THE METHOD

OF FINDING CURVES ENDOWED WITH THE

PROPERTY OF A MAXIMUM OR MINIMUM.

FIRST CHAPTER.

Concerning the Method of finding the maxima and minima of curved lines generally.

DEFINITION I.

1.*The Method of maxima and minima applied to curved lines being :* the method of finding curved lines, which are endowed with a certain proposed property of having a maximum or minimum.

COROLLARY 1.

2. Therefore curved lines are found by this method, in which a certain proposed quantity may obtain a maximum or minimum value.

COROLLARY 2

3. But since a curve may be put in place, itself similar to the same in an infinite number of ways, the problem shall be especially indeterminate and thus invalid, unless a certain restriction may be adhered to. For whatever curve may be proposed enjoying the proposed property of having a maximum or minimum, another may be shown always, whether indeed similar or dissimilar to that, either greater or lesser, but which within itself may retain that property.

COROLLARY 3

4. Therefore because a known equality of the curves demands that these may be referred to some given axis in place, and to whatever parts of that, which may be called the abscissas : and so in the first place that particular restriction will be required from the magnitude of the abscissa.

COROLLARY 4

5. Therefore problems relevant to this method must be proposed, thus so that the curved lines sought are related to the given axis in place, which shall have the prescribed property of a maximum or minimum among all the other curves corresponding to the same abscissas.

SCHOLIUM

6. And thus this method of maxima and minima differs from that, which we have explained elsewhere. For there, for a given and determined curved line, we have determined the place, where a certain proposed variable quantity pertaining to the curve becomes a maximum or minimum. But here the curved line itself is sought, in which a certain proposed quantity becomes a maximum or a minimum. Now this method had began to be developed by the celebrated Bernoulli brothers from the previous generation, soon after the discovery of infinitesimal analysis and from that time undertook a great advancement. In the first place a certain problem, which of this kind has been treated, was with regard to mechanics, and for that a curved line will be sought, upon which a weight released descending may fall the quickest ; on which was imposed the name *brachystochrone curves*, or *lines of the quickest descent*. In this problem it is now clear that the title of the question cannot be retained without an added condition : for it is clear, where the line may be taken shorter and more to the vertical situation, there the descent time to be shorter than that above. On which account a line cannot be sought absolutely, upon which a descending weight may fall down the quickest or in the shortest time ; for the magnitude of the abscissa, to which the curve found may correspond, must be defined likewise ; thus so that, among all the curves for the same corresponding abscissas selected on the given axis in position, that may be sought upon which the heavy body may fall the quickest. Nor indeed in this problem will this condition be sufficient towards effecting that determination: but in addition to that condition it is required to add this condition, that the curve requiring to be found shall pass through two given points ; and this same problem must be bound by these conditions, so that it may the determination may be done among all the curved lines, evidently passing through the two points, to determine that curve, upon which the descending body may resolve a given arc of the given abscissas in the corresponding shortest time. Yet meanwhile this condition is to be noted : the transition through the two points is not absolutely necessary, but in this problem is to be brought into this solution. For in the solution of this problem a differential equation of the second order is come upon immediately, which integrated twice will receive two arbitrary constants, for determining which there is a need for two points, through which the curve may be drawn, or from other similar properties : and this same condition, at once accedes as if by itself to the solution of all problems of this kind, the solution of which is deduced at once for the equation of the second order differential. But in problems, which are resolved by a differential equation of the fourth or of higher orders, indeed two points are not sufficient for determining the curve, for there is a need for just as many points, as the prevailing differential steps. Truly conversely, if the solution at once leads to an algebraic equation, then the problem will be determined perfectly without a condition of this kind ; provided the length of the abscissa may be defined. Truly all this will become more clear, when we arrive at the solutions to the problems below: and there we will explain these notations further. For here in the beginning it has been considered to mention only so many matters, so that we may remove perverse ideas about the determination of problems of this kind.

DEFINITION II

7. *The absolute method of maxima and minima* instructs how, between all the curves related to the same abscissa, to determine that, in which some proposed variable quantity may obtain a maximum or minimum value.

COROLLARY

8. Therefore in problems pertaining to this method with the position of the axis given ; and between all the curves which can refer to this axis, to determine a portion of that curve on which some variable quantity becomes a maximum or a minimum.

SCHOLIUM

9. Here in general we do not add a further condition towards the determination of the maximum or minimum, besides the magnitude of the abscissa. For problems are given, which are determined perfectly in this way; just as it will become more apparent below. And even if also problems of this kind occur, for which two or more points in addition are prescribed for the determination, through which the prescribed curve may pass, yet this finally may be noted from the solution of any problem itself. In as much as, if for an equation of this kind to be come upon for the curve sought, in which new constant quantities shall be introduced by integration, which were not present in the question itself, then the solution will be considered to be ambiguous and vague ; so that there will be innumerable curved lines, which can arise from the determination of these other constant and arbitrary quantities, included within it. Therefore in these cases it will be concluded that the problem from its nature not is to be completely determined, but for its full determination, besides the magnitude of the abscissas, it is necessary to add just as many new conditions, from which these arbitrary constants may be revoked by determined values. But points are assumed most conveniently for conditions of this kind, through which the curve sought shall pass ; truly just as many points, as there shall be arbitrary constants found, shall be present in the equation, and that will return the same determined equation. But in place of points, towards the curve sought being determined perfectly, also just as many tangents can be used, which touch the curve sought and, if the contact must be made at a given point of the tangent, this condition will be equivalent to two points. But indeed also in place of points some other conditions can be substituted, as long as these thus shall be in agreement, so that through these arbitrary quantities held in the equation found may be determined. Nor indeed is there a need before the solution is found at a bounary, as that may be undertaken by a judgment ; but below certain criteria will be examined, with the aid of which that variable quantity, which must be a maximum or minimum, will be able to be distinguished, in which new constants may be entering into the equation for the curve, which are not contained in the question. But these arbitrary constants arise from the order of the differential, for whatever order the differential equation produced for the curve sought, just as many arbitrary quantities being agreed upon, with the possibility of being present; and hence there will be a need for just as many conditions for the curve being determined. Likewise truly it comes into use also in the solution of all problems, when a differential equation either of first or of

higher order is found ; thus so that hence in the present circumstances no singular difficulty shall be considered to be present.

DEFINITION III

10. *The relative method of maxima or minima* does not set out instructions entirely between all the curves corresponding to the same abscissa, but only between these, which may have a certain common prescribed property, to determine that which shall be endowed with the property of being a maximum or minimum.

COROLLARY 1

11. Therefore towards solving problems of this kind, in the first place with every curve in general for curves with the same corresponding abscissas, these are to be removed in which the same prescribed property agree ; and then at last from these removed, that which is sought must be defined.

COROLLARY 2

12. But although the number of all the curves related to the same abscissas is strongly restricted by such a condition, yet this number will remain infinite even now. For indeed also, if not one, but several properties are prescribed with which all the curves must be provided, out of which the one sought is to be determined, to the extent still that the number of curves will remain infinite.

COROLLARY 3

13. And thus so that several properties are proposed, which for these curves, from which the curve sought is required to be defined, must be of the same form, there a greater number of curves, among which the chosen sought is to be put in place, even if it may be infinite.

SCHOLIOM 1

14. From this kind, in which we have established the Method of relative maxima or minima, initially at the beginning of this century, comes that famous *Isoperimetric Problem* advanced into the public eye by James Bernoulli*;* in which a curve with a given property of maximum or minimum was sought, not between all curves related to the same abscissa, but only amongst these which were of the same length ; from which these curves are called *isoperimetric*, and in turn a particular one sought was required to be elicited. Thus, if among all the curves corresponding to the same abscissas and equal in length, that sought which will enclose the maximum space [*i.e.* area] between the abscissa and the applied line, is found to be satisfied by a circular line ; which indeed had became known and demonstrated by the Geometers a long time before this method. But again in this case, according to nature of the problems, new conditions are to be added ; as in these which pertain to the Method of absolute maxima and minima, which with

arbitrary constants, which the solution induces, are required to be evaluated. Thus, in the solution of the problem in which a curve is sought, which amongst all the curves of the same length shall take the maximum area with the abscissa, two new constants are introduced ; from which, to bring about the solution of the problem, it thus being proposed, that amongst all the curves of the same length, which not only correspond to the same abscissas, but also may pass through the two given points, these are sought, which may return the greatest area for the given abscissas. And in a similar manner it can come about, that four points, or even more sometimes, must be assumed arbitrarily, so that the problem may become determined : the selection of these depends on the nature of the problem itself. But just as all curves of the same length are put in place in the isoperimetric problem , from which it is necessary to determine the curve sought ; thus, in place of this property, some other property can be proposed, which must be in common for all. Thus now maxima or minima curves are sought with a given property, amongst all these curves still related to the same abscissas, which conversely all generate equal surfaces about the same abscissa ; and in a similar manner all can have some proposed property. Then also not one but several properties of this kind can be prescribed, which all the curves must have in common ; among which these shall be required to be defined, which may contain some maximum or minimum. Thus if a curve shall be sought with the property of a maximum or minimum corresponding to some pre-given property among all the curves of the same abscissas, which still shall all be of equal lengths, so also equal areas shall included.

