as initially being small but finite linear quantities, from which relations are established, before undertaking a limiting process where *dx, dy*, etc are made infinitely small, while their ratio can remain finite.]

Corollary 5.

124. With the increment of the speed *dc* found due to the force, the angle *BAD* of the change of direction of the point from the original direction represented by *AB* should be considered also, which likewise is found from the force. Truly the sine of the angle is equal to $\frac{Bb}{AB} = \frac{kdz}{cdt}$.

Corollary 6.

125. Therefore there is a two-fold effect of the force affecting the motion of the point. One way is in agreement with the change in the speed, and the other with the change in the direction of this point. The first gives a change of the speed $dc = \frac{dz\sqrt{(1-kk)}}{dt}$, the second gives the declination of the sine of the angle $\frac{kdz}{cdt}$.

Corollary 7.

126. If the angle *BAC* is right then likewise $k = 1$, and $dc = 0$. Therefore in this case the speed remains unchanged by the force. Truly the sine of the angle of declination *BAD* will be $\frac{dz}{cdt}$.

Corollary 8.

127. If the angle *BAC* is obtuse or greater than a right angle, then the cosine of this angle $\sqrt{(1-kk)}$ is negative, and therefore the increment of the speed *dc* will be made negative, and equal to $-\frac{dz\sqrt{(1-kk)}}{dt}$. Which shows that the speed is diminished by the force. Declination $\frac{kdz}{cdt}$ remains the same as it was before.

Corollary 9.

128. If the direction of the force *AC* agrees with the direction *AB* of the point A, this makes $k = 0$. Therefore in this case [p. 50] the direction of the motion is not changed by the direction of the force. Truly the increment of the speed *dc* becomes equal to $\frac{dz}{dt}$, if the direction of the force agrees with the direction of the motion. But if moreover it should be in the opposite direction, then it becomes $dc = -\frac{dz}{dt}$.

Scholium 2.

129. Thus it is apparent from the solution to this proposition, how the absolute effect on a point in any kind of motion ought to be found, if the effect of the same force on the point at rest were known. Hence on account of this for the following propositions of this chapter, it will be sufficient for the force acting on a point to be placed either at rest or to be moving in the same direction as the force. For if the point *A* (Fig. 14) has the speed *c* and is moving along the direction *AB*; meanwhile truly it is acted upon by a force in the same direction *AB*, thus so that in the passing of a small interval of time *dt* it will not be at *B* that the body will be found, for only with the speed *c* will the width be traversed, but at *b*, and the effect of the force will be the small distance *Bb*. And by the same small distance *ao* the point *A*, if it were at rest at *a*, would have been treated in the same small increment of time *dt*. Therefore from the motion of the point A by the force the effect of the same force acting on the point at rest will be known, and again hence the effect of the force on any moton of the point.

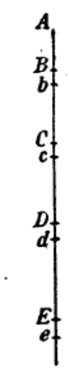


Fig. 14.

PROPOSITION 15.

PROBLEM.

130. *For a given increment of the speed, that a certain force produces on the point A in the small increment of time dt, to find the increment of the speed,*[p. 51] *that the same force produces on the same point in the time increment dτ.*

SOLUTION.

The point *A* (Fig. 14) has the speed *c* in the same direction *AB* as the force has acting on it, and *ao* is the small distance, through which the force pulls the point *A*, if it were at rest, in the small increment of time *dt*. If again the distance *AB* is taken, that the point *A* will traverse with speed *c* in the increment of time *dt*, then it will traverse the same distance beyond *Ab* with the force acting, by taking *Bb* = *ao* : and thus the distance, since it is infinitely small, is considered to be described by a uniform motion.
[In modern terms, while we do not follow Euler's development closely, the first distance gone without accelerating, is *cdt*; the acceleration *a* provides an extra distance gone *ao*, which is, if A starts from rest, given by $\frac{1}{2}adt^2 = vdt/2 = v_{av}.dt$, where v_{av} is the average speed over the increment, in which the force acting and the acceleration are considered to be constant. Thus we have *Ab* = *AB* + *ao*.] Therefore in the following small time *dt,* the body travels a distance *bC* = *Ab* with this speed, unless acted upon by a force; and with the force acting again, which it can be put to remain unchanged even through the infinitely short time it will arrive beyond *C* at *c*, by taking *Cc* = *ao*.
[Again, the second distance gone without further accelerating in the next equal time increment, is *Ab* = *AB* + *ao*; the acceleration *a* provides an extra distance gone *ao*, as above with the same conditions. Thus we have *Ab* = *AB* +2*ao*.]In the same manner in the third increment of the time *dt* it will traverse the distance *de* = *dE* + *Ee*, where again *dE* = *Cd* and *Ee* = *ao*. Indeed we have :

$$Ab = AB + ao; \quad bc = AB + 2ao; \quad cd = AB + 3ao; \quad de = AB + 4ao.$$

Hence $\frac{ao}{dt}$ is the increment of the speed produced by the force in the time *dt* ; $\frac{2ao}{dt}$ is the increment of the speed acquired in the time 2*dt* ; similarly $\frac{3ao}{dt}$ increment in the small time 3*dt*; and generally in the short time *ndt* the speed *c* of the point will increase by the element $\frac{nao}{dt}$. Put $ndt = d\tau$, then $n = \frac{d\tau}{dt}$. Therefore the increment of the speed acquired in the time increment $d\tau$ will be $\frac{ao.d\tau}{dt^2}$. Since indeed the increment of the speed in the short time *dt* is $\frac{ao}{dt}$, it will be produced in this ratio : (increment of the speed acquired in the increment of the time *dt*) is to (increment of the speed acquired in the time increment $d\tau$) as *dt* is to $d\tau$. Consequently the increments of the speed are in proportion to the times in which they are produced. Q. E. I.

Corollary 1.

131. It is apparent that this increment of the speed does not depend on the speed c itself, but it will have the same value, however large or small a value is put for c. And from this the nature of the absolute force is better understood, since they act equally on moving bodies and on bodies at rest.[p. 52]

Corollary 2.

132. If $c = 0$, and the point A at rest is urged into moving by a force, the speeds acquired from the motion itself will be as the times : obviously twice the speed in twice the time, with the time tripled, so it will obtain three times the speed.

Corollary 3.

133. If therefore from the start, the speed of the motion acquired in a small time t is called c, and the distance traversed is s, then $t = nc$. But also $t = \int \frac{ds}{c}$ (37). Hence the equation is produced $nc = \int \frac{ds}{c}$ or $ncdc = ds$ and hence $s = \frac{nc^2}{2} = \frac{t^2}{2n}$. Therefore the distances described from the start of the motion are in the square ratio of the times or of the speeds acquired in that distance. [p. 53; we see that the constant n is the inverse of the acceleration.]

Scholium 1.

134. The truth of this proposition, that the increments of the speed are in proportion with the increments of time in which they are generated, is also in agreement with finite quantities, as long as the force acting on the point stays the same, and always retains the same direction as the motion of the point itself. For infinitely small times it is not necessary to have this restriction; for no force of any variation greater than the smallest is to be considered. Moreover we are soon to show what the effect of the different forces may be, and also how points are influenced by forces of different kinds that we put in place, as another larger or smaller force can be put in the given ratio. Truly this does not adversely affect extremely small points, but indeed not only points that we understand to be mathematical but also the physical points that arise from the composition of bodies. For two or more points can be conceived to be merged into one, since the points still remain of infinitely small magnitude, and nevertheless it is better with single points.

Scholium 2.

135. GALILEO was the first person to use [this principle] in the investigation of a falling weight, for the solution of the problem found for this theorem. Indeed he did not give a demonstration of this, but yet on account of the conspicuous nature of this principle from many similar phenomena, he did not wish to be doubted. Indeed he had refuted other opinions regarding this matter, from which he had greatly confirmed his own views. [p. 54] Others indeed were of the opinion that increments not with time but with distance traversed were in proportion; truly the absurdity of this had been established by Galileo, and then by many philosophers. Moreover it appears, if the actions of the forces follow this law, that no body is able to lead in the motion at any time. Indeed it might be that *dc* = *nds* and *c* = *ns*, truly the time *t*, that is $\int \frac{ds}{c}$, that one body might evade the other is equal to $\frac{1}{n}\int \frac{ds}{c} = \frac{1}{n} l\, c = \frac{1}{n} l\, ns$ + const., which constant ought to be $= -\frac{1}{n} l\, 0n$. Clearly the times described by the logarithm of the distance divided by the zero distance should be in proportion, but is thus infinite. Therefore no body can ever lead in the motion from rest.

[In modern notation, $t = \frac{1}{n}\ln c = \frac{1}{n}\ln ns + const.$, and the constant should be $-\frac{1}{n}\ln(n0)$;leading to $t = \frac{1}{n}\ln c = \frac{1}{n}\ln(s/0)$. Euler had no qualms about writing down the log of zero! We should note that he has already used the argument that such a motion takes an infinite time to come to the distance 1 from negative infinity; it would seem better perhaps to have used purely physical arguments rather than the mathematical argument produced here, to show that all bodies fall at the same rate.]

Thus correctly Galileo responded to his opponents arguments, that instantaneous finite motion would have to arise from that [initial] position, otherwise forwards motion would not be possible. Although indeed in the beginning the speed at the point is made infinitely small, yet that speed from an imaginary force of this kind will never be able to be finite. Now from the given solution of the problem it is understood that it is necessary to find a law, so that neither can any other force with contradictions arise in the beginning.

[Roughly speaking, to refute opponents : if the body is at zero, then it will always be at zero, unless a speed can be produced spontaneously somehow; according to the other view, a sum of infinitesimally increasing speeds over time leads to a finite speed : a variation on Zeno's paradox, but using accelerated motion (see Book II of Gregorius' *Geometry* in this series of translations.)]

PROPOSITION 16.

THEOREM.

136. *The force q at the point b has the same effect that the force p has at the point a, if the ratio between the forces and distances is of the form q : p = b : a.*

DEMONSTRATION.

[To establish this, if] *q* is put equal to *np;* i. e. *q* = *np*, then *b* = *na*. Now it is understood that the point *na* is divided into *n* equal parts, any of which is equal to *a*; [p. 55] of which each of the parts is acted on by an n^{th} part of the force *np*, that is by the force *p*. With these put in place, any part is pulled in the same manner by its own force, as by which the point *a* itself is pulled by the force *p*. Neither are these points of *na* parts to be acted on by their own forces in turn on being separated; for they will always remain united, if they were indeed connected together initially. Moreover it is evident that these two cases revert to the same and do not disagree with each other, whether the point *na* is drawn by the force *np*, or if some part *a* of the point *na* is pulled by a similar part *p* of the force *np*, provided the parts are not separated from each other in turn. On account of which the proposition is agreed upon, that equally *na* parts are to be acted on by the force *np,* as *a* is acted on by the force *p*. Q. E. D.

Corollary 1.
137. Therefore the point *na* obtains the same acceleration from the force *np* as the point *a* from the force *p*.

Corollary 2.
138. In the same manner, it follows that for a point to have a greater speed induced than a smaller one, then it is necessary for a larger force, and with that force to be even so much greater, with that point so much greater than this one.

Scholium 1.
139. This Proposition embraces the foundation of measuring the inertial force, here indeed the ratio of all is advanced, whereby the matter or masses in Mechanics must be considered. For it is necessary that the number of points is attended to, from which the body to be moved has been agreed upon, [p.56], and the mass of the body must be made proportional to this. Truly the points must be taken amongst themselves as equal to each other, not in the sense that they are equally small, but in that the force exerts an equal effect on each. If therefore we consider that the whole body has been divided up into a number of equal points or elements in this manner, then it is necessary to estimate the quantity of the matter of each body from the number of points, from which it is composed. Moreover the force of inertia is proportional to this number of points or the quantity of material in the body, as we will show in the following proposition.

Corollary 3.
140. Therefore two bodies which have been made from the same number of points are equal, because each contains the same amount of matter. And two bodies are in the ratio *m* if *n*, if the numbers of the points, upon which they agree, keep the ratio *m* to *n*.

Scholium 2.
141. Truly it will be shown in the following propositions that this ratio of the quantity of the matter to be measured for the bodies themselves is to be put to use and to be undertaken in all work. For from the weight of each body it is usual to investigate the mass, and it is agreed that the weight and the quantity of matter are in proportion. Moreover it is agreed by experiments that all bodies in an empty space fall equally, and therefore all are accelerated equally by the force of gravity. Concerning which it is necessary that, in order that the force of gravity acting on individual bodies shall be proportional to their quantity of matter. [p.57] Truly the weight of the body indicates the force of gravity, by which that body is acted on. Whereby since that shall be proportional to the quantity of matter, with the weight of the quantity of matter known, from that itself considered, that we have divided here for the matter. [Thus, $\frac{wt._1}{wt._2} = \frac{mass_1}{mass_2}.$]

PROPOSITION 17.

THEOREM.

142. *The force of inertia of any body is proportional to the quantity of matter, upon which it depends.*

DEMONSTRATION.

