PROPOSITION IV. PROBLEM

40. *If Z were a function of x, y, p and q, thus so that there shall be*

$$dZ = Mdx + Ndy + Pdp + Qdq\,,$$

to find among all the curves corresponding to the same abscissa, that in which $\int Zdx$ *shall be a maximum or minimum.*

SOLUTION

The value of the formula of the integral $\int Zdx$ is expanded into these two series

$$Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx + \text{etc. and } Z_{/}dx + Z_{//}dx + Z_{///}dx + \text{etc.},$$

the sum of which will be a maximum or a minimum, if the values of the individual differential terms, which arise by increasing the particular applied line $y^{/}$ by the small amount *nv*, may be gathered together and equated to zero. But by such an increment of the applied line $y^{/}$ the letters $y^{/};\ p,\ p^{/};\ q_{/},\ q,\ q^{/}$ show a change and thus only these terms, in which these letters are present, that is the terms $Z_{/}dx,\ Zdx$ and $Z^{/}dx$ show a change. Towards finding the increase in these terms arising from the translation of the point *n* into *v*, these may be differentiated and there will be

$$d\cdot Z^{/}dx = dx(M^{/}dx + N^{/}dy^{/} + P^{/}dp^{/} + Q^{/}dq^{/}),$$
$$d\cdot Z\ dx = dx(M\ dx + N\ dy + P\ dp + Q\ dq\ ),$$
$$d\cdot Z_{/}dx = dx(M_{/}dx + N_{/}dy_{/} + P_{/}dp_{/} + Q_{/}dq_{/}).$$

Now truly, because the abscissa *x* is not affected by that translation, on putting $dx = 0$ everywhere, then the values of the remaining differentials arising from the translation of the point *n* into *v* thus themselves will be found from the first proposition of this chapter :

$dy^{/} = +nv$	$dp^{/} = -\dfrac{nv}{dx}$	$dq^{/} = +\dfrac{nv}{dx^2}$
$dy = 0$	$dp = +\dfrac{nv}{dx}$	$dq = -\dfrac{2nv}{dx^2}$
$dy_{/} = 0$	$dp_{/} = 0$	$dq_{/} = +\dfrac{nv}{dx^2}$

The following value of the differential will be produced from these differentials with the values of the expressions substituted through *nv*

$$nv\cdot dx\left(N^{/}-\frac{P^{/}}{dx}+\frac{P}{dx}+\frac{Q^{/}}{dx^2}-\frac{2Q}{dx^2}+\frac{Q_{/}}{dx^2}\right)$$
$$=nv\cdot dx\left(N^{/}-\frac{dP}{dx}+\frac{ddQ_{/}}{dx^2}\right)=nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}\right)$$

on account of $ddQ_{/}=ddQ$. On account of which the equation for this curve sought will be had

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0.$$

Q. E. I.

COROLLARY 1

41. Therefore if differentials of the second order shall be present in the formula of the maximum or minimum $\int Zdx$ or, which is the same, if Z were a function of x, y, p and q, thus so that there shall be

$$dZ=Mdx+Ndy+Pdp+Qdq\,,$$

then the equation for the curve sought will be

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0.$$

which will be formed easily from the differential of Z itself.

COROLLARY 2

42. If the magnitude Q itself involves q or the second order differential of y, then ddQ will contain a differential of the fourth order, and the equation found for the curve will be in this kind. From which the satisfying curve will be able to be drawn through four given points.

COROLLARY 3

43. Therefore if q may be contained in Q, then a determinate problem thus is being proposed, as that may be defined among all the curves drawn through four data points, in which $\int Zdx$ shall be a maximum or minimum.

SCHOLIUM I

44. We may consider q not to be contained in Q, so that we may investigate, the order into which the resulting differential equation is going to become. But this happens, if the proposed formula of the maximum or minimum were of this kind $\int Zqdx$, with the function Z being only of x, y and p, thus so that there shall be

$$dZ = Mdx + Ndy + Pdp.$$

Hence therefore there will be

$$d \cdot Zq = Mqdx + Nqdy + Pqdp + Zdq,$$

from which this equation will arise for the curve sought
[*i.e.* where $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0$ becomes $Nq - \frac{d}{dx}Pq + \frac{d(Mdx + Ndy + Pdp)}{dx^2} = 0$ etc.]

$$0 = Nq - \frac{Pdq + qdP}{dx} + \frac{dMdx + dNdy + Nddy + Pddp + dPdp}{dx^2}$$

or

$$0 = 2Nq + \frac{dM + pdN}{dx}$$

or

$$0 = 2Ndp + dM + pdN,$$

which is equivalent to a differential equation of the second order only on account of $dp = \frac{ddy}{dx}$, which is present. Therefore if a curve may be desired, in which $\int Zqdx$ shall be a maximum or minimum, with Z being some function of x, y and p, and

$$dZ = Mdx + Ndy + Pdp,$$

the equation for the curve sought will be had :

$$0 = dM + 2Ndp + pdN.$$

COROLLARY 4

45. So we may return to the equation found

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0,$$

it is apparent that it shall be integrable generally, if there shall be $N=0$, that is if *y* may not be contained in *Z* ; for on integrating it will produce

$$C-P+\frac{dQ}{dx}=0.$$

If in addition there shall be $P=0$, another integration succeeds, from which it will produce

$$Cx+D-Q=0.$$

COROLLARY 5

46. If there shall be $M=0$, equally it will succeed in being produced by a single integration ; for since there shall be

$$dZ=Ndy+Pdp+Qdq,$$

the equation

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0$$

may be multiplied by *dy* or *pdx,* there will be had

$$Ndy-pdP+\frac{pddQ}{dx}=0.$$

Adding

$$dZ-Ndy-Pdp-Qdq=0,$$

the equation becomes

$$dZ-pdP-Pdp+\frac{pddQ}{dx}-Qdq=0;$$

the integral of which is

$$Z-Pp+\frac{pdQ}{dx}-Qq=C.$$

COROLLARY 6

47. If both $M=0$ and $N=0$, in the first place on account of $N=0$ as above

$$C-P+\frac{dQ}{dx}=0$$

Then, since there shall be

$$dZ = Pdp + Qdq\,,$$

the former equation may be multiplied by *dp* or *qdx*, there will be

$$Cdp - Pdp + qdQ = 0\,;$$

adding $Pdp + Qdq - dZ = 0$, to produce

$$Cdp + Qdq + qdQ - dZ = 0\,,$$

of which the integral is

$$Cp + D + Qq - Z = 0\,.$$

SCHOLIUM 2

48. If we may examine the connection of the equation found to the curve sought with the differential of *Z*, by which $\int Zdx$ may have a maximum or minimum, it will be possible to determine a relation between the differentials *dy*, *dp* and *dq*, so that from the differential of *Z* put equal to zero, the relation may give rise to the equation for the curve sought. For since there shall be

$$dZ = Mdx + Ndy + Pdp + Qdq,$$

an equation with this form may be prepared for the curve

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0\,,$$

or this, on multiplying by $dy = pdx$, which will be

$$Ndy - pdP + \frac{pddQ}{dx} = 0\,;$$

[i.e. an alternative way of writing $dZ = Mdx + Ndy + Pdp + Qdq,$]

from which it is apparent in the differential of *Z* , 0 must be written in place of *Mdx*, but of the remaining terms *Ndy* is unchanged, again in place of *Pdp* writing $-pdP$, and in place of *Qdq* there must be placed $\frac{pddQ}{dx}$. Truly, as long as this may be apparent in the first place, it will be essential to retain the form of the equation found, certainly which can be remembered easily. Moreover it is required to observe that problems concerning this in general are new, nor have been examined by those who have written about this subject elsewhere. For other writers are not accustomed to consider formulas of maxima or minima, unless in which at most differentials of the coordinates of the first order are present. On account of which the more it will be worth the effort to investigate the nature of problems of this kind and to show in the first place, how four points satisfying a curve, through which they may pass, enable these curves themselves to be determined. Hence to this end, it has been considered to add the following examples, which will be able to provide a greater illustration.

EXAMPLE I

49. *To find the curve, in which* $\int \frac{y^n ddy}{x^m dy}$ *shall be a maximum or minimum.*

This formula of the maximum or minimum with the aid of the substitution

$$dy = pdx \text{ and } ddy = qdx^2$$

will be changed into this:

$$\int \frac{y^n qdx}{x^m p};$$

which since it shall be similar to the formula $\int Zqdx$ examined in paragraph 44, where in Z we have considered only x, y and p to be present, there becomes, with the comparison put in place

$$Z = \frac{y^n}{x^m p} \text{ and } dZ = -\frac{my^n dx}{x^{m+1} p} + \frac{ny^{n-1} dy}{x^m p} - \frac{y^n dp}{x^m p^2};$$

from which there shall be

$$M = -\frac{my^n}{x^{m+1} p} \text{ and } N = \frac{ny^{n-1}}{x^m p};$$

and hence

$$Np = \frac{ny^{n-1}}{x^m}.$$

Therefore since this equation will be found for the curve sought:

$$0 = dM + 2Ndp + pdN = dM + Ndp + d \cdot Np,$$

we will have in our case this equation :

$$0=\frac{m(m+1)y^n dx}{x^{m+2}p}-\frac{mny^{n-1}dy}{x^{m+1}p}+\frac{my^n dp}{x^{m+1}pp}+\frac{ny^{n-1}dp}{x^m p}+\frac{n(n-1)y^{n-2}dy}{x^m}-\frac{mny^{n-1}dx}{x^{m+1}},$$

which multiplied by $\frac{y^{n+2}p^2}{y^{n-1}}$ is changed into this:

$$0=m(m+1)ydy-mnxpdy+mxydp+nx^2pdp+\frac{n(n-1)x^2p^2dy}{y}-mnxpdy$$

or

$$0=m(m+1)y^2dy-2mnxypdy+n(n-1)x^2p^2dy\ +mxy^2dp+nx^2ypdp,$$

which is a differential equation of the second order, which on putting

$$y=e^{\int vdx},$$

will be reduced to that of first order

$$m(m+1)vdx+mxdv-m(2n-1)xv^2dx+nx^2vdv+n^2x^2v^3dx=0.$$

But if now we may put $m=0$, thus so that

$$\int\frac{y^n ddy}{dy},$$

must have a maximum or a minimum, this equation will be had

$$y(n-1)pdy+ydp=0,$$

which integrated will give

$$y^{n-1}p=C \text{ or } y^{n-1}dy=Cdx;$$

and this integrated again will give $y^n=Cx+D$. But if we may put $n=0$, thus so that this formula $\int\frac{ddy}{x^m dy}$ must be a maximum or minimum, there will be

$$(m+1)dy+xdp=0 \text{ or } (m+1)pdx+xdp=0,$$

of which the integral is

$$x^{m+1}p=C \text{ or } dy=Cx^{-m-1}dx,$$

which integrated again gives

$$y=\frac{C}{x^m}+D.$$

Moreover it is apparent from these curves found that the proposed formula will become a maximum, truly not a minimum; for if a right line be taken, on account of $ddy = 0$ it is evident the value of the proposed formula to be less for the right line than for the curves found.

SCHOLIUM 3

50. An account can be given here, why questions of this kind, in which $\int Zqdx$ must be a maximum or minimum, may be deduced for a differential equation of the second order only and thus may be detailed rather by the questions of the preceding problems, if indeed Z were a function of x, y and p. For by a reduction of the integral, $\int Zqdx$ or $\int \frac{Zddy}{dx}$ can be reduced to such a form $Y + \int Vdx$, in which Y and V shall be functions or x, y and p only, no longer involving q. Therefore since Y shall be an absolute magnitude and thus does not fall into the questioning of the maximum or minimum, the formula $\int Zqdx$ becomes a maximum or minimum, if $\int Vdx$ may be reduced to such; therefore so that formulas of this kind $\int Zqdx$ may be able to be reduced to the state of the preceding problem ; from which it is no wonder, that only the equation for a curve satisfying a differential of the second order may be found. But so that the reduction mentioned of the formula $\int Zqdx$ or $\int Zdp$ to

$$Y + \int Vdx$$

may be understood better, we may consider [the differential], since Y shall be a function of x, y and p, to become

$$dY = \rho dx + \sigma dy + \tau dp = (\rho + \sigma p)dx + \tau dp;$$

and from the equality $\int Zdp = Y + \int Vdx$ there will be

$$Zdp = (\rho + \sigma p)dx + \tau dp + Vdx;$$

from which it is concluded $\tau = Z$ and $V = -\rho - \sigma p$. On account of which, this reduction may be put in place in the following manner ; the formula Zdp may be integrated with x and y held constant and the integral will be a function of x, y and p, which may be called Y. Then this function Y will be differentiated by putting p constant and with the differential taken negative it will give Vdx, and V will be a function of x, y and p not containing q. Therefore as often as a formula of this kind $\int Zqdx$ must return a maximum or minimum and Z is a function of x, y and p, then the question, even if it may seem to relate to the present problem, yet may be reduced at once to the previous problem. Thus if we take the formula $\int \frac{y^n ddy}{dy}$ or $\int \frac{y^n dp}{p}$, this may be transformed easily into

$$y^n lp - n\int y^{n-1} dylp ,$$

from which the maximum or minimum will have to be this formula

$$\int y^{n-1} dylp \text{ or } \int y^{n-1} pdxlp ,$$

which treated by the preceding problem will give

$$Z = y^{n-1} plp \text{ and } dZ = (n-1) y^{n-2} dyplp + y^{n-1} dp(1+lp) ;$$

and there will be

$$M = 0, N = (n-1) y^{n-2} plp \text{ and } P = y^{n-1}(1+lp).$$

But on account of $M = 0$ from the above paragraph 30 this equation $Z + C = Pp$ has been found for the curve sought, which suited to our case gives

$$y^{n-1} plp + C = y^{n-1} p + y^{n-1} plp \text{ or } y^{n-1} p = C ;$$

which is that equation itself, as we have found before in the solution of the example for the same case. On account of this we may progress to appropriate examples for this problem.