SCHOLIUM 2

15. On account of this distinction between the absolute method of maxima and minima and the relative kinds of the same, our treatment shall be of two kinds. Clearly in the first place we treat the method among all curves generally corresponding to the same abscissas requiring to determine that, which shall be provided with the property of the maximum or minimum. Then truly we will progress to problems of this kind, in which a curve endowed with the maximum or minimum property is postulated among all the curves, which may have one or more common properties proposed ; and from the number of these properties treated a new subdivision may arise. Yet meanwhile there will be no need to progress further in this subdivision, since soon a method may be found, besides however many properties were proposed, the problems are able to be resolved easily. For the solutions of the problems at first face with the greatest intricacy beyond belief become extremely unencumbered and resolved by a light calculation.

HYPOTHESIS I

16. *In this treatment, to which we refer all the curves, we will designate the abscissa by the letter x always, and truly the applied line by the letter y. Then truly, with equal elements of the abscissas taken, there will be always*
$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx$ etc.

COROLLARY 1

17. Therefore with these substitutions all the differentials of *y* of whatever order will be removed and besides *dx* no other differentials will remain. But nevertheless in this way all the differentials besides the kind *dx* alone, are not actually removed, yet these remarkable substitutions help us in the present circumstances.

COROLLARY 2

18. So that also by an assumption of this kind, with the substitutions of the differential agreed upon, it is removed from the calculation completely : for whatever other kind of agreement may be assumed for the differential, after these substitutions the same formula must emerge always. Yet meanwhile, on account of the method used below, it is necessary to take *dx* for the differential as constant.

COROLLARY 3

19. But so that it may appear easier, how the differentials of each order of *y* may vanish by these substitutions, it will help to have added the following table:

$$dy = pdx$$
$$ddy = dpdx = qdx^2$$
$$d^3y = dqdx^2 = rdx^3$$
$$d^4y = drdx^3 = sdx^4$$
$$d^5y = dsdx^4 = tdx^5$$
etc. etc. etc.

COROLLARY 4

20. But if also the arc of a curve corresponding to the abscissa *x* with its differentials of any order appear, all these will be expressed by these letters thus, so that no other differentials besides *dx* shall be present. For with the arc put $= w$
there will be :

$$w = \int \sqrt{(dx^2 + dy^2)} = \int dx\sqrt{(1+pp)}$$
$$dw = dx\sqrt{(1+pp)}$$
$$ddw = \frac{pqdx^2}{\sqrt{(1+pp)}}$$
$$d^3w = \frac{prdx^2}{\sqrt{(1+pp)}} + \frac{qqdx^3}{(1+pp)^{\frac{3}{2}}}$$
etc.

COROLLARY 5

21. In a similar manner from these the radius of osculation or curvature at some place can be expressed at least in a form through finite quantities. Since indeed, on putting the element *dx* constant, the length of the radius of osculation
$=\frac{-dw^3}{dxddy}$, that becomes $=\frac{(1+pp)^{\frac{3}{2}}}{q}$.

COROLLARY 6

22. Again, from the same substitution, there will be as follows :

$$\text{Subtangent} = \frac{ydx}{dy} = \frac{y}{p},$$
$$\text{Subnormal} = \frac{ydy}{dx} = py,$$
$$\text{Tangent} = \frac{ydw}{dy} = \frac{y\sqrt{(1+pp)}}{p},$$
$$\text{Normal} = \frac{ydw}{dx} = y\sqrt{(1+pp)}.$$

And, in an equal manner, all the quantities pertaining to finite curves, unless they involve integrations, can thus be expressed by finite quantities of this kind, so that no further differentiation may be seen to be present.

DEFINITION IV

23. *The Maximum or minimum Formula*, for any problem, will be that quantity for us which in the curve sought must obtain a maximum or minimum value.

COROLLARY1

24. Because in all problems, to which this method is applied, the curve is sought which may be endowed with the property of a maximum or minimum, either among all or only among certain innumerable curves determined in a certain way, and this property itself, which must be the maximum or minimum in the curve sought, will be a quantity there expressed by a formula, that we may call here the maximum or minimum formula.

COROLLARY 2

25. But since the property of the maximum or minimum must be proposed, so that it may be referred to a given and known abscissa, the formula of the maximum or minimum also must be referred to that definite abscissa.

COROLLARY 3

26. Therefore the formula of the maximum or minimum will be some variable quantity depending on the length of the abscissa, to which it corresponds. And in some problem the curve is sought, for which, to define the abscissa, that formula of the maximum or minimum must obtain a maximum or minimum value.

COROLLARY 4

27. Nor indeed can the maximum or minimum formula depend only on the abscissa : for if this should be, for all curves with the same corresponding abscissas the formula would obtain the same value, and on that account all would be satisfied equally.

COROLLARY 5

28. On account of this also the maximum or minimum formula besides the abscissa for all curves, which come into consideration, in common, on which a particular curve may depend ; thus so that there shall be one, for which the maximum or minimum value may be assumed.

SCHOLIUM 1

29. So that everything may be understood more clearly and the condition of the questions sought in the following treatment may be grasped better, we may put in place that curve, either among all the curves entirely, or only among innumerable curves having a certain common property, and which may correspond to the same abscissa *AZ*, for which a maximum or a minimum value of the formula *W* may be determined. For this question to be satisfied we may consider the curve (Fig. 1) *amz*, thus so that, whatever other may be referred to the same abscissa *AZ*, the value of the formula *W* either becomes smaller or larger, as in the curve by satisfying *W* must become either a maximum or a minimum Therefore in this question clearly in the first place we have the extend of the abscissa determined by the length of *AZ* ; then a curve is sought, either between all the curves generally related to this same abscissa, or only between innumerable curves, from which one or more properties shall be in common, exactly as the question is applied to the method of maximum or minimum, either relative or absolute ; in the third place we have that quantity *W*, the value of which on the curve sought *amz* must be either a maximum or a minimum ; and therefore the quantity *W* will be the maximum or minimum formula, just as that has been defined. Now therefore it is apparent at once this formula *W* thus must be prepared, so that it may be applied to all the curves which indeed can be considered. In the first place it is clear that it must depend on the magnitude *AZ* of the abscissa defined, thus so as that may change with the value of *AZ* changed. Then also it must depend in a singular way on the nature of any curve,

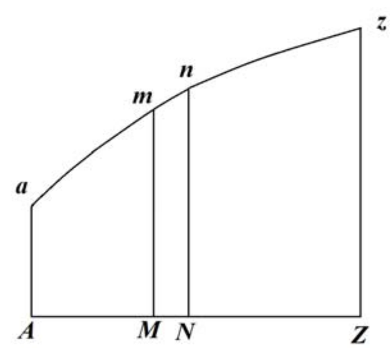

Fig. 1

which indeed can be considered ; for unless it may be prepared thus, for all the curves the same value might be chosen and the question would become void. On account of which the quantity *W*, in addition to the abscissa, must also contain within itself quantities relating to the curve itself. Therefore since each curve may be determined by a relation between the abscissa and the applied line, the quantity *W* must be constructed from the abscissas and the applied lines and from independent quantities. That is, if an indefinite abscissa were put $= x$ and the corresponding indefinite applied line $= y$, the quantity *W* must be a function of the two variables *x* and *y*. Which thus since there shall be, if some determined curve may be taken and from its nature the relation between *y* and *x* may be substituted into the formula *W*, that defined will produce a value for the given curve and relating to the defined abscissa of that. Because now, for with more and yet more curves, the formula *W* adopts different values, even if in all the abscissa may be taken the same, it is clear between these innumerable curves there must be one, in which the value of the formula *W* becomes a maximum or a minimum ; and for this curve to be discovered for any given determined question, the method to be expounded has been prepared.

COROLLARY 6

30. Therefore the maximum or minimum formula *W* will be a certain function of the two variables *x* and *y;* of which the one *x* will denote the abscissa, and the other *y* the applied line. Therefore in *W* not only will the variables *x* and *y* will be present, but also all the quantities depending on these, such as *p, q, r, s* etc., the significances of which we have examined above. Indeed integral formulas arising from these in some manner can be present in *W*, indeed also they must, if indeed a question must be determined, as we will show soon.

COROLLARY 7

31. Therefore with a proposed formula *W* of this kind or with a function of *x* and *y*, if the question may relate to the absolute method of maximum or minimum, an equation of this kind between *x* and *y* may be desired, so that, if the value of *y* determined by *x* may be substituted into *W* and a determined value of *x* itself may be granted, a greater or lesser quantity may be produced for *W*, than if any other equation between *x* and *y* were assumed.

COROLLARY 8

32. Therefore with this agreed on, questions relating to the teaching of curved lines according to pure analysis may be called upon. And in turn, if a question of this kind of pure analysis may be proposed, that can be referred to and resolved according to the theory of curved lines.