The force of inertia is a force in place in any body in its own state of rest or of uniform motion in a direction to be kept the same.(74). That therefore is to be estimated from the strength or the force applied to the body, with the aid of which it is to be disturbed from its state. Truly different bodies equally in their state are disturbed by forces which are as the quantities of material contained in these. Therefore the forces of inertia of these are proportional to these forces. Consequently also the quantities of matter are in proportion. Q. E. D.

Corollary 1.

143. Likewise it is evident by demonstration that the same body, either in a state of rest or of motion, always has the same force of inertia. For, either at rest or moving, clearly it is affected by the same absolute motion. [p.58]

Corollary 2.

144. Nor indeed is the force of inertia homogeneous with any force : for it is not able to become so, as any body of any great size is not affected by a small force, as is shown in the following.

Scholium.

145. Hence it is apparent that the origin of the said force of inertia, comes from that which we have introduced above (76), since the force of inertia resists the action of any kind of force. Which Newton too had decided on, and who in Definition III of the *Princ. Phil;. Nat.* joined together the force of inertia with the same idea of a force being resisted, and each was set up to be in proportion to the quantity of matter. [Thus, to change the state of motion of a free body, an external force has to be exerted, that overcomes the 'innate power of resisting' or the force of inertia present in the body.]

PROPOSITION 18.

PROBLEM.

146. *With the effect of one force on some point given, to find the effect of any number of forces acting on the same point.* [Or, how the distance moved in an element of time is related to the size of the force acting on a mass, and for which the parallelogram of forces is assumed. This development follows that of Daniel Bernoulli, who tried to give mechanics an axiomatic foundation, following along the line of Euclidian geometry. Part of this development was to show how any force could be decomposed into the sides of a rhombus: see *Die Werke von Daniel Bernoulli*, Band 3; *Examinen Principiorum Mechanicae ......* A commentary on this Latin paper is presented in English (there are also I believe French and German versions) in this book by David Speiser; part of an on-going edition of the works of the Bernoulli family. Birkhäuser (1987). One should note that no translations are actual presented, merely discussions of what the writer considers has been said, and many papers have not even been discussed, and are listed at the end of the book.]

SOLUTION.

The point is at rest at A (Fig. 15) and the effect of a given force AB on this point is agreed upon, which is that in an element of time dt it is drawn through a small distance Ab. Now it is required to be found, by what distance in the element of time dt, the same point is drawn by another force AC. The lines AB and AC are drawn thus, in order that the joining line BC is normal to AC, since that can always be done if $AC < AB$. [i. e. there is a condition placed on AC, which represents that force arising from the parallelogram of forces as the resultant of the two 'half' forces AE and AF acting symmetrically as shown] But if $AC > AB$, the solution can be easily deduced from the other condition. From the other side the line AD is drawn thus, in order that BAD is an isosceles triangle [p.59]. AB and AD are bisected in E and F, and half of the force AB may be represented by AE, and half the other force by AF. It is clear that the force AC is the greater [force] at the point A, because the two forces AE and AF act jointly (107), and since AC is equivalent, on account of the parallelogram $AECF$, to both AE and AF. Therefore in place of the force AC we consider the point A to be acted on by the forces AE and AF. Truly we can understand the matter in this way : as if any forces AE and AF should each affect half [the mass] of the point A. Truly these half forces themselves act for the element of time dt as if [for the two half masses] freed from each other, and these in turn finally we suddenly combine again. Because now, as the force AB draws the point A in the element of time dt through the distance Ab, then half the force AE draws half the point in the same time element dt through the same distance Ab (136). Similarly in the time element dt, the other half of the point A will be drawn by AF through the distance $Ad = Ab$. Therefore at the end of the element of time dt the one half point A will be at b, the other at d. Now they may suddenly fit together again with each other, or be drawn together by an infinite force of cohesion, and they come together at the mid-point c of the little line bd : indeed there is no reason why they should meet nearer to b rather than d . Therefore with the forces AE and AF jointly acting in the element of time dt , the point A will be drawn through the small distance Ac. On account of which, the force AC, also being equivalent to the forces AE and AF, acting for the element of time dt, draws the point through the small distance Ac. Indeed bd is parallel to BD and therefore $Ab : Ac = AB : AC$. Therefore with the small distance Ab given, through which the point A is pulled by the force AB, [p.60] the small distance Ac is given, through which the same point A is pulled by the other force AC in the same element of time. And likewise it is apparent, if

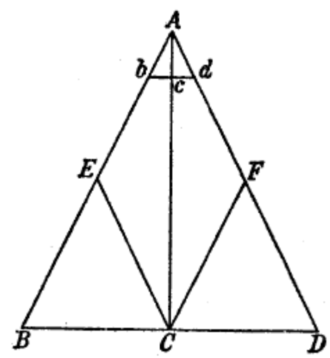


Fig. 15.

the effect *Ac* of the smaller force *AC* itself were given, so much greater will the effect *Ab* be of the greater force *AB*. Q. E. I.

Corollary 1.

147. Therefore the distances, through which equal points are to be pulled by any forces in equal time intervals, are as the forces themselves.

Corollary 2.

148. Since the distances moved from the beginning described by unequal time intervals are in the square ratio of the times (133), the distances will be, by which equal points by any forces for unequal time intervals will be dragged, in the ratio composed from simple forces and the square of the times.

Scholium.

149. From the first principles that we have used in the solution of this problem, it is well known in this respect that a body under the influence of many forces can be divided into an equal number of parts, any one part of which is pulled by the one force. Then when the individual parts are pulled forwards by their forces for an instant of time, it is understood that finally they are compelled to come together suddenly into a single point. The place where this happens, in which they gather together, is that same point to which the whole body would be pulled, acted upon likewise by all of the forces together for the same time for the whole body. The truth of this principle can be seen according to this, that the parts of bodies can be conceived to be connected together most strongly elastically [as by elastic threads] [p.61], and which, though the applied forces act incessantly during the interval, yet they fail at the end suddenly and the parts are able to contract as if by an infinite force put in place; thus in order that the time taken for the free parts to be reduced into one in turn shall be as nothing. Truly many other mechanics problems can now be solved by the use of this same principle. And many other problems have been adopted to non-separated bodies, in the case where the forces are not continuous, but are able to suddenly exercise their effect. Moreover with this principle admitted, it is clear that there are two equal lines upon which the points approach each other and they have to meet in the middle.

PROPOSITION 19.

THEOREM.

150. *A point can be moved along the direction AM* (Fig. 16) *and it is acted on, while it traverses the small distance Mm, by a force p pulling in the same direction ;the increase in the speed, that the point meanwhile acquires, is as the product of the force by the short time, in which the element of distance Mm is traversed.*

DEMONSTRATION.

Let the element of time be dt, and in this time the point completes the distance $M\mu$, if it is not acted on by a force, but has the speed that it had at M, and it can go on moving uniformly. Truly the effect of the force is in accordance with this : as the point is drawn further forwards by μm, this extra small distance is equal to that, by which the same point initially at rest, will be drawn forwards by the same force acting for the same element of time dt, as the force is absolute, [p. 62] (111). The increment of the speed is proportional to the time for this given distance. But if the force is the constant, then the increase of the speed is in proportion to the element of the time dt (130). [For if the force is not constant, then it will not have the same effect at different places.] Whereby when the small distance $m\mu$, or increment of the speed shall be as the force p for a given short time interval, then the increment of the speed for any short time and for any force shall be as pdt, i. e. as the product of the force taken with the short time.

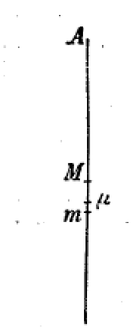


Fig. 16.

Q. E. D.

Corollary 1.

151. Let the speed of the point at M be c and the element of distance be $Mm = ds$, then $dt = \frac{ds}{c}$, since the time determined from the uniform motion in the element Mm can be put in place. Moreover since dc shall be as pdt, also dc will be as $\frac{pds}{c}$ or cdc will be as pds. Therefore the increment of the square of the speed is proportional to the product of the force by the length of the element traversed.
[Thus, the time to travel the distance Mm with speed c is simply $dt = Mm/c$; the extra speed dc generated in this time dt is equal to the acceleration a times by the time, or $a \times dt$; which is proportional to pdt, or to $\frac{pds}{c}$ as shown. Hence, $cdc \propto pds$; and on integrating for a constant force : $c^2 \propto p.s$. This would now be thought of as the conversion of work done by the constant force into kinetic energy. The reader should bear in mind that the terms work, power , strength, etc, did not have the specific physics-related meanings then that they now have; this makes the business of translating more difficult, as these words with special meaning should not be used without qualification. In addition, results of this nature cannot be referenced to a general principle; the work-energy principle still lay some time in the future, though there were rumblings about it in

the Bernoulli camp. Euler in sect. 153 then considers the extra distance as proportional to the force by the square of the time, from which the increase in the speed is again proportional to the force times the time increment.]

Corollary 2.

152. Therefore it is apparent that not only is this theorem true, but also it is true by necessity, as thus it would involve a contradiction to put $dc = p^2 dt$ or $p^3 dt$ or another function in place of p. All of which and equally commendable are considered by the most distinguished Daniel Bernoulli in *Comment. Tom.* I, and I have been greatly influenced with the rigor of the demonstration of these propositions.

[Daniel Bernoulli, *Examen principiorum mechanicae*. Comment. acad.sc.Petrop.I. (1726), 1728, p. 126 – 141; see in particular p. 127. See above.]

Scholium.

153. The demonstration of this proposition follows easily from (148), from which the element of distance $m\mu$ emerges as proportional to the [p.63] to the force p multiplied by the square of the time dt, thus so that $m\mu$ shall be as pdt^2 . But $m\mu$ divided by the time dt gives the increment of the speed, whereby the increment of the speed is as pdt, as was enunciated in the proposition.

PROPOSITION 20.

THEOREM.

154. *The motion of the point in a direction in agreement with the direction of the force, the increment of the speed will be as the force taken with the element of time, and divided by the quantity of matter of the point is composed.*

DEMONSTRATION.

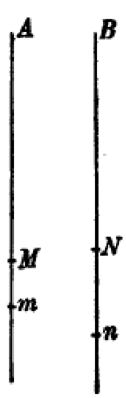


Fig. 17.

Let there be two points or unequal bodies A and B (Fig. 17) in motion along the line AM and BN. These are influenced by the forces p and π respectively, while they traverse the distances Mm and Nn, and the times in which these are traversed are dt and $d\tau$. It is clear that the point B is affected by the force π in the same manner as the point A by the force $\frac{A\pi}{B}$ (136). Whereby by putting in place of the point B equal to the point A, for the force π there must be substituted the force $\frac{A\pi}{B}$, and in this way we obtain the case of the preceding proposition, for which the points are put equal. Hence on account of this, the increment of the speed in traveling through the distance Mm is as the increase of the speed through the distance Nn as pdt is to $\frac{A\pi}{B}d\tau$ or as $\frac{pdt}{A}$ to $\frac{\pi d\tau}{B}$ (150). From which the proposition is agreed upon, that the increase of the speed is as the product of the force and the time divided by the mass or quantity of matter of the point .Q. E. D. [p.64]

Corollary 1.

155. If the speed of the point A were c, then $dc = \frac{npdt}{A}$, where n in all cases denotes the same number; that depends neither on the size of the force, on the element of the time, or on the quantity of matter. [There is now a constant of proportionality, enabling equations to be used rather than proportionalities. Eventually Euler sets this to ½ for convenience, as he is not required to adhere to a set or units as we now are. Later, when the need arises, he absorbs this constant into his equations to produce the correct experimental acceleration in the units chosen. In his later works, Euler does adopt standards of mass and length and time, and moves away from ratios : see e.g. his introduction to the motion of rigid bodies presented in these translations in vol. 3]

Corollary 2.

156. The quantity of matter comes under consideration here, in as much as , in as much as it resists the tendency to be disturbed by the force i. e. as much as it agrees with the force of inertia. On this account, the increment of the speed is directly as the force acting and the of the element of time acting, and inversely as the force of inertia of the body.

Corollary 3.

157. With the distance put as $Mm = ds$ the the time $dt = \frac{ds}{c}$. Hence the increment in the speed becomes $dc = \frac{npds}{Ac}$ or $cdc = \frac{npds}{A}$. Whereby the increment of the square of the speed is proportional to the product from the force by the distance traveled divided by the mass or the mass or the force of inertia of the body.

Scholium.

158. This proposition embraces all the principles expounded upon this far concerning the nature of motion to be defined and the laws of motion. On account of which if this proposition is joined with fourteen, (118), by which the effect of forces acting at angles are determined, all the principles will be put in place, from which the motion for any points can be found from any forces acting.

[p.65]

Corollary 4.

159. Since $dc = \frac{npdt}{A}$, the distance through which the point A is lead by the force p in the element of time dt will be equal to $\frac{npdt^2}{A}$. This distance is indeed the product of dc by dt. For by saying that this distance is dz, then $dc = \frac{dz}{dt}$ (128) and thus $dz = dcdt = \frac{npdt^2}{A}$.

