CHAPTER VI

A METHOD FOR DETERMINING THAT CURVE AMONGST ALL THE CURVES ENDOWED WITH SEVERAL COMMON PROPERTIES, WHICH SHALL BE PROVIDED WITH A MAXIMUM OR MINIMUM PROPERTY.

PROPOSITION I. THEOREM

I. *A curve, which amongst all the curves generally has a maximum or minimum expression* $\alpha A+\beta B$ *, the same will be prepared thus likewise, so that amongst all endowed with the same property A, the curve may contain a maximum or minimum value of the formula B.*

DEMONSTRATION

We may consider the curve to be found in which, amongst all the others corresponding to the same abscissa, the value of the expression $\alpha A+\beta B$ shall be a maximum; for what will be shown for a maximum, likewise with the necessary changes will be valid for a minimum. Moreover here the letters *A* and *B* denote for us formulas or indeterminate expressions of this kind, with which the question regarding maxima or minima may be consistent ; then truly α and β are some constant quantities. Now we designate that curve, on which $\alpha A+\beta B$ shall be a maximum, by the letter *Q*, by which we may be able to describe that more easily without troublesome words. Now some other curve *R* may be considered corresponding to the same abscissa, which may take the same value of the formula *A*, which the curve *Q* maintains ; therefore on this curve *R* the expression $\alpha A+\beta B$ will occupy a lesser value, than on the curve *Q*, because there on the curve *Q* the expression $\alpha A+\beta B$ is chosen the maximum value of all. Whereby, since on the curves *Q* and *R* the expression *A* may retain the same value and on *Q* the expression $\alpha A+\beta B$ shall be greater than on the curve *R*, it follows that the value of the expression *B* must be greater on the curve *Q* than on the curve *R*. Therefore since *R* may denote any curve, which may receive a common value of the formula *A* with *Q*, it is evident amongst all these curves *R* the curve *Q* to be that, in which the formula *B* may have a maximum value. From which that curve is constructed, which amongst all the curves generally may have a maximum or minimum value of the expression $\alpha A+\beta B$, the same curve likewise thus to be prepared so that amongst all the other curves endowed with the same common property *A*, it may be able to have a maximum or minimum value of the expression *B*. For although the demonstration has only been set out for the maximum, yet the same with the words transposed will be adapted for the minimum. Q. E. D.

[See Goldstine, p.99, sect. 2.6, for this prop. written in modern terms ; some care has to be taken with the sign of β, which Euler had overlooked.]

COROLLARY 1

2. And thus it is understood in turn, if the curve must be found, which amongst all the other curves endowed with the same common property *A*, *B* shall have the maximum or minimum expression ; moreover the question is satisfied, if that may be defined absolutely amongst all the curves, in which $\alpha A+\beta B$ shall be a maximum or minimum.

COROLLARY 2

3. Therefore in the solution of problems of this kind two new arbitrary constants are introduced α and β, which are not present in the expressions *A* and *B* themselves ; but these in turn will sustain a single constant only, because only the ratio of these will come into a calculation.

COROLLARY 3

4. But if therefore amongst all the curves endowed with the same common property *A* that may be required to be defined, in which *B* shall be a maximum or minimum, then the differential values of each expression *A* and *B* may be taken, which themselves multiplied by arbitrary constants and added together put equal to zero will give the equation for the curve sought.

COROLLARY 4

5. Likewise also it is evident in the same way, whether amongst all the curves endowed with the same common property *A* that may be sought, in which *B* shall be a maximum or minimum, or whether in turn amongst all the curves endowed with the common property *B* that may be sought, in which *A* shall be a maximum or minimum.

SCHOLIUM

6. Any matters we have treated, both in this proposition as well as in the attached corollaries, are now seen most plainly from the preceding chapter, certainly the method by which the inverse problems are to be resolved, in which amongst all the curves blessed with the same common property that may be sought, which shall be endowed with the property of some kind of maximum or minimum. Truly therefore it is not to be considered that we should only be repeating the same argument ; for the same truth, as before we have elicited at great length in a satisfactory manner, here at this point we have given in a very short and brief explanation. Accordingly, perhaps there the former method of explanation will be confirmed by the latter on account of most of each being agreed on, and if the former method may be considered not to be clear enough for this on account of so many infinitely small exceedingly slippery and uncertain parts connected together, the demonstration given here will remove all doubt from that. Then, if from which conversion made in corollary I concerning the present proposition even if doubt may remain, for that the former method may satisfy most fully. Meanwhile an account of the conversion can be taken from that safely enough. For since the curve *Q*, which amongst all the curves generally $\alpha A+\beta B$ may have a maximum or minimum, thus shall be prepared, so that amongst all the curves endowed with the same common property *A*, *B* may have a maximum or minimum, whatever may be taken in place of α and β, it is necessary, that the converse shall be equally apparent, if indeed the greatest range may be

attributed to the coefficients α and β. And indeed to mention this has seemed to declare the validity of this reasoning, so that in the following, where we will use the same, it will be free from doubt. For this proposition, even if it may belong particularly to the preceding chapter, here we have carried it across, so that we may handle the argument of this chapter more easily by the same special method ; certainly so that, if it were worked out by the other method, it would require the most lengthy calculations and the greatest difficulty of differentials of all orders. Yet meanwhile, to the extent it can be done, we will show everything clearly, which we will treat here, to be confirmed and also to be elicited by the above method.

PROPOSITION II. THEOREM

7. *Any curve amongst all the curves generally corresponding to the same abscissa has a maximum or minimum value of the expression* $\alpha A+\beta B+\gamma C$ *, the same curve likewise thus will be prepared, so that amongst all the curves, which both the expression A as well as the expression B have in common* [*i.e., have a common value*], *C may be able to possess a maximum or minimum value of the expression.*

DEMONSTRATION

Here the letters *A*, *B* and *C* denote for us the integral formulas or the indefinite expressions of this kind, which shall be capable of a maximum or minimum, but the letters α, β, γ designate arbitrary constant quantities. Now *Q* shall be the curve, which amongst all the curves generally, $\alpha A+\beta B+\gamma C$ may have a maximum or minimum value, and some other curve *R* may be considered, in which since both the expression *A* as well as *B* shall maintain the same value, which it maintains on the curve *Q* ; from which composite expression put in place $\alpha A+\beta B$ will have the same value in each curve *Q* and *R*. Because of this the whole expression $\alpha A+\beta B+\gamma C$ on the curve *R* will be allocated a smaller value than on the curve *Q*, if indeed $\alpha A+\beta B+\gamma C$ is a maximum on the curve *Q*; on the other hand, the value of the expression $\alpha A+\beta B+\gamma C$ on the curve *R* will be greater than on the curve *Q*, if $\alpha A+\beta B+\gamma C$ were a minimum on the curve *Q*. Therefore since the part $\alpha A+\beta B$ of this expression shall be common to each curve *Q* and *R*, the remaining part γC and thus the expression *C* in the case of a maximum will be greater on *Q* than on *R*, but in the case of a minimum the expression *C* on the curve *Q* will be less than on the curve *R*. From which it follows, if the curve *Q* amongst all the curves generally should have a maximum or minimum value of the expression $\alpha A+\beta B+\gamma C$, then likewise this curve *Q* is to be endowed with that nature, so that amongst all the curves *R*, which may be blessed with the same value both of the expression *A* as well as of the expression *B*, the value of the expression *C* will maintain a maximum or minimum. Q. E. D.

COROLLARY l

8. Because the expressions *A*, *B* and *C* are able to be interchanged among themselves as it pleases, a curve on which $\alpha A+\beta B+\gamma C$ is a maximum or minimum, there likewise either amongst all the curves *A* and *B* endowed with the same common properties, *C* will

have a maximum or minimum, or amongst all the curves *A* and *C* endowed with the same common properties, *B* will have a maximum or minimum, or finally *A* will have a maximum or minimum amongst all the curves, in which both the properties *B* and *C* are shared equally.

COROLLARY 2

9. Therefore any curve amongst all endowed with the same two common properties *A* and *B* has a maximum or minimum *C*, the same will be had amongst all the curves, either *B* will have a maximum or minimum with the properties *A* and *C*, or *A* with the properties *B* and *C,* given equally.

COROLLARY 3

10. Therefore if the curve must be sought, which amongst all the others with the two properties *A* and *B* equally provided, the expression *C* may have a maximum or minimum, then the question will be satisfied, if the curve may be sought, which absolutely amongst all the curves $\alpha A+\beta B+\gamma C$ may have a maximum or minimum expression.

COROLLARY 4

11. Because $\alpha,\ \beta,\gamma$ are constant arbitrary quantities, in the solution of problems of any kind three new arbitrary quantities are to be introduced, which were not present in the proposed formulas *A*, *B* and *C*; but these three constants $\alpha,\ \beta,$ et γ will be equivalent only to two.