SCHOLIUM 2

33. Although questions of this kind can be reduced to pure analysis, yet it is expedient to join these with the theory of curved lines. For if our thinking may be led away from curved lines and we wish only to prove absolute quantities, at first the questions become very abstruse and clumsy and the uses and worthiness of these may become less relevant. Then also the method of resolving questions of this kind, if it may be proposed only in abstract quantities, may become excessively abstruse and troublesome ; since yet the same, by inspection of the figures and of the quantities representing the lines, may be aided wonderfully and resolved easily by us. For this reason, even if questions of this kind can be applied both to abstract as well as concrete quantities, we can still treat and resolve these most conveniently according to curved lines. Clearly, as often as an equation of this kind is sought between x and y, so that a certain formula can be proposed and composed from x and y, if from that equation sought the value of y is proposed and the determined value of x granted, a maximum or a minimum arises, then we may transfer the question always towards finding a curved line, the abscissa of which shall be x and the applied line y, for which that formula W makes a maximum or a minimum, if the abscissa x may be taken of a given magnitude. Therefore with these noted, the nature of questions of this kind is seen clearly enough, unless perhaps some doubt still may create an ambiguity on talking about the maximum or minimum at the same time. Truly here indeed no ambiguity is present; for the method itself demonstrates equally maximum and minimum, yet it will be discerned easily in any case, whether the solution provides a maximum or minimum But occasionally it arises, that in a given question both a maximum as well as a minimum may arise, and in these cases the solution will be two-fold, with the one showing a maximum and the other a minimum But generally one or the other, either the maximum or the minimum, is usually impossible ; which happens, if the formula of the maxima or minima can increase or decrease indefinitely ; for in these cases either no maximum or no minimum will be given. Also it can happen in use, that the proposed formula W may be able to increase or decrease infinitely, and in these cases evidently no solution can be put in place. But all these distinctions will be shown after the calculation itself always.

PROPOSITION I. THEOREM

34. *So that the curve amz may be determined by the formula W of the maximum or minimum, which it may satisfy before all the rest, the formula W must be the magnitude of an indefinite integral, which unless a relation may be taken between x and y, may be unable to be integrated.*

DEMONSTRATION

For we may consider the formula W not to involve an indefinite integral ; that it will be a function of the quantities x and y and thus of the depending p, q, r, s etc., either algebraic or with such transcending, which it will be possible to show without an assumed relation between x and y ; which arises, if either the logarithms of these quantities, the circular arcs, or other definite transcending quantities of this kind may be introduced, which are considered equivalent to algebraic quantities. So that if now a function W may be considered of such of x and y only, it is evident the value of the formula W, which may be obtained for the given curve *amz* recalled to the given abscissa *AZ*, depends only on the final applied line *Zz* and for all the curves in Z we will have the applied line *Zz* to be the same ; and thus for such a formula W the nature of the whole curve will not be determined, but only the position of its extreme point z; if in W besides x and y the quantity p shall be present also, then besides the length of the applied line *Zz* the position of the tangent to the curve at z or the position of the final element at z will be determined. But if in addition q may be introduce, then the position of two contiguous elements of the curve at z will be determined, and thus henceforth. From which it follows, if W were a determined function of x, y, p, q, r etc. themselves, then only at that infinitely small part of the curve about the extremity z shall it be determined, and for all the curves ceasing at the same extremity to be producing the same value of W. So that therefore the whole curve *amz* may be defined by the formula W, to the extent that it may correspond to the whole of the abscissa *AZ*, thus the formula W is required to be prepared, so that its value for some determined curve *amz* may depend on the applied line with the position of the individual elements of this curve within the limits a and z. But this cannot arise, unless the quantity W shall be the formula of an indefinite integral, which integration generally cannot be admitted without an equation between x and y. Q. E. D.

COROLLARY 1

35. Therefore unless the maximum or minimum formula W shall be the magnitude of an indefinite integral, indeed the curved line, on which the value of W shall be a maximum or minimum, shall not be determined; and thus the question of finding the curve, on which W shall be a maximum or minimum, shall be void.

COROLLARY 2

36. Therefore so that the curve shall be able to be assigned, in which the value of W before all others shall be a maximum or minimum, the formula W must have such a form $\int Zdx$, and it is required that the quantity Z be prepared thus, so that the differential Zdx cannot be integrated, unless an equation be put in place between x and y.

SCHOLIUM

37. Because the maximum or minimum formula W must be an integral of the formula of an indefinite differential of the first order, that is its integral makes a finite quantity, that formula of the differential will be reduced always to a form of this kind Zdx with the aid of the letters p, q, r etc. And on account of this, in the following the maximum or minimum formula always will be indicated to us by $\int Zdx$. But Z will be a function not only of the quantities x and y, but also will contain the letters p, q, r etc. Thus if the area *AazZ* must be a maximum or min, the formula W will be changed into $\int ydx$; and, if the surface shall be of a round solid, which may be generated by the rotation of the curve *amz*

about the axis *AZ*, must be a maximum or a minimum, then W will be $=\int ydx\sqrt{(1+pp)}$; and thus again, whatever the formula must become for a maximum or a minimum in the curve sought, that will be of this form $\int Zdx$ always, clearly for the integral of a certain finite quantity *Z* taken by the differential *dx*. But *Z* must be a quantity of this kind, so that, for finding the curved lines if an equation may be put in place between *x* and *y*, the integral $\int Zdx$ may possess a determined value ; from which *Z* will be a function of the quantities *x*, *y* and thence of the depending *p*, *q*, *r* etc. either algebraic or determinate, or besides that by including indeterminate integral formulas ; which distinction is required to be understood properly. Thus if the maximum or minimum formula *W* were $\int ydx$ or $\int ydx\sqrt{(1+pp)}$, the quantity *Z* will be algebraic, but if it shall be $W=\int yxdx\int ydx$, then there will be $Z=yx\int ydx$, that is the quantity *Z* itself will be indeterminate, of which the value cannot be shown, unless a relation between *x* and *y* may be given. Also indeed it can eventuate, that a value of *Z* of this kind may be unable to be expressed by the formula established, but only must be expressed by a differential equation to be elicited finally, so that if there were $dZ = ydx + ZZdx$; from that equation the value of *Z* indeed cannot be shown by *x* and *y*. Hence therefore three kind of formulas $\int Zdx$ may arise,which must become the maximum or minimum of curves sought. The first of which includes these formulas, in which *Z* is an algebraic function, or from the determination of *x*, *y* and *p*, *q*, *r* etc. We will refer these formulas to the second kind, in which the magnitude *Z* will involve in addition integral formulas. But in the third kind those formulas may be encountered, in which the value of *Z* may be determined by a differential equation, of which the integration cannot be established.

PROPOSITION II. THEOREM

38. *If amz were a curve, in which the value of the formula* $\int Zdx$ *shall be a maximum or a minimum, and Z shall be algebraic or determined by x, y, p, q, r* etc., *then some portion of the same curve mn will be endowed with the same choice, so that for that part related to its own abscissa MN the value of* $\int Zdx$ *shall be equally a maximum or a minimum.*

DEMONSTRATION

The value of the formula $\int Zdx$ for the abscissa *AZ* is the sum of all the values of the same formula, which correspond to the individual parts of the abscissa *AZ*. But if therefore the abscissa *AZ* may be considered to be divided into some number of parts, one of which shall be *MN*, and for these individual parts this value of the formula $\int Zdx$ may be shown, the sum of all these values will provide the value of the formula $\int Zdx$, which the whole abscissa *AZ* agrees on, and which will be the maximum or minimum But because Z is put as an algebraic function of x, y, p, q etc., the value of the corresponding formula $\int Zdx$ for the part of the abscissa MN from the corresponding part mn of the curve alone will depend on the same kind and remain the same, in whatever way the remaining parts am and nz may be varied ; for the values of the individual letters x, y, p, q etc. are determined by the values of the part mn of the curve alone. Therefore if the values of the formula $\int Zdx$ are put P, Q and R,, which agree with the parts of the abscissas $AM, MN, NZ,$ these quantities P, Q and R do not depend on each other mutually. Whereby since the sum of these $P+Q+R$ shall be a maximum or a minimum, also each and every part by necessity shall be endowed with a maximum or a minimum On this account, if in the curve amz the formula $\int Zdx$ may have a maximum or minimum value and the quantity Z shall be an algebraic function of x, y, p, q etc., then also for any part of this curve, the formula $\int Zdx$ will be endowed with the same property of maximum or minimum Q. E. D.

COROLLARY 1

39. Therefore if a curve *amz* were found, which for the given abscissa *AZ* may have a maximum or minimum value of the formula $\int Zdx$, and Z shall be an algebraic or determined function, then also any part of the same curve, with respect to its corresponding abscissas, will be endowed with the same property of a maximum or minimum

COROLLARY 2

40. Therefore in problems of this kind, where it is found with such a maximum or minimum, there is no need for the magnitude of the abscissa to be defined, to which the maximum or minimum may correspond ; but if for some one abscissa the formula $\int Zdx$ shall be a maximum or a minimum, then likewise it will be endowed with the same property for any other abscissa.

COROLLARY 3

41. Therefore problems of this kind will be resolved, if particular individual curves thus shall be determined, so that for these the value of the formula $\int Zdx$ becomes a maximum or minimum. For then at the same time the whole curve and equally some part of it will be prepared with the same maximum or minimum property.

