CHAPTER V

A METHOD OF FINDING THAT CURVE AMONG ALL THE CURVES WITH A GIVEN PROPERTY, WHICH MAY BE ENDOWED WITH THE PROPERTY OF MAXIMUM OR MINIMUM

DEFINITION

1. The common property is the formula of the integral or the indefinite expression, which is in equal agreement in all the curves, from which it is required to determine the question.

SCHOLIUM 1

2. Up to this point we have treated the method of absolute maxima or minima, in which amongst all the curves in general it was usual to require always a single curve corresponding to the same abscissas, which was endowed with a certain property of being a maximum or minimum. But now we may progress to the relative method, in which we will demonstrate how to determine a single maximum or minimum line with the given property, not from entirely all the lines corresponding to the same abscissas, but truly from these innumerable curved lines only , in which either a certain single proposed quality or several may be in common. And indeed in the first place in this chapter we will consider innumerable curves corresponding to the same abscissas, which may have a single property in common ; and from these we will investigate a single line, in which some indefinite expression may reach a maximum or minimum value. Of such a kind is the celebrated *Isoperimetric Problem*, publicly proposed at the start of this century, in which among all the lines of equal length, which may correspond to the same certain abscissas, it will be required to define that, which may contain a certain maximum or minimum property. But after this question has been taken in the widest sense, so that that determination itself may happen not only among all the curves of the same length, but truly also between all the curves provided with some other common property ; which question itself we have undertaken in this chapter. Therefore since a curve shall not be selected from all the curves generally corresponding to the same abscissas, truly only from these innumerable curves, in which a certain proposed property may be agreed upon equally, that property itself will be required to be considered, which here we may indicate by the common name of the property. Therefore this common property, just as the equality of the lengths of the curves, must affect all common points and because of that it will be an indefinite function, which will not be determined by a single element of the curve, but truly from the whole curve in place. On account of which a common property of this kind will be either an indefinite simple integral or an expression including several formulas of this kind. Therefore in general it will be prepared in a like manner, by which the maximum or minimum formula or expression is prepared. Therefore these same varieties and divisions which we have made and treated before about a maximum or minimum, will pertain equally to the common property.

COROLLARY 1

3. Therefore if a common property were proposed, which shall be *B*, then all the curves are required to be considered, which contain the same value of *B* for the same given abscissas, and from these that must be defined, which may have a maximum or minimum.

COROLLARY 2

4. Therefore in problems relating to it is necessary to be given two things, the common property *B* and the expression of the maximum or minimum *A*. With which given between all the curves for a given abscissa containing the value *B* that must be defined, which for the same abscissa may have a maximum or minimum value of *A*.

COROLLARY 3

5. But not only an infinitude of curves are given, which for a given abscissa may have the same common property, but also they may be given in an infinite number of ways. For with some curve assumed for argument's sake, that will have a determined value of th proposed common property ; but besides that innumerable others will be given holding the same value of the common property for the same abscissa.

COROLLARY 4

6. Therefore with any proposed expression innumerable indefinite kinds of infinite curves will be given, which may retain the same value of the expression for the same given abscissa.

COROLLARY 5

7. Therefore since an infinitude of kinds will be given, of which the individual kinds comprise innumerable curved lines, in which the expression for the common property may be equally matched; in one kind also one curve will be given, which with the rest of the curves of the same kind may contain another expression in the maximum or minimum position.

COROLLARY 6

8. Therefore because from any kind a single maximum or minimum curves with the given property may be found, in general infinitely many satisfying curves of this kind may be found, of which a certain one may be prepared thus, so that among all the others with the same property it shall be provided with the property of maximum or minimum.

SCHOLIUM 2

9. All these will be illustrated more, if we may define the common property, about which we have been talking about in general. Therefore the common property shall be the formula expressing the length of the arc of a curve, but the expression of the maximum or minimum shall be $\int Zdx$; thus so that between all the curves, which may have an arc corresponding to the same equal among themselves, that must be determined, in which for the same abscissa makes $\int Zdx$ a maximum or minimum. But it is evident not only

infinitely many curved lines equal in length to be given for the same abscissa, truly this also can happen in an infinite number of ways. For the common abscissa shall be = a and some length c greater than a may be taken, infinitely many lines both right as well as curved can be shown, of which the length of the individual lines shall be c; and amongst these a single one can be defined, in which $\int Zdx$ shall be a maximum or a minimum. But in place of c an infinitude of quantities can be accepted, because for that no other condition is present, except that there shall be $c > a$; and whatever value assumed for c will give a single curve endowed with the property of maximum or minimum. On account of which for the infinitude of values of c an infinitude of curved lines satisfying the question will be found. Nor yet therefore is the question about indeterminacy being considered, for the solution presents infinitely satisfying curves thus being interpreted, so that whatever of these curves found among all the other curves of equal length may possess the value of the formula $\int Zdx$ in the maximum or minimum position. But it is seen, because here we have shown concerning the equality of curved arcs, the same must be valid for any other formula or indeterminate expression. Thus so that between all the curves, which for a given abscissa $x = a$ may contain the same value of the formula $\int Ydx$, that may be required, in which $\int Zdx$ shall be a maximum or minimum, then indeed an infinitude of satisfying lines may be found ; truly these will disagree between themselves, so that whatever amongst all the other possible curved lines from that having a common value of the formula $\int Ydx$ may contain a maximum or minimum value of the formula $\int Zdx$.

PROPOSITION I. THEOREM

10. *The curve amongst all the curves in general corresponding to the same abscissa which shall be endowed with the maximum or minimum of a certain proposed property, likewise the same curve will be endowed with the property of a maximum or minimum amongst all curves with the same given common property.*

DEMONSTRATION

Let the expression of the maximum or minimum be $= A$, moreover the expression of the common property shall be $= B$, and both A as well as B shall be either an indefinite integral or an expression composed from several formulas of this kind. Now we may consider the curve to be found, which amongst all the curves generally corresponding to the same abscissas shall retain a maximum or minimum expression A; that curve certainly will contain some value of the expression B; but besides that innumerable others will be given, in which the same value of the expression B will be shared and all these innumerable curves now may be retained in all these curves in general, from which that is found, in which the expression A is a maximum or minimum. Therefore since this curve amongst all the curves generally may be endowed with the property of a maximum or minimum, likewise also from that amongst those infinitudes of curves having that common value of the expression B, A will possess a maximum or minimum. Q. E. D.

COROLLARY 1

11. Therefore besides the absolute method the method of resolving the relative method of a problem is looked after, as long as one it shows one satisfying curve. Yet truly a complete solution is not presented.

COROLLARY 2

12. Therefore the curve, which among all the expressions *A* has a maximum or minimum, will be one from those infinitudes of curves, of which the individuals among all the others endowed with that common property *B* have the same maximum or minimum expression *A.*

COROLLARY 3

13. Therefore the solution of the problem, in which amongst all the given curves with the same common property *B,* that is found in which *A* shall be a maximum or minimum, will be more apparent, than if that may be sought absolutely among all the curves, in which *A* is a maximum or minimum; and that solution itself may be considered just as this special case.

PROPOSITION II. PROBLEM

14. *The method of resolving problems is to be outlined in general, in which among all the curves endowed with a certain common property that one is required, which may be endowed with the property of some proposed maximum or minimum,.*

SOLUTION

Every maximum or minimum is prepared thus, so that (Fig.15) with an infinitely small change made the value of this generally will not be changed. On account of which, if the curve *az* amongst all the curves corresponding to the same abscissa *AZ*, which indeed may be endowed with a certain common property *B*, may have a maximum or minimum value of the expression *A*, the same value will be retained, if an infinitely small change of such itself may be brought about, by which the common property *B* may not be disturbed. But it will not suffice for that, as we have done before, to consider a single applied line *Nn,* to be increased by the infinitely small amount *nv*; because indeed in this manner all the change is determined by a single condition, cannot be effected by that, so that just as the common property *B* equally will be shared and unchanged in the curve itself, so the expression of the maximum or minimum *A*. On account of which it will be necessary for two conditions to be used in the determination ; that which will be obtained if two applied lines *Nn* and *Oo* may be increased by the infinitely small amounts nv and $o\omega$. But

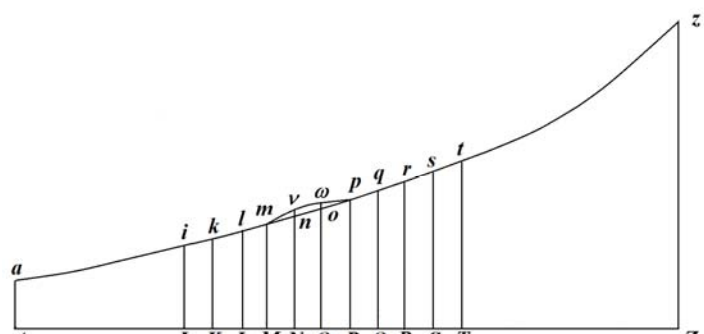

Fig. 15

therefore if the curve may be considered to be changed in this manner, the first change being effected so that the common property may be shared equally both by the curve itself and the change; then also the expression of the maximum or minimum will have to be retained the same value in each curve. As before it will be better, if the value of the differential of the expression may be investigated, in which the common property is contained, arising from the translation of the two [points] *n* and *o* into ν and ω and this may be put to be vanishing; truly for the latter condition to be satisfied, if the value of the differential expression, which must be a maximum or minimum, may be sought arising from the two small parts $n\nu$ and $o\omega$ and may be put equal to zero. With this agreed on two equations will be obtained, the one from the common property, the other from the expression of the maximum or minimum ; but each of this kind will have the form $S \cdot n\nu + T \cdot o\omega = 0$; in which *S* and *T* will be quantities pertaining to the curve. Moreover from the two equations of this kind the small amounts $n\nu$ and $o\omega$ will be eliminated and the equation will be come upon for the curve sought, which amongst all the others given with the same common property *B* may have a maximum or minimum value of the expression *A*. Q. E. I.

COROLLARY 1

15. Therefore the solution of problems of this kind also is reduced to the finding of differential values ; but the differential values themselves from these, which we have given before, disagree in this case, because they must be defined from the translation of two points of the curve.

COROLLARY 2

16. Therefore differential values of this kind arise from the two small parts $n\nu$ and $o\omega$ in whatever problem it is required to investigate the two, the one for the common property, the other for the expression of the maximum or minimum.

COROLLARY 3

17. But in whatever problem from the two differential values found each must be put equal to zero in an equation, from which two equations will be produced, which with th assumed particles $n\nu$ and $o\omega$ eliminated one equation expresses the nature of the curve sought.

COROLLARY 4

18. Therefore if among all the curves corresponding to the same abscissa, which are provided equally with the same common property *B*, that may be required, in which the expression *A* becomes a maximum or minimum, then the differential values of each expression *A* and *B* to be sought arise from the two particles $n\nu$ et $o\omega$ and must be put equal to zero ; from which two equations if the particles $n\nu$ and $o\omega$ may be eliminated, the equation will emerge for the curve sought.

COROLLARY 5

19. And thus in this operation both the expressions *A* and *B* generally are treated equally nor does it come into consideration, whether each may denote the common property or the maximum or minimum. From which it is understood the same solution must be produced, if the expressions *A* and *B* may be exchanged with each other.

COROLLARY 6

20. Therefore the same solution will have a place, whether amongst all the curves endowed with the common property *B* that may be sought, in which *A* shall be a maximum or minimum, or in turn amongst all the curves endowed with the common property *A* that may be sought, in which *B* shall be a maximum or minimum.

SCHOLIUM

21. Both the expressions *A* and *B*, although generally observed to signify diverse things, are interchangeable between themselves thus the nature of the solution is apparent at once. But if indeed we may look at the two particles nv and $o\omega$, by which the applied lines *Nn* and *Oo* are increased, first it is necessary to prepare these thus, so that the common property *B* both on the curve as well as in the changed curve may retain the same value ; clearly the common property *B* must be shared equally on the curve *amnopz* and on the curve $amv\omega pz$; then in a like manner it is required to be effected by the same particles nv and $o\omega$, so that the expression *A*, which must be a maximum or a minimum, may take the same value for the curve *amnopz* as for $amv\omega pz$. And thus both the common property as well as the nature of the maximum or minimum plainly leads to the same condition in the calculation ; from which it is evident both given expressions, of which one is the common property, and the other contains an account of the maximum or minimum, can be interchanged between each other and combined, with the solution unharmed. On this account therefore in the solution of problems of this kind it will suffice to know both these expressions ; nor is there a need to know for resolving the solution, whether it may signify the common property or the maximum or minimum. Thus if amongst all the curves with equal length that may be sought, which may take the greatest area, the same curve will be found, which appears, if amongst all the curves that include equal areas that may be sought, which shall be the shortest or may have the minimum length. These themselves thus may be had, if the nature of the maximum or minimum were prepared thus, so that the value of the differential shall be $=0$. But now above we have become aware of maxima and minima of a two-fold kind, in one of which the value of the differential shall be $=0$, and in the other truly $=\infty$. Truly here we may consider only maxima and minima of the first kind ; for in this relative method no place can be found generally for the second kind. But if indeed the value of the differential, which agrees with the expression of the maximum or minimum, may be considered infinitely great, then the equation for the curve is found from this alone nor thus is the common property introduced into the calculation. Whereby, if a maximum or minimum of this kind has a place in the absolute method, the same curve will be endowed with the same property in the relative method, whatever common property may be added on. Therefore since the whole motivation of the solution of problems of this kind may depend on the finding of differential values, which arise from the two particles nv and $o\omega$, the method we relate for finding the differential values of this kind for any indeterminate

expression, will be in that manner, which we have used above for finding the differential values arising from a single particle *nv*.

PROPOSITION III. PROBLEM

22. *With some indeterminate expression proposed* (Fig. 15), *which may refer to the given abscissa AZ, to find the value of differential value of this, arising from the translation of the two points of the curve n and o to* v *et* ω.

SOLUTION

We may put the abscissa $AI = x$ and the applied line $Ii = y$, there will be $Kk = y^{/}, Ll = y^{//}$, $Mm = y^{///}$, $Nn = y^{IV}$, $Oo = y^{V}$, $Pp = y^{VI}$ etc. Of these applied lines only two, namely y^{IV} and y^{V} will experience a change from the particles nv and $o\omega$ added to themselves. Therefore the value of the differential of the applied line y^{IV} is $= nv$ and the value of the differential of the applied line y^{V} is $= o\omega$, truly the value of the differential of all the remaining applied lines will be $= 0$. Hence the differential values of the remaining quantities pertaining to the curve *p, q, r, s* etc. will be found, just as these depend on these two applied lines y^{IV} and y^{V}. Thus

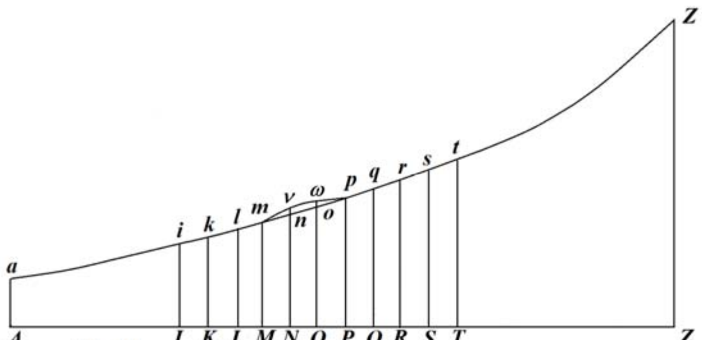

Fig. 15

since there shall be $p = \frac{y^{/} - y}{dx}$ the value of the differential of this $p = 0$ and similarly of $p^{/}$ itself and $p^{//}$; but since there shall be $p^{///} = \frac{y^{IV} - y^{///}}{dx}$,

the value of the differential of $p^{///}$ itself $= \frac{nv}{dx}$; and on account of $p^{IV} = \frac{y^{V} - y^{IV}}{dx}$ the value of the differential of $p^{IV} = \frac{o\omega}{dx} - \frac{nv}{dx}$ and again of p^{V} it will be $= -\frac{o\omega}{dx}$. Then,

since there shall be $q = \frac{p^{/} - p}{dx}$, the value of the differential of $q^{//} = \frac{nv}{dx^2}$, of

$q^{///} = \frac{o\omega}{dx^2} - \frac{2nv}{dx^2}$, or $q^{IV} = -\frac{o\omega}{dx^2} + \frac{nv}{dx^2}$, and of $q^{V} = \frac{o\omega}{dx^2}$. And it is allowed to progress similarly in this manner to the following quantities *r, s* etc. with their derivatives ; and hence the following table may be produced, in which the values of these individual quantities may be shown :

$d\cdot y^{IV} = nv$ $d\cdot y^{V} = o\omega$	$d\cdot q^{//} = \dfrac{nv}{dx^2}$
$d\cdot p^{///} = \dfrac{nv}{dx}$ $d\cdot p^{IV} = -\dfrac{nv}{dx} + \dfrac{o\omega}{dx}$ $d\cdot p^{V} = \quad -\dfrac{o\omega}{dx}$	$d\cdot q^{///} = -\dfrac{2nv}{dx^2} + \dfrac{o\omega}{dx^2}$ $d\cdot q^{IV} = \dfrac{nv}{dx^2} - \dfrac{2o\omega}{dx^2}$ $d\cdot q^{V} = \quad \dfrac{o\omega}{dx^2}$

$d\cdot r^{/} = +\dfrac{nv}{dx^3}$
 $d\cdot r^{//} = -\dfrac{3nv}{dx^3} + \dfrac{o\omega}{dx^3}$
 $d\cdot r^{///} = +\dfrac{3nv}{dx^3} - \dfrac{3o\omega}{dx^3}$
 $d\cdot r^{IV} = -\dfrac{nv}{dx^3} + \dfrac{3o\omega}{dx^3}$
 $d\cdot r^{V} = \quad -\dfrac{o\omega}{dx^3}$
	$d\cdot s = +\dfrac{nv}{dx^4}$
	 $d\cdot s^{/} = -\dfrac{4nv}{dx^4} + \dfrac{o\omega}{dx}$
	 $d\cdot s^{//} = +\dfrac{6nv}{dx^4} - \dfrac{4o\omega}{dx^4}$
	 $d\cdot s^{///} = -\dfrac{4nv}{dx^4} + \dfrac{6o\omega}{dx^4}$
	 $d\cdot s^{IV} = +\dfrac{nv}{dx^4} - \dfrac{4o\omega}{dx^4}$
	 $d\cdot s^{V} = \quad +\dfrac{o\omega}{dx^4}$

etc.

From this table it is seen that just as many terms in the differential values are come upon affected by the particle $o\omega$, as by the particle *nv*, and equal coefficients are present in each; truly the distinction consists of this, so that for any term affected by the particle $o\omega$ a quantity following that may correspond immediately, to which a similar term will correspond affected by the particle *nv*. Thus, while the term $-\dfrac{2nv}{dx^2}$ is found in the value of the differential quantity $q^{///}$, thus the term $-\dfrac{2o\omega}{dx^2}$ is present in the value of the following differential quantity q^{IV}. Successively on account of the two-fold generation, terms occur in the differentials values, one kind of which involve the particle *nv*, the other the particle $o\omega$, and the value of any indeterminate differential expression of this kind will have the form $nv\cdot I + o\omega\cdot K$; from which it is evident the first member $nv\cdot I$ is the differential value of the same expression, which arises, if only the particle *nv* may be considered, and thus the value of the differential itself will be $nv\cdot I$, which we have set out to define for any expression offered, thus so that this member by the precepts treated above may be shown for any indeterminate expression. What concerns the other member $o\omega\cdot K$, because the individual terms, in which $o\omega$ is present, always correspond to these following sequences, in which the similar terms involving the particle *nv*, it is plain that the quantity K is the value, that the quantity I in the next following place adopts, and thus there is $K = I' = I + dI$. Whereby since we may be able to assign the member $nv \cdot I$ from the above precepts, from that again the other member $o\omega \cdot K = o\omega(I + dI)$ will become known. Therefore V shall be some indeterminate expression, the differential value arising of which may be required to be defined from the two particles nv and $o\omega$. We may put the differential value of this arising from the single particle nv to be $= nv \cdot I$, the value of the differential, which arises from both the particles nv and $o\omega$, to be $= nv \cdot I + o\omega \cdot I'$ or $nv \cdot I + o\omega \cdot (I + dI)$; which therefore will be able to be assigned easily, with the help of the rules given above. Q. E. I.

COROLLARY 1

23. Therefore of all the expressions we have shown how to find, the differential values of which arise from the single particle *nv*, we are now able to define the differential values of the same arising from the two particles nv and $o\omega$.

COROLLARY 2

24. Therefore this method will be valid both for the values of differential expressions requiring to be found, which do not depend on the magnitude of the abscissa proposed *AZ*, as well as those which depend on the length of its abscissa.

COROLLARY 3

25. But also, if the proposed expression, which either may contain a common property or which must be a maximum or a minimum, were a function of two or more integral formulas, the value of its differential will be defined from the two particles *nv* and *ow* arising from the same rule.

SCHOLIUM

26. In the above chapters we have seen the differential value of any expression, which arises from a single particle *nv*, to have a form of this kind $nv \cdot dx \cdot T$ or $nv \cdot Tdx$, where T may denote a finite quantity ; whereby the differential value of the same expression arising from two particles *nv* and $o\omega$ will be $= nv \cdot Tdx + o\omega \cdot T' dx$, just as we have shown in the solution. But the same form will be able to be shown easily according to this manner: Evidently if there may be put $o\omega = 0$, then the value of the differential must come from the single particle *nv* present, that we have shown how to find above, and it will be $nv \cdot Tdx$. But if there may be put $nv = 0$ and only the particle $o\omega$ may be considered, the value of the differential may be found in a similar manner, as we have used above; but it will not be $= o\omega \cdot Tdx$*;* because now the particle $o\omega$ is accepted in the following place, in place of T the following value of that equally must be taken, thus so that the value of the differential will become truly $= o\omega \cdot T' dx$. But if therefore each particle *nv* and $o\omega$ may be considered jointly, the value of the differential will be $= nv \cdot Tdx + o\omega \cdot T' dx$, because there in the calculation the particles *nv* and $o\omega$ nowhere will be combined together, but each may be able always to be treated on its own. But so that we may apply this in the usual manner noted in the above chapter, we may consider V to be some indeterminate expression, which for the definite abscissa $AZ = a$ may receive the value $= A$, and its differential value arising from the particle *nv* to be

$= nv \cdot dA$, where *dA* may denote the same for us, as *Tdx* did before; and this value *dA* will be able to be found from the expression *V* in the same manner as set out in the previous chapter. With this found the differential value of the same expression *V* arising from the two particles *nv* and $o\omega = nv \cdot dA + ow \cdot dA'$, where dA' may denote the value *dA* increased by its differential. But nevertheless that distinction of the values of the differentials arising from the two particles is necessary generally for our set up, yet the solution itself of problems pertaining to this there may be reduced again, so that only single valued differentials found in the manner set out above are able to be resolved, which clearly arise from one particles *nv*; which will be set out now in the following proposition.

PROPOSITION IV. PROBLEM

27. *Amongst all the curves related to the same given abscissa* $AZ = a$, *in which the same value of the indefinite expression W agree, to determine that in which the expression V shall be a maximum or a minimum.*

SOLUTION

We may consider the curve sought *az* to satisfy the question and the expression *W* in that to possess the determined value $= B$; therefore this curve *az* amongst all the other curves will be related to the same abscissa *AZ*, in which the expression *W* possesses the same value, as in that expression *V* may receive a maxima or minimum value, which shall be $= A$. Therefore towards finding this curve on putting the indefinite abscissa $AI = x$ and the corresponding applied line $Ii = y$, two applied lines *Nn* and *Oo* may be considered to be increased by the infinitely small particles *nv* and $o\omega$: with which done both the differential value of *W* as well as of *V*, which may arise from these two particles *nv* and $o\omega$ joined together, will have to be put equal to zero, as we have shown in the second proposition. Now the value of the differential expression *V* arising from the single particle *nv* $= nv \cdot dA$ and the value of the differential of the other expression *W* arising from the same single particle *nv* shall be $= nv \cdot dB$, which differential values will be allowed to be found from the precepts given in the above chapter. Therefore now, while we may consider the two particles *nv* and $o\omega$, the value of the differential expression *V* will be $= nv \cdot dA + o\omega \cdot dA'$, truly the value of the other differential expression *W* will be $= nv \cdot dB + o\omega \cdot dB'$. On account of which to discovering the curve sought it will be necessary to come about, that $nv \cdot dA + o\omega \cdot dA' = 0$ as well as $nv \cdot dB + o\omega \cdot dB' = 0$.

[Note that in one case there is an extreme value, while in the other there is a common value, each of which satisfies these equations.]

Both equations may be multiplied by some quantities, so that the equations may arise

$$nv \cdot \alpha dA + o\omega \cdot \alpha dA' = 0,$$
$$nv \cdot \beta dB + o\omega \cdot \beta dB' = 0 .$$

And there arises on the particles *nv* and $o\omega$ being eliminated both $\alpha dA + \beta dB = 0$ as well as $\alpha dA' + \beta dB' = 0$; and α and β will be constant or variable quantities of this kind, which satisfy each equation. Because truly there is $\alpha dA + \beta dB = 0$, there will be also $\alpha' dA' + \beta' dB' = 0$; which equation compared with

$\alpha dA^{\prime} + \beta dB^{\prime} = 0$ shows there must be $\alpha^{\prime} = \alpha$ and $\beta^{\prime} = \beta$; from which these quantities α and β shall become constants and indeed any at all. And thus with any constants taken for the constant quantities α and β, the equation for the curve will be $\alpha dA + \beta dB = 0$. This same equation will emerge, if we may eliminate *nv* and $o\omega$ by the accustomed method. Certainly there will be

$$\frac{nv}{o\omega} = -\frac{dA^{\prime}}{dA} = -\frac{dB^{\prime}}{dB}$$

and thus

$$\frac{dA^{\prime}}{dA} = \frac{dB^{\prime}}{dB} \text{ or } \frac{ddA}{dA} = \frac{ddB}{dB}$$

as well as $dA^{\prime} = dA + ddA$ and $dB^{\prime} = dB + ddB$. But the equation integrated $\frac{ddA}{dA} = \frac{ddB}{dB}$ gives $ldA = ldB + LC$ or $dA = CdB$; which, on putting $C = -\frac{\beta}{\alpha}$, will change into $\alpha dA + \beta dB = 0$, as before we have itself found. On account of which for resolving these problems it is necessary that both the common property of the expression containing *W*, as well as of the expression, which must be a maximum or minimum, to investigate the *V* differential values treated by the method in the above chapters and to multiply these by some constant quantities and to put the sum to be $= 0$; with which done the equation expressing the nature of the curve sought will come about. Q. E. I.

[H. Goldstine considers this proposition in p.93 of the book *His.Cal.Var.*]

COROLLARY I

28. Now therefore for questions contained in this proposition being resolved it will suffice to know the differential values arising from the single particle *nv*; which we have shown above now how to find expeditely.

COROLLARY 2

29. Whereby for this calculation the preceding chapter IV must be called in to help, and from that both paragraph 7 as well as paragraph 31. For in the former the precepts are contained for finding differential values, if the expressions proposed were individual integral formulas, truly in the other, if they shall be functions of two or more integral formulas of this kind.

COROLLARY 3

30. Therefore with the property proposed of a common *W* and a maximum or minimum from the expression *V*, each value of the differential expression is required to be sought from these precepts ; with these found and multiplied by arbitrary constants the sum of which put equal to zero will give the equation for the curve sought.

COROLLARY 4

31. If amongst all the curves generally corresponding to the same abscissa *AZ* that may be sought, in which the expression *V* may reach a maximum or minimum value, for that this equation itself $dA = 0$ may be had, with *dA* the differential value of the expression *V*.

COROLLARY 5

32. But if moreover amongst all the curves corresponding to the same abscissa *AZ*, in which the expression *W* may be shared equally, that may be sought, in which the expression *V* may have a maximum or minimum value, this equation $\alpha dA + \beta dB = 0$ itself is found for that.

COROLLARY 6

33. Therefore it is evident the curve, which amongst all the curves generally *V* may have a maximum or minimum, of which the equation is $dA = 0$ to be contained in the equation $\alpha dA + \beta dB = 0$, by which the curve is expressed, which amongst all the same endowed with the common property *W, V* may have a maximum or minimum.

COROLLARY 7

34. Therefore in the first equation itself, as the solution given, $\alpha dA + \beta dB = 0$ now has a single arbitrary constant present, which moreover must be determined by that, so that the value of the expression *W* may obtain a given value.

COROLLARY 8

35. And thus the problem will be able to be solved, so that amongst all the curves corresponding to the same abscissa *AZ*, in which the expression *W* may obtain the same given value, that may be defined, in which the value of *V* shall be a maximum or a minimum.

COROLLARY 9

36. From these finally it is understood the solution of the problem proposed agrees with the solution of this problem, so that amongst all the curves generally corresponding to the same abscissa *AZ* that may be required, which $\alpha V + \beta W$ may have a maximum or minimum. Which sought, even if it may belong to the absolute method, yet gives the equation $\alpha dA + \beta dB = 0$, as we have found that itself.

SCHOLIUM 1

37. From these therefore not only an easy and convenient method is deduced for resolving all the questions pertaining to this, truly also the nature of this kind of problem is understood thoroughly. For in the first place it is apparent, as we have now shown above, the solution to be the same, either amongst all the curves endowed with the common property *W* that may be sought, which may have *V* a maximum or minimum, or inversely amongst all the curves endowed with the common property *V* that may be required, in which *W* shall be a maximum or minimum. Then also it is understood the question proposed thus be able to be such that its solution may belong to the method of absolute maxima or minima ; for the proposed problem agrees with that, so that amongst

all the curves generally related to the same abscissa *AZ* that may be required, in which this expression $\alpha V+\beta W$ shall be a maximum or minimum; and this transformation of the problem is the cause so that the solution by differential values arising from the single particle *nv* may be able to be perfected nor shall there be further need to consider two particles of this kind, provided the nature of the question may be seen by intuition. But we may consider that agreement afterwards by itself and without that method, by which two particles may be considered ; so that this truth, to be the greatest importance in this matter, may be confirmed more. Towards solving the remaining questions of this kind it is necessary to look at the precepts rendered in the summary in the preceding chapter ; with the help of which the differential values of any expression will be able to be found. Indeed in the first place, in paragraph 7 of that chapter the cases are reviewed, in which the values of the individual integral formulas are shown, while truly in paragraph 31 the method of finding the differential values of expressions is treated, which shall be composed from two or more integral formulas. And thus with the help of these, the value of the differential will be able to be assigned for whatever question presented, both of the maximum or minimum as well as of the common property; but for each found the equation for the curve sought will be formulated without difficulty, since there shall be a need only to put the sum of whatever multiples of these two differential values equal to zero. And this equation found subsequently will be treated in a similar manner, as we have used above both in the reduction for the construction as well as in the integration.