COROLLARY 5

12. Truly these constants thus now will be present in the equation for the first curve found; truly besides these by integration just as many new constants will be introduced, as there is a need for in the integrations, before it may reach a finite equation.

COROLLARY 6

13. In a similar manner, by which we have demonstrated this proposition and the preceding one, a curve will be shown, which, absolutely amongst all the curves, the expression $\alpha A+\beta B+\gamma C+\delta D$ may have a maximum or minimum value, the same fourth expression *D* amongst all curves having the three common expressions, is going to have a maximum or minimum.

SCHOLIUM

14. From this Proposition the method of resolving problems of this kind now is understood well enough pertaining to the relative method, in which a curve is sought, which amongst all corresponding to the same abscissa and being endowed with two or more common properties, may have the maximum or minimum value of some expression. The curve sought of course will always be returned to the absolute method, thus so that amongst all the curves being sought generally there shall be a curve, which may have a certain maximum or minimum. And we find it suitable with that reduction, so that we may be able to resolve all the problems of this kind with the help of differential values, which now we have instructed above how to find. But the manner of resolving

itself will be reduced to this, so that all common properties, together with the expression of the maximum or minimum, are explained separately, the individual terms will be multiplied by arbitrary constants and the products gathered into one sum ; with which done it will be necessary to find that absolutely amongst all the curves, in which this same sum shall be a maximum of minimum. Truly this itself is perfected, as long as the value of this differential will be found and will be put equal to zero. On account of which the whole operation may be resolved, if both the common properties or the individual expressions contained as well as the differential values of the maxima or minima may be taken following the rules given above, the individuals separately may be multiplied by arbitrary constants and the sum of all these products may be put equal to zero ; from which the equation for the curve sought will arise. And thus this single precept may be able to suffice to solve questions of this kind. Truly, before we could explain the use of this, it will be appropriate to confirm this method itself by the way used before.

PROPOSITION III. PROBLEM

15. *Amongst all the curves related to the same abscissa, which shall be provided equally with two common properties A and B, to define that, in which the value of the expression C shall be a maximum or minimum.*

SOLUTION

Now from the preceding it is understood this problem to be solved, if amongst all the curves that may be sought absolutely, in which $\alpha A+\beta B+\gamma C$ shall be a maximum or a minimum. But for this it is necessary to know the differential values of the expressions *A*, *B* and *C*. Therefore the value of the differential expression shall be $A=nv\cdot dx\cdot P$, of the expression $B=nv\cdot dx\cdot Q$, of the expression $C=nv\cdot dx\cdot R$; from which the equation for the curve sought will be $\alpha P+\beta Q+\gamma R=0$.

Truly, so that the truth of this expression may be more manifest, we may undertake this same problem by the same method, as we have used in the preceding chapter. But in the first place it is understood (Fig. 15) towards solving this problem three applied lines must be augmented by infinitely small amounts, so that it shall be able to be satisfied by three prescribed conditions. For in the first place these three small parts adjoined, by which the satisfying curve *az* may be changed into a new curve by differing from itself minimally, thus will be necessary to be prepared, so that the expression *A*, which contains one common property, may be present equally in each curve. Then also the other with the common property *B* will have to maintain the same value in each curve. In the third place from the nature of the maximum or minimum, the expression *C* also must find the same value on that curve and with the same change ; from which three conditions [the condition is satisfied], as it cannot be satisfied by fewer than the three particles from the three applied lines. Whereby besides the two applied lines *Nn* and *Oo*, which have been augmented in the figure by the particles *nv* and $o\omega$, it may be considered the particle $p\pi$ to be adjoined to the following applied line *Pp*. And the first increment may be sought, which the expression *A* obtains from the three particles, which will be

$$=nv\cdot Pdx+o\omega\cdot P^{/}dx+p\pi\cdot P^{//}dx.$$

And if the increment $nv\cdot Pdx$ arises from the particle *nv*, agreeing with that value of the differential, which the expression *A* arrives at from the particle *nv* alone. Truly from following particle $o\omega$ the increment $o\omega\cdot P^{/}dx$ arises, evidently the same as before, increased by its differential ; because indeed $o\omega$ is added to the following applied line, all the quantities affecting $o\omega$ are the following of these, by which the particle *nv* is affected ; and from a similar reckoning the increment $p\pi\cdot P^{//}dx$ will arise from the particle $p\pi$; which altogether will be satisfied, and to be accomplished revealed and transparent, if for that one wished for a calculation in that manner, as we have used in proposition 3 of paragraph 22 of the preceding chapter. Therefore again in the same manner the expression *B*, whose differential value arising from the one particle *nv* we have put $= nv\cdot Qdx$, from the three particles $nv,\ o\omega$ and $p\pi$ the increment may be taken

$$= = nv\cdot Qdx + o\omega\cdot Q^{/}dx + p\pi\cdot Q^{//}dx.$$

In the third place the expression *C* augmented by the three particles may take this increment

$$nv\cdot Rdx + o\omega\cdot R^{/}dx + p\pi\cdot R^{//}dx.$$

Now it is required to put these three individual increments each equal to zero, so that it may be satisfied by all the prescribed conditions ; from which the three following equations will arise, with division made by *dx*,

$$0 = nv\cdot P + o\omega\cdot P^{/} + p\pi\cdot P^{//}$$
$$0 = nv\cdot Q + o\omega\cdot Q^{/} + p\pi\cdot Q^{//}$$
$$0 = nv\cdot R + o\omega\cdot R^{/} + p\pi\cdot R^{//}.$$

But if now the particles $nv,\ o\omega,\ p\pi$ being carried through only as an aid be called upon to be eliminated, this equation will arise between the appropriate quantities of the curve, from which hence the nature of the curve will be expressed. But towards eliminating these particles we will multiply the individual equations themselves separately by the new unknown quantities $\alpha,\ \beta,\ \gamma$, so that there may be had :

$$0 = nv\cdot \alpha P + o\omega\cdot \alpha P^{/} + p\pi\cdot \alpha P^{//}$$
$$0 = nv\cdot \beta Q + o\omega\cdot \beta Q^{/} + p\pi\cdot \beta Q^{//}$$
$$0 = nv\cdot \gamma R + o\omega\cdot \gamma R^{/} + p\pi\cdot \gamma R^{//},$$

and hence these equations will be formed :

$$0 = \alpha P \quad + \beta \cdot Q + \gamma \cdot R$$
$$0 = \alpha P' \ + \beta Q' \ + \gamma R'$$
$$0 = \alpha P'' + \beta Q'' + \gamma R'',$$

Here it is apparent at once, if constant quantities may be taken for $\alpha,\ \beta,\ \gamma$, then in the first equation the remaining two besides may be included ;

[Caratheodory (p.24 of his introduction to Vol. 24, Series 1 of the *O.O.*) and Goldstine as above, consider Euler's proof to be flawed at this stage, as he has not shown that $\alpha,\ \beta,\ \gamma$ are in fact the constants as shown; it appears that Euler took an invalid short cut to get final agreement between his two methods.]

if indeed there were $0 = \alpha P + \beta \cdot Q + \gamma \cdot R$, then likewise there will be

$$0 = \alpha dP + \beta \cdot dQ + \gamma \cdot dR \ \text{ and } \ 0 = \alpha ddP + \beta \cdot ddQ + \gamma \cdot ddR\,;$$

and because there is

$$P' = P + dP\ ,\ Q' = Q + dQ\ ,\ R' = R + dR$$

and

$$P'' = P + 2dP + ddP,\ Q'' = Q + 2dQ + ddQ \ \text{ et } \ R'' = R + 2dR + ddR,$$

there becomes also :

$$0 = \alpha P' + \beta Q' + \gamma R'$$

and

$$0 = \alpha P'' + \beta Q'' + \gamma R''\,.$$

On which account for solving the problem this equation is being formed

$$0 = \alpha P + \beta Q + \gamma R\,,$$

which, if in place of $\alpha,\ \beta$ and γ some arbitrary constant quantities may be written, will express the nature of the curve sought. But generally this equation will agree with that, as we have elicited by the other method, and each method may be confirmed by the other.
Q. E. I.

COROLLARY 1

16. Therefore all problems of this kind also can be solved with the help of the differentials arising from one change of the applied line, which above we have taught how to find fully enough.

COROLLARY 2

17. Therefore it is evident, if the curve must be found, which, amongst all the others related to the same abscissa and in which the two expressions *A* and *B* are present equally, *C* may have a maximum or minimum value of the expression, then the question returns to that, which may pertain to the absolute method, so that amongst all the curves generally that related to the same abscissa will be determined, in which the expression $\alpha A+\beta B+\gamma C$ shall be a maximum or a minimum.

COROLLARY 3

18. Likewise truly hence also it is apparent the method of resolving problems, in which amongst all the curves , in which more than two and thus any number of properties may be agreed on equally, that is required, which may be endowed with a certain property of maximum or minimum.