SCHOLIUM

42. This property, with which the curves are endowed, in which the formula of its kind $\int Zdx$, where Z is an algebraic function or determined by x, y, p, q etc., are maximum or

minimum, is of the greatest importance ; for this depends on a general method for problems of this kind to be resolved. So thus this proposition chiefly brings to view, not only that property which is appropriate for these formulas $\int Zdx$, where *Z* is either an algebraic or determined function, but it may be considered to be a general property of all common formulas, which can be proposed ; indeed in the following proposition we will demonstrate, if integral formulas shall be present in *Z*, then the same property must belong to the whole curve ; from which at once the nature of questions of this kind will be understood more clearly. But the demonstration of the present proposition from that sought is fundamental, so that the value of the formula $\int Zdx$, if indeed *Z* is a function either algebraic, or determined from *x, y, p, q, r* etc. themselves; which agrees with any part *MN* of the abscissas, from which a single part of *mn* may depend, and may not depend on the remainder of the curve, either in the former part *am* or in the latter part *nz*; which reasoning is remiss, if indeterminate integral formulas shall be present in *Z*. For the values of the magnitudes *x, y, p, q, r* etc., which prevail for the arc of the curve *mn*, depend only on the position of the elements of this arc *mn* and some neighbouring elements, which do not constitute the arc of a finite magnitude ; from which also the magnitude will be determined by the nature of the arc *mn* alone from these letters [*i.e.* variables] put together in some way, unless integral magnitudes were present, of this kind are $\int ydx$, which may introduce the whole prior area *AamM*, or $\int dx\sqrt{(1+pp)}$, which may involve the whole preceding arc *am*. Hence therefore it is understood more distinctly, what we may wish to denote by the determined function of *x, y, p, q, r* etc. : Evidently a determined function is prepared thus, so that for whatever in place may depend only on the present values of the letters *x, y, p, q* etc. and does not itself include the previous values of these. Moreover an indeterminate function is of such a kind, the value of which can be determined at some place not only from the values, which these letters *x, y, p, q* etc. maintain at the same place, but in addition all the values required for its determination, which letters in all anterior places will maintain. Thus it is apparent that all algebraic functions likewise are to be determinate ; truly thus also all transcending functions, which may not depend on a relation between *x* and *y*, are determined, $l\sqrt{(xx+yy)},\ e^{py}, \text{A}\sin\frac{py}{q}$ are of this kind, the values of which can be assigned at some place from the values of the letters, which they maintain at this place only. But when indeterminate integral formulas are present in certain formulas, which depend on a mutual relation between *x* and *y*, which holds everywhere, then the value of these at a given place does not depend on the place from the values, which these letters have in that place, then to be known, but also on all the previous values that it is necessary to know, this is a general relation between the coordinates *x* and *y;* and we call such functions indeterminate, certainly which are completely different from these, which we have called determinate.

PROPOSITION III. THEOREM

43. *If amz were a curve corresponding to the abscissa AZ, in which* $\int Zdx$ *shall be a maximum or a minimum, but in Z there may be contained indefinite integral formulas, then the same property of maximum or minimum does not lie in any part of the curve, but only the whole of the abscissa of the curve AZ will be appropriate.*

DEMONSTRATION

The whole curve *amz,* for which $\int Zdx$ is a maximum or a minimum, may be considered to be divided into two parts by the applied line *Mm* and the value of the agreeing formula $\int Zdx$ for the part *am* shall be $=P$, but the value of the same formula for the other part *mz* shall be $=Q$; therefore for the whole curve *amz* the value of the formula $\int Zdx$ will be $P+Q$, which we may put to be a maximum or a minimum But so that we may remove all ambiguity and we may be able to make the whole matter becomes more distinct, we may put $P+Q$ to be a maximum; for what will be shown for a maximum, likewise may be understood for a minimum. But if now the value of *Q* may not depend on the value of *P*, then the sum $P+Q$ shall not be a maximum, unless likewise each value of *P* and *Q* separately shall be a maximum. But in our case, in which the magnitude *Z* maintains in itself indeterminate integral formulas, the value of *Q* not only will depend on the part of the curve *mz,* to which it is referred, but likewise to the whole preceding curve *am* and thus to the value of *P* itself. Now we say for that, that $P+Q$ shall not be required to be a maximum, as the value of *P* shall be a maximum. For we may consider the part of the curve *am* to be prepared thus, so that for that *P* shall be a maximum, and the part of the curve *am* may be considered to be changed a very small amount, thus so that the value of the formula $\int Zdx$ may emerge a little less, for example, $=P-p$; certainly it will come about, so that from this change the value of *Q* may increase, which increment is put to be *q,* and with the part *am* changed a very small amount, thus so that for that $\int Zdx$ shall be a maximum no further, the value of the formula $\int Zdx$ for the whole curve *amz* $=P-p+Q+q$. Therefore since it may be able to come about, so that there shall be $q>p$, it is understood the formula $\int Zdx$ is able to be the maximum for the whole curve *amz*, even if it shall not be the maximum for some part *am.* Q. E. D.

COROLLARY 1

44. Therefore when a curve were found, which, for the given abscissa *AZ,* the value of the formula $\int Zdx$ may have a maximum or minimum, and *Z* shall be an indeterminate function, then it does not follow that any part of the curve shall have the provided maximum or minimum property.

COROLLARY 2

45. Therefore in the resolution of problems of this kind, in which a curve is sought, which for the given abscissa AZ, $\int Zdx$ may have a maximum or a minimum, always the magnitude will be with respect of the whole abscissa proposed, and the maximum or minimum will be able to be applied to that only, not truly to any part of that.

COROLLARY 3

46. Hence therefore a maximum distinction is apparent, which intervenes between the formulas $\int Zdx$, in which Z is a determinate or an indeterminate function ; and likewise moreover it is understood for which a diversity of methods will be required to be used for the resolution of questions, in which maximum or minimum values are required of formulas of this kind.

SCHOLIUM

47. From the demonstration of this proposition it does not indeed follow by necessity, if for a given abscissa AZ the curve $\int Zdx$ may have a maximum or minimum formula, then its individual parts are endowed with this same primary choice; but yet it is understood well enough, whenever the same property agrees in the individual parts, in that case it happens. And hence nevertheless by the greatest necessity it is always a solution to accommodate the whole proposed abscissa. Yet meanwhile in problems relating to the relative method it can happen, that the formulas $\int Zdx$, in which Z shall be an indeterminate function, may be allowed to be treated as if determinate. Evidently this occurs, only if among all curves that may be desired in which $\int Zdx$ shall be a maximum or a minimum, in which these indeterminate integral formulas which are present in Z maintain equal values,; but in this case these indeterminate integral formulas are considered to become determinate. Thus, if among all the curves of the same length that shall be determined, in which $\int Zdx$ shall be a maximum or a minimum, and in Z, besides the magnitude to be determined, an arc of the curve shall be present $\int dx\sqrt{(1+pp)}$, here, because the same value may be maintained, in all the curves from which the function sought is required to be defined, an equivalent form of the function to be determined may be treated. But this will be dealt with everything fully in following.

HYPOTHESIS II

48. *If the abscissa of the curve* (Fig. 2) *AZ may be cut into innumerable infinitely small elements equal between themselves, IK, KL, LM etc. are of this kind, and some part AM*

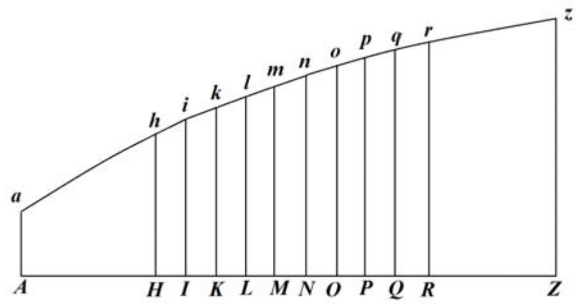

Fig. 2.

may be called x, to which some variable function F may correspond, the same function F, just as it may refer to the points of the abscissa either following N, O, P, Q etc. or before L, K, I etc. we will denote thus, so that the value of this same function, which for the point M is $= F$, *as follows :*

$$\left.\begin{array}{l} for\ N = F^{I} \\ for\ O = F^{II} \\ for\ P = F^{III} \\ for\ Q = F^{IV} \\ for\ R = F^{V} \\ \quad \text{etc.} \end{array}\right\} for\ the\ following\ points\ of\ the\ abscissa,$$

$$\left.\begin{array}{l} for\ L = F_{I} \\ for\ K = F_{II} \\ for\ I = F_{III} \\ for\ H = F_{IV} \\ \quad \text{etc.} \end{array}\right\} for\ the\ preceding\ points\ of\ the\ abscissa.$$

And with this agreed on, without writing many differentials, the value of any variable function, which is held at any point of the abscissa, will be indicated conveniently.