SCHOLIUM 2

38. Now we have observed in the equation $\alpha dA+\beta dB=0$, that in the solution supplied at once, one constant quantity not to be present, but which shall not be arbitrary in general, but must be determined from some condition proposed. Evidently, since in all the curves, in which it is necessary to define the question, the same expression *W* must be shared equally or to have the same value in everything, for example *B*, to be obtained this magnitude *B* can be considered as given ; and since it may not appear in the calculation itself, thus the constants α and β will be allowed to be defined, so that the value of the expression *W*, corresponding to the abscissa $AZ=a$, becomes equal to *B* itself ; and with this agreed on the question otherwise will be considered to be indeterminate. But it will be determined only to the extent, as far as through the integrations after putting in place new arbitrary constants may be defined also by just as many points. Without doubt clearly, as before, just as many points will be prescribed, through which the curve sought shall pass, as many new constants shall be agreed upon introduced by integration. But the number of these will be known from the greatest order of the integration, which will be present in the integration. Because truly the whole question can be recalled to the absolute method, the number of constants of this kind always will be even ; or the resulting equation $\alpha dA+\beta dB=0$ either will be finite, or a differential of the second, fourth, sixth, eighth order and thus so on. But if the equation emerging were finite, then the curve also will be determined completely, if the ratio between α and β were defined thus, so that the expression *W* may receive the given value *B* in the curve found, as we have always put in place to use in the determination. If a differential equation of the second order may be found, then it will be determined by two points ; but it is the agreed and customary manner to prescribe the ends themselves *a* et *z* of the curve, and from

these cases the problem will be determined, if that condition may be added, that the curve sought may be contained between the given ends *a* and *z*. But if a differential equation of the fourth order may be produced, then with four points assigned as it pleases, the satisfying curve will be determined ; therefore this may be agreed to be defined, so that besides the extreme points *a* et *z* likewise the position of the tangents at these ends may be prescribed. But if it may come to an equation of the sixth order, then the curve will be determined by some six points; but of these able to be prescribed in the first place both the end points *a* et *z*, then the position of the tangents at these ends, and in the third place by the curvature at these places or the magnitudes of the radii of osculation. Therefore with these noted it may be understood from that solution, a condition of some kind must be added to the proposition of each problem, so that it may be determined completely ; and this reminder has a place not only here, but also in the absolute method and in the remaining relative method.

SCHOLIUM 3

39. This distinction is to be noted also of the greatest importance, from which in the absolute method we have chosen the partition of the first treatment. Indeed that takes its place from the manner in which the curve found satisfies the question. For it can happen, that some part of that may be endowed to the indefinite abscissa related to the required property ; then also cases are given, in which that part cannot correspond to the definite abscissa $AZ = a$ unless it may satisfy the condition of the problem. Evidently that comes about, if this magnitude *a* in the equation, that the solution may supply, either generally is not present or can be taken in some arbitrary magnitudes α and β. From which it is evident, if both the formulas *W* and *V* in the first case of paragraph 7 of the preceding chapter may be retained on examination, then any part of the curve found is adapted to the question. Then indeed it can come about also, that the magnitude *a*, or the magnitudes depending on that, may be present in either value of the differential, or in both; yet these either mutually cancel each other out in the equation $\alpha dA + \beta dB = 0$, or may be able to be understood from the arbitrary α and β ; in which case it is required equally to satisfy some part of the curve found. But this only has a place, if the value given and determined is not prescribed, and which common property *W* ought to be obtain in the part satisfied ; for then that cannot come about, that in some part the same value may be chosen. But from the solution of any single question it is understood easily, by what condition either the whole curve *az* or some part may be able to be satisfied ; that which it will be able to be shown most conveniently in the examples.

EXAMPLE I

40. *Among all the curves related to the abscissa AZ, in which the formula* $\int yxdx$ *may retain the same value, to find that, in which the value of the formula* $\int yydx$ *shall be a minimum.*

Therefore the common property will be $W = \int xydx$, of which, on account of $dxy = ydx + xdy$, the value of the differential is $= nv \cdot dx \cdot x$. But the formula of the maxima or minima is $V = \int yydx$, of which, on account of $d \cdot yy = 2ydy$, the value of the differential is $= nv \cdot dx \cdot 2y$. Therefore this equation $\alpha x + 2\beta y = 0$ will be obtained with the division by $nv \cdot dx$ put in place ; from which it is apparent some right line passing through *A* making some angle with the axis *AZ* satisfies the question. And because the length of the abscissa $AZ = a$ [note : Euler uses the same letter *a* for the arc length and not to be confused with the left-hand end point] is not introduced into the calculation, some part of this right line may be equally satisfactory. So that if moreover it may be postulated, that for the given abscissa $AZ = a$ the formula $\int yxdx$ may posses a given value, such as *B*, then on account of $y = mx$ there becomes $\int yxdx = \frac{1}{3}mx^3$ and thus $\frac{1}{3}mx^3 = B$ *;* from which the position of the right line thus may be defined, so that it may become $y = \frac{3Bx}{a^3}$. Therefore this right line will now be endowed with that property, that amongst all the right lines or curves, which for a given abscissa $AZ = a$ may have the value of the formula $\int xydx = B$, it shall produce the minimum value of the formula $\int yydx$.

EXAMPLE II

41. *Amongst all the curves of the same length joining the points a and z to find that, which may comprise the maximum or minimum area aAZz* (Fig.l6).

Because the common property is the length of the arc $= \int dx\sqrt{(1+pp)}$, the value of its differential will be $-nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}$. Then the formula of the maximum or minimum is $\int ydx$, whose differential value is $nv \cdot dx$; from which the equation will be had for the curve sought $dx = bd \cdot \frac{p}{\sqrt{(1+pp)}}$ and by integrating,

$$x + c = \frac{bp}{\sqrt{(1+pp)}} \text{ and thus } p = \frac{x+c}{\sqrt{(b^2-(x+c)^2)}} = \frac{dy}{dx}.$$

Hence therefore by integration there shall become

$$y = f \pm \sqrt{\left(b^2 - (x+c)^2\right)} \text{ or } b^2 = (y-f)^2 + (x+c)^2,$$

which is the general equation for a circle. On account of which the arc of some circle drawn through the points *a* and *z* will enclose either a maximum or minimum area *aAZz* amongst all the other curved lines of the same length. But the arc of a given circle of a given length can be put in place between the ends *a* and *z* in a two-fold manner ; on the one hand, so that the concavity shall be turned towards the axis *AZ* , on the other hand, so that it shall be the convexity. In the first case it is evident the area is a maximum, truly in the latter a minimum. And hence, if the ends *a* and *z* may be given together with the length of a curve put in place between these end points, as indeed it is required to be greater than the straight line joining these ends, the solution will be determined fully ; for a single arc of the circle of this length can be described from these end points, which, provided the concavity or convexity shall be turned towards the axis *AZ* , will form either a maximum or minimum area.

COROLLARY

42. Hence also it is apparent the circular arc *az* (Fig. 16), drawn through the ends *a* and *z*, not only forms a maximum area *aAZz* among all the other lines of the same length, but also, whatever the line *aCEDza* that may be given drawn from the end point *a* to the end point *z*, with that the arc of the circle *az* shall include the maximum area. For if the area *aAZz* is a maximum, there will also be the maximum area

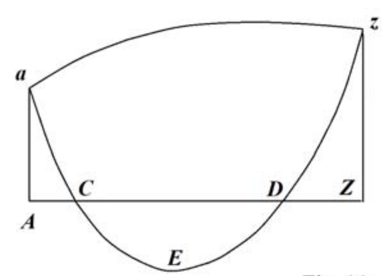

Fig. 16

$$aAZz - aAC - zZD + CED$$

on account of the areas *aAC, zZD* and *CED* of constant magnitude, whatever line may be taken for *az* to be a maximum, [*i.e.* the converse of the proposition].

EXAMPLE III

43. *Among all the curves of the same length* (Fig. 7) *joining the points A and M to find that, which with the right lines AC and MC drawn to the fixed point C, may comprise the maximum or minimum area ACM.*

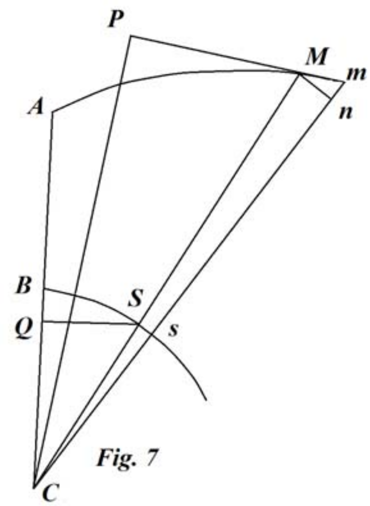

Fig. 7

Because, on account of the given points *A*, *C*, *M*, the right lines *AC* and *MC* are given in place, the angle ACM may be put $= x$, or with the arc of the circle *BS* described, with centre *C*, radius $CB = 1$, here the arc shall be $BS = x$ and there may be put $CM = y$; there will be $Ss = dx$, $Mn = ydx$ and the area $ACM = \frac{1}{2}\int yydx$. Again on account of $mn = dy$ there will be

$$Mm = \sqrt{(y^2dx^2 + dy^2)} = dx\sqrt{(yy + pp)},$$

on putting $dy = pdx$. Whereby amongst all the equations containing a relation between *x* and *y*, which for a given value of x give the same magnitude $\int dx\sqrt{(yy+pp)}$, it is required to define that, which for the same value of *x* may present a maximum or minimum magnitude of the formula $\frac{1}{2}\int yydx$. Therefore since the differential value of the formula $\int dx\sqrt{(yy+pp)}$ shall be

$$= nv\cdot dx\left(\frac{y}{\sqrt{(yy+pp)}} - \frac{1}{dx}d\cdot\frac{p}{\sqrt{(yy+pp)}}\right)$$

and the differential value of the formula $\frac{1}{2}\int yydx \; = nv\cdot dx\cdot y$, this equation will be had for the curve sought [for some constant *b* :]

$$ydx = \frac{bydx}{\sqrt{(yy+pp)}} - bd\cdot\frac{p}{\sqrt{(yy+pp)}},$$

which multiplied by *p* will be changed into this :

$$ydy = \frac{bydy}{\sqrt{(yy+pp)}} - bpd\cdot\frac{p}{\sqrt{(yy+pp)}}$$

$$= bd\cdot\sqrt{(yy+pp)} - \frac{bpdp}{\sqrt{(yy+pp)}} - bpd\cdot\frac{p}{\sqrt{(yy+pp)}};$$

of which the integral is :

$$\tfrac{1}{2}yy = b\sqrt{(yy+pp)} - \frac{bpp}{\sqrt{(yy+pp)}} + bc = \frac{byy}{\sqrt{(yy+pp)}} + bc.$$

From C the perpendicular $CP = u$ may be drawn to the tangent MP ; there will be

$$u = \frac{yy}{\sqrt{(yy+pp)}}$$

and there will be had $yy = 2bu + bc$; as we have now shown the equation above to be for a circle. On account of which the arc of the circle drawn through the end points A and M will have this property, so that amongst all the other curves themselves of the same length the end points A and M joined together may show either the maximum or minimum area ACM, provided that arc may turn through either a concavity or convexity within the angle ACM . From which itself that may be confirmed, as we have advised in the preceding paragraph.

EXAMPLE IV

44. *Amongst all the curves joining the points* (Fig. 15) *a and z, which rotated about the axis AZ generate the solids of the same surface, to determine that, which at the same time will produce a maximum volume of the sold in this way.*

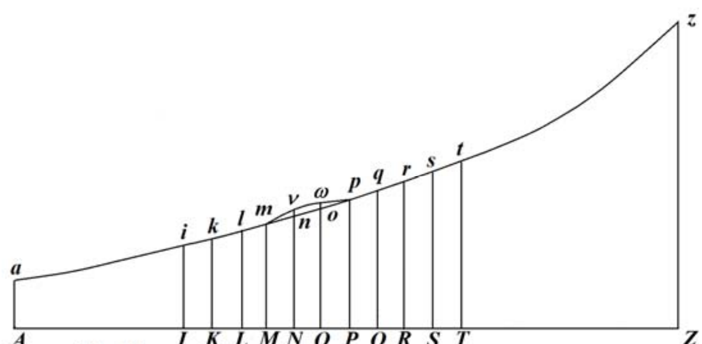

Fig. 15

The surface of the solid generated in this way is found to be proportional to this integral formula $\int ydx\sqrt{(1+pp)}$, of which the value of the differential is :

$$nv\cdot dx\left(\sqrt{(1+pp)} - \frac{1}{dx}d\cdot\frac{yp}{\sqrt{(1+pp)}}\right).$$

Truly the volume of the solid generated in this way is as $\int yydx$, of which the value of the differential is $= nv\cdot dx\cdot 2y$. On account of which this equation will result

$$2ydx = bdx\sqrt{(1+pp)} - bd\cdot\frac{yp}{\sqrt{(1+pp)}}.$$

This may be multiplied by *p*, so that there may be produced :

$$2ydy = bdy\sqrt{(1+pp)} - bpd \cdot \frac{yp}{\sqrt{(1+pp)}}$$

$$= bd \cdot y\sqrt{(1+pp)} - \frac{bypdp}{\sqrt{(1+pp)}} - bpd \cdot \frac{yp}{\sqrt{(1+pp)}},$$

of which the integral is

$$yy = by\sqrt{(1+pp)} - \frac{bypp}{\sqrt{(1+pp)}} - bc = \frac{bp}{\sqrt{(1+pp)}} + bc.$$

Therefore there will be

$$by = \left(yy - bc\right)\sqrt{(1+pp)} \text{ and } p = \frac{\sqrt{(b^2y^2 - (yy-bc)^2)}}{yy-bc} = \frac{dy}{dx}.$$

Whereby there will become :

$$dx = \frac{(yy-bc)dy}{\sqrt{(bbyy-(yy-bc)^2)}}.$$

In the first place being noted concerning this equation, if there were $c = 0$, there becomes

$$dx = \frac{ydy}{\sqrt{(bb-yy)}}$$

and thus the curve is a circle, of which the centre shall be placed on the axis *AZ* ; therefore that arc of the circle described with the centre taken on the axis *AN* and passing through the two given points *a* and *z* will satisfy the question ; but this will be unique, and thus it will generate a solid of a definite surface. Whereby if amongst all the curves, which generate a solid of another and diverse surface, that may be sought, which may produce a maximum volume, that will not be a circle, but another curve present in the equation

$$dx = \frac{(yy-bc)dy}{\sqrt{(bbyy-(yy-bc)^2)}}.$$

For not only can it come about that the curve may pass through two prescribed points *a* and *z*, on account of the two constants *b* and *c*, but also, so that the length of the curve *az* of a given magnitude may arise. Furthermore the length of the curve, on account of

$$\int dx\sqrt{(1+pp)}=\int\frac{bydx}{yy-bc}$$

becomes

$$=\int\frac{bydy}{\sqrt{(bbyy-(yy-bc)^2)}},$$

the integral of which depends on the quadrature of the circle, and is

$$=\frac{b}{2}\text{Acos.}\frac{b\left(2c+b\right)-2yy}{b\sqrt{(bb+4bc)}}+\text{Const.}$$

But if moreover b may be put $=\infty$, a singular case arises ; for this equation arises

$$dx=-\frac{cdy}{\sqrt{(yy-cc)}},$$

which is for a catenary curve with the convex side turned towards the axis *AZ*.

EXAMPLE V

45. *Among all the curves az containing equal areas aAZz to determine that, which rotated about the axis AZ, will generate the solid of minimum surface.*

Because the common property in the area $=\int ydx$ is put in place, the differential value of this will be $=nv\cdot dx$. Then the formula, which must be a minimum, is $\int ydx\sqrt{(1+pp)}$, of which the differential value is

$$=nv\cdot(dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}});$$

from which this equation arises for the curve sought :

$$ndx=dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}},$$

which, multiplied by p and integrated, will provide :

$$ny+b=\frac{y}{\sqrt{(1+pp)}}\text{ or }\sqrt{(1+pp)}=\frac{y}{ny+b};$$

from which there becomes :

$$p = \frac{(y^2 - (ny+b)^2)}{ny+b} = \frac{dy}{dx} \text{ and } dx = \frac{(ny+b)dy}{\sqrt{((1-n^2)y^2 - 2bny - bb)}}.$$

From which it is clear, if there shall be $b = 0$, then the curve is going to become a straight line joining the points *a* and *z*. Then if there shall be $n = 0$, on account of

$$dx = \frac{bdy}{\sqrt{(yy-bb)}}$$

the curve will be a concave catenary directed towards the axis *AZ*. But if moreover there shall be $n = -1$, the equation becomes

$$dx = \frac{(b-y)dy}{\sqrt{(2by-bb)}},$$

from which integrated there arises

$$x = c + \frac{2b-y}{3b}\sqrt{(2by-bb)};$$

which is the equation for an algebraic curve, and expressed in rationales gives

$$9b(x-c)^2 = \left(2b-y\right)^2\left(2y-b\right).$$

Therefore it is a line of the third order and refers to Newton's example 68.

EXAMPLE VI

46. *Amongst all the curves az of the same length to define that, which rotated about the axis AZ may produced the maximum solid.*

Therefore amongst all the curves endowed with the common property $\int dx\sqrt{(1+pp)}$ that is sought, in which $\int yydx$ shall be a maximum. Therefore because the value of the differential of the formula $\int dx\sqrt{(1+pp)}$ is

$$= -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}},$$

truly the value of the differential of the formula $\int yydx$ is $= 2nv \cdot ydx$, this equation will be had for the curve sought

$$2ydx = \pm bbd \cdot \frac{p}{\sqrt{(1+pp)}},$$

which multiplied by p and integrated will give

$$yy+bc = \frac{\pm bb}{\sqrt{(1+pp)}} \text{ or } \sqrt{(1+pp)} = \frac{\pm bb}{yy+bc};$$

and hence

$$p = \frac{\sqrt{\left(b^4-(yy+bc)^2\right)}}{yy+bc} = \frac{dy}{dx};$$

from which there becomes

$$x = \int \frac{(yy+bc)dy}{\sqrt{\left(b^4-(yy+bc)^2\right)}}.$$

This curve has this property, so that its radius of osculation, which generally is

$$= dx : d \cdot \frac{p}{\sqrt{(1+pp)}},$$

becomes $= \frac{bb}{2y}$, that is, it is proportional to the inverse of the applied line y ; from which it is apparent the curve to be elastic. But not only is the curve to be constructed through the arbitrary constants b and c, so that the curve may pass through the given end points a and z but also, so that its arc intercepted between these points may be made of a known magnitude. If there shall be $c = 0$, it will produce an elastic rectangle. In no case can the construction be resolved by the quadrature either of a circle or of a hyperbola, unless either b and c shall be infinite, in which case a right line az will be produced, or $b = c$. For in this case there will be found

$$x = \int \frac{(yy+bb)dy}{y\sqrt{(-\ 2bb-yy)}},$$

or on taking bb negative there will be

$$x = \int \frac{(yy-bb)dy}{y\sqrt{\left(2bb-yy\right)}} = -\sqrt{\left(2bb-yy\right)} - \ bb\int \frac{dy}{y\sqrt{\left(2bb-yy\right)}}$$

and by integrating by absolute logarithms there becomes

$$x = -\sqrt{\left(2bb-yy\right)} + b\sqrt{2l\,\frac{b\sqrt{2}+\sqrt{(2bb-yy)}}{y}}.$$

Indeed the length itself of the curve, which generally is

$$=\int \frac{bbdy}{\sqrt{\left(b^4-(yy+bc)^2\right)}},$$

in this case becomes

$$=g-b\sqrt{2l\frac{b\sqrt{2}+\sqrt{(2bb-yy)}}{y}}.$$

EXAMPLE VII

47. *To find the curve, which amongst all the other curves of the same length rotated about the axis AZ may produce a solid, of which the surface shall be a maximum or minimum.*

Because the common property is $\int dx\sqrt{(1+pp)}$, of which the differential value is $-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$, but the differential value of the maximum or minimum formula $\int ydx\sqrt{(1+pp)}$ is

$$=nv\cdot\left(dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}}\right),$$

this equation will be found for the curve sought

$$bd\cdot\frac{p}{\sqrt{(1+pp)}}=dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}},$$

which multiplied by p and integrated will provide

$$c-\frac{b}{\sqrt{(1+pp)}}=\frac{y}{\sqrt{(1+pp)}} \text{ or } c=\frac{b+y}{\sqrt{(1+pp)}}.$$

Hence the equation becomes

$$\sqrt{(1+pp)}=\frac{b+y}{c} \quad\text{and}\quad p=\frac{\sqrt{\left((b+y)^2-cc\right)}}{c}=\frac{dy}{dx}$$

and from that

$$dx=\frac{cdy}{\sqrt{\left((b+y)^2-cc\right)}},$$

which is the general equation for a catenary and is satisfied, provided the axis with respect to the suspended catenary may be held in a horizontal position. Therefore it can arise, that either a concave or convex curve may be turned towards the axis *AZ*, in the first case the surface of the solid becomes a minimum, in the latter a maximum.

EXAMPLE VIII

48. *Amongst all the curves* (Fig. 17**)** *passing through the points A and C, which all may comprise equal areas ABC, to define that, which in a fluid along the direction BA may experience the least resistance to the motion.*

With the abscissa put $AP = x$, the applied line $PM = y$, the common property is $\int ydx$, the differential value of this is $= nv\cdot dx$. But the total resistance of the curve, that the figure experiences in the direction *AB*, is as $\int\frac{p^3dx}{1+pp}$, of which the differential is

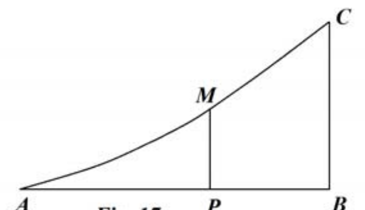

Fig. 17

$$-nv\cdot d\cdot\frac{3pp+p^4dx}{(1+pp)^2}.$$

From these the equation for this curve will emerge

$$dx=bd\cdot\frac{3pp+p^4}{(1+pp)^2};$$

which integrated gives

$$x=c+\frac{bpp(3+pp)}{(1+pp)^2}.$$

But the equation multiplied by *p* will change into this

$$dy=bpd\cdot\frac{3pp+p^4}{(1+pp)^2},$$

which changed in this form

$$dy=bpd\cdot\frac{3pp+p^4}{(1+pp)^2}+bdp\cdot\frac{3pp+p^4}{(1+pp)^2}-bdp\cdot\frac{3pp+p^4}{(1+pp)^2}$$

has the integral

$$y = f + \frac{bp^3(3+pp)}{(1+pp)^2} - \frac{bp^3}{1+pp} \text{ or } y = f + \frac{2bp^3}{(1+pp)^2};$$

therefore since there shall be

$$x = c + \frac{bpp(3+pp)}{(1+pp)^2},$$

the curve will be algebraic. But it is required to be effected, that, where in the case there becomes $x = 0$ (which cannot happen, unless either b or c may be taken negative) likewise y shall vanish. But so that the curve may be understood, putting $x - c = t$ and $y - f = u$, there will be

$$t = \frac{bpp(3+pp)}{(1+pp)^2} \text{ and } u = \frac{2bp^3}{(1+pp)^2};$$

from which there becomes

$$t + u\sqrt{3} = \frac{b(p^4 + 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}$$

and

$$t - u\sqrt{3} = \frac{b(p^4 - 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}.$$

Therefore with the square roots extracted the equation will be found :

$$\sqrt{\frac{t+u\sqrt{3}}{b}} = \frac{pp + p\sqrt{3}}{1+pp} \text{ and } \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{pp - p\sqrt{3}}{1+pp};$$

and hence

$$\sqrt{\frac{t+u\sqrt{3}}{b}} + \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2pp}{1+pp}$$

and

$$\sqrt{\frac{t+u\sqrt{3}}{b}} - \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2p\sqrt{3}}{1+pp}.$$

But there is

$$\frac{t}{b} = \tfrac{3}{2} \cdot \frac{2pp}{1+pp} - \tfrac{1}{2} \cdot \frac{4p^4}{(1+pp)^2} = \tfrac{3}{2}\sqrt{\frac{t+u\sqrt{3}}{b}} + \tfrac{3}{2}\sqrt{\frac{t-u\sqrt{3}}{b}} - \tfrac{1}{2}\left(\frac{2t}{b} + 2\sqrt{\frac{tt-3uu}{bb}}\right).$$

Therefore

$$\frac{4t}{b} = 3\sqrt{\frac{t+u\sqrt{3}}{b}} + 3\sqrt{\frac{t-u\sqrt{3}}{b}} - 2\frac{\sqrt{(tt-3uu)}}{b};$$

which made rational presents this equation of the fourth order

$$4t^4 + 8ttuu + 4u^4 = 4bt^3 + 36btuu - 27b^2uu,$$

or

$$4\left(tt+uu\right)^2 = 4bt^3 + 36btu^2 - 27b^2u^2.$$

But towards constructing this curve from an infinitude of points it is convenient to use these formulas

$$t = \frac{b\left(p^4+3pp\right)}{\left(1+pp\right)^2} \text{ and } u = \frac{2bp^3}{\left(1+pp\right)^2}.$$

Moreover in the first place it is clear the curve has a diameter placed in the position of the abscissas *t* and with two places becoming $u=0$, evidently in the case $p=0$, so that likewise there becomes $t=0$, and in the case $p=\infty$, where there becomes $t=b$. So that if there is put $b=4c$ and $t=3c+r$, this equation will arise

$$\left(rr+uu\right)^2 + 8c\left(r^3-3ru^2\right) + 18cc(r^2+u^2) - 27c^4 = 0,$$

which since it shall be a function of $rr+uu$ and r^3-3ruu, this curve may be indicated to have three diameters themselves crossing at the origin of the abscissas *r* of these. Therefore the curve sought (Fig.18) thus will be able to be inscribed in the equilateral triangle *ABC*, so that it may be constructed from the three branches *ADB*, *AEC* and *BFC* similar and equal to each other, which form the most acute cusps at the points *A* , *B* and *C*. Therefore the diameters of this are three right lines *AI*, *BH* and *CG* crossing each other at the centre of the triangle *O*. But there will be $AO=3c$, $OF=c$ and $OI=\frac{3}{2}c$, thus so that

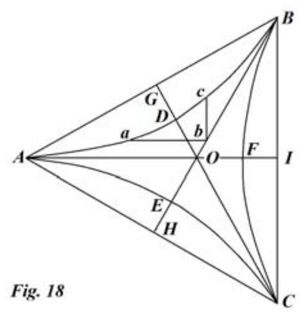

Fig. 18

there shall be $AI=\frac{9}{2}c$ and $FI=\frac{1}{2}c=\frac{1}{2}OF$. Now any portion *abc* of this curve thus will be prepared, taken from the right lines *ab* and *bc* parallel to *AI* and *BI* themselves and with the arc of the curve *ac*, so that the arc *ac* amongst all the other connecting points *a* and *c* and containing the equal area *abc* may experience the minimum resistance in the motion of the fluid moved along the direction *ba*. But again this curve will be rectifiable and the arc is being found $ADB=\frac{16}{3}c$; from which there will be

$$ADB : AI = \frac{16}{3} : \frac{9}{2} = 32 : 27$$

and

$$ADB : AB = 32 : 18\sqrt{3} = 16 : 9\sqrt{3}.$$

EXAMPLE IX

49. *Amongst all the equal area curves* (Fig.l9) *AM enclosing equal areas APM to find that, which may be prepared thus, so that, if always from the centre of the circle of oscillation O to the applied line MP produced the perpendicular ON may be drawn, the curve formed from the points N may comprise a minimum area APN.*

On putting the abscissa $AP = x$ and the applied line $PM = y$, the area will be $APM = \int ydx$, which is the common property, and the value of its differential $= nv \cdot dx$. Then, since the radius of osculation will be $MO = -(1+pp)^{3:2}$, there becomes $MN = -\frac{(1+pp)}{q}$ and $PN = -\frac{(1+pp)}{q} - y$;

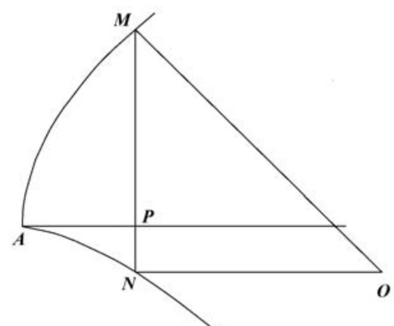

Fig. 19

from which the area *APN* will be

$$-\int ydx - \int \frac{(1+pp)}{q} dx,$$ which must be a minimum, of which the differential value is

$$= nv \cdot \left(-dx + d \cdot \frac{2p}{q} + \frac{1}{dx} dd \cdot \frac{(1+pp)}{qq} \right);$$

from which this equation arises

$$ndx^2 = dxd \cdot \frac{2p}{q} + dd \cdot \frac{(1+pp)}{qq};$$

which integrated gives

$$nxdx = \frac{2pdx}{q} + d \cdot \frac{(1+pp)}{qq} + bdx.$$

[Or $nxdy = \frac{2pdy}{q} + pd \cdot \frac{(1+pp)}{qq} + bdy.$]

Truly that same equation [before the above integration] multiplied by *p* gives

$$ndxdy = dyd \cdot \frac{2p}{q} + pdd \cdot \frac{1+pp}{qq};$$

of which the integral is

$$nydx = cdx - \frac{2dx}{q} + pd \cdot \frac{1+pp}{qq}.$$

[$nydx = cdx - \frac{2dx}{q} + pd \cdot \frac{1+pp}{qq}$; diff.: $ndydx = 0 + \frac{2}{q^2}dqdx + pdd \cdot \frac{1+pp}{qq} + qdxd \cdot \frac{1+pp}{qq}$;
but $\frac{2}{q^2}dq + qd \cdot \frac{1+pp}{qq} = \frac{2}{q^2}dq - \frac{2}{q^2}dq + 2q\frac{p}{q}d \cdot \frac{p}{q} = 2q\frac{p}{q}d \cdot \frac{p}{q} = pd \cdot \frac{2p}{q}$;
hence $ndydx = pdxd \cdot \frac{2p}{q} + pdd \cdot \frac{1+pp}{qq} = dyd \cdot \frac{2p}{q} + pdd \cdot \frac{1+pp}{qq}$.]