COROLLARY 4

19. Because if indeed amongst all the curves, in which the expressions *A*, *B*, *C, D* may obtain equal values, that must be investigated, in which the expression *E* shall be a maximum or minimum, then the question may be satisfied, if amongst all the curves generally that may be found, in which $\alpha A+\beta B+\gamma C+\delta D+\varepsilon E$ shall be a maximum or a minimum, with the letters $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ some constant and arbitrary quantities.

COROLLARY 5

20. Therefore when more properties may be proposed, which with these curves, from which it is required to investigate the nature of the maximum or minimum endowed, must be common, there more arbitrary constant quantities must enter into the equation for the curve and there also thus more satisfying curves may be dealt with in that.

SCHOLIUM 1

21. Why there more constants may be entering into the solution, where more common properties may be proposed, can be readily deduced from the preceding. For we may consider among all the curves endowed with the same property *A* it is necessary to investigate that, in which *B* shall be a maximum or minimum; and indeed in the first place it will be agreed for this question going to be satisfied by that curve, which amongst all the curves generally *B* may have a maximum or minimum; indeed this amongst all these also, which will be endowed with the same common property *A*, *B* will have a maximum or minimum. But then an innumerable kinds of curves of this kind can be taken, which receive the same value of the expression *A* ; also in one kind truly there will be a single curve, which before the rest may contain a maximum or minimum value of the expression *B*. But it is necessary all these satisfying curves must be contained in the general solution. Therefore since on account of one common prescribed property, the number of satisfying curves becomes infinite, this will be increased much more, on

account of the same reasoning, if several common properties may be proposed. Yet meanwhile, if the values, which the individual common properties have on curves, from which it will be necessary to elicit the question, actually may be defined, then certainly the solution will be provided by a single satisfying curve. Evidently these constants will be looked after there, so that the values, which the common properties will obtain on the curve found, will be determined by choice ; thus a curve will be able to be assigned from these constants in the case of two common properties *A* and *B*, which the given values of the expressions *A* and *B* may receive, and in addition shall be prepared thus, so that amongst all the infinitely many other *A* and *B* receiving the same values of these expressions, *C* may have another maximum or minimum value of some expression. And this same warning may be in place, if several common properties were prescribed ; from which it is evident enough, whatever shall be made from these constants entering into the solution and how these may be required to be adapted for use, that which will be able to be made clearer in the following examples.

EXAMPLE I

22. *Amongst all the curves* (Fig. 14) *related to the same abscissa* $AC = a$, *which both amongst themselves shall be of the same length as well as which may comprise equal areas DAD, to determine that, which rotated about the axes AC will generate a solid of the maximum or minimum capacity.*

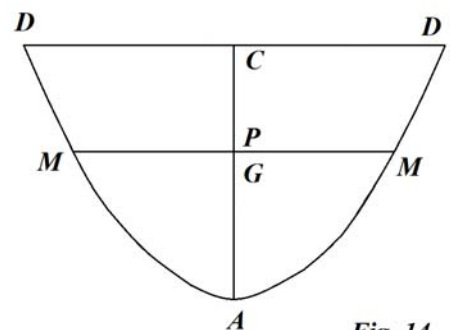

Fig. 14

With the abscissa $AP = x$, the applied line $PM = y$ and $dy = pdx$ the two common properties proposed are $\int ydx$ and $\int dx\sqrt{(1+pp)}$; but the formula of the maximum or minimum is $\int yydx$. Now the differential values of these three formulas are sought. And for the first indeed the value of the differential formula $\int ydx$ will be $= nv\cdot dx$, then of the formula $\int dx\sqrt{(1+pp)}$ the value of the differential is $-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$ and for the third the value of the differential of the formula $\int yydx$ is $= 2nv\cdot ydx$. From which three differential values this equation may be put in place for the curve sought :

$$0=\alpha dx-\beta d\cdot\frac{p}{\sqrt{(1+pp)}}+2\gamma ydx$$

or

$$ccd\cdot\frac{p}{\sqrt{(1+pp)}}=bdx+2ydx=\frac{ccdp}{(1+pp)^{3:2}}.$$

This equation may be multiplied by *p* and integrated ; there will be found :

$$ff + by + 2yy = \frac{-cc}{\sqrt{(1+pp)}},$$

where both cc as well as ff can be taken by choice either positive or negative. Hence again there becomes:

$$(ff + by + 2yy)^2 (1+pp) = c^4 \text{ and } p = \frac{\sqrt{\left(c^4 - (ff + by + 2yy)^2\right)}}{ff + by + 2yy} = \frac{dy}{dx};$$

and thus

$$dx = \frac{(ff + by + 2yy)dy}{\sqrt{\left(c^4 - (ff + by + 2yy)^2\right)}},$$

which is the equation for an elastic curve. Moreover through the integration a single new remaining constant may enter arbitrarily; and from these four constants it will be able to be effected, so that a given curve may pass through two given points; then for the two remaining constants it will be obtained, so that on putting $x = a$ both the area of the curve as well as its length may be followed a given magnitude. But above with the ambiguity of the signs, by which the root is affected by the sign, the one sign will give a maximum of the curve, the other endowed with the property of the minimum. But because in the equation found that magnitude of the abscissa a is not present, it follows that some portion of the curve found corresponding to whatever abscissa must also be endowed, so that amongst the other curves corresponding to the same abscissa and passing through the same two points, which likewise since with that curve both the equal length as well as the equal area are including, so that, I say, that curve rotated about its own abscissa will generate a solid of maximum or minimum capacity. Certainly with the two points, through which the curve sought may pass, thus here are to be led into consideration, because the calculation provides a differential equation of the second order, which by itself requires a two-fold determination. Also truly the two remaining points, which at once are present in the equation found, are able to be determined by the points, and with this agreed upon a determined solution of this kind will emerge, which will show how to describe a curve through four given points, which amongst all the others passing through the same four points and containing both equal lengths as well as equal areas may produce, rotated about its axis, either a maximum or minimum solid. Without doubt always a number of arbitrary constants will be declared, which both actually found in the equation as well as potentially present, will be declared, as many determinations shall be used, so that the curve may be determined ; and this then amongst all the other curves with the same determinations provided will satisfy the question.

EXAMPLE II

23. *Amongst all the lines corresponding to the same abscissa, which in the first place may contain equal areas* $\int ydx$ *and besides rotated about the axis equally will generate equal solids* $\int yydx$, *to determine that, which may have a maximum or minimum height of its centre of gravity, that is, in which* $\frac{\int yxdx}{\int ydx}$ *shall be a maximum or minimum.*

Let the length of the abscissa prescribed, to which the solution will be required to be adapted, $=a$ and for this abscissa the value of the formula becomes $\int ydx = A$, of the formula $\int yydx = B$ and of the formula $\int yxdx = C$. Again the value of the differential of the formula $\int ydx = dA = dx$, of the formula $\int yydx = dB = 2ydx$, and of the formula $\int yxdx = dC = xdx$, without doubt with the values of the differentials of these formulas taken following the rules given above, by omitting only the particle *nv*, clearly which is removed always by division. Now since the expression of the maximum or minimum shall not be a simple formula, but the fraction $\frac{\int yxdx}{\int ydx}$, its differential value will be

$$\frac{AdC - CdA}{A^2} = \frac{Axdx - Cdx}{A^2} ;$$

and on account of the common property of the two $\int ydx$ and $\int yydx$ the differential values given, namely $dA = dx$ and $dB = 2ydx$, the following equation will result for the curve sought :

$$\alpha dx + 2\beta ydx + \frac{\gamma Axdx - \gamma Cdx}{A^2} = 0 \text{ or } \left(\alpha A^2 - \gamma C\right)dx + 2\beta A^2 ydx + \gamma Axdx = 0 ;$$

in which equation since $\alpha,\ \beta,\ \gamma$ shall be arbitrary constants, by a transformation of these likewise the determined constants *A* and *C* are removed from the computation thus, so that the solution found becomes adapted equally to all the abscissas. Moreover this equation may be arrived at : $bdx = mydx + nxdx$ or divided by dx, $b = my + nx$ put in place, which is the equation for some right line. Therefore a right line placed somehow to the vertical axis will have its centre of gravity area either at a maximum or minimum height, amongst all the other lines inclined to the axis, containing both the same area $\int ydx$ as well as the same volume $\int yydx$. But the centre of gravity will be raised the least, if the right line may be inclined upwards to the axis, and moreover the elevation will be greatest, if it may be inclined downwards to the axis [*i.e.* the slope *m* is either positive or negative]; and these are both the cases, in which either the maximum or minimum height of the centre of gravity may have a place. Between these cases is the mean, in which that right line shall be parallel to the axis ; about which doubt can still remain, whether the centre of gravity shall be maximally lowered or maximally raised. Truly this case has not been answered in the question. For with the right line put parallel to the axis, thus so that it shall be $y = b$, then generally no other line can be shown, which for the same abscissa may contain both an equal area $\int ydx$ as well as an equal volume $\int yydx$, and this will arise thus, because this right line amongst all the other lines may include the same area $\int ydx$ comprising the minimum volume $\int yydx$.

