CHAPTER IV

CONCERNING THE USE OF THE METHOD NOW TREATED IN THE RESOLUTION OF VARIOUS KINDS OF QUESTIONS

PROPOSITION I. PROBLEM

1. *To find the equation between the two variables x and y, thus so that for a given value of x, for example on putting* $x = a$, *the formula* $\int Zdx$ *shall obtain a maximum or minimum value, with Z being a function of x, y, p, q, r etc. either determinate or indeterminate.*

SOLUTION

From whatever consideration the variables *x* and *y* may have arisen, these can be considered always as the orthogonal coordinates of a certain curve and on that account the proposed question may be recalled to this, so that a curve may be determined having the abscissa $= x$ and the applied line $= y$, if it may be applied to an abscissa of a given magnitude, for example $x = a$,to which the value $\int Zdx$ becomes the maximum or minimum of all values. But if moreover the problem may be prepared in this manner, then the solution of this problem has been treated in a satisfactory manner in the above chapters. On account of which it is required to take the differential value of the proposed $\int Zdx$, according to the method set out before, which may be appropriate for the given abscissa $x = a$, and putting this equal to nothing, it will give the desired equation between *x* and *y*, which will give rise to a maximum or minimum value for the formula $\int Zdx$, for the given abscissa $x = a$. Q. E. I.

COROLLARY l

2. Therefore the method treated previously may be extended much wider, as far as equations between the coordinates of curves are to be found, in order that some expression $\int Zdx$ becomes a maximum or a minimum. Evidently it extends to any two variables, either these that may pertain to some curve of any kind, or that may be present only in an analytical abstraction.

COROLLARY 2

3. But a huge distinction is present between the two proposed variables, by which the proposed formula $\int Zdx$ must obtain a maximum or minimum, from a certain value of the other variable. Therefore this same variable is agreed to be denoted always by the letter *x*, the other truly by the letter *y*.

COROLLARY3

4. Therefore with the letters *x* and *y* assigned in the same due manner for the two variable magnitudes, there will be $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$, $s = \frac{dr}{dx}$ etc. From these letters evidently differentials of any order will be introduced, which perhaps may be present in the formula of the maximum or minimum, thus so that *Z* may become a function of the letters *x, y, p, q, r* etc.

COROLLARY 4

5. Therefore when the formula $\int Zdx$ has been reduced to such a form of the maximum or minimum, in which *Z* shall be a function of *x, y, p, q, r* etc. either definite or indefinite, then from the above precepts of the formula $\int Zdx$ the value of the differential must be investigated, corresponding to the whole of the proposed abscissa $x = a$, which put equal to zero will provided the equation sought between *x* and *y*.

COROLLARY 5

6. If *Z* is a definite function of *x, y, p, q, r* etc., then the value of the differential of the formula $\int Zdx$ will not depend on the prescribed value of the abscissa $x = a$; and on this account the equation found between *x* and *y* for any abscissa will provide the maximum or minimum of the formula $\int Zdx$.

SCHOLIUM 1

7. Because it is required to make the differential values apparent in this treatment, which previously we have elicited set out individually for any kind of formula, here we will produce these to be seen jointly, so that the differential values, for which there shall be a need in any case considered, can be set out and investigated. Therefore we will show the differential values of the formula $\int Zdx$ for various kinds of the function *Z*, which may correspond always to the magnitude of the variable quantity *x*, considered to be $x = a$.

I. MAXIMUM OR MINIMUM FORMULA
$\int Zdx$

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc.}$$

The value of the differential will be

$$nv \cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}),$$

which value of the differential prevails for any magnitude of the variable *x*.

II. MAXIMUM OR MINIMUM FORMULA
$\int Zdx$

$$dZ = Ld\varPi + Mdx + Ndy + Pdp + Qdq + \text{ etc. and } \varPi = \int [Z]dx$$

with

$$d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.}$$

Now after integration on putting $x = a$ there shall be $\int Ldx = H$, and there may be put $H - \int Ldx = V$.

The value of the differential will be

$$nv \cdot dx\left(N + [N]V - \frac{d \cdot (P + [P]V)}{dx} + \frac{dd \cdot (Q + [Q]V)}{dx^2} - \frac{d^3 \cdot (R + [R]V)}{dx^3} + \frac{d^4 \cdot (S + [S]V)}{dx^4} - \text{etc.}\right)$$

III. MAXIMUM OR MINIMUM FORMULA
$\int Zdx$

$$\begin{aligned} dZ &= Ld\varPi + Mdx + Ndy + Pdp + Qdq + \text{etc. and } \varPi = \int [Z]dx \\ d[Z] &= [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc. and } \pi = \int [z]dx \\ d[z] &= [m]dx + [n]dy + [p]dp + [q]dq + [r]dr + \text{etc.} \end{aligned}$$

Again there shall be, on putting $x = a$ after integration as before, $\int Ldx = H$ and there may be put $H - \int Ldx = V$. Now $\int [L]Vdx$ may be integrated and in this case the integral in this case will be $= G$, when there is put $x = a$, and here may be put

$$G - \int [L]Vdx = [V] = G - \int [L]dx(H - \int Ldx).$$

With these in place the value of the differential will be

$$nv\cdot dx\left(\begin{array}{l} N+[N]V+[n]V-\dfrac{d\cdot(P+[P]V+[p]V)}{dx}+\dfrac{dd\cdot(Q+[Q]V+[q]V)}{dx^2} \\ -\dfrac{d^3\cdot(R+[R]V+[r]V)}{dx^3}+\dfrac{d^4\cdot(S+[S]V+[s]V)}{dx^4}-\text{etc.} \end{array}\right)$$

from which likewise the law of the progression is apparent, if more integrals may be involved at this stage.

IV. MAXIMUM OR MINIMUM FORMULA

$$\int Zdx$$

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc.} \text{ and } \Pi = \int Zdx.$$

On putting $x=a$, this expression $e^{\int Ldx}$ may be changed by dividing into *H*, with *e* denoting the base of natural logarithms, the logarithm of which is $=1$, and there shall be $He^{-\int Ldx}=V$, the value of the differential will become

$$nv\cdot dx\left(NV-\frac{d\cdot PV}{dx}+\frac{dd\cdot QV}{dx^2}-\frac{d^3\cdot RV}{dx^3}+\frac{d^4\cdot SV}{dx^4}-\text{etc.}\right).$$

V. MAXIMUM OR MINIMUM FORMULA

$$\int Zdx$$

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc. and } \Pi = \int [Z]dx$$
$$d[Z]=[L]d\Pi+[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+\text{ etc.}$$

There shall be after integration, if there is put $x=a$,

$$\int e^{\int [L]dx}Ldx = H$$

and there may be put

$$e^{-\int [L]dx}\left(H-\int e^{\int [L]dx}Ldx\right)=V.$$

The value of the differential will be

$$nv\cdot dx\left(\begin{array}{l} N+[N]V-\dfrac{d\cdot(P+[P]V)}{dx}+\dfrac{dd\cdot(Q+[Q]V)}{dx^2} \\ -\dfrac{d^3\cdot(R+[R]V)}{dx^3}+\dfrac{d^4\cdot(S+[S]V)}{dx^4}-\text{etc.}\end{array}\right)$$

Therefore all the rules will be contained in these five cases, which we have found in the preceding chapters. And these extend so widely, so that all cases which indeed may be able to occur, actually either may be contained in these cases or perhaps shall be able to be resolved by these without difficulty. Therefore from these summarized here, we will show the use of these in resolving questions, in which *x* and *y* do not denote orthogonal coordinates.

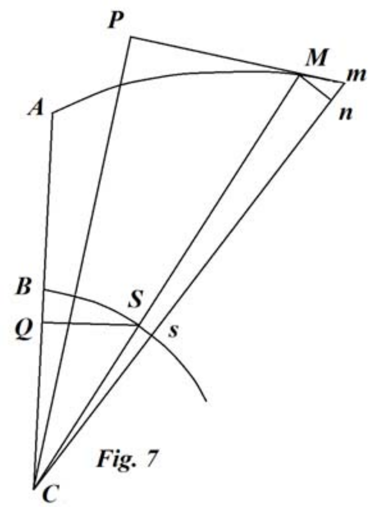

Fig. 7

EXAMPLE I

8. *From the given centre C* (Fig. 7) *with the radii CA, CM drawn, to find the line AM, which shall be the shortest among all the other lines drawn contained within the angle ACM.*

Indeed it is apparent this line sought is a straight line ; yet meanwhile this question may be agreed to be resolved following the given precepts, so that the agreement of the method with the truth may be seen more clearly. Therefore since the length of the line *AM* must be a minimum for the given angle *ACM*, we may put this angle *ACM* to be $=x$; or with the centre *C*, with the radius $CB=1$, we may describe a circle, the arc of which shall be $BS=x$. Then the other radius *CM* shall be the variable *y*; for with the equation found between these variables *x* and *y* the nature of the line sought *AM* will become known. But now with a nearby radius drawn *Cm* there will be $Ss=dx$ and $mn=dy$, on taking $Cn=CM$; truly on account of the similar triangles *CSs* and *CMn* there will be

$$1:dx=CM[y]:Mn[ydx].$$

And thus from these there will be $Mm=\sqrt{(dy^2+y^2dx^2)}$; and because always we have put $dy=pdx$, there will be $Mm=dx\sqrt{(yy+pp)}$; from which the length of the line *AM* will be $=\int dx\sqrt{(yy+pp)}$, which must become a minimum for a given value of *x*, for example $x=a$. But because this formula belongs to the first case, the satisfying line will be a minima for some value of *x*. Therefore since there shall be $Z=\sqrt{(yy+pp)}$, there will be

$$dZ=\frac{ydy}{\sqrt{(yy+pp)}}+\frac{pdp}{\sqrt{(yy+pp)}}$$

and in the first case there becomes

$$M = 0,\ N = \frac{y}{\sqrt{(yy + pp)}},\ P = \frac{p}{\sqrt{(yy + pp)}},\ Q = 0,\ R = 0 \text{ etc.}$$

and thus $dZ = Ndy + Pdp$. Therefore this value of the differential will be had

$$nv \cdot dx\left(N - \frac{dP}{dx}\right),$$

and thus for the solution this equation $0 = N - \frac{dP}{dx}$, which multiplied by $pdx = dy$, gives $Ndy = pdP$; which on substitution into the equation $dZ = Ndy + Pdp$ will produce $dZ = Pdp + pdP$ and on integrating

$$Z + C = Pp \text{ or } C + \sqrt{(yy + pp)} = \frac{pp}{\sqrt{(yy + pp)}}.$$

On account of which

$$\frac{yy}{\sqrt{(yy + pp)}} = \text{Const.} = b.$$

But there is

$$Mm\left[dx\sqrt{(yy + pp)}\right] : Mn\left[ydx\right] = MC\left[y\right] : \frac{yy}{\sqrt{(yy + pp)}};$$

which fourth proportional provides the perpendicular *CP*, because *MP* is sent from *C* into the tangent of the line sought. Therefore since this perpendicular *CP* shall be constant, it is understood the line sought to be a right line, and because in the first equation found $Ndx = dP$ two arbitrary constant powers are present, this condition of the question is to be added, that the line sought shall pass through two given points ; therefore then a right line drawn through these two points will satisfy the question.

EXAMPLE II

9. *On the axis AP* (Fig. 8) *to construct the line BM prepared thus, so that, with the abscissa area ABMP of a given magnitude, the arc of the curve BM corresponding to that area shall be the minimum of all.*

Because for the given area *ABMP* the minimum length of the arc *BM* is required, the area *ABMP* being assigned by us will be the variable *x*, but we may indicate the applied line of the curve *PM* for the other variable *y*. Now the abscissa shall be $AP = t$, there will be $x = \int ydt$ and thus $dt = \frac{dx}{y}$, and the length of the arc *BM* will be $= \int \sqrt{\left(dy^2 + \frac{dx^2}{yy} \right)}$. Therefore on putting $dy = pdx$, this formula must be a minimum :

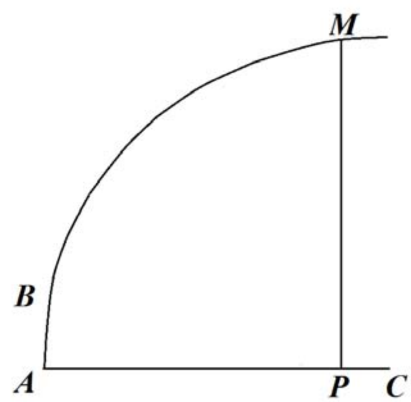

Fig. 8

$$\int dx \sqrt{\left(\frac{1}{yy} + pp \right)} = \int \frac{dx\sqrt{(1+yypp)}}{y}.$$

And thus there will be

$$Z = \frac{\sqrt{(1+yypp)}}{y} \text{ and } dZ = -\frac{dy}{yy\sqrt{(1+y^2p^2)}} + \frac{yypdp}{y\sqrt{(1+y^2p^2)}};$$

from which

$$M = 0 \text{ , } N = \frac{-1}{yy\sqrt{(1+y^2p^2)}}, \ P = \frac{yp}{\sqrt{(1+y^2p^2)}}, \ Q = 0 \text{ etc.}$$

Therefore this question belongs to the first case and the solution will present a curved line, which for some area the abscissa *APMB* will be the shortest. But this equation $Z = C + Pp$ will be come upon, as in the preceding example, and this curve sought will be able to be described by two given points. And thus there will be

$$\frac{\sqrt{(1+yypp)}}{y} = C + \frac{ypp}{\sqrt{(1+yypp)}}$$

or

$$1 = Cy\sqrt{(1+yypp)} \quad \text{or} \quad b = y\sqrt{(1+yypp)};$$

hence it becomes

$$bb = yy + y^4 pp \text{ and } p = \frac{\sqrt{(bb - yy)}}{yy} = \frac{dy}{dx} = \frac{dy}{ydt},$$

on account of $dx = ydt$. Therefore there will be

$$dt = \frac{ydy}{\sqrt{(bb - yy)}} \text{ and } t = c \pm \sqrt{(bb - yy)}.$$

Whereby the line sought will be a circle, with the centre somewhere on the axis *AP*, assumed, for example at *C*; and this among all the other curves drawn through the same two points, will have the shortest arc *BM*, for some given area cut *ABMP*.

EXAMPLE III

10. *With the radii CA, CM* (Fig.7) *drawn from some fixed point C, to describe the curve AM within these radii, which may have the shortest arc AM for a given area ACM.*

Because the arc *AM* must be the minimum, if the area *ACM* of a given magnitude may be cut off, this area may be put to be $ACM = x$ and the radius *CM* may be designated by the other variable *y*. Now the arc $BS = t$ may be put in place, described by the radius $CB = 1$; there will be, as we have seen before, $Mn = ydt$ and the area $MCm = \frac{1}{2} yydt = dx$, from which there becomes $dt = \frac{2dx}{yy}$. Because again there is

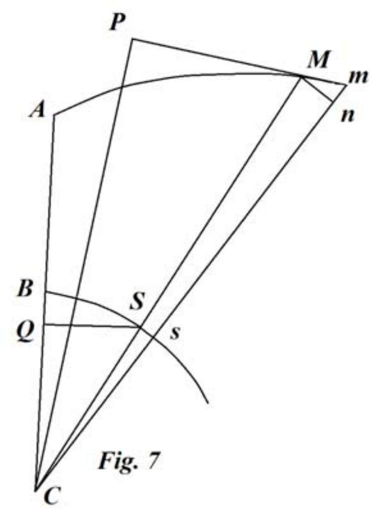

Fig. 7

$$Mm = \sqrt{(dy^2 + y^2 dt^2)} = \sqrt{(dy^2 + \frac{4dx^2}{yy})},$$

there shall be $dy = pdx$, and $\int \frac{dx}{y}\sqrt{(4 + ppyy)}$ must be a minimum. Therefore since there shall be $Z = \frac{\sqrt{(4 + y^2 p^2)}}{y}$, there becomes :

$$M = 0, \ N = -\frac{4}{yy\sqrt{(4 + y^2 p^2)}} \text{ and } P = \frac{yp}{\sqrt{(4 + y^2 p^2)}}, \ Q = 0 \text{ etc.}$$

Hence this equation results $Z = C + Pp$, because therefore there will be $M = 0$; and thus

$$\frac{\sqrt{(4+y^2p^2)}}{y}=C+\frac{ypp}{\sqrt{(4+y^2p^2)}}$$

or

$$4=Cy\sqrt{(4+yypp)} \text{ or } 2b=yy\sqrt{(4+yypp)};$$

and hence

$$p=\frac{2\sqrt{(bb-yy)}}{yy}=\frac{dy}{dx}=\frac{2dy}{yydt};$$

and $dt=\frac{dy}{\sqrt{(bb-yy)}}$; likewise on integrating:

$$t=\text{A}\sin\frac{y}{b}+\text{A}\sin\frac{c}{b}=\text{A}\sin\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}.$$

[Note that $\text{A}\sin x$ means $\arcsin x;$ while $\sin\text{A}\cdot x$ means sine arc x .]

From S to AC the perpendicular $QS=\sin At$ may be sent, and there will be

$$QS=\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}.$$

But it is deduced from the equation $t+\text{Const.}=\text{A}\sin\frac{y}{b}$ that the curve sought (Fig. 9) is the circle *AME* passing through the fixed point *C* . For the circle *CAME* may be described on some diameter *CE* ending in *C*, the arc *AM* intercepted between the radii *ACM* will be a minimum for the given area *ACM*. Evidently if some other curve may be described through any two points situated on this circle, and with the two radii drawn from *C* an area equal to the area *ACM* may be cut off, the corresponding arc of that curve will be greater always than the arc *AM*. So that which may be apparent, the normal *CD* may be drawn from *C* to *CE* and to that from *S* the perpendicular *SQ* may be sent; the triangle *SCQ* will be similar to the triangle *CEM* and hence

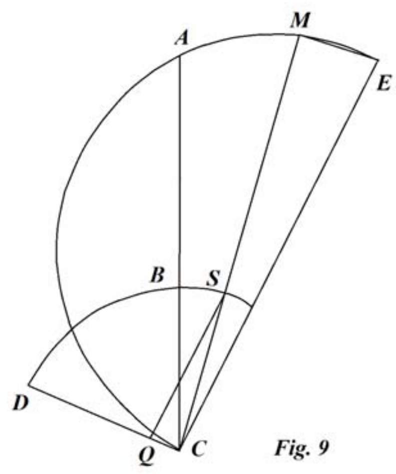

Fig. 9

$$CE:CM[y]=\text{CS}[1]:SQ \text{ or } SQ=\frac{y}{CE}=\sin\text{A}\cdot DBS,\text{or } DBS=\text{A}\sin\frac{y}{CE}.$$

Therefore on putting the diameter $CE=b$ and because there is

$$DBS=BS+BD=t+\text{Const.},$$

there will be $t+\text{Const.}=\text{A}\sin\frac{y}{b}$, which is that property itself we have found, that the curve sought is required to have.

EXAMPLE IV

11. *To draw a line on some curved surface* (Fig. 10) *either convex or concave, which shall be the shortest of all between its ends.*

Some plane *APQ* may be taken, to which the surface may be referred, and on that the right line *AP* may be taken for the axis. Now from the line sought perpendiculars may be considered to be sent from the individual points to this plane, from which the line *AQ* may be described, which will be the shortest projection into this plane; with which known, likewise the shortest line itself on the surface will become known. The coordinates may be called $AP=x, PQ=y$; and since the nature of the surface may be given, from a given $AP=x$ and $PQ=y$ it will be possible to define the length of the perpendicular *QM* to the plane *APQ*, that the surface will cut at *M*. So that therefore if there may be put $QM=z$, the length of this line *z* will be given by *x* and *y*, thus so that *z* will be a definite function of *x* and *y*. Therefore since *z* shall be a function of *x* and *y*, from which equation for the surface may be given nearby, we may put it to be $dz=Tdx+Vdy$; and both *T* and *V* will be functions of this *x* and *y*, so that $Tdx+Vdy$ shall be a definite differential formula ; clearly on putting $dT=Edx+Fdy$ there will be $dV=Fdx+Gdy$, with the letter *F* a common differential for each. Now an element of the line drawn on the surface is

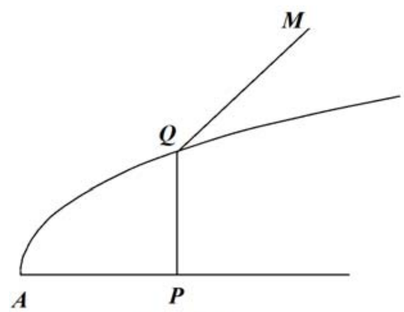

Fig. 10

$$=\sqrt{(dx^2+dy^2+dz^2)}=\sqrt{(dx^2+dy^2+(Tdx+Vdy)^2)}.$$

Therefore on putting $dy=pdx$ this formula must be a minimum

$$\int dx\sqrt{(1+p^2+T^2+2TVp+V^2p^2)}\ ,$$

thus so that there shall be

$$Z=\sqrt{(1+p^2+T^2+2TVp+V^2p^2)},$$

from which the equation arises

$$dZ = \frac{\begin{pmatrix} +TEdx & +TFdy & +pdp & +VEpdx & +VFpdy + TVdp \\ +TFpdx + TGpdy + V^2pdp + VFp^2dx + VGppdy \end{pmatrix}}{\sqrt{(1+p^2+T^2+2TVp+V^2p^2)}}$$

Which formula since it applies to the first case, this equation between x and y will arise :

[*i.e.* from $nv \cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.})$]

$$\frac{TFdx + VFpdx + TGpdx + VGppdx}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}} = d.\frac{p+TV+V^2p}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}}.$$

Truly there is

$$Fdx + Gpdx = Fdx + Gdy = dV ,$$

from which there will be

$$\frac{TdV + VpdV}{\sqrt{(1+pp+(T+Vp)^2)}} = d.\frac{p+TV+V^2p}{\sqrt{(1+pp+(T+Vp)^2)}}$$

$$= \frac{+dp(1+T^2+V^2) + dT(V-Tp) + dV(T+T^2+3T^2Vp+3TV^2p^2+V^3p^3+2Vp+Vp^3)}{(1+pp+(T+Vp)^2)^{3:2}}.$$

$$\left[i.e. \frac{TdV + VpdV}{\sqrt{(1+pp+(T+Vp)^2)}} = d.\frac{p+TV+V^2p}{\sqrt{(1+pp+(T+Vp)^2)}} \right.$$

$$= \frac{dp + VdT + TdV + V^2dp + 2pVdV}{\sqrt{(1+pp+(T+Vp)^2)}} - \frac{(p+TV+V^2p)(2pdp+2TdT+2VpdT+2TpdV+2TVdp+2VpdV+2V^2pdp)}{2\sqrt[3]{\sqrt{(1+pp+(T+Vp)^2)}}}$$

$$\left. = \frac{(1+pp+(T+Vp)^2)(dp+VdT+TdV+V^2dp+2pVdV) - (p+TV+V^2p)(pdp+TdT+VpdT+TpdV+TVdp+VpdV+V^2pdp)}{\sqrt[3]{\sqrt{(1+pp+(T+Vp)^2)}}} \; etc. \right]$$

Moreover it will result in this ordered equation

$$dp(1+T^2+V^2) + dT(V-Tp) + dV(Vp-Tpp) = 0$$

or

$$dp = \frac{(Tp-V)(dT+pdV)}{1+T^2+V^2}.$$

Truly since $p=\frac{ddy}{dx}$, there will be $dp=\frac{ddy}{dx}$; and hence there arises :

$$dxddy=\frac{(Tdy-Vdx)(dxdT+dydV)}{1+T^2+V^2},$$

which is a second order differential equation for the projection *AQ* of the shortest line on the surface sought and thus it will indicate that it can to be drawn through any two points. This equation found can be transformed into various forms, which will be able to be used more often with greater convenience. And in the first place it will be expedient to eliminate the differentials *dT* and *dV;* for since there shall be $dz=Tdx+Vdy$, there will be

$$ddz=dxdT+dydV+Vddy \text{ and thus } dxdT+dydV=ddz-Vddy,$$

with which value substituted this equation will be produced :

$$dxddy+T^2dxddy+V^2dxddy=Tdyddz-Vdxddz-TVdyddy+V^2dxddy$$

or

$$dxddy+Tdzddy=Tdyddz-Vdxddz$$

and hence

$$ddy:ddz=Tdy-Vdx:dx+Tdz.$$

The equation found may be multiplied by *dz* and in the first term there may be written $Tdx+Vdy$ in place of *dz*, and there will be

$$Tdx^2ddy+Vdxdyddy+Tdz^2ddy=Tdydzddz-Vdxdzddz.$$

$Tdy^2ddy-Vdxdyddy$ may be added to each side, and there will be

$$Tddy(dx^2+dy^2+dz^2)=(dzddz+dyddy)(Tdy-Vdx)$$

or

$$\frac{dyddy+dzddz}{dx^2+dy^2+dz^2}=\frac{Tddy}{Tdy-Vdx}=\frac{Tddz}{dx+Tdz}.$$

Or the equation may be multiplied by *dx* and $dz-Vdy$ may be written in place of *Tdx*, giving

$$dx^2ddy+dz^2ddy-Vdydzddy=dydzddz-Vdy^2ddz-Vdx^2ddz.$$

Again, $dy^2ddy-Vdz^2ddz$ may be added to each side, there will be

$$ddy(dx^2+dy^2+dz^2)-Vdz(dyddy+dzddz)=dy(dyddy+dzddz)-Vddz(dx^2+dy^2+dz^2)$$

and thus

$$\frac{dyddy+dzddz}{dx^2+dy^2+dz^2}=\frac{ddy+Vddz}{dy+Vdz};$$

which equations are all contained in the following expression:

$$\frac{dyddy+dzddz}{dx^2+dy^2+dz^2}=\frac{Tddy}{Tdy-Vdx}=\frac{Tddz}{dx+Tdz}=\frac{ddy+Vddz}{dy+Vdz}.$$

Here it is to be observed, because the differentials of the quantities T and V occur nowhere, likewise it shall be, z may be contained in T and V or less. Therefore in whatever case presented, it will be convenient to assume that equation, which allows the integration most easily. Just as if the surface proposed shall be a solid of revolution arising from the rotation of any figure about the axis AP, $yy+zz$ will be equal to the square of a function of x, which shall be $=X$, and the applied line of this curve generated corresponding to the abscissa x. And thus there will be

$$zdz=XdX-ydy \text{ et } dz=\frac{XdX}{z}-\frac{ydy}{z},$$

from which there becomes

$$T=\frac{XdX}{zdx} \text{ and } V=\frac{-y}{z}.$$

[Recall that $dz=Tdx+Vdy$.]