EXAMPLE II

51. *To find the curve Am* (Fig. 5), *which with its evolute AR and radius of osculation mR, the applied line may enclose a minimum space ARm in some place.*

With the abscissa $AM = x$, and with the applied line $Mm = y$ put in place, the radius of osculation will be $mR = -\frac{(1+pp)^{3:2}}{q}$;

moreover the area *ARm* is

$$= \int \tfrac{1}{2} mR \cdot dx\sqrt{(1+pp)} ;$$

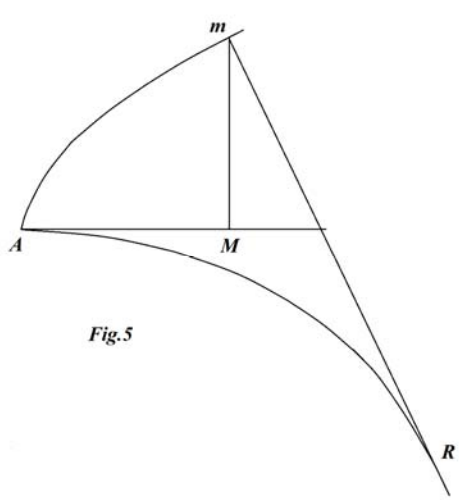

Fig.5

from which this formula

$$\int \frac{(1+pp)^2 dx}{q}$$

is required to be a minimum.
Thus there will be

$$Z = \frac{(1+pp)^2}{q} \text{ and } dZ = \frac{4(1+pp)pdp}{q} - \frac{(1+pp)^2 dq}{qq}.$$

from which it comes about that

$$M = 0,\ N = 0,\ P = \frac{4(1+pp)p}{q} \text{ and } Q = -\frac{(1+pp)^2}{qq}.$$

Now since there shall be $M = 0$ and $N = 0$, by Corollary 6 the equation for the curve sought will be

$$Z = D + Cp + Qq \text{ or } \frac{(1+pp)^2}{q} = D + Cp - \frac{(1+pp)^2}{q},$$

that is

$$2(1+pp)^2 = Dq + Cpq.$$

Again since there is $dp = qdx$ or $q = \frac{dp}{dx}$, there will be

$$2dx = \frac{(D+Cp)dp}{(1+pp)^2},$$

of which the integral is

$$x = \frac{a}{1+pp} + 2b\int \frac{dp}{(1+pp)^2} = \frac{a+bp}{1+pp} + b\int \frac{dp}{1+pp} + c;$$

with the constants changed for convenience, the equation will become

$$x = \frac{a+bp+cpp}{1+pp} + bA\operatorname{tang} p.$$

Then because there is $dy = pdx$, there will be

$$y = \int pdx = px - \int xdp$$

and thus

$$y=\frac{ap+bp^2+cp^3}{1+pp}+bp\,\text{Atang}\,p-\int\frac{(a+bp+cpp)dp}{1+pp}-b\int dp\,\text{Atang}\,p$$
$$=\frac{ap+bp^2+cp^3}{1+pp}-\int\frac{(a+cpp)dp}{1+pp},$$

on account of

$$b\int dp\,\text{Atang}\,p=bp\,\text{Atang}\,p-b\int\frac{pdp}{1+pp}.$$

Hence there will be

$$y=f+\frac{ap+bp^2+cp^3}{1+pp}+(c-a)\,\text{Atang}\,p-cp$$
$$=\frac{f+(a-c)p+(b+f)pp}{1+pp}-(c-a)\,\text{Atang}\,p.$$

And from these indeed the curve sought will be able to be constructed, with the values of x and y found through p, and drawn through four points. Truly, so that such a curve may become known, Atang p may be eliminated ; and there will be

$$\text{Atang}\,p=\frac{x}{b}-\frac{\frac{a}{b}+p+\frac{c}{b}pp}{1+pp}=\frac{y}{c-a}-\frac{\frac{f}{c-a}-p+\frac{(b+f)}{c-a}pp}{1+pp};$$

and hence

$$(c-a)x-by=\frac{(ac-aa-bf)+2b(c-a)p+(cc-ac-bb-bf)pp}{1+pp}.$$

But because the curve itself may not change, even if the coordinates may be increased or decreased by a constant quantity, there will be

$$(c-a)x-by=\frac{bb-(c-a)^2+2b(c-a)p}{1+pp};$$

and by putting a in place of $c-a$ there will be had

$$ax-by=\frac{bb-aa+2abp}{1+pp};$$

and with the constant bb subtracted there will be

$$ax-by=-\frac{aa-2abp+bbpp}{1+pp}$$

and hence

$$\sqrt{by-ax}=\frac{bp-a}{\sqrt{(1+pp)}}$$

The arc of the curve may be put $=w$; there will be $dw=dx\sqrt{(1+pp)}$; from which this equation itself arises

$$dw=\frac{bdy-adx}{\sqrt{by-ax}} \text{ and again } w=2\sqrt{(by-ax)}.$$

But $by-ax$ will express a multiple of the above abscissa assumed with a certain other fixed axis, to which thus the square of the corresponding arc is proportional. From which it is understood the corresponding curve sought to be a cycloid, which may be determined by the four given points, and thus the minimum area will be enclosed by its own evolute. This conclusion had been made a little more difficult, because the cycloid shall satisfy that sought corresponding to any right line assumed for the axis, and the equation for any axis may become exceedingly complicated. But if we were to put either a or $b=0$, for which indeed an extension of the solution were not restricted, the equation for a cycloid would be produced at once.

[Recall that the evolute of the cycloid is a similar displaced cycloid.]

EXAMPLE III

52. *To find the curve, in which* $\int \rho^n dw$ *shall be a maximum or minimum, with* ρ *denoting the radius of oscillation and dw an element of the curve.*

Through the relations established before there is

$$dw=dx\sqrt{(1+pp)} \text{ and } \rho=\frac{(1+pp)^{3:2}}{q};$$

from which the formula of the maximum or minimum will be

$$\int\frac{(1+pp)^{(3n+1):2}dx}{q^n};$$

and hence there will be

$$Z=\frac{(1+pp)^{(3n+1):2}}{q^n}$$

and

$$dZ = \frac{(3n+1)(1+pp)^{(3n-1):2}\, pdp}{q^n} - \frac{n(1+pp)^{(3n+1):2}\, dq}{q^{n+1}}.$$

On account of which there will be

$$M = 0, \quad N = 0, \quad P = \frac{(3n+1)(1+pp)^{(3n-1):2}\, p}{q^n} \text{ and } Q = -\frac{n(1+pp)^{(3n+1):2}}{q^{n+1}}.$$

But since there shall be $M = 0,\ N = 0,$ by paragraph 47, there will be

$$Z = Cp + D + Qq$$

and thus

$$\frac{(1+pp)^{(3n+1):2}}{q^n} = Cp + D - \frac{n(1+pp)^{(3n+1):2}}{q^n}$$

or

$$(n+1)(1+pp)^{(3n+1):2} = (Cp+D)q^n$$

and hence

$$q = \frac{(1+pp)^{(3n+1):2n}}{\sqrt[n]{(Cp+D)}} = \frac{dp}{dx};$$

therefore

$$dx = dp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$$

and

$$dy = pdp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$$

But here with due cause one can suspect the equation shall become simpler, if another axis may be taken [by rotation about the origin]. On this account we may consider another axis, in which the abscissa shall be *t*, the applied line *v*, and there shall be $dv = sdt$ and putting

$$x = \frac{\alpha t + \beta v}{\gamma} \text{ and } y = \frac{\beta t - \alpha v}{\gamma}$$

with $\gamma = \sqrt{(\alpha^2 + \beta^2)}$. Therefore there will be

$$dx = \frac{\alpha dt + \beta sdt}{\gamma} \text{ and } dy = \frac{\beta dt - \alpha sdt}{\gamma}$$

and

$$\frac{dy}{dx}=p=\frac{\beta-\alpha s}{\alpha+\beta s},\quad (1+pp)=\frac{\gamma^2(1+ss)}{(\alpha+\beta s)^2}\ \text{ and }\ dp=-\frac{\gamma\gamma ds}{(\alpha+\beta s)^2}.$$

Moreover, again there shall be

$$C+Dp=\frac{\alpha C+\beta D+s(\beta C-\alpha D)}{\alpha+\beta s}$$

and

$$(1+pp)^{(3n+1):2n}=\frac{\gamma^{(3n+1):n}(1+ss)^{(3n+1):2n}}{(\alpha+\beta s)^{(3n+1):n}}.$$

With these substitutions in place there will be

$$dx=\frac{\alpha dt+\beta dv}{\gamma}=\frac{a(\alpha+\beta s)ds}{\gamma(1+ss)^{(3n+1):2n}},$$

on putting $\beta C=\alpha D$ and with the constant changed. But again there becomes

$$dy=\frac{\beta dt-\alpha dv}{\gamma}=\frac{a(\beta-\alpha s)ds}{\gamma(1+ss)^{(3n+1):2n}}$$

and jointly it will produce

$$dt=\frac{ads}{(1+ss)^{(3n+1):2n}}\ \text{ and }\ dv=\frac{asds}{(1+ss)^{(3n+1):2n}}.$$

Now since we shall be able to call these coordinates equally *x* and *y* and preceding, there becomes $s=p$ and

$$dx=\frac{adp}{(1+pp)^{(3n+1):2n}}\ \text{ and }\ dy=\frac{apdp}{(1+pp)^{(3n+1):2n}},$$

which arises from the preceding, if there may be put $D=0$, from which it is evident for the breadth of the above solution, in which $C+Dp$ was present, to depart entirely from zero, even if there may be put $D=0$. Evidently the same will produce a curved line, whatever value may be attributed to the letter *D*, even if by another way an equation may come about between *x* and *y*, but yet related to the other axis. Meanwhile it will be agreed in several cases to satisfy an algebraic equation ; just as of which in the first place, if $n = \frac{1}{2}$, so that there will be

$$x = \int \frac{adp}{(1+pp)^{5:2}} = \frac{a(1+\frac{2}{3}pp)}{(1+pp)^{3:2}} \text{ and } y = \int \frac{apdp}{(1+pp)^{5:2}} = -\frac{\frac{1}{3}a}{(1+pp)^{3:2}}:$$

from which there becomes

$$\left(1+pp\right)^{3:2} = -\frac{a}{3y} \text{ and } pp = \sqrt[3]{\frac{aa}{9yy}} - 1,$$

with which substituted there becomes

$$x = -\left(2\sqrt[3]{\frac{aay}{9}} + y\right)\sqrt{\left(\sqrt[3]{\frac{aa}{9yy}} - 1\right)}$$

the algebraic equation for the curve, for the case $n = \frac{1}{2}$.

EXAMPLE IV

53. *To find the curve, in which the value of the formula* $\int \frac{ydydx^2}{ddy}$ *shall be the smallest of all.*

In the first place it is apparent a maximum cannot be present, because on a right line there becomes $ddy = 0$ and thus the value of the proposed formula becomes infinitely great. On account of which it may be considered, in what way a curved line becomes the minimum value of the formula $\int \frac{ydydx^2}{ddy}$. But this formula by our substitutions will change into this $\int \frac{ypdx}{q}$ and there will be

$$Z = \frac{yp}{q} \text{ and } dZ = \frac{pdy}{q} + \frac{ydp}{q} - \frac{ypdq}{qq};$$

therefore there will be

$$M = 0, \quad N = \frac{p}{q}, \quad P = \frac{y}{q} \text{ and } Q = -\frac{yp}{qq}.$$

But because $M = 0$, the curve sought may be expressed by the following equation

$$Z - Pp - Qq + \frac{pdQ}{dx} = C,$$

as has been shown by Corollary 5. On account of which this equation will arise :

$$\frac{yp}{q} - \frac{p}{dx} d.\frac{yp}{qq} = C$$

or

$$\frac{ydy}{pq} + \frac{adx}{p} = \frac{pdy}{qq} + \frac{ydp}{qq} - \frac{2ypdq}{q^3}$$

on account of $dy = pdx$, and setting $a = -C$. Indeed because $dp = qdx$, there will be

$$\frac{ydp}{qq} = \frac{ydx}{q} = \frac{ydy}{pq} \text{ and thus } \frac{adx}{p} = \frac{pdy}{qq} - \frac{2ypdq}{q^3}$$

or

$$\frac{adp}{pp} = \frac{dy}{q} - \frac{2ydq}{qq}.$$

If the constant may be put $a = 0$, this equation will become integrable and there will be

$$y = bqq \text{ and } q = \sqrt{\frac{y}{b}} = \frac{dp}{dx} = \frac{pdp}{dy},$$

from which there becomes $pdp = dy\sqrt{\frac{y}{b}}$ and on integrating

$$\frac{pp}{2} = \tfrac{2}{3} y\sqrt{\frac{y}{b}} + c$$

or, with the constants changed,

$$pp = \frac{y\sqrt{y} + a\sqrt{a}}{b\sqrt{b}} \quad \text{and} \quad p = \sqrt{\frac{y^{3:2} + a^{3:2}}{b^{3:2}}} = \frac{dy}{dx};$$

and hence

$$dx = dy\sqrt{\frac{b^{3:2}}{y^{3:2} + a^{3:2}}}.$$

Again putting $a=0$, there will become [$dx = dy\sqrt{\frac{b^{3:2}}{y^{3:2}}} = \frac{b^{3:4}}{y^{3:4}}$.]

$$x^4 = const.y \ ;$$

which is an especially specific equation for the curve satisfying the question.