[Recall that in Ch.I, Euler said that he would consider steady motion in increments or intervals, with the step in the speed occurring at the start of the interval, rather than a continuously acting force; in this he was of course just following Newton, who adopted this procedure in evaluating centripetal force by a sequence of small forces acting in succession in steps. If a Newtonian posture is adopted, as it were, then it can be said, as even Euler said in his preface, that he was merely re-inventing the *Principia* using analysis, which one must admit to being true; but it is not the whole truth, for the analytical method finally freed people from the shackles of geometry, and the whole subject of dynamics was given a re-birth and was enabled to move on. Thus, a glance at a formula reveals in a second what may take hours to appreciate geometrically. Euler was quite cynical in the preface, of the state of play of this early calculus, which he has also used in his earlier papers, and from which he was now free to apply and change as he saw fit, which could not always be done with the geometrical reasoning used initially.]

PROPOSITION 21.

PROBLEM.

160. *To determine the effect of any oblique forces acting on a moving point.*

SOLUTION.

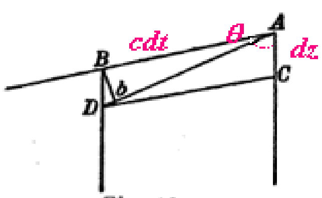


Fig. 13

Let the point *A* (Fig. 13 repeated, [in which *BAb* is isosceles and the angle *BAb* is incremental.]) have the speed *c* in the direction *AB*. [*A* also refers to the mass of the body, while *np/A* is the acceleration in some set of units with constant of proportionality *n*.] Indeed it is acted on by a force *p*, the direction of which *AC* makes an angle with *AB*, the sine of which is *k*. It is evident that the point *A* left to itself unless acted on by a force progresses along the line *AB* and in an element of time *dt* travels through the distance *AB* = *cdt* (30). Truly with the force *p* acting the point A will be deflect from the line *AB* and meanwhile travel along the element of distance *AD*, as has been shown in Prop. 14 (118). Moreover we have put AC or *BD* = *dz* there in the diagram, which is the element of distance through which the point A, if it should be at rest would be drawn forwards by the force *p* in the time *dt*. Hence it follows that $dz = \frac{npdt^2}{A}$ (159) [Recall that the final speed in the increment is taken as the speed throughout the increment]. Therefore the sine of the angle BAD which has been found, is equal to $\frac{kdz}{cdt}$ (124) [as $\sin\theta = k$ and $dz/\sin BAD \approx cdt/k$], which is equal to $\frac{nkpdt}{Ac}$. And the increment of the speed *dc* [= *Db/dt*] which was equal to $\frac{dz\sqrt{(1-kk)}}{dt}$ (123), is now equal to $\frac{npdt\sqrt{(1-kk)}}{A}$. Q. E. I. [p. 66]

Corollary 1.

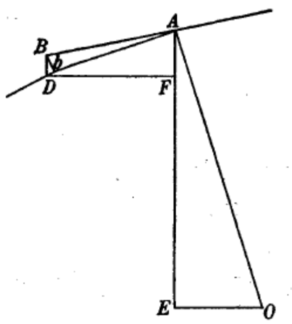


Fig. 18.

161. The distance AD is called ds (Fig. 18), and the element of time by $dt = \frac{ds}{c}$, then with $\frac{ds}{c}$ in place of dt above there is produced $dc = \frac{npds\sqrt{(1-kk)}}{Ac}$. The perpendicular DF is drawn from D to the direction of the force AE; let $AF = dy$ and $DF = dx$, then [the element of the distance squared] $ds^2 = dx^2 + dy^2$ and $k = \frac{dx}{ds}$ and $\sqrt{(1-kk)} = \frac{dy}{ds}$. Hence it comes about that $dc = \frac{npdy}{Ac}$ or $Acdc = npdy$.

Corollary 2.

162. [A circle] is drawn to the curve described by the small body in this way, at the point A with radius of osculation AO, and $Bb : AB = AD : AO$. [Essentially the law of the centripetal force at that point.] Whereby $AO = \frac{AB.AD}{Bb}$. Since $\frac{Bb}{AB}$ is the sine of the angle BAD, which was found to equal $\frac{nkpdt}{Ac}$. Hence $\frac{Bb}{AB} = \frac{npdxdt}{Acds}$, and on account of AD = ds it comes about that : $AO = \frac{Acds^2}{npdxdt}$.

Corollary 3.

163. Since it is the case that $dt = \frac{ds}{c}$, then $AO = \frac{Ac^2ds}{npdx}$. The radius of the osculating circle is taken as $AO = r$, and hence we have $nprdx = Ac^2ds$.

Corollary 4.

164. If the direction AE of the force p is incident along the normal AO, then there arises $AF = dy = 0$ and $DF = dx = AD = ds$. On account of which it follows that $cdc = 0$, and therefore the force does not change the speed.

Corollary 5.

165. Again in this case it follows that $npr = Ac^2$ on account of $dx = ds$, and $r = \frac{Ac^2}{np}$. Therefore this force, the direction of which is normal to the direction of the body [p. 67] results in the arc of the [equivalent circular] curve, as the body is not able to complete its rectilinear motion. [The centripetal force is $np = \frac{Ac^2}{r}$. This situation arises in projectile motion at the highest point, where dy is zero. We should perhaps recall that the force always acts downwards, while the initial speed is at any angle we choose, in Euler's derivation of the equations governing motion in two dimensions that he has presented here step by step, in a very careful manner, that I have tried to reproduce in this translation.]

Corollary 6.

166. If the direction of the force p is incident along the tangent AB, then $dx = 0$ and $dy = ds$. In this case we have $Acdc = npds$. Therefore the force acting in this direction will give the body the greatest increase in speed. [This corresponds to motion vertical downwards under gravity.]

Corollary 7.

167. If the direction of the force p is incident in the opposite direction to AB , thus in order that it is contrary to the direction of the motion of the body, the quantity p becomes negative, and we have $Acdc = -npds$. Therefore in this case the speed is decreased by the same amount as it was increased before. [Motion vertically upwards under gravity.]

Corollary 8.

168. Moreover in each case in which the direction of the force p is incident along the tangent, and $r = \frac{Ac^2ds}{np.0}$, on account of $dx = 0$. Therefore in that case the direction of the body will not then change, and it will accelerate in a straight line.

Corollary 9.

169. Therefore any case the value of the constant letter n determined by experiment will be put in place for all cases. Therefore than everything which could be wished for in the motion will be given absolute values.

Corollary 10.

170. From the first corollary there arises $A = \frac{npdy}{cdc}$. With this value substituted in the third corollary, we have $nprdx = \frac{npcdyds}{dc}$ or $rdxdc = cdyds$. [p. 68]
In which equation neither n, A, nor p is present, and this prevails for any motion of the body whatsoever, and for any force to be acting.

Corollary 11.

171. Nevertheless however, although the force p itself is not to be found in this equation, yet the direction of this, upon which the relation of the elements dx et dy depends, still remains. Therefore from the given direction of the force acting at any point on the curve and from the curve itself, along which the point may be moving, from these alone the speed at any point can be determined. For indeed it will be given by :

$\frac{dc}{c} = \frac{dyds}{rdx}$ or $c = e^{\int \frac{dyds}{rdx}}$ where e specifies the number the hyperbolic logarithm of which is 1.

[This was a great moment in mathematics: for it was the first time that e had been used as the inverse function of the hyperbolic logarithm, for which $\log e = 1$, and for some reason it stayed attached. We should note however that other letters had been used for the same constant quantity previously in Euler's early papers, and that he had already used the letters *b, c*, and *d* for other constant quantities here. There was thus no special meaning to be assigned to the choice ; as we have seen, he tended to use letters from the beginning of the alphabet to represent constant quantities, and letters from the end to represent variables. Euler also used π for the first time in his work in the next chapter : he was not the first to use π as the periphery to diameter ratio for the circle, but no doubt his adoption of the symbol made it popular. You find in Euler's work a sort of 'loving care' for symbols and numbers; he occasionally used verbs associated with human emotions to express happenings in the world of numbers.]

Corollary 12.

172. Again since $dt = \frac{ds}{c}$, then $t = \int e^{-\int \frac{dyds}{rdx}} ds$. Hence therefore it should be noted that likewise for the time in which any part of the motion is to be described, only the curve itself and the direction of the force needs to be given.

Corollary 13.

173. If from O the perpendicular OE is sent to the direction of the force AE, then $ds : dx = AO : AE$. [See Fig. 18.] Hence with the position $AE = q$, which line is called the co-radius by certain people, will be $\frac{rdx}{ds} = q$. Hence the speed becomes

$c = e^{\int \frac{dy}{q}}$ and $t = \int e^{-\int \frac{dy}{q}} ds$. [Joh. Bernoulli, *Concerning the motion of heavy bodies, pendula and projectiles.* Acta erud. Lips. 1713; *Opera Omnia,* Book I, Lausanne et Geneve 1742, p. 531.] [p. 69] [Thus, if $dx = ds$ as in (165) then $r = q$ and $dy = 0$ and we have circular motion with c given by a constant. Again, for motion vertically downwards (166), $dx = 0$ and $dy = ds$, which is a degenerate case.]

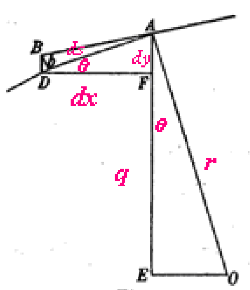


Fig. 18.

Scholium.

174. From the solution of this problem it appears to be possible to determine the motion of a small body, acted on by any kind of forces. For the motion is defined by two equations : both the speed of the body anywhere, and the curvature or the radius of osculation of the curve traversed. Indeed with these known, likewise the time can be found, in which the motion along some part of the curve is completed, which is sufficient for the motion to be determined. [As no examples are provided here, we are to imagine that Euler has just come across this result, and has not yet followed through with any consequences. He now moves on to another 'pet theory'.]

Definition 13.

175. *The force of restitution is that imaginary infinite force, which restores the separate parts of the body again to their previous state.* We considered a force of this kind to be present in the solution of Prop. 18 (146), by which the two parts of the small body, which momentarily we considered to be free, recombined again.

Corollary 1.

176. If a point is considered to be divided into two points and these were separated by forces, the restoring force draws these into the middle line, as was shown (146) with sufficient explanation at the start.

Corollary 2.

177. Since the effect of the restoring force must be produced instantaneously, the restoring force must be considered as provided by an infinite elastic force, by which the separated parts are again joined together.

[p. 70]

Scholium.

178. The use of this force of restitution now is clear in a certain way from proposition 18, yet the use of this will be most fully understood when we are to begin investigating the motion of bodies of finite magnitudes in what follows. Truly we will investigate the effect of this here with many separate parts of a point joined together, which will be of great use in what follows. Hence a certain principle of restitution is embraced, with the help of which many questions are easily resolved, that we call the principle of restitution.

PROPOSITION 22.

THEOREM.

179. *Let there be two parts of a point separated at b and d* (Fig. 19)*; I say that these are to be joined together by a force of restitution in the point c, at the centre of gravity of the particular b and d.*

DEMONSTRATION.

In the first place these parts are joined together at A, and these are then pulled apart by the forces *AB* and *AD* to the points *b* and *d* in the same element of time *dt*. Truly the force *AC* is equivalent to these two forces, which in the same element of time act on the point and pull the element to *c* from *A*. Therefore it is clear that the parts *b* and *d* must be drawn together by the force *c*, since the force *AC* has the same effect acting on the whole point *A* as both *AB* and *AD* acting on the two parts of this (149). Hence it is therefore to be understood that *c* is the point of concurrence, into which the small parts *b* and *d* are driven together by the resisting force. By which moreover [p. 71] the small mass *b* is pulled forwards by the force *AB* for the element of time *dt* through the small distance *Ab*, ought to be $Ab = \frac{n.AB.dt^2}{b}$ (159) or $AB = \frac{Ab.b}{ndt^2}$. Likewise on account of the ratio, $AD = \frac{Ad.d}{ndt^2}$ and $AC = \frac{Ac.(b+d)}{ndt^2}$. Hence truly AC is the diagonal of the parallelogram, that is constituted from the forces AB and AD, which is equivalent to these. Moreover from these equations it can be deduced that $\frac{AB}{Ab} + \frac{AD}{Ad} = \frac{AC}{Ac}$; truly AB, AD, and AC are between themselves as the sines of the angles DAC, BAC, and BAD. On account of which $\frac{\sin DAC}{Ab} + \frac{\sin BAC}{Ad} = \frac{\sin BAD}{Ac}$, from which property it follows that the points *b, c*, and *d* are in the directions given. From which, $bc : cd = \sin BAC.Ab : \sin DAC.Ad = AD.Ab : AB.Ad$. But *AD : AB = Ad.d : Ab.b*. Consequently it follows that *bc:cd = d:b* or *b.bc = d.dc*. From which it is understood that the point *c* is the centre of gravity of the particular *b* and *d*. Q. E. D.

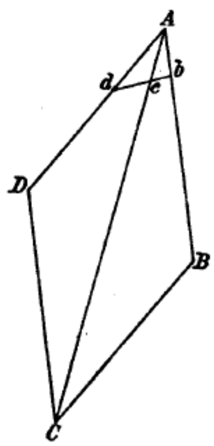


Fig. 19.