EXAMPLE III

24. *Amongst all the curves* (Fig. 21) *of the same length DAD with the given points DD joined to determine that, of which this shall be the property, so that, if between the vertical right lines DB, DB the space NDADN of a given magnitude may be cut by the horizontal NN, the centre of gravity of this space NDADN may obtain the lowest place.*

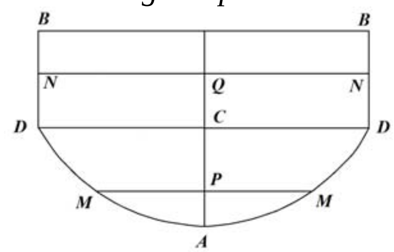

Fig. 21

The solution of this question is exceedingly useful in hydrostatics and may be solved with the help of this problem, in which the cloth figure *DAD* of the vessel *BDDB* joined at the points *DD* may be investigated, which it adopts, if a given amount of water may be poured into the vessel. For in the first place, while the cloth allows no extension , the length of the curve *DAD* will be given, then also the space *NDADN* will be given, so that the quantity of water poured in may be measured, and in the third place according to the general laws of hydrostatics and gravity it will be required to prepare the figure *DAD* thus, so that the lowest centre of gravity may occupy a place of the space *NDADN*. Towards resolving this problem there may be put $DC = CD = a$ and with some line drawn to the horizontal MPM there shall be $MP = PM = x$ and $AP = y$, therefore the arc will be

$$MAM = 2\int dx\sqrt{(1+pp)} \text{ on putting } dy = pdx.$$

But if now the length of the curve *DAD* may be put $= 2b$, the equation between x and y must be prepared thus, so that the integral of the formula $\int dx\sqrt{(1+pp)}$ becomes $= b$ by putting $x = a$. Again the area *MAM* shall become $= 2\int xdy = 2\int xpdx$; which may become $= 2ff$ in the case, where there is put $x = a$; thus so that then there shall be $\int xpdx = ff$. Truly this area has not itself been given, but that with the given area *NDDN* must produce the space, which shall be $= 2cc$. Therefore if there may be put $DN = z$, there will be

$$az + ff = cc \text{ and } z = \frac{cc - f}{a} = \frac{cc - \int xpdx}{a},$$

by putting $x = a$. And then the centre of gravity of the whole space *NDADN* will be distant from the point *A* by the interval

$$= \frac{\int xypdx + az\left(AC + \tfrac{1}{2}z\right)}{cc},$$

by putting $x = a$ after the integration ; therefore the centre of gravity will be placed below the point *C* by the interval

$$= \frac{AC\left(cc - az\right) - \tfrac{1}{2}azz - \int xypdx}{cc},$$

which must be a maximum. Therefore since there shall be $z = \frac{cc - \int xpdx}{a}$, the maximum must be this form :

$$AC\int xpdx - \frac{c^4}{2a} + \frac{cc\int xpdx}{a} - \frac{\left(\int xpdx\right)^2}{2a} - \int xypdx.$$

And thus the problem returns to this, so that amongst all the corresponding curves of the same given length of the abscissa $DC = a$ that may be defined, in which this expression shall be a maximum

$$h\int xpdx + \frac{cc}{a}\int xpdx - \frac{1}{2a}\left(\int xpdx\right)^2 - \int xypdx,$$

with $y = h$ on putting $x = a$. Now, because the length of the curve is $= \int dx\sqrt{(1+pp)}$, the value of its differential will be $= -d \cdot \frac{p}{\sqrt{(1+pp)}}$. Then the value of the differential of the formula $\int xpdx$ is $= -dx$ and the value of the differential of the formula

$$\int xypdx = xpdx - d \cdot xy = -ydx.$$

Hence the value of the differential of the whole expression emerges, which must be a maximum,

$$= -hdx - \frac{cc}{a}dx + \frac{ff}{a}dx + ydx,$$

which on account of the undetermined constants *h* and *ff* will change into this $kdx + ydx$, where *k* is an arbitrary constant. On account of which this equation will emerge for the curve sought :

$$kdx + ydx = -ggd \cdot \frac{p}{\sqrt{(1+pp)}},$$

which multiplied by *p* and integrated will give $m + 2ky + yy = \frac{2gg}{\sqrt{(1+pp)}}$; which is agreed to be an elastic curve, and this will remain invariant, whatever the value the quantity *cc* may retain. Therefore thus it satisfies the proposed question, so that through the given data points *D* and *D* the elastic curve may be drawn across, of which the axis or the orthogonal diameter shall be the vertical right line *AC*, and of which the given portion *DAD* may retain the length 2*b* ; and with this agreed the solution generally will be determined will result in a single satisfying curve. But because the magnitude of the space $NDADN = 2cc$, concerning which the centre of gravity is sought, will have passed away completely from the computation, that indeed shall be able to be foreseen easily ; with which agreed upon the solution may stand out much more easily. Truly we have added this condition to the given work, even if not useful, so that the rule may be apparent for other problems of this kind requiring to be resolved, where a place cannot be found for such a reduction.

SCHOLIUM 2

25. Therefore the universal method of indeterminate maxima and minima thus has been set out, from which a curved line is accustomed to be sought with a certain given property of a maximum or minimum. And this whole method has led to the discovery of values of differentials, which arise from the increment of one applied line only. Without doubt if a problem may postulate, amongst all the curves generally related to the same abscissa to determine that, in which the expression may obtain the value of a certain indefinite maximum or minimum, then the differential value of that expression is sought ; which will give an equation for the curve sought, equal to zero. Because if moreover among all the curves, which may be endowed with one or more common properties, that may be required to be defined, in which the value of any proposed expression becomes a maximum or a minimum, then the differential values of the expression must be sought, both of the single common properties as well as of the maxima or minima sought, and these multiplied by arbitrary constants, of which the sum of the products put equal to zero, will give the equation for the curve sought. But for the differential value of each indeterminate expression required to be found, the rules in the above chapters suffice and we have shown these very easily. For they are of this kind, either an indeterminate expression or a common property always containing either a maximum or minimum, or there is a simple integral formula, or a function of two or more integral formulas of this kind. Because truly it may extend to simple integral formulas, in chapter IV paragraph 7 we have set out the precepts, with the aid of which the differential values of the formulas may be able to be found ; where we have reduced this investigation to five cases. But just as following these same precepts, the differential value is agreed to be found of any function of two or more simple integral formulas, that we have indicated in the same proposition of chapter IV in proposition 4 , and we have set out the manner of differentiation similar and easy enough; thus so that nothing may be considered to remain in this kind, which requires to be added above. The End.

CAPUT VI

METHODUS INTER OMNES CURVAS PLURIBUS PROPRIETATIBUS COMMUNIBUS GAUDENTES EAM DETERMINANDI, QUAE MAXIMI MINIMIVE PROPRIETATE SIT PRAEDITA.

PROPOSITIO I. THEOREMA

I. *Curva, quae inter omnes omnino curvas habet expressionem* $\alpha A+\beta B$ *maximum vel minimum, eadem simul ita erit comparata, ut inter omnes eadem proprietate A praeditas contineat valorem formulae B maximum vel minimum.*

DEMONSTRATIO

Ponamus inventam esse curvam, in qua inter omnes alias eidem abscissae respondentes valor expressionis $\alpha A+\beta B$ sit maximus; quod enim de maximo demonstrabitur, idem mutatis mutandis de minimo valebit. Denotant autem litterae *A* et *B* hic nobis eiusmodi formulae vel expressiones indeterminatae, in quas Quaestio de maximis et minimis cadere queat; tum vero α et β sunt quantitates constantes quaecunque. Designemus iam istam curvam, in qua sit $\alpha A+\beta B$ maximum, littera *Q*, quo eam facilius sine molesta verborum descriptione indicare queamus. Nunc concipiatur alia quaecunque curva *R* eidem abscissae respondens, quae recipiat formulae *A* eundem valorem, quem tenet curva *Q* ; in hac igitur curva *R* expressio $\alpha A+\beta B$ minorem occupabit valorem, quam in curva *Q*, eo quod in curva *Q* expressio $\alpha A+\beta B$ omnium maximum valorem sortitur. Quare, cum in curvis *Q* et *R* expressio *A* eundem obtineat valorem atque in *Q* expressio $\alpha A+\beta B$ maior sit quam in curva *R*, sequitur in curva *Q* valorem expressionis *B* maiorem esse debere quam in curva *R*. Cum igitur *R* curvam quamcunque denotet, quae cum *Q* communem valorem formulae *A* recipiat, manifestum est inter omnes has curvas R curvam *Q* esse illam, in qua formula *B* maximum habeat valorem. Ex quibus conficitur eam curvam, quae inter omnes omnino curvas habeat expressionis $\alpha A+\beta B$ valorem maximum vel minimum, eandem curvam simul ita esse comparatam ut inter omnes alias curvas secum eadem communi proprietate *A* gaudentes possideat maximum minimumve valorem expressionis *B*. Quanquam enim Demonstratio tantum ad maximum est adornata, tamen eadem translatis verbis ad minimum accommodabitur. Q. E. D.