COROLLARY 1

49. Therefore since the value of the function in any place shall be equal to its value in the preceding place increased by its differential, there will be

$$\begin{array}{l||l} F' = F + dF & F = F_{/} + dF_{/}, \\ F'' = F' + dF' & F_{/} = F_{//} + dF_{//} \\ F''' = F'' + dF'' & F_{//} = F_{///} + dF_{///} \\ F^{IV} = F''' + dF''' & F_{///} = F_{IV} + dF_{IV} \\ \text{etc.} & \text{etc} . \end{array}$$

COROLLARY 2

50. If from the individual divisions of the abscissa the applied lines may be drawn and that, which corresponds to the abscissa $AM = x$, clearly *Mm*, is put $= y$, the remaining following as well as preceding thus will be denoted :

$$\begin{array}{l||l} Mm = y & Mm = y \\ Nn = y' & Ll = y_{/} \\ Oo = y'' & Kk = y_{//} \\ Pp = y''' & Ii = y_{///} \\ Qq = y^{IV} & Hh = Y_{IV} \\ \text{etc.} & \text{etc} \end{array}$$

COROLLARY 3

51. Since then the value of *p* shall be $= \frac{dy}{dx} = \frac{Nn - Mm}{dx}$, p will be $= \frac{y' - y}{dx}$; moreover the following and equally the preceding thus will have themselves have the values of *p*:

$p=\frac{y^{\prime}-y}{dx}$	$p=\frac{y^{\prime}-y}{dx}$
$p^{\prime}=\frac{y^{\prime\prime}-y^{\prime}}{dx}$	$p_{\prime}=\frac{y-y_{\prime}}{dx}$
$p^{\prime\prime}=\frac{y^{\prime\prime\prime}-y^{\prime\prime}}{dx}$	$p_{\prime\prime}=\frac{y_{\prime}-y_{\prime\prime}}{dx}$
$p^{\prime\prime\prime}=\frac{y^{IV}-y^{\prime\prime\prime}}{dx}$	$p_{\prime\prime\prime}=\frac{y_{\prime\prime}-y_{\prime\prime\prime}}{dx}$
etc.	etc.

COROLLARY 4

52. Then, because $q=\frac{dp}{dx}=\frac{p^{\prime}-p}{dx}$, there will be $q=\frac{y^{\prime\prime}-2y^{\prime}+y}{dx^2}$; from which the values of the quantity q, both following as well as before, thus will be had:

$q=\frac{y^{\prime\prime}-2y^{\prime}+y}{dx^2}$	$q=\frac{y^{\prime\prime}-2y^{\prime}+y}{dx^2}$
$q^{\prime}=\frac{y^{\prime\prime\prime}-2y^{\prime\prime}+y^{\prime}}{dx^2}$	$q_{\prime}=\frac{y^{\prime}-2y+y_{\prime}}{dx^2}$
$q^{\prime\prime}=\frac{y^{IV}-2y^{\prime\prime\prime}+y^{\prime\prime}}{dx^2}$	$q_{\prime\prime}=\frac{y-2y_{\prime}+y_{\prime\prime}}{dx^2}$
etc.	etc.

COROLLARY 5

53. Therefore in a similar manner by those signs of the applied lines the values of the quantities *r, s, t* etc. will be able to be determined, as we have taken above, and defined from the figure. Evidently there will be :

$$r=\frac{y^{\prime\prime\prime}-3y^{\prime\prime}+3y^{\prime}-y}{dx^3}$$
$$s=\frac{y^{IV}-4y^{\prime\prime\prime}+6y^{\prime\prime}-4y^{\prime}+y}{dx^4}$$
$$t=\frac{y^{V}-5y^{IV}+10y^{\prime\prime\prime}-10y^{\prime\prime}+5y^{\prime}-y}{dx^5}$$
etc.,

from which the values of these letters both preceding as well as following can be formed.

COROLLARY 6

54. But if moreover the $\int Zdx$ were related to the abscissa $AM = x$, the value of the following abscissa corresponding to the element $MN = dx$ will be $= Zdx$. And hence in a similar manner the values of the formula $\int Zdx$ corresponding to the elements will be denoted as follows :

for $MN = Zdx$	for $MN = Zdx$
for $NO = Z^{I}dx$	for $LM = Z_{I}dx$
for $OP = Z^{II}dx$	for $KL = Z_{II}dx$
for $PQ = Z^{III}dx$	for $IK = Z_{III}dx$
etc.	etc.

COROLLARY 7

55. Therefore if for the expression $\int Zdx$ pertaining to the abscissa of the curve $AM = x$, the value of the same expression, which will be appropriate for the proposed abscissa *AZ*, will be

$$= \int Zdx + Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.}$$

$$[i.e. \ = \int Zdx + \left[Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.} \right]]$$

indefinitely, until the final point *Z* may be arrived at.

COROLLARY 8

56. Therefore if the curve must be found, which for a given value of the abscissa *AZ* the value of the formula $\int Zdx$ may have a maximum or minimum, then with the position of any indefinite abscissa *A1Jf x* to be effected, so that this expression $\int Zdx + Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.}$ as far as *Z* becomes a maximum or minimum

SCHOLIUM

57. Though this hypothesis had been made only by choice, yet these signs [*i.e.* this new notation] afford the maximum usefulness to problems which relate to this method of maxima and minima, by resolving these quickly. For a suitable choice of notation prevails greatly in problems of this kind and with its help a calculation can not only be shortened but also can be accomplished more easily and expeditely. Moreover this

manner of adding signs will be an improvement on the other received method, by which the following values of functions of nearby variables are accustomed to be expressed by differentials, there, as in that method of resolving, differentials of other kinds occur, which may easily be confused with the natural differentials of variable quantities, unless perhaps these natural differentials may be removed by the method of signs.

PROPOSITION IV. THEOREM

58. *If amnoz were a curve* (Fig. 3) *related to the given abscissa AZ, in which formula* $\int Zdx$ *may obtain a maximum or a minimum value, and another curve amvoz may be considered differing from that curve by an infinitely small amount, then the value of the formula* $\int Zdx$ *for each curve will be the same.*

DEMONSTRATION

When in analysis a certain formula of the variable becomes a maximum, then in the first place by increasing continually it will approach the maximum value, then truly, since this has been reached, again it will recede by decreasing from that. But this accession to the maximum value and the recession from that shall be thus, so that, while the magnitude is changing near the maximum value, then its momentary increments and decrements may vanish ; and this is understood likewise for a minimum. Indeed also maxima and minima may be given of this kind, about which the increments shall be infinitely large; truly the maxima and minima of this kind will rarely find a place in the present instructions, and if they are found, that will be easy to determine. Therefore it may suffice to have observed that finite momentary changes cannot be given about a maximum and minimum. However if the expression $\int Zdx$ therefore may have a maximum or a minimum value in the curve *amnoz*, for another curve of the same expression, the value will recede more from that maximum or minimum, so that this other curve will differ more from that. But if another curve may differ by an infinitely small amount from that satisfied, then for each formula $\int Zdx$ the same value will be obtained. But we may consider a curve of this kind to be differing minimally, if the arc be varied only by the infinitely small *mno* and we may consider its arc *mvo* substituted in place. On this account from the curve *az*, for which $\int Zdx$ is a maximum or a minimum, the infinitely small part *mno* is to be removed completely and we can consider in its place the other infinitely small part to be inserted without any discrepancy ; then the value of the formula $\int Zdx$, which is appropriate for the curve *amnoz,* will be equal to the value, which is appropriate for the curve *amvoz.*
Q. E. D.

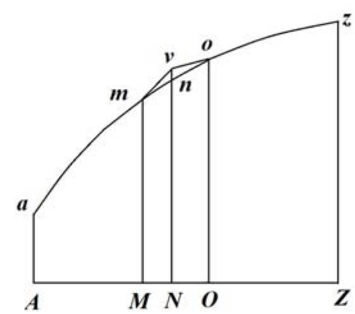

Fig.3

COROLLARY 1

59. Because the change must be put as minimal, the arc *mno*, which is considered unchanged, will not be sufficient to admit an infinitely small change, indeed also the deviation *nv* must be infinitely small in comparison to the length *mno*.

COROLLARY 2

60. Therefore with such a change placed on the curve, a change thence also may arise in the value of the formula $\int Zdx$, but which by the demonstration will be vanishing. And in this way from such an assumed change an equation may arise, which likewise will provide the nature of the curve sought.

SCHOLIUM

61. In this proposition a general method for solving problems is found, in which a curve may be desired, for which the value of a certain indeterminate formula, such as $\int Zdx$, shall be a maximum or a minimum For always an infinitely small part of the curve is considered, such as *mno,* to be varied a very small amount into *mvo,* and then the difference of the values is sought, which formula $\int Zdx$, since truly for the curve *amnoz* as well as for the curve *amvoz,* is chosen at will, and that difference put equal to zero gives the nature of the curve sought. But that indefinite change in the position must arise, so that it may extend to the whole curve and be apparent at the individual points. Moreover that change can be put in place in any manner, provided it shall be indefinitely small, and either be extended to two or more elements of the curve; for always the same final equation must result. Yet meanwhile it demands convenience of calculation, so that a change in just a few elements may be put in place, which may suffice for the solution to be become free. Thus, if between all the curves generally of the same corresponding abscissas, that must be determined in which $\int Zdx$ shall be a maximum or a minimum, then it may suffice to consider only two elements of the curve to be changed. But if not among all the curves, but only these, which may have one or several expressions in common, there it must be defined, in which a certain quantity shall be a maximum or minimum, then some change *mvo* cannot be allowed to be accepted, but it is required to put such in place, in order that these common properties from all the curves may be preserved. Therefore for these cases two elements may not suffice, but several must be accepted, so that all the conditions are able to be satisfied.

DEFINITION V

62. The *value of the differential* of a given maxima or minima corresponding to a formula is the difference between the values, which this formula may maintain both on the curve itself, as well as on the same changed by an infinitely small amount.