Corollary 1.

180. Therefore the point A taken in any place, always falls upon the position of the same point of concurrence *c*, from which it is apparent that the constant restoring force does not depend either on the position of the point *A* nor on the particular particles of A *b* and *d* acted upon [p. 72].

Scholium.

181. This force of restitution agrees uncommonly well with the effect of an elastic force, that it is permitted to put in its place. For an elastic string *bd* can join together the small parts *b* and *d*, that by acting together make *b* and *d* meet in *c*. Moreover this force acts to draw the particles *b* and *d* together equally, since each is trying to contract equally. For truly the distance, by which *b* and *d* are drawn together in the same time, vary inversely as the sizes of the particles themselves (159), since they are affected by the same force. Whereby if the point of concurrence is *c*, *bc* and *dc* will vary inversely as *b* to *d* or *b.bc* = *d.cd*. From which also it is understood that the point *c* is the centre of gravity of the particular masses *b* and *d*.

Corollary 2.

182. Therefore although the force of restitution is imaginary and only exists in the form of thoughts, yet the effect of this follows the real laws of motion. And from this we can be more sure with the aid of this principle of restitution to always arrive at the truth. [Note that this is not to be confused with the coefficient of restitution that is a more modern concept involving elastic and inelastic collisions.]

PROPOSITION 23.

THEOREM.

183. *Let a, b, c, and d be parts of a point mutually separated, which are to be joined together again by the force of restitution, and these are in agreement with a common centre of gravity g.*

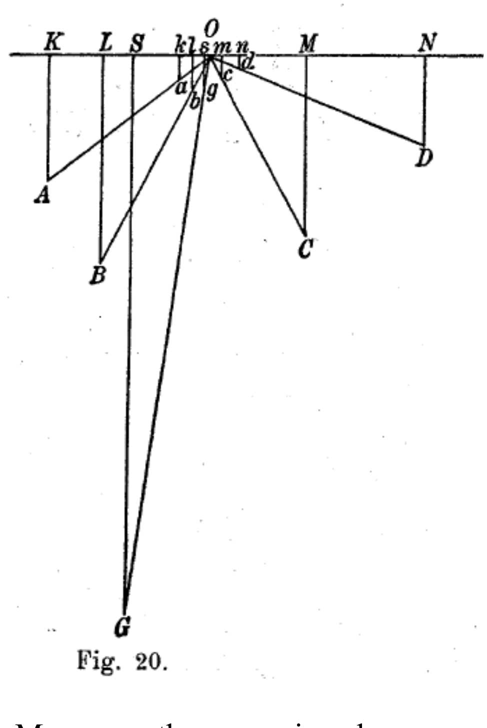


Fig. 20.

DEMONSTRATION.

Initially we put the whole mass at some point O (Fig. 20), from which these individual parts *a, b, c,* and *d* in the element of time *dt* by the forces *OA, OB, OC, OD* are to be drawn out into *a, b, c, d*. *OG* is taken as the equivalent of these forces, which in the same element of time, pulls the whole point mass forwards from *O* to *g*; and *g* is the point at which all the parts *a, b, c,* and *d* will be drawn together by the force of restitution (149). [p. 73] Through the point *O* some line *KN* is drawn, and to that line perpendiculars are sent from the points *A, a; B, b; C, c; D, d; G, g*. Moreover, these are given by :

$OA = \frac{Oa.a}{ndt^2}, OB = \frac{Ob.b}{ndt^2}, OC = \frac{Oc.c}{ndt^2}, OD = \frac{Od.d}{ndt^2}$ and $OA = \frac{Og.(a+b+c+d)}{ndt^2}$ (159). But from the similar triangles *OAK, Oak*; *OBL, Obl*, etc, it follows that :

$$AK = \frac{OA.ak}{Oa} = \frac{ak.a}{ndt^2}, BL = \frac{OB.bl}{Ob} = \frac{bl.b}{ndt^2}, CM = \frac{OC.cm}{Oc} = \frac{cm.c}{ndt^2},$$

$$DN = \frac{OD.dn}{Od} = \frac{dn.d}{ndt^2}, \text{and } GS = \frac{OG.gs}{Og} = \frac{gs.(a+b+c+d)}{ndt^2}, \text{ and}$$

$$OK = \frac{OA.Ok}{Oa} = \frac{Ok.a}{ndt^2}, OL = \frac{OB.Ol}{Ob} = \frac{Ol.b}{ndt^2}, OM = \frac{OC.om}{Oc} = \frac{Om.c}{ndt^2},$$

$$ON = \frac{OD.On}{Od} = \frac{On.d}{ndt^2}, \text{and } OS = \frac{OG.Os}{Og} = \frac{Os.(a+b+c+d)}{ndt^2}.$$

But since OG is the force equivalent to the forces *OA, OB, OC, OD*, it is agreed from statics that AK + BL + CM + DN = GS, et OK + OL – OM – ON = OS.
Hence we find $ak.a + bl.b + cm.c + dn.d = gs.(a+b+c+d)$ and
$Ok.a + Ol.b - Om.c - On.d = Os.(a+b+c+d)$.
From which properties it is understood that the point g is the centre of gravity of the particles *a, b, c, d*. Hence the force restoring these particles agrees with the common centre of gravity g. Q. E. D.

Corollary 1.

184. Therefore the effect of the force of restitution is in agreement with this, that bodies separated into any number of parts can be brought together at the common centre of gravity. [p. 74]

Corollary 2.

185. Therefore in this manner, the motion of a point acted on by many forces can be determined without considering equivalent forces, provided that from the individual forces some number of parts are put in place to be affected for any short time interval, and to be brought together again by the restituting force.

Scholium.

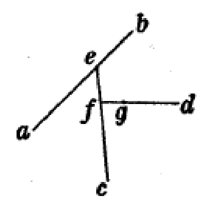


Fig. 21.

186. The demonstration of this theorem with the help of contracting elastic strings can be done in the same manner as we did before (181). For let the separate particles (Fig. 21) be at *a*, *b, c,* and *d*, and in the first place we only put a string to that draws the particles *a* and *b* together, these fit together with the centre of gravity *e*. Now we consider the particles *a* and *b* located at *e* to be joined with the particle at *c*, it will be the point of concurrence *f*, which is the centre of gravity of the three points *a, b*, and *c*. Now these three placed at *f* are joined to the fourth particle located at *d*, and it will be the point of concurrence *g,* the centre of gravity of the four points *a, b, c* and *d*. Whereby from the force of restitution all the particles come together at the common centre of gravity.

Corollary 3.

187. Therefore it is agreed once more correctly that the restoring force arising from the contraction of elastic fibers can be used to represent the join between each two particles. [p. 75]

GENERAL SCHOLIUM.

188. Therefore from these principles put in place, from which the motion of a free point can be determined from the action of some number of forces, we will progress to the motion of free points that have to be investigated. Indeed it will be convenient to separate this tract into two parts, in the first of which only rectilinear motion will be examined, and in the other any kind of curvilinear motion. The first rectilinear motion, as it is understood from what has been said, arises when the motion agrees with the direction of the force ; and truly the other, when these directions disagree. Truly we will deal with each part in two ways for the two kinds of forces involved, and then indeed for both the absolute and the relative forces taken together. Indeed, in place of the relative force we will substitute a medium with resistance, which as now we have reminded, are to be considered; for the relative forces are to be determined for the motion of bodies in fluids (116), on account of which these will be mainly for the kinds found in nature, and neither shall we say much about these which are only to be found in the imagination, and over which we will not tarry. First therefore we will handle the rectilinear motion of a free point acted on by absolute forces. Then we will investigate the rectilinear motion of a free point in a resisting medium. Thirdly we set out the curvilinear motion of a free point acted on by any absolute forces. Fourthly we set out the curvilinear motion of a free point in a resisting medium.

CAPUT SECUNDUM

DE EFFECTU POTENTIARUM IN PUNCTUM LIBRUM AGENTIUM

[p. 39]

DEFINITIO 10.

99. ***Potentia est vis corpus vel ex quiete in motum perducens vel motum eius alterans***.
Huiusmodi vis ideoque et potentia est gravitas; per eam enim corpora , remotis impedimentis, ex quiete deorsum delabuntur, motusque ipse descensus ab ea continuo acceleratur.

Corollarium.

100. Omne corpus sibi relictum vel in quiete perseverat vel motu aequabili in directum progreditur. Quoties igitur evenit, ut corpus liberum, quod quiescebat, moveri incipiat aut motum vel non aequabiliter vel non in directum progrediatur, causa est potentiae cuidam adscribenda : quicquid enim corpus de statu suo deturbature valet, potentiam appellamus.

Scholion 1.

101. Doctrina de potentiis, quatenus plures corpori applicatae in aequilibrio consistunt corpusque in quiete conservant, iam in Statica est exposita. Ibique potentia ita quoque est definita, ut denotet omne id, quod corpora movere valet. Motus vero ipse [p. 40] in Statica non consideratur, sed ii tantum casus investigantur, quibus plures potentiae sese destruunt, corpusque, in quod agunt, in quiete permanet. Nunc autem in Mechanica explicandum est, quomodo potentiae in corpus agentes, quae inter se non sunt contrariae, re ipsa motum producant in corpore quiescente, in motu vero motum immutent.

Scholion 2.

102. Utrum huiusmodi potentiae ex ipsis corporibus originem suam habeant, an vero per se tales in mundo, hic non definio. Sufficit enim hoc loco potentias in mundo revera existere, id quod vel sola vis gravitatis, qua omnia corpora terrestria deorsum delabi conantur, docet. Praeterea vero huiusmodi vires corpora sollicitantes conspiciuntur in motibus planetarum, qui nisi a quadam potentia essent affecti, uniformiter in lineis rectis progredi deberent. Similes etiam potentiae deprehenduntur in corporibus magneticis et electricis inesse, quae certa tantum corpora attrahunt. Quas omnes a motu materiae cuiusdam subtilis oriri alii putant, alii ipsis corporibus vim attrahendi et repellendi tribuunt. Quicquid autem sit, videmus certe ex corporibus elasticis et vorticibus huiusmodi potentias originem ducere posse, suoque loco inquiremus, num ex inde phaenomena haec potentiarum explicari possint. Interim vero potentiarum quarumvis in corpora effectus determinare conibimur, quo deinceps, [p. 41] cum ad eorum pleniorem cognitionem perventum fuerit, statim, quae eruta sunt, ad eas accommodari queant.

DEFINITIO 11.

103. ***Directio potentiae est linea recta, secundum quam ea corpus movere conatur.*** Ita gravitatis directio est linea recta verticalis, corpora enim gravia secundum eam delabi conantur.

Scholion 1.

104. In Statica, ubi omnia in quiete permanere ponuntur, omnes potentiae suas directiones perpetuo easdem servare statuuntur. At in Mechanica, cum corpus perpetuo in alium perveniat locum, potentiae in id agentis directio continuo mutabitur. Pro diversis enim corporis locis vel potentiae directiones erunt inter se parallelae vel ad fixum punctum convergentes, vel aliam tenebunt legem, ex quo tam multiplex potentiarum in Mechanica tractatio oritur.

Scholion 2.

105. Potentiarum diversarum comparatio et mensura ex Statica quoque est repetenda. In qua traditum est potentiam aliquam a se habere ad aliam b ut m ad n, quando potentia a puncto A (Fig. 11) n vicibus secundum directionem AB applicata, et potentia b m vicibus secundum directionem contrarium AC, puncturm A perseverat in aequilibrio. Tum enim potentia a, n vicibus sumta, aequavalet potentia b, m vicibus sumtae, eritque $na = mb$ seu $a : b = m : n$.

Fig. 11.

[p. 42]

Scholion 3.

106. In hoc vero differt mensura potentiarum Mechanica a Statica, quod in hac omnes magnitudinem eandem retinere ponuntur, in Mechanica vero, ut perveniente corpore in alium locum earum directiones mutabiles ponuntur, ita earum quantitas secundum certam legem variabilis esse potest.

PROPOSITIO 13.

THEOREMA.

107. *Quando punctum a pluribus potentiis est sollicitatum, eundem ab iis adipiscetur motum, ac si ab unica iis omnibus aequivalente fuisset sollicatum.*

DEMONSTRATIO.

Sit punctum A (Fig. 12) sollicitatum a potentiis *AB, AC, AD, AE,* quibus aequivaleat potentia *AM*. Sumatur huic aequalis et contrarie posita *AN*; haec, ut ex Statica notum est, destruet actionem potentiarum *AB, AC, AD, AE*. Primo igitur momento potentia *AN* tantum imprimeret puncto A motum secundum *AN*, quantum potentiae *AB, AC, AD, AE* simul agentes ei imprimerent secundum earum mediam directionem, quae est AM. Potentia vero *AM* sola, quia aequalis est potentiae *AN*, tantum quoque promovebit punctum *A* versus *AM*, quantum *AN* versus *AN*. Quare potentia *AM* tantum etiam puncto *A* imprimet motum secundum *AM*, quantum potentiae *AB, AC, AD, AE* simul agentes secundum [p. 43] eandem directionem AM. In utroque igitur casu effectus erit idem. Q. E. D.