Hence this suitable equation may be taken now, in which T does not occur,

$$\frac{dyddy+dzddz}{dx^2+dy^2+dz^2}=\frac{ddy+Vddz}{dy+Vdz},$$

which on account of $V=-\frac{y}{z}$ will be changed into this :

$$\frac{dyddy+dzddz}{dx^2+dy^2+dz^2}=\frac{zddy-yddz}{zdy-ydz},$$

of which the integral is

$$l\sqrt{(dx^2+dy^2+dz^2)}=l(\frac{zdy-ydz}{b}), \text{ or } zdy-ydz=b\sqrt{(dx^2+dy^2+dz^2)}.$$

[b is the integration constant.]

Because now there is $z = \sqrt{(X^2 - y^2)}$), putting $dX = vdx$, there will be

$$dz = \frac{Xvdx - ydy}{\sqrt{(X^2 - y^2)}} \quad \text{and} \quad zdy - ydz = \frac{X^2dy - Xyvdx}{\sqrt{(X^2 - y^2)}}$$

and

$$\sqrt{(dx^2 + dy^2 + dz^2)} = \frac{\sqrt{(X^2dx^2 - y^2dx^2 + X^2dy^2 + X^2v^2dx^2 - 2Xyvdxdy)}}{\sqrt{(X^2 - y^2)}}.$$

Hence

$$X^4dy^2 - 2X^3yvdxdy + X^2y^2v^2dx^2$$
$$= b^2X^2dx^2 - b^2y^2dx^2 + b^2X^2dy^2 + b^2X^2v^2dx^2 - 2b^2Xyvdxdy$$

or

$$dy^2 = \frac{2(b^2 - X^2)Xyvdxdy + X^2y^2v^2dx^2 - b^2X^2dx^2 + b^2y^2dx^2 - b^2X^2v^2dx}{X^2(bb - XX)},$$

which with the root extracted provides

$$dy = \frac{yvdx}{X} \pm \frac{bdx\sqrt{(1 + vv)(yy - XX)}}{X\sqrt{(bb - XX)}}.$$

So that if there is put $y = Xt$, so that there shall be $dy = Xdt + tvdx$, there becomes

$$\frac{dt}{\sqrt{(tt - 1)}} = \frac{bdx\sqrt{(1 + vv)}}{X\sqrt{(bb - XX)}} \ ;$$

in which equation, because *X* and *v* are functions of *x*, the variables *t* and *x* in turn are separated from each other.

EXAMPLE V

12. *To construct a curve AM of this kind on the axis APN* (Fig. 11)*, so that the abscissa by the normal MN area ANM shall be of a given magnitude, the arc AM shall be a minimum.*

Because for a definite magnitude of the area *AMN* the arc *AM* must be a minimum, the area may be put to be $AMN = ax$, and on putting $x = a$, in which case the area *AMN* shall be $= aa$, the arc *AM* may become a minimum. Again the orthogonal applied line shall be $MP = y$, the abscissa $AP = t$, and the subnormal $PN = u$; the area will be

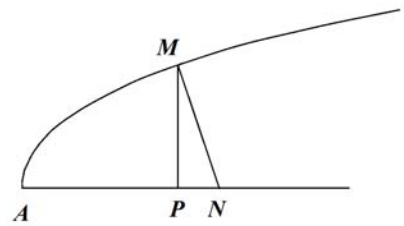

Fig. 11

$$ax = \int ydt + \tfrac{1}{2}uy \text{ and } u = \frac{ydy}{dt};$$

truly an element of the arc *AM* will be $= \dfrac{dy\sqrt{(yy+uu)}}{u}$. Again since there shall be

$$adx = ydt + \tfrac{1}{2}(udy + ydu) \text{ and } dt = \frac{ydy}{u},$$

there will be

$$audx = yydy + \tfrac{1}{2}uudy + \tfrac{1}{2}yudu \text{ and } du = \frac{2adx}{y} - \frac{2ydy}{u} - \frac{udy}{y}.$$

Now putting $dy = pdx$, the integral

$$\int \frac{pdx\sqrt{(yy+uu)}}{u}$$

must be a minimum, and *u* is a magnitude, the value of which must be defined from this equation

$$du = dx\left(\frac{2a}{y} - \frac{2yp}{u} - \frac{up}{y}\right).$$

And thus this equation belongs to the fifth case; since from which if a comparison may be put in place, there shall be $u = \Pi$ and $Z = \dfrac{p\sqrt{(yy+\Pi^2)}}{\Pi}$;

from which

$$L=\frac{-pyy}{\Pi^2\sqrt{(yy+\Pi^2)}},\ M=0,\ N=\frac{yp}{\Pi\sqrt{(yy+\Pi^2)}}\ \text{and}\ P=\frac{\sqrt{(yy+\Pi\Pi)}}{\Pi}.$$

Then since there shall be $\Pi=\int dx\left(\frac{2a}{y}-\frac{2yp}{\Pi}-\frac{\Pi p}{y}\right)$, there will become

$$[Z]=\frac{2a}{y}-\frac{2yp}{\Pi}-\frac{\Pi p}{y}$$

and on differentiation there will be

$$[L]=\frac{2yp}{\Pi^2}-\frac{p}{y},\ [M]=0,\ [N]=-\frac{2a}{yy}-\frac{2p}{\Pi}+\frac{\Pi p}{yy}\ \text{and}\ [P]=-\frac{2y}{\Pi}-\frac{\Pi}{y}.$$

Now there will be

$$\int[L]dx=\int\frac{2ydy}{\Pi^2}-ly\ \text{and}\ e^{\int[L]dx}=\frac{e^{\int 2ydy:\Pi\Pi}}{y};$$

but there is $Ldx=-\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}$; from which there becomes

$$\int e^{\int[L]dx}Ldx=-\int\frac{e^{\int 2ydy:\Pi\Pi}ydy}{\Pi^2\sqrt{(yy+\Pi^2)}},$$

the value of which on putting $x=a$ becomes $=H$, and there shall be

$$V=e^{-\int 2ydy:\Pi\Pi}y\left(H+\int\frac{e^{\int 2ydy:\Pi\Pi}ydy}{\Pi^2\sqrt{(yy+\Pi^2)}}\right).$$

With these preparations being satisfied there will be the equation

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi\Pi)}}-\frac{2aVdx}{yy}-\frac{2Vdy}{\Pi}+\frac{\Pi Vdy}{yy}=d\cdot\left(\frac{\sqrt{(yy+\Pi\Pi)}}{\Pi}-\frac{2Vy}{\Pi}-\frac{\Pi V}{y}\right).$$

But there is

$$2adx=yd\Pi+\frac{2yydy}{\Pi}+\Pi dy;$$

from which there becomes

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}}-\frac{Vd\Pi}{y}-\frac{4Vdy}{\Pi}=d\cdot\left(\frac{\sqrt{(yy+\Pi^2)}}{\Pi}-\frac{2Vy}{\Pi}-\frac{\Pi V}{y}\right)=$$

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}}-\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vdy}{\Pi}-\frac{2ydV}{\Pi}+\frac{2Vyd\Pi}{\Pi^2}-\frac{\Pi dV}{y}-\frac{Vd\Pi}{y}+\frac{\Pi Vdy}{yy};$$

and hence

$$\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vdy}{\Pi}+\frac{2ydV}{\Pi}-\frac{2Vyd\Pi}{\Pi^2}+\frac{\Pi dV}{y}-\frac{\Pi Vdy}{yy}=0.$$

Truly more generally there is $dV=-Ldx-V\left[L\right]dx$; from which there will be

$$dV=\frac{yydy}{\Pi^2\sqrt{(yy+\Pi\Pi)}}-\frac{2Vydy}{\Pi^2}+\frac{Vdy}{y}$$

and hence

$$\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi\Pi)}}=\frac{dVd\Pi}{dy}+\frac{2Vyd\Pi}{\Pi^2}-\frac{Vd\Pi}{y};$$

with which substituted the equation arises :

$$\frac{dVd\Pi}{dy}-\frac{2Vdy}{\Pi}+\frac{2ydV}{\Pi}+\frac{\Pi dV}{y}-\frac{Vd\Pi}{y}-\frac{\Pi Vdy}{yy}=0;$$

that is

$$dV\left(\frac{d\Pi}{dy}+\frac{2y}{\Pi}+\frac{\Pi}{y}\right)=V\left(\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}\right)=\frac{ydV}{dy}\left(\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}\right);$$

which equation, since it shall be divisible by $\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}$, gives a two-fold solution. The first of which will be $\frac{dV}{V}=\frac{dy}{y}$, which provides $V=cy$; because truly V must vanish in the case of a minimum, by the same case there will be $y=0$; evidently on putting $x=a$ it becomes $y=0$. Now since there shall be $V=cy$, with the substitution made in the equation

$$dV=\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vydy}{\Pi^2}+\frac{Vdy}{y}$$

there will be

$$\frac{yydy}{\Pi^2\sqrt{(y^2+\Pi^2)}}=\frac{2cyydy}{\Pi^2};$$

and hence either $y=0$ or $dy=0$, in which case the equation will produce a right line parallel to the axis, corresponding to $\Pi=\infty$, [while the case $\Pi=0$] produces a right line normal to the axis, or also $\sqrt{y^2+\Pi\Pi}=MN=\text{Const.}$, which equation gives a circle, and a whole semicircle, on account of $y=0$ in the case of a minimum, will satisfy the question. The second solution will be produced from the divisor

$$\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}=0 \text{ or } \Pi d\Pi+\frac{\Pi\Pi dy}{y}+2ydy=0,$$

which multiplied by yy becomes $yy\Pi d\Pi+\Pi\Pi ydy+2y^3dy=0$, of which the integral is $\Pi^2y^2+y^4=C$, and hence $\Pi=\frac{\sqrt{(b^4-y^4)}}{y}$; which equation, because it does not depend on *V*, may be satisfied by any value of *x*. But with the abscissa introduced $AP=t$ on account of $u=\Pi=\frac{ydy}{dt}$ this equation

$$\frac{ydy}{dt}=\frac{\sqrt{(b^4-y^4)}}{y}, \text{ from which } dt=\frac{yydy}{\sqrt{(b^4-y^4)}},$$

from which equation it is understood that an elastic rectangle satisfies the question ; thus so that for the area *ANM* between the normals *AN* and *MN*, the arc of the curve *AM* shall be the shortest. Moreover this curve through two given points, if indeed the axis *AP* shall be given in position, can be described.

SCHOLIUM 2

13. From these examples we have examined, the use which our method has in resolving problems is made abundantly clear also of diverse kinds; but the final example supplies circumstances worthy of the greatest respect, from which the nature of the solution can be illustrated. Indeed because the equation has arisen on account of two factors, the solution also will be two-fold ; the first satisfying line can be determined absolutely, thus so that cannot be drawn through two given points ; for it gives either a right line or a semicircle. The right line in the second way resolves the question, in as far as it is either perpendicular to the axis *AP* or parallel to the same ; and just as it is evident it may satisfy both : for in that , which is normal to the axis, the part is thus actually infinitely small and thus minimal, with the axis and the normal given defining a given space ; the other right line parallel to the axis appears a little more generality, since it may be drawn

through two points ; and because the applied lines to that themselves are normal and the interval of the abscissas shall be as the abscissas themselves, any right line will be the shortest with respect to that line. Then the semicircle, which is produced from the first solution, thus is satisfied completely, so that from the proposed magnitude of the interval being cut off, the semicircle itself will be determined, for the area of this must be $= aa$. But the second solution, which has produced a rectangular elastic curve, appears more general ; for through any two curves of this kind a curve can be drawn, and among all the other curves passing through the same points that one will be endowed with the right before the rest, so that, in all the curves equal areas may be cut off by the normals, the elastic arc shall become the minimum of all. Therefore with these established we may go on to the use of the method being treated in these investigations of maxima or minima, in which the formula of the maximum or minimum is not of such a simple integral expression $\int Zdx$, such a form as we have treated always until now, truly just as it is composed from two or more formulas of this kind. And certainly in the first place, if the maximum or minimum must be the sum of two or more integral formulas, for example $\int Zdx + \int Ydx - \int Xdx$, the operation will labour under no difficulty ; because indeed the formula of the maximum or minimum is $\int dx(Z + Y - X)$, this can be treated as a simple formula of the integral and the value of the differential can be assigned. But the operation will be compared with that, so that for the individual formulas $\int Zdx$, $\int Ydx$ and $\int Xdx$ the differential values of these may be sought, and may be substituted in place of these in the formula $\int Zdx + \int Ydx - \int Xdx$ and, what may arise may be put equal to nothing ; and thus the equation will be had satisfying the question sought.

PROPOSITION II. PROBLEM

14.*To find the equation between x and y, so that on putting $x = a$ this expression becomes $\int Zdx \cdot \int Ydx$, which is the maximum or minimum product from the two integral formulas* $\int Zdx$ and $\int Ydx$.

SOLUTION

Now we may consider this equation itself between *x* and *y* to be found, and from that the value of the formula from that on putting $x = a$ to become $\int Zdx = A$ and $\int Ydx = B$; these quantities *A* and *B* will be constants and the product of these *AB* will be a maximum or minimum. Now at the place of the indefinite *x* the variable *y* may be considered to increase by a small amount *nv*, from that each quantity *A* and *B* may take an increment, evidently each will be increased by a value of the differential requiring to be defined from the preceding. Therefore *dA* shall be the value of the differential of *A*, which will correspond to the integral of the formula $\int Zdx$ on putting $x = a$, and in a similar manner *dB* shall be the value of the differential of *B* arising from the formula $\int Ydx$ on putting $x = a$. Therefore since from adjoining the small amount *nv* to the variable *y*, it may be changed from *A* into $A + dA$ and B into $B + dB$, the product *AB* will be changed

into $AB + AdB + BdA + dAdB$; whereby, since *AB* must become a maximum or minimum, it is necessary that $AB = AB + AdB + BdA + dAdB$. And thus $0 = AdB + BdA$, of account of the term *dAdB* vanishing before the rest. And thus from these the solution of the following problem arises. The value of the differential of the formula $\int Zdx$ may be sought, which shall be *dA*, and thus the value *A* of the formula $\int Zdx$, which it obtains on putting $x = a$. Then the value of the differential of the formula $\int Ydx$ is sought, which shall be *dB*, and *B* will denote the value of the formula $\int Ydx$, which it receives on putting $x = a$; with which done this equation will be had $0 = AdB + BdA$, in which the satisfying relation between *x* and *y* will be contained.
Q. E. I.

COROLLARY 1

15. Although the constant quantities *A* and *B* are present in the equation $0 = AdB + BdA$, yet these are not arbitrary, but each may be defined by that equation itself. Evidently, if the values $\int Zdx$ and $\int Ydx$ may be elicited from this equation and there may be put $x = a$, these magnitudes *A* and *B* must be produced *;* from which these are determined by *a* and by the remaining arbitrary constants, which will be introduced by integration.

COROLLARY 2

16. If *Z* and *Y* were determinate functions of the magnitudes *x, y, p, q, r* etc., then the differential values *dA* and *dB* will not depend on *a;* yet meanwhile the magnitude *a* is introduced into the equation $0 = AdB + BdA$; from which the curve sought finally will be satisfied for a definite value of the abscissa *x* with the value $x = a$.

COROLLARY 3

17. But from the equation $0 = AdB + BdA$ the small amount *nv* is removed entirely, for, because each value of the differential *dA* and *dB* has been produced multiplied by *nv*, again *nv* will be removed by division and in this manner an equation between *x* , *y* and constants will arise, by which the problem may be satisfied.

SCHOLIUM 1

18. No one in the present circumstances should be offended by the form of the equation found $0 = AdB + BdA$, because from that a kind of definite differential formula itself may present itself, and hence also nor may any integral of the equation $0 = AdB + BdA$ be able to be taken containing this : Const. = *AB.* For now we have explained the meanings which we have attributed both to the letters *A* and *B* as well as to the forms of the differentials *dA* and *dB;* from which one may understand that the common way of writing cannot be used here. But thus this manner of writing, even if it differs from the ordinary, here has been considered to be applied, so that the connection of the equation $0 = AdB + BdA$ with the formula $\int Zdx \cdot \int Ydx$ of the maximum or minimum may be understood more clearly. For since the maximum or minimum must correspond to the value $x = a$, in this case we may put $\int Zdx$ into *A* and $\int Ydx$ into *B;* with which done, the

maximum or minimum will be *AB*. But hence at once the equation found arises $0 = AdB + BdA$, if indeed *AB* may be differentiated with the letters *A* and *B* regarded just as variables. Because when it has been done, it will be necessary to call to mind for the differentials being taken *dA* and *dB* to be these differential values, which agree with the integral formulas $\int Zdx$ and $\int Ydx$, from which the constant magnitudes *A* and *B* appear. Thus this tying together will help to record, as below the same will be apparent for whatever manner of composition, by which a maximum or minimum formula were composed from integrable formulas; and in a like manner from the expression itself of the maxima or minima we will show by differentiation how to obtain the equation sought.

EXEMPLUM I

19. *To find the equation between x and y, so that on putting* $x = a$ *this expression* $\int ydx \cdot \int xdy$ *becomes a maximum.*

There becomes $\int ydx = A$ and $\int xdy = B$ on putting $x = a$ and the differential values of the formulas $\int ydx$ and $\int xdy$ or $\int xpdx$ are sought; and the differential value of the formula $\int ydx$ is $nv \cdot dx \cdot 1$, moreover the differential of the formula $\int xdy$ or $\int xpdx$ is

$$nv \cdot dx(-\frac{d}{dx} d \cdot x) = -nv \cdot dx.$$

Therefore there will be

$$dA = nv \cdot dx \text{ and } dB = -nv \cdot dx,$$

[Following case I, $dZ = Mdx + Ndy + Pdp + Qdq + Rdr +$ etc.: for the function *A*, we have $Z = y$; $dZ = dy$; hence $M = 0; N = 1; P = 0$ etc.; while for the function *B*, $Z = xp$; $dZ = pdx + xdp$; $N = 0$; $P = x$; $Q = 0$ etc., while the value of the differential in each case is given by $nv \cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.})$]

from which the equation $0 = AdB + BdA$ will be changed into this

$$0 = -A \cdot nv \cdot dx + B \cdot nv \cdot dx \text{ or } A = B.$$

Therefore all the equations between equally satisfy the question, provided in the case $x = a$ there were $\int ydx = \int xdy$, that is the area of the curve $= \frac{1}{2}xy$.

EXAMPLE II

20. *To find the equation between x and y, so that in the case* $x = a$ *this expression* $\int ydx \cdot \int dx\sqrt{(1+pp)}$ *becomes a minimum.*

In the case $x=a$ there becomes $\int ydx = A$ and $\int dx\sqrt{(1+pp)} = B$. Again with the values of the differential taken there will be

$$dA = nv\cdot dx\cdot 1 \text{ and } dB = nv\cdot dx\left(-\frac{1}{dx}d\cdot\frac{p}{\sqrt{(1+pp)}}\right) = -nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}.$$

[Again following case I, $dZ = Mdx + Ndy + Pdp + Qdq + Rdr +$ etc.: for the function *A*, we have $Z = y$; $dZ = dy$; hence $M=0; N=1; P=0$ etc.; while for the second case, $Z=\sqrt{(1+pp)}$; $dZ = \frac{pdp}{\sqrt{(1+pp)}}$; $M = N = 0$; $P = \frac{p}{\sqrt{(1+pp)}}$; $Q = 0$ etc., while the value of the differential in each case is given by $nv\cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.})$]

Hence the following equation appears

$$0 = -A\cdot nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}} + B\cdot nv\cdot dx \text{ or } Bdx = Ad\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Which equation integrated gives

$$x+b = \frac{Ap}{B\sqrt{(1+pp)}}$$

where $\frac{A}{B}$ denotes the ratio, which $\int ydx$ then maintains to $\int dx\sqrt{(1+pp)}$, when there shall be $x=a$. For the sake of brevity let $\frac{A}{B}=c$, there will be

$$(x+b)\sqrt{(1+pp)} = cp \text{ and } p = \frac{x+b}{\sqrt{\left(cc-(x+b)^2\right)}} = \frac{dy}{dx}.$$

Therefore from this equation integrated there will result $y = f \pm\sqrt{\left(cc-(x+b)^2\right)}$, thus so that there shall be $(y-f)^2 + (x+b)^2 = c^2$, from which it is apparent the satisfying curve to be a circle described with radius *c*, with the axes taken generally. Truly not any arc may be satisfactory to a circle of this kind, truly only this, which multiplied by the radius *c* of the circle will produce an area; that is indeed $A = Bc$. Therefore either the radius of the circle *c* for argument's sake can be taken and from there that magnitude of the abscissa *x* determined by the magnitude *a ;* or, if *a* may be given, as we have considered, thence in turn the radius *c* will be determined. But it is evident the arc of the circle which is satisfactory, must be convex with respect to its axis ; for in this case the area will become less and thus the product from the area by the arc will be a minimum.

EXAMPLE III

21. *To find the curve, in which for a given abscissa $x = a$ this expression may become a minimum* $\int yxdx \cdot \int xdx\sqrt{(1+pp)}$.

On putting $x = a$ there becomes $\int yxdx = A$ and $\int xdx\sqrt{(1+pp)} = B$. Moreover there will be :

$$dA = nv \cdot dx \cdot x \text{ and } dB = -nv \cdot dx \cdot \frac{1}{dx} d \cdot \frac{xp}{\sqrt{(1+pp)}};$$

[For the function *A*, we have $Z = xy$; $dZ = ydx + xdy$; hence $M = y; N = x; P = 0$ etc.; while for the second case,

$Z = x\sqrt{(1+pp)}$; $dZ = dx\sqrt{(1+pp)} + \frac{xpdp}{\sqrt{(1+pp)}}$; $M = \sqrt{(1+pp)}$; $N = 0$; $P = \frac{xp}{\sqrt{(1+pp)}}$; $Q = 0$ etc.

while the value of the differential in each case is given by

$nv \cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.})$]

from which this equation will be obtained

$$Bxdx = Ad \cdot \frac{px}{\sqrt{(1+pp)}},$$

which integrated gives

$$xx \pm bb = \frac{2Apx}{B\sqrt{(1+pp)}} = \frac{2cpx}{\sqrt{(1+pp)}}$$

on putting $\frac{A}{B} = c$. Hence

$$p = \frac{xx \pm bb}{\sqrt{\left(4ccxx - \left(xx \pm bb\right)^2\right)}} = \frac{dy}{dx}$$

and thus for the equation this equation will be had

$$y = \int \frac{xx \pm bb}{\sqrt{\left(4ccxx - \left(xx \pm bb\right)^2\right)}}.$$

From which it should be observed, if there becomes $b=0$, then the equation for the circle $y=\int\frac{xdx}{\sqrt{(4cc-xx)}}$ will arise, the radius of which shall be 2*c*.