EXAMPLE V

54. *To find the curve, in which the value of this formula* $\int q^n dx$ *or* $\int \frac{ddy^n}{dx^{2n-1}}$ *shall be a maximum or a minimum.*

Therefore there will be had

$$Z = q^n \text{ and } dZ = nq^{n-1}dq \ ;$$

from which there will be

$$M = 0,\ N = 0,\ P = 0 \text{ and } Q = nq^{n-1}.$$

Therefore since the equation satisfying this curve shall be $\frac{ddQ}{dx^2} = 0$, there will be

$$dQ = \alpha dx \text{ and } Q = q^{n-1} = \alpha x + \beta \ ;$$

and hence

$$q = (\alpha x + \beta)^{1:(n-1)} = \frac{dp}{dx} \ ;$$

from which there becomes

$$p = (\alpha x + \beta)^{n:(n-1)} + \gamma = \frac{dy}{dx}$$

and finally

$$y = (\alpha x + \beta)^{(2n-1):(n-1)} + \gamma x + \delta,$$

where we have included the coefficients entering by integration into the constants. Therefore the satisfying curves are algebraic always, except in the case in which $n = \frac{1}{2}$; then indeed the final integration will give :

$$y = \frac{1}{\alpha} l(\alpha x + \beta) + \gamma x + \delta.$$

So that regarding the case $n=1$, that cannot occur in the investigation of maxima or minima, since the formula $\int qdx$ shall not be indeterminate, but may be referred to a determined value, e.g. p, on account of

$$qdx = dp\,.$$

Moreover it is apparent with the term $(\alpha x+\beta)^{(2n-1):(n-1)}$ vanishing on account of $y=\gamma x+\delta$ for the question to be satisfied by a right line. Evidently, if four points may be given, through which the desired curve may pass, they shall be placed in a line, then the right line itself before all the remaining lines through the same four points will satisfy the question.

EXAMPLE VI

55. *To find the curve, in which* $\int \frac{xpdx}{yq}$ *shall be a maximum or minimum.*

Because there is $Z=\frac{xp}{yq}$, there will be

$$dZ=\frac{pdx}{yq}-\frac{xpdy}{y^2q}+\frac{xdp}{yq}-\frac{xpdq}{yqq}$$

and thus

$$M=\frac{p}{yq},\quad N=-\frac{xp}{y^2q},\quad P=\frac{x}{yq}\quad\text{and}\quad Q=-\frac{xp}{yq^2}.$$

Of which terms since none may vanish, the equation for the curve sought will be

$$[\text{from } N-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0,]$$

$$-\frac{xp}{y^2q}-\frac{1}{dx}d\cdot\frac{x}{yq}-\frac{1}{dx^2}d^2\cdot\frac{xp}{yq^2}=0$$

or

$$0=\frac{xpdx^2}{y^2q}+\frac{dx^2}{yq}-\frac{xdxdy}{y^2q}-\frac{xdxdq}{yq^2}+d\cdot\left(\frac{pdx}{yq^2}-\frac{xpdy}{y^2q^2}+\frac{xdp}{yq^2}-\frac{2xpdq}{yq^3}\right)$$

or

$$0=q^2dx^2(3yq-2p^2)(y-xp)-4yqdxdq(xyq-\ xp^2+yp)+6xy^2pdq^2-2xy^2pqddq.$$

Which is a differential equation of the fourth order, which, whether it shall be integrable or not, is not readily apparent ; nor also is it worth the effort to inquire more carefully into the manner that may be integrated, because here no case of any use has arisen from the solution to the problem, but considered in a casual manner. But this example has been considered to be added thus, so that the case may be had, for which the solution not only rises to an equation of the fourth order, but also neither by the help brought in above may that lead to a lower order. For all the preceding examples have been prepared thus, so that by the general rules set out in the corollaries the equation for the curve sought at once will be able to be elicited of lower order differentials.

PROPOSITION V. PROBLEM

56. *To find the curve in which the value of the formula* $\int Zdx$ *shall be a maximum or minimum, with a function Z of this kind present, which involves differentials of any order, thus so that it shall be* $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt + \text{etc.}$

SOLUTION

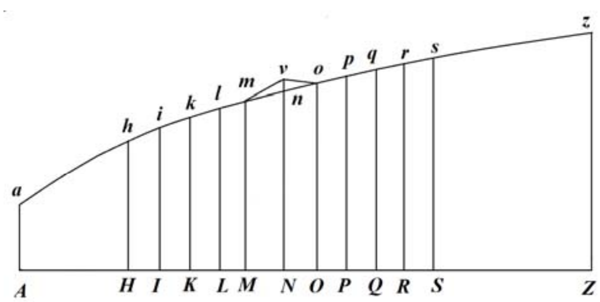

Fig. 4

Because (Fig. 4) the translation of the point *n* into *v* will affect the preceding elements more than the following, indeed it affects a single element following, but that is extended further in the preceding, so that differentials of higher orders may be present there, on account of this it will be convenient to select an element a little earlier, such as *Hh*, for the first to be accepted, thus so that the small change *nv* of the applied line *Nn* is not to be added before *Hh* is stretched out [see the table below to understand how this works]; as that will come about, if in *Z* the differentials do not rise beyond the sixth order. Moreover it will suffice to extend the values of *dZ* as far as to the term *Tdt*, because from that solution the manner may be deduced easily for however many further terms are to be put in place. In the first place likewise, because all the preceding problems will be contained in this one, it will be agreed to produce the same solution always, whatever applied line may be increased a certain infinitely small amount, such as *nv*. Therefore there shall be $AH = x$ and $Hh = y$, the values of the letters *p, q, r, s, t* etc. will correspond to the individual points of the abscissa *H, I, K, L, M, N, O* etc., as follows :

H	$y,$	$p,$	$q,$	$r,$	$s,$	t
I	$y^{/},$	$p,^{/}$	$q^{/},$	$r^{/},$	$s^{/},$	$t^{/}$
K	$y^{//},$	$p^{//},$	$q^{//},$	$r^{//},$	$s^{//},$	$t^{//}$
L	$y^{///},$	$p^{///},$	$q^{///},$	$r^{///},$	$s^{///},$	$t^{///}$
M	$y^{IV},$	$p^{IV},$	$q^{IV},$	$r^{IV},$	$s^{IV},$	t^{IV}

$$N \mid y^{V},\quad p^{V},\quad q^{V},\quad r^{V},\quad s^{V},\quad t^{V}$$

But these individual values in the translation *n* to *v* will be taken to increase, which by the first proposition, with the due customary change of sign, thus so that those themselves will be found :

$$dy = 0 \quad dy^{I} = 0 \quad dy^{II} = 0 \quad dy^{III} = 0 \quad dy^{IV} = 0 \quad dy^{V} = +nv$$

$$dp = 0 \quad dp^{I} = 0 \quad dp^{II} = 0 \quad dp^{III} = 0 \quad dp^{IV} = +\frac{nv}{dx} \quad dp^{V} = -\frac{nv}{dx}$$

$$dq = 0 \quad dq^{I} = 0 \quad dq^{II} = 0 \quad dq^{III} = +\frac{nv}{dx^2} \quad dq^{IV} = -\frac{2nv}{dx^2} \quad dq^{V} = +\frac{nv}{dx^2}$$

$$dr = 0 \quad dr^{I} = 0 \quad dr^{II} = +\frac{nv}{dx^3} \quad dr^{III} = -\frac{3nv}{dx^3} \quad dr^{IV} = +\frac{3nv}{dx^3} \quad dr^{V} = -\frac{nv}{dx^3}$$

$$ds = 0 \quad ds^{I} = +\frac{nv}{dx^4} \quad ds^{II} = -\frac{4nv}{dx^4} \quad ds^{III} = +\frac{6nv}{dx^4} \quad ds^{IV} = -\frac{4nv}{dx^4} \quad ds^{V} = +\frac{nv}{dx^4}$$

$$dt = +\frac{nv}{dx^5} \quad dt^{I} = -\frac{5nv}{dx^5} \quad dt^{II} = +\frac{10nv}{dx^5} \quad dt^{III} = -\frac{10nv}{dx^5} \quad dt^{IV} = +\frac{5nv}{dx^5} \quad dt^{V} = -\frac{nv}{dx^5}$$

Because again the value of the formula $\int Zdx$ will correspond to the abscissa *AH*, and that by the translation of the point *n* to *v* does not change, the values of the formula $\int Zdx$ will correspond to the following elements of the abscissas, which are shown in this table :

To the element	corresponds
HI	Zdx
IK	$Z^{I}dx$
KL	$Z^{II}dx$
LM	$Z^{III}dx$
MN	$Z^{IV}dx$
NO	$Z^{V}dx$

For the increments of these values requiring to be found, arising from the translation of the point *n* to *v*, it is required to substitute these individual values of the differences and for the derived values of these themselves assigned above, and expressed by *nv* in place of the differentials *dy, dp, dq, dr, ds, dt*; and thus there will be as follows :

[Recall from above that $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt + \text{etc.}$; the dashes and Roman numerals used throughout indicate merely evaluations of coefficients at subsequent points on the abscissa.]

$$d \cdot Zdx = nv \cdot dx\left(\frac{T}{dx^5}\right)$$

$$d \cdot Z^{/}dx = nv \cdot dx\left(\frac{S^{/}}{dx^4} - \frac{5T^{/}}{dx^5}\right)$$

$$d \cdot Z^{//}dx = nv \cdot dx\left(\frac{R^{//}}{dx^3} - \frac{4S^{/}}{dx^4} + \frac{10T^{/}}{dx^5}\right)$$

$$d \cdot Z^{///}dx = nv \cdot dx\left(\frac{Q^{///}}{dx^2} - \frac{3R^{///}}{dx^3} + \frac{6S^{///}}{dx^4} - \frac{10T^{///}}{dx^5}\right)$$

$$d \cdot Z^{IV}dx = nv \cdot dx\left(\frac{P^{IV}}{dx} - \frac{2Q^{IV}}{dx^2} + \frac{3R^{IV}}{dx^3} - \frac{4S^{IV}}{dx^4} + \frac{5T^{IV}}{dx^5}\right)$$

$$d \cdot Z^{V}dx = nv \cdot dx\left(N^{V} - \frac{P^{V}}{dx} + \frac{Q^{V}}{dx^2} - \frac{R^{V}}{dx^3} + \frac{S^{V}}{dx^4} - \frac{T^{V}}{dx^5}\right).$$

Because now these elements are altered only by the transposition of the point *n* to *v* and the increments taken, the sum of these increments will give the whole value of the differential, which the formula $\int Zdx$ takes for the whole extended abscissa *AZ* ; which therefore will be

$$nv \cdot dx \begin{cases} +N^{V} \\ -\dfrac{P^{V} - P^{IV}}{dx} \\ +\dfrac{Q^{V} - 2Q^{IV} + Q^{///}}{dx^2} \\ -\dfrac{R^{V} - 3R^{IV} + 3R^{///} - R^{//}}{dx^3} \\ +\dfrac{S^{V} - 4S^{IV} + 6S^{///} - 4S^{//} + S^{/}}{dx^4} \\ -\dfrac{T^{V} - 5T^{IV} + 10T^{///} - 10T^{//} + 5T^{/} - T}{dx^5}. \end{cases}$$

Moreover these individual members can be expressed conveniently and succinctly by differentials ; for there will be :

$$-P^{V}+P^{IV}=dP^{IV}$$
$$+Q^{V}-2Q^{IV}+Q^{///}=+ddQ^{///}$$
$$-R^{V}+3R^{IV}-3R^{///}+R^{//}=-d^{3}R^{//}$$
$$S^{V}-4S^{IV}+6S^{///}-4S^{//}+S^{/}=+d^{4}S^{/}$$
$$-T^{V}+5T^{IV}-10T^{///}+10T^{//}-5T^{/}+T=-d^{5}T.$$

On account of which the whole value of the differential of the formula $\int Zdx$ produced from the small increase *nv*, will be

$$=nv\cdot dx\left(N^{V}-\frac{dP^{IV}}{dx}+\frac{ddQ^{///}}{dx^{2}}-\frac{d^{3}R^{//}}{dx^{3}}+\frac{d^{4}S^{/}}{dx^{4}}-\frac{d^{5}T}{dx^{5}}\right);$$

But here, because all the terms are homogeneous, the distinguishing labels can be omitted without risk, for the difference between N^{V} and *N* and likewise between dP^{IV} and *dP* and the rest will vanish [in the limit]. On account of which this value of the differential of the formula $\int Zdx$ will be found from which likewise the value of the differential of the formula $\int Zdx$ can be deduced, if also other differentials may be present in *Z*. Whereby, if the curve may be sought, which may have a maximum or minimum $\int Zdx$ for the given abscissa, and there shall be $dZ=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+Tdt+\text{etc.}$, this will be the first value of the differential of the formula $\int Zdx$:

$$nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^{2}}-\frac{d^{3}R}{dx^{3}}+\frac{d^{4}S}{dx^{4}}-\frac{d^{5}T}{dx^{5}}+\text{ etc.}\right).$$

Hence this equation will arise for the curve sought:

$$0=N-\frac{dP}{dx}+\frac{ddQ}{dx^{2}}-\frac{d^{3}R}{dx^{3}}+\frac{d^{4}S}{dx^{4}}-\frac{d^{5}T}{dx^{5}}+\text{etc.}$$

Q. E. I.

COROLLARY 1

57. In the formula $\int Zdx$, as we have examined that, the magnitude *Z* contains differentials of the fifth order, if indeed in its differential

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt$$

the term *Tdt* is the final. Therefore since in *T* at this stage differentials of the fifth order or *t* are present, it is evident the equation for the curve sought to be a differential of the tenth order.

COROLLARY 2

58. Hence it is understood the differential equation for the curve always must rise to twice the higher order, as it were a formula itself of maxima or minima. For we may consider, in the final term *Tdt*, the magnitude *T* at this point to involve *t* squared ; for unless this shall be, the equation may be decreased by two orders, as can be deduced from paragraph 50.

COROLLARY 3

59. Therefore if the differentials may be contained in *Z* of order *n*, then the equation for the curve will be *2n* and on account of this just as many new arbitrary constants will be contained in the power.