COROLLARY 1

63. Therefore on a curve, for which a given formula, such as $\int Zdx$, must be a maximum or a minimum, the value of the corresponding differentials of this will vanish. And on this account, if the value of the differential may be placed equal to zero, an equation will be had, in which the nature of the curve sought is expressed.

COROLLARY 2

64. Therefore with the value of the differential found, which may correspond to the proposed maximum or minimum formula, an equation will be had at once expressing the nature of this curve, in which formula that maximum or minimum proposed may have a value.

COROLLARY 3

65. Therefore the whole task for finding curves, which are endowed with the property of a maximum or a minimum, has been reduced to this, so that for some maximum or minimum formula the value of its appropriate differential may be found.

SCHOLIUM

66. Therefore since in general the idea shall be related not only to the kind of question, in which curves are found with a given property of a maximum or minimum, but also we will progress to examine these methods, for which it may be required to be used for their resolution. And in the first place indeed we shall examine the absolute method, by which curves are sought, which shall be given among all the curves generally related to the same abscissa, with a certain maximum or minimum property. Then we will go on to the relative method of maximum or minimum, to which such questions are related, which are not between all the curves corresponding to a given abscissa, but these only, which are endowed with one or more common properties, that they will are asked to determine, for which some maximum or minimum initially demanded may be agreed upon. But in these treatments the nature of the formula $\int Zdx$, which must have a maximum or minimum, introduces a huge distinction, provided Z were a determinate or an indeterminate function, as we have observed just now.

METHODUS

INVENIENDI CURVAS

MAXIMI MINIMIVE PROPRIETATE

GAUDENTES.

CAPUT PRIMUM.

De Methodo maximorum & minimorum ad lineas curvas
invenienda applicata in genere.

DEFINITIO I.

1. *METHODUS maximorum & minimorum ad lineas curvas applicata* est: methodus inveniendi lineas curvas , quae maximi minimive proprietate quapiam proposita gaudeant.

COROLLARIUM 1.

2. Reperiuntur igitur per hanc methodum lineae curvae , in quibus proposita quaepiam quantitas maximum vel minimum obtineat valorem.

COROLLARIUM 2

3. Cum autem eadem curva infinitis modis sui similis effici queat, Problema, nisi quaedam restrictio adhibeatur, maxime esset indeterminatum, atque adeo nullum. Quaecunque enim curva praebeatur maximi minimive proprietate praedita, semper alia, illi quidem vel similis vel dissimilis, exhiberi posset, quae iliam proprietatem, vel maiorem, vel minorem, in se contineret.

COROLLARIUM 3

4. Quoniam igitur adaequata curvarum cognitio postulat, ut eae ad axem aliquem positione datum, eiusque portiones quascunque, quae abscissae vocantur, referantur: prima eaque praecipua restrictio ex quantitate abscissae petenda erit.

COROLLARIUM 4

5. Problemata ergo ad methodum hanc pertinentia ita proponi debent, ut quaerantur lineae curvae ad axem positione datum relatae, quae inter omnes alias curvas eidem abscissae respondentes maximi minimive proprietate sint praeditae.

SCHOLION

6. Haec itaque Methodus maximorum et minimorum maxime discrepat ab illa, quam alibi exposuimus. Ibi enim, pro data ac determinata linea curva, locum determinavimus, ubi proposita quaedam quantitas variabilis ad curvam pertinens fiat maxima vel minima. Hic autem ipsa linea curva quaeritur, in qua quantitas quaedam proposita fiat maxima vel minima. Methodus haec iam superiori Seculo, mox post inventam Analysin infinitorum, excoli coepit a Celeberrimis Fratribus BERNOULIJIIS, atque ex eo tempore maxima cepit incrementa. Primum quidem Problema, quod ex hoc genere est tractatum, ad Mechanicam respiciebat, eoque quaerebatur linea curva, super qua grave descendens citissime delabatur; cui *Curvae brachystochronae* seu *Lineae celerrimi descensus* nomen erat impositum. In hoc Problemate iam manifestum est id, sine adiuncta conditione, nequidem nomen quaestionis retinere posse: perspicuum enim est, quo brevior magisque ad situm verticalem accedens linea capiatur, eo fore tempus descensus super ea brevius. Quamobrem non absolute quaeri potest linea, super qua grave descendens celerrime seu brevissimo tempore delabatur; sed abscissae quantitas, cui curva invenienda respondeat, simul debuit definiri; ita ut, inter omnes curvas eidem abscissae in axe positione dato sumtae respondentes, quaereretur ea, super qua corpus grave citissime delaberetur. Neque vero in hoc Problemate ista conditio sufficiebat ad id determinatum efficiendum: sed insuper istam conditionem adiicere oportuit, ut curva invenienda per data duo puncta transeat; atque istud Problema his conditionibus adstringi debuit, ut fieret determinatum inter omnes, scilicet, lineas curvas per data duo puncta transeuntes eam determinare, super qua corpus descendens arcum datae abscissae respondentem brevissimo tempore absolvat. Interim tamen hic notandum est conditionem transitus per duo puncta non esse absolute necessariam, sed in hoc Problemate per ipsam solutionem esse illatam. In solutione enim huius Problematis immediate pervenitur ad aequationem differentialem secundi gradus, quae bis integrata duas recipit constantes arbitrarias, ad quas determinandas duobus opus est punctis, per quae curva traducatur, vel aliis similibus proprietatibus: atque haec eadem conditio, quasi sua sponte, ad omnia istiusmodi Problemata accedit, quarum solutio immediate ad aequationem differentialem secundi gradus deducit. In Problematibus autem, quae resolvuntur per aequationem differentialem quarti vel altioris ordinis, nequidem duo puncta ad curvam determinandam sufficiunt, sed tot opus est punctis, quot gradus differentialia obtinent. Contra vero, si solutio statim ad aequationem algebraicam perducat, tum sine huiusmodi conditione Problema perfecte erit determinatum; dummodo abscissae longitudo definiatur. Verum haec omnia clarius perspicientur, quando infra ad solutiones Problematum perveniemus: ibique has notationes fusius explicabimus. Hic enim in principio ista tantum commemorare visum est, ut perversas ideas circa determinationem huiusmodi Problematum toliamus.

DEFINITIO II

7. *Methodus maximorum ac minimorum absoluta* docet inter omnes omnino curvas, ad eandem abscissam relatas, determinare eam, in qua proposita quaedam quantitas variabilis maximum minimumve obtineat valorem.

COROLLARIUM

8. In Problematibus igitur ad hanc methodum pertinentibus datur axis positione; atque inter omnes curvas, quae ad hunc axem eiusque determinatam portionem referri possunt, determinatur ea, in qua quantitas quaedam variabilis fit maxima vel minima.

SCHOLION

9. Aliam conditionem ad maximi minimive determinationem praeter abscissae quantitatem, hic in genere non adiicimus. Dantur enim Problemata, quae hoc modo perfecte determinantur; quemadmodum infra distinctius patebit. Etsi enim etiam eiusmodi Problemata occurrunt, ad quae determinanda insuper duo plurave puncta praescribi possunt, per quae quaesita curva transeat, tamen hoc demum ex ipsa cuiuscunque Problematis solutione perspicietur. Namque si ad eiusmodi aequationem pro curva quaesita perveniatur, in qua per integrationem novae quantitates constantes sint ingressae, quae in ipsa quaestione non inerant, tum solutio censenda erit ambigua atque vaga; eo quod innumerabiles lineas curvas, quae ex determinatione illarum quantitatum constantium et arbitrariarum oriri possunt, in se complectitur. His igitur in casibus erit concludendum Problema ex sua natura non penitus esse determinatum, sed ad eius plenam determinationem, praeter abscissae quantitatem, tot novas conditiones adiungi oportere, quibus illae arbitrariae constantes ad determinatos valores revocentur. Pro huiusmodi autem conditionibus commodissime assumuntur puncta, per quae curvae quaesitae sit transeundum ; totidem vero puncta, quot insunt in aequatione inventa quantitates arbitrariae, ipsam aequationem determinatam reddent. Loco punctorum autem, ad curvam quaesitam perfecte determinandam, adhiberi etiam possunt totidem tangentes, quae curvam quaesitam tangant et, si contactus debeat fieri in dato tangentis puncto, haec conditio duobus punctis aequivalebit. Quin etiam in locum punctorum aliae quaecunque conditiones substitui possunt, dummodo eae ita sint comparatae, ut per eas quantitates arbitrariae in aequatione inventa contentae determinentur. Neque vero ante opus est solutionem ad finem perducere, quam ista diiudicatio suscipiatur; sed infra tradentur certa criteria, quorum ope statim ex illa quantitate variabili, quae maximum minimumve esse debet, dignosci poterit, quae novae constantes in aequationem pro curva ingrediantur, quae in quaestione non continebantur. Oriuntur autem istae constantes arbitrariae ex gradu differentialium, ad quem aequatio pro curva quaesita exsurgit; quoti enim gradus prodit aequatio differentialis pro curva quaesita, tot quantitates arbitrariae in illa censendae sunt potestate inesse ; hincque totidem conditionibus opus erit ad curvam determinandam. Idem vero etiam usu venit in solutione omnium Problematum, quando aequatio differentialis vel primi vel altioris gradus invenitur; ita ut hinc in praesenti instituto nulla peculiaris difficultas inesse censenda sit.