From these equations joined together there arises

$$nxdy - nydx = bdy - cdx + \frac{2pdy}{q} + \frac{2dx}{q} = bdy - cdx + \frac{2dx^2 + 2dy^2}{dp}.$$

Putting

$$nx - b = nt \text{ and } ny - c = nu;$$

there will be $dy = du$ et $dx = dt$, and

$$ndp = \frac{2dt^2 + 2du^2}{tdu - udt} = \frac{nddu}{dt},$$

or

$$dt^3 + 2dtdu^2 = ntduddu - nudtddu$$

with dt placed constant. Let $u = st$, there shall be

$$du = sdt + tds \text{ and } ddu = tdds + 2dtds;$$

and with these substituted this equation will be produced :

$$2(1+ss)dt^3 + 4stdt^2ds + 2(1-n)ttdtds^2 = nt^3dsdds.$$

Putting $t = e^{\int rds}$, there will be

$$dt = e^{\int rds}rds \text{ and } ddt = 0 = e^{\int rds}(rdds + drds + rrds^2);$$

from which there becomes

$$dds = -\frac{drds}{r} - rds^2;$$

from which there emerges finally :

$$2(1+ss)r^3ds+4sr^2ds+2(1-n)rds=-\frac{ndr}{r}-nrds$$

or

$$\frac{ndr}{r}+(2-n)rds+4sr^2ds+2r^3ds+2r^3s^2ds=0.$$

Let $s=v-\frac{1}{r}$, there becomes $dr+rrdv=\frac{ndv}{2(1+vv)}$; which equation can be integrated, as long as $n=2i(i-1)$ with *i* denoting some whole number ; thus if there shall be $n=4$, there becomes

$$r=\frac{2v}{1+vv}+\frac{1}{(1+vv)^2\int\frac{dv}{(1+vv)^2}};$$

from which the equation can be resolved by working backwards.

EXAMPLE X

50. *Amongst all the curves, in which $\int xTdx$ keeps the same value, to find that, in which $\int yTdx$ shall be a maximum or a minimum, with T being some function of p, so that thus there shall be $dT=Pdp$.*

For the differential value of the formula $\int xTdx$ being found is to be observed to be $d\cdot xT=Tdx+xPdp$, from which the value of that differential will be $=-nv\cdot d\cdot xP$. But from the other formula $\int yTdx$ there will be found $d\cdot yT=Tdy+yPdp$, from which the value of its differential will be $nv\cdot(Tdx-d\cdot yP)$. Whereby for the curve sought this equation will arise : $nd\cdot xP=Tdx-d\cdot yP$. Therefore $\int Tdx=nxP+yP+b.$ Again if that equation may be multiplied by *p*, there will be found :

$$npd\cdot xP=Tdy-pd\cdot yP=d\cdot yT-yP\cdot dp-pd\cdot yP=d\cdot yT-d\cdot yPp.$$

But there is

$$pd\cdot xP=Ppdx+pxdP+xPdp+Tdx-d\cdot xT=d\cdot xPp+Tdx-d\cdot xT.$$

On account of which there becomes

$$d\cdot yT-d\cdot yPp=nd\cdot xPp+nTdx-nd\cdot xT\ ;$$

and hence

$$\int nTdx = yT - yPp - nxPp + nxT + c .$$

Because indeed from the above integration we have

$$\int nTdx = nnxP + nyP + nb ,$$

by eliminating $\int nTdx$ there will be this equation

$$nnxP + nyP + nb = yT - yPp - nxPp + nxT + c$$

or

$$y = \frac{nx(nP + Pp - T) + c}{-nP - Pp + T}$$

or

$$y = -nx + \frac{c}{T - nP - Pp} .$$

Finally therefore the equation arises :

$$x = c\int \frac{dP}{(T - nP - Pp)^2},$$

and

$$y = c\int \frac{pdP}{(T - nP - Pp)^2} = \frac{c}{T - nP - Pp} - nc\int \frac{dP}{(T - nP - Pp)^2}.$$

EXAMPLE XI

51. *To find the curve, which among all the others contained between the same end points and containing the same value of the formula* $\int xdx\sqrt{(1+pp)}$, $\int ydx\sqrt{(1+pp)}$ *may have a maximum or minimum.*

This is an example of the preceding case and emanates from that on putting $T = \sqrt{(1+pp)}$, from which there will be

$$P = \frac{p}{\sqrt{(1+pp)}} \text{ and } dP = \frac{dp}{(1+pp)^{3:2}} .$$

Again indeed there will be

$$T - nP - Pp = \frac{1-np}{\sqrt{(1+pp)}} .$$

Now with these replaced, there will be produced :

$$x = c\int \frac{dp}{(1-np)^2\sqrt{(1+pp)}} \text{ and } y = \frac{c\sqrt{(1+pp)}}{1-np} - nx.$$

Moreover on integrating by logarithms there may be put in place :

$$x = \frac{nc(p+\sqrt{(1+pp)})-c}{(1+nn)(1-np)} + \frac{c}{(1+nn)^{3:2}} \cdot l\frac{n+(1+\sqrt{(1+nn)}(p+\sqrt{(1+pp)}}{n+(1-\sqrt{(1+nn)}(p+\sqrt{(1+pp)}} + b$$

and

$$x = \frac{nc+c(\sqrt{(1+pp)}-nnp)}{(1+nn)(1-np)} - \frac{nc}{(1+nn)^{3:2}} \cdot l\frac{n+(1+\sqrt{(1+nn)}(p+\sqrt{(1+pp)}}{n+(1-\sqrt{(1+nn)}(p+\sqrt{(1+pp)}} - nb\,;$$

from which values the curve can be constructed by logarithms. But generally, whatever function of p T may denote, the construction can always be resolved by quadrature. Moreover this example would be much more difficult to solve without the help of the preceding example; for it would not be seen so easily, how the integrable equation found might be returned, as in the general case.

PROPOSITION V. PROBLEM

52. *Amongst all the curves related to the same abscissa* $= a$, *which receive the same value of the formula* $\Pi = \int [Z]dx$, *to find that, in which* $\int Zdx$ *shall be a maximum or minimum, with the function* Z *likewise being a function of* Π, *thus so that there shall be*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + etc.$$

and

$$d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + \ etc.$$

SOLUTION

Because there is $d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc.}$, the value of the differential of the formula $\int [Z]dx$, which here represents a common magnitude for all the curves, the differential value

$$= nv \cdot dx\left([N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2} - \text{etc.}\right),$$

which follows from the first case of paragraph 7 of the preceding chapter. But the formula $\int Zdx$, expressing a maximum or minimum case, because Z involves the integral formula $\Pi = \int [Z] dx$, belongs to the second case *loc. cit.* and thus the differential value of this will be

$$= nv \cdot dx \left(\mathrm{N} + [N] - \frac{d\left(\mathrm{P} + [P]\right)}{dx} + \frac{dd\left(\mathrm{Q} + [Q]\right)}{dx^2} - \text{etc.} \right),$$

on calling $V = H - \int Ldx$, where H is the magnitude determined which arises, if in the integral $\int Ldx$ there may be put $x = a$. And on this account the value of the differential will depend on the magnitude H itself by the prescribed length of the abscissa $x = a$ itself. Therefore from these two differential values of both the formulas proposed, of which the one shows the common property, the other the maximum or minimum, according to the rule given the following equation arises:

$$\begin{aligned} 0 &= \alpha [N] - \frac{\alpha d[P]}{dx} + \frac{\alpha dd[Q]}{dx^2} - \text{etc.} \\ &+ \mathrm{N} + [N] - \frac{d\left(\mathrm{P} + [P]\right)}{dx} + \frac{dd\left(\mathrm{Q} + [Q]\right)}{dx^2} - \text{etc.}; \end{aligned}$$

which, on account of $V = H - \int Ldx$, will change into this :

$$\begin{aligned} 0 = \mathrm{N} + (\alpha + H - \int Ldx)[N] &- \frac{d\left(\mathrm{P} + (\alpha + H - \int Ldx)[P]\right)}{dx} \\ &+ \frac{dd\left(\mathrm{Q} + (\alpha + H - \int Ldx)[Q]\right)}{dx^2} - \text{etc.} \end{aligned}$$

Now since α shall be an arbitrary constant quantity, even if H shall be a determined constant quantity, yet $\alpha + H$ becomes an arbitrary quantity and thus no more will depend on the definite length a of the abscissa. Whereby, if we may write C in place of $\alpha + H$, we will have this equation for the curve sought :

$$\begin{aligned} 0 = \mathrm{N} + (C - \int Ldx)[N] &- \frac{d\left(\mathrm{P} + (C - \int Ldx)[P]\right)}{dx} \\ &+ \frac{dd\left(\mathrm{Q} + (C - \int Ldx)[Q]\right)}{dx^2} - \text{etc.}, \end{aligned}$$

which therefore will show the curve for any abscissa, which amongst all the other curves receiving the same value of the formula $\int[Z]dx$ will contain a maximum or minimum value of the formula $\int Zdx$. Q. E. I.

COROLLARY 1

53. Therefore if the common property were that integral formula itself, which is involved in the maximum or minimum formula, then the consideration of the magnitude of the abscissa determined is removed from the calculation and the curve found for whatever abscissa will satisfy the question.

COROLLARY 2

54. In this equation found at this stage two integral formulas are present ; evidently in the first place the formula $\int Ldx$ and then the formula $\Pi=\int[Z]dx$, which, since this may be contained in *Z,* will be present in the quantities *L, M, N, P* etc.

COROLLARY 3

55. Therefore if this integral may be willing to be removed by differentiation, it will come to differentials higher by two grades and it will likewise show the arbitrary constant *C*. Yet meanwhile the number of arbitrary constants will be less by one than this grade of the differential ; because from that the integral $\Pi=\int[Z]dx$ must obtain a definite value, clearly that itself, which it has in the formula of the maxima or minima $\int Zdx$.

COROLLARY 4

56. Hence therefore in the equation found on account of the arbitrary constant C the constants will be present with the ability to be one more, than the order of the differential indicates. Of which one will be determined from that, so that the value of the common formula $\Pi=\int[Z]dx$ becomes of a given magnitude for the curve sought ; truly the rest will be determined by given points or the given position of tangents.

COROLLARY 5

57. If *Z* were a function both of the quantities *x, y, p, q* etc. as well as the arc of the curve *s* and among all the isoperimetric curves that is sought, in which $\int Zdx$ shall be a maximum or minimum, then there becomes

$$\Pi=s=\int[Z]dx \text{ and } [Z]=\sqrt{(1+pp)},$$

thus so that there shall be

$$[M]=0\ ,\ [N]=0 \text{ and } [P]=\frac{p}{\sqrt{(1+pp)}}.$$

COROLLARY 6

58. Therefore in this case, if there were $dZ = Lds + Mdx + Ndy + Pdp + Qdq + \text{etc.}$, this equation there will be found for the curve, which amongst all the isoperimetric $\int Zdx$ may have a maximum or a minimum :

$$0 = N - \frac{1}{dx} d\left(\frac{\mathrm{P} + (C - \int Ldx)p}{\sqrt{(1+pp)}} \right) + \frac{ddQ}{dx^2} - \text{etc.}$$

or

$$N - \frac{P}{dx} + \frac{ddQ}{dx^2} = \frac{(C - \int Ldx)dp}{dx(1+pp)^{3:2}} - \frac{Lp}{\sqrt{(1+pp)}} + \text{etc.}$$

or

$$\frac{Lp}{\sqrt{(1+pp)}} + N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.} = \frac{\left(C - \int Ldx\right)dp}{dx\left(1+pp\right)^{3:2}}.$$

COROLLARY 7

59. Since C shall be an arbitrary quantity, in general it is convenient to note, that, if for C the value of that formula may be taken $\int Ldx$, which it adopts, if there may be put $x = a$, then the curve is going to be produced, which may have the maximum or minimum value of the formula $\int Zdx$, amongst all the curves generally corresponding to the same abscissa $x = a$.

SCHOLIUM 1

60. The case of corollary 6, because it usually has been treated chiefly by authors, deserves special attention, so that with its help problems, which may perhaps occur, shall be able to be resolved more easily and quickly. Therefore amongst all the isoperimetric curves or, which may have the same length $s = \int dx\sqrt{(1+pp)}$, that may be sought, in which $\int Zdx$ shall be a maximum or minimum, with Z being some function both of the defined quantities x, y, p, q etc. as well as the arc of the curve s ; thus so that there shall be

$$dZ = Lds + Mdx + Ndp + Pdp + \text{etc.}$$

This equation now has been found for a curve endowed with this property :

$$\frac{1}{dx} d \cdot \frac{\left(C - \int Ldx\right)p}{\sqrt{(1+pp)}} = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.}$$

which indeed in this widest sense neither can be integrated nor allows itself to be reduced to a simpler form. But it will be of help to note the cases, in which that will be allowed to be integrated. And in the first place indeed, if there shall be $N=0$, this equation is produced at once for the curve :

$$A+\frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}}=-P+\frac{dQ}{dx}-\text{etc.}$$

Now integrated once. In the second place we may put $M=0$; and the equation multiplied by $pdx=dy$ will be changed into this :

$$pd\cdot\frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}}=Ndy-pdP+\frac{pddQ}{dx}-\text{etc.},$$

to which if there is added

$$Lds=Ldx\sqrt{(1+pp)}=\ dZ-Ndy-Pdp-Qdq+\text{etc.},$$

with the integration established the equation will emerge :

$$\int\left(Ldx\sqrt{(1+pp)}+pd\cdot\frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}}\right)=-A+Z-pdP-Qq+\frac{pdQ}{dx}+\text{etc.}$$

Truly the first part, if it may be expanded out, will change into

$$\int\left(Ldx\sqrt{(1+pp)}+\frac{\left(C-\int Ldx\right)pdp}{(1+pp)^{3:2}}-\frac{Lppdx}{\sqrt{(1+pp)}}\right)$$
$$=\int\left(\frac{Ldx}{\sqrt{(1+pp)}}+\frac{\left(C-\int Ldx\right)pdp}{(1+pp)^{3:2}}\right),$$

or which the integral is

$$-\frac{C-\int Ldx}{\sqrt{(1+pp)}}.$$

Whereby, in the case where $M=0$, this equation will be had

$$\frac{C-\int Ldx}{\sqrt{(1+pp)}}=A-Z+Pp+Qq-\frac{pdQ}{dx}.$$

But if in the third place there was both $M=0$ as well as $N=0$, in the first place this equation will be found, on account of $N=0$:

$$A+\frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}}=-Pp+\frac{dQ}{dx};$$

which, multiplied by $dp=qdx$, will be changed into this :

$$Adp+\frac{\left(C-\int Ldx\right)pdp}{\sqrt{(1+pp)}}=-Pdp+qdQ.$$

But since there shall be

$$dZ=Ldx\sqrt{(1+pp)}+Pdp+Qdq,$$

there will be found

$$dZ+Adp-Ldx\sqrt{(1+pp)}+\frac{\left(C-\int Ldx\right)pdp}{\sqrt{(1+pp)}}=qdQ+Qdq;$$

which integrated will give

$$Z+B+Ap+\left(C-\int Ldx\right)\sqrt{(1+pp)}=Qq$$

or

$$C-\int Ldx=\frac{Qq-B-Ap-Z}{\sqrt{(1+pp)}}.$$

But from the former equation there is :

$$C-\int Ldx=-\frac{A\sqrt{(1+pp)}}{p}-\frac{P\sqrt{(1+pp)}}{p}+\frac{dQ\sqrt{(1+pp)}}{pdx};$$

from which joined together there is elicited :

$$Adx-Bdy=Zdy-Pdx-Ppdy+dQ+ppdQ-Qpdp,$$

in the formula of the integral $\int Ldx$ is no longer present. Therefore we will show the use of these cases in examples.

EXAMPLE I

61. *Amongst all the isoperimetric curves to define that, in which* $\int s^n dx$ *shall be a maximum or minimum, with s denoting the arc of the curve corresponding to the abscissa x.*

Because the common property is considered to be the length of the arc $s = \int dx\sqrt{(1+pp)}$ and the arc itself is present in the formula of the maximum or minimum $\int s^n dx$, the solution will belong to the case examined in the scholium. Therefore the formula $\int s^n dx$ is compared with the general $\int Zdx$, there becomes

$$Z = s^n \text{ and } dZ = ns^{n-1}ds$$

and hence

$$L = ns^{n-1},\ M = 0,\ N = 0,\ P = 0 \text{ etc.}$$

Whereby from the scholium in the final case, where we have put $M = 0$ and $N = 0$, this equation will be found

$$Adx - Bdy = Zdy = s^n dy,$$

from which there becomes :

$$Adx = dy\left(B+s^n\right) \text{ and } A^2dx^2 + A^2dy^2 = A^2ds^2 = dy^2\left(A^2 + \left(B+s^n\right)^2\right)$$

and thus

$$dy = \frac{Ads}{\sqrt{A^2 + \left(B+s^n\right)^2}}$$

and

$$dx = \frac{\left(B+s^n\right)ds}{\sqrt{A^2 + \left(B+s^n\right)^2}};$$

from which the construction of the curve can be completed. Or on putting $dy = pdx$, there will be

$$s^n = \frac{A-Bp}{p} \text{ and } s = \sqrt[n]{\frac{A-Bp}{p}}:$$

from which there becomes

$$ds = dx\sqrt{(1+pp)} = -\frac{Adp\left(A-Bp\right)^{(1-n):n}}{np^{(1+n):n}}.$$

And hence through p the coordinates of the curve x and y thus will be determined, so that there shall be :

$$x = -\frac{A}{n}\int \frac{dp\left(A-Bp\right)^{(1-n):n}}{p^{(1+n):n}\sqrt{(1+pp)}} \text{ and } y = -\frac{A}{n}\int \frac{dp\left(A-Bp\right)^{(1-n):n}}{p^{1:n}\sqrt{(1+pp)}}.$$

Here indeed four constants are considered, evidently two new, besides A and B, to be present, on account of the two-fold integration y and x. But since on putting $x = 0$, likewise the arc of the curve $s = \sqrt[n]{\frac{A-Bp}{p}}$ must vanish, hence in turn a constant may be found arising from the integration of x. Without doubt, if n were a positive number, the arc s vanishes on putting $p = \frac{B}{A}$; from which the value of x must be determined thus, so that on putting $p = \frac{B}{A}$ it is made $= 0$.

But if there may be put $n = 1$, from the first construction there will be had at once

$$dx = \frac{\left(B+s\right)ds}{\sqrt{A^2+\left(B+s\right)^2}}$$

and thus

$$x = \sqrt{\left(A^2+B^2+2Bs+ss\right)} - \sqrt{(A^2+B^2)},$$

or on putting $B = b$ and $\sqrt{\left(A^2+B^2\right)} = c$, there will be

$$x + c = \sqrt{\left(\ c^2+2bs+ss\right)}.$$

But from the latter being constructed there arises only

$$x = -\frac{A}{n}\int \frac{dp}{pp\sqrt{(1+pp)}} = \frac{A\sqrt{(1+pp)}}{p} + b \text{ or } (x-b)p = c\sqrt{(1+pp)};$$

and hence

$$p = \frac{c}{\sqrt{((x-b)^2-c^2)}} = \frac{dy}{dx}.$$

Whereby since there shall be

$$y=\int\frac{cdx}{\sqrt{((x-b)^2-c^2)}},$$

the satisfying curve will be a catenary.

EXAMPLE II

62. *Amongst all the curves of the same length to determine that, in which* $\int Sdx$ *shall be a maximum or minimum, with S being some function of the arc s*.

Because the common property is contained by the arc $s=\int dx\sqrt{(1+pp)}$, the solution may be demanded from the scholium. Evidently since there shall be $Z=S=$ to a function of *s*, there will be

$$Lds=dS \text{ and } M=N=P=Q \text{ etc.}=0.$$

Whereby, by the case of the third scholium, this equation will be found for the curve sought

$$Adx-Bdy=Sdy \text{ and } Adx=dy(B+S).$$

Hence therefore there will be

$$A^2dx^2+A^2dy^2=A^2ds^2=dy^2\left(A^2+\left(B+S\right)^2\right)$$

and

$$y=\int\frac{Ads}{\sqrt{\left(A^2+\left(B+S\right)^2\right)}};$$

moreover the abscissa will be

$$x=\int\frac{\left(B+S\right)ds}{\sqrt{\left(A^2+\left(B+S\right)^2\right)}};$$

from which the construction of the curve will be able to be resolved.

We may consider there to be $S=e^s$; and on putting $dy=pdx$ there will be

$$\frac{A-Bp}{p}=e^s \text{ and } e^sds=-\frac{Adp}{pp}=\frac{\left(A-Bp\right)dx\sqrt{(1+pp)}}{p},$$

and hence

$$dx=\frac{Adp}{\left(A-Bp\right)p\sqrt{(1+pp)}} \text{ and } dy=-\frac{Adp}{\left(A-Bp\right)\sqrt{(1+pp)}}.$$

Therefore by putting together there will be

$$dx = \frac{Bdy}{A} = -\frac{dp}{p\sqrt{(1+pp)}}$$

and on integration

$$Ax - By = Al\frac{1+\sqrt{(1+pp)}}{p} + C,$$

or

$$\frac{1+\sqrt{(1+pp)}}{p} = e^{(Ax-By-C):A}.$$

But since x must vanish on making $s = 0$, and on account of $\frac{A-Bp}{p} = e^{s}$, by making $s = 0$ there becomes $p = \frac{A}{B+1}$, being carried out by the integration, so that on making $p = \frac{A}{B+1}$ there is made $x = 0$.

EXAMPLE III

63. *Amongst all the curves of the same length to determine that, in which* $\int sydx$ *shall be a maximum or minimum, with s denoting the length of the curve.*

The solution of this question again is desired from that scholium ; and since there shall be $Z = sy$ and $dZ = yds + sdy$, from which there is made $L = y,\ M = 0$ and $N = s$, the remaining letters P, Q etc. vanish. Therefore since there shall be $M = 0$, the case of the second scholium supplies the solution :

$$\frac{C-\int ydx}{\sqrt{(1+pp)}} = A - ys\,;$$

truly at once there arises

$$sdx = d\cdot\frac{\left(C-\int ydx\right)p}{\sqrt{(1+pp)}} = \frac{\left(C-\int ydx\right)dp}{(1+pp)^{3:2}} - \frac{ypdx}{\sqrt{(1+pp)}}.$$

Whereby, since there shall be

$$C-\int ydx = A\sqrt{(\ 1+pp)} - ys\sqrt{(\ 1+pp)},$$

there will be

$$sdx = \frac{Adp - ysdp - ydy}{1 + pp}$$

or

$$sdx + spdy + ysdp + ydy = Adp.$$

But if it pleased to eliminate the arc s, from these two equations there is found :

$$s = \frac{A}{y} - \frac{\left(C - \int ydx\right)}{y\sqrt{(1+pp)}} = \frac{\left(C - \int ydx\right)dp}{dx(1+pp)^{3:2}} - \frac{yp}{\sqrt{(1+pp)}};$$

and hence

$$\frac{Adx}{y} + \frac{ypdx}{\sqrt{(1+pp)}} = \left(C - \int ydx\right)\left(\frac{dp}{(1+pp)^{3:2}} + \frac{dx}{y\sqrt{(1+pp)}}\right).$$

But in each case it is difficult to adapt the curve required to be constructed pertaining to the equation.

EXAMPLE IV

64. *Among all the curves containing the same area* $\Pi = \int ydx$ *to define that in which* $\int \frac{dx\sqrt{(1+pp)}}{\Pi}$ *shall be a maximum or minimum.*

If we compare this solution with the general, we will have

$$\int [Z]dx = \int ydx;$$

and hence $[Z] = y$ and $[N] = 1$; with the remaining letters $[M]$, $[P]$, $[Q]$ etc. vanishing. Again there will be

$$Z = \frac{\sqrt{(1+pp)}}{\Pi} \text{ and } dZ = -\frac{d\Pi\sqrt{(1+pp)}}{\Pi^2} + \frac{pdp}{\Pi\sqrt{(1+pp)}},$$

from which there will be

$$L = -\frac{\sqrt{(1+pp)}}{\Pi^2},\ M = 0,\ N = 0 \text{ and } P = \frac{p}{\Pi\sqrt{(1+pp)}}.$$

On account of which the following equation emerges for the curve sought :

$$0 = C + \int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \frac{1}{dx} d \cdot \frac{p}{\Pi\sqrt{(1+pp)}}.$$

This equation may be multiplied by $dy = pdx$, there will be

$$0 = Cdy + dy\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - pd \cdot \frac{p}{\Pi\sqrt{(1+pp)}},$$

which integrated will give :

$$0 = B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \int \frac{d\Pi\sqrt{(1+pp)}}{\Pi^2} - \frac{pp}{\Pi\sqrt{(1+pp)}} + \int \frac{pdp}{\Pi\sqrt{(1+pp)}}$$

$$= B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \frac{pp}{\Pi\sqrt{(1+pp)}} + \frac{\sqrt{(1+pp)}}{\Pi}.$$

And thus hence we will obtain that equation

$$0 = B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} + \frac{1}{\Pi\sqrt{(1+pp)}};$$

from which if the former may be multiplied by *y* may be subtracted, there will be

$$0 = B + \frac{1}{\Pi\sqrt{(1+pp)}} + \frac{y}{dx} d \cdot \frac{p}{\Pi\sqrt{(1+pp)}}$$

or

$$0 = Bdx + \frac{dx}{\Pi\sqrt{(1+pp)}} + \frac{ydp}{\Pi(1+pp)^{3:2}} - \frac{y^2 pdx}{\Pi^2\sqrt{(1+pp)}}$$

from which equation if again we may wish to remove $\Pi = \int ydx$, an equation of the third order will emerge, from which much less whatsoever may be able to be deduced towards knowing the curve.

SCHOLIUM 2

65. Although in this proposition we have considered [*Z*] to be a determined function of the quantities *x*, *y*, *p*, *q* etc., yet the method of solving is apparent, if this quantity [*Z*] were an indefinite function included in the integral formulas themselves. For we may consider in the formula $\Pi = \int [Z] dx$, which must be a common property for all the curves, to be present

$$d[Z] = [L] d\pi + [M] dx + [N] dy + [P] dp + [Q] dq + \text{etc.}$$

with $\pi = \int [z] dx$ present and

$$d[z] = [m] dx + [n] dy + [p] dp + [q] dq + \text{etc.}$$

But $\int Zdx$ is required to be a maximum or minimum formula, with

$$dZ = L\Pi + Mdx + Ndy + Pdp + \text{etc.}$$

Now the formula $\int [Z] dx$ is contained in the second case of paragraph 7 of the preceding chapter ; therefore thence, if the value corresponding to the abscissa $x = a$ may be taken in the integral $\int [L] dx$, to which the solution must be adapted, it may be put $= [H]$ and $[H] - \int [L] dx = [V]$, and the value of the differential of the formula $\int [Z] dx$ will be found

$$= nv \cdot dx \left([N] + [n][V] - \frac{d\left([P] + [p][V] \right)}{dx} + \frac{dd\left([Q] + [q][V] \right)}{dx^2} - \text{ etc.} \right).$$

[*i.e.* $[H] = \int_0^a [L] dx$ and $\int_0^a [L] dx - \int_0^x [L] dx = [V]$. See Goldstine p.77 : unless Euler states otherwise, the intervals of the integrations are between 0 and *x*.]

From that truly the formula $\int Zdx$ of the maximum or minimum will be contained in the third case of the place cited; towards finding its differential value the value of the formula $\int Ldx$ may be considered corresponding to the abscissa $x = a$, $= [H]$ and $H - \int Ldx = V$. Now putting the integral

$$\int [L] Vdx = H \int [L] dx - \int [L] dx \int Ldx$$

and $x=a$, the value of the formula shall be $\int[L]dx\int Ldx=K$, but in the same case the value of the formula $\int[L]dx$ is $=[H]$, from which the value of the formula $\int[L]Vdx$ in the case $x=a$ will be $=H[H]-K$, and calling

$$H[H]-K-H\int[L]dx+\int[L]dx\int Ldx=W,$$

thus so that there shall be

$$W=H[V]-K+\int[L]dx\int Ldx,$$

and the differential value of the proposed formula will be $\int Zdx$

$$==nv\cdot dx\left(N+[N]V+[n]W-\frac{d\cdot\left(P+[P]V+[p]W\right)}{dx}+\frac{dd(Q+[Q]V+[q]W}{dx^2}-\text{etc.}\right).$$

But if now to his differential value the preceding one may be added multiplied by an arbitrary constant α and the sum may be put equal to zero $=0$, this equation will be produced for the curve sought :

$$0=N+[N](\alpha+V)+[n](\alpha[V]+W)-\frac{1}{dx}d\cdot\left(P+[P](\alpha+V)+[p](\alpha[V]+W)\right)$$
$$+\frac{1}{dx^2}\cdot dd\left(Q+[Q](\alpha+V)+[q](\alpha[V]+W\right)-\text{etc.}$$

Now truly here there is $\alpha+V=\alpha+H-\int Ldx$; from which if there may be considered $\alpha+H=0$, C will be the arbitrary constant and $\alpha+V=C-\int Ldx$, and

$$\alpha[V]+W=C[H]-K-C\int[L]dx+\int[L]dx\int Ldx.$$

Therefore with this agreed the equation will arrive at the curve sought, in the equation of which, because on account of [*H*] and *K* being constant with *a* given, that sought will be satisfied only for the abscissa proposed $x=a$. So that of both the formulas one may pertain to case 4, the other to case 5, then again consideration of the given abscissa *a* departs from the calculation and the same curve will be satisfied for all the abscissa, that which it will suffice to be shown by the single following example.

EXAMPLE V

66. *Amongst all the curves corresponding to the same abscissa, which maintain the same value of the formula v, to find that, in which* $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$ *shall be a maximum or minimum, with* $dv = gdx + Wdx\sqrt{(1+pp)}$ *being present, and with W some function of* v.