COROLLARY 4

60. Therefore on account of so many arbitrary constants it will be necessary for just as many points for determining the proposed problem ; thus clearly a problem, so that it may be determined, must be enunciated : Between all the curves passing through *2n* given points to determine that, in which $\int Zdx$ shall be a maximum or minimum, if indeed the magnitude *Z* may include differentials of order *n*.

COROLLARY 5

61. Therefore on account of the whole number *n*, the number of points, from which the problem will be determined, will always be even. Thus either no point, two, four, six, or eight points and thus so on are required for the determination of the problem.

SCHOLIUM 1

62. Therefore from the order of the differentials, to which the equation for the curve found rises, or from the number of points through which the satisfying curve is required to pass, problems of this kind will be able to be distributed conveniently into classes. Therefore for the first class problems will be referred to that, in which an absolute curved line is sought, which for a given abscissa $\int Zdx$ may have a maximum or minimum ; such problems will be contained both in the second preposition as well as in the third, in these cases which we have set out in paragraphs 26 and 27 ; clearly in these cases the solution gives a determined curve satisfying the question. That second class includes problems, the solution of which leads to a differential equation of the second order ; and these two

points for its determination ask and thus they must propose, so that among all the curves passing through the same two points that may be defined, in which $\int Zdx$ shall be a maximum or a minimum; we have given solutions to problems of this kind in the third proposition. Again to the third class belong problems treated in the fourth proposition, which thus themselves may be found, so that among all the curves passing through four given points that may be determined, which may have $\int Zdx$ a maximum or minimum. In a similar manner the fourth class demands the determination of six points, the fifth eight, and thus so on , all which classes we have taken to be included in the present problem. Furthermore even if the equation found rises to such an order of differentiation, yet often generally it allows one or more cases, of which kind we have shown in several preceding problems. On this account we have seen also, in which cases our equation admits a general integration, either one or more; as it can be seen at once in the examples presented, whether that may be contained in these cases or otherwise. But chiefly here are two cases of this kind, in the first of which $N = 0$, and in the other $M = 0$, from which henceforth all cases depend, which we will present here.

CASE I

63. In the formula $\int Zdx$ of the maximum or minimum there shall be the term $N = 0$, thus so that there shall be

$$dZ = Mdx + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

Therefore in this case the equation for the curve will be this

$$0 = -\frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \text{etc.},$$

which multiplied by *dx* shall become integrable, and it will produce

$$0 = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \text{etc.}$$

CASE II

64. There shall be both $N = 0$ and $P = 0$, thus so that there shall be

$$dZ = Mdx + Qdq + Rdr + Sds + Tdt + \text{etc.}$$

Because $N = 0$, a single integration shall succeed and that equation found just now will be had for the curve sought, also on putting $P = 0$:

$$0 = A + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \text{etc.},$$

which multiplied by *dx* will be able to be integrated anew, and there will be

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.}$$

CASE III

65. If there were $N = 0$, $P = 0$ and $Q = 0$, thus so that there shall be

$$dZ = Mdx + Rdr + Sds + Tdt + \text{etc.}$$

The two vanishing values *N* and *P* now have provided this equation from two integrations

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.};$$

in which if there may be put $Q = 0$ and multiplied by *dx*, the following equation will be obtained integrated three times :

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \text{ etc.}$$

From which it is apparent now, if in addition there were $R = 0$, then also the fourth integration has a place and so on thus.

CASE IV

66. If there were $M = 0$, thus so that there shall be

$$dZ = Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

The equation for the curve sought produced before

$$0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.},$$

which multiplied by $dy = pdx$ and then there is added

$$dZ - Ndy - Pdp - Qdq - Rdr - Sds - \text{etc.}$$

will give

$$\begin{aligned}0 = dZ - pdP + \frac{pddQ}{dx} - \frac{pd^3R}{dx^2} + \frac{pd^4S}{dx^3} - \text{etc.}\\ - Pdp \quad - Qdq \quad - Rdr \quad - Sds \quad - \text{etc.,}\end{aligned}$$

the integral of which can be assigned ; indeed it shall be

$$\begin{aligned}0 = A + Z - pP + \frac{pdQ}{dx} - \frac{pddR}{dx^2} + \frac{pd^3S}{dx^3} - \text{etc.}\\ - Qq + \frac{qdR}{dx} - \frac{qddS}{dx^2}\\ - Rr + \frac{rdS}{dx}\\ - Ss\end{aligned}$$

or

$$0 = A + Z - pP + \frac{pdQ - Qdp}{dx} - \frac{pddR - dpdR + Rddp}{dx^2} + \frac{pd^3S - dpddS + dSddp - Sd^3p}{dx^3} - \text{etc.,}$$

of which the terms may progress further in some manner, if it is apparent at once that the following differentials shall be present in *dZ, Tdt, Udu* etc.

CASE V

67. If there shall be both $M = 0$ and $N = 0$, thus so that there shall be

$$dZ = Pdp + Qdq + Rdr + Sds + \text{etc.}$$

Because there is $N = 0$, a single integration may be put in place by the first case, and there will be found

$$0 = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.,}$$

which may be multiplied by $dp = qdx$ and added to that

$$0 = -dZ + Pdp + Qdq + Rdr + Sds + \text{etc.,}$$

with which done this integrable equation will be produced

$$\begin{aligned}0 = Adp - dZ + qdQ - \frac{qddR}{dx} + \frac{pd^3S}{dx^2} - \text{etc.}\\ + Qdq + Rdr \quad + Sds \quad + \text{etc.,}\end{aligned}$$

of which the integral will be

$$0 = Ap - B - Z + qQ - \frac{qdR}{dx} + \frac{pddS}{dx^2} - \text{etc.}$$
$$+ Rr \ - \frac{rdS}{dx} + Ss$$

or

$$0 = Ap - B - Z + Qq - \frac{qdR - Rdq}{dx} + \frac{qddS - dqdS + Sddq}{dx^2}$$
$$- \frac{qd^3T - dqddT + dTddq - Td^3q}{dx^3} + \text{ etc.}$$

CASE VI

68. If there shall be $M = 0$, $N = 0$ and $P = 0$, thus so that there shall be

$$dZ = Qdq + Rdr + Sds + Tdt + \text{etc.}$$

On account of $N = 0$ and $P = 0$ by the second case there is place for two integrations, and this will be the equation for the curve sought :

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.},$$

to which multiplied by $dq = rdx$ if there may be added

$$0 = dZ - Qdq - Rdr - Sds - Tdt - \text{etc.},$$

this integrable equation will be had anew :

$$0 = Axdq - Bdq + dZ - \ rdR + \frac{rddS}{dx} - \frac{rd^3T}{dx^2} + \text{etc.}$$
$$- Rdr - Sds - Tdt - \text{etc.} \ ,$$

the integral of which is

$$0 = Axq - Bq + C + Z - \ rR + \frac{rdS}{dx} - \frac{rddT}{dx^2} + \text{etc.}$$
$$- Ap \qquad\qquad - Ss + \frac{sdT}{dx} - Tt$$

or

$$0 = A(xq - p) - Bq + C + Z - rR + \frac{rdS - Sdr}{dx} - \frac{rddT - drdT + Tddr}{dx^2} + \text{etc.}$$

CASE VII

69. If there were $M = 0,\ N = 0,\ P = 0$ and $Q = 0$, thus so that there shall be

$$dZ = Rdr + Sds + Tdt + \text{ etc.}$$

On account of $N = 0$, $P = 0$ and $Q = 0$, the third case itself supplies the equation for the curve sough now integrated three times :

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \text{ etc.},$$

to which multiplied by $dr = sdx$ the equation may be added :

$$0 = -dZ + Rdr + Sds + Tdt + \text{etc.},$$

from which done this equation will be produced

$$\begin{aligned} 0 = Ax^2dr - Bxdr + Cdr - dZ + sdS - \frac{sddT}{dx} &+ \text{etc.} \\ + Sds + Tdt &+ \text{etc.}, \end{aligned}$$

which integrated hence will give

$$\begin{aligned} 0 = Ax^2r - Bxr + Cr - D - Z \quad &+ Ss - \frac{sdT}{dx} + \text{etc.} \\ -2Axq + Bq \quad &+ Tt \\ +2Ap \end{aligned}$$

or

$$0 = A(x^2r - 2xq + 2p) - B(xr - q) + Cr - D - Z \ + Ss - \frac{sdT - Tds}{dx} + \text{etc.}$$

SCHOLIUM 2

70. With the aid of these cases, the number of which may be allowed to increase further, if it may seem convenient, often problems will be able to be resolved very quickly. For if

indeed a certain problem may be contained in any of these cases, which themselves permit one or more integrations, the equation for the curve can be formed at once, now once or several times integrated, which thus will be more easy to be treated further. In order that it may appear clearer and likewise the use of this latter problem, so that it may be indicated in the formula $\int Zdx$ of maxima or minima present beyond differentials of the second order, it will help to present a single example.

EXAMPLE

71. *Among all the curves corresponding to the same abscissas to define that, of which the evolute, with its own evolute, may enclose a maximum or minimum space between the radii of the evolutes.*

On putting the abscissa $=x$ and the applied line $=y$ for the curve sought, an element of the curve shall be $=dw$ and its radius of oscillation $=\rho$; an element of its evolute will be $=d\rho$ and the radius of osculation of that $=\frac{\rho d\rho}{dw}$, from which the area taken between the evolute of the curve sought and its evolute will be $=\frac{1}{2}\int\frac{\rho d\rho^2}{dw}$; which expression therefore is required to return a maximum or minimum. But since there shall be

$$dw=dx\sqrt{(1+pp)}\ \text{ and }\ \rho=\frac{(1+pp)^{\frac{3}{2}}}{q}.$$

there will be

$$d\rho=3(1+pp)^{\frac{1}{2}}\,pdx-\frac{(1+pp)^{\frac{3}{2}}rdx}{qq}$$

and

$$d\rho^2=(1+pp)dx^2\left(9pp-\frac{6(1+pp)r}{qq}+\frac{(1+pp)^2rr}{q^4}\right)$$

and

$$\frac{\rho}{dw}=\frac{1+pp}{qdx}.$$

And thus the formula of the maximum or minimum is

$$\int\frac{(1+pp)dx}{q}\left(9pp-\frac{6(1+pp)r}{qq}+\frac{(1+pp)^2r^2}{q^4}\right)$$

$$\int dx\left(\frac{9pp(1+pp)^2}{q}-\frac{6(1+pp)^3r}{q^3}+\frac{(1+pp)^4r^2}{q^5}\right),$$

from which Z will be a function of p, q and r themselves ; from which by differentiation the expression will be produced :

$$dZ=\frac{18pdp(1+pp)(1+3pp)}{q}-\frac{9ppdq(1+pp)^2}{qq}-\frac{6dr(1+pp)^3}{q^3}$$
$$-\frac{36prdp(1+pp)^2}{q^3}+\frac{18rdq(1+pp)^3}{q^4}+\frac{2rdr(1+pp)^4}{q^5}$$
$$+\frac{8rrpdp(1+pp)^3}{q^5}-\frac{5r^2dq(1+pp)^4}{q^6}.$$

Therefore by comparing with the general form put in place there will be $M=0,\ N=0,$

$$Z=\frac{9pp(1+pp)^2}{q}-\frac{6(1+pp)^3r}{q^3}+\frac{(1+pp)^4r^2}{q^5},$$
$$P=\frac{18p(1+pp)\left(1+3pp\right)}{q}-\frac{36pr(1+pp)^2}{q^3}+\frac{8rrp(1+pp)^3}{q^5},$$
$$Q=\frac{-9pp(1+pp)^2}{qq}+\frac{18r(1+pp)^3}{q^4}-\frac{5rr(1+pp)^4}{q^6},$$
$$R=-\frac{6(1+pp)^3}{q^3}+\frac{2r(1+pp)^4}{q^5}.$$

Now since there shall be $M=0$ and $N=0$, the solution falls into the fifth case and this will be the equation for the curve sought :

$$0=Ap-B-Z+Qq+Rr-\frac{qdR}{dx},$$

which with the substitutions made will change into this :

$$0=Ap-B-\frac{18pp(1+pp)^2}{q}-\frac{16pr(1+pp)^3}{q^3}+\frac{6rr(1+pp)^4}{q^5}$$
$$-\frac{2dr(1+pp)^4}{q^4dx}+\frac{36p(1+pp)^2}{q};$$

which equation is exceedingly complicated, as far as its further integrations may be undertaken. Moreover it is apparent this equation is a differential of the fourth order, thus so that by the four remaining integrations at this stage four constants will be introduced ; from which six given points will be required, through which the curve may pass, as may be determined in the problem.

[Alas, as noted by Carathéodory in the *O.O*. edition, Series I, Vol. 24, p.79, Euler had made a slip of the pen in his definition of $d\rho^2$, which prevented him from finding the solution, which Carathéodory provides.]