COROLLARIUM 1

2. Vicissim itaque intelligitur, si curva debeat investigari, quae inter omnes alias eadem communi proprietate *A* praeditas expressionem *B* sit habitura maximum vel minimum, tum quaesito satisfieri, si absolute inter omnes curvas ea definiatur, in qua sit $\alpha A+\beta B$ maximum vel minimum.

COROLLARIUM 2

3. In solutionem igitur huiusmodi Problematum binae novae ingrediuntur constantes arbitrariae α et β, quae in ipsis expressionibus *A* et *B* non inerant; hae autem unius dumtaxat constantis vicem sustinebunt, quia earum ratio tantum in computum venit.

COROLLARIUM 3

4. Quodsi ergo inter omnes curvas eadem communi proprietate *A* gaudentes eam definiri oporteat, in qua sit *B* maximum minimumve, tum utriusque expressionis *A* et *B* capiantur valores differentiales, qui per constantes arbitrarias seorsim multiplicati et coniunctim nihilo aequales positi dabunt aequationem pro curva quaesita.

COROLLARIUM 4

5. Simul etiam perspicuum est perinde esse, sive inter omnes curvas eadem communi proprietate *A* gaudentes ea quaeratur, in qua sit *B* maximum vel minimum, sive vicissim inter omnes curvas eadem communi proprietate *B* gaudentes ea quaeratur, in qua sit *A* maximum vel minimum.

SCHOLION

6. Quae cum in hac Propositione tum in annexis Corollariis tradidimus, ex Capite praecedente iam sunt planissima, quippe quibus continetur inversa Methodus resolvendi Problemata, in quibus inter omnes curvas eadem communi proprietate gaudentes ea quaeritur, quae praedita sit maximi minimive alicuius indole. Neque vero idcirco idem argumentum nos tantum repetivisse censendum est; nam eandem veritatem, quam ante modo satis prolixo elicueramus, hic admodum succincte et breviter dedimus demonstratam. Quocirca eo fortius altera demonstrandi Methodus per alteram confirmabitur ob summum utriusque consensum, atque si cui prior Methodus non satis perspecta propter tantam infinite parvorum compagem nimis lubrica et incerta videatur, ei Demonstratio hic data omnem scrupulum adimet. Deinde, si quis de praesentis Propositionis conversione in Corollario 1 facta etiamnum dubitet, ei prior Methodus plenissime satisfaciet. Interim ratio conversionis ex se satis tuto inferri potest. Cum enim curva *Q*, quae inter omnes omnino curvas habeat $\alpha A+\beta B$ maximum vel minimum, ita sit comparata, ut inter omnes curvas eadem communi proprietate *A* gaudentes habeat *B* maximum vel minimum, quicquid loco α et β accipiatur, necesse est, ut conversio aeque pateat, siquidem coefficientibus α et β summa extensio tribuatur. Hocque adeo commemorare huiusque ratiocinii validitatem declarare visum est, ut in sequentibus, ubi eodem utemur, nullum dubium relinquatur. Hanc enim Propositionem, etsi proprie ad Caput praecedens pertinet, huc transtulimus, quo eadem Methodo proprium huius Capitis argumentum facilius pertractare possimus; quippe quod, si altera Methodo expediri deberet, prolixissimos requireret calculos maximasque differentialium omnium ordinum tricas. Interim tamen, quantum fieri potest, dilucide ostendemus omnia, quae hic trademus, per Methodum superiorem confirmari atque etiam elici posse.

PROPOSITIO II. THEOREMA

7. *Quae curva inter omnes omnino curvas eidem abscissae respondentes habet valorem expressionis* $\alpha A+\beta B+\gamma C$ *maximum vel minimum, eadem curva simul ita erit comparata, ut inter omnes curvas, quae tam expressionem A quam expressionem B communem habent, possideat valorem expressionis C maximum vel minimum.*

DEMONSTRATIO

Denotant hic nobis litterae *A*, *B* et *C* formulas integrales vel expressiones indefinitas eiusmodi, quae maximi minimive sint capaces, at litterae α, β, γ designant quantitates constantes arbitrarias. Sit nunc *Q* curva, quae inter omnes omnino curvas habeat valorem $\alpha A+\beta B+\gamma C$ maximum vel minimum, atque concipiatur alia quaecunque curva *R*, in qua cum expressio *A* tum *B* eundem obtineat valorem, quem obtinet in curva *Q* ; quo posito expressio composita $\alpha A+\beta B$ eundem habebit valorem in utraque curva *Q* et *R*. Hanc ob rem expressio tota $\alpha A+\beta B+\gamma C$ in curva *R* minorem sortietur valorem quam in curva *Q*, siquidem $\alpha A+\beta B+\gamma C$ in curva *Q* est maximum; contra expressionis $\alpha A+\beta B+\gamma C$ valor in curva *R* maior erit quam in curva *Q*, si $\alpha A+\beta B+\gamma C$ in curva *Q* fuerit minimum. Cum igitur expressionis portio $\alpha A+\beta B$ utrique curvae *Q* et *R* sit communis, reliqua portio γC atque adeo expressio *C* in casu maximi maior erit in *Q* quam in *R*, in casu minimi autem expressio *C* in curva *Q* minor erit quam in curva *R*. Ex quibus sequitur, si curva *Q* inter omnes omnino curvas habuerit valorem expressionis $\alpha A+\beta B+\gamma C$ maximum vel minimum, tum simul hanc curvam *Q* ea indole esse praeditam, ut inter omnes curvas *R*, quae eodem valore cum expressionis *A* tum expressionis *B* gaudeant, contineat valorem expressionis *C* maximum vel minimum.
Q. E. D.

COROLLARIUM 1

8. Quoniam expressiones *A*, *B* et *C* pro lubitu inter se commutari possunt, curva, in qua est $\alpha A+\beta B+\gamma C$ maximum vel minimum, ea simul vel inter omnes curvas iisdem proprietatibus *A* et *B* communibus gaudentes habebit *C* maximum vel minimum, vel habebit *B* maximum minimumve inter omnes curvas, quae proprietatibus *A* et *C* communibus gaudebunt, vel denique habebit *A* maximum minimumve inter omnes curvas, in quas ambae proprietates *B* et *C* aeque competunt.

COROLLARIUM 2

9. Quae igitur curva inter omnes iisdem binis proprietatibus *A* et *B* communibus gaudentes habet *C* maximum minimumve, eadem habebit inter omnes curvas, binis proprietatibus vel *A* et *C* vel *B* et *C* aeque praeditas, vel *B* vel *A* maximum minimumve.

COROLLARIUM 3

10. Si igitur curva quaeri debeat, quae inter omnes alias binis proprietatis *A* et *B* aequaliter praeditas habeat expressionem *C* maximam vel minimam, tum quaesito satisfiet, si curva quaeratur, quae absolute inter omnes curvas habeat expressionem $\alpha A+\beta B+\gamma C$ maximum vel minimum.

COROLLARIUM 4

11. Quoniam $\alpha,\ \beta, \gamma$ sunt quantitates constantes arbitrariae, in solutionem huiusmodi Problematum tres novae quantitates arbitrariae ingrediuntur, quae in formulis propositis *A*, *B* et C non inerant; aequivalebunt autem hae tres constantes $\alpha,\ \beta,$ et γ tantum duabus.

COROLLARIUM 5

12. Hae vero constantes adeo iam in aequatione pro curva primum inventa inerant; praeter eas vero per integrationes novae ingredientur constantes tot, quot integrationibus opus est, antequam ad aequationem finitam perveniatur.

COROLLARIUM 6

13. Simili modo, quo hanc Propositionem et praecedentem demonstravimus, ostendetur curvam, quae absolute inter omnes curvas habeat expressionem $\alpha A + \beta B + \gamma C + \delta D$ maximam vel minimam, eandem inter omnes curvas tres expressiones *A*, *B* et *C* communes habentes habituram esse quartam *D* maximam vel minimam.