DEFINITIO III

10. *Methodus maximorum ac minimorum relativa* docet non inter omnes omnino curvas eidem abscissae respondentes, sed inter eas tantum, quae praescriptam quandam proprietatem communem habeant, eam determinare, quae maximi minimive proprietate gaudeat.

COROLLARIUM 1

11. Ad huiusmodi igitur Problemata solvenda, primum ex omnibus omnino curvis eidem abscissae respondentibus eae sunt segregandae, in quas eadem praescripta proprietas competat; atque tum demum ex his segregatis ea, quae quaeritur, debebit definiri.

COROLLARIUM 2

12. Quanquam autem tall conditione numerus curvarum omnium ad eandem abscissam relatarum vehementer restringitur, tamen is etiamnum manebit infinitus. Quin etiam, si non una, sed plures proprietates praescribantur, quibus omnes curvae, ex quibus quaesita est determinanda, debeant esse praeditae, tamen usque numerus curvarum manebit infinitus.

COROLLARIUM 3

13. Quo plures itaque proponuntur proprietates, quae iis curvis, ex quibus quaesitam definiri oportet, communes esse debeant, eo magis numerus curvarum, inter quas electio quaesitae est instituenda, restringetur, etiamsi maneat infinitus.

SCHOLION 1

14. Ex hoc genere, in quo Methodum maximorum et minimorum relativam constituimus, initio huius Seculi, primum a JACOBO BERNOULLIO in medium prolatum est famosum illud *Problema Isoperimetricum;* in quo quaerebatur curva maximi minimive proprietate praedita, non inter omnes curvas ad eandem abscissam relatas, sed inter eas tantum, quae eiusdem essent longitudinis; ex quo istae curvae, ex quibus quaesitam erui oportebat, *isoperimetrae* sunt appellatae. Ita si inter omnes curvas eidem abscissae respondentes et longitudine aequales quaeratur ea, quae cum abscissa et applicata maximum spatium includat, reperitur quaesito linea circularis satisfacere; quod quidem iam diu ante inventam hanc methodum Geometris innotuerat ac demonstratum erat. At hoc casu iterum, ex ipsa Problematum natura, novae conditiones accedunt ; uti in iis, quae ad Methodum maximorum ac minimorum absolutam pertinent, quae ex constantibus arbitrariis, quas solutio inducit, sunt aestimandae. Ita in solutione Problematis, quo curva quaeritur, quae inter omnes eiusdem longitudinis maximam comprehendat aream cum abscissa, duae constantes novae ingrediuntur; ex quo, ad Problema determinatum efficiendum, id ita est proponendum, ut inter omnes curvas eiusdem longitudinis, quae non solum eidem abscissae respondeant, sed etiam per data duo puncta transeant, quaeratur ea, quae ad datam abscissam maximam aream referat.

Atque simili modo evenire potest, ut quatuor puncta, et plura etiam interdum, pro arbitrio assumi debeant, quo Problema fiat determinatum : cuius rei diiudicatio ex ipsa Problematum natura est petenda. Quemadmodum autem in Problemate isoperimetrico omnes curvae, ex quibus quaesitam determinari oportet, eiusdem longitudinis ponuntur, ita loco huius proprietatis alia quaecunque proponi potest, quae omnibus communis esse debeat. Sic iam quaesitae sunt curvae maximi minimive proprietate praeditae, inter omnes eas curvas ad eandem abscissam relatas tantum, quae circa eam abscissam conversae omnes aequales superficies generent; atque simili modo aliae quaecunque proprietates proponi possunt. Deinde etiam non una, sed plures huiusmodi proprietates praescribi possunt, quae omnibus curvis, inter quas ea, quae maximum minimumve aliquod contineat, definienda sit, communes esse debeant. Ita si quaereretur curva maximi vel minimi proprietate quapiam praedita inter omnes curvas eidem abscissae respondentes, quae tam essent omnes inter se longitudine aequales, quam etiam areas aequales concluderent.

SCHOLION 2

15. Propter hoc discrimen inter Methodum maximorum et minimorum absolutam ac relativam tractatio nostra erit bipartita. Primum scilicet methodum trademus inter omnes omnino curvas eidem abscissae respondentes eam determinandi, quae maximi minimive proprietate sit praedita. Deinde vero progrediemur ad eiusmodi Problemata, in quibus curva maximi minimive proprietate gaudens postulatur inter omnes curvas, quae unam pluresve propositas proprietates communes habeant; atque ex numero harum proprietatum istius tractationis denuo subdivisio orietur. Interim tamen non opus erit in hac subdivisione longius progredi, cum mox reperiatur methodus, quotcunque etiam propositae fuerint proprietates, Problemata facile resolvendi. Solutiones enim Problematum prima fronte maxime intricatorum praeter opinionem fient perquam expeditae ac levi calculo absolvendae.

HYPOTHESIS I

16. *In hac tractatione abscissam, ad quam omnes curvas referemus, perpetuo littera x, applicatam vero littera y designabimus. Tum vero, sumptis elementis abscissae aequalibus, semper erit* $dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx$ etc.

COROLLARIUM 1

17. His igitur substitutionibus omnia differentialia ipsius y cuiuscunque gradus ex expressionibus tollentur atque praeter differentiale dx nulla alia differentialia relinquentur. Quanquam autem hoc modo omnia differentialia praeter dx specie tantum, non revera tolluntur, tamen hae substitutiones ingens nobis in praesenti instituto afferent subsidium.

COROLLARIUM 2

18. Quin etiam huiusmodi substitutionibus differentialis constantis assumtio penitus de calculo tollitur: quodcunque enim differentiale aliud constans assumatur, post istas substitutiones perpetuo eadem formula emergere debet. Interim tamen, ob methodum infra adhibendam, necesse erit differentiale dx tanquam constans assumere.

COROLLARIUM 3

19. Ut autem facilius appareat, quomodo per has substitutiones differentialia cuiusque gradus ipsius y evanescant, iuvabit sequentem Tabeliam adiecisse:

$$
\begin{aligned}
&dy = pdx\\
&ddy = dpdx = qdx^2\\
&d^3y = dqdx^2 = rdx^3\\
&d^4y = drdx^3 = sdx^4\\
&d^5y = dsdx^4 = tdx^5\\
&\text{etc.} \quad \text{etc.} \quad \text{etc.}
\end{aligned}
$$

COROLLARIUM 4

20. Quodsi etiam arcus curvae abscissae x respondens cum suis differentialibus cuiuscunque gradus occurrat, ea omnia per istas litteras ita exprimi poterunt, ut nulla alia differentialia praeter dx adsint. Posito enim arcu $= w$ erit:

$$
\begin{aligned}
&w = \int \sqrt{(dx^2 + dy^2)} = \int dx\sqrt{(1+pp)}\\
&dw = dx\sqrt{(1+pp)}\\
&ddw = \frac{pqdx^2}{\sqrt{(1+pp)}}\\
&d^3w = \frac{prdx^2}{\sqrt{(1+pp)}} + \frac{qqdx^3}{(1+pp)^{\frac{3}{2}}}\\
&\qquad \text{etc.}
\end{aligned}
$$

COROLLARIUM 5

21. Simili modo ex his radius osculi seu curvedinis curvae, in quovis loco, per quantitates specie saltem finitas poterit exprimi. Cum enim, posito elemento dx constante, sit longitudo radii osculi $= \frac{-dw^3}{dxddy}$, fiet ea $= \frac{(1+pp)^{\frac{3}{2}}}{q}$.

COROLLARIUM 6

22. Porro ex iisdem substitutionibus erit, ut sequitur:

$$\text{Subtangens} = \frac{ydx}{dy} = \frac{y}{p},$$

$$\text{Subnormalis} = \frac{ydy}{dx} = py,$$

$$\text{Tangens} = \frac{ydw}{dy} = \frac{y\sqrt{(1+pp)}}{p},$$

$$\text{Normalis} = \frac{ydw}{dx} = y\sqrt{(1+pp)}.$$

Atque, pari modo, omnes quantitates finitae ad curvam pertinentes, nisi integralia involvant, per huiusmodi quantitates finitas ita exprimi poterunt, ut nulla differentialia amplius inesse videantur.

DEFINITIO IV

23. *Maximi minimive Formula*, pro quovis Problemate, nobis erit ea quantitas, quae in curva quaesita maximum minimumve valorem obtinere debet.

COROLLARIUM 1

24. Quoniam in omnibus Problematibus, ad quae haec Methodus est accommodata, curva quaeritur, quae vel inter omnes vel tantum inter innumeras curvas certo modo determinatas maximi minimive proprietate gaudeat, haec ipsa proprietas, quae in curva quaesita maxima vel minima esse debet, erit quantitas eaque exprimetur Formula, quam maximi minimive Formuiam hic appeliamus.

COROLLARIUM *2*

25. Cum autem maximi minimive proprietas ita proponi debeat, ut ad datam ac determinatam abscissam referatur, Formula maximi minimive quoque ad iliam definitam abscissam debet referri.