PROPOSITIO IV. PROBLEMA

40. *Si Z fuerit functio ipsarum x, y, p et q, ita ut sit* $dZ = Mdx + Ndy + Pdp + Qdq$, *invenire inter omnes curvas eidem abscissae respondentes eam, in qua sit* $\int Zdx$ *maximum vel minimum.*

SOLUTIO

Valor formulae integralis $\int Zdx$ evolvitur in binas has series

$$Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx + \text{etc.} \ \text{ et } \ Z_{/}dx + Z_{//}dx + Z_{///}dx + \text{etc.},$$

quarum aggregatum maximum erit vel minimum, si singulorum terminorum valores differentiales, qui oriuntur augendo applicatam $y^{/}$ particula *nv*, colligantur et nihilo aequentur. Tali autem applicatae $y^{/}$ incremento mutationem patiuntur litterae $y^{/};\ p,\ p^{/};\ q_{/},\ q,\ q^{/}$ adeoque ii tantum termini, in quibus istae litterae insunt, hoc est termini $Z_{/}dx,\ Zdx$ et $Z^{/}dx$. Ad horum terminorum augmenta ex translatione puncti *n* in *v* orta invenienda, differentientur ii eritque

$$d\cdot Z^{/}dx = dx(M^{/}dx + N^{/}dy^{/} + P^{/}dp^{/} + Q^{/}dq^{/}),$$
$$d\cdot Z\ dx = dx(M\ dx + N\ dy + Pdp + Qdq),$$
$$d\cdot Z_{/}dx = dx(M_{/}dx + N_{/}dy_{/} + P_{/}dp_{/} + Q_{/}dq_{/}).$$

Iam vero, quia abscissa *x* ab illa translatione non afficitur, ponendum est ubique $dx = 0$, deinde vero reliquorum differentialium valores ex translatione puncti *n* in *v* orti per primam huius Capitis Propositionem ita se habebunt:

$dy^{/} = +nv$	$dp^{/} = -\frac{nv}{dx}$	$dq^{/} = +\frac{nv}{dx^2}$
$dy = 0$	$dp = +\frac{nv}{dx}$	$dq = -\frac{2nv}{dx^2}$
$dy_{/} = 0$	$dp_{/} = 0$	$dq_{/} = +\frac{nv}{dx^2}$

His differentialium per *nv* expressorum valoribus substitutis prodibit sequens valor differentialis

$$nv \cdot dx\left(N^{/} - \frac{P^{/}}{dx} + \frac{P}{dx} + \frac{Q^{/}}{dx^2} - \frac{2Q}{dx^2} + \frac{Q_{/}}{dx^2} \right)$$
$$= nv \cdot dx\left(N^{/} - \frac{dP}{dx} + \frac{ddQ_{/}}{dx^2} \right) = nv \cdot dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} \right)$$

ob $ddQ_{/} = ddQ$. Quamobrem pro curva quaesita ista habebitur aequatio

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0.$$

Q. E. I.

COROLLARIUM 1

41. Quodsi ergo in maximi minimive formula $\int Zdx$ insint etiam differentialia secundi gradus seu, quod idem est, si Z fuerit functio ipsarum x, y, p et q, ita ut sit

$$dZ = Mdx + Ndy + Pdp + Qdq,$$

aequatio pro curva quaesita erit
$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0.$$
quae facile ex differentiali ipsius Z formabitur.

COROLLARIUM 2

42. Si quantitas Q ipsa involvit q vel differentio-differentiale ipsius y, tum ddQ continebit differentialia quarti ordinis, in hocque genere erit aequatio pro curva inventa. Ex quo curva satisfaciens per quatuor data puncta traduci poterit.

COROLLARIUM 3

43. Si igitur in Q contineatur q, tum Problema ita determinate proponendum erit, ut inter omnes curvas per quatuor data puncta ductas ea definiatur, in qua $\int Zdx$ sit maximum vel minimum.

SCHOLION I

44. Ponamus in Q non contineri q, ut investigemus, cuiusnam gradus futura sit aequatio differentialis resultans. Accidit autem hoc, si maximi minimive formula proposita fuerit huiusmodi $\int Zqdx$, existente Z functione tantum ipsarum x, y et p, ita ut sit

$$dZ = Mdx + Ndy + Pdp.$$

Hinc igitur erit

$$d\cdot Zq = Mqdx + Nqdy + Pqdp + Zdq\ ,$$

unde pro curva quaesita orietur aequatio haec

$$0 = Nq - \frac{Pdq + qdP}{dx} + \frac{dMdx + dNdy + Nddy + Pddp + dPdp}{dx^2}$$

seu

$$0 = 2Nq + \frac{dM + pdN}{dx}$$

vel

$$0 = 2Ndp + dM + pdN\ ,$$

quae aequipollet tantum aequationi differentiali secundi gradus propter $dp = \frac{ddy}{dx}$, quod inest. Si igitur curva desideretur, in qua sit $\int Zqdx$ maximum vel minimum, existente *Z* functione ipsarum *x*, *y* et *p*, atque

$$dZ = Mdx + Ndy + Pdp,$$

pro curva quaesita habebitur aequatio

$$0 = dM + 2Ndp + pdN.$$

COROLLARIUM 4

45. Ut revertamur ad aequationem inventam

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0.$$

patet eam fore generaliter integrabilem, si sit $N = 0$, hoc est si in *Z* non contineatur *y;* prodibit enim integrando

$$C - P + \frac{dQ}{dx} = 0\,.$$

Si insuper sit $P = 0$, altera integratio succedit, qua prodit

$$Cx + D - Q = 0\,.$$

COROLLARIUM 5

46. Si sit $M = 0$, pariter una integratio in genere succedit; cum enim sit

$$dZ = Ndy + Pdp + Qdq,$$

multiplicetur aequatio

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0.$$

per *dy* seu *pdx*, habebitur

$$Ndy - pdP + \frac{pddQ}{dx} = 0.$$

Addatur

$$dZ - Ndy - Pdp - Qdq = 0,$$

orietur

$$dZ - pdP - Pdp + \frac{pddQ}{dx} - Qdq = 0;$$

cuius integrale est

$$Z - Pp + \frac{pdQ}{dx} - Qq = C.$$

COROLLARIUM 6

47. Si fuerit et $M = 0$ et $N = 0$, erit primo ob $N = 0$ ut supra

$$C - P + \frac{dQ}{dx} = 0$$

Deinde, cum sit

$$dZ = Pdp + Qdq,$$

multiplicetur illa aequatio per *dp* seu *qdx*, erit

$$Cdp - Pdp + qdQ = 0;$$

addatur $Pdp + Qdq - dZ = 0$, prodibit

$$Cdp + Qdq + qdQ - dZ = 0,$$

cuius integralis est

$$Cp + D + Qq - Z = 0.$$

SCHOLION 2

48. Si nexum aequationis inventae pro curva quaesita, quae habeat $\int Zdx$ maximum minimumve, cum differentiali ipsius Z inspiciamus, determinare licebit relationem inter differentialia dy, dp et dq, ut differentiale ipsius Z nihilo aequale positum praebeat aequationem pro curva quaesita. Cum enim sit

$$dZ = Mdx + Ndy + Pdp + Qdq,$$

comparetur cum hac forma aequatio pro curva

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0,$$

seu haec per $dy = pdx$ multiplicata, quae erit

$$Ndy - pdP + \frac{pddQ}{dx} = 0;$$

unde patet in differentiali ipsius Z loco Mdx scribi debere 0, at terminum Ndy invariatum relinqui, porro loco Pdp scribendum esse $-pdP$ ac loco Qdq poni debere $\frac{pddQ}{dx}$. Verum quoad haec a priori pateant, praestabit formam aequationis inventae retinere, quippe quae facile memoria teneri potest. Caeterum notandum est Problemata huc pertinentia omnino esse nova neque adhuc ab iis, qui alias de hoc argumento scripserunt, pertractata. Alias enim Scriptares maximi minimive formulas contemplari non consueverunt, nisi in quibus ad summum differentialia coordinatarum primi gradus inessent. Quamobrem eo magis erit operae pretium naturam huiusmodi Problematum accuratius indagare atque inprimis ostendere, quomodo curvae satisfacientes quatuor puncta, per quae transeant, ad sui determinationem admittant. Hunc in finem sequentia Exempla adiicere visum est atque in singulis indicare, quae ad maiorem illustrationem facere poterunt.

EXEMPLUM I

49. *Invenire curvam, in qua sit* $\int \frac{y^n ddy}{x^m dy}$ *maximum vel minimum.*

Ista maximi minimive formula ope substitutionum

$$dy = pdx \text{ et } ddy = qdx^2$$

abit in hanc

$$\int \frac{y^n qdx}{x^m p};$$

quae cum sit similis formulae paragrapho 44 tractatae $\int Zqdx$, ubi in Z tantum x, y et p contineri posuimus, fiet, comparatione instituta

$$Z = \frac{y^n}{x^m p} \text{ et } dZ = -\frac{my^n dx}{x^{m+1} p} + \frac{ny^{n-1}dy}{x^m p} - \frac{y^n dp}{x^m p^2};$$

unde erit

$$M = -\frac{my^n}{x^{m+1} p} \text{ et } N = \frac{ny^{n-1}}{x^m p};$$

hincque

$$Np = \frac{ny^{n-1}}{x^m}.$$

Cum igitur pro curva quaesita inventa sit haec aequatio

$$0 = dM + 2Ndp + pdN = dM + Ndp + d \cdot Np,$$

habebimus pro nostro casu hanc aequationem

$$0 = \frac{m(m+1)y^n dx}{x^{m+2} p} - \frac{mny^{n-1}dy}{x^{m+1} p} + \frac{my^n dp}{x^{m+1} pp} + \frac{ny^{n-1}dp}{x^m p} + \frac{n(n-1)y^{n-2}dy}{x^m} - \frac{mny^{n-1}dx}{x^{m+1}},$$

quae multiplicata per $\frac{y^{n+2} p^2}{y^{n-1}}$ mutatur in hanc

$$0 = m(m+1)ydy - mnxpdy + mxydp + nx^2 pdp + \frac{n(n-1)x^2 p^2 dy}{y} - mnxpdy$$

seu

$$0 = m(m+1)y^2 dy - 2mnxypdy + n(n-1)x^2 p^2 dy + mxy^2 dp + nx^2 ypdp,$$

quae est aequatio differentialis secundi gradus, quae posito

$$y = e^{\int vdx}$$

reducetur ad istam primi gradus

$$m(m+1)vdx + mxdv - m(2n-1)xv^2 dx + nx^2 vdv + n^2 x^2 v^3 dx = 0.$$

Quodsi autem ponamus $m = 0$, ita ut maximum minimumve esse debeat

$$\int \frac{y^n ddy}{dy},$$

habebitur haec aequatio

$$y(n-1)pdy + ydp = 0,$$

quae integrata dabit

$$y^{n-1}p = C \text{ seu } y^{n-1}dy = Cdx;$$

haecque denuo integrata praebet $y^n = Cx + D$. Sin autem ponamus $n = 0$, ita ut maximum minimumve esse debet haec formula $\int \frac{ddy}{x^m dy}$, erit

$$(m+1)dy + xdp = 0 \text{ seu } (m+1)pdx + xdp = 0,$$

cuius integrale est

$$x^{m+1}p = C \text{ seu } dy = Cx^{-m-1}dx,$$

quae denuo integrata dat

$$y = \frac{C}{x^m} + D.$$

Patet autem in his curvis inventis formulam propositam fieri maximum, non vero minimum; nam si sumatur linea recta, ob $ddy = 0$ manifestum est valorem formulae propositae minorem fore pro recta linea quam pro curvis inventis.

SCHOLION 3

50. Ratio hic assignari potest, cur huiusmodi quaestiones, in quibus $\int Zqdx$ maximum minimumve esse debet, deducant tantum ad aequationem differentialem secundi gradus ideoque quaestionibus praecedentis Problematis potius sint adnumerandae, siquidem *Z* fuerit functio ipsarum *x* et *y* et *p*. Nam per reductionem integralium formula $\int Zqdx$ seu $\int \frac{Zddy}{dx}$ reduci potest ad talem formam $Y + \int Vdx$, in qua *Y* et *V* sint functiones ipsarum *x*, *y* et *p* tantum, non amplius involventes *q*. Cum igitur *Y* sit quantitas absoluta atque idcirco in maximi minimive inquisitionem non cadat, formula $\int Zqdx$ fiet maxima vel minima, si $\int Vdx$ talis reddatur; adeo ut huiusmodi formulae $\int Zqdx$ reduci queant ad praecedentis Problematis statum; unde mirum non est, quod pro curvis satisfacientibus aequatio differentialis secundi gradus duntaxat reperiatur. Quo autem memorata reductio formulae $\int Zqdx$ seu $\int Zdp$ ad

$$Y+\int Vdx$$

melius percipiatur, ponamus, cum *Y* sit functio ipsarum *x, y* et *p*, esse

$$dY=\rho dx+\sigma dy+\tau dp=(\rho+\sigma p)dx+\tau dp\,;$$

et ex aequalitate $\int Zdp=Y+\int Vdx$ erit

$$Zdp=(\rho+\sigma p)dx+\tau dp+Vdx\,;$$

unde concluditur $\tau=Z$ et $V=-\rho-\sigma p$. Quamobrem ipsa haec reductio sequenti modo instituetur; integretur formula *Z dp* positis *x* et *y* constantibus et integrale erit functio ipsarum *x, y* et *p*, quae vocetur *Y*. Deinde differentietur haec functio *Y* ponendo *p* constans et differentiale negative sumtum dabit *V dx,* eritque *V* functio ipsarum *x, y* et *p* non continens *q*. Quoties igitur reddi debet huiusmodi formula $\int Zqdx$ maximum minimumve ac *Z* est functio ipsarum *x* et *y* et *p*, tum quaestio, etiamsi videatur ad praesens Problema pertinere, tamen statim ad Problema praecedens reducetur. Ita si sumamus formulam $\int\frac{y^n ddy}{dy}$ seu $\int\frac{y^n dp}{p}$, haec facile transformatur in

$$y^n lp-n\int y^{n-1}dylp\,,$$

unde maximum vel minimum esse debebit haec formula

$$\int y^{n-1}dylp \text{ seu } \int y^{n-1}pdxlp\,,$$

quae per praecedens Problema tractata dabit

$$Z=y^{n-1}plp \text{ et } dZ=\left(n-1\right)y^{n-2}dyplp+y^{n-1}dp(1+lp)\,;$$

eritque

$$M=0, N=(n-1)y^{n-2}plp \text{ et } P=y^{n-1}(1+lp).$$

At ob $M=0$ supra paragrapho 30 pro curva quaesita inventa est haec aequatio $Z+C=Pp$, quae ad nostrum casum accommodata praebet

$$y^{n-1}plp+C=y^{n-1}p+y^{n-1}plp \text{ sive } y^{n-1}p=C\,;$$

quae est ea ipsa aequatio, quam ante pro eodem casu in solutione Exempli invenimus. Hanc ob rem ad Exempla huic Problemati propria progrediamur.