SCHOLION

14. Ex hac Propositione iam satis percipitur Methodus resolvendi eiusmodi Problemata ad Methodum relativam pertinentia, in quibus quaeritur curva, quae inter omnes eidem abscissae respondentes et duabus pluribusve proprietatibus communibus aeque gaudentes habeat valorem cuiuspiam expressionis maximum minimumve. Quaestio scilicet perpetuo revocabitur ad Methodum absolutam, ita ut inter omnes omnino curvas quaerenda sit curva, quae expressionem quampiam habeat maximam vel minimam. Hacque reductione id commodi nanciscimur, ut omnia huiusmodi Problemata ope valorum differentialium, quos iam supra investigare docuimus, resolvere queamus. Ipse autem resolvendi modus eo redibit, ut omnes proprietates communes, una cum maximi minimive expressione, seorsim explicentur, singulae per constantes arbitrarias multiplicentur et producta in unam summam colligantur; quo facto absolute inter omnes curvas eam quaeri oportebit, in qua ista summa sit maxima vel minima. Hoc vero ipsum perficietur, dum summae illius valor differentialis investigabitur nihiloque aequalis ponetur. Quocirca universa operatio absolvetur, si cum singularum expressionum proprietates communes continentium tum maximi minimive expressionis valores differentiales secundum regulas supra datas capiantur, singuli seorsim in constantes arbitrarias ducantur omniumque horum productorum aggregatum nihilo aequale ponatur; ex quo orietur aequatio pro curva quaesita. Sufficere itaque posset hoc unicum praeceptum ad Quaestiones huius generis solvendas. Verum, antequam huius usum exponamus, hanc ipsam Methodum via ante adhibita confirmari conveniet.

PROPOSITIO III. PROBLEMA

15. *Inter omnes curvas ad eandem abscissam relatas, quae binis proprietatibus communibus A et B aequaliter sint praedita, definire eam, in qua sit valor expressionis C maximus vel minimus.*

SOLUTIO

Ex praecedentibus iam intelligitur hoc Problema solvi, si inter omnes curvas absolute quaeratur ea, in qua sit $\alpha A+\beta B+\gamma C$ maximum vel minimum. Ad hoc autem nosse oportet valores differentiales expressionum *A*, *B* et C. Sit igitur valor differentialis expressionis $A=nv\cdot dx\cdot P$, expressionis $B=nv\cdot dx\cdot Q$, expressionis $C=nv\cdot dx\cdot R$; ex quibus aequatio pro curva desiderata erit $\alpha P+\beta Q+\gamma R=0$.

Verum, quo huius Solutionis veritas magis eluceat, idem hoc Problema eadem Methodo, qua supra in Capite praecedente usi sumus, aggrediamur. Primum autem intelligitur (Fig. 15) ad hoc Problema resolvendum ternas applicatas particulis infinite parvis augeri debere, ut tribus conditionibus praescriptis satisfieri possit. Primo enim tres has particulas adiunctas, quibus ipsa curva satisfaciens *az* in novam a se quam-minime discrepantem transmutatur, ita comparatas esse oportet, ut expressio *A*, quae unam proprietatem communem continet, in utramque curvam aequaliter competat. Deinde etiam altera proprietas communis *B* in utraque curva eundem valorem obtinere debebit. Tertio ex maximi minimive natura expressio quoque *C* eundem valorem in ipsa curva et eadem mutata nancisci debet; quibus tribus conditionibus per pauciores quam tres particulas tribus applicatis adiunctas satisfieri non potest. Quare praeter binas applicatas *Nn* et *Oo*, quae in figura particulis *nv* et $o\omega$ sunt auctae, concipiatur sequenti applicatae *Pp* particula $p\pi$ addiici. Ac quaeratur primum incrementum, quod expressio *A* ex his tribus particulis assequitur, quod erit

$$=nv\cdot Pdx+o\omega\cdot P^{/}dx+p\pi\cdot P^{//}dx.$$

Namque ex particula *nv* nascitur incrementum $nv\cdot Pdx$, congruens cum ipso valore differentiali, quem expressio *A* ex sola particula *nv* adipiscitur. Ex sequenti vero particula $o\omega$ oritur incrementum $o\omega\cdot P^{/}dx$, scilicet idem, quod ante, suo differentiali auctum; quia enim $o\omega$ sequenti applicatae adiungitur, omnes quantitates $o\omega$ afficientes erunt sequentes earum, quibus particula *nv* afficitur; atque simili ratione ex particula $p\pi$ nascetur incrementum $p\pi\cdot P^{//}dx$; quae omnia, si cui libuerit calculum eo modo, quo in Capitis praecedentis Propositione 3 paragraphi 22 usi sumus, persequi, satisfient manifesta ac perspicua. Eodem igitur porro modo expressio *B*, cuius valorem differentialem ex unica particula *nv* oriundum posuimus $=nv\cdot Qdx$, ex tribus particulis *nv*, $o\omega$ et $p\pi$ incrementum accipiet

$$==nv\cdot Qdx+o\omega\cdot Q^{/}dx+p\pi\cdot Q^{//}dx.$$

Tertio expressio *C* ex his tribus particulis augmentum capiet hoc

$$=nv\cdot Rdx+o\omega\cdot R^{/}dx+p\pi\cdot R^{//}dx.$$

Singula iam haec tria incrementa seorsim nihilo aequalia poni oportet, ut omnibus conditionibus praescriptis satisfiat; unde tres sequentes aequationes orientur, facta divisione per dx,

$$0 = nv \cdot P + o\omega \cdot P' + p\pi \cdot P''$$
$$0 = nv \cdot Q + o\omega \cdot Q' + p\pi \cdot Q''$$
$$0 = nv \cdot R + o\omega \cdot R' + p\pi \cdot R''.$$

Quodsi nunc particulae $nv,\ o\omega,\ p\pi$ ad solutionem peragendam tantum in subsidium vocatae eliminentur, orietur aequatio inter quantitates curvae proprias, quibus proin natura curvae exprimetur. Ad has autem particulas eliminandas singulas aequationes per novas incognitas $\alpha,\ \beta,\ \gamma$ seorsim multiplicemus, ut habeatur

$$0 = nv \cdot \alpha P + o\omega \cdot \alpha P' + p\pi \cdot \alpha P''$$
$$0 = nv \cdot \beta Q + o\omega \cdot \beta Q' + p\pi \cdot \beta Q''$$
$$0 = nv \cdot \gamma R + o\omega \cdot \gamma R' + p\pi \cdot \gamma R'',$$

atque formentur hinc istae aequationes

$$0 = \alpha P \quad + \beta \cdot Q + \gamma \cdot R$$
$$0 = \alpha P' \ + \beta Q' \ + \gamma R'$$
$$0 = \alpha P'' + \beta Q'' + \gamma R'',$$

Hic statim patet, si pro $\alpha,\ \beta,\ \gamma$ accipiantur quantitates constantes, tum primam aequationem reliquas binas ultro in se complecti; si enim fuerit $0 = \alpha P + \beta \cdot Q + \gamma \cdot R$, tum simul erit

$$0 = \alpha dP + \beta \cdot dQ + \gamma \cdot dR \ \text{ et } \ 0 = \alpha ddP + \beta \cdot ddQ + \gamma \cdot ddR\,;$$

et quia est

$$P' = P + dP\ ,\ Q' = Q + dQ\ ,\ R' = R + dR$$

atque

$$P'' = P + 2dP + ddP,\ Q'' = Q + 2dQ + ddQ \ \text{ et } \ R'' = R + 2dR + ddR,$$

fiet quoque

$$0 = \alpha P' + \beta Q' + \gamma R'$$

et

$$0 = \alpha P^{//} + \beta Q^{//} + \gamma R^{//}.$$

Quocirca ad Problema solvendum formanda est haec aequatio

$$0 = \alpha P + \beta Q + \gamma R,$$

quae, si loco α, β et γ quantitates quaecunque constantes arbitrariae scribantur, exprimet naturam curvae quaesitae. Congruit autem omnino haec aequatio cum ea, quam altera Methodo elicuimus, alteraque Methodus per alteram confirmatur. Q. E. I.

COROLLARIUM 1

16. Omnia ergo huius quoque generis Problemata resolvi possunt ope valorum differentialium ex unius applicatae mutatione oriundorum, quos supra satis ampliter invenire docuimus.

COROLLARIUM 2

17. Manifestum igitur est, si curva debeat inveniri, quae, inter omnes alias ad eandem abscissam relatas atque in quas binae expressiones *A* et *B* aequaliter competant, habeat valorem expressionis *C* maximum minimumve, tum quaestionem redire ad hanc, quae ad Methodum absolutam pertineat, ut inter omnes omnino curvas ad eandem abscissam relatas determinetur ea, in qua sit expressio $\alpha A + \beta B + \gamma C$ maximum vel minimum.

COROLLARIUM 3

18. Simul vero etiam hinc Methodus patet resolvendi Problemata, in quibus inter omnes Curvas, in quas plures duabus atque adeo quotcunque proprietates aequaliter conveniant, ea requiritur, quae maximi minimive cuiusdam proprietate gaudeat.