SCHOLIUM 2

22. Likewise all these examples can be resolved also by the method now treated above ; whereby, since the same solution may be obtained by each way, it will help to show the solution by the other way by one example. Therefore we may take a third example, in which the formula of the maximum or minimum $\int yxdx\cdot\int xdx\sqrt{(1+pp)}$, by differentiation and by integrating by parts, is reduced to this form

$$\int yxdx\int xdx\sqrt{(1+pp)}+\int xdx\sqrt{(1+pp)}\int yxdx,$$

each part of which will be present in the second case set out above in paragraph 7.
[Recall for the second case : $\int Zdx$

$$dZ=Ld\Pi+Mdx+Ndy+Pdp+Qdq+\text{ etc. and }\Pi=\int[Z]dx\text{ with}$$

$$d[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+\text{etc.}$$

Now after integration on putting $x=a$ there shall be $\int Ldx=H$, and putting $H-\int Ldx=V$. The value of the differential will be

$$nv\cdot dx\left(N+[N]V-\frac{d\cdot(P+[P]V)}{dx}+\frac{dd\cdot(Q+[Q]V)}{dx^2}-\frac{d^3\cdot(R+[R]V)}{dx^3}+\frac{d^4\cdot(S+[S]V)}{dx^4}-\text{etc.}\right)]$$

And thus the value of each differential, for the sum of these put equal to $=0$ will give the equation for the curve sought. But the formula $\int yxdx\cdot\int xdx\sqrt{(1+pp)}$ when deduced from the second case will give $\Pi=\int xdx\sqrt{(1+pp)}$ and $Z=yx\Pi$; from which there becomes

$$L=yx,\ M=y\Pi,\ N=x\Pi,\ P=0\ \text{ etc.}$$

Then there will be $[Z]=x\sqrt{(1+pp)}$; and thence

$$[M]=\sqrt{(1+pp)},\ \ [N]=\ 0\ \text{ and }\ [P]=\frac{xp}{\sqrt{(1+pp)}}.$$

Again there is $\int Ldx=\int yxdx$, the value of which on putting $x=a$, generally we have put *H*, this in the solution of the example is *A*, thus so that there shall be $V=A-\int yxdx$. Whereby the value of the differential of this formula will be

$$= nv\cdot dx\left(x\Pi - \frac{1}{dx} d\cdot \frac{xp\left(A - \int yxdx\right)}{\sqrt{(1+pp)}} \right)$$

$$= nv\cdot dx\left(x\int xdx\sqrt{(1+pp)} - \frac{1}{dx} d\cdot \frac{xp}{\sqrt{(1+pp)}} + \frac{1}{dx} d\cdot \frac{xp\int yxdx}{\sqrt{(1+pp)}} \right).$$

The other $\int xdx\sqrt{(1+pp)}\int yxdx$, when deduced from the second case of paragraph 7, will give $\Pi = \int yxdx$ and $Z = x\Pi\sqrt{(1+pp)}$, from which there will be

$$L = x\sqrt{(1+pp)},\ M = \Pi\sqrt{(1+pp)},\ N = 0 \text{ and } P = \frac{x\Pi p}{\sqrt{(1+pp)}};$$

and hence $\int Ldx = \int xdx\sqrt{(1+pp)}$; whereby, since H shall be the value of $\int Ldx$ on putting $x = a$, will be $H = B$ and $V = B - \int xdx\sqrt{(1+pp)}$. Again there shall be $[Z] = yx$, and hence $[M] = y$, $[N] = x$ and $[P] = 0$. From these the value of the differential will be produced

$$= nv\cdot dx\left(Bx - x\int xdx\sqrt{(1+pp)} - \frac{1}{dx} d\cdot \frac{xp\int yxdx}{\sqrt{(1+pp)}} \right).$$

Therefore the composition of this expression

$$\int yxdx\int xdx\sqrt{(1+pp)} + \int xdx\sqrt{(1+pp)}\int yxdx$$

emerges from both these differential values added together, or of this :

$$\int yxdx\cdot\int xdx\sqrt{(1+pp)},$$

which was proposed in the example, the value of the differential

$$= nv\cdot dx\left(Bx - \frac{A}{dx} d\cdot \frac{xp}{\sqrt{(1+pp)}} \right),$$

from which for the curve, the equation will be this

$$Bxdx = Ad \cdot \frac{xp}{\sqrt{(1+pp)}},$$

as we have found in the same solution to the example. Moreover a similar agreement will be understood in general, if whatever expression $\int Zdx \cdot \int Ndx$ were wished to be treated in the same manner.

PROPOSITION III. PROBLEM

23. *To find the equation between x and y of this condition, so that on putting* $x = a$ *this fraction* $\frac{\int Zdx}{\int Ydx}$ *may reach a maximum or minimum value, with some functions Z and Y present of x, y, p, q, r etc. either determinate or indeterminate.*

SOLUTION

In the case, in which there shall be $x = a$, there becomes $\int Zdx = A$ and $\int Ydx = B$ and $\frac{A}{B}$ will be a maximum or a minimum, if indeed the relation between *x* and *y* were correctly assigned. Therefore the fraction $\frac{A}{B}$ will be equal to this same fraction $\frac{\int Zdx}{\int Ydx}$, in the case where $x = a$, if somewhere an applied line *y* may be increased by a little amount *nv*. Then truly $\int Zdx$ may be equal to *A* itself, together with the value of the differential of the formula $\int Zdx$, which shall be $= dA$; and in a similar manner $\int Ydx$ will change into *B* increased by the value of the differential of the formula $\int Ydx$, which shall be $= dB$; and thus on adding the small amounts *nv* to the applied line *y*, in the case when $x = a$, the fraction $\frac{\int Zdx}{\int Ydx}$ will be changed into this $\frac{A+dA}{B+dB}$; which must be equal to the fraction $\frac{A}{B}$; from which this equation arises $BdA = AdB$, which will give the equation sought between *x* and *y*. Q. E. I.

COROLLARY 1

24. Therefore it must be effected that the equation between *x* and *y* be found, so that the differential values of $\int Zdx$ and $\int Ydx$ themselves become proportionals of these formulas with these values, which are found on putting $x = a$.

COROLLARY 2

25. Just as in this equation found $BdA = AdB$ two unknown constants *A* and *B* may be considered to be present, yet both may be allowed to be joined into one. For on putting $\frac{A}{B} = C$ there becomes $dA = CdB$; and with the equation found, from which the value *a* substituted in place of *x* will determine the value of *C*.

SCHOLIUM

26. If the solutions of this and of the preceding problems may be brought together among themselves, a huge agreement will be understood in these. For if the product $\int Zdx \cdot \int Ydx$ must be a maximum or a minimum, this equation will arise $0 = AdB + BdA$; but if the quotient $\frac{\int Zdx}{\int Ydx}$ must be a maximum or minimum, this equation has been found $0 = AdB - BdA$; but in each case the letters *A*, *B* and *dA*, *dB* retain the same values. Whereby, since *A* and *B* shall be constant quantities, both equations differ only in the sign of the constant ; for on putting $\frac{A}{B} = C$ in the first case there will be had $dA = -CdB$, truly in the latter $dA = +CdB$. From which for each case almost the same solution will appear, because the whole distinction will be placed in the sign of the constant quantity *C*. Therefore if the equation between *x* and *y* were found the $x = a$, which product $\int Zdx \cdot \int Ydx$ may contain a maximum or minimum, the same equation brought to light with the change $\frac{\int Zdx}{\int Ydx}$ likewise will contain the maximum or minimum quotient. But it is evident, whether $\frac{\int Zdx}{\int Ydx}$ should be a maximum or minimum, or $\frac{\int Ydx}{\int Zdx}$, in each case evidently the same equation is going to arise. Truly the nature of the matter itself demands this agreement ; for if $\frac{\int Zdx}{\int Ydx}$ is a maximum, then for that itself $\frac{\int Ydx}{\int Zdx}$ will be a minimum and vice versa, from which it is necessary that each question will be satisfied by the same solution. It will help to have noted this other connection also between the maxima or minima formula $\frac{\int Zdx}{\int Ydx}$, which on putting $x = a$ will be changed into $\frac{A}{B}$, and between the equation found $BdA - AdB = 0$, for this equation arises from the differentiation of the formula $\frac{A}{B}$ by putting its differential $= 0$; as we will show in the following proposition, a connection of this kind always has a place.

EXAMPLE I

27. *To find the curve, the area of which with coordinates orthogonal to the abscissa may maintain a maximum ratio to the arc of the curve, if a given value a may be attributed to the abscissa.*

With the abscissa of the curve sought $= x$, the applied line $= y$, the area will be $= \int ydx$ and the arc $= \int dx\sqrt{(1+pp)}$ on putting $dy = pdx$; therefore the ratio $\frac{\int ydx}{\int dx\sqrt{(1+pp)}}$ must be a maximum, for the case in which there may be put $x = a$.
Therefore in the case $x = a$ the value of the formula $\int ydx$, or the area, $= A$ and $\int dx\sqrt{(1+pp)}$, or the arc corresponding to the abscissa a, $= B$. Then the value of the differential of the formula $\int ydx$, dA will be $= nv\cdot dx\cdot 1$ and of the formula $\int dx\sqrt{(1+pp)}$ or

$$dB = nv\cdot dx\left(-\frac{1}{dx}d\cdot\frac{p}{\sqrt{(1+pp)}}\right) = -nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}.$$

With which values substituted into the equation $BdA = AdB$ the following equation will emerge for the curve sought

$$Bdx = -Ad\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Putting $\frac{A}{B} = c$, thus so that for the abscissa $x = a$ the area of the curve may become equal to the product from the arc into this constant c. Therefore there will be
$dx = -cd\cdot\frac{p}{\sqrt{(1+pp)}}$ and on integrating,

$$x = b - \frac{cp}{\sqrt{(1+pp)}} \quad \text{or} \quad cp = (b-x)\sqrt{(1+pp)},$$

and hence $p = \frac{b-x}{\sqrt{(c^2-(b-x)^2)}} = \frac{dy}{dx}$. Therefore there will be

$$y = \int\frac{(b-x)dx}{\sqrt{(c^2-(b-x)^2)}} = f \pm\sqrt{(c^2-(b-x)^2)}$$

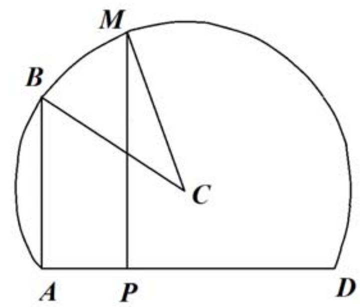

Fig. 12

or $(y-f)^2+(b-x)^2=cc$; from which it is agreed the curve sought to be a circle with radius *c* described relative to some rectangular axes. But of this circle only that part will satisfy the question, which corresponds to the abscissa *a*, on which the value *c* may depend, thus so that on taking the abscissa $=a$ the equal area may arise from the arc multiplied by the radius of the circle. So that if therefore in turn the radius *c* may be given, so much must be cut from the abscissa axis, so that the arc multiplied by the radius gives the area. Therefore the question can be satisfied in an infinite number of ways ; moreover the question will be determined , if two points may be prescribed, through which the prescribed curve shall pass. Therefore we may take the radius *c* as known and we may describe from that the circle (Fig. 12) *BMD* with centre *C*. Again some line *APD* may be taken for the axis and on that *A* for the origin of the abscissas. Now with this done it will satisfy the question, if the applied line *PM* may cut only an interval *ABMP*, thus so that it shall be equal to the product from the arc *BM* by the radius of the circle *BC*. Moreover because the sector *BCM* is $=\frac{1}{2}BM\cdot BC$, it will be required that the area *ABMP* be double the greater sector *BCM*. But it is apparent, with both the axis as well as its beginning taken as it pleases, how often the prescribed condition cannot be satisfied. For if the axis *AD* may pass through the centre, then the area *ABMP* always will be less than twice the sector *BCM*, unless, with the arc *BM* infinitely small, the first applied line *BA* likewise may pass through the centre ; but if the axis *AD* shall pass above the centre, then in no manner can the condition found be satisfied. Whereby it is necessary, that the axis *AD* be drawn below the centre *C*, from which fact many unusual geometrical observations are able to be made, if the ratio put in place will allow that. Otherwise, if this solution may be compared with the second example of the preceding proposition of paragraph 20, it will be apparent that exactly the same equation to be found, whether $\int ydx\cdot\int dx\sqrt{(1+pp)}$ must be a minimum, or

$$\frac{\int ydx}{\int dx\sqrt{(1+pp)}}$$

a maximum. Yet the distinction consists of this, because the radius of the circle $c=\frac{A}{B}$ in the one case shall be positive, it must be taken negative in the other. Evidently, if $\int ydx\cdot\int dx\sqrt{(1+pp)}$ must be a minimum, the arc *BM* must enclose the interval *ABMP* by its convexity, but in the other case by its concavity.

EXAMPLE II

28. *Within the given angle ACM* (Fig. 7) *to construct a curve AM to be compared thus, so that the area ACM divided by the arc AM shall be the greatest of all.*

The angle *ACM* may be considered, or $= x$ with the arc of the circle *BS* described with the radius $CB = 1$, which in the case proposed becomes $= a$, because $\frac{ACM}{AM}$ must become a maximum. Again there may be put $CM = y$ and there shall be $dy = pdx$, there will be $Mn = ydx$ and the area $ACM = \frac{1}{2}\int yydx$; but the arc *AM* will be found $= \int dx\sqrt{(yy+pp)}$; from which the fraction $\frac{\int yydx}{2\int dx\sqrt{(yy+pp)}}$ of the double of this $\frac{\int yydx}{\int dx\sqrt{(yy+pp)}}$ must be a maximum. There shall be in the case, in which $x = a$, $\int yydx = A$ and $\int dx\sqrt{(yy+pp)} = B$; there will be, if $x = a$, the area $ACM = \frac{1}{2}A$ and the arc $AM = B$.

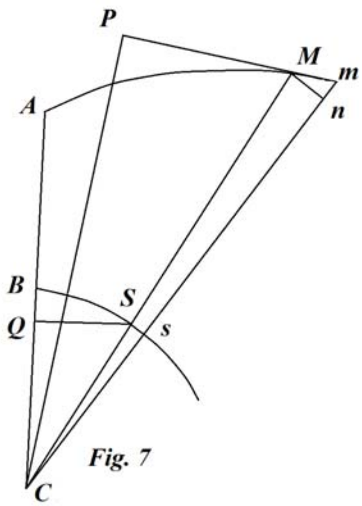

Fig. 7

Now the value of the differential of the formula $\int yydx = A$, dA is $= nv \cdot dx \cdot 2y$ and the value of the differential of the formula $\int dx\sqrt{(yy+pp)}$, dB is $nv \cdot dx\left(\frac{y}{\sqrt{(yy+pp)}} - \frac{1}{dx} \cdot \frac{p}{\sqrt{(yy+pp)}}\right)$. Whereby, since generally we have come across this equation $BdA = AdB$ for the curve, there will be on with the division by *nv* put in place :

$$2Bydx = \frac{Aydx}{\sqrt{(yy+pp)}} - Ad \cdot \frac{p}{\sqrt{(yy+pp)}}.$$

That may be multiplied by *p*, on account of $pdx = dy$ and there will be

$$2Bydy = A\left(\frac{ydx}{\sqrt{(yy+pp)}} - pd\,\frac{p}{\sqrt{(yy+pp)}}\right).$$

But there is

$$d\cdot\sqrt{(yy+pp)}=\frac{ydy}{\sqrt{(yy+pp)}}+\frac{pdp}{\sqrt{(yy+pp)}}$$

and

$$\frac{ydy}{\sqrt{(yy+pp)}}=d\cdot\sqrt{(yy+pp)}-\frac{p}{\sqrt{(yy+pp)}}dp\,;$$

from which

$$2Bydy=A\left(d\cdot\sqrt{(yy+pp)}-d\cdot\frac{pp}{\sqrt{(yy+pp)}}\right),$$

on account of

$$pd\frac{p}{\sqrt{(yy+pp)}}+dp\frac{p}{\sqrt{(yy+pp)}}=d\cdot p\cdot\frac{p}{\sqrt{(yy+pp)}}=d\cdot\frac{pp}{\sqrt{(yy+pp)}}.$$

Whereby on integrating there will be had, if there may be put $\frac{A}{B}=c$, this equation :

$$yy\pm bb=c\sqrt{(yy+pp)}-\frac{cpp}{\sqrt{(yy+pp)}}=\frac{cyy}{\sqrt{(yy+pp)}},$$

or

$$p=\frac{y\sqrt{(c^2y^2-(yy\pm bb)^2)}}{yy\pm bb}=\frac{dy}{dx}$$

and hence

$$dx=\frac{(yy\pm bb)dy}{y\sqrt{(c^2y^2-(yy\pm bb)^2)}}\,;$$

from which equation it can be deduced easily, if $cc+4bb$ shall be a positive magnitude, the construction to be resolved by the quadrature of the circle. But the same will be apparent easier, if in place of *dy* or *pdx* we may introduce the perpendicular *CP*, sent from *C* to the tangent *MP*. Because if moreover this perpendicular *CP* may be put to be *u*, there will be $y:u=dx\sqrt{(yy+pp)}:ydx$ and hence $\frac{yy}{\sqrt{(yy+pp)}}=u$; on account of which since there shall be $yy\pm bb=\frac{cyy}{\sqrt{(yy+pp)}}$; as it is agreed the equation to be that of a circle itself. In order that we may show this , we may

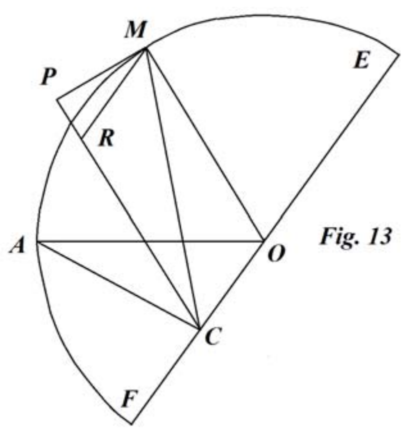

Fig. 13

take some circle (Fig. 13), described with centre *O*, radius $OM = g$, and the point shall be taken at *C*, thus so that there shall be $OC = h$. Now with the right line drawn $CM = y$ and $CP = u$, perpendicular to the tangent *MP*, *CP* will be parallel to the radius OM. From *M* the diameter *EF* may be drawn parallel to *MR*, there will be $MR = CO = h,\ CR = OM = g$ and $PR = u - g$; therefore because there is $MR^2 = MP^2 + PR^2 = CM^2 - CP^2 + PR^2$, there will be

$$h^2 = y^2 - u^2 + (u-g)^2 = y^2 - 2gu + gg$$

and hence $yy + gg - kh = 2gu$; which compared with that found $yy \pm bb = cu$, will become $g = \frac{1}{2}c$ and $\pm bb = \frac{1}{2}cc - hh$ or $hh = \frac{1}{4}cc + bb$. And thus the curve sought will be a circle described with radius $= \frac{1}{2}c$, with the point *C* taken where wished. The arc *AM* will be satisfied in such a circle sought, if there were

$$\frac{ACM}{AM} = \frac{A}{2B} = \tfrac{1}{2}c = \text{radius } OM;$$

that is, if the area $ACM = \text{arc.}AM \cdot AO =$ twice the sector AOM.
But this cannot happen, unless the point *C* may be taken outside the circle ; in which case this condition can be satisfied in an infinite number of ways and thus it can be effected that the satisfying curve may pass through two given points.

EXAMPLE III

29. *To find the curve DAD* (Fig.14) *relative to the axis AC, on which for a given abscissa AC = a ,* $\dfrac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$ *shall be a minimum.*

If the indefinite abscissa may be put $AP = x$, the applied line $PM = y$ and $dy = pdx$, $\dfrac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$ will express the distance of the centre of gravity of the curve *MAM*, just as it were of a weight considered uniformly upwards from the lowest point *A;* therefore which distance, by moving from *P* to *C*, must be a minimum. Towards finding this, on putting $x = a$ there shall be

$$\int xdx\sqrt{(1+pp)} = A \text{ and } \int dx\sqrt{(1+pp)} = B;$$

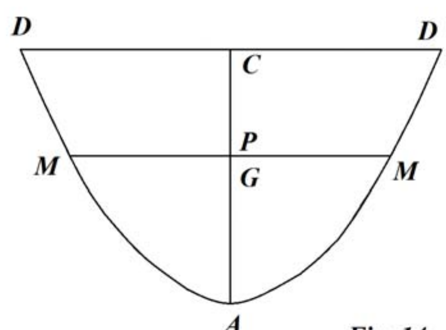

Fig. 14

but the value of the differential of the formula $\int xdx\sqrt{(1+pp)}$ is found $dA = -nv \cdot d \cdot \dfrac{xp}{\sqrt{(1+pp)}}$

and the value of the differential of the formula $\int dx\sqrt{(1+pp)}$ will be

$$dB = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}};$$

from which substituted into the equation $BdA = AdB$ there will emerge

$$Bd \cdot \frac{xp}{\sqrt{(1+pp)}} = Ad \cdot \frac{p}{\sqrt{(1+pp)}}$$

and on putting $\frac{A}{B} = c$ there will be

$$d \cdot \frac{xp}{\sqrt{(1+pp)}} = cd \cdot \frac{p}{\sqrt{(1+pp)}};$$

from which on integrating there arises

$$\frac{xp}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}} - b \text{ or } b\sqrt{(1+pp)} = (c-x)p;$$

and hence there is elicited :

$$p = \frac{b}{\sqrt{((c-x)^2 - b^2)}} = \frac{dy}{dx};$$

Therefore there will be

$$y = \int \frac{bdx}{\sqrt{((c-x)^2 - b^2)}};$$

which equation shows the curve to be a catenary, the start of the abscissas for *x* to be taken at some point on the axis *AC*; it is also possible for some right axis to be taken parallel to diameter of the centenary *AC* and on that some point for the start of the abscissas. But however the axis and its start may be put in place, that will satisfy only a part of the curve, where there shall be

$$\int xdx\sqrt{(1+pp)} = c\int dx\sqrt{(1+pp)}.$$

We may take for the axis the diameter *AC* itself and the vertex *A* for the start of the abscissas. Because at *A*, where there is $x = 0$, there becomes $\frac{dy}{dx} = p = \infty$, it is necessary that there shall be $cc - bb = 0$ and thus $b = c$. Truly in this case there becomes

$y=\int\frac{cdx}{\sqrt{(xx-2cx)}}$, which curve directly upwards shall become imaginary, as then there becomes $x>2c$. Therefore there shall be $x=2c+t$, and $t=$ to the abscissa *AP* and $y=PM=\int\frac{cdt}{\sqrt{(2ct+tt)}}$; and the curve *DAD* will be the ordinary catenary. From which moreover it may be apparent, how great a part of this may satisfy the question, it is to be noted on account of $dx=dt$ to be

$$p=\frac{c}{\sqrt{(2ct+tt)}} \text{ and } \sqrt{(1+pp)}=\frac{c+t}{\sqrt{(2ct+tt)}};$$

and hence

$$\int dx\sqrt{(1+pp)}=\int\frac{(c+t)dt}{\sqrt{(2ct+tt)}}=\sqrt{(2ct+tt)}.$$

But that expression

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} \text{ becomes } = 2c+\frac{\int tdt\sqrt{(1+pp)}}{\sqrt{(2ct+tt)}},$$

which in no manner can become equal to *c* itself. From which it is concluded no part of this curve sought is more satisfactory than the rest. On account of which the start of the abscissas cannot be taken from the vertex *A*. Therefore it may be taken from some other point, and putting $AP=t$ there must become $2bt+tt=(c-x)^2-bb$; from which there becomes either $b+t=x-c$ or $b+t=c-x$. The first equation $x=b+c+t$ cannot have a place, because on account of $dx=dt$ there cannot occur

$$\frac{\int xdt\sqrt{(1+pp)}}{\int dt\sqrt{(1+pp)}} \text{ or } \left(b+c+\frac{\int tdt\sqrt{(1+pp)}}{\int dt\sqrt{(2ct+tt)}}\right)=c.$$

Therefore there may become $x=c-b-t$, in which case the abscissas descend downwards from some superior point of the axis *AC* and there must become

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} \text{ or } c-b-\frac{\int tdt\sqrt{(1+pp)}}{\int dt\sqrt{(2ct+tt)}}=c,$$

which equally cannot occur; from which it is to be concluded no part is to be more satisfactory than any other part. But this comes to be seen thence, because the catenary has two parts joined together just as the conic hyperbola and hence it happens always :

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}=0,$$

which is the minimum value. This can be confirmed more clearly from the value found

$$p=\frac{b}{\sqrt{((c-x)^2-b^2)}};$$

from which there becomes

$$\sqrt{(1+pp)}=\frac{c-x}{\sqrt{((c-x)^2-b^2)}}=(c-x)r$$

on putting for brevity

$$r=\frac{1}{\sqrt{((c-x)^2-b^2)}}.$$

Therefore there will be necessary in the case sought to be

$$\frac{\int(c-x)xrdx}{\int(c-x)rdx}=c \quad \text{or} \quad \int(c-x)^2rdx=0,$$

which must vanish in the case $x=0$, as well as with the other case above.
But there is

$$\int(c-x)^2rdx=\int\frac{(c-x)^2dx}{\sqrt{((c-x)^2-b^2)}}=$$

$$=-\tfrac{1}{2}(c-x)\sqrt{((c-x)^2-b^2)}-\frac{bb}{2}l\frac{c-x+\sqrt{((c-x)^2-b^2)}}{c+\sqrt{(c^2-b^2)}}+\tfrac{1}{2}c\sqrt{(c^2-b^2)},$$

which expression, since once it were $=0$, after, on account of $(c-x)^2$ being positive always, continually increases and cannot again become $=0$. On account of which both terms of the integration formula

$$\int\frac{(c-x)^2dx}{\sqrt{((c-x)^2-b^2)}}$$

must agree between themselves, if there were $x = c$; in which case the satisfying curve will change into a right line normal to the axis, which certainly has the centre of gravity removed minimally.