EXEMPLUM II

51. *lnvenire curvam Am* (Fig. 5), *quae cum sua evoluta AR et radio osculi mR in quovis loco applicato minimum spatium ARm includat.*

Positis abscissa $AM = x$, applicata $Mm = y$, erit radius osculi

$$mR = -\frac{(1+pp)^{3:2}}{q};$$

area autem *ARm* est

$$= \int \tfrac{1}{2} mR \cdot dx\sqrt{(1+pp)};$$

ex qua minimum esse oportet hanc formulam

$$\int \frac{(1+pp)^2 dx}{q}.$$

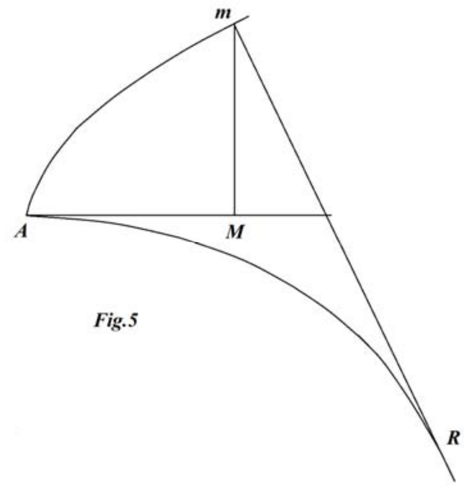

Fig.5

Erit itaque

$$Z = \frac{(1+pp)^2}{q} \text{ et } dZ = \frac{4(1+pp)pdp}{q} - \frac{(1+pp)^2 dq}{qq}.$$

unde fit

$$M = 0,\ N = 0,\ P = \frac{4(1+pp)p}{q} \text{ et } Q = -\frac{(1+pp)^2}{qq}.$$

Cum nunc sit $M = 0$ et $N = 0$, erit per Corollarium 6 aequatio pro curva quaesita

$$Z = D + Cp + Qq \text{ seu } \frac{(1+pp)^2}{q} = D + Cp - \frac{(1+pp)^2}{q},$$

hoc est

$$2(1+pp)^2 = Dq + Cpq.$$

Quoniam porro est $dp = qdx$ seu $q = \frac{dp}{dx}$, erit

$$2dx = \frac{(D+Cp)dp}{(1+pp)^2},$$

cuius integrale est

$$x = \frac{a}{1+pp} + 2b\int \frac{dp}{(1+pp)^2} = \frac{a+bp}{1+pp} + b\int \frac{dp}{1+pp} + c;$$

mutatis pro lubitu constantibus habebitur

$$x=\frac{a+bp+cpp}{1+pp}+b\,\text{Atang}\,p.$$

Deinde quia est $dy=pdx$, erit

$$y=\int pdx=px-\int xdp$$

ideoque

$$y=\frac{ap+bp^2+cp^3}{1+pp}+bp\,\text{Atang}\,p-\int\frac{(a+bp+cpp)dp}{1+pp}-b\int dp\,\text{Atang}\,p$$
$$=\frac{ap+bp^2+cp^3}{1+pp}-\int\frac{(a+cpp)dp}{1+pp},$$

ob

$$b\int dp\,\text{Atang}\,p=bp\,\text{Atang}\,p-b\int\frac{pdp}{1+pp}.$$

Hinc erit

$$y=f+\frac{ap+bp^2+cp^3}{1+pp}+(c-a)\,\text{Atang}\,p-cp$$
$$=\frac{f+(a-c)p+(b+f)pp}{1+pp}-(c-a)\,\text{Atang}\,p.$$

Atque ex his quidem ipsarum *x* et *y* valoribus per *p* inventis curva quaesita per data quatuor puncta duci atque construi potest. Verum, ut ipsa curva qualis sit cognoscatur, eliminetur Atang *p;* eritque

$$\text{Atang}\,p=\frac{x}{b}-\frac{\frac{a}{b}+p+\frac{c}{b}pp}{1+pp}=\frac{y}{c-a}-\frac{\frac{f}{c-a}-p+\frac{(b+f)}{c-a}pp}{1+pp};$$

atque hinc

$$(c-a)x-by=\frac{(ac-aa-bf)+2b(c-a)p+(cc-ac-bb-bf)pp}{1+pp}.$$

Quoniam autem ipsa curva non mutatur, etiamsi coordinatae constante quantitate vel augeantur vel deminuantur, erit

$$(c-a)x-by=\frac{bb-(c-a)^2+2b(c-a)p}{1+pp};$$

positoque *a* loco $c-a$ habebitur

$$ax - by = \frac{bb - aa + 2abp}{1 + pp};$$

et subtracta constante *bb* erit

$$ax - by = -\frac{aa - 2abp + bbpp}{1 + pp}$$

hincque

$$\sqrt{by - ax} = \frac{bp - a}{\sqrt{(1 + pp)}}$$

Ponatur arcus curvae $= w$; erit $dw = dx\sqrt{(1 + pp)}$; unde emerget ista aequatio

$$dw = \frac{bdy - adx}{\sqrt{by - ax}} \text{ atque porro } w = 2\sqrt{(by - ax)}.$$

Exprimit autem $by - ax$ multiplum abscissae super alio quodam axe fixo assumtae, cui adeo quadratum arcus respondentia est proportionale. Ex quo intelligitur curvam quaesito respondentem esse Cycloidem, quae per quatuor data puncta determinatur, atque sic descripta inter omnes alias curvas per eadem quatuor puncta ductas minimum cum sua evoluta concludit spatium. Conclusio haec ideo aliquantum difficilior facta est, quod Cyclois pro recta quacunque instar axis assumpta quaesito satisfaciat, atque aequatio pro axe quocunque admodum fiat intricata. Si autem vel *a* vel *b* posuissemus $= 0$, quo quidem extensio solutionis non fuisset restricta, aequatio pro Cycloide statim prodiisset.

EXEMPLUM III

52. *Invenire curvam, in qua sit* $\int \rho^n dw$, *denotante* ρ *radium osculi et dw elementum curvae, maximum vel minimum.*

Per positiones ante factas est

$$dw = dx\sqrt{(1 + pp)} \text{ et } \rho = \frac{(1 + pp)^{3:2}}{q};$$

unde maximi minimive formula erit

$$\int \frac{(1 + pp)^{(3n+1):2} dx}{q^n};$$

hincque fit

$$Z = \frac{(1+pp)^{(3n+1):2}}{q^n}$$

et

$$dZ = \frac{(3n+1)(1+pp)^{(3n-1):2}\, pdp}{q^n} - \frac{n(1+pp)^{(3n+1):2}\, dq}{q^{n+1}}.$$

Quamobrem erit

$$M = 0, \quad N = 0, \quad P = \frac{(3n+1)(1+pp)^{(3n-1):2}\, p}{q^n} \text{ et } Q = -\frac{n(1+pp)^{(3n+1):2}}{q^{n+1}}.$$

Cum autem sit $M = 0,\ N = 0,$ erit per paragraphum 47

$$Z = Cp + D + Qq$$

ideoque

$$\frac{(1+pp)^{(3n+1):2}}{q^n} = Cp + D - \frac{n(1+pp)^{(3n+1):2}}{q^n}$$

seu

$$(n+1)(1+pp)^{(3n+1):2} = (Cp+D)q^n$$

atque hinc

$$q = \frac{(1+pp)^{(3n+1):2n}}{\sqrt[n]{(Cp+D)}} = \frac{dp}{dx};$$

ergo

$$dx = dp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$$

et

$$dy = pdp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$$

Hic autem merito suspicari licet aequationem futuram esse simpliciorem, si alius axis accipiatur. Hanc ob rem concipiamus alium axem, in quo abscissa sit *t*, applicata *v*, sitque $dv = sdt$ ac ponatur

$$x = \frac{\alpha t + \beta v}{\gamma} \text{ et } y = \frac{\beta t - \alpha v}{\gamma}$$

posito $\gamma = \sqrt{(\alpha^2 + \beta^2)}$. Erit ergo

$$dx=\frac{\alpha dt+\beta sdt}{\gamma} \text{ et } dy=\frac{\beta dt-\alpha sdt}{\gamma}$$

atque

$$\frac{dy}{dx}=p=\frac{\beta-\alpha s}{\alpha+\beta s} \text{ et } (1+pp)=\frac{\gamma^2(1+ss)}{(\alpha+\beta s)^2} \text{ et } dp=-\frac{\gamma\gamma ds}{(\alpha+\beta s)^2}.$$

Porro autem erit

$$C+Dp=\frac{\alpha C+\beta D+s(\beta C-\alpha D)}{\alpha+\beta s}$$

et

$$(1+pp)^{(3n+1):2n}=\frac{\gamma^{(3n+1):n}(1+ss)^{(3n+1):2n}}{(\alpha+\beta s)^{(3n+1):n}}$$

His substitutis erit

$$dx=\frac{\alpha dt+\beta dv}{\gamma}=\frac{a(\alpha+\beta s)ds}{\gamma(1+ss)^{(3n+1):2n}},$$

posito $\beta C=\alpha D$ et mutata constante. Porro autem fit

$$dy=\frac{\beta dt-\alpha dv}{\gamma}=\frac{a(\beta-\alpha s)ds}{\gamma(1+ss)^{(3n+1):2n}}$$

et coniunctim prodit

$$dt=\frac{ads}{(1+ss)^{(3n+1):2n}} \text{ et } dv=\frac{asds}{(1+ss)^{(3n+1):2n}}.$$

Cum nunc has coordinatas aeque *x* et *y* appellare possimus ac praecedentes, fiet $s=p$ atque

$$dx=\frac{adp}{(1+pp)^{(3n+1):2n}} \text{ et } dy=\frac{apdp}{(1+pp)^{(3n+1):2n}},$$

quae ex praecedentibus oriuntur, si ibi ponatur $D=0$, ex quo perspicuum est latitudini solutionis superioris, in qua inerat $C+Dp$, nihil omnino decedere, etsi ponatur $D=0$. Eadem scilicet prodit linea curva, quicunque valor litterae *D* tribuatur, etiamsi alia aequatio inter *x* et *y* proveniat, verumtamen ad alium axem relata. Notare interim convenit pluribus casibus curvam algebraicam satisfacere; quorum quasi primus est, si $n = \frac{1}{2}$, quo erit

$$x = \int \frac{adp}{(1+pp)^{5:2}} = \frac{a(1+\frac{2}{3}pp)}{(1+pp)^{3:2}} \text{ et } y = \int \frac{apdp}{(1+pp)^{5:2}} = -\frac{\frac{1}{3}a}{(1+pp)^{3:2}}:$$

unde fiet

$$\left(1+pp\right)^{3:2} = -\frac{a}{3y} \text{ et } pp = \sqrt[3]{\frac{aa}{9yy}} - 1,$$

quibus substitutis resultat

$$x = -\left(2\sqrt[3]{\frac{aay}{9}} + y\right)\sqrt{\left(\sqrt[3]{\frac{aa}{9yy}} - 1\right)}$$

aequatio algebraica pro curva, casu quo est $n = \frac{1}{2}$.

EXEMPLUM IV

53. *Invenire curvam, in qua sit valor huius formulae* $\int \frac{ydydx^2}{ddy}$ *omnium minimus.*

Patet primo maximum locum habere non posse, quia in linea recta fit $ddy = 0$ ideoque valor formulae propositae infinite magnus. Quamobrem videndum est, in quanam linea curva fiat valor formulae $\int \frac{ydydx^2}{ddy}$ minimus. Haec autem formula per substitutiones nostras abit in hanc $\int \frac{ypdx}{q}$ eritque

$$Z = \frac{yp}{q} \text{ et } dZ = \frac{pdy}{q} + \frac{ydp}{q} - \frac{ypdq}{qq};$$

erit ergo

$$M = 0, \quad N = \frac{p}{q} \text{ et } P = \frac{y}{q} \text{ et } Q = -\frac{yp}{qq}.$$

Quoniam autem est $M = 0$, curva quaesita sequenti exprimetur aequatione

$$Z - Pp - Qq + \frac{pdQ}{dx} = C,$$

ut Corollario 5 est ostensum. Quamobrem ista proveniet aequatio

$$\frac{yp}{q}-\frac{p}{dx}d.\frac{yp}{qq}=C$$

seu

$$\frac{ydy}{pq}+\frac{adx}{p}=\frac{pdy}{qq}+\frac{ydp}{qq}-\frac{2ypdq}{q^3}$$

ob $dy=pdx$. Quia vero est $dp=qdx$, erit

$$\frac{ydp}{qq}=\frac{ydx}{q}=\frac{ydy}{pq} \text{ ideoque } \frac{adx}{p}=\frac{pdy}{qq}-\frac{2ypdq}{q^3}$$

vel

$$\frac{adp}{pp}=\frac{dy}{q}-\frac{2ydq}{qq}.$$

Si ponatur constans $a=0$, haec aequatio fiet integrabilis eritque

$$y=bqq \text{ et } q=\sqrt{\frac{y}{b}}=\frac{dp}{dx}=\frac{pdp}{dy},$$

unde fit $pdp=dy\sqrt{\frac{y}{b}}$ atque integrando

$$\frac{pp}{2}=\tfrac{2}{3}y\sqrt{\frac{y}{b}}+c$$

seu, mutatis constantibus,

$$pp=\frac{y\sqrt{y}+a\sqrt{a}}{b\sqrt{b}} \quad \text{et} \quad p=\sqrt{\frac{y^{3:2}+a^{3:2}}{b^{3:2}}}=\frac{dy}{dx};$$

hincque

$$dx=dy\sqrt{\frac{b^{3:2}}{y^{3:2}+a^{3:2}}}.$$

Ponatur denuo $a=0$, erit

$$x=\frac{b\sqrt{c}}{\sqrt{y}} \text{ et } xxy=b^2c;$$

quae est aequatio maxime specialis pro curva quaestioni satisfaciente.