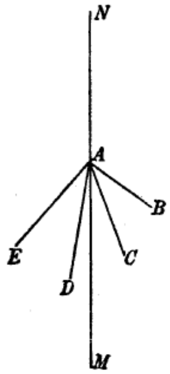


Fig. 12.

Corollarium 1.

108. Si igitur punctum a pluribus potentiis sollicitetur, poterit id tanquam ab unica sollicitatum considerari, quae iis omnibus est aequivalens.

Corollarium 2.

109. Atque vicissim loco unius potentiae in punctum agentis possunt plures in id agentes considerari, quibus illa aequivaleat; id quod, ut ex Statica manifestum est, infinitis modis fieri potest.

Scholion.

110. Quia vero, quam primum corpus de loco suo est motum, potentiae in id agentes directiones suas et magnitudines mutant vel mutare ponuntur, potentia quoque aequivalens quovis momento erit alia. Hanc ob rem quovis momento potentiarum punctum sollicitantium aequivalens debet investigari, neque id diutius ab eadem potentia affici ponendum est quam per temporis elementum infinite parvum.

DEFINITIO 12.

111. ***Potentia absoluta est potentia, quae in corpus sive motum sive quiescens aequaliter agit.*** Huiusmodi potentia absoluta est vis gravitatis, quae corpora, sive moveantur sive quiescant, aequaliter deorsum trahit. [p. 44]

Corollarium.

112. Si igitur cognitus fuerit potentiae absolutae effectus in corpus quiescens, innotescet quoque effectus in corpus utcunque motum.

DEFINITIO 13.

113. ***Potentia relativa est, quae aliter agit in corpus quiescens, aliter in motum.*** Huiusmodi potentia est vis fluvii corpus secum abripientis; quo enim celerius corpus movetur, eo vis fluvii in id fit minor : eaque prorsus evanescit, quando corpus iam eandem, quam habet fluvius, celeritatem est adeptum.

Corollarium 1.

114. Si igitur data sit corporis celeritas una cum lege potentiae relativae, inveniri poterit vis, quantum potentia in corpus agit. Haecque deinde ut potentia absoluta poterit considerari, quamdiu corpus eandem habet celeritatem, eiusque effectus ex potentiarum absolutarum actione determinari. Vim enim potentiae relativae in corpus data celeritate motum determinare.

Corollarium 2.

115. Hoc igitur differunt a se invicem potentiae absolutae et relativae, quod potentiae absolutae quantitas et directio a solo corporis, in quod agit, loco pendeat; relativae vero quantitas et directio insuper a corporis, in quod agit, celeritate. [p. 45]

Scholion 1.

116. Respiciunt potissimum potentiae relativae ad motum corporum in fluidis; horum enim actio in corpora a celeritate eorum pendet relativa; quae quo est maior, eo quoque maiorem vim corpus a fluido patitur. Praeter alios autem casus motuum in fluidis. qui maiorem cognitionem fluidorum requirunt, duo sunt tractatu faciliores; alter, quando fluidum quiescit, alter, quando movetur uniformiter in directum. Poterit vero iste ad illum substituendo motum relativum loco absoluti semper reduci; fluidum scilicet ut quiescens considerandum est, in quo statu quoque vi propria permanebit. Quae igitur in sequentibus de potentiis relativis proferentur, ea ad motum corporum in fluidis quiescentibus potissimum pertinebunt. Actio vero fluidorum in corpora mota consistit tota in motu eorum diminuendo et propterea resistentia appellatur, quae, quo celerius corpora moventur, maior est quoque, et evanescit omnino, quando corpora quiescunt. Hanc ob rem in posterum loco potentiarum relativarum media resistentia substituemus; motus vero, qui a solis potentiis absolutis afficiuntur, in vacuo fieri ponemus.

Scholion 2.

117. Motus quidem in mediis resistentibus, si maxime ordinem sequi vellemus, ad ultimam partem, quae fluidis est destinata, esset referenda, cum etiam nunc non constet, qua lege fluida corporibus in iis motis resistant. [p. 46] Verum quia haec materia a plerisque ita tractari est solita, ut prorsus a fluidorum natura sit revocata et uti hypothesis pure mathematica considerata : hanc methodum retinere malui quam plurima elegantia problemata praeterire, quae in tractatine de fluidis etiam locum non inveniunt. Attamen hanc medii resistentiam non nisi punctorum motui accommodabo; pro corporibus enim finitae magnitudinis calculis fieret insuperabilis. Quando autem corpora instar punctorum considerari possunt, hoc inde nascitur commodum, quod directio vis resistentis congruat cum motus directione, si quidem ea a fluido quiescent oriatur. Hanc ob rem hac de motu punctorum tractatione potentiis relativis eandem semper directionem tribuemus, quam habet ipsum punctum, eamque semper ut motum diminuentem considerabimus.

PROPOSITIO 14.

PROBLEMA.

118. *Dato effectu potentiae absolutae in punctum quiescens, invenire effectum eiusdem potentiae in punctum idem quomodocunque motum.*

SOLUTIO.

Sit punctum in A (Fig. 13) positum, unde moveatur celeritate *c* secundum directionem *AB*, potentiae vero in id agentis directio sit *AC*. Assumatur temporis aliquod elementum *dt*, hocque tempusculo protrahatur punctum *A*, si quiesceret in *A*, per spatiolum *AC*, [p. 47]quod vocetur *dz*, ita ut post tempus *dt* non amplius sit in *A*, sed in *C*. Hic motus igitur per *AC* erit effectus potentiae in punctum quiescens. Effectus vero eiusdem potentiae,

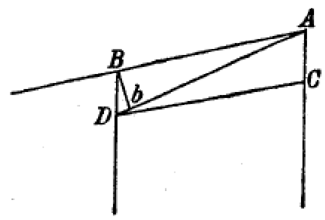


Fig. 13.

quae ponitur absoluta, in idem punctum motum aequalis esse debet effectui in quiescens (111). Abscindatur nunc in puncti *A* directione, quam habet secundum *AB*, spatium *AB*, quod celerate sua *c* tempusculo *dt* percurret, si a nulla potentia sollicitaretur, erit *AB* = *cdt* (30). Agente vero potentia post temporis elementum *dt*, punctum non reperietur in *B*, sed alibi in *D*, ita ut effectus, qui mensurandus est deviatione a puncto *B*, quae est spatium *BD*, aequalis sit effectui eiusdem potentiae in punctum quiescens (111), i. e. *AC*. Erit ergo *BD* = *AC*. Praeterea vero erit *BD* ipsi *AC* parallela, quia *BD* est effectus potentiae, ideoque in eius directionem incidere debet, quae durante tempusculo infinite parvo *dt* non mutator. Quamobrem punctam *A* celeritatem *c* habens secundum directionem *AB* et sollicitatum a potentia absoluta, elapso tempuscul *dt*, non in *B*, set *D* reperietur, ducta *BD* aequali et parallela ipsi *AC*. Spatia vero infinite parvo tempusculo percursa ut lineolae rectae possunt considerari; propterea punctum tempusculo *dt* spatium *AD* percurrisse censendum est. Q. E. I.

Corollarium 1.

119. Quia etiam motus per spatiola infinite parva pro aequabilibus haberi possunt (33), erit celeritas, qua elementum AD percurritur, $=\frac{AD}{dt}$ (30) [p. 48]

Corollarium 2.

120. Ponatur celeritas per AD = c + dc, quia praecedens erat c (35), erit $c + dc = \frac{AD}{dt}$; at ante erat $AB = cdt$, ex quo sit $c = \frac{AB}{dt}$. Prodit ergo $dc = \frac{AD - AB}{dt}$. Abscindatur igitur in AD portio $Ab = AB$, erit $dc = \frac{Db}{dt}$.

Scholion 1.

121. Notandum autem est AC vel BD infinities esse minorem quam AB, nam AB est spatium celeritate finita tempora dt percursum, at AC spatiolum celeritate infinite parva eodem tempore absolutum; corpori enim quiescenti nulla potentia finitam celeritatem tempusculo infinite parvo potest inferre.

Corollarium 3.

122. Hanc ob rem angulus BAD erit infinite parvus, et iunctis punctis B et b lineola Bb erit in AD perpendicularis. Vocetur sinus anguli BAC, quippe qui datur, k, posito sinu toto 1, erit sinus ang. BDb etiam k, quia est $BD = AC = dz$, erit $Db = dz\sqrt{(1-kk)}$ et $Bb = kdz$.

Corollarium 4.

123. Incrementum igitur celeritatis dc, quod ante inveneramus $= \frac{Db}{dt}$, erit $\frac{dz\sqrt{(1-kk)}}{dt}$.
Intelligitur vero dz esse infinities minus quam dt; est enim dz infinite parvum respectu AB, i. e. cdt, ideoque etiam respectu ipsius dt, quia c ponitur finitae magnitudinis. [p.49]

Corollarium 5.

124. Invento celeritatis incremento dc a potentia illato considerandus quoque est angulus BAD declinationem puncti ab insita directione AB repraesentans, quae itidem a potentia producitur. Est vero eius sinus $= \frac{Bb}{AB} = \frac{kdz}{cdt}$.

Corollarium 6.

125. Duplex igitur est effectus potentiae punctum motum sollicitantis. Aliter in celeritate immutanda consistit, alter in eius directione. Ille dat incrementum celeritatis $dc = \frac{dz\sqrt{(1-kk)}}{dt}$, hic vero anguli declinationis sinum $= \frac{kdz}{cdt}$.

Corollarium 7.

126. Si angulus BAC fuerit rectus ideoque $k = 1$, erit $dc = 0$. Hoc igitur casu celeritas a potentia manet immutata. Anguli vero declinationis BAD sinus fit $= \frac{dz}{cdt}$.

Corollarium 8.

127. Si angulus *BAC* fit obtusus seu recto maior, erit eius cosinus $\sqrt{(1-kk)}$ negativus, et propterea celeritatis incrementum *dc* prodibit negativum $=-\frac{dz\sqrt{(1-kk)}}{dt}$. Id quod indicat celeratem a potentia diminui. Declinatio vero $\frac{kdz}{cdt}$ eadem manet, quae ante.

Corollarium 9.

128. Si potentiae directio *AC* cum motus puncti A directione *AB* congruit, sit $k = 0$. Hoc [p. 50] igitur casu motus directo a potentia non immutatur. Celeritatis vero incrementum *dc* fiet $=\frac{dz}{dt}$, si potentiae directio conspirat cum directione motus. Sin autem ei fuerit contratia, fit $dc=-\frac{dz}{dt}$.

Scholion 2.

129. Apparet itaque ex huius propositionis solutione, quomodo poterntiae absolutae effectus in punctum quomodocunque motum inveniri debeat, si cognitus fuerit effectus eiusdem potentiae in idem punctum quiescens. Hanc ob rem in sequentibus huius capitis propositionibus sufficiet punctum a potentiis sollicitatam quiescens ponere vel motum in eadem, quam habet potentia, directione. Nam si punctum *A* (Fig. 14) habeat celeritatem *c* eaque moveatur secundum directionem *AB*; interea vero sollicitetur a potentia eandem habente directionem *AB*, ita ut elapso tempusculo *dt* non in *B*, quo sola celeritate *c* latum perveniret, sed in *b* reperiatur, erit potentiae effectus spatiolum *Bb*. Atque per tantumdem spatiolum *ao* punctum *A*, si in *a* quiesceret, fuisset eodem tempusculo *dt* pertractatum. Innotescit ergo ex motu puncti *A* a potentia sollicitati effectus eiusdem potendiae in idem punctum quiescens, porroque hinc effectus potentiae in punctum utcunque motum.

Fig. 14.

PROPOSITIO 15.

PROBLEMA.

130. *Dato celeritatis incremento, quod quaedam potentia in puncto A tempusculo dt producit, invenire incrementum celeritatis,*[p. 51] *quod eadem potentia in eodem puncto tempusculo dτ producit.*

SOLUTIO.

Habeat punctum *A* (Fig. 14) celeritatem *c* eandemque directionem *AB*, quam habet potentia id sollicitans, sitque *ao* spatiolum, per quod potentia punctum *A*, si quiesceret, tempusculo *dt* traheret. Sit porro *AB* spatium, quod punctum *A* celeritate *c* tempusculo *dt* percurrit, percurret idem insuper a potentia sollicitatum spatium *Ab*, sumto *Bb* = *ao* : hocque spatium, quia est infinite parvum, aequabili motu descripsisse aestimandum est. Sequente igitur tempusculo *dt* hac celeritate percurreret spatium *bC* = *Ab*, nisi a potentia

sollicitetur; at agente iterum potentia, quae immutata manere ponitur saltem per tempus infinite parvum perveniet id ultra *C* in *c*, sumto $Cc = ao$. Simili modo tertio tempusculo dt percurret spatium $de = dE + Ee$, ubi rursus est $dE = Cd$ et $Ee = ao$. Est vero
$Ab = AB + ao; \quad bc = AB + 2ao; \quad cd = AB + 3ao; \quad de = AB + 4ao.$
Erit ergo $\frac{ao}{dt}$ incrementum celeritatis tempusculo *dt* productum a potentia; $\frac{2ao}{dt}$ erit celeritatis incrementum tempusculo 2*dt* acquisitum; similiter $\frac{3ao}{dt}$ incrementum tempusculi 3*dt*; et generaliter tempusculo *ndt* crescet celeras puncti *c* elemento $\frac{nao}{dt}$. Ponatur $ndt = d\tau$, erit $n = \frac{d\tau}{dt}$. Celeritatis igitur incrementum tempusculo $d\tau$ asquisitum erit $\frac{ao.d\tau}{dt^2}$. Quia vero pro tempusculo *dt* incrementum celeritatis est $\frac{ao}{dt}$, prodibit ista analogia: celeritatis incrementum temp. *dt* est ad celeritatis incrementum temp. $d\tau$ acquisitum ut *dt* ad $d\tau$. Consequenter celeritatis incrementa sunt temporibus, quibus produntur, proportinalia. Q. E. I.