The solution of this question will show the curve amongst all the other curves, upon which a body descending downwards with a uniform gravity g in the direction of the abscissas, will fall the quickest in a medium with some resistance acting, upon which it will acquire the same speed. For $\sqrt{v}$ is the speed of the body at some point of the curve and W expresses the resistance of the medium. Because now in the first place for the common property

$$v = \int dx\left(g + W\sqrt{(1+pp)}\right),$$

we may put $dW = Udv$, and this will belong to the fourth case ; for now there will be

$$\Pi = v \text{ and } Z = g + W\sqrt{(1+pp)}, \text{ also } dZ = Udv\sqrt{(1+pp)} + \frac{Wpdp}{\sqrt{(1+pp)}};$$

from which there will be

$$L = U\sqrt{(1+pp)},\ M = 0,\ N = 0 \text{ and } P = \frac{Wp}{\sqrt{(1+pp)}}.$$

Therefore $\int Udx\sqrt{(1+pp)}$ may be taken for the integral and there shall be, for the case in which there may be put $x = a$,

$$e^{\int Udx\sqrt{(1+pp)}} = H$$

and there may be put

$$V = He^{-\int Udx\sqrt{(1+pp)}}.$$

From these the value of the differential formula v will be

$$= nv \cdot dx\left(-\frac{1}{dx}d \cdot \frac{WVp}{\sqrt{(1+pp)}}\right) = -nv \cdot dx \cdot d \cdot \frac{WVp}{\sqrt{(1+pp)}}.$$

Again the formula of the maximum or minimum $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$ will pertain to the fifth case and there will be

$$Z=\frac{\sqrt{(1+pp)}}{\sqrt{v}} \text{ and } dZ=-\frac{dv\sqrt{(1+pp)}}{2v\sqrt{v}}+\frac{pdp}{\sqrt{v(1+pp)}}$$

and thus

$$\varPi=v \text{ and } L=-\frac{\sqrt{(1+pp)}}{2v\sqrt{v}},\ M=0\ ,\ N=0 \text{ and } P=\frac{p}{\sqrt{v(1+pp)}}.$$

$$Z=\frac{\sqrt{(1+pp)}}{\sqrt{v}} \text{ and } dZ=+\frac{pdp}{\sqrt{v(1+pp)}}$$

Then truly on account of $v=\int dx(g+W\sqrt{(1+pp)})$ there will be

$$[Z]=g+W\sqrt{(1+pp)} \text{ and } d[Z]=Udv\sqrt{(1+pp)}+\frac{Wpdp}{\sqrt{(1+pp)}},$$

from which

$$[\mathrm{L}]=U\sqrt{(1+pp)},\ [M]=0,\ [N]=0 \text{ and } [P]=\frac{Wp}{\sqrt{(1+pp)}}.$$

There may be put, if after integration there becomes $x=a$,

$$-\int e^{\int Udx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}=K,$$

and there shall be

$$e^{-\int Udx\sqrt{(1+pp)}}\left(K+\int e^{\int Udx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}\right)=T\ ;$$

and the differential value of the formula $\int\frac{dx\sqrt{(1+pp)}}{v}$ will be

$$=-nv\cdot dx\left(\frac{1}{dx}d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right)\right)=-nv\cdot d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right).$$

From these two differential values found the following equation for the curve sought arises

$$0=\alpha\cdot d\frac{WVp}{\sqrt{(1+pp)}}+d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right)$$

and on integrating

$$B=\frac{p}{\sqrt{v(1+pp)}}+\frac{Wp(\alpha V+T)}{\sqrt{(1+pp)}}.$$

But there is

$$\alpha V+T=e^{-\int Udx\sqrt{(1+pp)}}\left(\alpha H+K+\int e^{\int Udx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}\right).$$

But if therefore there may be put $\alpha H+K=C$, C will be an arbitrary constant and the magnitude *a* defined generally from the equation will vanish ; and thus the curve sought will possess the desired property for whatever abscissa. Therefore for the curve sought this equation will be found :

$$e^{\int Udx\sqrt{(1+pp)}}\left(\frac{B\sqrt{(1+pp)}}{Wp}-\frac{1}{W\sqrt{v}}\right)=C+\int e^{\int Udx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}$$

and by differentiation,

$$-\frac{Bdp}{Wp^2\sqrt{(1+pp)}}-\frac{BUdv\sqrt{(1+pp)}}{W^2p}+\frac{Udv}{W^2\sqrt{v}}+\frac{dv}{2Wv\sqrt{v}}+\frac{BUdx(1+pp)}{Wp}$$

$$-\frac{Udx\sqrt{(1+pp)}}{W\sqrt{v}}=\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}.$$

But since there shall be $dv=gdx+Wdx\sqrt{(1+pp)}$, we will have this equation with the substitution made

$$\frac{Bdp}{Wp^2\sqrt{(1+pp)}}=\frac{gdx}{2Wv\sqrt{v}}+\frac{gUdx}{W^2\sqrt{v}}-\frac{gBUdv\sqrt{(1+pp)}}{W^2p}$$

or this

$$\frac{2BWdp}{\sqrt{(1+pp)}}=\frac{gWp^2dx}{v\sqrt{v}}+\frac{2gUp^2dx}{\sqrt{v}}-gBUpdx\sqrt{(1+pp)}.$$

This equation may be multiplied by *dv* and in the first term in place of *dv* there may be written
$dv = gdx + Wdx\sqrt{(1+pp)}$, and *dW* in place of *Udv*; with which done this equation will be found :

$$\frac{2gBdp}{W\sqrt{(1+pp)}} + 2Bdp - \frac{gp^2dv}{Wv\sqrt{v}} = \frac{2gppdW}{W^2\sqrt{v}} - \frac{2gBpdW\sqrt{(1+pp)}}{W^2},$$

which divided by p^2 shall become integrable ; and this equation integrated will be

$$2C - \frac{2B}{p} = \frac{2gB\sqrt{(1+pp)}}{Wp} - \frac{2g}{W\sqrt{v}} \quad \text{or} \quad W = \frac{gB\sqrt{v}\sqrt{(1+pp)} - gp}{Cp\sqrt{v} - B\sqrt{v}} = \frac{dv - gdx}{dx\sqrt{(1+pp)}}.$$

From which this equation free from the resistance *W* arises :

$$(Cp - B)dv = gCpdx + gBppdx - \frac{gpdx\sqrt{(1+pp)}}{\sqrt{v}}.$$

But since *W* shall be a given function of *v*, with the aid of the equation

$$W\sqrt{v} = \frac{gB\sqrt{v(1+pp)} - gp}{Cp - B},$$

p will be given by *v* ; which value if it may be substituted into the preceding equation, will give *dx* by *v* and *dv*, and hence the curve sought will be able to be constructed.

PROPOSITION VI. PROBLEM

67. *Amongst all the curves provided with the common property A to determine that, in which some function shall be a maximum or minimum both of this expression A as well as of some other B.*

SOLUTION

The value of the differential of the expression *A* shall be *dA*, and the value of the differential expression *B* shall be *dB*; the value of the differentials of that function of *A* and of *B* themselves will have a form of this kind $\alpha dA + \beta dB$, as it is necessary to be a maximum or minimum, in which the constants α and β shall depend on the ratio of composition, by which the expressions *A* and *B* in that function may be added together among themselves, thus so that the values determined may be obtained depending on the magnitude of the abscissa, to which the solution is required to be adapted. Because truly the value of the differential of the expression *A* which includes the common property is *dA*, of which some multiple γdA may be added to the differential value $\alpha dA + \beta dB$ of the expression, which must be a maximum or minimum, and the sum $(\alpha + \gamma)dA + \beta dB$ put equal to zero will give the equation for the curve sought. Therefore this equation will be found :

$$(\alpha+\gamma)dA+\beta dB=0 \text{ or } (\alpha+\gamma)\delta A+\beta\delta dB=0,$$

in which, even if α and β shall be constant determined quantities, still, on account of the arbitrary constant quantities γ and δ, the coefficients of the values *dA* and *dB*, which are $(\alpha+\gamma)\delta$ and $\beta\delta$, go through constants of arbitrary magnitude. Therefore if the letters ξ and η may be written in place of these, this equation $\xi dA+\eta dB=0$ will be found for the curve sought. On account of which for solving problems of the expressions *A* and *B*, of which one contains the common property, but some function of each must be a maximum or minimum, it is required to take the differential values *dA* and *dB* one by one and these, multiplied by some arbitrary constant quantities, to be put equal to zero, with which understood this equation $\xi dA+\eta dB=0$ will result, which will express the nature of the curve sought. Q. E. I.
[See *e.g.* Goldstine, p.98]

COROLLARY l

68. Therefore the nature of the satisfying curve depends only on the expressions *A* and *B* and not on the ratio of the function of *A* and *B*, which must be a maximum or minimum, remains in the calculation by any way ; for whatever the function shall be, it will produce the same solution.

COROLLARY 2

69. And thus whatever the function of *A* and *B* amongst all the curves endowed with the same property in which *A* must be a maximum or a minimum , a solution will be found likewise, and if amongst all the curves endowed with the same common property *A* that may be required, in which the other expression *B* may obtain a maximum or minimum value.

COROLLARY 3

70. But if therefore the expressions *A* and *B* were formulas of this kind, the differential values of which *dA* and *dB* may not depend on the magnitude of the abscissa *x*, to which they correspond, because it happens, if these formulas may belong to the first or fourth case, following our enumeration made in paragraph 7 of the previous chapter, then the curve will be satisfied equally by any abscissa.

COROLLARY 4

71. The same solution will have a place, if amongst all the curves, of which there shall be a common property, some function of *A* and *B* there may be required, in which some other function of the same *A* and *B* shall be a maximum or minimum. For in this case also it arrives at the equation $\xi dA+\eta dB=0$, in which ξ and η shall be constant quantities to be chosen freely.

EXAMPLE I

72. *Amongst all the curves* (Fig. 20) *aMb with the axis AB containing the same area* $\int ydx$ *to determine that, in which* $\frac{\int yydx}{\int ydx}$ *shall be a minimum*.

This question is entered, if amongst all the equal areas, which can be formed between the extreme coordinates *Aa* and *Bb* and to be formed from the base *AB*, that may be desired, which may have its centre of gravity in the lowest position. For with some curve *aMb* taken and with the position of the abscissa $AP = x$, the applied line $PM = y$, the centre of gravity of the part *aAPM* will be removed from the base *AP* by an interval $= \frac{\int yydx}{2\int ydx}$, which thus will become a minimum, if this expression $\frac{\int yydx}{\int ydx}$ may

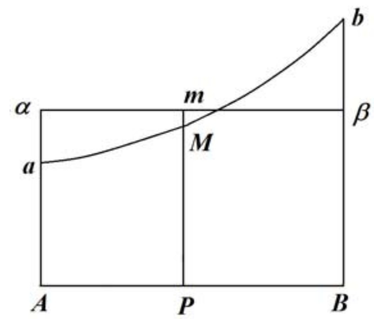

Fig. 20

be returned a minimum. Therefore we have these two formulas $\int ydx$ and $\int yydx$, of which the differential values are $nv\cdot dx\cdot 1$ and $nv\cdot dx\cdot 2y$, from which for the curve sought this equation may be deduced $\xi + 2\eta y = 0$ or $y = c$. Therefore the question is satisfied by the right line $\alpha\beta$ parallel to the base *AB* or horizontal and the right-angled parallelogram $A\alpha\beta B$ before all other figures so that *AabB* will be endowed with this area indicated, so that its centre of gravity may approach nearest to the base *AB*. But if therefore $\alpha AB\beta$ may be a vessel filled with water, if the upper surface of the water $\alpha\beta$ itself will be placed horizontal, then the water will have it centre of gravity placed deeper, than if its upper surface maintained some other position.

EXAMPLE II

73. *Amongst all the curves of the same length DAD* (Fig.14) *to find that, which may have it centre of gravity placed as the deepest, or in which*

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$$

shall be a minimum.

Now it is understood the solution of this question is going to give the curve of a catenary [*i.e.* a hanging chain]; in as much as according to the statics laws a chain suspended from the points *D* and *D* will adopt a figure of this kind, so that its centre of gravity may descend maximally. On account of which among all the figures, which the chain can adopt, which indeed are all of the same length, the catenary curve shall arise, if that may be sought, in which

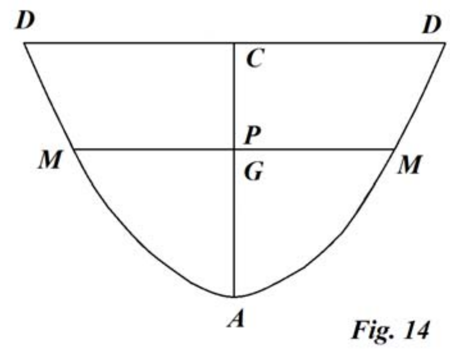

Fig. 14

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$$

shall be a minimum; which certainly gives the distance of the centre of gravity *G* from the start of the abscissas *A*. Therefore since these two formulas $\int dx\sqrt{(1+pp)}$ and $\int xdx\sqrt{(1+pp)}$ may be found, the differential values of these may be sought; which will be

$$\text{the first} = -\,nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}} \quad \text{and the second} = -\,nv\cdot d\cdot\frac{xp}{\sqrt{(1+pp)}}$$

from which this equation arises for the curve sought

$$cd\cdot\frac{p}{\sqrt{(1+pp)}} = d\cdot\frac{xp}{\sqrt{(1+pp)}}$$

and by integration,

$$\frac{xp}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}} + b \quad \text{or} \quad x-c = \frac{b\sqrt{(1+pp)}}{p} \quad \text{and} \quad dx = \frac{-bdp}{pp\sqrt{(1+pp)}}.$$

Hence therefore there becomes :

$$y=\int pdx=-b\int\frac{dp}{p\sqrt{(1+pp)}};$$

from which equations the curve may be constructed and the length of the curve will be :

$$\int dx\sqrt{(1+pp)}=s=\frac{b}{p}+Const.=\frac{b}{p}+f.$$

Hence a different construction for defining x and y by s can be formed; certainly there shall be $p=\frac{b}{s-f}$ and, if the start may be taken at A, where there becomes $p=\infty$, it is required to put $f=0$, thus so that there shall be $p=\frac{b}{s}$; from which there becomes

$$\sqrt{(1+pp)}=\sqrt{(bb+ss)} \text{ and } dx\sqrt{(1+pp)}=ds=\frac{dx\sqrt{(bb+ss)}}{s};$$

and hence

$$dx=\frac{sds}{\sqrt{(bb+ss)}} \text{ and } x=\sqrt{(bb+ss)}-b.$$

Again there will be

$$dy=pdx=\frac{bds}{\sqrt{(bb+ss)}} \text{ and } y=bl\frac{s+\sqrt{(bb+ss)}}{b}.$$

But the equation between the orthogonal coordinates x and y may be deduced from the equation

$$x-c=\frac{b\sqrt{(1+pp)}}{p};$$

which if it may be wished on the axis AP, which is the diameter and for the start of the abscissas selected at A, where there is $pp=\infty$, it will be required to put $c=-b$; and there will be $(x+b)p=b\sqrt{(1+pp)}$ and hence

$$(x+b)^2 pp=bb+bbpp \text{ and } p=\frac{b}{\sqrt{(xx+2bx)}};;$$

and thus $dy=\frac{bdx}{\sqrt{(xx+2bx)}}$, which is the equation for the catenary noted.

EXAMPLE III

74. *Amongst all the curves of the same length to determine that, in which*

$$\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$$

shall be a minimum with S denoting some function of the arc of the curve

$$s=\int dx\sqrt{(1+pp)}.$$

The discovery of a catenary curve may be contained in this example, if the catenary were not everywhere uniformly thick, but its thickness for the corresponding arc *s* is as a function *S* of *s*. For then $\int Sdx\sqrt{(1+pp)}$) expresses the weight of this catenary and $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ the height of the centre of gravity above the start of the abscissas; which is to be a minimum. Indeed from the beginning it may be seen that this case is not contained in the preceding problem, because the formula expressing the arc itself $\int dx\sqrt{(1+pp)}$ is not present in the expression of the maximum or minimum

$\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$, certainly which is a function of two other integral formulas. But since *S* shall be a function of the arc of the curve *s* and $ds=dx\sqrt{(1+pp)}$, there will be $\int Sdx\sqrt{(1+pp)}=\int Sds$, and thus a function of *s*; from which expression $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ will be a function of the formulas $\int dx\sqrt{(1+pp)}$ and $\int Sxd\sqrt{(1+pp)}$, in which that common property is contained. Therefore likewise it is the case, if we must search, amongst all the curves equal in length, for that in which $\int Sxd\sqrt{(1+pp)}$ shall be a minimum. Now since *S* shall be a function of $s=\int dx\sqrt{(1+pp)}$, this question will belong to the previous proposition and to that case, which has been treated in paragraph 60. Evidently there shall be $Z=Sx\sqrt{(1+pp)}$; from which, if we may put $dS=Tds$, there becomes

$$dZ=xTds\sqrt{(1+pp)}+Sdx\sqrt{(1+pp)}+\frac{Sxpdx}{\sqrt{(1+pp)}},$$

thus so that there shall be

$$L = xT\sqrt{(1+pp)},\ M = S\sqrt{(1+pp)},\ N = 0 \text{ and } P = \frac{Sxp}{\sqrt{(1+pp)}}.$$

Now on account of this equation $N = 0$, we will obtain at once this equation cited in the same place

$$A + \frac{\left(C - \int xTdx\sqrt{(1+pp)}\right)p}{\sqrt{(1+pp)}} = -\frac{Sxp}{\sqrt{(1+pp)}}$$

or

$$\frac{A\sqrt{(1+pp)}}{p} + C - \int xTdx\sqrt{(1+pp)} + Sx = 0.$$

But there is $Tdx\sqrt{(1+pp)} = Tds = dS$; from which there is found:

$$\frac{A\sqrt{(1+pp)}}{p} + C + Sx - \int xdS = 0,$$

where A and C are arbitrary quantities. This equation may be differentiated and there becomes

$$\frac{-Adp}{pp\sqrt{(1+pp)}} + Sdx = 0 \text{ or } Sdx\sqrt{(1+pp)} = -\frac{cdp}{pp} = Sds.$$

Whereby, since S shall be a function of s, Sds may be integrated and the integral will be , because it shall be $= R$, the weight of the length of the corresponding catenary of length s. Therefore by integrating it becomes $\frac{c}{p} = R + C$; and, if it may please to take the start of the curve at the place A, where the tangent of the curve is horizontal p, there will be $C = 0$ and $p = \frac{c}{R}$. Hence again there will be

$$\sqrt{(1+pp)} = \frac{\sqrt{(cc+RR)}}{R} = \frac{ds}{dx}\ ;$$

and thus

$$dx = \frac{Rds}{\sqrt{(cc+RR)}} \text{ and } dy = \frac{cds}{\sqrt{(cc+RR)}},$$

from which equations the curve thus will be able to be constructed, so that at once for some length of the catenary both the abscissa as well as the corresponding applied line may be defined. But it is evident in the case, where $R = s$, that this is where the catenary may be put of uniform thickness, and then the curve of the ordinary catenaries appears.

SCHOLIUM

75. Unless the agreements of this chapter should be observed both for this proposition as well as for the preceding, then the solution could be resolved by the general rule, yet truly it would emerge much longer. But so that the use of the general method likewise may be shown more clearly, this same example has been seen to be resolved following the general precepts. Therefore amongst all the curves of the same length $s=\int dx\sqrt{(1+pp)}$ that may be sought, which may have a maximum or minimum value of this expression

$$\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}},$$

with *S* being some function of the arc *S*. And because it is not yet allowed to consider an examination of the given abscissas, on which the value of the differential expression depends $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$, to be coming from the calculation, we may put for this question only to be required to be satisfied for the abscissa given by the length $x=a$. Indeed the differential value of the formula $\int dx\sqrt{(1+pp)}$ containing the common property does not depend on this length, certainly which is always $=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$; but we may put in the case of the maximum or minimum in the expression $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$, because $x=a$, to become

$$\int Sxdx\sqrt{(1+pp)}=A \text{ and } \int Sdx\sqrt{(1+pp)}=B,$$

truly the value of the differential of the numerator $\int Sxdx\sqrt{(1+pp)}$ is $=dA$, and of the denominator $\int Sdx\sqrt{(1+pp)}$ the differential value is $=dB$. Hence therefore the differential value of the maximum or minimum of the expression, which in the case $x=a$ becomes $=\frac{A}{B}$, that will be $-\frac{BdA-AdB}{BB}$, which put equal to some multiple of the common value of the differential formula $=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$ will give the equation for the curve sought. Now for the differential values requiring to be found *dA* and *dB* we will consider the first formula $\int Sxdx\sqrt{(1+pp)}$, which following the enumeration made in paragraph 7 of the preceding chapter concerning the second case ; where there will be $Z = S\sqrt{(1+pp)}$ and on putting $dS = Tds$ there becomes

$$dZ = Tds\sqrt{(1+pp)} + \frac{Spdp}{\sqrt{(1+pp)}}.$$

Therefore with the comparison made, there will be

$$\Pi = s,\ L = T\sqrt{(1+pp)},\ M = 0,\ N = 0 \text{ and } P = \frac{Sp}{\sqrt{(1+pp)}};$$

then truly, on account of $\Pi = s = \int dx\sqrt{(1+pp)}$, there will be

$$[Z] = \sqrt{(1+pp)} \text{ and } [M] = 0,\ [N] = 0 \text{ and } [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Now the integral may be taken $\int Ldx = \int Tdx\sqrt{(1+pp)} = \int Tds = S$, the value of which in the case $x = a$ becomes $= G$, and there will be $V = G - S$.
[i.e. as before, we have $\int_0^x Ldx = \int_0^x Tdx\sqrt{(1+pp)} = \int_0^x Tds = S$ and $G = \int_0^a Ldx$, etc.]
On account of which the differential value of the formula $\int Sxdx\sqrt{(1+pp)}$ will be found

$$dB = -nv\cdot d\left(\frac{Sp}{\sqrt{(1+pp)}} + \frac{p(G-S)}{\sqrt{(1+pp)}}\right) = -nv\cdot d\cdot\frac{Gp}{\sqrt{(1+pp)}}.$$

Again the other formula $\int Sxdx\sqrt{(1+pp)}$ is understood equally in the same second case and there will be

$$Z = Sx\sqrt{(1+pp)} \text{ and } dZ = Txds\sqrt{(1+pp)} + Sdx\sqrt{(1+pp)} + \frac{Sxpdp}{\sqrt{(1+pp)}},$$

from which there becomes

$$\Pi = s,\ L = Tx\sqrt{(1+pp)},\ M = S\sqrt{(1+pp)},\ N = 0 \text{ and } P = Sxp\sqrt{(1+pp)}.$$

Then on account of $\Pi = s = \int dx\sqrt{(1+pp)}$ there will be as before

$$\left[Z\right]=\sqrt{\left(1+pp\right)},\ \left[M\right]=0,\ \left[N\right]=0 \text{ and } \left[P\right]=\frac{p}{\sqrt{\left(1+pp\right)}}.$$

Now the integral may be taken $\int Ldx=\int Txdx\sqrt{\left(1+pp\right)}=\int Txds=\int xds$, the value of which on putting $x=a$ shall be $=H$; there will be $V=H-\int xdS$ and hence the differential value of this formula will come about

$$dA=-nv\cdot d\left(\frac{Sxp+\ p\left(H-\int xdS\right)}{\sqrt{\left(1+pp\right)}}\right)=-nv\cdot d\cdot\frac{p\left(H+\int Sdx\right)}{\sqrt{\left(1+pp\right)}}.$$

Therefore with the values dA and dB found the equation for the curve sought will be

$$\alpha B^{2}d\cdot\frac{p}{\sqrt{\left(1+pp\right)}}-\beta Ad\cdot\frac{Gp}{\sqrt{\left(1+pp\right)}}+\ \beta Bd\cdot\frac{p\left(H+\int Sdx\right)}{\sqrt{\left(1+pp\right)}}=0,$$

and on integrating

$$\frac{\alpha B^{2}p-\beta AGp+\beta BHp+\beta Bp\int Sdx}{\sqrt{\left(1+pp\right)}}=C;$$

in which α,β and C are arbitrary constants and both the constants G and H determined. Because if therefore there may be put $\frac{\alpha B}{\beta}-\frac{AG}{B}+H=b$ and $\frac{C}{\beta B}=c$, b and c will be arbitrary constants, and the determined constants G and H depending on the definition of the abscissa with the value $x=a$ generally vanish from the equation ; thus so that the curve found for some abscissa shall become endowed with the desired property, and the equation of that will be this

$$c=\frac{bp+p\int Sdx}{\sqrt{\left(1+pp\right)}} \text{ or } \frac{c\sqrt{\left(1+pp\right)}}{p}=b+\int Sdx;$$

which differentiated will give

$$Sdx=-\frac{cdp}{pp\sqrt{\left(1+pp\right)}} \text{ or } Sdx\sqrt{\left(1+pp\right)}=Sds=-\frac{cdp}{pp}.$$

There may be put $\int Sds = R$ as above, thus so that *R* may represent the weight of a length *s* of the catenary, there will be $R = \frac{c}{p} + Const.$, which is that equation, which we have elicited in the previous method. And thus from this solution it is understood, how through this general method questions of this kind shall be able to be resolved, if the common property may not be present in the expression of the maximum or minimum ; which so that it may be understood more clearly, it will suffice at this stage to put in place a single example of this kind.

EXAMPLE IV

76. *Amongst all the curves of the same length DAD* (Fig. 14) *corresponding to the given abscissa AC= a to define that, which comprises the area DAD, of which the centre of gravity G shall be the highest or lowest position or in which* $\frac{\int yxdx}{\int ydx}$ *shall be a maximum or minimum.*

Therefore the common property is $\int dx\sqrt{(1+pp)}$, of which the differential value corresponding to any abscissa *x* is $= -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}$. But the value of the maximum or minimum of the expression $\frac{\int yxdx}{\int ydx}$ will depend on prescribed abscissa with the length $x = a$; which so that it may be found in the case, where $x = a$, there becomes $\int yxdx = A$ and the differential value of this formula shall be $= dA$, which by the rules given above is found $= nv \cdot dx \cdot x = nv \cdot xdx$. Again in the same case $x = a$ the other formula $\int ydx$ will be changed into *B* and the differential value of its differential shall be $= dB$, which by the rules given is found $= nv \cdot dx$; thus so that there shall be $dA = nv \cdot xdx$ and $dB = nv \cdot dx$. From these the differential value of the maximum or minimum of the expression $\frac{\int yxdx}{\int ydx}$, which in the case $x = a$ will change into $\frac{A}{B}$, will be

$$= \frac{BdA - AdB}{BB} = \frac{nv(Bxdx - Adx)}{BB},$$

which put equal to a multiple of the value of the differential $-nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}$, which has emerged from the common property, will give this equation for the curve sought

$$\alpha d\cdot\frac{p}{\sqrt{(1+pp)}}=\frac{Bxdx-Adx}{BB}.$$

Let $\frac{A}{B}=h$, h will be the constant quantity determined, which the formula $\frac{\int yxdx}{\int ydx}$ gives, if there ma be put $x=a$, and αB may be put $=cc$, cc will be an arbitrary quantity. Hence this equation will be found for the curve sought

$$ccd\cdot\frac{p}{\sqrt{(1+pp)}}=xdx-hdx,$$

which integrated gives

$$\frac{2ccp}{\sqrt{(1+pp)}}=xx-2hx+bb\,;$$

therefore

$$4c^4pp=(xx-2hx+bb)^2(1+pp)\text{ and } p=\frac{xx-2hx+bb}{\sqrt{(4c^4-(xx-2hx+bb)^2)}}=\frac{dy}{dx}.$$

On account of which there will be

$$y=\int\frac{(xx-2hx+bb)dx}{\sqrt{(4c^4-(xx-2hx+bb)^2)}},$$

where it is possible for the arbitrary constant bb to be taken either as positive or negative. But this curve only satisfies the question in the case, where $x=a$; and so that it may be satisfied, a value of the letter h must be attributed to this, which the expression $\frac{\int yxdx}{\int ydx}$ receives in the case $x=a$, from which the value h will be determined. Moreover it is agreed to note this curve to be that, which commonly is known under the name elastic.

CAPUT V

METHODUS INTER OMNES CURVAS EADEM PROPRIETATE PRAEDITAS INVENIENDI EAM QUAE MAXIMI MINIMIVE PROPRIETATE GAUDEAT

DEFINITIO

1. Proprietas communis est Formula integralis seu expressio indefinita, quae in omnes curvas, ex quibus quaesitam determinari oportet, aequaliter competit.

SCHOLION 1

2. Hactenus Methodum maximorum ac minimorum tradidimus absolutam, in qua perpetuo inter omnes omnino curvas eidem abscissae respondentes una requiri solebat, quae maximi minimive cuiuspiam proprietate gauderet. Nunc autem progredimur ad Methodum relativam, in qua unam lineam maximi minimive proprietate praeditam determinare docebimus, non ex omnibus omnino lineis eidem abscissae respondentibus, verum ex illis, innumerabilibus quidem, lineis curvis tantum, quibus una quaedam proprietas proposita pluresve sint communes. Ac primo quidem in hoc Capite innumerabiles curvas eidem abscissae respondentes contemplabimur, quae unam quandam proprietatem habeant communem; ex hisque unam lineam investigabimus, in qua expressio quaecunque indefinita maximum minimumve obtineat valorem. Hoc in genere inprimis celebre est *Problema lsoperimetricum*, initio huius saeculi publice propositum, in quo inter omnes curvas eiusdem longitudinis, quae quidem eidem abscissae respondeant, eam definiri oportebat, quae contineret maximi minimive cuiuspiam proprietatem. Postmodum autem haec Quaestio in latiori sensu est accepta, ut ista determinatio non solum inter omnes curvas eiusdem longitudinis fieret, verum etiam inter omnes curvas alia quacunque proprietate communi praeditas; quam ipsam quaestionem in hoc Capite pertractare suscepimus. Cum igitur curva sit eligenda non ex omnibus omnino curvis eidem abscissae respondentibus, verum ex iis innumerabilibus

duntaxat, in quas proprietas quaepiam proposita aequaliter competat, hanc ipsam proprietatem ante omnia considerari oportet, quam hic nomine proprietatis communis indicamus. Haec igitur proprietas communis, veluti aequalitas longitudinis curvarum, omnia puncta media afficere debet et hanc ob rem erit functio indefinita, quae non ex unici curvae elementi, verum ex totius curvae positione determinetur. Quamobrem istiusmodi proprietas communis erit vel formula integralis indefinita simplex vel expressio plures eiusmodi formulas integrales complectens. Omnino igitur pari modo erit comparata, quo ipsa maximi minimive formula seu expressio. Eaedem igitur varietates atque divisiones, quas ante circa maximi minimive expressionem fecimus et tractavimus, aeque ad proprietatem communem pertinebunt.

COROLLARIUM 1

3. Si igitur proprietas communis fuerit proposita, quae sit *B*, tum omnes curvae sunt considerandae, quae pro eadem data abscissa eundem valorem ipsius *B* continent, atque ex his ea debet definiri, quae habeat maximum vel minimum.

COROLLARIUM 2

4. In Problematis ergo huc pertinentibus duas res datas esse oportet, proprietatem communem *B* ac maximi minimive expressionem *A*. Quibus datis inter omnes curvas pro data abscissa eundem valorem *B* continentes ea definiri debebit, quae pro eadem abscissa valorem ipsius *A* habeat maximum vel minimum.

COROLLARIUM 3

5. Dantur autem non solum infinitae curvae, quae pro data abscissa eandem proprietatem communem habeant, sed etiam dantur infinitis modis. Assumta enim curva quacunque pro lubitu, ea determinatum habebit valorem proprietatis communis propositae; praeter eam autem dabuntur innumerabiles aliae eundem valorem proprietatis communis pro eadem abscissa continentes.

COROLLARIUM 4

6. Proposita igitur expressione quacunque indefinita innumerabilia infinitarum curvarum dabuntur genera, quorum quodlibet genus infinitas in se complectitur curvas, quae pro eadem data abscissa eundem illius expressionis valorem contineant.