EXAMPLE IV

30. *To find the curve, in which for a given abscissa* $x = a$ *this expression* $\dfrac{\int xydx}{\int dx\sqrt{(1+pp)}}$ *shall be a maximum or minimum.*

On putting $x = a$ there becomes $\int xydx = A$ and $\int dx\sqrt{(1+pp)} = B$. Now the value of the differential of the formula $\int xydx$ is $dA = nv \cdot dx \cdot x = nv \cdot xdx$ and of the formula $\int dx\sqrt{(1+pp)}$ the value of the differential is

$$dB = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}.$$

Whereby since there shall be $BdA = AdB$, this equation will be had

$$Bxdx = -Ad \cdot \frac{p}{\sqrt{(1+pp)}} \quad \text{or} \quad xdx = -ccd \cdot \frac{p}{\sqrt{(1+pp)}}$$

on putting $A = Bc^2$. From which on integrating there will be found $xx = bc - \dfrac{2ccp}{\sqrt{(1+pp)}}$

and hence $p = \dfrac{bc - xx}{\sqrt{(4c^4 - (bc - xx)^2)}} = \dfrac{dy}{dx}$;

which will give

$$y = \int \frac{(bc - xx)dx}{\sqrt{(4c^4 - (bc - xx)^2)}},$$

which is the general equation for an elastic curve, of which this is a property, because the radius of osculation everywhere shall be inversely proportional to the abscissa *x*, which is apparent from the equation

$$xdx = -ccd \cdot \frac{p}{\sqrt{(1+pp)}},$$

which will change into this and becomes

$$-\frac{dx}{d\cdot\frac{p}{\sqrt{(1+pp)}}}=\frac{cc}{x},$$

and there shall be

$$-\frac{dx}{d\cdot\frac{p}{\sqrt{(1+pp)}}}=-\frac{dx(1+pp)^{3:2}}{dp}$$

the radius of osculation on the curve. But so large a portion of this curve from the start of the computation is satisfied, in which there shall be

$$\int yxdx=cc\int dx\sqrt{(1+pp)}=2c^4\int\frac{dx}{\sqrt{(4c^4-(bc-xx)^2)}};$$

which on being determined may be recalled to that, so that there must be effected

$$\int dx\sqrt{(4c^4-(bc-xx)^2)}=(aa-bc)\int\frac{(bc-xx)dx}{\sqrt{(4c^4-(bc-xx)^2)}},$$

if after integration each may be put $x=a$. And in this manner that constant c will be determined by a.

PROPOSITION IV. PROBLEM

31. *To find the equation between the two variables x and y prepared thus, so that on putting the variable* $x=a$ *to become a maximum or minimum the expression W, which shall be some function of the integrals* $\int Zdx,\ \int Ydx,\ \int Xdx$ *etc., in which Z, Y, X etc. may denote some functions of x, y, p, q etc. themselves, either determinate or indeterminate.*

SOLUTION

We may consider a suitable equation between x and y now to be found and on putting $x=a$ there becomes $\int Zdx=A,\ \int Ydx=B,\ \int Xdx=C$ etc.; and with these variables substituted into the expression, W will have actually a maximum or minimum. But if therefore one variable y at one place may be considered to be increased by the small amount nv and hence the changes arising may be introduced into the individual formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc., likewise the value for W must be shown. But by that small amount nv all the formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. will be increased by their own differential values. If therefore the value of the differential of the formula $\int Zdx = dA$, of the formula $\int Ydx = dB$, of the formula $\int Xdx = dC$ etc., in place of the magnitudes *A*, *B*, *C* etc. from the small amount *nv* these increases will arise $A + dA,\ B + dB,\ C + dC$ etc., which substituted into *W* must produce the same value, as *A*, *B*, *C* etc. themselves. We may put $A + dA,\ B + dB,\ C + dC$ etc. substituted in place of $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. for $W + dW$ to appear; and there will be $W + dW = W$ and thus $dW = 0$. But here the value *dW* is found, as the nature may become clear from differentiation, if the magnitude *W* may be differentiated, after the letters *A*, *B*, *C* etc. have been substituted in place of the integral formulas, with these letters *A*, *B*, *C* etc. treated as variables ; and in this from the differentials *dA*, *dB*, *dC* etc. the differential values of the corresponding formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. may be designated. Therefore with this meaning taken for the differential of the proposed quantity *W*, if that may be put equal to zero, it will give the equation sought between *x* and *y*. Q. E. I.

COROLLARY I

32. If therefore and expression *W* were proposed of this kind, a function of the integral formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc., which for the determination of *x* with the value $= a$ must be a maximum or minimum, then in place of the formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. the letters *A*, *B*, *C* etc. may be written, with which done the expression *W* may be differentiated with these letters *A*, *B*, *C* etc. treated only as variables and with the differential may be put $= 0$.

COROLLARY 2

33. In this differential, in which the letters *A* , *B*, *C* etc. will be present with their differentials *dA*, *dB*, *dC* etc., the letters *A*, *B*, *C* etc. will denote the respective values of the formulas $\int Zdx,\ \int Ydx,\ \int Xdx$ etc., which they adopt on putting $x = a$; but the differentials themselves *dA*, *dB*, *dC* etc. express the differential values of the same formulas corresponding to the abscissa $x = a$.

COROLLARY 3

34. But it may be apparent from the preceding, if *Z*, *Y*, *X* etc. were determinate functions of the magnitudes *x*, *y*, *p*, *q* etc., then the differential values *dA*, *dB*, *dC* etc. do not depend on the value *a* ; truly on the other hand, if *Z*, *Y*, *X* etc. were indefinite functions, then the differential values *dA*, *dB*, *dC* etc. likewise must depend on the value *a*.

COROLLARY 4

35. Therefore since in this manner *W* may become a function of the letters *A, B, C* etc., a differential of this kind will have the form $FdA + GdB + HdC + \text{etc.}$ and hence the equation sought will be $0 = FdA + GdB + HdC + \text{etc.}$, where *F, G, H* etc. will be constant quantities determined by *A, B, C* etc.

COROLLARY 5

36. Therefore an equation satisfying the problem will be agreed upon from the differential values of the individual integral formulas in the expression *W* of the contained maximum or minimum, with the individuals multiplied by determined constant quantities; evidently the sum of the product of these put equal to zero will give the desired equation.

SCHOLIUM 1

37. We could have included a method of resolving this proposed problem from the solutions of the two previous problems by induction, certainly from which it may be apparent now, if the formula *W* of the maximum or minimum either were a product from two integrable formulas, or a quotient arising from the division of one by the other, then the differential of the expression *W* set out in the accepted manner gives a fitting equation for the problem. But this problem has excelled on account of its main extension to provide a singular solution. For in this problem all the questions are present entirely, which generally in this, from which some expression for the maximum or minimum may be wished, to be proposed at some time and can be thought out, and thus by that proposition itself the method of absolute maxima and minima has been completely exhausted, that in the first place we have undertaken to handle. Besides here it is to be noted, if the expression *W* may include not only integral formulas, as we have put in place, but truly also determinate functions of *x, y, p, q* etc., then the solution is returned without more difficulty. For in a like manner, in place of these determined functions constant magnitudes must be put in place, into which clearly they will change on putting $x = a$; but afterwards in the differentiation of *W* it is necessary to treat these quantities also as constants, because no determinate functions receive differential values from that. But so that it may be made clearer, how expressions of this kind may be agreed upon, in the following examples several will occur, in will illustrate this argument thoroughly.

EXAMPLE I

38. *To find the curve containing orthogonal coordinates, in which this expression* $\left(1+pp\right)^{1:2}\int ydx+y\int dx\sqrt{(1+pp)}$ *shall be a maximum or minimum, if the abscissa may be put* $x=a$.

We may consider the equation between *x* and *y* satisfying the question now to be found and on putting $x=a$ it becomes $y=f$ and $\sqrt{(1+pp)}=g$, likewise $\int ydx=A$ and $\int dx\sqrt{(1+pp)}=B$*;* there will be

$$dA=nv\cdot dx \text{ and } dB=-nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Therefore the expression, which will be a maxima or minima, in this case is $gA+fB$, of which the differential is $gdA+fdB$*;* which on putting $=0$ will give the desired equation for the curve. Evidently here it is understood the letters *g* and *f*, which have come from determinate functions, are to be treated as constants in the differentiation. Now with the due values substituted for *dA* and *dB* and with the division made by *nv* this equation will arise for the curve sought

$$gdx=fd\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Putting $\frac{f}{g}=c$, thus so that there shall be $\frac{y}{\sqrt{(1+pp)}}=c,$ in the case in which $x=a$*;* on integrating there will be

$$x+b=\frac{cp}{\sqrt{(1+pp)}} \text{ and } p=\frac{x+b}{\sqrt{(cc-(x+b)^2)}}=\frac{dy}{dx};$$

from which there becomes

$$y=h\pm\sqrt{(c^2-(x+b)^2)}.$$

Therefore the satisfying curve is a circle described with radius *c*, with the abscissas above assumed for some right lines and equally with the start of the abscissas put in place somewhere. But the magnitude *c,* which established the radius of the circle, may be determined from the definite abscissa $x=a$, because there must be $\frac{y}{\sqrt{(1+pp)}}=c$ for the case in which $x=a$. But in this case there becomes

$$y = h \pm \sqrt{(c^2 - (x+b)^2)} \text{ and } \sqrt{(1+pp)} = \frac{c}{\sqrt{(cc-(x+b)^2)}},$$

from which there arises

$$cc = h\sqrt{(cc-(x+b)^2)} \ \pm \ (cc-(x+b)^2),$$

by which either c can be determined by a, or a by c in turn. We may put in place to be $h = 0,\ b = -c$, thus so that the axis shall be a diameter of the circle, and the start of the abscissas may be established at the vertex ; there will be $y = \sqrt{(2cx - xx)}$ and there becomes $(a-c)^2 = 0$ or $c = a$. From which it is understood in this case to satisfy a quadrant of the circle sought. But if the start of the abscissas may be taken at some place of the diameter, only $h = 0$, and if a positive applied line may be taken, there becomes $(a+b)^2 = 0$ or $b = -a$. Therefore the diameter of the circle remains indeterminate and a part of the circle taken in this manner satisfies the question, which corresponds to an abscissa from its origin produced as far as to the centre of the circle.

EXAMPLE II

39. *To find the equation between x and y, so that for the value defined* $x = a$ *this expression* $y^{\int dx\sqrt{(1+pp)}}\int ydx$ *becomes a maximum or minimum.*

On putting $x = a$ there becomes

$$y = f,\ \int dx\sqrt{(1+pp)} = A \text{ and } \int ydx = B,$$

there will be

$$dA = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}} \text{ and } dB = nv \cdot dx.$$

Therefore this function $f^A B$ is required to be a maximum or minimum, of which the differential is $f^A BdAlf + f^A dB$; which on putting $= 0$ will give $BdAlf = -dB$. Therefore for the equation sought there will be had

$$Blfd \cdot \frac{p}{\sqrt{(1+pp)}} = dx.$$

and on integrating,

$$x + b = \frac{pBplf}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}}$$

on putting $Blf = c$. Therefore there will be had

$$p = \frac{b+x}{\sqrt{(cc-(b+x)^2)}} \quad \text{and} \quad y = h \pm \sqrt{(\,c^2-(b+x)^2)}\ .$$

Therefore there will be $f = h \pm \sqrt{(\,c^2-(b+a)^2)}$ on putting $x = a$, and

$$B = \int ydx = ha \pm \int dx\sqrt{(\,c^2-(b+x)^2)}$$

on putting $x = a$ after the integration. Therefore on making $Blf = c$ the value *a* may be known, for which if x may be taken in a circle of radius *c*, a part may be cut off satisfying the problem. The remainder can be deduced from these and from corollary 5, whenever the formula of the maxima or minima should be some function of the two formulas $\int ydx$ and $\int dx\sqrt{(1+pp)}$, the satisfying curve always being a circle ; from the solution only a part of the satisfying magnitude must be diligently investigated and determined.

EXAMPLE III

40. *To find the equation between x and y, so that on putting* $x = a$ *this expression* $e^{-n\int dx\sqrt{(1+pp)}}\int e^{n\int dx\sqrt{(1+pp)}}dx$ *becomes a maximum or minimum.*

We may consider in the case proposed, for which $x = a$, to become

$$n\int dx\sqrt{(1+pp)} = A \quad \text{and} \quad \int e^{n\int dx\sqrt{(1+pp)}}dx = B\,;$$

thus so that this quantity $e^{-A}B$ thus may become a maximum or a minimum, of which the differential is $e^{-A}dB - e^{-A}BdA$; because on putting $=0$ it will give this equation $dB = BdA$.
But *dA* is the value of the differential of the formula $n\int dx\sqrt{(1+pp)}$, from which

$$dA = -nv \cdot d \cdot \frac{np}{\sqrt{(1+pp)}}$$

and *dB* is the value of the differential of the formula $\int e^{n\int dx\sqrt{(1+pp)}}dx$, which is present in the second case of paragraph 7, where there is

$$Z = \int e^{n\int dx\sqrt{(1+pp)}}dx \quad \text{and} \quad \Pi = \int dx\sqrt{(1+pp)}\,,$$

so that thus there shall be $Z = e^{n\Pi}$ and $dZ = e^{n\Pi} nd\Pi$, from which there will be $L = e^{n\Pi} n$ and the remaining letters M, N, P etc. become $= 0$. Again on account of $\Pi = \int dx\sqrt{(1+pp)}$ there will be

$$[Z] = \sqrt{(1+pp)} \text{ and } d[Z] = \frac{pdp}{\sqrt{(1+pp)}},$$

from which there will be

$$[M] = 0,\ [N] = 0 \text{ and } [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Now there is

$$\int Ldx = n\int e^{n\int dx\sqrt{(1+pp)}} dx,$$

the value of which on putting $x = a$ will be $= nB$; and hence

$$V = n(B - \int e^{n\int dx\sqrt{(1+pp)}} dx)$$

Therefore by the rule given there becomes

$$dB = nv\cdot dx\left(-\frac{d\cdot[P]V}{dx}\right) = -nv\cdot d\cdot\frac{np(B - \int e^{n\int dx\sqrt{(1+pp)}} dx)}{\sqrt{(1+pp)}}$$

$$= -nv\cdot d\cdot\frac{nBp}{\sqrt{(1+pp)}} \text{ on account of } dB = BdA.$$

And thus on integrating there will be

$$\frac{np(B - \int e^{n\int dx\sqrt{(1+pp)}} dx)}{\sqrt{(1+pp)}} = \frac{nBp}{\sqrt{(1+pp)}} - nb$$

and hence

$$\frac{b\sqrt{(1+pp)}}{p} = \int e^{n\int dx\sqrt{(1+pp)}} dx.$$

From which equation, because the determined value *a* has gone, it is evident the equation found to be equally valid for some value of *x*. But so that we may set out this equation, there will be with the differentials taken,

$$-\frac{bdp}{p^2\sqrt{(1+pp)}}=e^{n\int dx\sqrt{(1+pp)}}dx,$$

which multiplied by $n\sqrt{(1+pp)}$ and integrated, gives

$$\frac{nb}{p}+c=e^{n\int dx\sqrt{(1+pp)}},$$

which value with the exponential magnitude substituted into that equation will give

$$\frac{nbdx}{p}+cdx=-\frac{bdp}{p^2\sqrt{(1+pp)}} \text{ or } dx=-\frac{bdp}{p(nb+cp)\sqrt{(1+pp)}}.$$

But a more convenient equation arises, if there may be put $\int dx\sqrt{(1+pp)}=s$, and s will be the arc of the curve, if *x* and *y* were normal coordinates. Whereby this equation will be had $nb+cp=e^{ns}p$, which multiplied by *dx* on account of $dy=pdx$ will be changed into this equation $nbdx+cdy=e^{ns}dy$. But since on putting $x=0$ the arc *s* may vanish, as it shall be in the case $\frac{nb}{p}+c=0$; and thus hence either the constant *c* will be determined from the given start of the curve *p* or in turn from *c* in place the position of the first tangent will become known. The remainder, if we may consider this question more carefully, we may take that now to be contained in a certain example of paragraph 45 of the preceding chapter. For since our expression, which must be a maximum or minimum, shall be

$$e^{-n\int dx\sqrt{(1+pp)}}\int e^{n\int dx\sqrt{(1+pp)}}dx,$$

that may be considered to be *W*, and there will be

$$e^{n\int dx\sqrt{(1+pp)}}W=\int e^{n\int dx\sqrt{(1+pp)}}dx,$$

and by differentiation it becomes

$$dW+nWdx\sqrt{(1+pp)}=dx.$$

The expression of the maximum or minimum of W may be given by the differential of the equation, which may be contained in case four of paragraph 7 and treated conveniently by the method and the same equation emerges, as we have found here. But we have treated that question included in paragraph 45 (1°) of the preceding chapter, to which this case itself can be seen to be permitted to be added. But by comparison the greatest agreement may be seen of the various solutions of the same problem, which indeed they shall be able to hold.

EXAMPLE IV

41. *To find the curve, in which for a given abscissa* $=a$ *this expression*
$\dfrac{\int dx\sin \mathrm{A}y\cdot\sqrt{(1+pp)}}{\int dx\cos \mathrm{A}y\cdot\sqrt{(1+pp)}}$ *may become a maximum or minimum.*

On putting $x=a$ there becomes

$$\int dx(1+pp)^{1:2}\sin \mathrm{A}y=A \quad\text{and}\quad \int dx(1+pp)^{1:2}\cos \mathrm{A}y=B;$$

through the differential values there will be

$$dA=nv\cdot dx\left((1+pp)^{1:2}\cos \mathrm{A}y-\frac{1}{dx}d\cdot\frac{p\sin \mathrm{A}y}{\sqrt{(1+pp)}}\right)$$

and

$$dB=nv\cdot dx\left(-(1+pp)^{1:2}\sin \mathrm{A}y-\frac{1}{dx}d\cdot\frac{p\cos \mathrm{A}y}{\sqrt{(1+pp)}}\right).$$

Therefore since $\dfrac{A}{B}$ must be a maximum or minimum, there will be $BdA=AdB$*;*
Therefore on putting $\dfrac{A}{B}=m$ there becomes

$$\left(1+pp\right)^{1:2}dx\cos \mathrm{A}y-d\cdot\frac{p\sin \mathrm{A}y}{\sqrt{(1+pp)}}=-m(1+pp)^{\frac{1}{2}}dx\sin \mathrm{A}y-md\cdot\frac{p\cos \mathrm{A}y}{\sqrt{(1+pp)}}$$

The equation may be multiplied by *p*, there will be on account of

$$d\cdot\left(1+pp\right)^{1:2}\sin \mathrm{A}y=dy\left(1+pp\right)^{1:2}\cos \mathrm{A}y+\frac{pdp\sin \mathrm{A}y}{\sqrt{(1+pp)}}$$

and

$$d\cdot\left(1+pp\right)^{1:2}\cos \mathrm{A}y=-dy\left(1+pp\right)^{1:2}\sin \mathrm{A}y+\frac{pdp\cos \mathrm{A}y}{\sqrt{(1+pp)}}:$$

$$d\cdot\left(1+pp\right)^{1:2}\sin \mathrm{A}y-d.\frac{pp\sin \mathrm{A}y}{\sqrt{(1+pp)}}=md\cdot\left(1+pp\right)^{1:2}\cos \mathrm{A}y-md\cdot\frac{pp\cos \mathrm{A}y}{\sqrt{(1+pp)}};$$

which integrated and reduced gives

$$\frac{\sin \mathrm{A}y}{\sqrt{(1+pp)}}=\frac{m\cos \mathrm{A}y}{\sqrt{(1+pp)}}+b \quad \text{or} \quad b\sqrt{(1+pp)}=\sin \mathrm{A}y-m\cos \mathrm{A}y\,;$$

where it must be noted to become, if there may be put $x=a$,

$$m=\frac{\int dx(1+pp)^{1:2}\sin \mathrm{A}y}{\int dx(1+pp)^{1:2}\cos \mathrm{A}y}$$

Let $m=\frac{\sin\mathrm{A}n}{\cos \mathrm{A}n}=\text{tang }\mathrm{A}n$, there becomes

$$b\sqrt{(1+pp)}=\frac{\sin \mathrm{A}(y-n)}{\cos \mathrm{A}n} \text{ and } y=n+\mathrm{Asin}\, b\left(1+pp\right)^{1:2}\cos \mathrm{A}n.$$

Because truly there is $dy=pdx$, there will be $dx=\frac{dy}{p}$. But there is

$$dy=\frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$$

on putting $b\cos \mathrm{A}n=c$. From which there is produced

$$x=\int\frac{cdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$$

and

$$y=\int\frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}}\,;$$

but the length of the curve will be

$$=\int\frac{cdp}{\sqrt{(1-cc-ccpp)}}=\mathrm{Asin}\frac{cp}{\sqrt{(1-cc)}}.$$

Whereby, if the arc of the curve may be called s, this suitably arranged equation will be had

$$dx \sin As = \frac{cdy}{\sqrt{(1-cc)}}.$$

Truly the construction follows at once from the previous formulas.

SCHOLIUM 2

42. Therefore from these chapters we have thoroughly resolved that part of the method of adapting maxima and minima for finding curved lines, which we have called absolute, in which a curved line is accustomed to be required always, which may have for a certain given abscissa or rather with the value of another variable x some indeterminate expression of the maximum or minimum. For that expression, which must be a maximum or minimum, either will be a single integral formula of the form $\int Zdx$, thus so that Z shall be some function of x, y, p, q etc. either definite or indefinite, for which cases we have treated a method in the preceding chapters ; or an expression of the maxima or minima that will contain in itself several integral formulas of this kind, thus so that there shall be some function of two or more integral formulas ; and in this case a suitable method has been set out in this chapter and illustrated with examples. But the general method, which we have given here, depends on the discovery of the values of the differentials, which from the individual integrable formulas, which either themselves must be maxima or minima or may be contained in some expression of the maxima or minima, and thus the whole method of resolving is reduced to those cases, which we have represented together in paragraph 7 of this chapter. Therefore he who holds those cases in mind or has at the ready, will be prepared for resolving all problems of this kind quickly. Nor truly only the cases to be put in place there to enumerate the absolute method of maxima or minima, but truly also the other relative method, which we will undertake in the following, the cases will be resolved ; from which the greatest uses of these cases in each method may be seen in abundance. But we have resolved this treatment in two chapters, in the first of which for all the curves, from which a question must be elicited, to which we have attributed a certain single common property, and in the latter, several.

CAPUT IV

DE USU METHODI HACTENUS TRADITAE IN RESOLUTIONE VARII GENERIS QUAESTIONUM

PROPOSITIO I. PROBLEMA

1. *Invenire aequationem inter binas variabiles x et y, ita ut pro dato ipsius x valore, puta posito* $x = a$, *formula* $\int Zdx$ *obtineat maximum minimumve valorem, existente Z functione ipsarum x, y, p, q, r etc. sive determinata sive indeterminata.*

SOLUTIO

Ex quacunque consideratione variabiles x et y sint natae, eae semper tanquam coordinatae orthogonales cuiuspiam curvae considerari possunt atque hanc ob rem quaestio proposita huc revocatur, ut determinetur curva abscissam habens $= x$ et applicatam $= y$, in qua valor $\int Zdx$, si ad abscissam datae magnitudinis, puta $x = a$, applicetur, fiat omnium maximus vel minimus. Quodsi autem Problema hoc modo proponatur, tum eius solutio in praecedentibus Capitibus satis superque est tradita. Quamobrem formulae propositae $\int Zdx$, secundum Methodos ante expositas, capi oportet valorem differentialem, qui datae abscissae $x = a$ conveniat, isque nihilo aequalis positus dabit aequationem inter x et y desideratam, quae pro data abscissa $x = a$ producet formulae $\int Zdx$ maximum minimumve valorem. Q. E. I.