EXEMPLUM V

54. *Invenire curvam, in qua sit valor huius formulae* $\int q^n dx$ *seu* $\int \frac{ddy^n}{dx^{2n-1}}$ *maximus vel minimus.*

Habetur ergo

$$Z = q^n \text{ et } dZ = nq^{n-1}dq\,;$$

unde erit

$$M = 0, N = 0\,, P = 0 \text{ et } Q = nq^{n-1}.$$

Cum igitur aequatio pro curva satisfaciente sit haec $\frac{ddQ}{dx^2} = 0$, erit

$$dQ = \alpha dx \text{ et } Q = q^{n-1} = \alpha x + \beta\,;$$

hincque

$$q = (\alpha x + \beta)^{1:(n-1)} = \frac{dp}{dx}\,;$$

ex quo fiet

$$p = (\alpha x + \beta)^{n:(n-1)} + \gamma = \frac{dy}{dx}$$

et tandem

$$y = (\alpha x + \beta)^{(2n-1):(n-1)} + \gamma x + \delta,$$

ubi coefficientes per integrationes ingressos in constantibus sumus complexi. Curvae igitur satisfacientes perpetuo sunt algebraicae, excepto casu, quo est $n = \frac{1}{2}$; tum enim postrema integratio praebebit

$$y = \frac{1}{\alpha} l(\alpha x + \beta) + \gamma x + \delta.$$

Quod ad casum $n = 1$ attinet, ille in investigationem maximorum et minimorum nequidem incurrit, cum formula $\int qdx$ non sit indeterminata, sed determinatum valorem, puta *p*, ob

$$qdx = dp$$

referat. Caeterum patet evanescente termino $(\alpha x + \beta)^{(2n-1):(n-1)}$ lineam rectam quaesito satisfacere ob $y = \gamma x + \delta$. Scilicet, si quatuor puncta data, per quae curva quaesita transire debeat, sint in directum posita, tum ipsa linea recta prae omnibus reliquis lineis per eadem quatuor puncta transeuntibus quaesito satisfaciet.

EXEMPLUM VI

55. *lnvenire curvam, in qua sit* $\int \frac{xpdx}{yq}$ *maximum vel minimum.*

Quia est $Z = \frac{xp}{yq}$, erit

$$dZ = \frac{pdx}{yq} - \frac{xpdy}{y^2q} + \frac{xdp}{yq} - \frac{xpdq}{yqq}$$

ideoque

$$M = \frac{p}{yq},\ \ N = -\frac{xp}{y^2q},\ \ P = \frac{x}{yq}\ \text{ et }\ Q = -\frac{xp}{yq^2}.$$

Quorum terminorum cum nullus evanescat, aequatio pro curva quaesita erit

$$-\frac{xp}{y^2q} - \frac{1}{dx} d \cdot \frac{x}{yq} - \frac{1}{dx^2} d^2 \cdot \frac{xp}{yq^2} = 0$$

seu

$$0 = \frac{xpdx^2}{y^2q} + \frac{dx^2}{yq} - \frac{xdxdy}{y^2q} - \frac{xdxdq}{yq^2} + d \cdot \left(\frac{pdx}{yq^2} - \frac{xpdy}{y^2q^2} + \frac{xdp}{yq^2} - \frac{2xpdq}{yq^3} \right)$$

vel

$$0 = q^2dx^2(3yq - 2p^2)(y - xp) - 4yqdxdq(xyq - \ xp^2 + yp) + 6xy^2pdq^2 - 2xy^2pqddq.$$

Quae est aequatio differentialis quarti ordinis, quae, utrum integrari possit an non, haud facile patet; neque etiam operae pretium est in modum eam integrandi diligentius inquirere, quoniam hic casus non ex solutione Problematis alicuius utilis est natus, sed fortuito excogitatus. Hoc autem Exemplum ideo adiicere visum est, ut casus habeatur, quo solutio non solum ad aequationem differentialem quarti ordinis ascendit, sed etiam neque per subsidia generalia supra allata ad gradum inferiorem perduci queat. Praecedentia enim Exempla cuncta ita sunt comparata, ut per regulas generales in Corollariis expositas statim aequatio pro curva quaesita inferioris gradus differentialis erui potuerit.

PROPOSITIO V. PROBLEMA

56. *Invenire curvam, in qua sit valor formulae* $\int Zdx$ *maximus vel minimus, existente Z eiusmodi functione, quae differentialia cuiusvis gradus involvat, ita ut sit* $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt + \text{etc.}$

SOLUTIO

Quoniam (Fig. 4) translatio puncti *n* in *v* praecedentia elementa magis afficit quam sequentia, unicum enim sequens elementum afficit, at in praecedentia eo ulterius extenditur, quo altiorum ordinum differentialia adsint, hanc ob rem expediet aliquam anteriorem applicatam, uti *Hh*, pro prima accipere, ita ut mutatio ex particula *nv* applicatae *Nn* adiecta non citra *Hh* porrigatur; id quod eveniet, si in *Z* differentialia non ultra sextum ordinem ascendant. Sufficiet autem valorem ipsius *dZ* ad terminum *Tdt* usque extendere, quia ex ipsa solutione modus facile colligetur eam ad quotcunque ulteriores terminos accommodandi. Praeterea, quia in hoc Problemate praecedentia omnia continentur, constabit simul solutionem perpetuo eandem prodire, quaecunque applicata particula quadam infinite parva, uti *nv*, augeatur. Sit igitur $AH = x$ et $Hh = y$, respondebunt singulis punctis abscissae *H, I, K, L, M, N, O* etc. valores litterarum *p, q, r, s, t* etc., ut sequitur:

H	$y,$	$p,$	$q,$	$r,$	$s,$	t
I	$y^{I},$	$p,^{I}$	$q^{I},$	$r^{I},$	$s^{I},$	t^{I}
K	$y^{II},$	$p^{II},$	$q^{II},$	$r^{II},$	$s^{II},$	t^{II}
L	$y^{III},$	$p^{III},$	$q^{III},$	$r^{III},$	$s^{III},$	t^{III}
M	$y^{IV},$	$p^{IV},$	$q^{IV},$	$r^{IV},$	$s^{IV},$	t^{IV}
N	$y^{V},$	$p^{V},$	$q^{V},$	$r^{V},$	$s^{V},$	t^{V}

Hi autem singuli valores a translatione *n* in *v* sequentia augmenta accipient, quae ex Propositione prima, debita mutatione signorum adhibita, ita se habebunt:

$$dy = 0 \quad dy^{I} = 0 \quad dy^{II} = 0 \quad dy^{III} = 0 \quad dy^{IV} = 0 \quad dy^{V} = +nv$$

$$dp = 0 \quad dp^{I} = 0 \quad dp^{II} = 0 \quad dp^{III} = 0 \quad dp^{IV} = +\frac{nv}{dx} \quad dp^{V} = -\frac{nv}{dx}$$

$$dq = 0 \quad dq^{I} = 0 \quad dq^{II} = 0 \quad dq^{III} = +\frac{nv}{dx^2} \quad dq^{IV} = -\frac{2nv}{dx^2} \quad dq^{V} = +\frac{nv}{dx^2}$$

$$dr = 0 \quad dr^{I} = 0 \quad dr^{II} = +\frac{nv}{dx^3} \quad dr^{III} = -\frac{3nv}{dx^3} \quad dr^{IV} = +\frac{3nv}{dx^3} \quad dr^{V} = -\frac{nv}{dx^3}$$

$$ds = 0 \quad ds^{I} = +\frac{nv}{dx^4} \quad ds^{II} = -\frac{4nv}{dx^4} \quad ds^{III} = +\frac{6nv}{dx^4} \quad ds^{IV} = -\frac{4nv}{dx^4} \quad ds^{V} = +\frac{nv}{dx^4}$$

$$dt = +\frac{nv}{dx^5} \quad dt^{I} = -\frac{5nv}{dx^5} \quad dt^{II} = +\frac{10nv}{dx^5} \quad dt^{III} = -\frac{10nv}{dx^5} \quad dt^{IV} = +\frac{5nv}{dx^5} \quad dt^{V} = -\frac{nv}{dx^5}$$

Quoniam porro valor formulae $\int Zdx$ abscissae *AH* respondet, isque a translatione puncti *n* in *v* non mutatur, sequentibus abscissae elementis valores formulae $\int Zdx$ respondent, qui in hac Tabula exhibentur:

Elemento	respondet
HI	Zdx
IK	$Z^{/}dx$
KL	$Z^{//}dx$
LM	$Z^{///}dx$
MN	$Z^{IV}dx$
NO	$Z^{V}dx$

Ad horum valorum incrementa, ex translatione puncti *n* in *v* oriunda, invenienda, singulos hos valores differentiari locoque differentialium *dy, dp, d q, dr, ds, dt* cum ipsorum derivativis valores supra assignatos et per *nv* expressos substitui oportet ; eritque ut sequitur:

$$d\cdot Zdx = nv\cdot dx\left(\frac{T}{dx^5}\right)$$

$$d\cdot Z^{/}dx = nv\cdot dx\left(\frac{S^{/}}{dx^4}-\frac{5T^{/}}{dx^5}\right)$$

$$d\cdot Z^{//}dx = nv\cdot dx\left(\frac{R^{//}}{dx^3}-\frac{4S^{/}}{dx^4}+\frac{10T^{/}}{dx^5}\right)$$

$$d\cdot Z^{///}dx = nv\cdot dx\left(\frac{Q^{///}}{dx^2}-\frac{3R^{///}}{dx^3}+\frac{6S^{///}}{dx^4}-\frac{10T^{///}}{dx^5}\right)$$

$$d\cdot Z^{IV}dx = nv\cdot dx\left(\frac{P^{IV}}{dx}-\frac{2Q^{IV}}{dx^2}+\frac{3R^{IV}}{dx^3}-\frac{4S^{IV}}{dx^4}+\frac{5T^{IV}}{dx^5}\right)$$

$$d\cdot Z^{V}dx = nv\cdot dx\left(N^{V}-\frac{P^{V}}{dx}+\frac{Q^{V}}{dx^2}-\frac{R^{V}}{dx^3}+\frac{S^{V}}{dx^4}-\frac{T^{V}}{dx^5}\right)$$

Quia iam haec sola elementa a transpositione puncti *n* in *v* alterantur et incrementa capiunt, summa horum incrementorum dabit integrum valorem differentialem, quem formula $\int Zdx$ ad totam abscissam *AZ* extensa accipit; qui igitur erit

$$nv\cdot dx\begin{cases} +N^{V} \\ -\dfrac{P^{V}-P^{IV}}{dx} \\ +\dfrac{Q^{V}-2Q^{IV}+Q^{///}}{dx^{2}} \\ -\dfrac{R^{V}-3R^{IV}+3R^{///}-R^{//}}{dx^{3}} \\ +\dfrac{S^{V}-4S^{IV}+6S^{///}-4S^{//}+S^{/}}{dx^{4}} \\ -\dfrac{T^{V}-5T^{IV}+10T^{///}-10T^{//}+5T^{/}-T}{dx^{5}}. \end{cases}$$

Singula autem haec membra per differentialia commode et succincte exprimi poterunt; erit enim

$$\begin{aligned} &-P^{V}+P^{IV}=dP^{IV} \\ &+Q^{V}-2Q^{IV}+Q^{///}=+ddQ^{///} \\ &-R^{V}+3R^{IV}-3R^{///}+R^{//}=-d^{3}R^{//} \\ &S^{V}-4S^{IV}+6S^{///}-4S^{//}+S^{/}=+d^{4}S^{/} \\ &-T^{V}+5T^{IV}-10T^{///}+10T^{//}-5T^{/}+T=-d^{5}T. \end{aligned}$$

Quamobrem formulae $\int Zdx$ integer valor differentialis ex particula nv ortus, erit

$$=nv\cdot dx\left(N^{V}-\frac{dP^{IV}}{dx}+\frac{ddQ^{///}}{dx^{2}}-\frac{d^{3}R^{//}}{dx^{3}}+\frac{d^{4}S^{/}}{dx^{4}}-\frac{d^{5}T}{dx^{5}}\right);$$

Hic autem, quia omnes termini sunt homogenei, signaturae tuto omitti possunt, evanescit enim discrimen inter N^{V} et N itemque inter dP^{IV} et dP reliquaque. Quocirca habebitur formulae $\int Zdx$ iste valor differentialis ex quo simul valor differentialis formulae $\int Zdx$ colligi potest, si in Z altiora etiam differentialia inessent. Quare, si curva quaeratur, quae habeat $\int Zdx$ maximum vel minimum pro data abscissa, fueritque $dZ=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+Tdt+\text{etc.}$, erit primum formulae $\int Zdx$ valor differentialis hic:

$$nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^{2}}-\frac{d^{3}R}{dx^{3}}+\frac{d^{4}S}{dx^{4}}-\frac{d^{5}T}{dx^{5}}+\text{ etc.}\right).$$

Hincque pro curva quaesita orietur ista aequatio

$$0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \text{etc.}$$

Q. E. I.

COROLLARIUM 1

57. In formula $\int Zdx$, uti eam tractavimus, quantitas Z continet differentialia quinti gradus, siquidem in differentiali ipsius

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt$$

terminus Tdt est ultimus. Cum igitur in T adhuc insint differentialia quinti gradus seu t, perspicuum est aequationem pro curva quaesita fore differentialem decimi gradus.

COROLLARIUM 2

58. Hinc intelligitur perpetuo aequationem differentialem pro curva ad gradum duplo altiorem ascendere debere, quam fuerit ipsa formula maximi minimive. Ponimus enim, in ultimo termino $T\,dt$, quantitatem T adhuc t in se complecti; nisi enim hoc esset, duobus gradibus aequatio deprimeretur, uti ex paragrapho 50 colligere licet.