COROLLARIUM 4

19. Quodsi enim inter omnes Curvas, in quibus expressiones *A*, *B*, *C*, *D* aequales obtineant valores, ea debeat investigari, in qua sit expressio *E* maximum vel minimum, tum quaesito satisfiet, si inter omnes omnino Curvas ea quaeratur, in qua sit $\alpha A + \beta B + \gamma C + \delta D + \varepsilon E$ maximum vel minimum, denotantibus litteris $\alpha, \beta, \gamma, \delta, \varepsilon$ quantitates quascunque constantes et arbitrarias.

COROLLARIUM 5

20. Quo plures igitur proponantur proprietates, quae iis curvis, ex quibus quaesitam maximi minimive indole praeditam indagare oportet, communes esse debeant, eo plures in aequationem pro curva ingredientur quantitates constantes arbitrariae atque adeo eo plures curvae satisfacientes in ea comprehendentur.

SCHOLION 1

21. Cur eo plures constantes in Solutionem ingrediantur, quo plures proponantur proprietates communes, ex praecedentibus facile colligi potest. Ponamus enim inter omnes curvas eadem proprietate *A* gaudentes eam investigari oportere, in qua sit *B* maximum vel minimum; ac primo quidem constabit huic Quaestioni eam Curvam esse

satisfacturam, quae inter omnes omnino Curvas habeat *B* maximum vel minimum; haec enim inter omnes quoque illas, quae secum eadem communi proprietate *A* gaudebunt, habebit *B* maximum vel minimum. Deinde autem licet innumerabilia istiusmodi curvarum genera concipi, quae singula eundem valorem expressionis *A* recipiant; in uno quoque vero genere una erit curva, quae prae reliquis valorem expressionis *B* contineat maximum vel minimum. Necesse autem est has Curvas satisfacientes omnes in Solutione generali contineri debere. Cum igitur ob unam proprietatem communem praescriptam numerus curvarum satisfacientium fiat infinitus, multo magis is augebitur, propter eandem rationem, si plures proprietates communes proponantur. Interim tamen, si valores, quos habent singulae proprietates communes in curvis, ex quibus quaesitam erui oportet, actu definiantur, tum utique solutio unicam Curvam satisfacientem praebebit. Constantes scilicet illae eo inservient, ut valores, quos proprietates communes in curva inventa obtinebunt, pro arbitrio determinentur; sic per has constantes in casu duarum proprietatum communium *A* et *B* curva poterit assignari, quae datos expressionum *A* et *B* recipiat valores atque insuper ita sit comparata, ut inter omnes infinitas alias eosdem illarum expressionum *A* et *B* valores recipientes habeat valorem alius cuiuscunque expressionis *C* maximum vel minimum. Atque haec eadem admonitio locum habet, si plures proprietates communes fuerint praescriptae; ex quo satis perspicuum est, quid hisce constantibus in Solutionem ingredientibus sit faciendum et quomodo eas ad usum traduci oporteat, id quod in sequentibus Exemplis clarius declarari poterit.

EXEMPLUM I

22. *Inter omnes curvas* (Fig. 14) *ad eandem abscissam* $AC = a$ *relatas, quae cum inter se eiusdem sint longitudinis tum etiam aequales areas DAD comprehendant, determinare eam, quae circa axem AC rotata generet solidum maximae vel minimae capacitatis.*

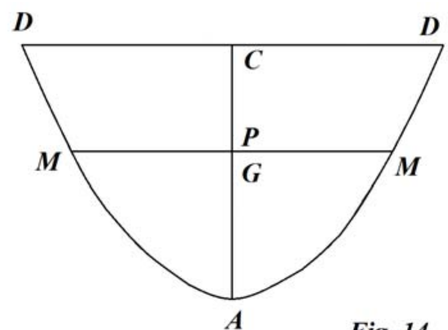

Fig. 14

Positis abscissa $AP = x$, applicata $PM = y$ et $dy = pdx$ binae proprietates communes propositae sunt $\int ydx$ et $\int dx\sqrt{(1+pp)}$; at maximi minimive formula est $\int yydx$. Quaerantur iam harum trium formularum valores differentiales. Ac primo quidem erit formulae $\int ydx$ valor differentialis $= nv \cdot dx$, deinde formulae $\int dx\sqrt{(1+pp)}$ valor differentialis est $-nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}$ et tertio formulae $\int yydx$ valor differentialis est $= 2nv \cdot ydx$. Ex quibus tribus valoribus differentialibus conficietur pro curva quaesita ista aequatio

$$0 = \alpha dx - \beta d \cdot \frac{p}{\sqrt{(1+pp)}} + 2\gamma ydx$$

seu

$$ccd\cdot\frac{p}{\sqrt{(1+pp)}}=bdx+2ydx=\frac{ccdp}{(1+pp)^{3:2}}.$$

Multiplicetur haec aequatio per p et integretur; habebitur

$$ff+by+2yy=\frac{-cc}{\sqrt{(1+pp)}},$$

ubi tam cc quam ff pro arbitrio sive affirmative sive negative accipere licet.
Hinc porro fiet

$$(ff+by+2yy)^2(1+pp)=c^4 \text{ et } p=\frac{\sqrt{\left(c^4-(ff+by+2yy)^2\right)}}{ff+by+2yy}=\frac{dy}{dx};$$

ideoque

$$dx=\frac{(ff+by+2yy)dy}{\sqrt{\left(c^4-(ff+by+2yy)^2\right)}},$$

quae est aequatio pro curva Elastica. Ingredietur autem per integrationem unam reliquam nova quarta constans arbitraria; atque hisce quatuor constantibus effici poterit, ut curva per data duo puncta transeat; deinde binis reliquis constantibus obtinebitur, ut posito $x=a$ tam area curvae quam eius longitudo datam magnitudinem consequantur. Insuper autem ambiguitate signorum, qua signum radicale afficitur, alterum signum praebebit curvam maximi, alterum minimi proprietate gaudentem. Quoniam autem in aequatione inventa data illa abscissae magnitudo a non inest, sequitur curvae inventae portionem quamvis cuicunque abscissae respondentem hac quoque gaudere praerogativa, ut inter omnes alias curvas eidem illi abscissae respondentes et per eadem duo puncta transeuntes, quae simul cum illa curva tam aequalem longitudinem quam aequalem aream complectantur, ut illa, inquam, curva circa abscissam suam rotata generet solidum maximae minimaeve capacitatis. Duo scilicet puncta, per quae curva quaesita transeat, ideo hic in considerationem sunt ducenda, quia calculus praebuit aequationem differentialem secundi gradus, quae per se duplicem determinationem requirit. Poterunt vero etiam binae reliquae constantes, quae statim in aequatione inventa inerant, per puncta determinari hocque pacto determinata solutio huiusmodi emerget, quae docebit per quatuor data puncta curvam describere, quae inter omnes alias per eadem quatuor puncta transeuntes atque cum aeque longas tum aequales areas continentes producat circa axem rotata solidum vel maximum vel minimum. Perpetuo nimirum numerus constantium arbitrariarum, quae in aequatione inventa cum actu tum potentia insunt, declarabit, quot determinationes sint adhibendae, ut curva determinetur; haecque deinde inter omnes alias curvas iisdem determinationibus praeditas quaesito satisfaciet.

EXEMPLUM II

23. *Inter omnes lineas eidem abscissae respondentes, quae primo aequales contineant areas* $\int ydx$ *atque praeterea circum axem rotatae aequalia generent solida* $\int yydx$, *determinare eam, quae suum gravitatis centrum vel maxime vel minime habeat elevatum, hoc est, in qua sit* $\frac{\int yxdx}{\int ydx}$ *vel maximum vel minimum.*