COROLLARIUM 5

7. Cum igitur infinita dentur genera, quorum singula innumerabiles lineas curvas comprehendunt, in quas proposita pro proprietate communi expressio aequaliter competat; in uno quoque genere dabitur una curva, quae pro reliquis eiusdem generis curvis alteram expressionem in maximo minimove gradu contineat.

COROLLARIUM 6

8. Quoniam ergo ex quolibet genere una curva maximi minimive proprietate praedita invenitur, omnino eiusmodi curvae satisfacientes infinitae invenientur, quarum quaevis ita erit comparata, ut inter omnes alias eadem proprietate communi gaudentes maximi minimive proprietate sit praedita.

SCHOLION 2

9. Haec omnia magis illustrabuntur, si proprietatem communem, de qua hactenus in genere sumus locuti, definiamus. Sit igitur proprietas communis formula longitudinem arcus curvae exprimens, maximi minimive expressio autem sit $\int Zdx$; ita ut inter omnes curvas, quae habeant arcus eidem abscissae respondentes inter se aequales, ea debeat determinari, in qua pro eadem abscissa fiat $\int Zdx$ maximum vel minimum. Manifestum autem est non solum infinitas lineas curvas dari pro eadem abscissa longitudine aequales, verum hoc etiam infinitis modis fieri posse. Sit enim abscissa communis = a sumaturque quaecunque longitudo c maior quam a, infinitae exhiberi poterunt lineae tum rectae tum curvae, quarum singularum longitudo sit c; atque inter has definiri poterit una, in qua sit $\int Zdx$ maximum vel minimum. Loco c autem infinitae accipi possunt quantitates, eo quod alia non adest conditio, nisi ut sit $c > a$; atque quilibet valor pro c assumtus dabit unam curvam maximi minimive proprietate praeditam. Quamobrem pro infinitis ipsius c valoribus infinitae reperientur lineae curvae quaestioni satisfacientes. Neque tamen idcirco Quaestio pro indeterminata est habenda, nam solutio ipsa infinitas curvas satisfacientes praebens ita est interpretanda, ut unaquaeque harum curvarum inventarum inter omnes alias aeque Iongas possideat valorem formulae $\int Zdx$ in maximo minimove gradu. Perspicuum autem est, quod hic de aequalibus arcubus curvarum ostendimus, idem de alia quacunque formula seu expressione indeterminata valere debere. Ita si inter omnes curvas, quae pro data abscissa $x = a$ valorem formulae $\int Ydx$ eundem continent, ea requiratur, in qua sit $\int Zdx$ maximum vel minimum, tum infinitae quidem reperientur lineae satisfacientes; verum hae inter se ita discrepabunt, ut quaelibet inter omnes alias possibiles lineas curvas secum valorem formulae $\int Ydx$ communem habentes contineat formulae $\int Zdx$ valorem maximum vel minimum.

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PROPOSITIO I. THEOREMA

10. *Quae curva inter omnes omnino curvas eidem abscissae respondentes maximi minimive cuiuspiam propositi proprietate gaudet, eadem curva simul inter omnes curvas communi quacunque cum ipsa proprietate praeditas eadem maximi minimive proprietate gaudebit.*

DEMONSTRATIO

Sit maximi minimive expressio A, proprietas autem communis B, eritque tam A quam B vel formula integralis indefinita vel expressio ex huiusmodi pluribus formulis composita. Ponamus iam curvam esse inventam, quae inter omnes omnino curvas eidem abscissae respondentes expressionem A contineat maximam vel minimam; ea curva certum quemdam expressionis *B* continebit valorem; praeter eam autem dabuntur innumerabiles aliae, in quas idem expressionis *B* valor competet haeque innumerabiles curvae omnes iam continentur in illis omnibus omnino curvis, ex quibus ea, in qua expressio *A* est maximum minimumve, est inventa. Cum igitur haec curva inter omnes omnino curvas proposita maximi minimive proprietate gaudeat, eadem quoque inter illas infinitas curvas secum expressionem *B* communem habentes valorem expressionis *A* maximum minimumve possidebit. Q. E. D.

COROLLARIUM 1

11. Methodus igitur absoluta etiam Problematis Methodi relativae resolvendis inservit, dum unam semper curvam satisfacientem exhibet. Verum tamen solutionem completam non largitur.

COROLLARIUM 2

12. Curva ergo, quae inter omnes expressionem *A* habet maximam vel minimam, erit una ex infinitis illis curvis, quarum singulae inter omnes alias secum communi proprietate *B* gaudentes eandem expressionem *A* maximam habent minimamve,

COROLLARIUM 3

13. Solutio igitur Problematis, quo inter omnes curvas eadem communi proprietate *B* praeditas ea quaeritur, in qua sit *A* maximum vel minimum, latius patebit, quam si absolute inter omnes curvas ea quaereretur, in qua est *A* maximum vel minimum; illaque solutio hanc tanquam casum specialem in se comprehendet.

PROPOSITIO II. PROBLEMA

14. *Methodum resolvendi Problemata, in quibus inter omnes curvas communi quadam proprietate gaudentes ea requiritur, quae maximi minimive cuiuspiam propositi proprietate gaudeat, in genere adumbrare.*

SOLUTIO

Omne maximum vel minimum ita est comparatum, ut (Fig.15) facta nutatione infinite parva valor eius omnino non immutetur. Quamobrem, si curva *az* inter omnes curvas eidem abscissae *AZ* respondentes, quae quidem communi proprietate *B* gaudeant, habeat valorem expressionis *A* maximum vel minimum, eundem valorem

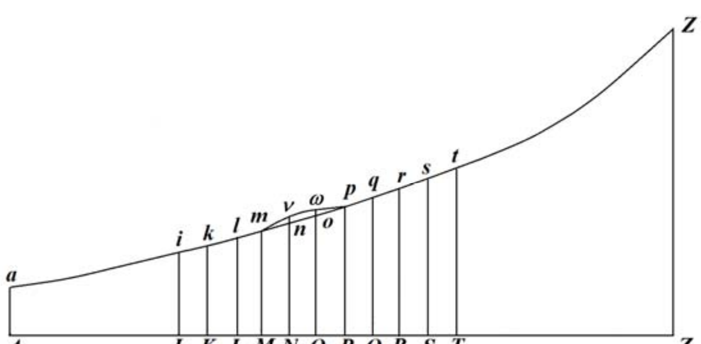

Fig. 15

retinebit, si ipsi talis mutatio infinite parva inferatur, qua communis proprietas *B* non turbetur. Ad hoc autem non sufficit, ut ante fecimus, unicam applicatam, puta *Nn*, particula infinite parva *nν* auxisse; quoniam enim hoc modo tota mutatio unica conditione determinatur, per eam effici nequit, ut tam proprietas communis *B* in ipsam curvam et immutatam aequaliter competat, quam maximi minimive expressio *A* . Quocirca mutationem adhibendam binis conditionibus determinatum esse oportet; id quod obtinebitur, si binae applicatae *Nn* et *Oo* particulis infinite parvis $n\nu$ et $o\omega$ augeantur. Quodsi ergo curva hoc modo immutari concipiatur, primum efficiendum est, ut proprietas communis cum in ipsam curvam tum in mutatam aeque competat; deinde etiam maximi minimive expressio in utraque curva eundem valorem retinere debebit. Prius praestabitur, si expressionis, qua proprietas communis continetur, valor differentialis investigetur oriundus ex translatione binorum *n* et *o* in ν et ω isque evanescens ponatur; posteriori vero conditioni satisfiet, si pari modo valor differentialis expressionis, quae maximum minimumve esse debet, quaeratur oriundus ex binis particulis $n\nu$ et $o\omega$ atque nihilo aequalis ponatur. Hoc pacto duae obtinebuntur aequationes, altera ex proprietate communi, altera ex maximi minimive expressione; utraque autem eiusmodi habebit formam $S\cdot n\nu + T\cdot o\omega = 0$;; in qua *S* et *T* erunt quantitates ad curvam pertinentes. Ex binis autem eiusmodi aequationibus eliminabuntur particulae $n\nu$ et $o\omega$ provenietque aequatio pro curva quaesita, quae inter omnes alias eadem communi proprietate *B* praeditas habeat valorem expressionis *A* maximum vel minimum. Q. E. I.

COROLLARIUM 1

15. Solutio igitur huiusmodi Problematum quoque reducitur ad inventionem valorum differentialium; ipsi autem valores differentiales ab iis, quos ante dedimus, in hoc discrepant, quod ex translatione duorum curvae punctorum definiri debeant.

COROLLARIUM 2

16. Eiusmodi valores differentiales ergo ex duabus particulis $n\nu$ et $o\omega$ oriundos in quovis Prablemate binos investigari oportet, alterum pro proprietate communi, alterum pro maximi minimive expressione.

COROLLARIUM 3

17. Inventis autem in quovis Prablemate his duobus valoribus differentialibus uterque nihilo aequalis poni debet, ex quo binae nascentur aequationes, quae eliminandis particulis assumtis $n\nu$ et $o\omega$ praebebunt unam aequationem naturam curvae quaesitae exprimentem.

COROLLARIUM 4

18. Si ergo inter omnes curvas eidem abscissae respondentes, quae communi proprietate *B* aequaliter sunt praeditae, ea requiratur, in qua expressio *A* fiat maximum vel minimum, tum utriusque expressionis *A* et *B* valores differentiales ex binis particulis $n\nu$ et $o\omega$ oriundi quaeri et nihilo aequales poni debent; ex quibus duabus aequationibus si eliminentur particulae $n\nu$ et $o\omega$, emerget aequatio pro curva quaesita.

COROLLARIUM 5

19. In hac itaque operatione ambae expressiones A et B omnino pariter tractantur neque in considerationem venit, utra vel proprietatem communem vel maximum minimumve denotet. Ex quo perspicuum est eandem solutionem prodire debere, si expressiones A et B inter se commutentur.

COROLLARIUM 6

20. Eadem ergo solutio locum habebit, sive inter omnes curvas communi proprietate B gaudentes ea quaeratur, in qua sit A maximum vel minimum, sive vicissim inter omnes curvas communi proprietate A gaudentes ea quaeratur, in qua sit B maximum vel minimum.

SCHOLION

21. Ambas expressiones A et B, licet in se spectatae res omnino diversas significent, inter se commutabiles esse ipsa solutionis natura sponte patet. Quodsi enim ad binas particulas nv et $o\omega$ respiciamus, quibus applicatae Nn et Oo augentur, primum eas ita comparatas esse oportet, ut proprietas communis B tam in ipsa curva quam in mutata eundem valorem obtineat; scilicet proprietas communis B in curvam *amnopz* et in $amv\omega pz$ aeque competere debet; deinde pari modo per easdem particulas nv et $o\omega$ efficiendum est, ut expressio A, quae maximum minimumve esse debet, tam pro curva *amnopz* quam pro $amv\omega pz$ eundem valorem recipiat. Atque adeo tam proprietas communis quam maximi minimive natura eandem plane conditionem in calculum inducit; ex quo manifestum est ambas expressiones datas, quarum altera proprietatem communem, altera maximi minimive rationem continet, inter se commutari atque confundi posse, salva Solutione. Hane ob rem ergo in Solutione huiusmodi Problematum sufficit nosse ambas illas expressiones; neque ad Solutionem absolvendam nosse opus est, utra proprietatem communem aut maximum minimumve significet. Sic si inter omnes curvas longitudine aequales quaeratur ea, quae maximam aream comprehendat, eadem reperitur curva, quae prodit, si inter omnes curvas aequales areas includentes ea quaeratur, quae sit brevissima vel minimam longitudinem habeat. Haec ita se habent, si maximi minimive, quod quaeritur, natura ita fuerit comparata, ut eius valor differentialis sit $=0$. Iam supra autem animadvertimus duplicis generis dari maxima et minima, in quorum altero valor differentialis sit $=0$, in altero vero $=\infty$. Hic vero tantum maxima ae minima prioris generis contemplamur; nam in hac Methodo relativa posterius genus locum omnino habere nequit. Quodsi enim valor differentialis, qui convenit maximi minimive expressioni, infinite magnus ponatur, tum ex hoc solo aequatio pro curva reperitur neque ideo proprietas communis in computum ingreditur. Quare, si huius generis maximum vel minimum in Methodo absoluta locum habet, eadem curva in Methodo relativa eadem proprietate gaudebit, quaecunque proprietas communis adiungatur. Cum igitur totum Solutionis huiusmodi Problematum momentum versetur in inventione valorum differentialium, qui ex binis particulis nv et $o\omega$ oriuntur, Methodum trademus eiusmodi valores differentials pro quacunque expressione indeterminata inveniendi, eo modo, quo supra usi sumus ad inveniendos valores differentiales ex unica particula nv oriundos.

PROPOSITIO III. PROBLEMA

22. *Proposita* (Fig. 15) *quacunque expressione indeterminata, quae ad datam abscissam AZ referatur, invenire eius valorem differentialem, ortum ex translatione binorum curvae punctorum n et o in* ν *et* ω.

SOLUTIO

Ponamus abscissam $AI = x$ et applicatam $Ii = y$, erit $Kk = y^{/}, Ll = y^{//}$, $Mm = y^{///}$, $Nn = y^{IV}$, $Oo = y^{V}$, $Pp = y^{VI}$ etc. Harum applicatarum duae tantum, nempe y^{IV} et y^{V} patiuntur alterationem a particulis $n\nu$ et $o\omega$ ipsis adiunctis. Erit igitur applicatae y^{IV} valor differentialis $= n\nu$ et applicatae y^{V} valor differentialis $= o\omega$, reliquarum vero applicatarum omnium valor differentialis erit $= 0$. Hinc reliquarum quantitatum ad curvam pertinentium *p, q, r, s* etc. valores differentiales habebuntur, quatenus eae ab his binis applicatis y^{IV} et y^{V} pendent. Sic cum sit $p = \frac{y^{/} - y}{dx}$ erit valor differentialis ipsius $p = 0$ similiterque ipsius $p^{/}$ et $p^{//}$; at cum sit $p^{///} = \frac{y^{IV} - y^{///}}{dx}$, erit ipsius $p^{///}$ valor differentialis $= \frac{n\nu}{dx}$; et ob $p^{IV} = \frac{y^{V} - y^{IV}}{dx}$ erit valor differential ipsius $p^{IV} = \frac{o\omega}{dx} - \frac{n\nu}{dx}$ porroque ipsius p^{V} erit $= -\frac{o\omega}{dx}$. Deinde, cum sit $q = \frac{p^{/} - p}{dx}$, erit valor differentialis ipsius $q^{//} = \frac{n\nu}{dx^2}$, ipsius $q^{///} = \frac{o\omega}{dx^2} - \frac{2n\nu}{dx^2}$, ipsius $q^{IV} = -\frac{o\omega}{dx^2} + \frac{n\nu}{dx^2}$, ipsius $q^{V} = \frac{o\omega}{dx^2}$. Hocque modo similiter progredi licet ad sequentes quantitates *r*, *s* etc. cum suis derivativis; hincque nascetur sequens Tabella, qua singularum harum quantitatum valores differentiales exhibentur:

$d \cdot y^{IV} = n\nu$	$d \cdot q^{//} = \frac{n\nu}{dx^2}$
$d \cdot y^{V} = o\omega$	
$d \cdot p^{///} = \frac{n\nu}{dx}$	$d \cdot q^{///} = -\frac{2n\nu}{dx^2} + \frac{o\omega}{dx^2}$
$d \cdot p^{IV} = -\frac{n\nu}{dx} + \frac{o\omega}{dx}$	$d \cdot q^{IV} = \frac{n\nu}{dx^2} - \frac{2o\omega}{dx^2}$
$d \cdot p^{V} = -\frac{o\omega}{dx}$	$d \cdot q^{V} = \frac{o\omega}{dx^2}$

$d\cdot r^{I} = +\frac{nv}{dx^3}$	$d\cdot s = +\frac{nv}{dx^4}$
$d\cdot r^{II} = -\frac{3nv}{dx^3} + \frac{o\omega}{dx^3}$	$d\cdot s^{I} = -\frac{4nv}{dx^4} + \frac{o\omega}{dx}$
$d\cdot r^{III} = +\frac{3nv}{dx^3} - \frac{3o\omega}{dx^3}$	$d\cdot s^{II} = +\frac{6nv}{dx^4} - \frac{4o\omega}{dx^4}$
$d\cdot r^{IV} = -\frac{nv}{dx^3} + \frac{3o\omega}{dx^3}$	$d\cdot s^{III} = -\frac{4nv}{dx^4} + \frac{6o\omega}{dx^4}$
$d\cdot r^{V} = \qquad -\frac{o\omega}{dx^3}$	$d\cdot s^{IV} = +\frac{nv}{dx^4} - \frac{4o\omega}{dx^4}$
	$d\cdot s^{V} = \qquad +\frac{o\omega}{dx^4}$

etc.

Ex hac Tabella perspicitur in valoribus differentialibus totidem terminus particula $o\omega$ affectos occurrere, ac particula *nv*, atque in utrisque pares adesse coefficientes; discrimen vero in hoc consistere, ut cuilibet termino particula $o\omega$ affecto respondeat quantitas immediate sequens eam, cui respondet similis terminus particula *nv* affectus. Sic, dum terminus $-\frac{2nv}{dx^2}$ reperitur in valore differentiali quantitatis q^{III}, ita terminus $-\frac{2o\omega}{dx^2}$ adest in valore differentiali quantitatis sequentis q^{IV}. Deinceps ob duplicis generis terminus in valoribus differentialibus occurrentes, quorum alteri particulam *nv*, alteri particulam $o\omega$ involvunt, valor differentialis cuiuscunque expressionis indeterminatae huiusmodi habebit formam $nv\cdot I + o\omega\cdot K$; de qua primum manifestum est membrum prius $nv\cdot I$ esse eiusdem expressionis valorem differentialem, qui oritur, si sola particula *nv* consideretur, eritque ideo $nv\cdot I$ ille ipse valor differentialis, quem supra pro quavis expressione oblata definire docuimus, ita ut hoc membrum per praecepta supra tradita pro quavis expressione indeterminata exhibere liceat. Quod ad alterum membrum $o\omega\cdot K$ attinet, quia singuli termini, in quibus $o\omega$ inest, perpetuo respondent quantitatibus sequentibus eas, quibus respondent similes termini particulam *nv* involventes, palam est quantitatem *K* fore valorem, quem quantitas *I* in proximo sequente loco induit, atque idcirco esse $K = I^{I} = I + dI$. Quare cum membrum $nv\cdot I$ ex praeceptis iam supra datis assignare queamus, ex eo porro alterum membrum $o\omega\cdot K = o\omega(I + dI)$ innotescet. Sit igitur *V* expressio quaecunque indeterminata, cuius valorem differentialem ex duabus particulis *nv* et $o\omega$ oriundum definiri oporteat. Ponamus eius valorem differentialem ex unica particula *nv* oriundum esse $= nv\cdot I$, eritque valor differentialis, qui ex ambabus particulis *nv* et $o\omega$ oritur, $= nv\cdot I + o\omega\cdot I^{I}$ seu $nv\cdot I + o\omega\cdot(I + dI)$; qui igitur ope regularum supra datarum facile assignari poterit. Q. E. I.

COROLLARIUM 1

23. Omnium ergo expressionum, quarum valores differentiales ex unica particula *nv* oriundos invenire docuimus, earundem valores differentiales ex binis particulis *nv* et $o\omega$ oriundos nunc definire possumus.

COROLLARIUM 2

24. Haec igitur Methodus valebit tam ad expressionum valores differentiales inveniendos, qui non pendent a quantitate abscissae propositae *AZ*, quam qui ab istius abscissae longitudine pendent.

COROLLARIUM 3

25. Quin etiam, si expressio proposita, quae vel proprietatem communem continet vel maximum minimumve esse debet, fuerit functio duarum pluriumve formularum integralium, eius valor differentialis ex binis particulis *nv* et *ow* oriundus eadem lege definietur.

SCHOLION

26. In Capitibus superioribus vidimus valorem differentialem cuiuscunque expressionis, qui ex unica particula *nv* oritur, huiusmodi habere formam $nv\cdot dx\cdot T$ seu $nv\cdot Tdx$, ubi *T* denotat quantitatem finitam; quare eiusdem expressionis valor differentialis ex binis particulis *nv* et $o\omega$ ortus erit $= nv\cdot Tdx + o\omega\cdot T^{/}dx$, quemadmodum in Solutione ostendimus. Eadem autem forma facile ad hunc modum potest evinci: Scilicet si ponatur $o\omega = 0$, tum prodire debet ipse valor differentialis ex unica particula *nv* ortus, quem supra invenire docuimus, eritque $nv\cdot Tdx$. Sin autem ponatur $nv = 0$ ac sola particula $o\omega$ consideratur, valor differentialis simili modo reperietur, quo supra usi sumus; non autem erit $= o\omega\cdot Tdx$; nam quia particula $o\omega$ in situ sequente accipitur, loco *T* eius valor sequens pariter sumi debet, ita ut valor differentialis verus futurus sit $= o\omega\cdot T^{/}dx$. Quodsi ergo utraque particula *nv* et $o\omega$ coniunctim consideretur, erit valor differentialis $= nv\cdot Tdx + o\omega\cdot T^{/}dx$, eo quod in ipso calculo particulae *nv* et $o\omega$ nusquam inter se permiscentur, sed utraque perpetuo seorsim tractari possit. Ut autem haec ad notandi modum in superiori capite receptum accommodemus, ponamus *V* esse expressionem quamcunque indeterminatam, quae pro abscissa definita $AZ = a$ valorem recipiat $= A$, eiusque valorem differentialem ex particula *nv* ortum esse $= nv\cdot dA$, ubi *dA* nobis denotet idem, quod ante *Tdx;* poteritque iste valor *dA* ex expressione *V* modo in Capitibus praecedentibus exposito inveniri. Hoc invento erit eiusdem expressionis *V* valor differentialis ex binis particulis *nv* et $o\omega$ oriundus $= nv\cdot dA + ow\cdot dA^{/}$, ubi $dA^{/}$ denotat valorem *dA* suo differentiali auctum. Quanquam autem ista valorum differentialium ex binis particulis oriundorum distinctio ad nostrum institutum omnino est necessaria, tamen solutio ipsa Problematum huc pertinentium eo iterum reducetur, ut per solos valores differentiales modo supra exposito inventos, qui scilicet ex unica particula *nv* nascuntur, absolvi queat; id quod ex sequente Propositione mox patebit.

PROPOSITIO IV. PROBLEMA

27. *Inter omnes curvas ad eandem datam abscissam $AZ = a$ relatas, in quas idem valor expressionis indefinitae W competit, determinare eam, in qua sit expressio V maximum vel minimum.*

SOLUTIO

Ponamus curvam *az* quaesito satisfacere atque expressionem *W* in ea obtinere valorem determinatum $= B$; erit ergo haec curva *az* inter omnes alias curvas ad eandem abscissam *AZ* relatas, in quibus expressio *W* eundem obtinet valorem, ita comparata, ut in ea expressio *V* maximum minimumve valorem recipiat, qui sit $= A$. Ad curvam ergo hanc inveniendam positis abscissa indefinita $AI = x$ et applicata respondente $Ii = y$, binae applicatae *Nn* et *Oo* particulis infinite parvis *nv* et $o\omega$ augeri concipiantur: quo facto tam ipsius *W* quam ipsius *V* valor differentialis, qui ex his duabus particulis *nv* et $o\omega$ adiunctis nascetur, nihilo aequalis poni debebit, uti in Propositione secunda ostendimus. Sit iam expressionis *V* valor differentialis ex unica particula *nv* ortus $= nv \cdot dA$ atque expressionis alterius *W* valor differentialis ex eadem unica particula *nv* ortus $= nv \cdot dB$, quos valores differentiales ex praeceptis in superioribus Capitibus datis invenire licebit. Nunc igitur, dum binas particulas *nv* et $o\omega$ contemplamur, erit expressionis *V* valor differentialis $= nv \cdot dA + o\omega \cdot dA'$, alterius vero expressionis *W* valor differentialis erit $= nv \cdot dB + o\omega \cdot dB'$. Quocirca ad quaesitam curvam inveniendam fieri oportet enim $nv \cdot dA + o\omega \cdot dA' = 0$, tum etiam $nv \cdot dB + o\omega \cdot dB' = 0$. Multiplicentur ambae aequationes per quantitates quascunque, ita ut prodeat

$$nv \cdot \alpha dA + o\omega \cdot \alpha dA' = 0,$$
$$nv \cdot \beta dB + o\omega \cdot \beta dB' = 0 .$$

Fiatque ad particulas *nv* et $o\omega$ eliminandas tam $\alpha dA + \beta dB = 0$ quam $\alpha dA' + \beta dB' = 0$; eruntque α et β eiusmodi quantitates sive constantes sive variabiles, quae utrique aequationi satisfaciunt. Quoniam vero est $\alpha dA + \beta dB = 0$, erit quoque $\alpha' dA' + \beta' dB' = 0$; quae aequatio cum $\alpha dA' + \beta dB' = 0$ comparata monstrat esse debere $\alpha' = \alpha$ et $\beta' = \beta$; ex quo quantitates hae α et β debebunt esse constantes et quidem quaecunque. Sumtis itaque pro α et β quantitatibus quibuscunque constantibus, aequatio pro curva erit $\alpha dA + \beta dB = 0$. Haec eadem aequatio prodit, si methodo consueta particulas *nv* et $o\omega$ eliminemus. Erit nempe

$$\frac{nv}{o\omega} = -\frac{dA'}{dA} = -\frac{dB'}{dB}$$

ideoque

$$\frac{dA'}{dA} = \frac{dB'}{dB} \text{ seu } \frac{ddA}{dA} = \frac{ddB}{dB}$$

et $dA' = dA + ddA$ et $dB' = dB + ddB$. Aequatio autem $\frac{ddA}{dA} = \frac{ddB}{dB}$

integrata dat $ldA = ldB + LC$ seu $dA = CdB$; quae, posito $C = -\frac{\beta}{\alpha}$, transit in $\alpha dA + \beta dB = 0$, quam ipsam ante invenimus. Quamobrem ad Problema resolvendum oportet tam expressionis proprietatem communem continentis *W*, quam expressionis, quae maximum minimumve esse debet, *V* valores differentiales methodo in superioribus Capitibus tradita investigare eosque per quantitates constantes quascunque multiplicare summamque $= 0$ ponere; quo facto resultabit aequatio naturam curvae quaesitae exprimens. Q. E. I.

COROLLARIUM I

28. Nunc igitur ad Quaestiones in hac Propositione contentas resolvendas sufficit nosse valores differentiales ex unica particula *nv* oriundos; quos supra iam expedite invenire docuimus.

COROLLARIUM 2

29. Quare ad hoc negotium in subsidium vocari debebit Caput praecedens IV, ex eoque cum paragraphus 7 tum paragraphus 31. In loco priore enim continentur praecepta valores differentiales inveniendi, si expressiones indeterminatae propositae fuerint formulae integrales singularis, in altero vero, si sint functiones duarum pluriumve eiusmodi formularum integralium.

COROLLARIUM 3

30. Proposita ergo proprietate communi *W* et maximum minimumve expressione *V*, utriusque expressionis valorem differentialem ex his praeceptis quaeri oportet; quibus inventis et per constantes arbitrarias mulplicatis eorum aggregatum nihilo aequale positum dabit aequationem pro curva quaesita.

COROLLARIUM 4

31. Si inter omnes omnino curvas eidem abscissae *AZ* respondentes quaeratur ea, in qua expressio *V* maximum minimumve obtineat valorem, pro ea habetur ista aequatio $dA = 0$, denotante *dA* valorem differentialem expressionis *V*.

COROLLARIUM 5

32. Quodsi autem inter omnes curvas eidem abscissae *AZ* respondentes, in quas expressio *W* aequaliter competat, quaeratur ea, in qua expressio *V* maximum minimumve habeat valorem, invenitur pro ea ista aequatio
$\alpha dA + \beta dB = 0$.

COROLLARIUM 6

33. Perspicuum ergo est curvam, quae inter omnes omnino curvas habeat *V* maximum vel minimum, cuius aequatio est $dA = 0$, contineri in aequatione $\alpha dA + \beta dB = 0$, qua

exprimitur curva, quae inter omnes eadem communi proprietate *W* gaudentes habeat *V* maximum vel minimum.

COROLLARIUM 7

34. In ipsa igitur prima aequatione, quam solutio praebet, $\alpha dA + \beta dB = 0$ iam inest una constans arbitraria, quae autem per id determinari debet, ut valor expressionis *W* datum obtineat valorem.

COROLLARIUM 8

35. Problema itaque Sic solvi poterit, ut inter omnes curvas eidem abscissae *AZ* respondentes, in quibus expressio *W* eundem datum obtineat valorem, definiatur ea, in qua sit valor ipsius *V* maximus vel minimus.

COROLLARIUM 9

36. Ex his denique intelligitur Solutionem Problematis propositi convenire cum Solutione huius Problematis, quo inter omnes omnino curvas eidem abscissae *AZ* respondentes requiratur ea, quae habeat $\alpha V + \beta W$ maximum vel minimum. Quae quaestio, etsi ad Methodum absolutam pertineat, tamen dat aequationem $\alpha dA + \beta dB = 0$, quam ipsam invenimus.

SCHOLION 1

37. Ex his igitur non solum Methodus facilis atque expedita colligitur omnes Quaestiones huc pertinentes resolvendi, verum etiam natura huius generis Problematum penitius cognoscitur. Primo enim apparet, quod iam supra demonstravimus, Solutionem eandem fore, sive inter omnes curvas communi proprietate *W* praeditas quaeratur ea, quae habeat *V* maximum vel minimum, sive inverse inter omnes curvas communi proprietate *V* praeditas ea requiratur, in qua sit *W* maximum vel minimum. Deinde etiam intelligitur Quaestionem ita proponi posse, ut eius Solutio ad Methodum maximorum ac minimorum absolutam pertineat; congruit enim Problema propositum cum hoc, quo inter omnes omnino curvas ad eandem abscissam *AZ* relatas requiritur ea, in qua sit ista expressio $\alpha V + \beta W$ maximum vel minimum; atque haec Problematis transformatio in causa est, quod Solutio per valores differentiales ex unica particula *nv* oriundos perfici queat neque amplius opus sit duas huiusmodi particulas considerare, prouti primo intuitu natura Quaestionis postulare videbatur. Hanc autem convenientiam postmodum per se ac sine ista Methodo, qua binae particulae considerantur, demonstrabimus; quo veritas haec, summi in isto negotio momenti, magis confirmetur. Ad solvendas caeterum huiusmodi Quaestiones ante oculos habere oportet praecepta Capite praecedente in compendium redacta; quorum ope valores differentiales quarumcunque expressionum inveniri poterunt. Primo enim, paragrapho 7 illius Capitis recensentur Casus, quibus formularum integralium solitariarum valores exhibentur, tum vero paragrapho 31 traditur Methodus inveniendi valores differentiales expressionum, quae ex duabus pluribusve formulis integralibus utcunque sint compositae. Ex his itaque subsidiis pro quavis Quaestione oblata tam maximi minimive expressionis quam proprietatis communis valor differentialis assignari poterit; utroque autem invento aequatio pro Curva quaesita nullo negotio formabitur, cum tantum opus sit aggregatum quorumcunque multiplorum illorum binorum valorum differentialium nihilo aequale poni. Haecque aequatio inventa deinceps

pari modo erit tractanda, quo supra cum in reductione ad construendum tum in integratione usi sumus.