COROLLARIUM 1

2. Methodus ergo ante tradita multo latius patet, quam ad aequationes inter coordinatas curvarum inveniendas, ut quaepiam expressio $\int Zdx$ fiat maximum minimumve. Extenditur scilicet ad binas quascunque variabiles, sive eae ad curvam aliquam pertineant quomodocunque, sive in sola analytica abstractione versentur.

COROLLARIUM 2

3. Inter binas autem variabiles propositas discrimen ingens intercedit, eo quod proposita formula $\int Zdx$ pro determinato quodam alterius variabilis valore maximum minimumve obtinere debeat. Isthanc ergo variabilem constanter littera x, alteram vero littera y denotari convenit.

COROLLARIUM 3

4. Litteris igitur *x* et *y* debito modo binis quantitatibus variabilibus impositis, erit $p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx},\ s = \frac{dr}{dx}$ etc. His scilicet litteris differentialia cuiuscunque gradus, quae forte in maximi minimive formula insint, tolli poterunt, ita ut *Z* futura sit functio litterarum *x*, *y, p, q, r* etc.

COROLLARIUM 4

5. Cum ergo maximi minimive formula ad talem formam $\int Zdx$ fuerit reducta, in qua *Z* sit functio ipsarum *x, y, p, q, r* etc. sive definita sive indefinita, tum ex superioribus praeceptis formulae $\int Zdx$ valor differentialis, respondens toti abscissae propositae $x = a$, debet investigari, qui nihilo aequalis positus praebebit aequationem inter *x* et *y* quaesitam.

COROLLARIUM 5

6. Si *Z* est functio definita ipsarum *x, y, p, q, r* etc., tum valor differentialis formulae $\int Zdx$ non pendet a praescripto abscissae valore $x = a$; et hanc ob rem aequatio inter *x* et *y* inventa pro qualibet abscissa praebebit maximum vel minimum formulae $\int Zdx$.

SCHOLION 1

7. Quia in hoc negotio valores differentiales, quos ante pro omni genere formularum sparsim eruimus, in promtu esse oportet, eos hic coniunctim in conspectum producemus, ut sit, unde quovis casu oblato valores differentiales, quibus opus fuerit, conquiri ac depromi queant. Exhibebimus igitur formulae $\int Zdx$ pro varia functionis *Z* indole valorem differentialem, qui perpetuo determinatae variabilis *x* quantitati, puta $x = a$, respondeat.

I. MAXIMI MINIMIVE FORMULA
$\int Zdx$

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc.}$$

Valor differentialis erit

$$nv \cdot dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}),$$

qui valor differentialis pro omni variabilis *x* magnitudine aeque valet.

II. MAXIMI MINIMIVE FORMULA
$$\int Zdx$$

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{ etc. et } \Pi = \int [Z]dx$$

existente

$$d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.}$$

Iam posito post integrationem $x = a$ sit $\int Ldx = H$, ponaturque $H - \int Ldx = V$.
Valor differentialis erit

$$nv\cdot dx\left(N + [N]V - \frac{d\cdot(P + [P]V)}{dx} + \frac{dd\cdot(Q + [Q]V)}{dx^2} - \frac{d^3\cdot(R + [R]V)}{dx^3} + \frac{d^4\cdot(S + [S]V)}{dx^4} - \text{etc.}\right)$$

III. MAXIMI MINIMIVE FORMULA
$$\int Zdx$$

$$\begin{aligned} dZ &= Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc. et } \Pi = \int [Z]dx \\ d[Z] &= [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc. et } \pi = \int [z]dx \\ d[z] &= [m]dx + [n]dy + [p]dp + [q]dq + [r]dr + \text{etc.} \end{aligned}$$

Sit iterum, posito post integrationem $x = a$ ut ante, $\int Ldx = H$ ac ponatur $H - \int Ldx = V$.
Iam integretur $\int [L]Vdx$ sitque integrale eo casu, quo $x = a$ ponitur, $= G$ ac ponatur

$$G - \int [L]Vdx = [V] = G - \int [L]dx(H - \int Ldx) .$$

His positis valor differentialis erit

$$nv\cdot dx\left(\begin{aligned} & N + [N]V + [n]V - \frac{d\cdot(P + [P]V + [p]V)}{dx} + \frac{dd\cdot(Q + [Q]V + [q]V)}{dx^2} \\ & - \frac{d^3\cdot(R + [R]V + [r]V)}{dx^3} + \frac{d^4\cdot(S + [S]V + [s]V)}{dx^4} - \text{etc.} \end{aligned} \right)$$

unde simul lex progressionis patet, si adhuc plura integralia involvantur.

IV. MAXIMI MINIMIVE FORMULA
$$\int Zdx$$

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc. et } \Pi = \int Zdx.$$

Abeat posito $x = a$ haec expressio $e^{\int Ldx}$ in H denotante e numerum, cuius logarithmus est $=1$, sitque $He^{-\int Ldx} = V$, valor differentialis erit
$$nv \cdot dx\left(NV - \frac{d \cdot PV}{dx} + \frac{dd \cdot QV}{dx^2} - \frac{d^3 \cdot RV}{dx^3} + \frac{d^4 \cdot SV}{dx^4} - \text{etc.} \right).$$

V. MAXIMI MINIMIVE FORMULA
$$\int Zdx$$

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc. et } \Pi = \int [Z] dx$$
$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{ etc.}$$

Sit, si ponatur $x = a$ post integrationem,

$$\int e^{\int [L]dx} Ldx = H$$

atque ponatur
$$e^{-\int [L]dx} \left(H - \int e^{\int [L]dx} Ldx \right) = V.$$

Valor differentialis erit

$$nv \cdot dx \left(\begin{array}{l} N + [N]V - \dfrac{d \cdot (P + [P]V)}{dx} + \dfrac{dd \cdot (Q + [Q]V)}{dx^2} \\ - \dfrac{d^3 \cdot (R + [R]V)}{dx^3} + \dfrac{d^4 \cdot (S + [S]V)}{dx^4} - \text{etc.} \end{array} \right)$$

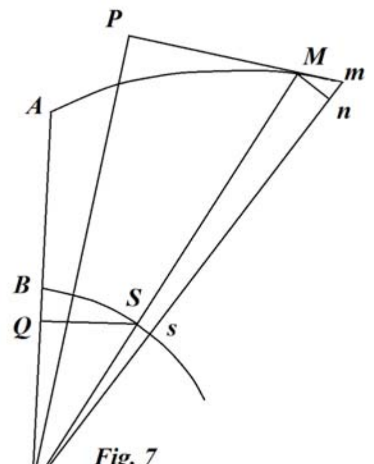

Fig. 7

In his igitur quinque casibus continentur omnes regulae, quas in Capitibus praecedentibus invenimus. Iique tam late patent, ut omnes casus, qui quidem occurrere queant, in iis vel actu contineantur vel saltem per eos non difficulter resolvi possint. Iis igitur hic in compendium redactis eorum usum monstrabimus in resolvendis quaestionibus, in quibus x et y non denotant coordinatas orthogonales.

EXEMPLUM I

8. *Ex dato centro C* (Fig. 7) *ductis radiis CA, CM invenire lineam AM, quae inter omnes alias lineas intra angulum ACM contentas sit brevissima.*

Patet quidem hanc lineam quaesitam esse rectam; interim tamen hanc quaestionem secundum praecepta data resolvi conveniet, ut consensus Methodi cum veritate luculentius perspiciatur. Cum igitur longitudo lineae *AM* pro dato angulo *ACM* debeat esse minima, ponamus angulum hunc *ACM* esse $= x$; seu centro *C*, radio $CB = 1$, describamus circulum, sitque arcus $BS = x$. Tum sit radius *CM* altera variabilis *y;* aequatione enim inter has variabiles *x* et *y* inventa innotescet natura lineae quaesitae *AM*. Iam autem ducto radio proximo *Cm* erit $Ss = dx$ et $mn = dy$, sumto $Cn = CM$; ob triangula vero similia *CSs* et *CMn* erit

$$1 : dx = CM\left[y\right] : Mn[ydx].$$

Ex his itaque erit $Mm = \sqrt{(dy^2 + y^2dx^2)}$; et quia perpetuo ponimus $dy = pdx$, erit $Mm = dx\sqrt{(yy + pp)}$; unde lineae *AM* longitudo erit $= \int dx\sqrt{(yy + pp)}$, quae debet esse minima pro dato ipsius *x* valore, puta $x = a$. At quia haec formula ad casum primum pertinet, linea satisfaciens erit pro quovis valore ipsius *x* minima. Cum igitur sit $Z = \sqrt{(yy + pp)}$, erit

$$dZ = \frac{ydy}{\sqrt{(yy + pp)}} + \frac{pdp}{\sqrt{(yy + pp)}}$$

et in casu primo fiet

$$M = 0,\ N = \frac{y}{\sqrt{(yy + pp)}},\ P = \frac{p}{\sqrt{(yy + pp)}},\ Q = 0,\ R = 0 \text{ etc.}$$

ideoque $dZ = Ndy + Pdp$. Habebitur ergo iste valor differentialis

$$nv \cdot dx\left(N - \frac{dP}{dx}\right),$$

indeque pro solutione ista aequatio $0 = N - \dfrac{dP}{dx}$, quae multiplicata per $pdx = dy$, dat $Ndy = pdP$; quo in aequatione $dZ = Ndy + Pdp$ substitute prodibit $dZ = Pdp + pdP$ et integrando

$$Z + C = Pp \text{ seu } C + \sqrt{(yy + pp)} = \frac{pp}{\sqrt{(yy + pp)}}.$$

Quocirca erit

$$\frac{yy}{\sqrt{(yy+pp)}}=\text{Const.}=b.$$

At est

$$Mm\left[dx\sqrt{(yy+pp)}\right]:Mn\left[ydx\right]=MC\left[y\right]:\frac{yy}{\sqrt{(yy+pp)}};$$

quae quarta proportionalis praebet perpendiculum *CP*, quod ex *C* in tangentem lineae quaesitae *MP* demittitur. Cum igitur hoc perpendiculum *CP* sit constans, intelligitur lineam quaesitam esse rectam, et quia in aequatione inventa prima $Ndx=dP$ duae insunt potentia constantes arbitrariae, conditio haec quaestioni est addenda, ut linea quaesita per data duo puncta transeat; tum igitur linea recta per illa duo puncta ducta quaesito satisfaciet.

EXEMPLUM II

9. *Super axe AP* (Fig. 8) *construere lineam BM ita comparatam, ut, abscissa area ABMP datae magnitudinis, arcus curvae BM illi areae respondens sit omnium minimus.*

Quia pro data area *ABMP* minima longitudo arcus *BM* requiritur, area *ABMP* nobis designanda Fig. 8 erit variabili *x*, altera variabili *y* autem indicemus applicatam curvae *PM*. Iam sit abscissa $AP=t$, erit $x=\int ydt$ ideoque $dt=\frac{dx}{y}$, atque arcus *BM* longitudo erit $=\int\sqrt{\left(dy^2+\frac{dx^2}{yy}\right)}$. Posito ergo $dy=pdx$, minimum esse debet haec formula

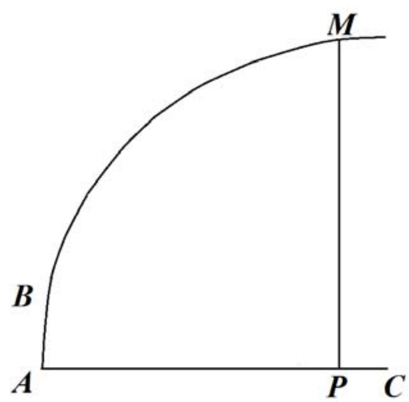

Fig. 8

$$\int dx\sqrt{\left(\frac{1}{yy}+pp\right)}=\int\frac{dx\sqrt{(1+yypp)}}{y}..$$

Erit itaque

$$Z=\frac{\sqrt{(1+yypp)}}{y}\ \text{ et }\ dZ=-\frac{dy}{yy\sqrt{(1+y^2p^2)}}+\frac{yypdp}{y\sqrt{(1+y^2p^2)}};$$

unde

$$M=0\ ,\ N=\frac{-1}{yy\sqrt{(1+y^2p^2)}},\ P=\frac{yp}{\sqrt{(1+y^2p^2)}},\ Q=0\ \text{ etc.}$$

Pertinet ergo haec quaestio ad casum primum ac solutio praebebit lineam curvam, quae pro area quacunque *APMB* abscissa erit brevissima. Prevenietur autem, uti in praecedente Exemplo, ad aequationem hanc $Z = C + Pp$ atque curva quaesita per data duo puncta describi poterit. Erit itaque

$$\frac{\sqrt{(1+yypp)}}{y} = C + \frac{ypp}{\sqrt{(1+yypp)}}$$

seu

$$1 = Cy\sqrt{(1+yypp)} \quad \text{vel} \quad b = y\sqrt{(1+yypp)}\,;$$

hinc fit

$$bb = yy + y^4 pp \quad \text{and} \quad p = \frac{\sqrt{(bb-yy)}}{yy} = \frac{dy}{dx} = \frac{dy}{ydt},$$

ob $dx = ydt$. Erit igitur

$$dt = \frac{ydy}{\sqrt{(bb-yy)}} \quad \text{et} \quad t = c \pm \sqrt{(bb-yy)}.$$

Quare linea quaesita erit Circulus, centro alicubi in axe *AP*, puta in *C*, assumto; isque inter omnes alias curvas per eadem duo quaecunque puncta ductas, pro data resecta area *ABMP*, habebit arcum *BM* brevissimum.

EXEMPLUM III

10. *Eductis* (Fig. 7) *ex puncto fixo C radiis CA, CM intra eos describere curvam AM, quae pro dato spatio ACM habeat arcum AM brevissimum.*

Quia arcus *AM* minimus esse debet, si spatium *ACM* datae magnitudinis abscindatur, ponatur area haec $ACM = x$ atque radius *CM* designetur altera variabili *y*. Iam ponatur arcus *BS*, radio $CB = 1$ descriptus, $= t$; erit, ut ante vidimus, $Mn = ydt$ et area $MCm = \frac{1}{2}yydt = dx$, unde fit $dt = \frac{2dx}{yy}$. Quia porro est

$$Mm = \sqrt{(dy^2 + y^2dt^2)} = \sqrt{(dy^2 + \frac{4dx^2}{yy})},$$

sit $dy = pdx$, minimumque esse debet $\int \frac{dx}{y}\sqrt{(4+ppyy)}$.

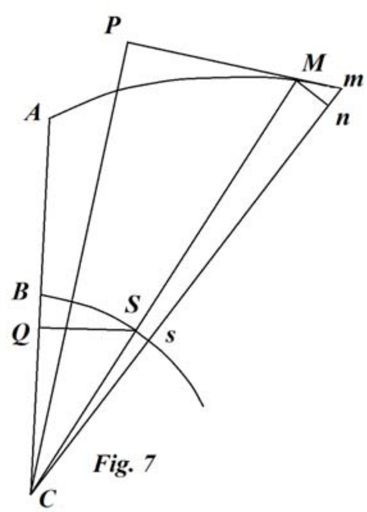

Fig. 7

Cum igitur sit

$$Z = \frac{\sqrt{(4+y^2p^2)}}{y}, \text{erit}$$

$$M=0\ ,\ N=-\frac{4}{yy\sqrt{(4+y^2p^2)}}\ \text{ et }\ P=\frac{yp}{\sqrt{(4+y^2p^2)}},\ Q=0\ \text{ etc.}$$

Hinc resultat ista aequatio $Z=C+Pp$, propterea quod sit $M=0$; ideoque

$$\frac{\sqrt{(4+y^2p^2)}}{y}=C+\frac{ypp}{\sqrt{(4+y^2p^2)}}$$

seu

$$4=Gy\sqrt{(4+yypp)}\ \text{ vel }\ 2b=yy\sqrt{(4+yypp)}\ ;$$

hincque

$$p=\frac{2\sqrt{(bb-yy)}}{yy}=\frac{dy}{dx}=\frac{2dy}{yydt};$$

ac $dt=\dfrac{dy}{\sqrt{(bb-yy)}}$; itemque integrando

$$t=\text{A}\sin\frac{y}{b}+\text{A}\sin\frac{c}{b}=\text{A}\sin\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}.$$

In *AC* ex *S* demittatur perpendiculum $QS=\sin At$, erit

$$QS=\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}.$$

At ex aequatione $t+\text{Const.}=\text{A}\sin\dfrac{y}{b}$ colligitur curva quaesita (Fig. 9) esse Circulus *AME* per punctum fixum *C* transiens. Describatur enim super diametro quacunque *CE* in *C* terminata Circulus *CAME,* arcus *AM* interceptus inter radios *ACM* pro data area *ACM* erit minimus. Scilicet si alia quaecunque curva per duo quaecunque puncta in hoc Circulo sita describatur binisque radiis ex *C* ductis area aequalis areae *ACM* abscindatur, arcus illius curvae respondens perpetuo maior erit quam arcus *AM.* Quod ut appareat, ducatur ex *C* ad *CE* normalis *CD* in eamque ex *S* perpendiculum *SQ* demittatur; erit triangulum *SCQ* simile triangulo *CEM* hincque

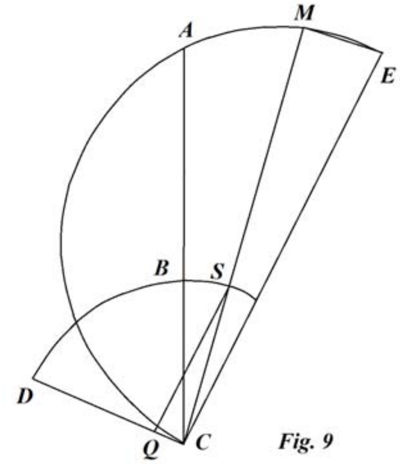

Fig. 9

$$CE : CM[y] = CS[1] : SQ \text{ seu } SQ = \frac{y}{CE} = \sin A \cdot DBS, \text{vel } DBS = A\sin\frac{y}{CE}.$$

Posita ergo diametro $CE = b$ et quia est $DBS = BS + BD = t + \text{Const.}$,
erit

$$t + \text{Const.} = A\sin\frac{y}{b},$$

quae est ipsa illa proprietas, qua curvam quaesitam praeditam esse oportere invenimus.

EXEMPLUM IV

11. *In superficie quacunque* (Fig. 10) *sive convexa sive concava ducere lineam, quae sit intra suos terminos omnium brevissima.*

Sumatur planum quodcunque, ad quod superficies referatur, *APQ* in eoque capiatur recta *AP* pro axe. Iam ex lineae quaesitae singulis punctis concipiantur perpendicula in hoc planum demitti, quibus describatur linea *AQ*, quae erit proiectio lineae brevissimae in hoc planum ; qua cognita, simul ipsa linea brevissima in superficie proposita innotescet. Vocetur $AP = x, PQ = y$; atque cum natura superficiei detur, ex datis $AP = x$ et $PQ = y$ definiri poterit longitudo perpendicularis *QM* in planum *APQ*, donec superficiem in *M* secet. Quodsi ergo ponatur $QM = z$, longitudo huius lineae *z* dabitur per *x* et *y*, ita ut *z* sit functio definita ipsarum *x* et *y*. Cum igitur sit *z* functio ipsarum *x* et *y*, quae ex aequatione locali ad superficiem datur, ponamus esse $dz = Tdx + Vdy$; eruntque *T* et *V* eiusmodi functiones ipsarum *x* et *y*, ut $Tdx + Vdy$ sit formula differentialis definita; posito nempe $dT = Edx + Fdy$ erit $dV = Fdx + Gdy$ existente littera *F* utrique differentiali communi. Nunc elementum lineae in superficie ductae est

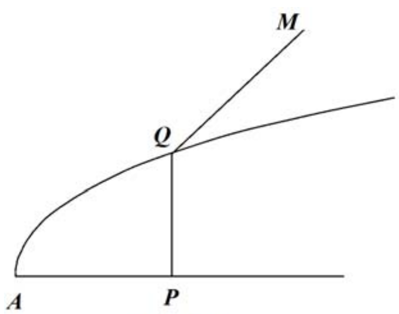

Fig. 10

$$= \sqrt{(dx^2 + dy^2 + dz^2)} = \sqrt{(dx^2 + dy^2 + (Tdx + Vdy)^2)}.$$

Posito ergo $dy = pdx$ minimum esse debet haec formula

$$\int dx\sqrt{(1 + p^2 + T^2 + 2TVp + V^2p^2)} \ ,$$

ita ut sit

$$Z = \sqrt{(1 + p^2 + T^2 + 2TVp + V^2p^2)},$$

unde fit

$$dZ = \frac{\begin{pmatrix} +TEdx & +TFdy & +pdp & +VEpdx & +VFpdy + TVdp \\ +TFpdx + TGpdy + V^2pdp + VFp^2dx + VGppdy \end{pmatrix}}{\sqrt{(1+p^2+T^2+2TVp+V^2p^2)}}$$

Quae formula cum ad casum primum pertineat, proveniet ista aequatio inter *x* et *y*:

$$\frac{TFdx + VFpdx + TGpdx + VGppdx}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}} = d.\frac{p+TV+V^2p}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}}.$$

Est vero

$$Fdx + Gpdx = Fdx + Gdy = dV \ ,$$

unde erit

$$\frac{TdV + VpdV}{\sqrt{(1+pp+(T+Vp)^2)}} = d.\frac{p+TV+V^2p}{\sqrt{(1+pp+(T+Vp)^2)}}$$

$$= \frac{+dp(1+T^2+V^2) + dT(V-Tp) + dV(T+T^2+3T^2Vp+3TV^2p^2+V^3p^3+2Vp+Vp^3)}{(1+pp+(T+Vp)^2)^{3:2}}.$$

Aequatione autem ordinata resultabit haec

$$dp(1+T^2+V^2) + dT(V-Tp) + dV(Vp-Tpp) = 0$$

seu

$$dp = \frac{(Tp-V)(dT+pdV)}{1+T^2+V^2}.$$

Cum vero sit $p = \frac{ddy}{dx}$, erit $dp = \frac{ddy}{dx}$; hincque fiet

$$dxddy = \frac{(Tdy-Vdx)(dxdT+dydV)}{1+T^2+V^2},$$

quae est aequatio differentio-differentialis pro proiectione *AQ* lineae brevissimae in superficie quaesita ideoque indicat eam per duo quaeque puncta duci posse. Aequatio haec inventa invarias formas transmutari potest, quae saepius maiore commodo usurpari poterunt. Ac primo quidem expediet eliminari differentialia *dT* et *dV;* cum enim sit $dz = Tdx + Vdy$, erit

$$ddz = dxdT + dydV + Vddy \quad \text{ideoque} \quad dxdT + dydV = ddz - Vddy,$$

quo valore substituto prodibit ista aequatio

$$dxddy + T^2 dxddy + V^2 dxddy = Tdyddz - Vdxddz - TVdyddy + V^2 dxddy$$

seu

$$dxddy + Tdzddy = Tdyddz - Vdxddz$$

hincque

$$ddy : ddz = Tdy - Vdx : dx + Tdz.$$

Multiplicetur aequatio inventa per *dz* ac in primo termino scribatur $Tdx + Vdy$ loco *dz,* erit

$$Tdx^2 ddy + Vdxdyddy + Tdz^2 ddy = Tdydzddz - Vdxdzddz.$$

Addatur utrimque $Tdy^2 ddy - Vdxdyddy$, erit

$$Tddy(dx^2 + dy^2 + dz^2) = (dzddz + dyddy)(Tdy - Vdx)$$

seu

$$\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{Tddy}{Tdy - Vdx} = \frac{Tddz}{dx + Tdz}.$$

Vel multiplicetur aequatio per *dx* ac loco *Tdx* scribatur $dz - Vdy$, obtinebitur

$$dx^2 ddy + dz^2 ddy - Vdydzddy = dydzddz - Vdy^2 ddz - Vdx^2 ddz.$$

Addatur utrimque $dy^2 ddy - Vdz^2 ddz$, erit

$$ddy(dx^2 + dy^2 + dz^2) - Vdz(dyddy + dzddz) = dy(dyddy + dzddz) - Vddz(dx^2 + dy^2 + dz^2)$$

ideoque

$$\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{ddy + Vddz}{dy + Vdz};$$

quae aequationes omnes in sequenti expressione continentur:

$$\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{Tddy}{Tdy - Vdx} = \frac{Tddz}{dx + Tdz} = \frac{ddy + Vddz}{dy + Vdz}.$$

Hic notandum est, quia quantitatum *T* et *V* differentialia nusquam occurrunt, perinde esse, sive in *T* et *V* contineatur *z* sive minus. Quovis igitur casu oblato conveniet eam aequationem assumere, quae facillime integrationem admittat. Veluti si superficies proposita sit solidi rotundi conversione cuiuscunque figurae circa axem *AP* nati, erit $yy+zz$ quadrato functionis ipsius *x*, quae sit $=X$, estque applicata illius curvae genitricis abscissae *x* respondens. Erit itaque

$$zdz = XdX - ydy \ \text{ et } \ dz = \frac{XdX}{z} - \frac{ydy}{z},$$

unde fiet

$$T = \frac{XdX}{zdx} \ \text{ et } \ V = \frac{-y}{z}.$$

Sumatur iam, commodi ergo, aequatio, in qua *T* non occurrit, haec

$$\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{ddy + Vddz}{dy + Vdz},$$

quae ob $V = -\frac{y}{z}$ transit in hanc

$$\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{zddy - yddz}{zdy - ydz},$$

cuius integrale est

$$l\sqrt{(dx^2 + dy^2 + dz^2)} = l(\frac{zdy - ydz}{b}), \ \text{seu} \ zdy - ydz = b\sqrt{(dx^2 + dy^2 + dz^2)}.$$

Quoniam nunc est $z = \sqrt{(X^2 - y^2)}$), ponatur $dX = vdx$, erit

$$dz = \frac{Xvdx - ydy}{\sqrt{(X^2 - y^2)}} \quad \text{et} \ zdy - ydz = \frac{X^2dy - Xyvdx}{\sqrt{(X^2 - y^2)}}$$

et

$$\sqrt{(dx^2 + dy^2 + dz^2)} = \frac{\sqrt{(X^2dx^2 - y^2dx^2 + X^2dy^2 + X^2v^2dx^2 - 2Xyvdxdy)}}{\sqrt{(X^2 - y^2)}}.$$

Ergo

$$X^4dy^2 - 2X^3yvdxdy + X^2y^2v^2dx^2$$
$$= b^2X^2dx^2 - b^2y^2dx^2 + b^2X^2dy^2 + b^2X^2v^2dx^2 - 2b^2Xyvdxdy$$

seu

$$dy^2 = \frac{2(b^2 - X^2)Xyvdxdy + X^2y^2v^2dx^2 - b^2X^2dx^2 + b^2y^2dx^2 - b^2X^2v^2dx}{X^2(bb - XX)},$$

quae extracta radice praebet

$$dy = \frac{yvdx}{X} \pm \frac{bdx\sqrt{(1+vv)(yy-XX)}}{X\sqrt{(bb-XX)}}.$$

Quodsi ponatur $y = Xt,$ ut sit $dy = Xdt + tvdx$, fiet

$$\frac{dt}{\sqrt{(tt-1)}} = \frac{bdx\sqrt{(1+vv)}}{X\sqrt{(bb-XX)}} \; ;$$

in qua aequatione, quia X et v sunt functiones ipsius x, variabiles t et $x = a$ se invicem sunt separatae.