Corollarium 1.

131. Apparet haec incrementa celeritatis non pendere ab ipsa celeritate *c*, sed eundem habitura esse valorem, quantumvis magna aut parva ponatur *c*. Hocque natura potentiarum absolutarum melius intelligitur, quod aequaliter agant in corpora mota et quiescentia. [p. 52]

Corollarium 2.

132. Si ergo fuerit $c = 0$, punctumque A quiescensa a potentia ad motum sollicitetur, erunt ipso motus initio celeritates acquisitae ut tempora : scilicet duplo tempore duplam, triplo triplam adipiscetur celeritatem.

Corollarium 3.

133. Si igitur ipso motus initio celeritas tempusculo *t* acquisita dicatur *c*, et spatium percursum sit *s*, erit $t = nc$. Sed est etiam $t = \int \frac{ds}{c}$ (37). Ergo prodit $nc = \int \frac{ds}{c}$ seu $ncdc = ds$ hincque $s = \frac{nc^2}{2} = \frac{t^2}{2n}$. Spatia igitur primo motus initio descripta sunt in duplicata ratione temporum sive celeritatum per ea spatia acquisita. [p. 53]

Scholion 1.

134. Veritas huius propositionis, quod celeritatis incrementa temporibus, quibus generantur, sint proportionalia, etiam in quantitatibus finitis constat, si modo potentia punctum sollicitans manet eadem et eadem perpetuo retinet directionem, quam habet ipse puncti motus. In infinite parvis hac restrictione non est opus; potentia enim utcunque variabilis per tempusculum quam minimum nullius mutationes capax est consideranda . Quomodo autem se habeant diversarum potentiarum effectus, mox sumus expondituri, atque etiam in punctis, quae a potentiis sollicitantur, diversitatem ponemus, ut aliud in data ratione maius minusve esse possit. Neque vero haec inaequalitas adversatur extremae punctorum parvitati, non enim puncta mathematica intelligimus, sed physica, ex quorum compositione corpora oniuntur. Possunt enim duo plurave in unum coalescere concipi. quod, quanquam simplicibus est maius, infinite tamen exiguae manet magnitudinis.

Scholion 2.

135. Theoremate hoc ex solutione problematis invento primus est usus GALILAEUS ad motum gravium delabentium investigandum. Eius quidem demonstrationem non dedit, sed tamen propter insignem eius cum phaenomenis congruentiam de eo amplis dubitari noluit. Refutavit vero etiam alias hac de re opiniones, quo suam sententiam non parum confirmavit. [p. 54] Alii enim statuebant celeritatis incrementa non temporibus, sed spatiis percursis esse proportionalia; huius vero absurditas a GALILAEO iam tunc plerisque Philosophis erat persuasa. Apparet autem, si potentiarum actiones hanc sequerentur legem, nulla corpora ad motum perduci unquam posse. Foret enim $dc = nds$ et $c = ns$, tempus vero t, quod est $\int \frac{ds}{c}$, evaderet $= \frac{1}{n}\int \frac{ds}{c} = \frac{1}{n} lc = \frac{1}{n} lns$ + const., quae constans esse debet $= -\frac{1}{n} l0n$. Tempus scilicet logarithmo spatii descriptio per divisi esset proportionale et propterea infinitum. Nullum igitur corpus ex quiete unquam ad motum posset perduci. Recte itaque GALILAEUS adversariis respondit, quod in instanti motus finitus hoc posito generari deberet, alioquin motum produci prorsus non posse. Etsi enim in initio infinite parva in puncto ponatur celeritas, ea tamen ab huiusmodi potentia imaginaria nunquam effici poterit finita. Ex data vero problematis solutione intelligitur legem inventam necessariam esse, neque ullam aliam, vi principii contradictionis, existere posse.

PROPOSITIO 16.

THEOREMA.

136. *Potentia q in punctum b eundem habet effectum, quem potentia p in punctum a, si fuerit $q : p = b : a$..*

DEMONSTRATIO.

Ponatur $q = np$, erit $b = na$. Concipiatur iam punctum na in n partes aequales divisum, quarum quaelibet erit = a; [p. 55] harum partium unaquaeque sollicitata sit a parte n-sima ipsius potentiae np, id est a potentia p. His positis quaevis pars eodem modo trahetur a sua potentia, quo punctum ipsum a a potentia p. Neque vero hae puncti na partes a suis potentiis sollicitatae a se invicem segregabuntur; sed perpetuo unitae manebunt, si quidem initio fuerint coniunctae. Perspicuum autem est hos duos casus eodem redire nec a se invicem discrepare, sive punctum na a potentia np trahatur, sive quaevis puncti na pars a simili parte p potentiae np trahatur, dummodo partes non a se invicem divellantur. Quapropter constat propositum na aeque a potentia np urgeri ac punctum a a potentia p. Q. E. D.

Corollarium 1.

137. Punctum igitur na a potentia np easdem adipiscetur accelerationes, quas punctum a a potentia p.

Corollarium 2.

138. Ad eandem ergo maiori puncto celeritatem inducendam quam minori, opus est maiori potentia, idque tanto maiori, quanto illud punctum maius est quam hoc.

Scholion 1.

139. Propositio ista fundamentum complectitur ad vim inertiae metiendam, hac enim nititur omnis ratio, quare corporum materia seu massa in Mechanicis considerari debeat. Attendi enim oportet ad punctorum numerum, ex quibus corpus movendum [p.56]est constatum, eique massa corporis proportionalis est ponenda. Puncta vero ea inter se aequalia censeri debent, non quae aequae sunt parva, sed in quae eadem potentia aequales exerit effectus. Si igitur universam materiam in huiusmodi aequalia puncta seu elementa concipiamus divisam, quantitatem materiae cuiusque corporis ex numero punctorum, ex quibus est compositum, aestimari necesse est. Vim autem inertiae proportionalem esse huic punctorum numero seu quantitati materiae in sequenti proportione demonstrabimus.

Corollarium 3.

140. Aequalia ergo sunt, quod ad quantitatem materiae attinet, duo corpora, quae ex aequali punctorum numero sunt composita. Atque duo corpora sunt in ratio m ad n, si punctorum, ex quibus constant, numeri teneant rationem hanc m ad n.

Scholion 2.

141. Ostendetur vero in sequentibus hanc ipsam quantitatis materiae mensurandae rationem re ipsa ahviberi et apud omnes esse receptam. Ex pondere enim cuiusque corporis materia solet investigari, ponderique materiae quantitas proportionalis censetur. Corpora autem omnia aequaliter in spatio vacuo descendere per experimenta constat, et propterea omnia a vi gravitatis aequaliter accelerantur. Quo circa necesse est, ut vis gravitatis in singula corpora agens eorum quantitati materiae sit proportionalis. [p.57] Pondus vero corporis indicat vim gravitatis, qua illlud sollicitatur. Quare cum illa sit proportionalis materiae quantitati, ponderatione simul quantitas materiae innotescit, eo ipso sensu, quem hic materiae tribuimus.

PROPOSITIO 17.

THEOREMA.

142. *Vis inertiae cuiuscunque corporis proportionis est quantitati materiae, ex qua constat.*

DEMONSTRATIO.

Vis inertiae est vis in quovis corpore insita in statu suo quietis vel motus aequabilis in directum permanendi (74). Ea igitur aestimanda est ex vi vel potentia, qua opus est ad corpus ex statu suo deturbandum. Diversa vero corpora aequaliter in statu suo perturbantur a potentiis, quae sunt ut quantitates materiae in illis contentae. Eorum igitur vires inertiae proportionales sunt his potentiis. Consequenter etiam materiae quantitatibus sunt proportionales. Q. E. D.

Corollarium 1.

143. Perspicitur simul ex demonstratione idem corpus, sive quiescat sive moveatur, eandem habere semper vim inertiae. Nam, sive quiescat sive moveatur, aequaliter ab eadem potentia afficitur scilicet absoluta. [p.58]

Corollarium 2.

144. Neque vero vis inertiae homogenea est cum ulla potentia : fieri enim non potest, ut corpus quantumvis magnum a minima potentia non afficiatur, ut in sequentibus demonstrabitur.

Scholion.

145. Apparet hinc origo vocis vis inertiae, quam supra (76) innvimus, ex eo, quod vis inertiae actioni potentiarum quodammodo resistat. NEUTONUS quoque qui in Princ. Phil;. Nat. Definitione III cum vi inertiae et haec vi resistendi eandem ideam coniungit et utramque quantitati materiae proportionalem statuit.

PROPOSITIO 18.

PROBLEMA.

146. *Dato effectu unius potentiae in punctum aliquod, invenire effectum cuiusvis alius potentiae in idem punctum*

SOLUTIO.

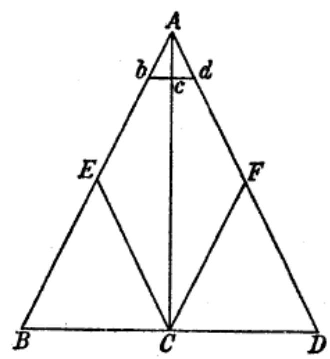


Fig. 15.

Quiscat punctum in *A* (Fig. 15) et consistat potentiae datae *AB* in id effectus in hoc, quod ab ea tempusculo *dt* per spatiolum *Ab* deducatur. Quaeritur iam, per quantum spatium idem hoc punctum tempusculo *dt* ab alia potentia *AC* protrahatur. Ducantur lineae *AB* et *AC*, ita ut iuncta *BC* sit in *AC* normalis, id quod semper fieri potest, si $AC < AB$. At si $AC > AB$, solutio ex illa facile deducetur. Ex altera parte ducatatur recta *AD*, ita ut *BAD* sit triangulum isosceles [p.59]. Bisecentur *AB* et *AD* in *E* et *F*, et per *AE* repraesentetur dimidium potentiae *AB* et per *AF* tantadem potentia. Manifestum est potentiam *AC* idem preastare in punctum *A*, quod duae potentiae *AE* et *AF* coniunctim (107), quia *AC* aequivalet ob parallelogrammum *AECF* ambabus *AE* et *AF*. Loco igitur potentiae *AC* ponamus punctum *A* sollicitari a potentiis *AE* et *AF*. Hoc vero modo rem concipiamus, quasi quaelibet potentia *AE* et *AF* dimidium puncti *A* afficiat. Medietates vero istae sint ad tempusculum *dt* saltem a se invicem solutae, hocque finito eas subito ad se invicem rursus accedere ponemus. Quia nunc potentia *AB* punctum *A* tempusculo *dt* per spatium *Ab* protrahit, protrahet dimidia potentia *AE* punctum dimidium eodem tempusculo *dt* per idem spatium *Ab* (136). Similiter tempusculo *dt* altera medietas puncti *A* a potentia *AF* protrahetur per spatium $Ad = Ab$. Finito igitur tempuslculo *dt* altera medietas puncti *A* erit in *b*, altera in *d*. Coeant nunc rursus subito ad se mutuo seu contrahantur vi cohaesionis infinita, convenient in puncto medio *c* lineolae *bd* : nulla enim est ratio, quare propius ad *b* quam *d* conveniant. *A* potentiis ergo *AE* et *AF* coniunctim agentibus punctum *A* tempusculo *dt* per spatiolum *Ac* protrahetur. Quamobrem etiam potentia *AC* aequivalens potentiis *AE* et *AF* tempusculo *dt* protrahet per spatiolum *Ac*. Est vero *bd* parallela ipsi *BD* et propterea $Ab : Ac = AB : AC$. Dato igitur spatiolo *Ab*, per quod punctum *A* a potentia *AB* protrahitur, [p.60] dabitur spatiolum *Ac*, per quod idem punctum *A* ab alia potentia *AC* protrahitur eodem tempusculo. Atque simul patet, se effectus *Ac* minoris potentiae *AC* fuerit datus, quantus sit maioris *AB* effectus *Ab.* Q. E. I.

Corollarium 1.

147. Spatia igitur, per quae aequalia puncta a quibuscunque potentiiis protrahuntur aequalibus temporibus, sunt ut ipsae potentiae.

Corollarium 2.

148. Quia spatia motus initio descripta inaequalibus temporibus sunt in duplicata ratione temporum (133), erunt spatia, per quae aequalia puncta a quibuscunque potentiis inaequalibus temporibus protrahuntur, in ratione composita ex ratione simplici potentiarum et duplicata temporum.