SCHOLION 2

38. Iam observavimus in aequatione $\alpha dA + \beta dB = 0$, quam Solutio immediate suppeditat, unam inesse quantitatem constantem, quae autem non omnino sit arbitraria, sed ex conditione proposita debeat determinari. Scilicet, cum in omnes curvas, ex quibus quaesitam definiri oportet, eadem expressio *W* aequaliter competere debeat seu in omnibus eundem valorem, puta *B*, obtinere, haec quantitas *B* tanquam data spectari potest; atque cum ipsa in calculum non ingrediatur, ita constantes α et β definire licebit, ut valor expressionis *W*, abscissae $AZ = a$ respondens, ipsi *B* aequalis fiat; hocque pacto Quaestio alioquin indeterminata determinabitur. Eatenus autem tantum determinabitur, quatenus per integrationes post instituendas novae constantes arbitrariae etiam per totidem puncta definiuntur. Prorsus nimirum, ut ante, totidem puncta praescribi poterunt, per quae curva quaesita transeat, quot novae constantes per integrationes ingredi censendae sunt. Horum autem numerus innotescet ex gradu differentialium summo, qui in aequatione inerit. Quoniam vero tota Quaestio ad Methodum absolutam revocari potest, numerus istiusmodi constantium perpetuo erit par; seu aequatio resultans $\alpha dA + \beta dB = 0$ erit vel finita vel differentialis secundi gradus vel differentialis quarti gradus vel differentialis sexti gradus vel octavi vel ita porro. Quodsi aequatio prodit finita, tum quoque curva penitus iam erit determinata, siquidem ratio inter α et β ita definiatur, ut expressio *W* datum recipiat valorem *B* in curva inventa, quam determinationem perpetuo adhiberi ponimus. Si aequatio inveniatur differentialis secundi gradus, tum duobus punctis curva inventa determinabitur; congruum autem ac more receptum est ipsos curvae terminus *a* et *z* praescribi, hisque casibus Problema determinabitur, si conditio ista adiungatur, ut curva quaesita intra datos terminus *a* et *z* contineatur. Sin autem aequatio prodeat differentialis quarti gradus, tum quatuor punctis pro lubitu assignatis, curva satisfaciens determinabitur; haec igitur definiri ita conveniet, ut praeter terminus extremos *a* et *z* simul positio tangentium in his terminis praescribatur. Sin perveniatur ad aequationem differentialem sexti gradus, tum curva per sex quaecunque puncta determinabitur; eorum autem loco praescribi poterunt primo ambo termini *a* et *z*, tum positio tangentium in his terminis, ac tertio curvedo in his ipsis locis seu radii osculi quantitas. His igitur notatis intelligetur ex ipsa Solutione, cuiusmodi conditio ad Problematis cuiusque propositionem adiungi debeat, ut id fiat penitus determinatum; haecque admonitio non solum hic, sed etiam in Methodo absoluta atque reliqua Methodo relativa locum habet.

SCHOLION 3

39. Discrimen hic etiam maximi momenti inprimis est notandum, ex quo in Methodo absoluta primariam tractationis partitionem desumsimus. Consistit id autem in modo, quo curva inventa Quaestioni satisfacit. Fieri enim potest, ut eius quaecunque portio ad abscissam indefinitam relata requisita proprietate gaudeat; deinde etiam dantur casus, quibus nonnisi ea portio, quae definitae abscissae $AZ = a$ respondet, conditioni Problematis satisfaciat. Illud scilicet evenit, si quantitas haec *a* in aequationem, quam

Solutio suppeditat, vel omnino non ingreditur vel in quantitates arbitrarias α et β comprehendi queat. Ex quo manifestum est, si ambae formulae *W* et *V* in casu primo paragraphi 7 Capitis praecedentis recensito contineantur, tum curvae inventae quamlibet portionem ad Quaestionem esse acommodatam. Deinde vero etiam fieri potest, ut licet quantitas *a* seu quantitates ab ea pendentes vel in alterutro valore differentiali insint vel in utroque; tamen eae vel se mutuo tollant in aequatione $\alpha dA + \beta dB = 0$ vel sub arbitrariis α et β comprehendi queant; quo casu pariter quamvis curvae inventae portionem satisfacere oportet. Hoc autem tantum locum habet, si non datus ac determinatus praescribatur valor, quem proprietas communis *W* in portione satisfaciente obtinere debeat; tum enim fieri nequit, ut in quavis portione eundem valorem sortiatur. Ex Solutione autem unius cuiusque Quaestionis facile intelligetur, qua conditione sive tota curva *az* sive quaevis portio satisfacere queat; id quod commodissime in Exemplis ostendi poterit.

EXEMPLUM I

40. *Inter omnes curvas ad abscissam AZ relatas, in quibus formula* $\int yxdx$ *eundem obtinet valorem, invenire eam, in qua sit valor formulae* $\int yydx$ *minimus.*

Erit igitur proprietas communis $W = \int xydx$, cuius, ob $dxy = ydx + xdy$, valor differentialis est $= nv \cdot dx \cdot x$. Maximi autem minimive formula est $V = \int yydx$, cuius, ob $d \cdot yy = 2ydy$, valor differentialis est $= nv \cdot dx \cdot 2y$. Obtinebitur ergo, divisione per $nv \cdot dx$ instituta, haec aequatio $\alpha x + 2\beta y = 0$; ex qua patet Quaestioni satisfacere lineam rectam in *A* cum axe *AZ* angulum quemcunque constituentem. Et quia longitudo abscissae $AZ = a$ non in computum ingreditur, quaevis huius rectae portio aeque satisfaciet. Quodsi autem postuletur, ut pro data abscissa $AZ = a$ formula $\int yxdx$ datum obtineat valorem, puta *B*, tum ob $y = mx$ fiet $\int yxdx = \frac{1}{3}mx^3$ ideoque $\frac{1}{3}mx^3 = B$ *;* ex quo positio lineae rectae ita definietur, ut esse debeat $y = \dfrac{3Bx}{a^3}$. Haec igitur recta iam ista proprietate gaudebit, ut ea inter omnes lineas sive rectas sive curvas, quae pro data abscissa $AZ = a$ habeant formulae $\int yxdx$ valorem $= B$, producat formulae $\int yydx$ minimum valorem.

EXEMPLUM II

41. *Inter omnes curvas eiusdem longitudinis puncta a et z iungentes invenire eam, quae maximam vel minimam aream aAZz* (Fig.l6) *comprehendat.*

Quoniam proprietas communis est longitudo arcus $= \int dx\sqrt{(1+pp)}$, erit eius valor differentialis $-nv \cdot d \cdot \dfrac{p}{\sqrt{(1+pp)}}$. Deinde maximi minimive formula est $\int ydx$, cuius valor differentialis est $nv \cdot dx$; unde pro curva quaesita ista habebitur aequatio

$dx = bd \cdot \frac{p}{\sqrt{(1+pp)}}$ et integrando

$$x + c = \frac{bp}{\sqrt{(1+pp)}} \text{ ideoque } p = \frac{x+c}{\sqrt{(b^2-(x+c)^2)}} = \frac{dy}{dx}.$$

Hinc ergo integrando fit $y = f \pm \sqrt{\left(b^2 - \left(x+c\right)^2\right)}$ seu $b^2 = \left(y-f\right)^2 + \left(x+c\right)^2$,

quae est aequatio generalis pro Circulo. Quamobrem arcus Circuli quicunque per puncta *a* et *z* ductus inter omnes alias lineas curvas eiusdem longitudinis vel maximam vel minimam aream *aAZz* includet. Duplici autem modo Circuli arcus datae longitudinis intra terminus *a* et *z* constitui potest; altero, quo concavitatem axi *AZ* obvertit, altero, quo convexitatem. Priori casu manifestum est aream fore maximam, posteriore vero minimam. Atque hinc, si dentur termini *a* et *z* una cum longitudine curvae intra hos terminus constitutae, quam maiorem quidem esse oportet lineam rectam hos terminus iungentem, Solutio penitus erit determinata; arcus Circuli enim huius longitudinis per hos terminus poterit describi unicus, qui, prout vel concavitatem vel convexitatem axi *AZ* obvertat, aream formabit vel maximam vel minimam.

COROLLARIUM

42. Hinc etiam patet arcum circularem *az* (Fig. 16), per terminus *a* et *z* ductum, non solum maximam aream *aAZz* inter omnes alias lineas eiusdem longitudinis formare, sed etiam, quaecunque linea *aCEDza* termino *a* ad terminum *z* ducta detur, cum ea arcus circularis *az* maximam includet aream. Nam si area *aAZz* est maxima, erit quoque area

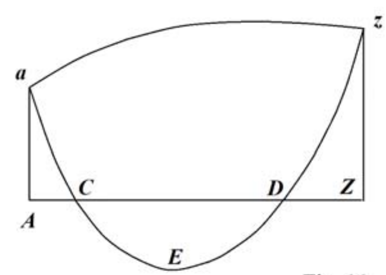

Fig. 16

$$aAZz - aAC - zZD + CED$$

ob areas *aAC, zZD* et *CED* constantis magnitudinis, quaecunque linea pro *az* capiatur, maxima.

EXEMPLUM III

43. *Inter omnes curvas* (Fig. 7) *eiusdem longitudinis puncta A et M iungentes invenire eam, quae cum rectis AC et MC ad punctum fixum C ductis maximam vel minimam comprehendat aream ACM.*

Quoniam, ob data puncta *A, C, M,* rectae *AC* et *MC* positione dantur, ponatur angulus $ACM = x$, seu descripto, centro *C*, radio $CB = 1$, arcu circulari *BS,* sit hic arcus $BS = x$ et ponatur $CM = y$; erit $Ss = dx$, $Mn = ydx$ et area $ACM = \frac{1}{2}\int yydx$. Porro ob $mn = dy$ erit

$$Mm = \sqrt{(y^2dx^2 + dy^2)} = dx\sqrt{(yy + pp)}$$

posito $dy = pdx$. Quare inter omnes aequationes relationem ipsarum *x* et *y* continentes, quae pro dato ipsius *x* valore eandem praebent quantitatem $\int dx\sqrt{(yy + pp)}$, eam definiri oportet, quae pro eodem ipsius *x* valore praebeat formulae $\frac{1}{2}\int yydx$ quantitatem vel maximam vel minimam. Cum igitur formulae $\int dx\sqrt{(yy + pp)}$ valor differentialis sit

$$= nv \cdot dx\left(\frac{y}{\sqrt{(yy + pp)}} - \frac{1}{dx}d \cdot \frac{p}{\sqrt{(yy + pp)}}\right)$$

et formulae $\frac{1}{2}\int yydx$ valor differentialis $= nv \cdot dx \cdot y$ habebitur pro curva quaesita ista aequatio

$$ydx = \frac{by}{\sqrt{(yy + pp)}} - bd \cdot \frac{p}{\sqrt{(yy + pp)}},$$

quae per *p* multiplicata abit in hanc

$$ydy = \frac{bydy}{\sqrt{(yy + pp)}} - bpd \cdot \frac{p}{\sqrt{(yy + pp)}}$$

$$= bd \cdot \sqrt{(yy + pp)} - \frac{bpdp}{\sqrt{(yy + pp)}} - bpd \cdot \frac{p}{\sqrt{(yy + pp)}};$$

cuius integrale est

$$\tfrac{1}{2}yy = b\sqrt{(yy + pp)} - \frac{bpp}{\sqrt{(yy + pp)}} + bc = \frac{byy}{\sqrt{(yy + pp)}} + bc.$$

Ducatur in tangentem *MP* ex *C* perpendiculum $CP = u$; erit

$$u = \frac{yy}{\sqrt{(yy + pp)}}$$

habebiturque $yy = 2bu + bc$; quam aequationem supra iam ostendimus esse ad Circulum. Quamobrem arcus Circuli per terminus *A* et *M* ductus hanc habebit proprietatem, ut inter omnes alias curvas eiusdem secum longitudinis terminus *A* et *M* iungentes aream *ACM* exhibeat vel maximam vel minimam, prout ille arcus vel concavitatem vel convexitatem intra angulum *ACM* vertat. Quo ipso id confirmatur, quod paragrapho praecedente in genere adnotavimus.

EXEMPLUM IV

44. *Inter omnes curvas* (Fig. 15) *puncta a et z iungentes, quae circa axem AZ rotatae generant solida eiusdem superficiei, determinare eam, quae simul producat volumen solidi hoc modo generati maximum.*

Superficies solidi hoc modo generati proportionalis invenitur formulae integrali huic $\int ydx\sqrt{(1+pp)}$, cuius valor differentialis est

$$nv\cdot dx\left(\sqrt{(1+pp)}-\frac{1}{dx}d\cdot\frac{yp}{\sqrt{(1+pp)}}\right).$$

Volumen vero solidi hoc modo generati est ut $\int yydx$, cuius valor differentialis est $=nv\cdot dx\cdot 2y$. Quocirca resultabit ista aequatio

$$2ydx=bdx\sqrt{(1+pp)}-bd\cdot\frac{yp}{\sqrt{(1+pp)}}.$$

Multiplicetur haec per *p*, ut prodeat

$$2ydy=bdy\sqrt{(1+pp)}-bpd\cdot\frac{yp}{\sqrt{(1+pp)}}$$

$$=bd\cdot y\sqrt{(1+pp)}-\frac{bypdp}{\sqrt{(1+pp)}}-bpd\cdot\frac{yp}{\sqrt{(1+pp)}},$$

cuius integrale est

$$yy=by\sqrt{(1+pp)}-\frac{bypp}{\sqrt{(1+pp)}}-bc=\frac{bp}{\sqrt{(1+pp)}}+bc.$$

Erit ergo

$$by=\left(yy-bc\right)\sqrt{(1+pp)}\ \text{ et }\ p=\frac{\sqrt{(b^2y^2-(yy-bc)^2)}}{yy-bc}=\frac{dy}{dx}.$$

Quare erit

$$dx=\frac{(yy-bc)dy}{\sqrt{(bbyy-(yy-bc)^2)}}.$$

De hac aequatione primo notandum est, si fuerit $c=0$, fore

$$dx = \frac{ydy}{\sqrt{(bb-yy)}}$$

ideoque curvam esse Circulum, cuius centrum in axe *AZ* sit positum; ille igitur arcus circularis centro in axe *AN* sumpto descriptus et per data duo puncta *a* et *z* transiens Quaestioni satisfaciet; erit autem is unicus, ideoque solidum definitae superficiei generabit. Quare si inter omnes curvas, quae solida alius atque diversae superficiei generant, quaeratur ea, quae maximum volumen producat, ea non erit Circulus, sed alia curva in aequatione

$$dx = \frac{(yy-bc)dy}{\sqrt{(bbyy-(yy-bc)^2)}}.$$

contenta. Non solum enim, ob binas constantes *b* et *c*, effici potest, ut curva per praescripta duo puncta *a* et *z* transeat, sed etiam, ut longitudo curvae *az* existat datae magnitudinis. Caeterum longitudo curvae, ob

$$\int dx\sqrt{(1+pp)} = \int \frac{bydx}{yy-bc}$$

fiet

$$= \int \frac{bydy}{\sqrt{(bbyy-(yy-bc)^2)}},$$

cuius integrale a quadratura Circuli pendet, estque

$$= \frac{b}{2}\text{Acos.}\frac{b\left(2c+b\right)-2yy}{b\sqrt{(bb+4bc)}} + \text{Const.}$$

Quodsi autem *b* ponatur $= \infty$, casus oritur singularis; aequatio namque prodit haec

$$dx = -\frac{cdy}{\sqrt{(yy-cc)}},$$

quae est pro curva Catenaria convexitatem axi *AZ* obvertente.

EXEMPLUM V

45. *Inter omnes curvas az aequales areas aAZz continentes determinare eam, quae circa axem AZ rotata generet solidum minimae superficiei.*

Quoniam proprietas communis in area $= \int ydx$ constituitur, erit eius valor differentialis $= nv \cdot dx$. Deinde formula, quae minimum esse debet, est $\int ydx\sqrt{(1+pp)}$, cuius valor differentialis est

$$= nv \cdot (dx\sqrt{(1+pp)} - d \cdot \frac{yp}{\sqrt{(1+pp)}});$$

unde orietur, pro curva quaesita, ista aequatio

$$ndx = dx\sqrt{(1+pp)} - d \cdot \frac{yp}{\sqrt{(1+pp)}},$$

quae, per p multiplicata et integrata, praebet

$$ny + b = \frac{y}{\sqrt{(1+pp)}} \text{ seu } \sqrt{(1+pp)} = \frac{y}{ny+b};$$

unde fit

$$p = \frac{(y^2 - (ny+b)^2)}{ny+b} = \frac{dy}{dx} \text{ ac } dx = \frac{(ny+b)dy}{\sqrt{((1-n^2)y^2 - 2bny - bb)}}.$$

Ex qua patet, si sit $b = 0$, tum curvam esse abituram in lineam rectam puncta a et z iungentem. Deinde si sit $n = 0$, ob

$$dx = \frac{bdy}{\sqrt{(yy - bb)}}$$

curva erit Catenaria concavitatem axi AZ obvertens. Quodsi autem sit $n = -1$, fiet

$$dx = \frac{(b-y)dy}{\sqrt{(2by - bb)}},$$

ex qua integrando oritur

$$x = c + \frac{2b-y}{3b}\sqrt{(2by - bb)};$$

quae est pro curva algebraica et in rationalibus praebet

$$9b(x-c)^2 = \left(2b - y\right)^2 \left(2y - b\right).$$

Est ideo linea tertii ordinis et pertinet ad speciem 68 NEWTONI.

EXEMPLUM VI

46. *Inter omnes curvas az eiusdem longitudinis definire eam, quae circa axem AZ rotata producat maximum solidum.*

Inter omnes igitur curvas proprietate communi $\int dx\sqrt{(1+pp)}$ gaudentes ea quaeritur, in qua sit $\int yydx$ maximum. Quoniam ergo formulae $\int ydx\sqrt{(1+pp)}$ valor differentialis est

$$=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}},$$

formulae vero $\int yydx$ valor differentialis est $=2nv\cdot ydx$, habebitur pro curva quaesita ista aequatio

$$2ydx=\pm bbd\cdot\frac{p}{\sqrt{(1+pp)}},$$

quae multiplicata per p et integrata dabit

$$yy+bc=\frac{\pm bb}{\sqrt{(1+pp)}}\ \text{ seu }\ \sqrt{(1+pp)}=\frac{\pm bb}{yy+bc};$$

hincque

$$p=\frac{\sqrt{\left(b^4-(yy+bc)^2\right)}}{yy+bc}=\frac{dy}{dx};$$

ex qua fit

$$x=\int\frac{(yy+bc)dy}{\sqrt{\left(b^4-(yy+bc)^2\right)}}.$$

Haec curva hanc habet proprietatem, ut eius radius osculi, qui generaliter est

$$=dx:\ d\cdot\frac{p}{\sqrt{(1+pp)}},$$

fiat $=\dfrac{bb}{2y}$, hoc est, proportionalis est applicatae y inversae; unde patet curvam quaesitam esse Elasticam. Non solum autem per constantes b et c arbitrarias effici potest, ut curva per datos terminus a et z transeat, sed etiam, ut eius arcus intra hos terminus interceptus fiat datae magnitudinis. Si sit $c=0$, prodit Elastica rectangula. Caeterum nullo casu constructio per quadraturam vel Circuli vel Hyperbolae absolvi potest, nisi sint vel b et c infinita, quo quidem casu linea az prodit recta, vel $b=c$. Hoc enim casu habebitur

$$x=\int\frac{(yy+bb)dy}{y\sqrt{(-2bb-yy)}},$$

seu sumto bb negativo erit

$$x=\int\frac{(yy-bb)dy}{y\sqrt{\left(2bb-yy\right)}}=-\sqrt{\left(2bb-yy\right)}-\ bb\int\frac{dy}{y\sqrt{\left(2bb-yy\right)}}$$

et integratione per logarithmos absoluta fiet

$$x=-\sqrt{\left(2bb-yy\right)}+b\sqrt{2l\frac{b\sqrt{2}+\sqrt{(2bb-yy)}}{y}}.$$

Ipsa vero curvae longitudo, quae generaliter est

$$=\int\frac{bbdy}{\sqrt{\left(b^{4}-(yy+bc)^{2}\right)}},$$

erit hoc casu

$$=g-b\sqrt{2l\frac{b\sqrt{2}+\sqrt{(2bb-yy)}}{y}}.$$

EXEMPLUM VII

47. *Invenire curvam, quae inter omnes alias eiusdem longitudinis circa axem AZ rotata producat solidum, cuius superficies sit vel maxima vel minima.*

Quoniam proprietas communis est $\int dx\sqrt{(1+pp)}$, cuius valor differentialis est $-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$, maximi minimive formulae autem $\int dx\sqrt{(1+pp)}$ valor differentialis est

$$=nv\cdot\left(dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}}\right),$$

habebitur pro curva quaesita ista aequatio

$$bd\cdot\frac{p}{\sqrt{(1+pp)}}=dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}},$$

quae per p multiplicata et integrata praebet

$$c-\frac{b}{\sqrt{(1+pp)}}=\frac{y}{\sqrt{(1+pp)}}\ \text{ seu }\ c=\frac{b+y}{\sqrt{(1+pp)}}.$$

Hinc fiet

$$\sqrt{(1+pp)} = \frac{b+y}{c} \quad \text{et} \quad p = \frac{\sqrt{\left((b+y)^2 - cc\right)}}{c} = \frac{dy}{dx}$$

ex hacque

$$dx = \frac{cdy}{\sqrt{\left((b+y)^2 - cc\right)}},$$

quae est aequatio generalis pro Catenaria et satisfacit, dummodo axis respectu catenae suspensae situm teneat horizontalem. Fieri igitur potest, ut curva vel convexitatem vel concavitatem axi *AZ* obvertat, priori casu superficies solidi fiet minima, posteriori maxima.

EXEMPLUM VIII

48. *Inter omnes curvas* (Fig. 17**)** *per puncta A et C transeuntes, quae omnes aequales areas ABC comprehendant, definire eam, quae in fluido secundum directionem axis BA mota minimam patiatur resistentiam.*

Positis abscissa $AP = x$, applicata $PM = y$, proprietas communis est $\int ydx$, eiusque valor differentialis $= nv \cdot dx$. Resistentia autem totalis, quam figura in directione *AB* sentit, est ut $\int \frac{p^3 dx}{1+pp}$, cuius differentialis $-nv \cdot d \cdot \frac{3pp + p^4 dx}{(1+pp)^2}$. Ex his emergit pro curva ista aequatio

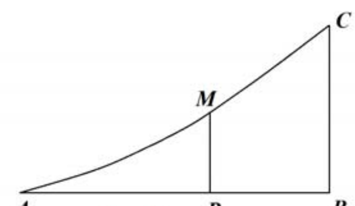

Fig. 17

$$dx = bd \cdot \frac{3pp + p^4}{(1+pp)^2};$$

quae integrata dat

$$x = c + \frac{bpp(3+pp)}{(1+pp)^2}.$$

Aequatio autem differentialis per *p* multiplicata abit in hanc

$$dy = bpd \cdot \frac{3pp + p^4}{(1+pp)^2},$$

quae in hanc formam

$$dy = bpd \cdot \frac{3pp + p^4}{(1+pp)^2} + bdp \cdot \frac{3pp + p^4}{(1+pp)^2} - bdp \cdot \frac{3pp + p^4}{(1+pp)^2}$$

transmutata habet integrale

$$y = f + \frac{bp^3(3+pp)}{(1+pp)^2} - \frac{bp^3}{1+pp} \text{ seu } y = f + \frac{2bp^3}{(1+pp)^2};$$

cum igitur sit

$$x = c + \frac{bpp(3+pp)}{(1+pp)^2},$$

curva erit algebraica. Efficiendum est autem, ut, quo casu fit $x = 0$ (quod fieri nequit, nisi vel *b* vel *c* capiatur negativum) simul *y* evanescat. Quo autem curva cognoscatur, ponatur $x - c = t$ et $y - f = u$, erit

$$t = \frac{bpp(3+pp)}{(1+pp)^2} \text{ et } u = \frac{2bp^3}{(1+pp)^2};$$

unde fit

$$t + u\sqrt{3} = \frac{b(p^4 + 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}$$

atque

$$t - u\sqrt{3} = \frac{b(p^4 - 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}$$

Extrahendis igitur radicibus quadratis habebitur

$$\sqrt{\frac{t+u\sqrt{3}}{b}} = \frac{pp + p\sqrt{3}}{1+pp} \text{ et } \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{pp - p\sqrt{3}}{1+pp};$$

hincque

$$\sqrt{\frac{t+u\sqrt{3}}{b}} + \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2pp}{1+pp}$$

et

$$\sqrt{\frac{t+u\sqrt{3}}{b}} - \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2p\sqrt{3}}{1+pp}.$$

At est

$$\frac{t}{b} = \tfrac{3}{2} \cdot \frac{2pp}{1+pp} - \tfrac{1}{2} \cdot \frac{4p^4}{(1+pp)^2} = \tfrac{3}{2}\sqrt{\frac{t+u\sqrt{3}}{b}} + \tfrac{3}{2}\sqrt{\frac{t-u\sqrt{3}}{b}} - \tfrac{1}{2}\left(\frac{2t}{b} + 2\sqrt{\frac{tt-3uu}{bb}}\right).$$

Ergo

$$\frac{4t}{b}=3\sqrt{\frac{t+u\sqrt{3}}{b}}+3\sqrt{\frac{t-u\sqrt{3}}{b}}-2\frac{\sqrt{(tt-3uu)}}{b};$$

quae rationalis facta praebet aequationem hanc quarti ordinis

$$4t^4+8ttuu+4u^4=4bt^3+36btuu-27b^2uu,$$

seu

$$4\left(tt+uu\right)^2=4bt^3+36btu^2-27b^2u^2.$$

Ad curvam autem per infinita puncta construendam expedit adhibere has formulas

$$t=\frac{b\left(p^4+3pp\right)}{\left(1+pp\right)^2} \text{ et } u=\frac{2bp^3}{\left(1+pp\right)^2}.$$

Primum autem patet curvam habere diametrum in positione abscissarum t sitam duobusque locis fieri $u=0$, nempe casu $p=0$, quo simul fit $t=0$, et casu $p=\infty$, quo fit $t=b$. Quodsi ponatur $b=4c$ atque $t=3c+r$, orietur ista aequatio

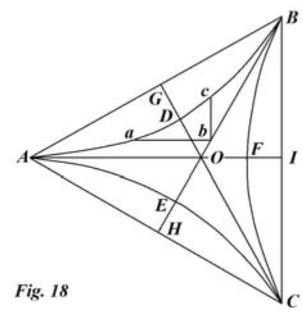

Fig. 18

$$\left(rr+uu\right)^2+8c\left(r^3-3ru^2\right)+18cc(r^2+u^2)-27c^4=0,$$

quae cum sit functio ipsarum $rr+uu$ et r^3-3ruu, declarat curvam hanc habere tres diametros sese in initio abscissarum harum r decussantes. Curva ergo quaesita (Fig.18) triangulo aequilatero *ABC* ita erit inscriptibilis, ut constet ex tribus ramis *ADB, AEC* et *BFC* inter se similibus et aequalibus, qui in punctis *A* , *B* et *C* cuspides forment acutissimos. Eius igitur diametri erunt tres rectae *AI, BH* et CG sese in centro trianguli *O* decussantibus. Erit autem $AO=3c,\ OF=c$ et $OI=\frac{3}{2}c$, ita ut sit $AI=\frac{9}{2}c$ et $FI=\frac{1}{2}c=\frac{1}{2}OF$. Huius iam curvae quaecunque portio *abc* rectis *ab* et bc parallelis ipsis *AI* et *BI* et arcu curvae ac comprehensa ita erit comparata, ut arcus *ac* inter omnes alios puncta *a* et *c* iungentes et aequalem aream *abc* continentes in fluido secundum directionem *ba* mota minimam patiatur resistentiam. Porro autem haec curva erit rectificabilis reperiturque arcus $ADB=\frac{16}{3}c$; ex quo erit

$$ADB:AI=\frac{16}{3}:\frac{9}{2}=32:27$$

atque

$$ADB:AB=32:18\sqrt{3}=16:9\sqrt{3}.$$

EXEMPLUM IX

49. *Inter omnes curvas (Fig.l9) AM aequales areas APM includentes invenire eam, quae sit ita comparata, ut, si perpetuo a centro circuli osculantis O ad applicatam MP productam ducatur perpendicularis ON, curva a punctis N formata minimam comprehendat aream APN.*

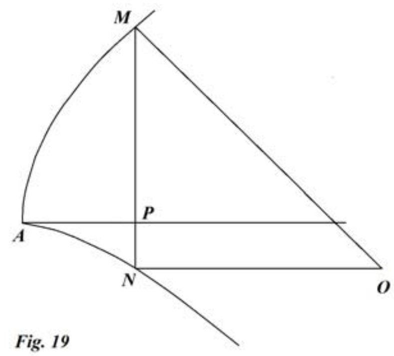

Fig. 19

Positis abscissa $AP = x$ et applicata $PM = y$, erit area $APM = \int ydx$, quae est proprietas communis, eiusque valor differentialis $= nv \cdot dx$. Deinde, cum sit radius osculi

$MO = -(1+pp)^{3:2}$, fiet $MN = -\frac{(1+pp)}{q}$ et

$PN = -\frac{(1+pp)}{q} - y$;

ex quo area APN erit $-\int ydx - \int \frac{(1+pp)}{q} dx,$ quae debet esse minima, cuius valor differentialis est
(

$$= nv \cdot \left(-dx + d \cdot \frac{2p}{q} + \frac{1}{dx} dd \cdot \frac{(1+pp)}{qq} \right);$$

unde ista nascitur aequatio

$$ndx^2 = dxd \cdot \frac{2p}{q} + dd \cdot \frac{(1+pp)}{qq};$$

quae integrata dat

$$nxdx = \frac{2pdx}{q} + d \cdot \frac{(1+pp)}{qq} + bdx.$$

Illa vero eadem aequatio per p multiplicata dat

$$ndx = \frac{2pdx}{q} + d \cdot \frac{1+pp}{qq} + bdx.$$

cuius integrale est

$$nydx = cdx - \frac{2dx}{q} + pd \cdot \frac{1+pp}{qq}.$$

His aequationibus coniungendis oritur

$$nxdy - nydx = bdy - cdx + \frac{2pdy}{q} + \frac{2dx}{q} = bdy - cdx + \frac{2dx^2 + 2dy^2}{dp}.$$

Ponatur

$$nx - b = nt \text{ et } ny - c = nu;$$

erit $dy = du$ et $dx = dt$, atque

$$ndp = \frac{2dt^2 + 2du^2}{tdu - udt} = \frac{nddu}{dt},$$

seu

$$dt^3 + 2dtdu^2 = ntduddu - nudtddu$$

posito *dt* constante. Sit $u = st$, erit

$$du = sdt + tds \text{ et } ddu = tdds + 2dtds;$$

hisque substitutis prodibit ista aequatio:

$$2(1+ss)dt^3 + 4stdt^2ds + 2(1-n)ttdtds^2 = nt^3dsdds.$$

Ponatur $t = e^{\int rds}$, erit

unde fit

$$dt = e^{\int rds}rds \text{ et } ddt = 0 = e^{\int rds}(rdds + drds + rrds^2);$$

unde fit

$$dds = -\frac{drds}{r} - rds^2;$$

ex quibus tandem emergit

$$2(1+ss)r^3ds + 4sr^2ds + 2(1-n)rds = -\frac{ndr}{r} - nrds$$

seu

$$\frac{ndr}{r} + (2-n)rds + 4sr^2ds + 2r^3ds + 2r^3s^2ds = 0.$$

Sit $s = v - \frac{1}{r}$, fiet $dr + rrdv = \frac{ndv}{2(1+vv)}$; quae aequatio integrationem admittit, quoties est $n = 2i(i-1)$ denotante *i* numerum integrum quemcunque ; ut si sit $n = 4$, fiet

$$r = \frac{2v}{1+vv} + \frac{1}{(1+vv)^2\int\frac{dv}{(1+vv)^2}};$$

ex qua retrogrediendo constructio absolvi poterit.