EXEMPLUM V

12. *Super axe APN* (Fig. 11) *construere curvam AM eiusmodi, ut abscissa per normalem MN area ANM datae magnitudinis arcus AM sit minimus.*

Quia pro definita areae *AMN* magnitudine arcus *AM* minimus esse debet, ponatur area $AMN = ax$, positoque $x = a$, quo casu area *AMN* sit $= aa$, fiat arcus *AM* minimus. Ponatur porro applicata orthogonalis $MP = y$, abscissa $AP = t$ et subnormalis $PN = u$; erit

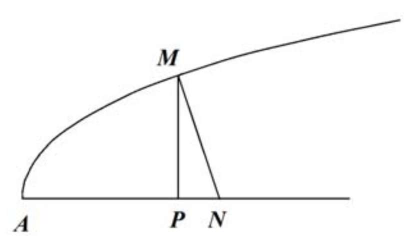

Fig. 11

$$ax = \int ydt + \tfrac{1}{2}uy \text{ et } u = \frac{ydy}{dt};$$

elementum vero arcus *AM* erit $= \dfrac{dy\sqrt{(yy+uu)}}{u}$. Porro cum sit

$$adx = ydt + \tfrac{1}{2}(udy + ydu) \text{ et } dt = \frac{ydy}{u},$$

erit

$$audx = yydy + \tfrac{1}{2}uudy + \tfrac{1}{2}yudu \text{ et } du = \frac{2adx}{y} - \frac{2ydy}{u} - \frac{udy}{y}.$$

Iam ponatur $dy = pdx$, minimum esse debebit

$$\int \frac{pdx\sqrt{(yy+uu)}}{u},$$

atque u est quantitas, cuius valor ex hac aequatione

$$du = dx\left(\frac{2a}{y} - \frac{2yp}{u} - \frac{up}{y}\right)$$

definiri debet. Pertinet itaque haec quaestio ad Casum quintum; cum quo si comparatio instituatur, sit $u = \Pi$ et $Z = \frac{p\sqrt{(yy+\Pi^2)}}{\Pi}$;
unde

$$L = \frac{-pyy}{\Pi^2\sqrt{(yy+\Pi^2)}},\ M = 0,\ N = \frac{yp}{\Pi\sqrt{(yy+\Pi^2)}} \text{ et } P = \frac{\sqrt{(yy+\Pi\Pi)}}{\Pi}.$$

Deinde cum sit $\Pi = \int dx\left(\frac{2a}{y} - \frac{2yp}{\Pi} - \frac{\Pi p}{y}\right)$, fit

$$[Z] = \frac{2a}{y} - \frac{2yp}{\Pi} - \frac{\Pi p}{y}$$

et differentiando erit

$$[L] = \frac{2yp}{\Pi^2} - \frac{p}{y},\ [M] = 0,\ [N] = -\frac{2a}{yy} - \frac{2p}{\Pi} + \frac{\Pi p}{yy} \text{ et } [P] = -\frac{2y}{\Pi} - \frac{\Pi}{y}.$$

Iam erit

$$\int [L]dx = \int \frac{2ydy}{\Pi^2} - ly \text{ et } e^{\int [L]dx} = \frac{e^{\int 2ydy:\Pi\Pi}}{y};$$

at est $Ldx = -\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}$; unde fiet

$$\int e^{\int [L]dx} Ldx = -\int \frac{e^{\int 2ydy:\Pi\Pi} ydy}{\Pi^2\sqrt{(yy+\Pi^2)}},$$

cuius valor posito $x = a$ fiat $= H$, sitque

$$V = e^{-\int 2ydy:\Pi\Pi} y\left(H + \int \frac{e^{\int 2ydy:\Pi\Pi} ydy}{\Pi^2\sqrt{(yy+\Pi^2)}}\right).$$

His praeparatis erit aequatio satisfaciens $(N + [N]V)dx = d\cdot(P + [P]V)$

sive substitutionibus factis

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi\Pi)}}-\frac{2aVdx}{yy}-\frac{2Vdy}{\Pi}+\frac{\Pi Vdy}{yy}=d\cdot\left(\frac{\sqrt{(yy+\Pi\Pi)}}{\Pi}-\frac{2Vy}{\Pi}-\frac{\Pi V}{y}\right).$$

At est

$$2adx=yd\Pi+\frac{2yydy}{\Pi}+\Pi dy\,;$$

unde erit

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}}-\frac{Vd\Pi}{y}-\frac{4Vdy}{\Pi}=d\cdot\left(\frac{\sqrt{(yy+\Pi^2)}}{\Pi}-\frac{2Vy}{\Pi}-\frac{\Pi V}{y}\right)=$$

$$\frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}}-\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vdy}{\Pi}-\frac{2ydV}{\Pi}+\frac{2Vyd\Pi}{\Pi^2}-\frac{\Pi dV}{y}-\frac{Vd\Pi}{y}+\frac{\Pi Vdy}{yy}\,;$$

hincque

$$\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vdy}{\Pi}+\frac{2ydV}{\Pi}-\frac{2Vyd\Pi}{\Pi^2}+\frac{\Pi dV}{y}-\frac{\Pi Vdy}{yy}=0.$$

Verum est generaliter $dV=-Ldx-V[L]dx$; unde erit

$$dV=\frac{yydy}{\Pi^2\sqrt{(yy+\Pi\Pi)}}-\frac{2Vydy}{\Pi^2}+\frac{Vdy}{y}$$

hincque

$$\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi\Pi)}}=\frac{dVd\Pi}{dy}+\frac{2Vyd\Pi}{\Pi^2}-\frac{Vd\Pi}{y}\,;$$

quo substituto oritur

$$\frac{dVd\Pi}{dy}-\frac{2Vdy}{\Pi}+\frac{2ydV}{\Pi}+\frac{\Pi dV}{y}-\frac{Vd\Pi}{y}-\frac{\Pi Vdy}{yy}=0\,;$$

hoc est

$$dV\left(\frac{d\Pi}{dy}+\frac{2y}{\Pi}+\frac{\Pi}{y}\right)=V\left(\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}\right)=\frac{ydV}{dy}\left(\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}\right);$$

quae aequatio, cum sit divisibilis per $\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}$, duplicem dat solutionem. Quarum prima erit $\frac{dV}{V}=\frac{dy}{y}$, quae praebet $V=cy$; quoniam vero V evanescere debet in casu minimi, eodem casu erit $y=0$; scilicet posito $x=a$ fiet $y=0$. Cum nunc sit $V=cy$, facta substitutione aequatione

$$dV=\frac{yydy}{\Pi^2\sqrt{(yy+\Pi^2)}}-\frac{2Vydy}{\Pi^2}+\frac{Vdy}{y}$$

erit

$$\frac{yydy}{\Pi^2\sqrt{(y^2+\Pi^2)}}=\frac{2cyydy}{\Pi^2};$$

hincque vel $y=0$ vel $dy=0$, quo casu prodit linea recta axi parallela, vel $\Pi=\infty$, quo casu prodit linea recta ad axem normalis, vel etiam $\sqrt{y^2+\Pi\Pi}=MN=\text{Const.}$, quae aequatio dat Circulum; atque integer semicirculus, ob $y=0$ in casu minimi, quaesito satisfaciet. Secunda solutio prodit ex divisore

$$\frac{d\Pi}{y}+\frac{2dy}{\Pi}+\frac{\Pi dy}{yy}=0 \text{ seu } \Pi d\Pi+\frac{\Pi\Pi dy}{y}+2ydy=0,$$

quae multiplicata per yy fit $yy\Pi d\Pi+\Pi\Pi ydy+2y^3dy=0$, cuius integrale est $\Pi^2y^2+y^4=C$, hincque $\Pi=\frac{\sqrt{(b^4-y^4)}}{y}$; quae aequatio, quia non pendet ab V, pro quocunque valore ipsius x satisfaciet. Erit autem introducta abscissa $AP=t$ ob $u=\Pi=\frac{ydy}{dt}$ ista aequatio

$$\frac{ydy}{dt}=\frac{\sqrt{(b^4-y^4)}}{y}, \text{ unde } dt=\frac{yydy}{\sqrt{(b^4-y^4)}},$$

ex qua aequatione intelligitur Elasticam rectangulam quaesito satisfacere; ita ut pro area *ANM* inter normales *AN* et *MN* arcus curvae *AM* sit brevissimus. Haec autem curva per data duo puncta, siquidem axis *AP* sit positione datus, describi potest.

SCHOLION 2

13. Ex his Exemplis eximius usus, quem habet nostra Methodus in Problematis etiam diversi generis resolvendis, abunde patet; inprimis autem ultimum Exemplum nonnullas

notatu maxime dignas suppeditat circumstantias, ex quibus natura solutionis illustrari poterit. Quoniam enim duplex aequatio ob factores duos nata est, duplex quoque solutio prodiit; quarum prior lineam satisfacientem absolute determinat, ita ut ea per data duo puncta duci nequeat; dat enim vel lineam rectam vel semicirculum. Linea recta duplici modo quaestionem solvit, dum est vel normalis ad axem AP vel eidem parallela; et quemadmodum utraque satisfaciat, manifestum est: nam in ea, quae est normalis ad axem, portio, quae cum axe et normali datum spatium comprehendit, perpetuo est infinite parva ideoque revera minima; altera recta axi parallela aliquanto latius patet, cum ea per datum punctum duci possit; et quia ipsae applicatae ad eam sunt normales ac spatium abscissum sit ut ipsa abscissa, eius respectu linea illa recta utique erit brevissima. Semicirculus deinde, qui ex prima solutione prodiit, ita absolute satisfacit, ut, proposita spatii abscindendi quantitate, ipse semicirculus determinetur, eius enim area esse debet $= aa$. Secunda autem solutio, quae curvam Elasticam rectangulam praebuit, latius patet; nam per data duo quaecunque puncta eiusmodi curva traduci potest eaque inter omnes alias curvas per eadem puncta transeuntes hac gaudebit praerogativa, ut, si in omnibus curvis per normales areae aequales abscindantur, arcus Elasticae futurus sit omnium minimus. His igitur expositis pergamus ad usum Methodi traditae ostendendum in iis maximi minimive investigationibus, in quibus maximi minimive formula non est talis expressio integralis simplex $\int Zdx$, qualem formam hactenus perpetuo tractavimus, verum est composita ex duabus pluribusve huiusmodi formulis quomodocunque. Ac prima quidem, si maximum minimumve esse debeat aggregatum duarum pluriumve formularum integralium, puta $\int Zdx + \int Ydx - \int Xdx$, operatio nulla difficultate laborat; quia enim formula maximi minimive est $\int dx(Z + Y - X)$, haec tanquam simplex formula integralis tractari eiusque valor differentialis assignari poterit. Operatio autem eo redibit, ut pro singulis formulis $\int Zdx$, $\int Ydx$ et $\int Xdx$ earum valores differentiales quaerantur earumque loco in formula $\int Zdx + \int Ydx - \int Xdx$ substituantur et, quod oritur, nihilo aequale ponatur; sicque habebitur aequatio quaesito satisfaciens.

PROPOSITIO II. PROBLEMA

14. *Invenire aequationem inter x et y, ut posito $x = a$ fiat hac expressio $\int Zdx \cdot \int Ydx$, quae est productum ex duabus formulis integralibus $\int Zdx$* et *$\int Ydx$, maximum vel minimum.*

SOLUTIO

Ponamus istam aequationem inter x et y iam esse inventam foreque ex ea posito $x = a$ valorem Formulae $\int Zdx = A$ et $\int Ydx = B$; erunt hae quantitates A et B constantes atque earum productum AB maximum vel minimum. Iam ponatur apud Valorem indefinitum x variabilem y augeri particula nv, ex ea utraque quantitas A et B incrementum accipiet, unaquaeque scilicet augebitur valore differentiali ex praecedentibus definiendo. Sit igitur *dA* valor differentialis ipsius *A*, qui respondet formulae integrali $\int Zdx$ posito $x = a$, similique modo sit *dB* valor differentialis ipsius *B* oriundus ex formula $\int Ydx$ posito $x = a$. Cum ergo ex adiecta particula *nv* variabili *y*, abeat *A* in $A + dA$ et B in $B + dB$, productum *AB* transmutabitur in $AB + AdB + BdA + dAdB$; quare, cum *AB* esse debeat maximum vel minimum, oportebit esse $AB = AB + AdB + BdA + dAdB$. Ideoque $0 = AdB + BdA$, ob evanescentem terminum *dAdB* prae reliquis. Ex his itaque oritur sequens Problematis solutio. Quaeratur formulae $\int Zdx$ valor differentialis, qui sit *dA*, sitque *A* valor formulae $\int Zdx$, quem obtinet posito $x = a$. Deinde quaeratur formulae $\int Ydx$ valor differentialis, qui sit *dB*, ac *B* denotet valorem formulae $\int Ydx$, quem recipit posito *x a;* quibus factis habebitur ista aequatio $0 = AdB + BdA$, in qua relatio satisfaciens inter *x* et *y* continebitur. Q. E. I.

COROLLARIUM 1

15. Quanquam in aequatione $0 = AdB + BdA$ insunt quantitates constantes *A* et *B*, tamen eae non sunt arbitrariae, sed utraque per ipsam hanc aequationem definietur. Scilicet, si ex hac aequatione eliciantur valores $\int Zdx$ et $\int Ydx$ ponaturque $x = a$, prodire debent illae quantitates *A* et *B;* unde hae determinabuntur per *a* et per reliquas constantes arbitrarias, quae per integrationem ingredientur.

COROLLARIUM 2

16. Si *Z* et *Y* fuerint functiones determinatae quantitatum *x, y, p, q, r* etc., tum valores differentiales *dA* et *dB* non pendebunt ab *a;* interim tamen quantitas *a* ingreditur in aequationem $0 = AdB + BdA$; ex quo curva inventa tantum pro definito abscissae *x* valore $x = a$ quaesito satisfaciet.

COROLLARIUM 3

17. Ex aequatione autem $0 = AdB + BdA$ particula *nv* omnino egredietur, nam, quia uterque valor differentialis *dA* et *dB* per *nv* multiplicatus prodiit, iterum *nv* per divisionem exterminabitur hocque modo aequatio inter *x* et *y* atque constantes nascetur, qua Problemati satisfiet.

SCHOLION 1

18. Neminem hic forma aequationis $0 = AdB + BdA$ inventae offendat, eo quod speciem formulae differentialis definitae prae se ferat, neque hinc etiam quisquam concludat aequationis $0 = AdB + BdA$ integralem sumi posse hanc: Const. = *AB.* Iam enim significationes explicavimus, quas tribuimus cum litteris *A* et *B* tum etiam formis differentialibus *dA* et *dB;* ex quo intelligere licet vulgarem notandi modum hic non locum habere. Ideo autem hunc notandi modum, etsi a consueto dissentientem, hic adhibere visum est, ut nexus aequationis $0 = AdB + BdA$ cum formula maximi minimive $\int Zdx \cdot \int Ydx$ melius perspiciatur. Cum enim maximum minimumve respondere debeat valori $x = a$, ponamus hoc casu abire $\int Zdx$ in *A* et $\int Ydx$ in *B;* quo facto, maximum

minimumve erit *AB*. Hinc autem sponte nascitur aequatio inventa $0 = AdB + BdA$, siquidem *AB* litteris *A* et *B* tanquam variabilibus spectatis differentietur. Quod cum fuerit factum, in memoriam revocari oportet pro differentialibus *dA* et *dB* accipiendos esse valores differentiales eos, qui conveniunt formulis integralibus $\int Zdx$ et $\int Ydx$, ex quibus ipsae quantitates *A* et *B* constantes prodiere. Hunc nexum ideo annotasse iuvabit, quod infra eundem ad quemcunque compositionis modum, quo formula maximi minimive ex formulis integralibus composita fuerit, aeque patere; similique modo ex ipsa maximi minimive expressione per differentiationem aequationem quaesitam obtineri ostendemus.

EXEMPLUM I

19. *Invenire aequationem inter x et y, ut posito $x = a$ fiat ista expressio $\int ydx \cdot \int xdy$ maximum.*

Fiat $\int ydx = A$ et $\int xdy = B$ posito $x = a$ et quaerantur formularum $\int ydx$ et $\int xdy$ seu $\int xpdx$ valores differentiales; ac formulae $\int ydx$ valor differentialis est $nv \cdot dx \cdot 1$, formulae autem $\int xdy$ seu $\int xpdx$ est

$$nv \cdot dx(-\frac{d}{dx} d \cdot x) = -nv \cdot dx.$$

Erit ergo

$$dA = nv \cdot dx \text{ et } dB = -nv \cdot dx,$$

unde aequatio $0 = AdB + BdA$ abibit in hanc

$$0 = -A \cdot nv \cdot dx + B \cdot nv \cdot dx \text{ seu } A = B.$$

Quaesito ergo omnes aequationes inter *x* et *y* aeque satisfaciunt, dummodo casu $x = a$ fuerit $\int ydx = \int xdy$, hoc est area curvae $= \frac{1}{2} xy$.

EXEMPLUM II

20. *Invenire aequationem inter x et y, ut casu $x = a$ fiat minimum haec expressio $\int ydx \cdot \int dx\sqrt{(1+pp)}$.*

Casu $x = a$ fiat $\int ydx = A$ et $\int dx\sqrt{(1+pp)} = B$. Porro sumendis valoribus differentialibus erit

$$dA = nv \cdot dx \cdot 1 \text{ et } dB = nv \cdot dx\left(-\frac{1}{dx} d \cdot \frac{p}{\sqrt{(1+pp)}}\right) = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}.$$

Hinc prodit sequens aequatio

$$0=-A\cdot nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}+B\cdot nv\cdot dx \text{ seu } Bdx=Ad\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Quae integrata dat

$$x+b=\frac{Ap}{B\sqrt{(1+pp)}}$$

ubi $\frac{A}{B}$ denotat rationem, quam tenet $\int ydx$ ad $\int dx\sqrt{(1+pp)}$ tum, cum sit $x=a$. Sit brevitatis gratia $\frac{A}{B}=c$, erit

$$(x+b)\sqrt{(1+pp)}=cp \text{ et } p=\frac{x+b}{\sqrt{\left(cc-(x+b)^2\right)}}=\frac{dy}{dx}.$$

Integrata ergo hac aequatione resultabit $y=f\pm\sqrt{\left(cc-(x+b)^2\right)}$, ita ut sit $(y-f)^2+(x+b)^2=c^2$, unde patet curvam satisfacientem esse Circulum radio c descriptum, axe ubicunque accepto. Huiusmodi vero Circuli non quivis arcus satisfaciet, verum is tantum, qui per c radium Circuli multiplicatus producit aream; est enim $A=Bc$. Ergo vel radius Circuli c pro lubitu accipi potest ex eoque definietur illa abscissae x magnitudo determinata a; vel, si a detur, ut ponimus, inde vicissim radius c determinabitur. Perspicuum autem est arcum Circuli, qui satisfacit, convexitate sua axem respicere debere; hoc enim casu area fit minor ideoque productum ex area in arcum minimum.

EXEMPLUM III

21. *Invenire curvam, in qua pro data abscissa $x=a$ minimum fiat haec expressio* $\int yxdx\cdot\int xdx\sqrt{(1+pp)}$.

Posito $x=a$ fiat $\int yxdx=A$ et $\int xdx\sqrt{(1+pp)}=B$. Erit autem

$$dA=nv\cdot dx\cdot x \text{ et } dB=-nv\cdot dx\cdot\frac{1}{dx}d\cdot\frac{xp}{\sqrt{(1+pp)}};$$

unde obtinebitur ista aequatio

$$Bxdx=Ad\cdot\frac{px}{\sqrt{(1+pp)}},$$

quae integrata dat

$$xx \pm bb = \frac{2Apx}{B\sqrt{(1+pp)}} = \frac{2cpx}{\sqrt{(1+pp)}}$$

posito $\frac{A}{B} = c$. Hinc

$$p = \frac{xx \pm bb}{\sqrt{\left(4ccxx - (xx \pm bb)^2\right)}} = \frac{dy}{dx}$$

ideoque pro curva habebitur haec aequatio

$$y = \int \frac{xx \pm bb}{\sqrt{\left(4ccxx - (xx \pm bb)^2\right)}}.$$

De qua notandum est, si fiat $b = 0$, tum prodire aequationem pro Circulo
$y = \int \frac{xdx}{\sqrt{(4cc - xx)}}$, cuius radius sit 2*c*.

SCHOLION 2

22. Eadem haec Exempla omnia quoque resolvi possunt per Methodum supra iam traditam; quare, cum utraque via eadem solutio obtineatur, iuvabit solutionem per alteram viam uno Exemplo exhiberi. Sumamus igitur tertium Exemplum, in quo maximi minimive formula $\int yxdx \cdot \int xdx\sqrt{(1+pp)}$, differentiando iterumque integrando per partes, reducitur ad hanc formam

$$\int yxdx \int xdx\sqrt{(1+pp)} + \int xdx\sqrt{(1+pp)} \int yxdx,$$

cuius utrumque membrum in Casu secundo supra paragrapho 7 exposito continetur. Quaeratur itaque utriusque valor differentialis, eorum enim summa posita $= 0$ dabit aequationem pro curva quaesita. Formula autem $\int yxdx \cdot \int xdx\sqrt{(1+pp)}$ cum Casu secundo collata dabit $\Pi = \int xdx\sqrt{(1+pp)}$ et $Z = yx\Pi$; unde fit

$$L = yx,\ M = y\Pi,\ N = x\Pi,\ P = 0 \text{ etc.}$$

Deinde erit $[Z] = x\sqrt{(1+pp)}$; indeque

$$[M] = \sqrt{(1+pp)},\ \ [N] = 0 \text{ et } [P] = \frac{xp}{\sqrt{(1+pp)}}.$$

Porro est $\int Ldx = \int yxdx$, cuius valor posito $x = a$, quem generaliter posuimus H, hic in solutione Exempli est A, ita ut sit $V = A - \int yxdx$. Quare huius formulae valor differentialis erit

$$= nv \cdot dx\left(x\Pi - \frac{1}{dx} d \cdot \frac{xp\left(A - \int yxdx \right)}{\sqrt{(1+pp)}} \right)$$

$$= nv \cdot dx\left(x\int xdx\sqrt{(1+pp)} - \frac{1}{dx} d \cdot \frac{xp}{\sqrt{(1+pp)}} + \frac{1}{dx} d \cdot \frac{xp\int yxdx}{\sqrt{(1+pp)}} \right).$$

Altera formula $\int xdx\sqrt{(1+pp)}\int yxdx$, cum Casu secundo paragraphi 7 collata, dat $\Pi = \int yxdx$ et $Z = x\Pi\sqrt{(1+pp)}$, unde erit

$$L = x\sqrt{(1+pp)},\ M = \Pi\sqrt{(1+pp)},\ N = 0 \ \text{ et } \ P = \frac{x\Pi p}{\sqrt{(1+pp)}};$$

hincque $\int Ldx = \int xdx\sqrt{(1+pp)}$; quare, cum H sit valor ipsius $\int Ldx$ posito $x = a$, erit $H = B$ et $V = B - \int xdx\sqrt{(1+pp)}$. Porro est $[Z] = yx$, hincque $[M] = y$, $[N] = x$ et $[P] = 0$. Ex his prodit valor differentialis

$$= nv \cdot dx\left(Bx - x\int xdx\sqrt{(1+pp)} - \frac{1}{dx} d \cdot \frac{xp\int yxdx}{\sqrt{(1+pp)}} \right).$$

His igitur valoribus differentialibus ambobus additis emerget huius expressionis compositae

$$\int yxdx\int xdx\sqrt{(1+pp)} + \int xdx\sqrt{(1+pp)}\int yxdx$$

seu huius

$$\int yxdx \cdot \int xdx\sqrt{(1+pp)},$$

quae in Exemplo erat proposita, valor differentialis

$$= nv \cdot dx\left(Bx - \frac{A}{dx} d \cdot \frac{xp}{\sqrt{(1+pp)}} \right),$$

ex quo pro curva aequatio erit haec

$$Bxdx = Ad \cdot \frac{xp}{\sqrt{(1+pp)}},$$

quam eandem in solutione Exempli invenimus. Similis autem consensus in genere deprehendetur, si quis expressionem $\int Zdx \cdot \int Ndx$ eodem modo tractare voluerit.