Scholion.

149. Principium, quo in huius problematis solutione sumus usi, in hoc constit, ut corpus a pluribus potentiis sollicitatum in totidem partes concipiatur divisum, quarum qualibet ab una tantum potentia trahatur. Deinde cum singulae a potentiis suis momento temporis fuerint protractae, subito ad se mutuo compelli in unumque congredi intelligantur, quo facto locus, in quo convenerunt, erit is, ad quem integrum corpus ab omnibus potentiis simul agentibus eodem tempore fuisset pertractum. Veritas huius principii ex hoc potest perspici, quod corporis partes elastris fortissimis coniunctae possint concipi, [p.61]quae, quanquam indesinenter agunt, tamen per intervalla cedere moxque se subito contrahere vi infinita poni possunt, ita ut tempus, quo partes sulutae ad se invicem reducuntur, sit nullum. Eodem vero hoc principio iam alii in pluribus mechanicis problematibus solvendis sunt usi. Atque plerique adoptaverunt istud re non diversum, quod potentias non indesinenter, sed per saltus effectum suum exercere posuerunt. Hoc autem principio admisso manifestum est duas partes aequales recta ad se mutuo accedere et intervalli sui medio congredi debere.

PROPOSITIO 19.

THEOREMA.

150. *Moveatur punctum in directionem AM* (Fig. 16) *et sollicitetur, dum per spatiolum Mm percurrit, a potentia p secundum eandem directionem trahente ;erit incrementum celeritatis, quod interea punctum acquirit, ut potentia sollicitans ducta in tempusculum, quo elementum Mm percurritur.*

DEMONSTRATIO.

Sit tempusculorum dt, et absolvat punctum hoc tempore spatium $M\mu$, si a potentia non sollicitaretur, sed celeritate, quam in M habuit, uniformiter progredi pergeret. Effectus vero potentiae in hoc consistit, ut punctum ab ea ulterius per μm protrahatur, quod spatiolum aequale est illi, per quod idem punctum, si quiesceret, ab eadem potentia eodem tempusculo dt protraheretur, quia potentia ponitur absoluta. [p. 62] (111). Huic spatio dato tempore proportionale est celeritas incrementum. At si potentia est eadem, celeritatis incrementum est ut tempusculum *dt (130).* Quare cum spatiolum $m\mu$ seu incrementum celeritatis sit dato tempusculo ut potentia p, erit celeritatis incrementum pro quocunque tempusculo et quibuscunque potentiis ut pdt, i. e. ut potentia ducta in tempusculum. Q. E. D.

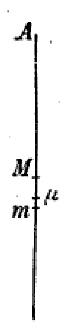


Fig. 16.

Corollarium 1.

151. Sit puncti in *M* celeritas c et spatiolum $Mm = ds$, erit $dt = \frac{ds}{c}$, quia ad tempus determinandum elementum *Mm* motu aequabili describi potendum est. Cum autem sit dc ut pdt, erit quoque dc ut $dt = \frac{pds}{c}$ seu cdc ut pds. Incrementum ergo quadrati celeritatis est ut potentia ducta in spatii elementum percursum.

Corollarium 2.

152. Apparet igitur non solum verum esse hoc theorema, sed etiam necessario verum, ita ut contradictionem involveret ponere $dc = p^2 dt$ vel $p^3 dt$ aliamve functionem loco p. Quae omnes cum Clar. DAN. BERNOULLIO in Comment. Tom. I. aeque probabiles videantur, de rigidis harum propositionum demonstrationibus maxime eram sollicitus.

[Daniel Bernoulli, *Examen principiorum mechanicae*. Comment. acad.sc.Petrop.I. (1726), 1728, p. 126 – 141; vide praecipue p. 127.]

Scholion.

153. Propositionis huius demonstratio facilius sequitur ex (148), unde prodit spatiolum $m\mu$ proportionale [p.63] potentiae p ductae in quadratum tempusculi dt, ita ut sit $m\mu$ ut pdt^2. At $m\mu$ divisum per tempus dt dat incrementum celeritatis; quare celeritatis incrementum erit ut pdt, quemadmodum in propositione erat enunciatum.

PROPOSITIO 20.

THEOREMA.

154. *Congruente puncti directione motus cum potentiae directione erit incrementum celeritatis ut potentia ducta in tempusculum et divisa per materiam seu quantitatem puncti.*

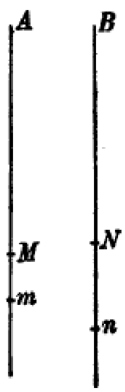


Fig. 17.

DEMONSTRATIO.

Sint duo puncta seu corpuscula inaequalia A et B (Fig. 17) mota in rectis AM, BN. Sollicitentur ea a potentiis p et π respective, dum percurrant spatiola *Mm, Nn*, et sint tempora, quibus ea percurruntur, dt, $d\tau$. Manifestum est punctum B a potentia π eodem modo affici ac punctum A a potentia $\frac{A\pi}{B}$ (136). Quare substituto loco B puncto ipsi A aequali, pro potentia π substitui debet potentia $\frac{A\pi}{B}$, hocque modo obtinemus casum propositionis praecedentis, quo puncta ponuntur aequalia. Hanc ob rem incrementum celeritatis per *Mm* est ad incrementum celeritatis per *Nn* ut pdt ad $\frac{A\pi}{B} d\tau$ seu ut $\frac{pdt}{A}$ ad $\frac{\pi d\tau}{B}$ (150).

Ex quo constat propositum, quod celeritatis incrementum sit ut factum ex potentia et tempusculo divisum per puncti materiam seu quantitatem. Q. E. D. [p.64]

Corollarium 1.

155. Si igitur celeritas puncti *A* fuerit *c*, erit $dc = \frac{npdt}{A}$, ubi *n* in omnibus casibus eundem denotat numerum; neque enim a potentia neque a tempusculo neque a puncti quantitate pendet.

Corollarium 2.

156. Quantitas materiae A hic in considerationem venit, quatenus potentiae sollicitanti reluctatur, i. e. quatenus congruit cum vi inertiae. Hanc ob rem est celeritatis incrementum ut potentiae sollicitans et tempusculorum directe, atque inverse ut vis corporis inertiae.

Corollarium 3.

157. Positio spatio $Mm = ds$ erit $dt = \frac{ds}{c}$. Hinc fiet $dc = \frac{npds}{Ac}$ seu $cdc = \frac{npds}{A}$. Quare incrementum quadrati celeritatis proportionale est facto ex potentia in spatiolum percursum diviso per massam seu vim inertiae corpusculi.

Scholion.

158. Propositio ista complectitur omnia principia hactenus tradita motus naturum definientia omnesquae leges motus, si quidem potentiae directio cum motus directione congruit. Quamobrem si haec propositio cum decima quarta (118) coniungitur, qua effectus potentiarum oblique agentium determinatur, omnia habebuntur principia, ex quibus punctorum a quibuscunque potentiss sollicitatorum motus possunt inveniri.

[p.65]

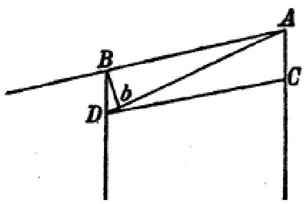
Corollarium 4.

159. Quia est $dc = \frac{npdt}{A}$, erit spatiolum, per quod punctum *A* a potentia *p* tempusculo *dt* perducitur, $= \frac{npdt^2}{A}$. Est enim hoc spatiolum factum ex *dc* in *dt*. Nam dicto hoc spatio *dz* est $dc = \frac{dz}{dt}$ (128) adeoque $dz = dcdt = \frac{npdt^2}{A}$.

PROPOSITIO 21.

PROBLEMA.

160. *Potentiae cuiuscunque in punctum motum oblique agentis effectum determinare.*



SOLUTIO.

Habeat punctum A (Fig. 13) celeritatem c directionemque AB. Sollicitetur vero a potentia p, cuius directio AC cum AB faciat angulum, cuius sinus est k. Perspicuum est punctum A sibi relictum neque a potentia sollicitatum in recta AB esse progressurum tempusculoque dt percursurum spatium $AB = cdt$ (30). Agente vero potentia p declinabit punctum A a recta AB percurretque interea spatiolum AD, ut in prop. 14 (118) est ostensum. Posuimus autem ibi AC seu BD = dz, quod est spatiolum, per quod punctum A, si quiesceret, a potentia p tempore dt pertraheretur. Est ergo $dz = \frac{npdt^2}{A}$ (159). Anguli igitur BAD sinus, qui inventus est $= \frac{kdz}{cdt}$ (124), erit $= \frac{nkpdt}{Ac}$. Atque celeritatis incrementum dc, quod erat $= \frac{dz\sqrt{(1-kk)}}{dt}$ (123), sit $= \frac{npdt\sqrt{(1-kk)}}{A}$. Q. E. I. [p. 66]

Corollarium 1.

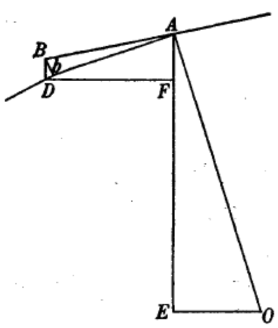


Fig. 18.

161. Vocetur spatium $AD = ds$ (Fig. 18), erit $dt = \frac{ds}{c}$ loco dt prodibit $dc = \frac{npds\sqrt{(1-kk)}}{Ac}$. Ducatur ex D in directionem potentiae AE perpendicularis DF, sitque $AF = dy$ et $DF = dx$, erit $ds^2 = dx^2 + dy^2$ et $k = \frac{dx}{ds}$ et $\sqrt{(1-kk)} = \frac{dy}{ds}$. Proveniet ergo $dc = \frac{npdy}{Ac}$ seu $Acdc = npdy$.

Corollarium 2.

162. Ducatur ad curvam, quam corpusculum hoc modo describit, in A radius osculi AO, erit $Bb : AB = AD : AO$. Est vero $\frac{Bb}{AB}$ sinus anguli BAD, qui inventus est $= \frac{nkpdt}{Ac}$. Erit ergo $\frac{Bb}{AB} = \frac{npdxdt}{Acds}$, and on account of AD = ds there comes about : $AO = \frac{Acds^2}{npdxdt}$.

Corollarium 3.

163. Quia vero est $dt = \frac{ds}{c}$, erit $AO = \frac{Ac^2ds}{npdx}$. Vocetur radius osculi $AO = r$, habebitur $nprdx = Ac^2ds$.

Corollarium 4.

164. Si potentiae p directio AE incidat in normalem AO, fiet $AF = dy = 0$ et $DF = dx = AD = ds$. Quamobrem erit $cdc = 0$, atque idcirco haec potentia celeritatem non immutabit.

Corollarium 5.

165. Hoc porro casu erit $npr = Ac^2$ ob $dx = ds$, atque $r = \frac{Ac^2}{np}$. Haec igitur potentia, cuius directio [p. 67] est normalis in corporis directionem, efficit, ut corpus non in recta motum suum absolvat, sed in arcu curvae.

Corollarium 6.

166. Si potentiae p directio incidat in tangentem AB, fiet $dx = 0$ et $dy = ds$. Habebitur ergo $Acdc = npds$. In hac igitur directione potentia p celeritatem corporis maxime augebit.

Corollarium 7.

167. Si potentiae p directio in oppositam ipsi AB directionem incidat, ita ut motui corporis sit contraria, fiet p quantitas negativa, habebiturque $Acdc = -npds$. Tantum igitur casu minuetur celeritas, quantum ante augebatur.

Corollarium 8.

168. In utroque autem casu, quo directio potentiae p in tangentem incidit, erit $r = \frac{Ac^2 ds}{np.0}$ ob $dx = 0$. Tum igitur directio corporis non mutabitur, sed in recta moveri perget.

Corollarium 9.

169. Determinato ergo in unico casu ex experimento valore litterae n inserviet is pro omnibus casibus. Tum igitur omnium, quae in motibus possunt desiderari, poterunt assignari valores absoluti.

Corollarium 10.

170. Ex corollario primo prodit $A = \frac{npdy}{cdc}$. Quo valore in tertio substituto habebitur

$$nprdx = \frac{npcdyds}{dc} \text{ seu } rdxdc = cdyds. \text{ [p. 68]}$$

In qua aequatione neque n neque A neque p inest, haec valet pro motu puncti cuiuscunque a quacunque potentia sollicitati.

Corollarium 11.

171. Quanquam autem in ista aequatione ipsa potentia p non inest, tamen eius directio, a qua relatio elementorum dx et dy pendet, adhuc superest. Data igitur directione potentiae punctum in quovis loco sollocitantis et ipsa curva, in qua punctum movetur, poterit ex his solis datis determinari puncti celeritas in quovis loco. Erit enim

$\frac{dc}{c} = \frac{dyds}{rdx}$ seu $c = e^{\int \frac{dyds}{rdx}}$ ubi e denotat numerum,cuius logarithmus hyperbolicus est 1.

Corollarium 12.