EXEMPLUM X

50. *Inter omnes curvas, in quibus* $\int xTdx$ *eundem obtinet valorem, invenire eam, in qua sit* $\int yTdx$ *maximum vel minimum, existente T functione quacunque ipsius p, ita ut sit* $dT = Pdp$.

Ad formulae $\int xTdx$ valorem differentialem inveniendum notandum est esse $d \cdot xT = Tdx + xPdp$, ex quo illius valor differentialis erit $= -nv \cdot d \cdot xP$. Ex altera autem formula $\int yTdx$ habetur $d \cdot yT = Tdy + yPdp$, unde eius valor differentialis erit $nv \cdot (Tdx - d \cdot yP)$. Quare pro curva quaesita orietur ista aequatio $nd \cdot xP = Tdx - d \cdot yP$. Ergo $\int Tdx = nxP + yP + b.$ Porro si illa aequatio per p multiplicetur, habebitur

$$npd \cdot xP = Tdy - pd \cdot yP = d \cdot yT - yP \cdot dp - pd \cdot yP = d \cdot yT - d \cdot yPp.$$

At est

$$pd \cdot xP = Ppdx + pxdP + xPdp + Tdx - d \cdot xT = d \cdot xPp + Tdx - d \cdot xT .$$

Quamobrem orietur

$$d \cdot yT - d \cdot yPp = nd \cdot xPp + nTdx - nd \cdot xT \;;$$

hincque

$$\int nTdx = yT - yPp - nxPp + nxT + c .$$

Quia vero ex superiori integratione habemus

$$\int nTdx = nnxP + nyP + nb ,$$

erit eliminando $\int nTdx$ ista aequatio

$$nnxP + nyP + nb = yT - yPp - nxPp + nxT + c$$

seu

$$y = \frac{nx(nP + Pp - T) + c}{-nP - Pp + T}$$

vel

$$y = -nx + \frac{c}{T - nP - Pp} .$$

Ergo prodit tandem

$$x = c\int \frac{dP}{(T-nP-Pp)^2},$$

atque

$$y = c\int \frac{pdP}{(T-nP-Pp)^2} = \frac{c}{T-nP-Pp} - nc\int \frac{dP}{(T-nP-Pp)^2}.$$

EXEMPLUM XI

51. *Invenire curvam, quae inter omnes alias intra eosdem terminus contentas et eundem formulae* $\int xdx\sqrt{(1+pp)}$ *valorem continentes habeat* $\int ydx\sqrt{(1+pp)}$ *maximum vel minimum.*

Exemplum hoc est Casus praecedentis atque ex illo manat ponendo $T = \sqrt{(1+pp)}$, ex quo erit

$$P = \frac{p}{\sqrt{(1+pp)}} \text{ et } dP = \frac{dp}{(1+pp)^{3:2}}.$$

Porro vero erit

$$T-nP-Pp = \frac{1-np}{\sqrt{(1+pp)}}.$$

Ex his iam surrogatis prodibit

$$x = c\int \frac{dp}{(1-np)^2\sqrt{(1+pp)}} \text{ et } y = \frac{c\sqrt{(1+pp)}}{1-np} - nx.$$

Integratione autem per logarithmos instituta fiet

$$x = \frac{nc(p+\sqrt{(1+pp)})-c}{(1+nn)(1-np)} + \frac{c}{(1+nn)^{3:2}}\cdot l\frac{n+(1+\sqrt{(1+nn)}(p+\sqrt{(1+pp)}}{n+(1-\sqrt{(1+nn)}(p+\sqrt{(1+pp)}} + b$$

et

$$x = \frac{nc+c(\sqrt{(1+pp)}-nnp)}{(1+nn)(1-np)} - \frac{nc}{(1+nn)^{3:2}}\cdot l\frac{n+(1+\sqrt{(1+nn)}(p+\sqrt{(1+pp)}}{n+(1-\sqrt{(1+nn)}(p+\sqrt{(1+pp)}} - nb;$$

ex quibus valoribus curva construi poterit per logarithmos. Generaliter autem, quamcunque T functionem ipsius p denotet, constructio semper per quadraturas absolvi potest. Caeterum hoc Exemplum sine subsidio praecedentis multo difficilius solutu fuisset; non tam facile enim perspicere licuisset, quomodo aequatio inventa integrabilis redderetur, quam in casu generali.

PROPOSITIO V. PROBLEMA

52. *Inter omes curvas ad eandem abscissam* $=a$ *relatas, quae eundem formulae* $\Pi=\int[Z]dx$ *valorem recipiunt, invenire eam, in qua sit* $\int Zdx$ *maximum vel minimum, existente Z functione simul ipsius* Π *, ita ut sit*

$$dZ=Ld\Pi+Mdx+Ndy+Pdp+Qdq+etc.$$

atque

$$d[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+\ etc.$$

SOLUTIO

Quoniam est $d[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+\text{etc.}$, erit formulae $\int[Z]dx$, quae hic quantitatem omnibus curvis communem repraesentat, valor differentialis

$$=nv\cdot dx\left([N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}-\text{etc.}\right),$$

qui ex Casu primo paragraphi 7 Capitis praecedentis sequitur. At formula $\int Zdx$, maximum minimumve exprimens, quia Z involvit formulam integralem $\Pi=\int[Z]dx$, pertinet ad Casum secundum loci citati eiusque adeo valor differentialis erit

$$=nv\cdot dx\left(\mathrm{N}+[N]-\frac{d(\mathrm{P}+[P])}{dx}+\frac{dd(\mathrm{Q}+[Q])}{dx^2}-\text{etc.}\right),$$

denotante $V=H-\int Ldx$, ubi H est quantitas determinata, quae oritur, si in integrali $\int Ldx$ ponatur $x=a$. Atque ob hanc ipsam quantitatem H iste valor differentialis a praescripta longitudine abscissae $x=a$ pendet. Ex his igitur duobus valoribus differentialibus ambarum formularum propositarum, quarum altera proprietatem communem, altera maximum minimumve exponit, secundum regulam datam nascitur aequatio pro curva sequens:

$$0=\alpha[N]-\frac{\alpha d[P]}{dx}+\frac{\alpha dd[Q]}{dx^2}-\text{etc.}$$
$$+\mathrm{N}+[N]-\frac{d(\mathrm{P}+[P])}{dx}+\frac{dd(\mathrm{Q}+[Q])}{dx^2}-\text{etc.};$$

quae, ob $V = H - \int Ldx$, transit in hanc

$$0 = \mathrm{N} + (\alpha + H - \int Ldx)[N] - \frac{d\left(\mathrm{P} + (\alpha + H - \int Ldx)[P]\right)}{dx} + \frac{dd\left(\mathrm{Q} + (\alpha + H - \int Ldx)[Q]\right)}{dx^2} - \text{etc.}$$

Cum iam α sit quantitas constans arbitraria, etiamsi *H* sit quantitas constans determinata, tamen $\alpha + H$ fiet quantitas arbitraria ideoque non amplius *a* definita abscissae longitudine a pendet. Quare, si loco $\alpha + H$ scribamus *C*, habebimus pro curva quaesita hanc aequationem:

$$0 = \mathrm{N} + (C - \int Ldx)[N] - \frac{d\left(\mathrm{P} + (C - \int Ldx)[P]\right)}{dx} + \frac{dd\left(\mathrm{Q} + (C - \int Ldx)[Q]\right)}{dx^2} - \text{etc.},$$

quae ergo pro quacunque abscissa exhibet Curvam, quae inter omnes alias eundem formulae $\int [Z] dx$ valorem recipientes continebit formulae $\int Zdx$ maximum minimumve valorem. Q. E. I.

COROLLARIUM 1

53. Si igitur proprietas communis fuerit ea ipsa formula integralis, quae in maximi minimive formula implicatur, tum consideratio determinatae abscissae magnitudinis ex calculo egreditur et Curva inventa pro quavis abscissa quaesito satisfaciet.

COROLLARIUM 2

54. In hac aequatione inventa duae adhuc inerunt formulae integrales; primo nempe formula $\int Ldx$ ac deinde formula $\Pi = \int [Z] dx$, quae, cum ea in *Z* contineatur, inerit in quantitatibus *L*, *M*, *N*, *P* etc.

COROLLARIUM 3

55. Si igitur haec integralia per differentiationem tollere lubeat, pervenietur ad differentialia binis gradibus altiora simulque exibit constans arbitraria *C*. Interim tamen numerus constantium arbitrariarum unitate minor erit quam gradus iste differentialium; eo quod integrale $\Pi = \int [Z] dx$ definitum obtinere debet valorem, eum ipsum scilicet, quem in maximi minimive formula $\int Zdx$ habet.

COROLLARIUM 4

56. Hinc igitur in aequatione inventa ob constantem arbitrariam C potestate una plures inerunt constantes, quam differentialium gradus indicat. Quarum una eo determinabitur,

ut valor formulae communis $\Pi = \int [Z] dx$ fiat pro curva inventa datae magnitudinis; reliquae vero per data puncta vel tangentium positionem datam determinabuntur.

COROLLARIUM 5

57. Si Z fuerit functio cum quantitatum x, y, p, q etc. tum arcus curvae s atque inter omnes Curvas isoperimetras quaeratur ea, in qua sit $\int Zdx$ maximum vel minimum, tum fiet

$$\Pi = s = \int [Z] dx \text{ et } [Z] = \sqrt{(1+pp)},$$

ita ut sit

$$[M] = 0 \text{ , } [N] = 0 \text{ et } [P] = \frac{p}{\sqrt{(1+pp)}}.$$

COROLLARIUM 6

58. Hoc igitur casu, si fuerit $dZ = Lds + Mdx + Ndy + Pdp + Qdq + \text{etc.}$, habebitur pro Curva, quae inter omnes isoperimetras habeat $\int Zdx$ maximum vel minimum, ista aequatio:

$$0 = N - \frac{1}{dx} d\left(\frac{\mathrm{P} + (C - \int Ldx)p}{\sqrt{(1+pp)}} \right) + \frac{ddQ}{dx^2} - \text{etc.}$$

seu

$$N - \frac{P}{dx} + \frac{ddQ}{dx^2} = \frac{(C - \int Ldx)dp}{dx(1+pp)^{3:2}} - \frac{Lp}{\sqrt{(1+pp)}} + \text{etc.}$$

sive

$$\frac{Lp}{\sqrt{(1+pp)}} + N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.} = \frac{\left(C - \int Ldx\right)dp}{dx\left(1+pp\right)^{3:2}}.$$

COROLLARIUM 7

59. Cum sit C quantitas arbitraria, in genere notari convenit, quod, si pro C accipiatur ille formulae $\int Ldx$ valor, quem inducit, si ponatur $x = a$, tum prodituram esse curvam, quae inter omnes omnino curvas eidem abscissae $x = a$ respondentes habeat valorem formulae $\int Zdx$ maximum vel minimum.

SCHOLION 1

60. Casus Corollarii 6, quia is ab Auctoribus potissimum tractari est solitus, peculiarem evolutionem meretur, ut eius ope Problemata, quae forte occurrere queant, facilius et expeditius resolvi possint. Inter omnes igitur Curvas isoperimetricas seu, quae eandem

habeant longitudinem $s=\int dx\sqrt{(1+pp)}$, quaeratur ea, in qua sit $\int Zdx$ maximum vel minimum, existente Z functione cum quantitatum definitarum x, y, p, q etc. tum arcus curvae s ; ita ut sit

$$dZ = Lds + Mdx + Ndp + Pdp + \text{etc.}$$

Pro curva hac proprietate gaudente iam inventa est haec aequatio:

$$\frac{1}{dx} d\cdot \frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}} = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.}$$

quae quidem in hoc latissimo sensu nec integrari nec ad simpliciorem formam se reduci patitur. At casus notasse iuvabit, quibus eam integrare licebit. Ac primo quidem, si sit $N=0$, sponte prodit ista pro curva aequatio:

$$A + \frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}} = -P + \frac{dQ}{dx} - \text{etc.}$$

Iam semel integrata. Secundo ponamus esse $M=0$; atque aequatio per $pdx = dy$ multiplicata abibit in hanc

$$pd\cdot \frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}} = Ndy - pdP + \frac{pddQ}{dx} - \text{etc.}$$

ad quam si addatur

$$Lds = Ldx\sqrt{(1+pp)} = dZ - Ndy - Pdp - Qdq + \text{etc.},$$

integratione instituta prodibit

$$\int\left(Ldx\sqrt{(1+pp)} + pd\cdot \frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}} \right) = -A + Z - pdP - Qq + \frac{pdQ}{dx} + \text{etc.}$$

Prius vero membrum, si evolvatur, transit in

$$\int\left(Ldx\sqrt{(1+pp)} + \frac{\left(C-\int Ldx\right)pdp}{(1+pp)^{3:2}} - \frac{Lppdx}{\sqrt{(1+pp)}} \right)$$
$$= \int\left(\frac{Ldx}{\sqrt{(1+pp)}} + \frac{\left(C-\int Ldx\right)pdp}{(1+pp)^{3:2}} \right),$$

cuius integrale est

$$-\frac{C-\int Ldx}{\sqrt{(1+pp)}}.$$

Quare, casu quo $M=0$, habebitur ista aequatio

$$\frac{C-\int Ldx}{\sqrt{(1+pp)}} = A - Z + Pp + Qq - \frac{pdQ}{dx}.$$

Sin autem tertio fuerit tam $M=0$ quam $N=0$, habebitur primum, ob $N=0$, haec aequatio:

$$A + \frac{\left(C-\int Ldx\right)p}{\sqrt{(1+pp)}} = -Pp + \frac{dQ}{dx};$$

quae, multiplicata per $dp = qdx$, abit in hanc

$$Adp + \frac{\left(C-\int Ldx\right)pdp}{\sqrt{(1+pp)}} = -Pdp + qdQ.$$

Cum autem sit

$$dZ = Ldx\sqrt{(1+pp)} + Pdp + Qdq,$$

habebitur

$$dZ + Adp - Ldx\sqrt{(1+pp)} + \frac{\left(C-\int Ldx\right)pdp}{\sqrt{(1+pp)}} = qdQ + Qdq;$$

quae integrata dabit

$$Z+B+Ap+\left(C-\int Ldx\right)\sqrt{(1+pp)}=Qq$$

seu

$$C-\int Ldx=\frac{Qq-B-Ap-Z}{\sqrt{(1+pp)}}.$$

At ex priore aequatione est

$$C-\int Ldx=-\frac{A\sqrt{(1+pp)}}{p}-\frac{P\sqrt{(1+pp)}}{p}+\frac{dQ\sqrt{(1+pp)}}{pdx};$$

ex quibus coniungendis elicitur:

$$Adx-Bdy=Zdy-Pdx-Ppdy+dQ+ppdQ-Qpdp,$$

in qua non amplius inest formula integralis $\int Ldx$. Usum igitur horum casuum in Exemplis monstrabimus.

EXEMPLUM I

61. *Inter omnes curvas isoperimetricas definire eam, in qua sit* $\int s^n dx$ *maximum vel minimum, denotante s arcum curvae abscissae x respondentem.*

Quoniam proprietas communis longitudinem arcus $s=\int dx\sqrt{(1+pp)}$ respicit atque in maximi minimive formula $\int s^n dx$ inest ipse arcus, solutio pertinebit ad Casum in Scholio pertractatum. Comparata ergo formula $\int s^n dx$ cum generali $\int Zdx$, fiet

$$Z=s^n \text{ et } dZ=ns^{n-1}ds$$

hincque

$$L=ns^{n-1},\ M=0,\ N=0,\ P=0 \text{ etc.}$$

Quare ex Scholii Casu ultimo, quo posueramus $M=0$ et $N=0$, habebitur ista aequatio

$$Adx-Bdy=Zdy=s^n dy,$$

ex qua oritur

$$Adx=dy\left(B+s^n\right) \text{ and } A^2dx^2+A^2dy^2=A^2ds^2=dy^2\left(A^2+\left(B+s^n\right)^2\right)$$

ideoque

$$dy = \frac{Ads}{\sqrt{A^2 + \left(B + s^n\right)^2}}$$

atque

$$dx = \frac{\left(B + s^n\right)ds}{\sqrt{A^2 + \left(B + s^n\right)^2}};$$

unde Curvae constructio perfici poterit. Vel posito $dy = pdx$, erit

$$s^n = \frac{A - Bp}{p} \text{ atque } s = \sqrt[n]{\frac{A - Bp}{p}};$$

ex quo fiet

$$ds = dx\sqrt{(1 + pp)} = -\frac{Adp\left(A - Bp\right)^{(1-n):n}}{np^{(1+n):n}}.$$

Atque hinc per *p* coordinatae curvae *x* et *y* ita determinabuntur, ut sit

$$x = -\frac{A}{n}\int \frac{dp\left(A - Bp\right)^{(1-n):n}}{p^{(1+n):n}\sqrt{(1 + pp)}} \text{ et } y = -\frac{A}{n}\int \frac{dp\left(A - Bp\right)^{(1-n):n}}{p^{1:n}\sqrt{(1 + pp)}}.$$

Videntur hic quidem quatuor constantes, duae scilicet novae, praeter *A* et *B*, ingredi, ob duplicem integrationem *y* et *x*. At cum posito $x = 0$ simul arcus curvae $s = \sqrt[n]{\frac{A - Bp}{p}}$ evanescere debeat, hinc vicissim constans integratione ipsius *x* orta definietur. Nimirum, si *n* fuerit numerus affirmativus, arcus *s* evanescit posito $p = \frac{B}{A}$; ex quo valor ipsius *x* ita determinari debet, ut posito $p = \frac{B}{A}$ fiat $= 0$.

Quodsi ponatur $n = 1$, habebitur ex priore constructione statim

$$dx = \frac{\left(B + s\right)ds}{\sqrt{A^2 + \left(B + s\right)^2}}$$

ideoque

$$x = \sqrt{\left(A^2 + B^2 + 2Bs + ss\right)} - \sqrt{(A^2 + B^2)},$$

seu posito $B = b$ et $\sqrt{\left(A^2 + B^2\right)} = c$, erit

$$x + c = \sqrt{\left(c^2 + 2bs + ss\right)}.$$

Ex posteriore autem construendi modo oritur

$$x=-\frac{A}{n}\int\frac{dp}{pp\sqrt{(1+pp)}}=\frac{A\sqrt{(1+pp)}}{p}+b \text{ seu } (x-b)p=c\sqrt{(1+pp)};$$

hincque

$$p=\frac{c}{\sqrt{((x-b)^2-c^2)}}=\frac{dy}{dx}.$$

Quare cum sit

$$y=\int\frac{cdx}{\sqrt{((x-b)^2-c^2)}},$$

curva satisfaciens erit Catenaria.

EXEMPLUM II

62. *Inter omnes curvas eiusdem longitudinis eam determinare, in qua sit* $\int Sdx$ *maximum vel minimum, existente S functione quacunque arcus s.*

Quia proprietas communis arcu $s=\int dx\sqrt{(1+pp)}$ continetur, solutio ex Scholio peti poterit. Scilicet cum sit $Z=S=$ functioni ipsius *s*, erit

$$Lds=dS \text{ et } M=N=P=Q \text{ etc.}=0.$$

Quare, per tertium Scholii Casum, habebitur pro curva quaesita ista aequatio

$$Adx-Bdy=Sdy \text{ et } Adx=dy(B+S).$$

Hinc ergo erit

$$A^2dx^2+A^2dy^2=A^2ds^2=dy^2\left(A^2+(B+S)^2\right)$$

et

$$y=\int\frac{Ads}{\sqrt{\left(A^2+(B+S)^2\right)}};$$

erit autem abscissa

$$x=\int\frac{(B+S)ds}{\sqrt{\left(A^2+(B+S)^2\right)}};$$

unde curvae constructio absolvi poterit.

Ponamus esse $S=e^s$; positoque $dy=pdx$ erit

$$\frac{A-Bp}{p}=e^{s} \quad \text{et} \quad e^{s}ds=-\frac{Adp}{pp}=\frac{(A-Bp)dx\sqrt{(1+pp)}}{p},$$

hincque

$$dx=\frac{Adp}{(A-Bp)\,p\sqrt{(1+pp)}} \quad \text{et} \quad dy=-\frac{Adp}{(A-Bp)\sqrt{(1+pp)}}.$$

Componendo vero fiet

$$dx=\frac{Bdy}{A}=-\frac{dp}{p\sqrt{(1+pp)}}$$

et integrando

$$Ax-By=Al\frac{1+\sqrt{(1+pp)}}{p}+C,$$

seu

$$\frac{1+\sqrt{(1+pp)}}{p}=e^{(Ax-By-C):A}.$$

Cum autem facto $s=0$ evanescere debeat *x*, atque ob $\frac{A-Bp}{p}=e^{s}$ facto $s=0$ fiat $p=\frac{A}{B+1}$, per integrationes efficiendum est, ut facto $p=\frac{A}{B+1}$ fiat $x=0$.

EXEMPLUM III

63. *Inter omnes curvas eiusdem longitudinis determinare eam, in qua sit* $\int sydx$ *maximum vel minimum, denotante s arcum curvae.*

Solutio huius Quaestionis iterum petenda est ex Scholio; erit namque $Z=sy$ et $dZ=yds+sdy$, ex quo fit $L=y$, $M=0$ et $N=s$, reliquae litterae *P*, *Q* etc. evanescent. Cum igitur sit $M=0$, Casus Scholli secundus hanc suppeditabit solutionem:

$$\frac{C-\int ydx}{\sqrt{(1+pp)}}=A-ys;$$

immediate vero prodit

$$sdx=d\cdot\frac{\left(C-\int ydx\right)p}{\sqrt{(1+pp)}}=\frac{\left(C-\int ydx\right)dp}{(1+pp)^{3:2}}-\frac{ypdx}{\sqrt{(1+pp)}}.$$

Quare, cum sit

$$C-\int ydx=A\sqrt{(1+pp)}-ys\sqrt{(1+pp)},,$$

erit

$$sdx = \frac{Adp - ysdp - ydy}{1 + pp}$$

seu

$$sdx + spdy + ysdp + ydy = Adp.$$

Sin autem lubuerit arcum s eliminare, habebitur ex binis aequationibus

$$s = \frac{A}{y} - \frac{\left(C - \int ydx\right)}{y\sqrt{(1+pp)}} = \frac{\left(C - \int ydx\right)dp}{dx(1+pp)^{3:2}} - \frac{yp}{\sqrt{(1+pp)}};$$

hincque

$$\frac{Adx}{y} + \frac{ypdx}{\sqrt{(1+pp)}} = \left(C - \int ydx\right)\left(\frac{dp}{(1+pp)^{3:2}} + \frac{dx}{y\sqrt{(1+pp)}} \right).$$

In utroque autem casu difficile est ad aequationem ad curvam construendum accommodatam pertingere.

EXEMPLUM IV

64. *Inter omnes curvas eandem aream* $\Pi = \int ydx$ *continentes definire eam, in qua sit* $\int \frac{dx\sqrt{(1+pp)}}{\Pi}$ *maximum vel minimum.*

Si hanc Quaestionem cum Solutione generali comparemus, habebimus

$$\int [Z]dx = \int ydx;$$

hincque $[Z] = y$ et $[N] = 1$; reliquis litteris $[M]$, $[P]$, $[Q]$ etc. evanescentibus. Porro erit

$$Z = \frac{\sqrt{(1+pp)}}{\Pi} \quad \text{et} \quad dZ = -\frac{d\Pi\sqrt{(1+pp)}}{\Pi^2} + \frac{pdp}{\Pi\sqrt{(1+pp)}},$$

unde erit

$$L = -\frac{\sqrt{(1+pp)}}{\Pi^2},\ M = 0,\ N = 0 \ \text{et} \ P = \frac{p}{\Pi\sqrt{(1+pp)}}.$$

Quocirca pro curva quaesita sequens emerget aequatio:

$$0 = C + \int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \frac{1}{dx} d \cdot \frac{p}{\Pi\sqrt{(1+pp)}}.$$

Multiplicetur haec aequatio per $dy = pdx$, erit

$$0 = Cdy + dy\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - pd \cdot \frac{p}{\Pi\sqrt{(1+pp)}},$$

quae integrata dabit:

$$0 = B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \int \frac{d\Pi\sqrt{(1+pp)}}{\Pi^2} - \frac{pp}{\Pi\sqrt{(1+pp)}} + \int \frac{pdp}{\Pi\sqrt{(1+pp)}}$$
$$= B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} - \frac{pp}{\Pi\sqrt{(1+pp)}} + \frac{\sqrt{(1+pp)}}{\Pi}.$$

Hinc itaque istam obtinebimus aequationem

$$0 = B + Cy + y\int \frac{dx\sqrt{(1+pp)}}{\Pi^2} + \frac{1}{\Pi\sqrt{(1+pp)}};$$

a qua si prior per *y* multiplicata subtrahatur, erit

$$0 = B + \frac{1}{\Pi\sqrt{(1+pp)}} + \frac{y}{dx} d \cdot \frac{p}{\Pi\sqrt{(1+pp)}}$$

seu

$$0 = Bdx + \frac{dx}{\Pi\sqrt{(1+pp)}} + \frac{ydp}{\Pi(1+pp)^{3:2}} - \frac{y^2 pdx}{\Pi^2\sqrt{(1+pp)}}$$

ex qua aequatione si denuo $\Pi = \int ydx$ exterminare velimus, prodiret aequatio differentialis tertii ordinis, ex qua multo minus quicquam ad Curvam cognoscendam deduci posset.

SCHOLION 2

65. Quanquam in hac Propositione posuimus $[Z]$ esse functionem determinatam quantitatum *x*, *y*, *p*, *q* etc., tamen Methodus solvendi patet, si haec ipsa quantitas $[Z]$ fuerit functio indefinita formulas integrales in se complectens. Ponamus enim in formula $\Pi = \int [Z] dx$, quae omnibus curvis debet esse communis, esse

$$d[Z]=[L]d\pi+[M]dx+[N]dy+[P]dp+[Q]dq+\text{etc.}$$

existente $\pi=\int[z]dx$ et

$$d[z]=[m]dx+[n]dy+[p]dp+[q]dq+\text{etc.}$$

Maximum minimumve autem esse oportere formulam $\int Zdx$, existente

$$dZ=L\Pi+Mdx+Ndy+Pdp+\text{etc.}$$

Iam formula $\int[Z]dx$ continetur in Casu secundo paragraphi 7 Capitis praecedentis; inde ergo, si capiatur integrale $\int[L]dx$ eiusque valor respondens abscissae $x=a$, ad quam solutio debet accommodari, ponatur $=[H]$ atque $[H]-\int[L]dx=[V]$, habebitur formulae $\int[Z]dx$ valor differentialis

$$=nv\cdot dx\left([N]+[n][V]-\frac{d\left([P]+[p][V]\right)}{dx}+\frac{dd\left([Q]+[q][V]\right)}{dx^2}-\text{ etc.}\right).$$

Deinde vero maximi minimive formula $\int Zdx$ continetur in Casu tertio loci citati; ad eiusque valorem differentialem inveniendum ponatur formulae $\int Ldx$ valor abscissae $x=a$ respondens $=H$, ac $H-\int Ldx=V$. Iam capiatur integrale

$$\int[L]Vdx=H\int[L]dx-\int[L]dx\int Ldx$$

sitque posito $x=a$ valor formulae $\int[L]dx\int Ldx=K$, eodem autem casu formulae $\int[L]dx$ valor est $=[H]$, ex quo formulae $\int[L]Vdx$ casu $x=a$ valor erit $=H[H]-K$, et vocetur

$$H[H]-K-H\int[L]dx+\int[L]dx\int Ldx=W,$$

ita ut sit

$$W=H[V]-K+\int[L]dx\int Ldx,$$

eritque formulae propositae $\int Zdx$ valor differentialis

$$==nv\cdot dx\left(N+[N]V+[n]W-\frac{d\cdot\left(P+[P]V+[p]W\right)}{dx}+\frac{dd(Q+[Q]V+[q]W}{dx^2}-\text{etc.}\right).$$

Quodsi iam ad hunc valorem differentialem addatur praecedens per quantitatem constantem arbitrariam α multiplicatus summaque ponatur $=0$, prodibit aequatio pro curva quaesita haec:

$$0=N+[N](\alpha+V)+[n](\alpha[V]+W)-\frac{1}{dx}d\cdot\left(P+[P](\alpha+V)+[p](\alpha[V]+W)\right)$$
$$+\ \frac{1}{dx^2}\cdot dd\left(Q+[Q](\alpha+V)+[q](\alpha[V]+W\right)-\text{etc.}$$

Est vero hic $\alpha+V=\alpha+H-\int Ldx$; unde si ponatur $\alpha+H=0$, erit C constans arbitraria et $\alpha+V=C-\int Ldx$, atque

$$\alpha[V]+W=C[H]-K-C\int[L]dx+\int[L]dx\int Ldx.$$

Hoc igitur pacto pervenietur ad curvam quaesitam, in cuius aequatione, quia ob [*H*] et *K* adhuc inest constans data *a*, ea quaesito satisfaciet tantum pro proposita abscissa $x=a$. Quodsi autem formularum ambarum altera ad Casum 4, altera ad Casum 5 pertineat, tum iterum consideratio datae abscissae *a* ex calculo egreditur eademque curva pro omni abscissa satisfaciet, id quod unico sequenti Exemplo declarasse sufficiet.