PROPOSITIO III. PROBLEMA

23. *Invenire aequationem inter x et y eius conditionis, ut posito* $x = a$ *ista fractio* $\frac{\int Zdx}{\int Ydx}$ *obtineat maximum minimumve valorem, existentibus Z et Y functionibus quibuscunque ipsarum x, y, p, q, r etc. sive determinatis sive indeterminatis.*

SOLUTIO

Casu, quo fit $x = a$, sit $\int Zdx = A$ atque $\int Ydx = B$ eritque $\frac{A}{B}$ maximum vel minimum, siquidem relatio inter *x* et *y* recte fuerit assignata. Erit igitur fractio $\frac{A}{B}$ aequalis eidem huic fractioni $\frac{\int Zdx}{\int Ydx}$, casu quo $x = a$, si alicubi una applicata *y* augeatur particula *nv*. Tum vero fiet $\int Zdx$ aequalis ipsi *A*, una cum valore differentiali formulae $\int Zdx$, qui sit $= dA$; similique modo $\int Ydx$ abibit in *B* auctum valore differentiali formulae $\int Ydx$, qui sit $= dB$; sicque ex adiecta particula *nv* ad applicatam *y*, casu quo $x = a$, transibit fractio $\frac{\int Zdx}{\int Ydx}$ in hanc $\frac{A+dA}{B+dB}$; quae aequalis esse debet fractioni $\frac{A}{B}$; unde nascitur ista aequatio $BdA = AdB$, quae praebebit aequationem inter *x* et *y* quaesitam. Q. E. I.

COROLLARIUM 1

24. Ad hanc igitur aequationem inter *x* et *y* inveniendam effici debet, ut valores differentiales ipsarum $\int Zdx$ et $\int Ydx$ proportionales fiant ipsis harum formularum valoribus, quos obtinent posito $x = a$.

COROLLARIUM 2

25. Quanquam in hac aequatione inventa $BdA = AdB$ duae in esse videantur constantes incognitae *A* et *B*, tamen ambas in unam compingere licet. Posito enim $\frac{A}{B} = C$ erit

$dA = CdB$; inventaque aequatione, ex valore a loco x substituto determinabitur valor ipsius C.

SCHOLION

26. Si huius et praecedentis Problematis solutiones inter se conferantur, ingens in iis deprehendetur consensus. Nam si maximum minimumve esse debeat factum $\int Zdx \cdot \int Ydx$, orta est ista aequatio $0 = AdB + BdA$; sin autem quotus $\frac{\int Zdx}{\int Ydx}$ debeat esse vel maximus vel minimus, inventa est ista aequatio $0 = AdB - BdA$; utroque autem casu litterae A, B et dA, dB eosdem retinent valores. Quare, cum A et B sint quantitates constantes, ambae aequationes tantum ratione signi constantis differunt; posito enim $\frac{A}{B} = C$ priore casu habetur $dA = -CdB$, posteriore vero $dA = +CdB$. Ex quo pro utroque casu etiam eadem fere prodibit solutio, quia totum discrimen tantum in signo quantitatis constantis C situm erit. Quodsi ergo aequatio inter x et y fuerit inventa, quae contineat pro $x = a$ factum $\int Zdx \cdot \int Ydx$ maximum vel minimum eadem aequatio levi adhibita mutatione simul continebit quotum $\frac{\int Zdx}{\int Ydx}$ maximum vel minimum. Perspicuum autem est, sive $\frac{\int Zdx}{\int Ydx}$ debeat esse maximum ve minimum, sive $\int Zdx$, utroque casu eandem plane esse prodituram aequationem. Hanc vero convenientiam ipsa rei natura postulat ; nam $\frac{\int Zdx}{\int Ydx}$ est maximum, tum eo ipso erit $\frac{\int Ydx}{\int Zdx}$ minimum et vicissim, unde utrique quaestioni eandem solutionem satisfacere necesse est. Caeterum hunc quoque nexum observasse iuvabit inter maximi minive formulam $\frac{\int Zdx}{\int Ydx}$, quae posito $x = a$ abibit in $\frac{A}{B}$, et inter aequationem inventam $BdA - AdB = 0$, haec enim aequatio oritur ex differentiatione formulae A ponendo eius differentiale $= 0$; istiusmodi autem nexum perpetuo locum haberein sequente Propositione demonstrabimus.

EXEMPLUM I

27. *Invenire curvam, cuius area coordinatis orthogonalibus abscissa ad arcum curvae maximam teneat rationem, si abscissae datus valor a tribuatur.*

Posita curvae quaesitae abscissa $= x$, applicata $= y$, erit area $= \int ydx$ et arcus $= \int dx\sqrt{(1+pp)}$ posito $dy = pdx$; maximum ergo esse debet $\frac{\int ydx}{\int dx\sqrt{(1+pp)}}$, casu quo ponitur $x = a$. Sit igitur casu $x = a$ valor formulae $\int ydx$ seu area $= A$ et $\int dx\sqrt{(1+pp)}$ seu arcus abscissae *a* respondens $= B$. Deinde formulae $\int ydx$ valor differentialis *dA* erit $= nv \cdot dx \cdot 1$ et formulae $\int dx\sqrt{(1+pp)}$ seu

$$dB = nv \cdot dx\left(-\frac{1}{dx} d \cdot \frac{p}{\sqrt{(1+pp)}}\right) = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}.$$

Quibus valoribus in aequatione $BdA = AdB$ substitutis prodibit pro curva quaesita sequens aequatio

$$Bdx = -Ad \cdot \frac{p}{\sqrt{(1+pp)}}.$$

Ponatur $\frac{A}{B} = c$, ita ut pro abscissa $x = a$ area curvae fiat aequalis producto ex arcu in hanc constantem *c*. Erit ergo $dx = -cd \cdot \frac{p}{\sqrt{(1+pp)}}$ et integrando

$$x = b - \frac{cp}{\sqrt{(1+pp)}} \quad \text{seu} \quad cp = (b-x)\sqrt{(1+pp)},$$

hincque $p = \frac{b-x}{\sqrt{(c^2-(b-x)^2)}} = \frac{dy}{dx}$. Erit ergo

$$y = \int \frac{(b-x)dx}{\sqrt{(c^2-(b-x)^2)}} = f \pm \sqrt{(c^2-(b-x)^2)}$$

seu $(y-f)^2+(b-x)^2=cc$; unde constat curvam quaesitam esse Circulum radio *c* descriptum ad rectam quamcunque tanquam axem relatum. Huius autem Circuli ea tantum portio quaesito satisfacit, quae respondet abscissae *a*, a quo valore pendet *c*, ita ut sumpta abscissa $=a$ area aequalis fiat producto ex arcu in radium Circuli multiplicato. Quodsi ergo vicissim radius *c* detur, tanta in axe abscissa abscindi debet, ut arcus per radium multiplicatus praebeat aream. Infinitis igitur modis quaesito satisfieri potest; quaestio autem erit determinata, si duo praescribantur puncta, per quae curva quaesita sit transeunda. Sumamus igitur radium *c* tanquam cognitum eoque describamus (Fig. 12) Circulum *BMD* centro *C*. Porro sumatur linea quaecunque *APD* pro axe in eaque *A* pro origine abscissarum. Hoc iam facto quaestioni satisfiet, si applicata *PM* tantum spatium *ABMP* abscindatur, ut id sit aequale producto ex arcu *BM* in radium Circuli *BC*. Quia autem sector *BCM* est $=\frac{1}{2}BM\cdot BC$, oportet aream *ABMP* esse duplo maiorem sectore *BCM*. Apparet autem, sumto pro lubitu cum axe tum eius initio, saepe-numero conditionem praescriptam nequidem impleri posse. Nam si axis *AD* per centrum transeat, tum area *ABMP* perpetuo minor erit quam duplum sectoris *BCM*, nisi, arcu *BM* infinite parvo, prima applicata *BA* simul per centrum transeat; sin autem axis *AD* supra centrum transiret, tum nullo modo conditioni inventae satisfieri potest. Quare necesse est, ut axis *AD* infra centrum *C* , ducatur, qua de re multae egregiae observationes geometricae fieri possent, si ratio instituti id permitteret. Caeterum, si haec Solutio cum Exemplo secundo praecedentis Propositionis paragraphi 20 comparetur, apparebit eandem prorsus aequationem esse inventaro, sive $\int ydx\cdot\int dx\sqrt{(1+pp)}$ debeat esse minimum, sive $\dfrac{\int ydx}{\int dx\sqrt{(1+pp)}}$ maximum. Discrimen tamen in hoc consistit, quod radius Circuli $c=\dfrac{A}{B}$ altero casu affirmative, altero negative debeat accipi. Scilicet, si $\int ydx\cdot\int dx\sqrt{(1+pp)}$ debeat esse minimum, arcus *BM* convexitate sua spatium *ABMP*, altero autem casu concavitate claudere debet.

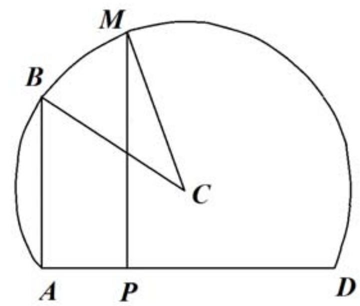

Fig. 12

EXEMPLUM II

28. *Intra datum angulum* (Fig. 7) *ACM curvam AM construere ita comparatam, ut area ACM per arcum AM divisa sit omnium maxima.*

Ponatur angulus ACM seu arcus circuli BS radio $CB=1$ descriptus $=x$, qui in casu proposito fiat $=a$, quo $\frac{ACM}{AM}$ fieri debet maximum.

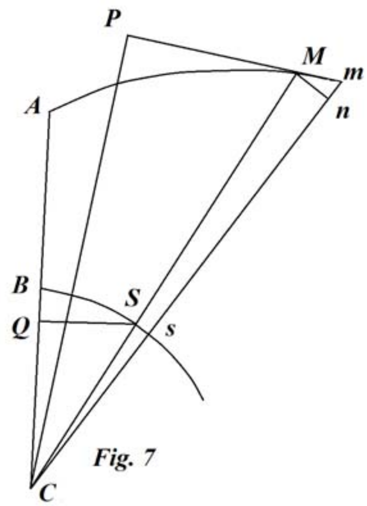

Fig. 7

Ponatur porro $CM=y$ sitque $dy=pdx$, erit $Mn=ydx$ et area

$ACM=\frac{1}{2}\int yydx$; arcus autem AM reperitur $=\int dx\sqrt{(yy+pp)}$; unde haec fractio $\frac{\int yydx}{2\int dx\sqrt{(yy+pp)}}$ seu eius duplum $\frac{\int yydx}{\int dx\sqrt{(yy+pp)}}$ debebit esse maximum. Sit casu, quo $x=a$ est, $\int yydx=A$ et $\int dx\sqrt{(yy+pp)}=B$; erit, si $x=a$, area $ACM=\frac{1}{2}A$ et arcus $AM=B$. Iam formulae $\int yydx=A$ valor differentialis dA est $=nv\cdot dx\cdot 2y$ et formulae $\int dx\sqrt{(yy+pp)}$ valor differentialis dB est $nv\cdot dx\left(\frac{y}{\sqrt{(yy+pp)}}-\frac{1}{dx}\cdot\frac{p}{\sqrt{(yy+pp)}}\right)$. Quare, cum generaliter invenerimus pro curva hanc aequationem $BdA=AdB$, erit divisione per nv instituta

$$2Bydx=\frac{Aydx}{\sqrt{(yy+pp)}}-Ad\cdot\frac{p}{\sqrt{(yy+pp)}}.$$

Multiplicetur ea per p, ob $pdx=dy$ erit

$$2Bydy=A\left(\frac{ydx}{\sqrt{(yy+pp)}}-pd\,\frac{p}{\sqrt{(yy+pp)}}\right).$$

At est

$$d\cdot\sqrt{(yy+pp)}=\frac{ydy}{\sqrt{(yy+pp)}}+\frac{pdp}{\sqrt{(yy+pp)}}$$

et

$$\frac{ydy}{\sqrt{(yy+pp)}}=d\cdot\sqrt{(yy+pp)}-\frac{p}{\sqrt{(yy+pp)}}dp\,;$$

unde fiet

$$2Bydy=A\left(d\cdot\sqrt{(yy+pp)}-d\cdot\frac{pp}{\sqrt{(yy+pp)}}\right),$$

ob

$$pd\frac{p}{\sqrt{(yy+pp)}}+dp\frac{p}{\sqrt{(yy+pp)}}=d\cdot p\cdot\frac{p}{\sqrt{(yy+pp)}}=d\cdot\frac{pp}{\sqrt{(yy+pp)}}.$$

Quare integrando habebitur, si $\frac{A}{B}=c$ ponatur, ista aequatio

$$yy\pm bb=c\sqrt{(yy+pp)}-\frac{cpp}{\sqrt{(yy+pp)}}=\frac{cyy}{\sqrt{(yy+pp)}}$$

seu

$$p=\frac{y\sqrt{(c^2y^2-(yy\pm bb)^2)}}{yy\pm bb}=\frac{dy}{dx}$$

hincque

$$dx=\frac{(yy\pm bb)dy}{y\sqrt{(c^2y^2-(yy\pm bb)^2)}}\,;$$

ex qua aequatione facile deduci potest, si sit $cc+4bb$ quantitas positiva, constructionem per quadraturam Circuli absolvi posse. At idem facilius patebit, si loco *dy* vel *pdx* introducamus perpendiculum *CP*, ex *C* in tangentem *MP* demissum. Quodsi autem hoc perpendiculum *CP* ponatur *u*, erit $y:u=dx\sqrt{(yy+pp)}:ydx$ hincque $\frac{yy}{\sqrt{(yy+pp)}}=u$; quamobrem cum esset $yy\pm bb=\frac{cyy}{\sqrt{(yy+pp)}}$; quam constat esse aequationem ad ipsum Circulum. Hoc ut ostendamus,

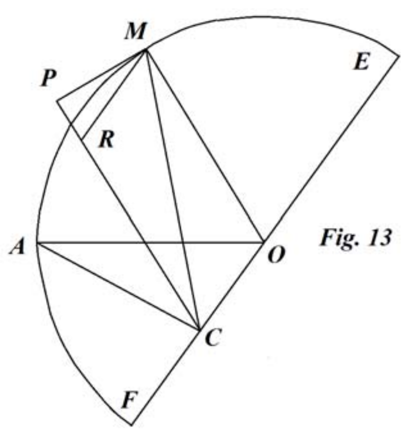

Fig. 13

sumatur Circulus (Fig. 13) quicunque, centro *O*, radio $OM = g$ descriptus, punctumque *C* sumtum sit in *C*, ita ut sit $OC = h$. Iam ducta recta $CM = y$ et $CP = u$, perpendiculari in tangentem *MP*, erit *CP* parallela radio *OM* . Ex *M* ducatur diametro *EF* parallela *MR*, erit $MR = CO = h$, $CR = OM = g$ et $PR = u - g$; quia igitur est $MR^2 = MP^2 + PR^2 = CM^2 - CP^2 + PR^2$, erit

$$h^2 = y^2 - u^2 + (u-g)^2 = y^2 - 2gu + gg$$

hincque $yy + gg - kh = 2gu$; quae comparata cum inventa $yy \pm bb = cu$, fiet $g = \frac{1}{2}c$ et $\pm bb = \frac{1}{2}cc - hh$ seu $hh = \frac{1}{4}cc + bb$. Erit itaque curva quaesita Circulus radio $= \frac{1}{2}c$ descriptus, puncto *C* ubi libuerit accepto. In tali Circulo quaesito satisfaciet arcus *AM*, si fuerit

$$\frac{ACM}{AM} = \frac{A}{2B} = \frac{1}{2}c = \text{radio } OM;$$

hoc est, si fuerit

area $ACM = \text{arc.}AM \cdot AO =$ duplici sectori AOM. Hoc autem fieri nequit, nisi punctum *C* extra Circulum accipiatur; quo casu haec conditio infinitis modis adimpleri potest atque adeo effici, ut curva satisfaciens per data duo puncta transeat.

EXEMPLUM III

29. *Invenire curvam* (Fig.14) *DAD ad axem AC relatam, in qua pro data abscissa AC = a sit* $\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$ *minimum.*

Si ponatur abscissa indefinita $AP = x$, applicata $PM = y$ et $dy = pdx$, exprimit $\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$ distantiam centri gravitatis curvae *MAM*, tanquam uniformiter gravis spectatae a puncto infimo *A;* quae ergo distantia, translato *P* in *C*, debet esse minima. Ad hoc inveniendum, posito $x = a$ sit

$$\int xdx\sqrt{(1+pp)} = A \text{ et } \int dx\sqrt{(1+pp)} = B;$$

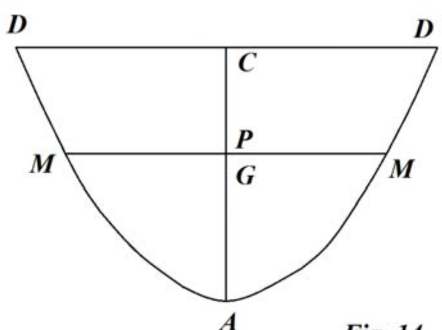

Fig. 14

formulae autem $\int xdx\sqrt{(1+pp)}$ reperitur valor differentialis $dA = -nv \cdot d \cdot \frac{xp}{\sqrt{(1+pp)}}$

et formulae $\int dx\sqrt{(1+pp)}$ valor differentialis *dB*

$$dB = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}};$$

quibus in aequatione $BdA = AdB$ substitutis prodibit

$$Bd \cdot \frac{xp}{\sqrt{(1+pp)}} = Ad \cdot \frac{p}{\sqrt{(1+pp)}}$$

et posito $\frac{A}{B} = c$ erit

$$d \cdot \frac{xp}{\sqrt{(1+pp)}} = cd \cdot \frac{p}{\sqrt{(1+pp)}};$$

unde integrando oritur

$$\frac{xp}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}} - b \text{ seu } b\sqrt{(1+pp)} = (c-x)p;$$

hincque elicitor

$$p = \frac{b}{\sqrt{((c-x)^2 - b^2)}} = \frac{dy}{dx};$$

Erit ergo

$$y = \int \frac{bdx}{\sqrt{((c-x)^2 - b^2)}};$$

quae aequatio indicat curvam quaesitam esse Catenariam, initio abscissarum pro x in loco axis AC quocunque accepto; quin etiam pro axe sumi potest recta quaecunque diametro Catenariae AC parallela in eaque punctum quodcunque pro axis initio. Quomodocunque autem axis eiusque initium constituatur, quaestioni satisfiet ea tantum curvae portio, ubi sit

$$\int xdx\sqrt{(1+pp)} = c\int dx\sqrt{(1+pp)}.$$

Ponamus pro axe ipsam diametrum AC et verticem A pro initio abscissarum accipi. Quia in A, ubi est $x = 0$, fit $\frac{dy}{dx} = p = \infty$, necesse est, ut sit $cc - bb = 0$ ideoque $b = c$. Verum hoc casu fit $y = \int \frac{cdx}{\sqrt{(xx - 2cx)}}$, quae curva sursum directa fit imaginaria, donec fiat

$x > 2c$. Sit ergo $x = 2c + t$, erit $t =$ abscissae AP et $y = PM = \int \frac{cdt}{\sqrt{(2ct+tt)}}$; curvaque *DAD* erit catenaria ordinaria. Quo autem appareat, quanta eius portio quaestioni satisfaciat, notandum est ob $dx = dt$ esse

$$p = \frac{c}{\sqrt{(2ct+tt)}} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{c+t}{\sqrt{(2ct+tt)}};$$

hincque

$$\int dx\sqrt{(1+pp)} = \int \frac{(c+t)dt}{\sqrt{(2ct+tt)}} = \sqrt{(2ct+tt)}.$$

At ipsa expressio

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} \text{ fit } = 2c + \frac{\int tdt\sqrt{(1+pp)}}{\sqrt{(2ct+tt)}},$$

quae ipsi *c* aequalis fieri nullo modo potest. Ex quo concluditur nullam curvae huius portionem quaesito prae reliquis magis satisfacere. Quamobrem initium abscissarum non sumi potest invertice *A*. Sumatur ergo in alio quocunque puncto, positaque $AP = t$ fieri debet $2bt + tt = (c-x)^2 - bb$; unde fit vel $b + t = x - c$ vel $b + t = c - x$. Prior aequatio $x = b + c + t$ locum habere nequit, quia ob $dx = dt$ fieri non potest

$$\frac{\int xdt\sqrt{(1+pp)}}{\int dt\sqrt{(1+pp)}} \text{ seu } \left(b + c + \frac{\int tdt\sqrt{(1+pp)}}{\int dt\sqrt{(2ct+tt)}} \right) = c.$$

Ergo fiat $x = c - b - t$, quo casu abscissae ab aliquo puncto axis *AC* superiori deorsum descendent fierique deberet

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} \text{ seu } \quad c - b - \frac{\int tdt\sqrt{(1+pp)}}{\int dt\sqrt{(2ct+tt)}} = c,$$

quod pariter fieri nequit; ex quo concludendum est nullam portionem magis quam aliam quamvis satisfacere. Hoc autem inde venire videtur, quod Catenaria duas habet partes coniugatas veluti Hyperbola conica hincque semper fieri potest

$$\frac{\int xdx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} = 0,$$

qui est valor minimus. Hoc clarius confirmari potest ex valore invento

$$p=\frac{b}{\sqrt{((c-x)^2-b^2)}};$$

unde fit

$$\sqrt{(1+pp)}=\frac{c-x}{\sqrt{((c-x)^2-b^2)}}=(c-x)r$$

posito brevitatis gratia

$$r=\frac{1}{\sqrt{((c-x)^2-b^2)}}.$$

Oporteret ergo in casu quaesito esse

$$\frac{\int(c-x)xrdx}{\int(c-x)rdx}=c \quad \text{seu} \quad \int(c-x)^2 rdx=0,$$

quod cum casu $x=0$ evanescere debeat, alio insuper casu evanescere deberet.
At est

$$\int(c-x)^2 rdx=\int\frac{(c-x)^2 dx}{\sqrt{((c-x)^2-b^2)}}=$$

$$=-\tfrac{1}{2}(c-x)\sqrt{((c-x)^2-b^2)}-\frac{bb}{2}l\frac{c-x+\sqrt{((c-x)^2-b^2)}}{c+\sqrt{(c^2-b^2)}}+\tfrac{1}{2}c\sqrt{(c^2-b^2)},$$

quae expressio, cum semel fuit $=0$, post, ob $(c-x)^2$ perpetuo affirmativum, continuo crescet neque denuo fieri potest $=0$. Quamobrem ambos terminos integrationis formulae

$$\int\frac{(c-x)^2 dx}{\sqrt{((c-x)^2-b^2)}}$$

inter se congruere oportet; quod evenit, si fuerit $x=c$; quo casu curva satisfaciens abit in lineam rectam axi normalem, quae utique centrum suum gravitatis a se minime habet remotum.