172. Quia porro est $dt = \frac{ds}{c}$, erit $t = \int e^{-\int \frac{dyds}{rx}} ds$. Hinc igitur simul innotescit tempus, quo quaevis portio describitur, neque ad hoc pluribus opus est datis quam ipsa curva et directione potentiae.

Corollarium 13.

173. Si ex *O* in potentiae directionem *AE* dimittatur perpendicularis *OE*, erit $ds : dx = AO : AE$. Posito ergo $AE = q$, quae linea a quibusdam co-radius appellatur, erit $\frac{rdx}{ds} = q$. Fiet igitur $c = e^{\int \frac{dy}{q}}$ et $t = \int e^{-\int \frac{dy}{q}} ds$. [Joh. Bernoulli, *De motu corpurum gravium, pendulorum et projectilium.* Acta erud. Lips. 1713; *Opera Omnia,* Tomus I, Lausanne et Geneve 1742, p. 531.] [p. 69]

Scholion.

174. Ex solutione huius problematis apparet posse eius beneficio puncti a quibuscunque potentiis sollicitati motum determinari. Ex duabus enim aequationibus definitur et puncti celeritas in loco quovis et curvatura seu radius osculi ipsius curvae percursae. His vero cognitis simul reperietur tempus, quo quaevis curvae portio absolvitur, quae abunde sufficiunt ad motum determinandum.

Definitio13.

175. *Vis restituens est vis illa imaginaria et infinita, quae partes corporis separatas momento rursus congregat et in statum presitinum restituit.* Huiusmodi vim in solutione Prop. 18 (146) adesse concepimus, qua duae puncti partes, quae ad momentum solutae concipiebantur, rursus contrahebantur.

Corollarium 1.

176. Si punctum in duas partes aequales concipiatur divisum eaeque a potentiis fuerint separatae, vis restitutens eas contrahet in medio rectae illas iungentis, ut (146) ex principio sufficientis rationis est ostensum.

Corollarium 2.

177. Quia effectus vis restistuentis in instanti debet produci, poterit vis restituens considerari ut elastrum vi infinita praeditum, quo partes separatae iterum coniunguntur. [p. 70]

Scholion.

178. Usus huius vis restituentis iam elucet quodammodo ex propositione 18, usus vero eius adhuc erit amplissimus in sequentibus, quando motus corporum finitae magnitudinis sumus investigaturi. Hic vero effectum eius indagabimus in coniungendis pluribus puncti partibus separatis, quae inquisitio in sequentibus magnam habebit utilitatem. Complectitur ergo restituens principium aliquod, cuius ope plurimae quaestiones facile resolvi poterunt, idque principium restitutionis vocabimus.

PROPOSITIO 22.

THEOREMA.

179. *Sint duae puncti partes in b et d separatae* (Fig. 19)*; dico eas a vi restituente coniunctum iri in puncto c, centro gravitatis particularum b et d.*

DEMONSTRATIO.

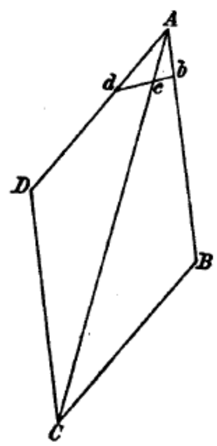


Fig. 19.

Fuerint hae partes primo coniunctae in A, sintque eae a potentiis *AB, AD* pertractae in *b* et *d* eodem tempusculo *dt*. Harum potentiarum vero aequivalens sit potentia *AC*, quae eodem tempusculo integrum punctum ex A pertrahere valeat in *c*. Manifestum igitur est partes *b* et *d* a vi resistente in *c* contrahi debere, quia potentia *AC* eundem in punctum integrum *A* edit effectum ac ambae *AB* et *AD* in duas eius partes (149). Hinc igitur innotescit punctum concursus *c*, in quod particulae *b* et *d* a vi resistuente compellentur. Quo autem [p. 71] particula *b* a potentia *AB* tempusculo *dt* per spatiolum *AB* protrahatur, debet esse $Ab = \frac{n.AB.dt^2}{b}$ (159) seu $AB = \frac{Ab.b}{ndt^2}$. Similem ob rationem erit $AD = \frac{Ad.d}{ndt^2}$ et $AC = \frac{Ac.(b+d)}{ndt^2}$. Est vero AC diagonalis parallelogrammi, quod a potentiis AB et AD constituitur, quia his aequivalet. Ex illis autem aequationibus deducitur $\frac{AB}{Ab} + \frac{AD}{Ad} = \frac{AC}{Ac}$; sunt vero AB, AD, et AC inter se ut sinus angulorum DAC, BAC, et BAD. Quamobrem erit $\frac{\sin DAC}{Ab} + \frac{\sin BAC}{Ad} = \frac{\sin BAD}{Ac}$, ex qua proprietate sequitur puncta *b, c* , et *d* esse in directum posita. Hoc cum sit, erit $bc : cd = \sin BAC.Ab : \sin DAC.Ad = AD.Ab : AB.Ad$. At est $AD : AB = Ad.d : Ab.b$. Consequenter prodibit $bc:cd = d:b$ seu $b.bc = d.dc$. Ex quo intelligitur punctum *c* esse centrum gravitatis particularum *b* et *d*. Q. E. D.

Corollarium 1.

180. Ubicunque ergo accipiatur punctum *A*, semper in eundem incidit locum punctum concursus *c*, ex quo apparet vim restituentem constantem habere effectum neque a loco puncti *A* nec a potentiis [p. 72] particulas *b* et *d* sollicitantibus pendere.

Scholion.

181. Egregie convenit hic vis restituentis effectus cum effectu vis elasticae, quam eius loco substituere licet. Iungat enim particulas *b* et *d* filum elasticum *bd*, quod sese contrahendo *b* et *d* congreget in *c*. Vis autem haec contrahens aequaliter aget in particulas *b* et *d*, cum utrinque se aequaliter contrahere conetur. Spatio vero, per quae *b* et *d* eodem tempore protrahuntur, sunt reciproce ut ipsae particulae (159), quia ab eadem potentia afficiuntur. Quare si punctum concursus est *c*, erit *bc* et *dc* reciproce ut *b* ad *d* seu $b.bc = d.cd$. Ex quo etiam intelligitur punctum *c* esse centrum gravitatis particularum *b* et *d*.

Corollarium 2.

182. Quamvis igitur vis restituens sit imaginaria et in sola cogitatione formata, tamen eius effectus sequitur motus leges reales. Hocque magins erimus certi ope principii restitutionis semper ad veritatem perveniri.

PROPOSITIO 23.

THEOREMA.

183. *Sint a, b, c, d partes puncti a se invicem separatae, quae a vi restituente rursus congregentur, convenient eae in communi centro gravitatis g.*

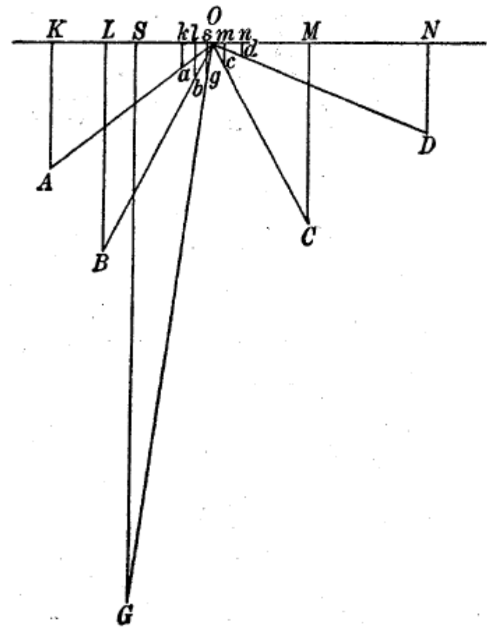


Fig. 20.

DEMONSTRATIO.

Ponamus initio integrum punctum fuisse in puncto quocunque (Fig. 20), ex quo singulae hae partes *a, b, c, d* *t*empusculo *dt* a potentiis *OA, OB, OC, OD* pertractae sint in *a, b, c, d.* Accipiatur harum potentiarum aequaivalens *OG*, quae eodem tempusculo ex *O* pertraxisset in *g*; erit *g* punctum, in quod partes *a, b, c, d* a vi restituente congregabuntur (149). [p. 73] Per punctum *O* ducatur recta quaevis *KN*, in eamque ex punctis *A, a, B, b, C, c, D, d, G, g* demittantur perpendicula. Erunt autem

$$OA = \frac{Oa.a}{ndt^2}, OB = \frac{Ob.b}{ndt^2}, OC = \frac{Oc.c}{ndt^2}, OD = \frac{Od.d}{ndt^2}$$

et $OA = \frac{Og.(a+b+c+d)}{ndt^2}$ (159). At ob triangula simila *OAK, Oak; OBL, Obl*, etc, erit

$$AK = \frac{OA.ak}{Oa} = \frac{ak.a}{ndt^2}, BL = \frac{OB.bl}{Ob} = \frac{bl.b}{ndt^2}, CM = \frac{OC.cm}{Oc} = \frac{cm.c}{ndt^2},$$

$$DN = \frac{OD.dn}{Od} = \frac{dn.d}{ndt^2}, \text{et } GS = \frac{OG.gs}{Og} = \frac{gs.(a+b+c+d)}{ndt^2}, \text{ atque}$$

$$OK = \frac{OA.Ok}{Oa} = \frac{Ok.a}{ndt^2}, OL = \frac{OB.Ol}{Ob} = \frac{Ol.b}{ndt^2}, OM = \frac{OC.om}{Oc} = \frac{Om.c}{ndt^2},$$

$$ON = \frac{OD.On}{Od} = \frac{On.d}{ndt^2}, \text{et } OS = \frac{OG.Os}{Og} = \frac{Os.(a+b+c+d)}{ndt^2}.$$

Sed quoniam OG est potentia aequivalens potentiis *OA, OB, OC, OD*, constat ex Statica esse AK + BL + CM + DN = GS, et OK + OL – OM – ON = OS.
Fiet ergo $ak.a + bl.b + cm.c + dn.d = gs.(a+b+c+d)$ et

$$Ok.a + Ol.b - Om.c - On.d = Os.(a+b+c+d).$$

Ex quibus proprietatibus intelligitur punctum g esse centrum gravitatis particularum a, b, c, d. Vis ergo restituens has particulas in centro communi gravitatis g congregat. Q. E. D.

Corollarium 1.

184. In hoc igitur vis restituentis effectus consistet, quod corpusculi quotcunque partes separatas in ipsarum communi centro gravitatis congreget. [p. 74]

Corollarium 2.

185. Hoc igitur modo puncti a pluribus potentiis sollicitati motus poterit determinari sine potentiae aequivalentis consideratione, dum a singulis potentiis partes quaecunque affici ponantur quolibetque tempusculo iterum a vi restituente congregari.

Scholion.

186. Demonstratio huius theorematis ope filorum elasticorum sese contrahentium eodem modo potest perfici, quo ante fecimus (181). Sint enim (Fig. 21) particulae separatae in *a*, *b, c, d*, et ponamus primo filum particulas *a* et *b* tantum contrahere, coibunt eae in centrum gravitatis *e*. Nunc concipiamus particulas *a* et *b* in *e* locatas cum particula *c* coniungi, erit punctum concursus in *f*, quod est centrum gravitatis trium particularum *a, b*, et *c*. Iam hae tres in *f* positae cum quarta *d* coniungantur, erit punctum concursus in *g,* centro gravitatis omnium quatuor *a, b, c* et *d*. Quare a vi resistuente omnes particulae in commune centrum gravitatis congregantur.

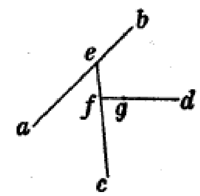


Fig. 21.

Corollarium 3.

187. Denuo igitur constat vim restituentem recte per contractionem filorum elasticorum binas quasque particulas iungentium repraesentari. [p. 75]

SCHOLION GENERALE.

188. His igitur positis principiis, ex quibus puncti liberi a quibuscunque potentiis sollicitati motus determinari poterit, progrediemur ad punctorum liberorum motus investigandos. Hanc vero tractationem in duas partes dispesci conveniet, in quarum prima motus tantum rectilinei examinabuntur, in altera vero curvilinei cuicunque. Illi rectilinei, ut ex dictis intelligitur, oriuntur, quando motus directo cum potentiae directione conveniet; hi vero, quando hae directiones discrepant. Utramque vero partem duplici modo tractabimus, pro duplici potentiarum natura; deinde vero ab absolutis et relativis coniunctim. Loco relativarum quidem substituemus media resistentia, quia, ut iam monuimus, potentiae relativae ad motus corporum in fluidis determinandos considerantur (116), quamobrem eos potissimum casus, qui in rerum natura existunt, evolvemus neque multum iis, qui non nisi in imaginatione reperiuntur, immorabimur. Primum ergo de motu rectilineo puncti liberi a potentiis absolutis sollicitati tractabimus. Deinde investigabimus motus rectilineos puncti liberi in medio resistante. Tertio motus curvilineos puncti liberi a potentiis absolutis utcunque sollicitati evolvemus. Quarto denique motus curvilineos puncti liberi in medio resistente exponemus.