Sit abscissae longitudo praescripta, ad quam Solutionem accommodari oportet, $= a$ atque pro hac abscissa fiat valor formulae $\int ydx = A$, formulae $\int yydx = B$ et formulae $\int yxdx = C$. Porro sit valor differentialis formulae $\int ydx = dA = dx$, formulae $\int yydx = dB = 2ydx$ et formulae $\int yxdx = dC = xdx$, sumptis nimirum harum formularum valoribus differentialibus secundum regulas supra datas, omittendo tantum particulam *nv*, quippe quae perpetuo per divisionem tollitur. Cum iam maximi minimive expressio sit non simplex formula, sed fractio $\frac{\int yxdx}{\int ydx}$, eius valor differentialis erit

$$\frac{AdC - CdA}{A^2} = \frac{Axdx - Cdx}{A^2} \;;$$

atque ob proprietatum binarum communium $\int ydx$ et $\int yydx$ valores differentiales datos, nempe $dA = dx$ et $dB = 2ydx$, resultabit pro curva quaesita sequens aequatio

$$\alpha dx + 2\beta ydx + \frac{\gamma Axdx - \gamma Cdx}{A^2} = 0 \text{ vel } \left(\alpha A^2 - \gamma C\right)dx + 2\beta A^2 ydx + \gamma Axdx = 0\,;$$

in qua aequatione cum $\alpha,\ \beta,\ \gamma$ sint constantes arbitrariae, earum transformatione simul constantes determinatae *A* et *C* ex computo expelli possunt ita, ut Solutio inventa ad omnes abscissas aeque fiat accommodata. Pervenietur autem ad hanc aequationem $bdx = mydx + nxdx$ seu per *dx* divisione instituta $b = my + nx$, quae aequatio est pro linea recta quacunque. Linea recta igitur ad axem verticalem utcunque sita inter omnes alias lineas cum axe tam eandem aream $\int ydx$ quam idem volumen $\int yydx$ continentes habebit suae areae centrum gravitatis vel maxime vel minime elevatum. Erit autem centrum gravitatis minime elevatum, si linea recta sursum cum axe convergat, maxime autem erit elevatum, si deorsum cum axe convergat; hique sunt ambo casus, quibus vel maxima vel minima centri gravitatis elevatio locum habet. Inter hos casus est medius, quo linea illa recta sit axi parallela; de quo dubium superesse potest, utrum centrum gravitatis sit vel maxime depressum vel maxime elevatum. Verum iste casus nequidem in Quaestione locum invenit. Nam posita linea recta axi parallela, ita ut sit $y = b$, tum omnino nulla alia exhiberi potest linea, quae pro eadem abscissa cum aequalem aream $\int ydx$ tum aequale volumen $\int yydx$ contineat, hocque ideo evenit, quod ista linea recta inter omnes alias lineas eandem aream $\int ydx$ comprehendentes minimum volumen $\int yydx$ includat.

EXEMPLUM III

24. *Inter omnes curvas* (Fig. 21) *eiusdem longitudinis DAD puncta data DD iungentes determinare eam, cuius haec sit proprietas, ut, si inter rectas verticales DB, DB per horizontalem NN spatium NDADN datae magnitudinis abscindatur, huius spatii NDADN Centrum gravitatis imum obtineat locum.*

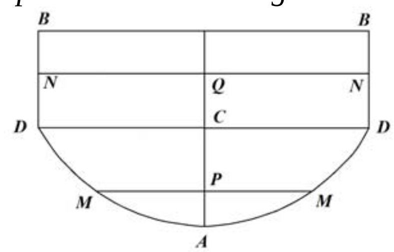

Fig. 21

Quaestionis huius Solutio eximium Fig. 21 habet usum in Hydrostatica eiusque ope solvetur Problema, quo figura lintei *DAD* vasi *BDDB* in punctis *DD* annexi investigatur, quam induit, si vasi data aquae copia infundatur. Primo enim, dum linteum extensionem non admittit, longitudo curvae *DAD* erit data, deinde etiam spatium *NDADN*, quo quantitas aquae infusae mensuratur, erit data, ac tertio secundum generales Hydrostaticae et gravitationis leges figuram *DAD* ita comparatam esse oportet, ut spatii *NDADN* centrum gravitatis infimum occupet locum. Ad hoc Problema resolvendum ponatur $DC = CD = a$ et ducta horizontali quacunque *MPM* sit $MP = PM = x$ et $AP = y$, erit arcus

$$MAM = 2\int dx\sqrt{(1+pp)} \text{ posito } dy = pdx.$$

Quodsi iam longitudo curvae *DAD* ponatur $= 2b$, aequatio inter x et y ita debet esse comparata, ut formula integralis $\int dx\sqrt{(1+pp)}$ fiat $= b$ posito $x = a$. Porro area *MAM* fit $= 2\int xdy = 2\int xpdx$; quae fiat $= 2ff$ casu, quo ponitur $x = a$; ita ut tum sit $\int xpdx = ff$. Haec vero area non ipsa est data, sed ea cum area *NDDN* datum spatium producere debet, quod sit $= 2cc$. Si igitur ponatur $DN = z$, erit

$$az + ff = cc \text{ et } z = \frac{cc - f}{a} = \frac{cc - \int xpdx}{a},$$

posito $x = a$. Denique centrum gravitatis totius spatii *NDADN* a puncto *A* distabit intervallo

$$= \frac{\int xypdx + az\left(AC + \frac{1}{2}z\right)}{cc},$$

posito post integrationem $x = a$; infra punctum *C* igitur centrum gravitatis situm erit intervallo

$$= \frac{AC(cc - az) - \frac{1}{2}azz - \int xypdx}{cc},$$

quod debet esse maximum. Cum vero sit $z = \frac{cc - \int xpdx}{a}$, maximum esse debebit haec forma

$$AC\int xpdx - \frac{c^4}{2a} + \frac{cc\int xpdx}{a} - \frac{\left(\int xpdx\right)^2}{2a} - \int xypdx.$$

Problema itaque huc redit, ut inter omnes curvas eiusdem longitudinis datae abscissae $DC = a$ respondentes definiatur ea, in qua sit haec expressio

$$h\int xpdx + \frac{cc}{a}\int xpdx - \frac{1}{2a}\left(\int xpdx\right)^2 - \int xypdx$$

maximum, existente $y = h$ posito $x = a$. Iam, quia longitudo curvae est $= \int dx\sqrt{(1+pp)}$, erit eius valor differentialis $= -d \cdot \frac{p}{\sqrt{(1+pp)}}$. Deinde formulae $\int xpdx$ valor differentialis est $= -dx$ et valor differentialis formulae

$$\int xypdx = xpdx - d \cdot xy = -ydx.$$

Hinc totius expressionis, quae maximum esse debet, valor differentialis prodit

$$= -hdx - \frac{cc}{a}dx + \frac{ff}{a}dx + ydx,$$

a a quae ob h et ff constantes non determinatas transit in hanc $kdx + ydx$, ubi k est constans arbitraria. Quocirca prodibit ista aequatio pro curva quaesita

$$kdx + ydx = -ggd \cdot \frac{p}{\sqrt{(1+pp)}},$$

quae per p multiplicata et integrata dabit $m + 2ky + yy = \frac{2gg}{\sqrt{(1+pp)}}$; quam curvam constat esse Elasticam, manebitque ea invariata, quemcunque valorem obtineat quantitas cc. Quaestioni ergo propositae ita satisfiet, ut per data puncta D et D curva Elastica traducatur, cuius axis seu diameter orthogonalis sit recta verticalis AC et cuius portio DAD datam obtineat longitudinem $2b$; hocque pacto Solutio omnino erit determinata unicaque curva satisfaciens resultabit. Quod autem quantitas spatii $NDADN = 2cc$, de cuius centro gravitatis quaestio est, prorsus ex computo excesserit, id quidem facile praevidere licuisset; quo pacto Solutio multo facilior extitisset. Verum data opera hanc conditionem, etsi inutilem, adiecimus, ut modus pateret alia istiusmodi Problemata, ubi talis reductio locum non invenit, resolvendi.

SCHOLION 2

25. Sic igitur exposita est universa Methodus maximorum et minimorum indeterminata, qua linea curva quaeri solet maximi minimive proprietate quapiam praedita. Istaque Methodus tota perducta est ad inventionem valorum differentialium, qui ex unius tantum applicatae incremento oriuntur. Scilicet si Problema postulet inter omnes omnino curvas ad eandem abscissam relatas eam determinare, in qua expressio quaepiam indefinita maximum minimumve obtineat valorem, tum illius expressionis quaerendus est valor differentialis; qui nihilo aequalis positus dabit aequationem pro Curva quaesita. Quodsi autem inter omnes curvas, quae una pluribusve proprietatibus communibus gaudeant, eam definiri oporteat, in qua valor cuiuspiam expressionis propositae fiat maximus vel minimus, tum, tam singularum proprietatum communium quam maximi minimive, expressionis quaeri debent valores differentiales hique singuli per constantes arbitrarias multiplicari, quorum productorum summa nihilo aequalis posita dabit aequationem pro Curva quaesita. Ad valorem autem differentialem cuiusque expressionis indeterminatae inveniendum Regulas in superioribus Capitibus sufficientes atque admodum faciles tradidimus. Eiusmodi enim expressio indeterminata sive proprietatem communem continens sive maximum minimumve perpetuo vel est formula integralis simplex vel functio duarum pluriumve huiusmodi formularum integralium. Quod vero ad formulas integrales simplices attinet, in Capitis IV paragrapho 7 praecepta exposuimus, quorum ope eiusmodi formularum valores differentiales reperiri queant; ubi hanc indagationem ad quinque casus reduximus. Quemadmodum autem secundum haec eadem praecepta cuiuscunque functionis duarum pluriumve formularum integralium simplicium valor differentialis convieniens definiri queat, id in eiusdem Capitis IV Propositione 4 indicavimus modumque differentiationis similem atque satis facilem exposuimus; ita ut in hoc genere nihil superesse videatur, quod insuper sit adiiciendum.

FINIS