EXEMPLUM IV

30. *Invenire curvam, in qua pro data abscissa* $x = a$ *sit hac expressio* $\frac{\int xydx}{\int dx\sqrt{(1+pp)}}$ *maximum vel minimum.*

Posito $x = a$ fiet $\int xydx = A$ et $\int dx\sqrt{(1+pp)} = B$. Iam formulae $\int xydx$ valor differentialis est $dA = nv \cdot dx \cdot x = nv \cdot xdx$ et formulae $\int dx\sqrt{(1+pp)}$ valor differentialis est

$$dB = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}}.$$

Quare cum sit $BdA = AdB$, habebitur ista aequatio

$$Bxdx = -Ad \cdot \frac{p}{\sqrt{(1+pp)}} \quad \text{seu} \quad xdx = -ccd \cdot \frac{p}{\sqrt{(1+pp)}}$$

posito $A = Bc^2$. Unde integrando obtinebitur $xx = bc - \frac{2ccp}{\sqrt{(1+pp)}}$ hincque

$$p = \frac{bc - xx}{\sqrt{(4c^4 - (bc - xx)^2)}} = \frac{dy}{dx};$$

quae praebet

$$y = \int \frac{(bc - xx)dx}{\sqrt{(4c^4 - (bc - xx)^2)}},$$

quae est aequatio generalis pro curva Elastica, cuius haec proprietas, quod radius osculi ubique abscissae x sit reciproce proportionalis, id quod patet ex aequatione

$$xdx = -ccd \cdot \frac{p}{\sqrt{(1+pp)}},$$

quae abit in estque

$$-\frac{dx}{d \cdot \frac{p}{\sqrt{(1+pp)}}} = \frac{cc}{x},$$

estque

$$-\frac{dx}{d\cdot\frac{p}{\sqrt{(1+pp)}}}=-\frac{dx(1+pp)^{3:2}}{dp}$$

radius osculi in curva. Huius autem curvae tanta portio ab initio computando satisfacit, in qua erit

$$\int yxdx=cc\int dx\sqrt{(1+pp)}=2c^4\int\frac{dx}{\sqrt{(4c^4-(bc-xx)^2)}};$$

quae determinatio eo revocatur, ut effici debeat

$$\int dx\sqrt{(4c^4-(bc-xx)^2)}=(aa-bc)\int\frac{(bc-xx)dx}{\sqrt{(4c^4-(bc-xx)^2)}},$$

si post integrationem utramque ponatur $x=a$. Hoc itaque modo constan illa c per a determinabitur.

PROPOSITIO IV. PROBLEMA

31. *Invenire aequationem inter binas variabiles x et y ita comparatam, ut posita variabili* $x=a$ *maximum minimumve fiat expressio W, quae sit functio quaecunque formularum integralium* $\int Zdx,\ \int Ydx,\ \int Xdx$ *etc., in quibus denotent Z, Y, X etc. functiones quascunque ipsarum x, y, p, q etc. sive determinatas sive indeterminatas.*

SOLUTIO

Ponamus idoneam aequationem inter x et y iam esse inventam positoque $x=a$ fieri $\int Zdx=A,\ \int Ydx=B,\ \int Xdx=C$ etc.; hisque valoribus in expressione W substitutis habebitur revera maximum vel minimum. Quodsi igitur altera variabilis y in uno loco particula nv augeri ponatur atque nascentes hinc mutationes in singulis formulis $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. introducantur, idem pro W valor prodire debet. At ab illa particula nv formulae $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. quaeque suis valoribus differentialibus augebuntur. Si ergo ponatur formulae $\int Zdx$ valor differentialis $=dA$, formulae $\int Ydx=dB$, formulae $\int Xdx=dC$ etc., loco quantitatum A, B, C etc. orientur a particula nv istae auctae $A+dA,\ B+dB,\ C+dC$ etc., quae in W substitutae eundem valorem producere debent, quem ipsae A, B, C etc. Ponamus $A+dA,\ B+dB,\ C+dC$ etc. loco $\int Zdx,\ \int Ydx,\ \int Xdx$ etc. substitutis prodire $W+dW$; eritque $W+dW=W$ ideoque

$dW = 0$. Hic autem valor *dW*, ut ex differentiationis natura liquet, invenitur, si quantitas *W*, postquam in illa loco formularum integralium litterae *A*, *B*, *C* etc. sunt substitutae, differentietur, his ipsis litteris *A*, *B*, *C* etc. tanquam variabilibus tractatis; in hocque differentiali *dA*, *dB*, *dC* etc. valores differentiales formularum respondentium $\int Zdx$, $\int Ydx$, $\int Xdx$ etc. designent. Hac igitur significatione sumptum differentiale quantitatis propositae *W*, si id nihilo aequale ponatur, dabit aequationem inter *x* et *y* quaesitam. Q. E. I.

COROLLARIUM I

32. Si ergo proposita fuerit eiusmodi expressio *W* functio formularum integralium $\int Zdx$, $\int Ydx$, $\int Xdx$ etc., quae pro determinato ipsius *x* valore $= a$ debeat esse maximum vel minimum, tum loco formularum $\int Zdx$, $\int Ydx$, $\int Xdx$ etc. scribantur litterae *A*, *B*, *C* etc., quo facto expressio *W* differentietur his litteris *A*, *B*, *C* etc. solis tanquam variabilibus tractatis atque differentiale ponatur $= 0$.

COROLLARIUM 2

33. In hoc differentiali, in quo inerunt litterae *A* , *B*, *C* etc. cum suis differentialibus *dA*, *dB*, *dC* etc., litterae *A*, *B*, *C* etc. denotabunt respective valores formularum $\int Zdx$, $\int Ydx$, $\int Xdx$ etc., quos induunt posito $x = a$; at differentialia *dA*, *dB*, *dC* etc. exprimunt valores differentiales earundem formularum integralium abscissae $x = a$ respondentes.

COROLLARIUM 3

34. Ex praecedentibus autem apparet, si *Z*, *Y*, *X* etc. fuerint functiones determinatae quantitatum *x*, *y*, *p*, *q* etc., tum valores differentiales *dA*, *dB*, *dC* etc. non a valore *a* pendere; contra vero, si *Z*, *Y*, *X* etc. fuerint functiones indefinitae, tum valores differentiales *dA*, *dB*, *dC* etc. simul a valore *a* pendere debere.

COROLLARIUM 4

35. Cum igitur hoc modo *W* fiat functio litterarum *A*, *B*, *C* etc., eius differentiale huiusmodi habebit formam $FdA + GdB + HdC + \text{etc.}$ hincque aequatio quaesita erit $0 = FdA + GdB + HdC + \text{etc.}$, ubi *F*, *G*, *H* etc. erunt quantitates constantes per *A*, *B*, *C* etc. determinatae.

COROLLARIUM 5

36. Aequatio ergo Problemati satisfaciens constabit ex valoribus differentialibus singularum formularum integralium in maximi minimive expressione *W* contentarum, singulis per constantes quantitates determinatas multiplicatis; horum scilicet productorum aggregatum nihilo aequale positum dabit aequationem desideratam.

SCHOLION 1

37. Potuissemus hanc Problema propositum resolvendi methodum ex solutionibus binorum Problematum praecedentium per inductionem iam concludere, quippe ex quibus iam patebat, si fuerit maximi minimive formula W vel productum ex duabus formulis integralibus vel quotus ex divisione unius per alteram ortus, tum differentiale expressionis W modo exposito sumtum praebere aequationem Problemati convenientem. Praestitit autem hoc Problema ob summam eius extensionem singulari solutione munire. In hoc enim Problemate continentur omnes omnino Quaestiones, quae in hoc genere, quo expressio quaepiam maxima minimave desideratur, unquam proponi atque excogitari possunt, ideoque per istam Propositionern penitus exhausta est methodus maximorum ac minimorum absoluta, quam primo pertractandam suscepimus. Praeterea hic notandum est, si expressio W non tantum formulas integrales, uti posuimus, complectatur, verum etiam functiones determinatas ipsarum x, y, p, q etc., tum solutionem nihilo difficiliorem reddi. Nam pari modo loco harum functionum determinatarum quantitates constantes poni debent, in quas scilicet abeunt posito $x = a$; at postmodum in differentiatione ipsius W has quantitates etiam tanquam constantes tractari oportet, eo quod functiones determinatae nullos valores differentiales recipiunt. Quo autem clarius appareat, quomodo istiusmodi expressiones tractari conveniat, in sequentibus Exemplis nonnulla occurrent, quae hoc argumentum penitus illustrabunt.

EXEMPLUM I

38. *Invenire curvam coordinatis orthogonalibus contentam, in qua sit maximum vel minimum ista expressio* $(1+pp)^{1:2}\int ydx + y\int dx\sqrt{(1+pp)}$, *si ponatur abscissa* $x = a$.

Ponamus aequationem inter x et y quaesito satisfacientem iam esse inventam atque posito $x = a$ fieri $y = f$ et $\sqrt{(1+pp)} = g$ itemque $\int ydx = A$ et $\int dx\sqrt{(1+pp)} = B$; erit

$$dA = nv\cdot dx \text{ et } dB = -nv\cdot d\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Expressio igitur, quae maxima erit vel minima, hoc casu est $gA + fB$, cuius differentiale est $gdA + fdB$; quod positum $= 0$ dabit aequationem desideratam pro curva. Hic scilicet intelligitur litteras g et f, quae ex functionibus determinatis sunt ortae, in differentiatione tanquam quantitates constantes esse tractatas. Substitutis iam pro dA et dB valoribus debitis divisioneque per nv facta orietur ista aequatio pro curva quaesita

$$gdx = fd\cdot\frac{p}{\sqrt{(1+pp)}}.$$

Ponatur $\frac{f}{g}=c$, ita ut sit $\frac{y}{\sqrt{(1+pp)}}=c,$ casu quo est $x=a$; erit integrando

$$x+b=\frac{cp}{\sqrt{(1+pp)}} \text{ atque } p=\frac{x+b}{\sqrt{(cc-(x+b)^2)}}=\frac{dy}{dx};$$

ex qua fit

$$y=h\pm\sqrt{(c^2-(x+b)^2)}.$$

Curva igitur satisfaciens est Circulus radio *c* descriptus, abscissis super recta quacunque assumptis pariter que abscissarum initio ubicumque statuto. Quantitas autem *c*, quae radium Circuli constituit, ex definita abscissa $x=a$ determinatur, quia esse debet $\frac{y}{\sqrt{(1+pp)}}=c$ casu quo $x=a$. Fit autem hoc casu

$$y=h\pm\sqrt{(c^2-(x+b)^2)} \text{ et } \sqrt{(1+pp)}=\frac{c}{\sqrt{(cc-(x+b)^2)}},$$

unde oritur

$$cc=h\sqrt{(cc-(x+b)^2)} \pm (cc-(x+b)^2),$$

per quam vel *c* per *a* vel vicissim *a* per *c* determinari potest. Ponamus esse $h=0,$ $b=-c$, ita ut axis sit Circuli diameter, initiumque abscissarum invertice constituatur; erit $y=\sqrt{\left(2cx-xx\right)}$ atque, fiet $\left(a-c\right)^2=0$ seu $c=a$. Ex quo intelligitur hoc casu quadrantem Circuli quaesito satisfacere. Sin autem initium abscissarum in loco diametri quocunque capiatur, fiet tantum $h=0$, et si applicatae positivae sumantur, fiet $\left(a+b\right)^2=0$ seu $b=-a$. Diameter Circuli ergo manet indeterminatus portioque Circuli hoc modo sumti quaestioni satisfaciet, quae abscissae a sua origine ad centrum Circuli usque productae respondet.

EXEMPLUM II

39. *Invenire aequationem inter x et y, ut pro valore definito* $x=a$ *haec expressio* $y^{\int dx\sqrt{(1+pp)}}\int ydx$ *fiat maximum vel minimum.*

Posito $x=a$ fiat

$$y=f,\ \int dx\sqrt{(1+pp)}=A \text{ et } \int ydx=B,$$

erit

$$dA = -nv \cdot d \cdot \frac{p}{\sqrt{(1+pp)}} \quad \text{et} \quad dB = nv \cdot dx.$$

Maximum ergo minimumve esse oportet hanc quantitatem $f^{A}B$, cuius differentiale est $f^{A}BdAlf + f^{A}dB$; quod positum $= 0$ dabit $BdAlf = -dB$. Pro aequatione quaesita igitur habetur

$$Blfd \cdot \frac{p}{\sqrt{(1+pp)}} = dx.$$

et integrando

$$x + b = \frac{pBplf}{\sqrt{(1+pp)}} = \frac{cp}{\sqrt{(1+pp)}}$$

posito $Blf = c$. Habetur ergo

$$p = \frac{b+x}{\sqrt{(cc-(b+x)^2)}} \quad \text{et} \quad y = h \pm \sqrt{(c^2-(b+x)^2)}\ .$$

Erit igitur $f = h \pm \sqrt{(c^2-(b+a)^2)}$ posito $x = a$, atque

$$B = \int ydx = ha \pm \int dx\sqrt{(c^2-(b+x)^2)}$$

posito post integrationem $x = a$. Facto igitur $Blf = c$ innotescet valor *a*, cui si *x* aequalis capiatur in Circulo radii *c*, portio abscindetur Problemati satisfaciens. Caeterum ex his et Corollario 5 colligere licet, quoties formula maximi minimive fuerit functio quaecunque binarum harum formularum $\int ydx$ et $\int dx\sqrt{(1+pp)}$, curvam satisfacientem perpetuo esse Circulum; tantum ex solutione quantitas portionis satisfacientis debet diligenter investigari ac determinari.

EXEMPLUM III

40. *Invenire aequationem inter x et y, ut posito* $x = a$ *maximum minimumve fiat ista expressio* $e^{-n\int dx\sqrt{(1+pp)}}\int e^{n\int dx\sqrt{(1+pp)}}dx$.

Ponamus casu proposito, quo $x = a$, fieri

$$n\int dx\sqrt{(1+pp)} = A \quad \text{atque} \quad \int e^{n\int dx\sqrt{(1+pp)}}dx = B;$$

ita ut maximum minimumve sit haec quantitas $e^{-A}B$, cuius differentiale est $e^{-A}dB - e^{-A}BdA$; quod positum $= 0$ dabit aequationem hanc $dB = BdA$.

At est dA valor differentialis formulae $n\int dx\sqrt{(1+pp)}$, unde erit

$$dA = -nv \cdot d \cdot \frac{np}{\sqrt{(1+pp)}}$$

atque dB est valor differentialis formulae $\int e^{n\int dx\sqrt{(1+pp)}}dx$, quae continetur in Casu secundo paragraphi 7, ubi est

$$Z = \int e^{n\int dx\sqrt{(1+pp)}}dx \quad \text{et} \quad \Pi = \int dx\sqrt{(1+pp)}\,,$$

ita ut sit $Z = e^{n\Pi}$ et $dZ = e^{n\Pi} nd\Pi$, unde erit $L = e^{n\Pi} n$ et reliquae litterae M, N, P etc. fient $= 0$. Porro ob $\Pi = \int dx\sqrt{(1+pp)}$ erit

$$[Z] = \sqrt{(1+pp)} \quad \text{et} \quad d[Z] = \frac{pdp}{\sqrt{(1+pp)}},$$

ex quo erit

$$[M] = 0,\ [N] = 0 \quad \text{et} \quad [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Iam est

$$\int Ldx = n\int e^{n\int dx\sqrt{(1+pp)}}dx,$$

cuius valor posito $x = a$ erit $= nB$; hincque

$$V = n(B - \int e^{n\int dx\sqrt{(1+pp)}}dx)$$

Per Regulam ergo datam fiet

$$dB = nv \cdot dx\left(-\frac{d \cdot [P]V}{dx}\right) = -nv \cdot d \cdot \frac{np(B - \int e^{n\int dx\sqrt{(1+pp)}}dx)}{\sqrt{(1+pp)}}$$

$$= -nv \cdot d \cdot \frac{nBp}{\sqrt{(1+pp)}} \quad \text{ob} \quad dB = BdA.$$

Integrando itaque erit

$$\frac{np(B - \int e^{n\int dx\sqrt{(1+pp)}}dx)}{\sqrt{(1+pp)}} = \frac{nBp}{\sqrt{(1+pp)}} - nb$$

hincque

$$\frac{b\sqrt{(1+pp)}}{p}=\int e^{n\int dx\sqrt{(1+pp)}}dx.$$

Ex qua aequatione, quia valor determinatus *a* excessit, perspicuum est aequationem inventam pro quovis ipsius *x* valore aeque valere. Ut autem hanc aequationem evolvamus, erit differentialibus sumtis

$$-\frac{bdp}{p^2\sqrt{(1+pp)}}=e^{n\int dx\sqrt{(1+pp)}}dx,$$

quae per $n\sqrt{(1+pp)}$ multiplicata atque integrata, dat

$$\frac{nb}{p}+c=e^{n\int dx\sqrt{(1+pp)}},$$

qui exponentialis quantitatis valor in illa aequatione substitutus dabit

$$\frac{nbdx}{p}+cdx=-\frac{bdp}{p^2\sqrt{(1+pp)}} \quad \text{seu} \quad dx=-\frac{bdp}{p(nb+cp)\sqrt{(1+pp)}}.$$

Commodior autem aequatio oritur, si ponatur $\int dx\sqrt{(1+pp)}=s$, eritque *s* arcus curvae, si fuerint *x* et *y* coordinatae normales. Quare habebitur ista aequatio $nb+cp=e^{ns}p$ quae per dx multiplicata ob $dy=pdx$ abit in hanc $nbdx+cdy=e^{ns}dy$. Cum autem posito $x=0$ arcus *s* evanescere debeat, necesse est, ut sit hoc casu $\frac{nb}{p}+c=0$; hinc itaque vel dato curvae *p* initio constans *c* determinabitur vel vicissim ex *c* positio primae tangentis innotescet. Caeterum, si hanc quaestionem attentius contemplemur, deprehendemus eam iam contineri in Exemplo quodam Capitis praecedentis paragraphi 45. Cum enim nostra expressio, quae maximum minimumve esse debeat, sit

$$e^{-n\int dx\sqrt{(1+pp)}}\int e^{n\int dx\sqrt{(1+pp)}}dx,$$

ponatur ea *W*, erit

$$e^{n\int dx\sqrt{(1+pp)}}W=\int e^{n\int dx\sqrt{(1+pp)}}dx,$$

atque differentiando fiet

$$dW+nWdx\sqrt{(1+pp)}=dx.$$

Maximi igitur minimive expressio *W* datur per aequationem differentialem, quae in Casu quarto paragraphi 7 continetur atque methodo convenienti tractata ad eandem perducit aequationem, quam hic invenimus. Quaestionem autem illam in se complectentem supra in Capitis praecedentis paragrapho 45 (1°) tractavimus, in quo hunc ipsum casum adiunctum spectare licet. Comparatione autem instituta summus perspicietur consensus solutionum variarum eiusdem Problematis, quae quidem tentari queant.

EXEMPLUM IV

41. *Invenire curvam, in qua pro data abscissa* $=a$ *fiat ista expressio* $\dfrac{\int dx \sin \mathrm{A}y \cdot \sqrt{(1+pp)}}{\int dx \cos \mathrm{A}y \cdot \sqrt{(1+pp)}}$ *maximum vel minimum.*

Posito $x=a$ fiat

$$\int dx(1+pp)^{1:2} \sin \mathrm{A}y = A \quad \text{et} \quad \int dx(1+pp)^{1:2} \cos \mathrm{A}y = B;$$

erit per valores differentials

$$dA = nv \cdot dx\left((1+pp)^{1:2} \cos \mathrm{A}y - \frac{1}{dx} d \cdot \frac{p \sin \mathrm{A}y}{\sqrt{(1+pp)}} \right)$$

=

et

$$dB = nv \cdot dx\left(-(1+pp)^{1:2} \sin \mathrm{A}y - \frac{1}{dx} d \cdot \frac{p \cos \mathrm{A}y}{\sqrt{(1+pp)}} \right).$$

Cum igitur $\frac{A}{B}$ debeat esse maximum vel minimum, erit $BdA = AdB$*;* posito ergo $\frac{A}{B} = m$:fiet

$$\left(1+pp\right)^{1:2} dx\cos \mathrm{A}y - d \cdot \frac{p\sin \mathrm{A}y}{\sqrt{(1+pp)}} = -m(1+pp)^{\frac{1}{2}}\ dx\sin \mathrm{A}y - md \cdot \frac{p\cos \mathrm{A}y}{\sqrt{(1+pp)}}$$

Multiplicetur per *p*, erit ob

$$d \cdot \left(1+pp\right)^{1:2} \sin \mathrm{A}y = dy\left(1+pp\right)^{1:2} \cos \mathrm{A}y + \frac{pdp\sin \mathrm{A}y}{\sqrt{(1+pp)}}$$

et

$$d \cdot \left(1+pp\right)^{1:2} \cos \mathrm{A}y = -dy\left(1+pp\right)^{1:2} \sin \mathrm{A}y + \frac{pdp\cos \mathrm{A}y}{\sqrt{(1+pp)}}:$$

$$d\cdot\left(1+pp\right)^{1:2}\sin\mathrm{A}y-d.\frac{pp\sin\mathrm{A}y}{\sqrt{(1+pp)}}=md\cdot\left(1+pp\right)^{1:2}\cos\mathrm{A}y-md\cdot\frac{pp\cos\mathrm{A}y}{\sqrt{(1+pp)}};$$

quae integrata et reducta praebet

$$\frac{\sin\mathrm{A}y}{\sqrt{(1+pp)}}=\frac{m\cos\mathrm{A}y}{\sqrt{(1+pp)}}+b \quad \text{sive} \quad b\sqrt{(1+pp)}=\sin\mathrm{A}y-m\cos\mathrm{A}y;;$$

ubi notandum est fieri debere, si $x=a$ ponatur,

$$m=\frac{\int dx(1+pp)^{1:2}\sin\mathrm{A}y}{\int dx(1+pp)^{1:2}\cos\mathrm{A}y}$$

Sit $m=\dfrac{\sin\mathrm{A}n}{\cos\mathrm{A}n}=\mathrm{tang}\,\mathrm{A}n$, fiet

$$b\sqrt{(1+pp)}=\frac{\sin\mathrm{A}(y-n)}{\cos\mathrm{A}n} \quad \text{atque} \quad y=n+\mathrm{A}\sin b\left(1+pp\right)^{1:2}\cos\mathrm{A}n.$$

Quia vero est $dy=pdx$, erit $dx=\dfrac{dy}{p}$. At est

$$dy=\frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$$

posito $b\cos\mathrm{A}n=c$. Ex quibus conficitur

$$x=\int\frac{cdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$$

atque

$$y=\int\frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}};$$

longitudo autem curvae erit

$$=\int\frac{cdp}{\sqrt{(1-cc-ccpp)}}=\mathrm{A}\sin\frac{cp}{\sqrt{(1-cc)}}.$$

Quare, si arcus curvae dicatur *s*, habebitur ista concinna aequatio

$$dx \sin As = \frac{cdy}{\sqrt{(1-cc)}}.$$

Constructio vero ex anterioribus formulis sponte consequitur.

SCHOLION 2

42. His igitur Capitibus penitus absolvimus eam Methodi maximorum ac minimorum ad lineas curvas inveniendas accommodatae partem, quam absolutam vocavimus, in qua semper linea curva requiri solet, quae habeat pro dato quodam abscissae seu alterius variabilis x valore expressionem quamcunque indeterminatam maximum minimumve. Nam ista expressio, quae maximum minimumve esse debet, vel erit una quaedam formula integralis formae $\int Zdx$, ita ut Z sit functio quaecunque ipsarum x, y, p, q etc. sive definita sive indefinita, pro quibus casibus Methodum tradidimus in Capitibus praecedentibus; vel maximi minimive expressio illa continebit in se plures eiusmodi formulas integrales, ita ut sit duarum pluriumve formularum integralium functio quaecunque; pro hocque casu Methodus idonea in isto Capite est exposita atque Exemplis illustrata. Universa autem Methodus, quam hic dedimus, nititur inventione valorum differentialium, qui singulis formulis integralibus, quae vel ipsae maximum minimumve esse debeant vel in maximi minimive expressione contineantur, atque ideo tota solvendi Methodus reducitur ad Casus illos, quos paragrapho 7 huius Capitis coniunctim repraesentavimus. Qui igitur illos casus in memoria tenet vel in promtu habet, is ad omnia huius generis Problemata expedite resolvenda erit paratus. Neque vero solum Casus ibi enumerati Methodum maximorum ac minimorum absolutam constituunt, verum etiam Methodum alteram relativam, quam in sequentibus aggrediemur, absolvent; ex quo illorum Casuum summus usus in utraque Methodo abunde perspicietur. Hanc autem tractationem duobus Capitibus absolvemus, in quorum priori omnibus curvis, ex quibus quaesita debet erui, unam quandam proprietatem communem, in posteriori vero plures tribuemus.