EXEMPLUM V

66. *Inter omnes curvas eidem abscissae respondentes, quae eundem formulae v valorem recipiunt, invenire eam, in quae sit* $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$ *maximum vel minimum, existente* $dv=gdx+Wdx\sqrt{(1+pp)}$ *et W functione quacunque ipsius v.*

Solutio huius Quaestionis exhibebit curvam, super qua corpus descendens a gravitate uniformi *g* deorsum in directione abscissarum sollicitatum in medio quocunque resistente celerrime delabitur inter omnes alias curvas, super quibus descendendo eandem acquirit celeritatem. Est enim $\sqrt{v}$ celeritas corporis in quocunque curvae puncto et *W* exprimit resistentiam medii. Quod nunc primum ad proprietatem communem

$$v=\int dx\left(g+W\sqrt{(1+pp)}\right),$$

ponamus esse $dW=Udv$, atque haec formula ad Casum quartum pertinebit ; erit namque

$$\Pi=v \text{ et } Z=g+W\sqrt{(1+pp)} \text{ ac } dZ=Udv\sqrt{(1+pp)}+\frac{Wpdp}{\sqrt{(1+pp)}};$$

unde erit

$$L=U\sqrt{(1+pp)},\ M=0,\ N=0 \text{ et } P=\frac{Wp}{\sqrt{(1+pp)}}.$$

Sumatur ergo integrale $\int U dx\sqrt{(1+pp)}$ sitque, casu quo $x=a$ ponitur,

$$e^{\int U dx\sqrt{(1+pp)}} = H$$

ac ponatur

$$V = He^{-\int U dx\sqrt{(1+pp)}}.$$

Ex his erit formulae *v* valor differentialis

$$= nv\cdot dx\left(-\frac{1}{dx}d\cdot\frac{WVp}{\sqrt{(1+pp)}}\right) = -nv\cdot dx\cdot d\cdot\frac{WVp}{\sqrt{(1+pp)}}.$$

Porro maximi minimive formula $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$ pertinebit ad Casum quintum eritque

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{v}} \text{ et } dZ = -\frac{dv\sqrt{(1+pp)}}{2v\sqrt{v}} + \frac{pdp}{\sqrt{v(1+pp)}}$$

ideoque

$$\Pi = v \text{ et } L = -\frac{\sqrt{(1+pp)}}{2v\sqrt{v}},\ M=0,\ N=0 \text{ et } P = \frac{p}{\sqrt{v(1+pp)}}.$$

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{v}} \text{ et } dZ = +\frac{pdp}{\sqrt{v(1+pp)}}$$

Deinde vero ob $v = \int dx(g + W\sqrt{(\ 1+pp)})$ erit

$$[Z] = g + W\sqrt{(1+pp)} \text{ et } d[Z] = Udv\sqrt{(1+pp)} + \frac{Wpdp}{\sqrt{(1+pp)}},$$

unde

$$[\mathrm{L}] = U\sqrt{(1+pp)},\ [M]=0,\ [N]=0 \text{ et } [P] = \frac{Wp}{\sqrt{(1+pp)}}.$$

Ponatur, si post integrationem fiat $x=a$,

$$-\int e^{\int U dx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}} = K,$$

sitque

$$e^{-\int Udx\sqrt{(1+pp)}}\left(K+\int e^{\int Udx\sqrt{(1+pp)}}\,\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}\right)=T\,;$$

atque erit formulae $\int\frac{dx\sqrt{(1+pp)}}{v}$ valor differentialis

$$=-nv\cdot dx\left(\frac{1}{dx}d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right)\right)=-nv\cdot d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right).$$

Ex his duobus valoribus differentialibus inventis nascitur pro curva quaesita sequens aequatio

$$0=\alpha\cdot d\,\frac{WVp}{\sqrt{(1+pp)}}+d\left(\frac{p}{\sqrt{v(1+pp)}}+\frac{WTp}{\sqrt{(1+pp)}}\right)$$

et integrando

$$B=\frac{p}{\sqrt{v(1+pp)}}+\frac{Wp(\alpha V+T)}{\sqrt{(1+pp)}}.$$

At est

$$\alpha V+T=e^{-\int Udx\sqrt{(1+pp)}}\left(\alpha H+K+\int e^{\int Udx\sqrt{(1+pp)}}\,\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}\right).$$

Quodsi ergo ponatur $\alpha H+K=C$, erit C constans arbitraria atque quantitas definita a omnino ex aequatione evanescet; ideoque curva quaesita desideratam proprietatem pro quavia abscissa possidebit. Pro curva quaesita habebitur ergo ista aequatio:

$$e^{\int Udx\sqrt{(1+pp)}}\left(\frac{B\sqrt{(1+pp)}}{Wp}-\frac{1}{W\sqrt{v}}\right)=C+\int e^{\int Udx\sqrt{(1+pp)}}\,\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}$$

et differentiando

$$-\frac{Bdp}{Wp^2\sqrt{(1+pp)}}-\frac{BUdv\sqrt{(1+pp)}}{W^2p}+\frac{Udv}{W^2\sqrt{v}}+\frac{dv}{2Wv\sqrt{v}}+\frac{BUdx(1+pp)}{Wp}$$

$$-\frac{Udx\sqrt{(1+pp)}}{W\sqrt{v}}=\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}}.$$

Cum autem sit $dv=gdx+Wdx\sqrt{(1+pp)}$, habebimus facta substitutione hanc aequationem

$$\frac{Bdp}{Wp^2\sqrt{(1+pp)}}=\frac{gdx}{2Wv\sqrt{v}}+\frac{gUdx}{W^2\sqrt{v}}-\frac{gBUdv\sqrt{(1+pp)}}{W^2p}$$

sive istam

$$\frac{2BWdp}{\sqrt{(1+pp)}}=\frac{gWp^2dx}{v\sqrt{v}}+\frac{2gUp^2dx}{\sqrt{v}}-gBUpdx\sqrt{(1+pp)}.$$

Multiplicetur haec aequatio per *dv* et in primo termino loco *dv* scribatur $dv=gdx+Wdx\sqrt{(1+pp)}$, ac *dW* loco *Udv*; quo facto habebitur ista aequatio

$$\frac{2gBdp}{W\sqrt{(1+pp)}}+2Bdp-\frac{gp^2dv}{Wv\sqrt{v}}=\frac{2gppdW}{W^2\sqrt{v}}-\frac{2gBpdW\sqrt{(1+pp)}}{W^2},$$

quae divisa per p^2 fit integrabilis; eritque aequatio integrata haec

$$2C-\frac{2B}{p}=\frac{2gB\sqrt{(1+pp)}}{Wp}-\frac{2g}{W\sqrt{v}}\quad\text{sive}\quad W=\frac{gB\sqrt{v}\sqrt{(1+pp)}-gp}{Cp\sqrt{v}-B\sqrt{v}}=\frac{dv-gdx}{dx\sqrt{(1+pp)}}.$$

Unde nascitur aequatio a resistentia *W* libera haec

$$(Cp-B)dv=gCpdx+gBppdx-\frac{gpdx\sqrt{(1+pp)}}{\sqrt{v}}$$

Cum autem *W* sit functio ipsius *v* data, ope aequationis

$$W\sqrt{v}=\frac{gB\sqrt{v(1+pp)}-gp}{Cp-B},$$

dabitur *p* per *v* ; qui valor si in praecedente aequatione substituatur, dabitur *dx* per *v* et *dv* hincque curva quaesita poterit construi.

PROPOSITIO VI. PROBLEMA

67. *Inter omnes curvas proprietate communi A praeditas determinare eam, in qua sit functio quaecunque cum ipsius illius expressionis A tum alius cuiuscunque B maximum vel minimum.*

SOLUTIO

Sit *dA* valor differentialis expressionis *A* atque *dB* valor differentialis expressionis *B* ; habebit functionis illius ipsarum *A* et *B*, quam maximum minimumve esse oportet, valor differentialis huiusmodi formam $\alpha dA+\beta dB$, in qua constantes α et β a ratione compositionis, qua expressiones *A* et *B* in illa functione inter se permiscentur, pendent, ita ut valores obtineant determinatos ab abscissae quantitate, cui solutionem accomodatam esse oportet, pendentes. Quoniam vero expressionis *A*, quae proprietatem communem complectitur, valor differentialis est *dA*, huius multiplum quodcunque *ydA* addatur ad valorem differentialem $\alpha dA+\beta dB$ expressionis, quae maximum minimumve esse debet, ac summa $(\alpha+\gamma)dA+\beta dB$ nihilo aequalis posita dabit aequationem pro curva quaesita. Habebitur igitur ista aequatio

$$\left(\alpha+\gamma\right)dA+\beta dB=0 \text{ seu } \left(\alpha+\gamma\right)\delta A+\beta\delta dB=0,$$

in qua, etiamsi α et β sint quantitates constantes determinatae, tamen, ob γ et δ quantitates constantes arbitrarias, coefficientes valorum *dA* et *dB*, qui sunt $\left(\alpha+\gamma\right)\delta$ et $\beta\delta$, evadent constantes arbitrariae magnitudinis. Harum igitur loco si scribantur litterae ξ et η, habebitur pro curva quaesita ista aequatio $\xi dA+\eta dB=0$. Quocirca ad Problema solvendum expressionum *A* et *B*, quarum altera proprietatem communem continet, utriusque autem functio quaecunque maximum minimumve esse debet, singulatim valores differentiales *dA* et *dB* capi oportet eosque, per quantitates constantes arbitrarias quasque multiplicatos, nihilo aequales poni, quo pacto resultabit ista aequatio $\xi dA+\eta dB=0$, quae naturam curvae quaesitae exprimet. Q. E. I.

COROLLARIUM l

68. Natura igitur curvae satisfacientis tantum ab expressionibus *A* et *B* pendet neque ratio functionis ipsarum *A* et *B*, quae maximum minimumve esse debet, ullo modo in computo manet; sed quaecunque sit functio, eadem solutio prodibit.

COROLLARIUM 2

69. Quaecunque itaque ipsarum *A* et *B* functio inter omnes curvas eadem proprietate *A* gaudentes debeat esse maximum vel minimum, solutio perinde se habebit, ac si inter omnes curvas eadem communi proprietate *A* gaudentes ea requiratur, in qua expressio altera B maximum minimumve obtineat valorem.

COROLLARIUM 3

70. Quodsi ergo expressiones *A* et *B* eiusmodi fuerint formulae, quarum valores differentiales *dA* et *dB* non pendeant a magnitudine abscissae *x*, cui respondent, quod evenit, si illae formulae pertineant ad Casum vel primum vel quartum, secundum nostram enumerationem Capite praecedente paragrapho 7 factam, tum curva inventa pro quacunque abscissa aeque satisfaciet.

COROLLARIUM 4

71. Eadem Solutio locum habebit, si inter omnes curvas, quarum communis sit proprietas, functio quaecunque ipsarum *A* et *B* ea requiratur, in qua alia quaepiam earundem *A* et *B* functio sit maximum vel minimum. Hoc enim quoque casu pervenitur ad aequationem $\xi dA + \eta dB = 0$, in qua ξ et η sint quantitates constantes ad arbitrium accipiendae.

EXEMPLUM I

72. *Inter omnes curvas (Fig. 20) aMb cum axe AB eandem aream* $\int ydx$ *continentes invenire eam, in qua sit* $\frac{\int yydx}{\int ydx}$ *minimum.*

Quaestio haec initur, si inter omnes areas aequales, quae intra ordinatas extremas *Aa* et *Bb* atque basi *AB* formari possunt, desideretur ea, quae habeat suum centrum gravitatis in loco infimo positum. Sumpta enim curva quacunque *aMb* positisque abscissa $AP = x$, applicata $PM = y$, erit portionis *aAPM* centrum gravitatis a basi *AP* remotum intervallo $= \frac{\int yydx}{2\int ydx}$, quod adeo fiet minimum, si reddatur haec expressio $\frac{\int yydx}{\int ydx}$ minima. Habemus ergo

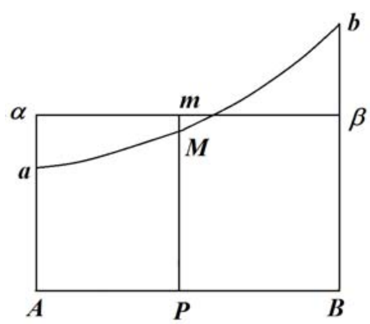

Fig. 20

binas has formulas $\int ydx$ et $\int yydx$, quarum valores differentiales sunt $nv \cdot dx \cdot 1$ et $nv \cdot dx \cdot 2y$, ex quibus pro curva quaesita ista colligitur aequatio $\xi + 2\eta y = 0$ seu $y = c$. Quaestioni igitur satisfacit linea recta $\alpha\beta$ basi *AB* parallela seu horizontalis atque parallelogrammum rectangulum $A\alpha\beta B$ prae omnibus aliis figuris ut *AabB* eiusdem areae hac gaudebit praerogativa, ut eius centrum gravitatis ad basin *AB* proxime accedat. Quodsi ergo $\alpha AB\beta$ concipiatur tanquam vas aqua repletum, si suprema aquae superficies $\alpha\beta$ sese ad situm horizontalem composuerit, tum aqua habebit suum centrum gravitatis profundins situm, quam si eius suprema superficies alium quemcunque situm teneret.

EXEMPLUM II

73. *Inter omnes curvas* (Fig. 14) *eiusdem longitudinis DAD invenire eam, quae habeat suum gravitatis centrum quam profundissime situm seu in qua sit*

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} \quad \textit{minimum.}$$

Iam intelligitur Solutio huius Quaestionis datura esse curvam Catenariam; namque secundum leges Staticas catena ex punctis *D* et *D* suspensa eiusmodi induet figuram, ut eius centrum gravitatis maxime descendat. Quamobrem inter omnes figuras, quas catena inducere potest, quae quidem omnes eiusdem sunt longitudinis, curva Catenaria orietur, si quaeratur ea, in qua sit

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$$

minimum; quippe quae expressio dat distantiam centri gravitatis *G* ab abscissarum initio *A*. Cum igitur habeantur binae istae formulae $\int dx\sqrt{(1+pp)}$ et $\int xdx\sqrt{(1+pp)}$, quaerantur earum valores differentiales; qui erunt

$$\text{primae} = -\,nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}} \quad \text{et alterius} = -\,nv\cdot d\cdot\frac{xp}{\sqrt{(1+pp)}}$$

ex quibus nascitur pro curva quaesita ista aequatio

$$cd\cdot\frac{p}{\sqrt{(1+pp)}} = d\cdot\frac{xp}{\sqrt{(1+pp)}}$$

et integrando

$$\frac{xp}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}} + b \quad \text{seu} \quad x-c = \frac{b\sqrt{(1+pp)}}{p} \quad \text{et} \quad dx = \frac{-bdp}{pp\sqrt{(1+pp)}}.$$

Hinc ergo fiet

$$y = \int pdx = -b\int\frac{dp}{p\sqrt{(1+pp)}};$$

ex quibus aequationibus curva construetur eritque curvae longitude

$$\int dx\sqrt{(1+pp)} = s = \frac{b}{p} + Const. = \frac{b}{p} + f.$$

Hinc alia Constructio definiendis *x* et *y* per *s* formari poterit; erit nempe $p = \frac{b}{s-f}$ et, si initium capiatur in *A*, ubi fit $p = \infty$, ponendum est $f = 0$, ita ut sit $p = \frac{b}{s}$; unde fit

$$\sqrt{(1+pp)} = \sqrt{(bb+ss)} \text{ et } dx\sqrt{(1+pp)} = ds = \frac{dx\sqrt{(bb+ss)}}{s} ;$$

hincque

$$dx = \frac{sds}{\sqrt{(bb+ss)}} \text{ et } x = \sqrt{(bb+ss)} - b.$$

Porro erit

$$dy = pdx = \frac{bds}{\sqrt{(bb+ss)}} \text{ atque } y = bl\frac{s+\sqrt{(bb+ss)}}{b}.$$

Aequatio autem inter coordinatas orthogonales *x* et *y* deducetur ex aequatione

$$x - c = \frac{b\sqrt{(1+pp)}}{p};$$

quae si desideretur super axe *AP*, qui est diameter et pro initio abscissarum in *A* sumpto, ubi est $pp = \infty$, poni oportet $c = -b$; eritque $(x+b)p = b\sqrt{(1+pp)}$ hincque

$$(x+b)^2 pp = bb + bbpp \text{ et } p = \frac{b}{\sqrt{(xx+2bx)}};;$$

ideoque $dy = \frac{bdx}{\sqrt{(xx+2bx)}}$, quae est aequatio pro Catenaria nota.

EXEMPLUM III

74. *Inter omnes curvas eiusdem longitudinis determinare eam, in qua sit* $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ *minimum denotante S functionem quamcunque arcus curvae* $s = \int dx\sqrt{(1+pp)}$.

In hoc Exemplo continetur inventio curvae Catenariae, si catena non fuerit ubique uniformiter crassa, sed cuius crassities arcui *s* respondens est ut *s* functio ipsius *s*. Tum enim exprimet $s = \int dx\sqrt{(1+pp)}$) huius catenae pondus

$\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ et altitudinem centri gravitatis supra abscissarum initium; quae esse debet minima. Principio quidem hic casus in Problemate praecedente non contineri videtur, quia formula arcum exprimens ipsa $s=\int dx\sqrt{(1+pp)}$ non

inest in maximi minimive expressione $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$, quippe quae est functio duarum aliarum formularum integralium. At cum sit S functio arcus curvae s atque $ds=dx\sqrt{(1+pp)}$, erit $\int Sdx\sqrt{(1+pp)}=\int Sds$, ideoque functio ipsius s; ex quo expressio $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ erit functio formularum $\int dx\sqrt{(1+pp)}$ et $\int Sxd\sqrt{(1+pp)}$, quarum illa proprietatem communem continet. Idem igitur est, ac si quaerere deberemus inter omnes curvas aeque longas eam, in qua sit $\int Sxd\sqrt{(1+pp)}$ minimum. Cum iam S sit functio ipsius $s=\int dx\sqrt{(1+pp)}$, pertinebit haec Quaestio ad Propositionem praecedentem eumque casum, qui paragrapho 60 est pertractatus. Scilicet erit $Z=Sx\sqrt{(1+pp)}$; unde, si ponamus $dS=Tds$, fiet

$$dZ=xTds\sqrt{(1+pp)}+Sdx\sqrt{(1+pp)}+\frac{Sxpdx}{\sqrt{(1+pp)}},$$

ita ut sit

$$L=xT\sqrt{(1+pp)},\ M=S\sqrt{(1+pp)},\ N=0 \ \text{ et } \ P=\frac{Sxp}{\sqrt{(1+pp)}}.$$

Iam ob $N=0$, obtinemus ex eodem loco citato statim hanc aequationem

$$A+\frac{\left(C-\int xTdx\sqrt{(1+pp)}\right)p}{\sqrt{(1+pp)}}=-\frac{Sxp}{\sqrt{(1+pp)}}$$

seu

$$\frac{A\sqrt{(1+pp)}}{p}+C-\int xTdx\sqrt{(1+pp)}+Sx=0.$$

At est $Tdx\sqrt{(1+pp)}=Tds=dS$; unde habetur

$$\frac{A\sqrt{(1+pp)}}{p}+C+Sx-\int xdS=0,$$

ubi *A* et *C* sunt quantitates arbitrariae. Differentietur haec aequatio fietque

$$\frac{-Adp}{pp\sqrt{(1+pp)}}+Sdx=0 \text{ seu } Sdx\sqrt{(1+pp)}=-\frac{cdp}{pp}=Sds.$$

Quare, cum sit *S* functio ipsius *s*, integretur *Sds* eritque integrale, quod sit $=R$, pondus catenae longitudini *s* respondens. Fiet ergo integrando $\frac{c}{p}=R+C$; et, si initium curvae capere placeat in loco *A*, ubi curvae tangens *p* est horizontalis, erit $C=0$ atque $p=\frac{c}{R}$.
Hinc ergo porro erit

$$\sqrt{(1+pp)}=\frac{\sqrt{(cc+RR)}}{R}=\frac{ds}{dx};$$

ideoque

$$dx=\frac{Rds}{\sqrt{(cc+RR)}} \text{ atque } dy=\frac{cds}{\sqrt{(cc+RR)}},$$

ex quibus aequationibus curva ita poterit construi, ut statim ad quamvis catenae longitudinem tam abscissa quam applicata respondens definiatur. Manifestum autem est casu, quo $R=s$, hoc est quo catena ponitur uniformis crassitiei, tum prodire Catenariam curvam ordinariam.

SCHOLION

75. Nisi huius Exempli convenientia tam cum ista Propositione quam cum praecedente esset observata, tum Solutio quidem per regulam generalem absolvi potuisset, verum tamen multo prolixior evasisset. Quo autem nihilominus Methodi generalis usus clarius ob oculos ponatur, idem hoc Exemplum secundum generalia praecepta resolvere visum est. Quaeratur igitur inter omnes curvas eiusdem longitudinis $s=\int dx\sqrt{(1+pp)}$ ea, quae habeat valorem expressionis huius

$$\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$$

maximum vel minimum, existente *S* functione quacunque arcus curvae *S*. Et quoniam nondum suspicari licet considerationem datae abscissae, a qua valor differentialis expressionis $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ pendet, ex calculo esse egressuram, ponamus huic Quaestioni tantum pro data abscissae longitudine $x=a$ satisfieri oportere. Ab hac longitudine quidem formulae communem proprietatem continentis $\int dx\sqrt{(1+pp)}$ valor differentialis non pendet, quippe qui constanter est $=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$; at in maximi minimive $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ expressione ponamus casu, quo $x=a$, fore

$$\int Sxdx\sqrt{(1+pp)}=A \text{ et } \int Sdx\sqrt{(1+pp)}=B,$$

illius vero numeratoris $\int Sxdx\sqrt{(1+pp)}$ valorem differentialem esse $=dA$, denominatoris vero $\int Sdx\sqrt{(1+pp)}$ valorem differentialem esse $=dB$. Hinc igitur maximi minimive expressionis, quae casu $x=a$ fit $=\frac{A}{B}$, valor differential is erit , $-\frac{BdA-AdB}{BB}$, qui multiplo cuicunque formulae communis valoris differentialis $=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$ aequalis positus dabit aequationem pro curva quaesita. Iam ad valores differentiales dA et dB inveniendos consideremus primum formulam $\int Sxdx\sqrt{(1+pp)}$, quae secundum enumerationem paragrapho 7 Capitis praecedentis factam pertinet ad Casum secundum; quo erit $Z=S\sqrt{(1+pp)}$ et posito $dS=Tds$ erit

$$dZ=Tds\sqrt{(1+pp)}+\frac{Spdp}{\sqrt{(1+pp)}}.$$

Comparatione ergo facta, erit

$$\Pi=s,\ L=T\sqrt{(1+pp)},\ M=0,\ N=0 \text{ et } P=\frac{Sp}{\sqrt{(1+pp)}};$$

tum vero ob $\Pi=s=\int dx\sqrt{(1+pp)}$ erit

$$[Z]=\sqrt{(1+pp)} \text{ et } [M]=0,\ [N]=0 \text{ et } [P]=\frac{p}{\sqrt{(1+pp)}}.$$

Iam sumatur integrale $\int Ldx=\int Tdx\sqrt{(1+pp)}=\int Tds=S$, cuius valor casu $x=a$ fiat $=G$, eritque $V=G-S$. Quamobrem habebitur formulae $\int Sxdx\sqrt{(1+pp)}$ valor differentialis

$$dB = -nv \cdot d\left(\frac{Sp}{\sqrt{(1+pp)}} + \frac{p(G-S)}{\sqrt{(1+pp)}}\right) = -nv \cdot d \cdot \frac{Gp}{\sqrt{(1+pp)}}.$$

Altera porro formula $\int Sxdx\sqrt{(1+pp)}$ pariter in eodem Casu secundo comprehenditur eritque

$$Z = Sx\sqrt{(1+pp)} \text{ et } dZ = Txds\sqrt{(1+pp)} + Sdx\sqrt{(1+pp)} + \frac{Sxpdp}{\sqrt{(1+pp)}},$$

unde fit

$$\Pi = s,\ L = Tx\sqrt{(1+pp)},\ M = S\sqrt{(1+pp)},\ N = 0 \text{ et } P = Sxp\sqrt{(1+pp)}.$$

Deinde ob $\Pi = s = \int dx\sqrt{(1+pp)}$ erit ut ante

$$[Z] = \sqrt{(1+pp)},\ [M] = 0,\ [N] = 0 \text{ et } [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Nunc sumatur integrale $\int Ldx = \int Txdx\sqrt{(1+pp)} = \int Txds = \int xds$, cuius valor posito $x = a$ sit $= H$; erit $V = H - \int xdS$ hincque prodibit istius formulae valor differentialis

$$dA = -nv \cdot d\left(\frac{Sxp + p\left(H - \int xdS\right)}{\sqrt{(1+pp)}}\right) = -nv \cdot d \cdot \frac{p\left(H + \int Sdx\right)}{\sqrt{(1+pp)}}.$$

Inventis ergo valoribus *dA* et *dB* aequatio pro curva quaesita erit

$$\alpha B^2 d \cdot \frac{p}{\sqrt{(1+pp)}} - \beta Ad \cdot \frac{Gp}{\sqrt{(1+pp)}} + \beta Bd \cdot \frac{p\left(H + \int Sdx\right)}{\sqrt{(1+pp)}} = 0$$

et integrando

$$\frac{\alpha B^2 p - \beta AGp + \beta BHp + \beta Bp\int Sdx}{\sqrt{(1+pp)}} = C;$$

in qua α, β et C sunt constantes arbitrariae et *G* et *H* constantes determinatae. Quodsi ergo ponatur $\frac{\alpha B}{\beta} - \frac{AG}{B} + H = b$ et $\frac{C}{\beta B} = c$, erunt *b* et *c* constantes arbitrariae, atque constantes determinatae *G* et *H* a definito abscissae valore $x = a$ pendentes omnino ex aequatione evanescent; ita ut Curva inventa pro quavis abscissa gavisura sit desiderata proprietate, eiusque aequatio erit haec

$$c=\frac{bp+p\int Sdx}{\sqrt{(1+pp)}} \quad \text{seu} \quad \frac{c\sqrt{(1+pp)}}{p}=b+\int Sdx\,;$$

quae differentiata dabit

$$Sdx=-\frac{cdp}{pp\sqrt{(1+pp)}} \quad \text{seu} \quad Sdx\sqrt{(1+pp)}=Sds=-\frac{cdp}{pp}.$$

Ponatur ut supra $\int Sds=R$, ita ut R pondus longitudinis catenae s repraesentet, erit $R=\frac{c}{p}+Const.$, quae est ipsa aequatio, quam praecedenti Methodo elicuimus. Ex hac itaque solutione intelligitur, quemadmodum per Methodum generalem huiusmodi Quaestiones resolvi possint, si proprietas communis non ingrediatur in maximi minimive expressionem; quod ut clarius intelligatur, unum adhuc huiusmodi Exemplum apposuisse sufficiet.

EXEMPLUM IV

76. *Inter omnes curvas (Fig. 14) eiusdem longitudinis DAD datae abscissae AC= a respondentes eam definire, quae comprehendat aream DAD, cuius centrum gravitatis G sit vel altissime vel profundissime positum seu in qua sit* $\frac{\int yxdx}{\int ydx}$ *maximum vel minimum.*

Proprietas igitur communis est $\int dx\sqrt{(1+pp)}$, cuius valor difierentialis cuicunque abscissae x respondens est $=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}$. Maximi autem minimive expressionis $\frac{\int yxdx}{\int ydx}$ valor differentialis pendebit a praescripta abscissae longitudine $x=a$; qui ut inveniatur casu, quo $x=a$, fiat $\int yxdx=A$ huiusque formulae valor differentialis sit $=dA$, qui per Regulas supra datas invenitur $=nv\cdot dx\cdot x=nv\cdot xdx$. Porro eodem casu $x=a$ abeat altera formula $\int ydx$ in B sitque eius valor differentialis $=dB$, qui per Regulas datas reperitur $=nv\cdot dx$; ita ut sit $dA=nv\cdot xdx$ et $dB=nv\cdot dx$. Ex his maximi minimive expressionis $\frac{\int yxdx}{\int ydx}$, quae casu $x=a$ abit in $\frac{A}{B}$, valor differentialis erit $=\frac{BdA-AdB}{BB}=\frac{nv(Bxdx-Adx)}{BB}$, qui multiplo valoris differentialis $-nv\cdot\ d\cdot\frac{p}{\sqrt{(1+pp)}}$, qui ex proprietate communi prodiit, aequalis positus dabit pro curva quaesita istam. aequationem

$$\alpha d\cdot\frac{p}{\sqrt{(1+pp)}}=\frac{Bxdx-Adx}{BB}.$$

Sit $\frac{A}{B}=h$ erit h quantitas constans determinata, quam praebet formula $\frac{\int yxdx}{\int ydx}$, si ponature ponatur $x=a$, et αB ponatur $=cc$, erit cc quantitas arbitraria. Hinc habebitur ista pro curva aequatio

$$ccd\cdot\frac{p}{\sqrt{(1+pp)}}=xdx-hdx,$$

quae integrata dat

$$\frac{2ccp}{\sqrt{(1+pp)}}=xx-2hx+bb\,;$$

ergo

$$4c^4pp=(xx-2hx+bb)^2(1+pp)\ \text{ atque }\ p=\frac{xx-2hx+bb}{\sqrt{(4c^4-(xx-2hx+bb)^2)}}=\frac{dy}{dx}.$$

Quocirca erit

$$y=\int\frac{(xx-2hx+bb)dx}{\sqrt{(4c^4-(xx-2hx+bb)^2)}},$$

ubi constantem bb pro arbitrio sive affirmativam sive negativam accipere licet. Haec autem curva Quaestioni satisfacit tantum casu, quo $x=a$; atque ut satisfaciat, litterae h is tribui debet valor, quem casu $x=a$ recipiet expressio $\frac{\int yxdx}{\int ydx}$, ex quo valor h determinabitur. Caeterum notari convenit hanc curvam esse eam, quae vulgo sub nomine Elasticae est cognita.