COROLARIUM 3

59. Si igitur in Z differentialia gradus n contineantur, tum aequatio pro curva differentialis erit gradus $2n$ et hanc ob rem totidem novas constantas arbitrarias potestate in se continet.

COROLLARIUM 4

60. Ob tot igitur constantes arbitrarias totidem puncta ad Problema determinandum proposita esse oportet; ita scilicet Problema, ut sit determinatum, enuntiari debebit: Inter omnes curvas per data $2n$ puncta transeuntes determinare eam, in qua sit $\int Zdx$ maximum vel minimum, siquidem quantitas Z complectatur differentialia n gradus.

COROLLARIUM 5

61. Ob *n* igitur numerum integrum, numerus punctorum, quo Problema determinabitur, semper erit par. Sic vel nullum punctum vel duo vel quatuor vel sex vel octo puncta et ita porro ad Problematis determinationem requiruntur.

SCHOLION 1

62. Ex gradu *differentialitatis* igitur, ad quem aequatio pro curva inventa assurgit, vel ex numero punctorum, per quae curvam satisfacientem transire oportet, huiusmodi Problemata commode in Classes distribui poterunt. Ad primam igitur Classem ea referentur Problemata, in quibus absolute quaeritur linea curva, quae pro data abscissa habeat valorem $\int Zdx$ maximum vel minimum; talia Problemata cum in Propositione secunda continentur, tum etiam in tertia, iis casibus, quos paragraphia 26 et 27 exposuimus; his scilicet casibus solutio praebet curvam determinatam quaesito satisfacientem. Classis secunda ea complectitur Problemata, quorum solutio ad aequationem differentialem secundi gradus perducit; haecque duo puncta ad sui determinationem poscunt et ita proponi debent, ut inter omnes curvas per data duo puncta transeuntes ea definiatur, in qua sit $\int Zdx$ maximum vel minimum; cuiusmodi Problemata in Propositione tertia soluta dedimus. Porro ad tertiam Classem pertinent Problemata in Propositione quarta tractata, quae ita se habent, ut inter omnes curvas per quatuor data puncta transeuntes determinetur ea, quae habeat $\int Zdx$ maximum vel minimum. Simili modo quarta Classis postulat ad determinationem sex puncta, quinta octo et ita porro, quas Classes omnes in praesente Problemate sumus complexi. Caeterum etsi aequatio inventa ad tantum differentialium gradum ascendit, tamen saepius generaliter integrationem unam vel plures admittit, cuiusmodi casus in praecedentibus Problematibus nonnullos exhibuimus. Hanc ob rem videamus etiam, quibus casibus aequatio nostra generalis integrationem, vel unam vel plures, admittat; ut in allatis Exemplis statim videre liceat, utrum ea in his casibus contineantur an secus. Huiusmodi autem casus potissimum sunt duo, in quorum altero est $N=0$, in altero $M=0$, a quibus deinceps alii casus pendent, quos hic evolvemus.

CASUS I

63. Sit in maximi minimive formula $\int Zdx$ terminus $N=0$, ita ut sit

$$dZ = Mdx + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

Hoc ergo casu aequatio pro curva erit haec

$$0 = -\frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \text{etc.},$$

quae per *dx* multiplicata fit integrabilis, prodibitque

$$0 = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \text{etc.}$$

CASUS II

64. Sit et $N = 0$ et $P = 0$, ita ut sit

$$dZ = Mdx + Qdq + Rdr + Sds + Tdt + \text{etc.}$$

Quoniam est $N = 0$, una integratio iam successit habeturque pro curva quaesita ista aequatio modo inventa, posito etiam $P = 0$:

$$0 = A + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \text{etc.},$$

quae per *dx* multiplicata denuo integrari poterit eritque

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.}$$

CASUS III

65. Si fuerit et $N = 0$ et $P = 0$ et $Q = 0$, ita ut sit

$$dZ = Mdx + Rdr + Sds + Tdt + \text{etc.}$$

Bini valores *N* et *P* evanescentes iam praebuerunt hanc aequationem bis integratam

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.}$$

in qua si ponatur $Q = 0$ et multiplicetur per *dx*, obtinebitur sequens aequatio ter integrata:

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \text{ etc.}$$

Ex quo iam apparet, si insuper fuerit $R = 0$, tum etiam quartam integrationem locum habere et ita porro.

CASUS IV

66. Si fuerit $M = 0$, ita ut sit

$$dZ = Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

Aequatio pro curva quaesita ante prodiit

$$0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.},$$

quae multiplicetur per $dy = pdx$ et tum addatur

$$dZ - Ndy - Pdp - Qdq - Rdr - Sds - \text{etc.}$$

prodibit

$$\begin{aligned} 0 = dZ - pdP + \frac{pddQ}{dx} - \frac{pd^3R}{dx^2} + \frac{pd^4S}{dx^3} - \text{etc.} \\ - Pdp \quad - Qdq \quad - Rdr \quad - Sds \quad - \text{etc.,} \end{aligned}$$

cuius integrale assignari potest ; erit enim

$$\begin{aligned} 0 = A + Z - pP + \frac{pdQ}{dx} - \frac{pddR}{dx^2} + \frac{pd^3S}{dx^3} - \text{etc.} \\ - Qq + \frac{qdR}{dx} - \frac{qddS}{dx^2} - Rr + \frac{rdS}{dx} - Ss \end{aligned}$$

vel

$$0 = A + Z - pP + \frac{pdQ - Qdp}{dx} - \frac{pddR - dpdR + Rddp}{dx^2} + \frac{pd^3S - dpddS + dSddp - Sd^3p}{dx^3} - \text{etc.,}$$

cuius termini quomodo ulterius progrediantur, si in dZ insint sequentia differentialia *Tdt, Udu* etc., sponte patet.

CASUS V

67. Si sit et $M = 0$ et $N = 0$, ita ut sit

$$dZ = Pdp + Qdq + Rdr + Sds + \text{etc.}$$

Quia est $N = 0$, una integratio per Casum primum instituatur habebiturque

$$0 = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.,}$$

quae multiplicetur per $dp = qdx$ ad eamque addatur

$$0 = -dZ + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

quo facto prodibit ista aequatio integrabilis

$$\begin{aligned} 0 = Adp - dZ + qdQ - \frac{qddR}{dx} + \frac{pd^3S}{dx^2} - \text{etc.} \\ + Qdq + Rdr \quad + Sds \quad + \text{ etc.}, \end{aligned}$$

cuius integrale erit

$$\begin{aligned} 0 = Ap - B - Z + qQ - \frac{qdR}{dx} + \frac{pddS}{dx^2} - \text{etc.} \\ + Rr \quad - \frac{rdS}{dx} + Ss \end{aligned}$$

seu

$$\begin{aligned} 0 = Ap - B - Z + Qq - \frac{qdR - Rdq}{dx} + \frac{qddS - dqdS + Sddq}{dx^2} \\ - \frac{qd^3T - dqddT + dTddq - Td^3q}{dx^3} + \text{ etc.} \end{aligned}$$

CASUS VI

68. Sit et $M = 0$ et $N = 0$ et $P = 0$, ita ut sit

$$dZ = Qdq + Rdr + Sds + Tdt + \text{etc.}$$

Ob $N = 0$ et $P = 0$ per Casum secundum duae integrationes locum habent eritque aequatio pro curva quaesita haec

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{etc.},$$

ad quam per $dq = rdx$ multiplicatam si addatur

$$0 = dZ - Qdq - Rdr - Sds - Tdt - \text{etc.},$$

habebitur ista aequatio denuo integrabilis:

$$\begin{aligned} 0 = Axdq - Bdq + dZ - rdR + \frac{rddS}{dx} - \frac{rd^3T}{dx^2} + \text{etc.} \\ - Rdr - Sds - Tdt - \text{etc.}, \end{aligned}$$

cuius integrale est

$$0 = Axq - Bq + C + Z - rR + \frac{rdS}{dx} - \frac{rddT}{dx^2} + \text{etc.}$$
$$- Ap \qquad - Ss + \frac{sdT}{dx} - Tt$$

seu

$$0 = A(xq - p) - Bq + C + Z - rR + \frac{rdS - Sdr}{dx} - \frac{rddT - drdT + Tddr}{dx^2} + \text{etc.}$$

CASUS VII

69. Si fuerit $M = 0$, $N = 0$, $P = 0$ et $Q = 0$, ita ut sit

$$dZ = Rdr + Sds + Tdt + \text{ etc.}$$

Ob $N = 0$, $P = 0$ et $Q = 0$ Casus tertius istam suppeditat aequationem pro curva iam ter integratam

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \text{ etc.},$$

ad quam per $dr = sdx$ multiplicatam addatur

$$0 = -dZ + Rdr + Sds + Tdt + \text{etc.},$$

quo facto prodibit ista aequatio

$$0 = Ax^2dr - Bxdr + Cdr - dZ + sdS - \frac{sddT}{dx} + \text{ etc.}$$
$$+ Sds + Tdt \quad + \text{etc.},$$

quae integrata dabit hanc

$$0 = Ax^2r - Bxr + Cr - D - Z + Ss - \frac{sdT}{dx} + \text{etc.}$$
$$- 2Axq + Bq \qquad + Tt$$
$$+ 2Ap$$

seu

$$0 = A(x^2r - 2xq + 2p) - B(xr - q) + Cr - D - Z + Ss - \frac{sdT - Tds}{dx} + \text{etc.}$$

SCHOLION 2

70. Horum Casuum ope, quorum numerum ulterius augere liceret, si commodum videretur, saepe-numero Problemata admodum expedite resolvi poterunt. Quodsi enim Problema quodpiam contineatur in aliquo istorum Casuum, qui unam pluresve integrationes per se admittat, statim formari poterit aequatio pro curva, semel vel aliquoties iam integrata, quae propterea ulterius facilius tractari poterit. Quod ut distinctius pateat simulque usus huius postremi Problematis, quo in maximi minimive formula $\int Zdx$ differentialia secundum gradum superantia insunt, declaretur, unicum Exemplum afferre iuvabit.

EXEMPLUM

71. *Inter omnes curvas eidem abscissae respondentes definire eam, cuius evoluta, cum sua ipsius evoluta, intra radios evolutae maximum minimumve spatium complectatur.*

Positis, pro curva quaesita, abscissa $= x$ et applicata $= y$ sit elementum curvae $= dw$ et eius radius osculi $= \rho$; erit elementum ipsius evolutae $= d\rho$ et huius radius osculi $= \frac{\rho d\rho}{dw}$, unde area comprehensa inter evolutam curvae quaesitae ipsiusque evolutam erit $= \frac{1}{2}\int \frac{\rho d\rho^2}{dw}$; quae ergo expressio maxima minimave est reddenda. Cum autem sit

$$dw = dx\sqrt{(1+pp)} \text{ et } \rho = \frac{(1+pp)^{\frac{3}{2}}}{q}.$$

erit

$$d\rho = 3(1+pp)^{\frac{1}{2}}\, pdx - \frac{(1+pp)^{\frac{3}{2}} rdx}{qq}$$

et

$$d\rho^2 = (1+pp)dx^2\left(9pp - \frac{6(1+pp)r}{qq} + \frac{(1+pp)^2 rr}{q^4}\right)$$

atque

$$\frac{\rho}{dw} = \frac{1+pp}{qdx}.$$

Maximi minimive formula itaque est

$$\int \frac{(1+pp)dx}{q}\left(9pp - \frac{6(1+pp)r}{qq} + \frac{(1+pp)^2 r^2}{q^4}\right)$$

$$\int dx\left(\frac{9pp(1+pp)^2}{q} - \frac{6(1+pp)^3 r}{q^3} + \frac{(1+pp)^4 r^2}{q^5}\right),$$

ex quo Z erit functio ipsarum p, q et r; unde differentiando prodibit:

$$dZ = \frac{18pdp(1+pp)(1+3pp)}{q} - \frac{9ppdq(1+pp)^2}{qq} - \frac{6dr(1+pp)^3}{q^3}$$
$$-\frac{36prdp(1+pp)^2}{q^3} + \frac{18rdq(1+pp)^3}{q^4} + \frac{2rdr(1+pp)^4}{q^5}$$
$$+\frac{8rrpdp(1+pp)^3}{q^5} - \frac{5r^2dq(1+pp)^4}{q^6}.$$

Comparatione ergo cum forma generali instituta erit

$$M = 0,\ N = 0,$$

$$Z = \frac{9pp(1+pp)^2}{q} - \frac{6(1+pp)^3}{q^3}r + \frac{(1+pp)^4 r^2}{q^5},$$
$$P = \frac{18p(1+pp)(1+3pp)}{q} - \frac{36pr(1+pp)^2}{q^3} + \frac{8rrp(1+pp)^3}{q^5},$$
$$Q = \frac{-9pp(1+pp)^2}{qq} + \frac{18r(1+pp)^3}{q^4} - \frac{5rr(1+pp)^4}{q^6},$$
$$R = -\frac{6(1+pp)^3}{q^3} + \frac{2r(1+pp)^4}{q^5}.$$

Cum nunc sit $M = 0$ et $N = 0$, solutio cadit in Casum quintum eritque aequatio pro curva quaesita haec:

$$0 = Ap - B - Z + Qq + Rr - \frac{qdR}{dx},,$$

quae factis substitutionibus transit in hanc

$$0 = Ap - B - \frac{18pp(1+pp)^2}{q} - \frac{16pr(1+pp)^3}{q^3} + \frac{6rr(1+pp)^4}{q^5}$$
$$-\frac{2dr(1+pp)^4}{q^4dx} + \frac{36p(1+pp)^2}{q};$$

quae aequatio nimis est complicata, quam ut eius ulteriores integrationes suscipi queant. Caeterum apparet hanc aequationem esse differentialem quarti ordinis, ita ut per quatuor residuas integrationes quatuor constantes adhuc ingrediantur; ex quo sex data oportebit esse puncta, per quae curva transeat, ut Problema determinetur.