COROLLARIUM 3

26. Erit igitur maximi minimive Formula quantitas variabilis a longitudine abscissae cuiuscunque, cui respondet, pendens. Atque in quovis Problemate quaeretur curva, pro qua, ad definitam abscissam, illa maximi minimive Formula maximum minimumve obtineat valorem.

COROLLARIUM 4

27. Neque vero maximi minimive Formula a sola abscissa pendere potest: hoc enim si esset, pro omnibus curvis eidem abscissae respondentibus eundem obtineret valorem atque idcirco omnes aequaliter satisfacerent.

COROLLARIUM 5

28. Hanc ob rem quoque maximi minimive Formula praeter abscissam omnibus curvis, quae in considerationem veniunt, communem, a qualibet curva peculiariter debet pendere; ita ut una sit, pro qua maximum minimumve valorem induere queat.

SCHOLION 1

29. Quo haec omnia clarius intelligantur atque status Quaestionum in sequenti pertractandarum melius comprehendatur, ponamus vel inter omnes omnino curvas vel tantum inter innumerabiles certam quamdam proprietatem communem habentes, quae eidem abscissae *AZ* respondeant, eam determinari debere, pro qua valor formulae *W* sit maximus vel minimus. Ponamus huic Quaestioni satisfacere (Fig. I) curvam *amz,* ita ut, quaecunque alia curva ad abscissam definitam *AZ* referatur, valor formulae *W* vel fiat minor quam pro hac curva vel maior: prout in curva satisfaciente *W* vel maximum esse debet vel minimum. In hac igitur quaestione latissime patente habemus primo abscissam determinatae longitudinis *AZ* ; deinde curva est quaerenda, vel inter omnes ommino curvas ad eandem hanc abscissam relatas vel tantum inter innumerabiles, quibus una pluresve proprietates sint communes, prout quaestio ad methodum maximorum et minimorum vel ahsolutam vel relativam est accommodata; tertio habemus eam quantitatem *W,* cuius valor in curva quaesita *amz* maximus esse debet vel minimus; eritque igitur quantitas *W* maximi minimive formula, sicut ea est definita. Nunc igitur statim apparet hanc formulam *W* ita esse debere comparatam, ut ad omnes curvas, quae quidem concipi possunt, accommodari queat. Primo scilicet a quantitate abscissae definitae *AZ* debebit pendere, ita ut ea mutetur valore ipsius *AZ* mutato. Deinde etiam a natura cuiusvis curvae, quae quidem concipi potest, peculiari modo debet pendere; nisi enim ita esset comparata, pro omnibus curvis eundem valorem sortiretur quaestioque foret nulla. Quamobrem quantitas *W* praeter abscissam in se quoque complecti debebit quantitates ad curvam ipsam pertinentes. Cum igitur omnis curva determinetur per relationem inter abscissam et applicatam, quantitas *W* debet esse constata ex abscissa et applicata et quantitatibus independentibus. Hoc est, si abscissa indefinita ponatur $= x$ et applicata respondens indefinita $= y$, quantitas *W* esse debet functio binarum variabilium x et y. Quod cum ita sit, si curva quaecunque determinata concipiatur atque ex eius natura relatio inter y et x in formula *W* substituatur, ea definitum impetrabit valorem ad datam iliam curvam atque eius definitam abscissam pertinentem. Quoniam iam, pro aliis atque allis curvis, formula *W* diversos valores induit, etiamsi in omnibus abscissa eadem capiatur, manifestum est inter innumerabiles illas curvas unam esse debere, in qua valor formulae *W* maximus fiat

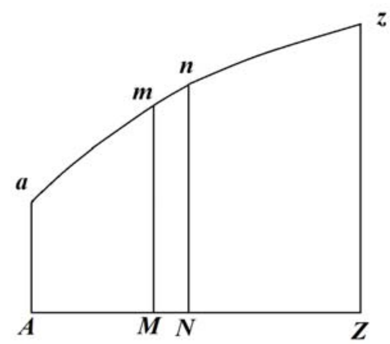

Fig. 1

vel minimus; atque ad hanc curvam pro data quacunque determinata quaestione inveniendam Methodus tradenda est comparata.

COROLLARIUM 6

30. Erit igitur maximi minimive formula W functio quaedam binarum variabilium x et y; quarum altera x abscissam, altera y applicatam denotat. In W inesse igitur poterunt non solum ipsae variabiles x et y, sed etiam omnes quantitates ab iis pendentes, cuiusmodi sunt p, q, r, s etc., quarum significationes supra tradidimus. Quinetiam formulae integrales ex his ortae quaecunque in W inesse possunt, imo etiam debent, siquidem quaestio debeat esse determinata, uti mox ostendemus.

COROLLARIUM 7

31. Proposita igitur eiusmodi formula W seu functione ipsarum x et y, *si* quaestio ad methodum maximorum et minimorum absolutam pertineat, eiusmodi aequatio inter x et y desideratur, ut, si in W valor ipsius y per x determinatus substituatur atque ipsi x valor definitus tribuatur, maior prodeat quantitas pro W vel minor, quam si ulla alia aequatio inter x et y assumta fuisset.

COROLLARIUM 8

32. Hoc ergo pacto quaestiones ad doctrinam linearum curvarum pertinentes ad Analysin puram revocari possunt. Atque vicissim, si huius generis quaestio in Analysi pura sit proposita, ea ad doctrinam de lineis curvis poterit referri ac resolvi.

SCHOLION 2

33. Quanquam huius generis quaestiones ad puram Analysin reduci possunt, tamen expedit eas cum doctrina linearum curvarum coniungere. Quodsi enim animum a lineis curvis abducere atque ad solas quantitates absolutas firmare velimus, quaestiones primum ipsae admodum fierent abstrusae et inelegantes ususque earum ac dignitas minus conspiceretur. Deinde etiam methodus resolvendi huiusmodi quaestiones, si in solis quantitatibus abstractis proponeretur, nimium foret abstrusa et molesta; cum tamen eadem, per inspectionem figurarum et quantitatum repraesentationem linearem, mirifice adiuvetur atque intellectu facilis reddatur. Hanc ob causam, etsi huius generis quaestiones cum ad quantitates abstractas tum concretas applicari possunt, tamen eas ad lineas curvas commodissime traducemus et resolvemus. Scilicet, quoties aequatio eiusmodi inter x et y quaeritur, ut formula quaedam proposita et composita ex x et y, si ex illa aequatione quaesita valor ipsius y subrogetur et ipsi x determinatus valor tribuatur, maxima fiat vel minima, tum semper quaestionem transferemus ad inventionem lineae curvae, cuius abscissa sit x et applicata y, pro qua illa formula W fiat maxima vel minima, si abscissa x datae magnitudinis capiatur. His igitur notatis natura huiusmodi quaestionum satis luculenter perspicitur, nisi forte cuiquam adhuc dubium creat ambigua locutio de maximo et minimo simul. Verum ne hic quidem ulla adest ambiguitas; nam

etsi methodus ipsa aeque monstrat maxima et minima, tamen in quovis casu facile erit discernere, utrum solutio praebeat maximum an minimum. Saepenumero autem evenire potest, ut in data quaestione tam maximum quam minimum locum obtineat, atque his casibus solutio erit duplex, altera monstrante maximum altera minimum. Plerumque autem alterutrum scilicet vel maximum vel minimum solet esse impossibile; quod evenit, si maximi minimive formula in infinitum vel crescere vel decrescere potest; his enim casibus vel non dabitur maximum vel non minimum. Usu venire etiam potest, ut formula proposita W in infinitum tam crescere quam decrescere queat, atque his casibus nulla prorsus solutio locum habebit. Haec autem discrimina cuncta ipse calculus post solutionem perpetuo monstrabit.

PROPOSITIO I. THEOREMA

34. *Ut per maximi minimive formulam W curva determinetur amz, quae prae omnibus reliquis satisfaciat, formula W debet esse quantitas integralis indefinita, quae, nisi data assumatur relatio inter x et y, integrari nequeat.*

DEMONSTRATIO

Ponamus enim formulam W integralia indefinita non involvere; erit ea functio quantitatum x et y indeque pendentium p, q, r, s etc. vel algebraica vel talis transcendens, quae sine assumta relatione inter x et y exhiberi possit; quod evenit, si vel logarithmi harum quantitatum vel arcus circulares vel aliae huiusmodi quantitates transcendentes definitae ingrediantur, quae algebraicis aequivalentes sunt censendae. Quodsi iam W ponatur functio talis ipsarum x et y tantum, manifestum est valorem formulae W, quem pro data curva *amz* ad datam abscissam *AZ* relata obtinet, tantum ab ultima applicata *Zz* pendere atque pro omnibus curvis in Z eandem applicatam *Zz* habentibus fore eundem; atque adeo tali formula W indoles totius curvae non determinabitur, sed tantum positio extremi eius puncti z; si in W praeter x et y etiam quantitas p insit, tum praeter longitudinem applicatae *Zz* positio tangentis curvae in z seu positio ultimi elementi in z determinabitur. Sin autem insuper q ingrediatur, tum positio binorum elementorum curvae contiguorum in z determinabitur, et ita porro. Ex quibus sequitur, si fuerit W functio determinate ipsarum x, y, p, q, r etc., tum per iliam tantum curvae portionem infinite parvam circa extremitatem z determinari atque pro omnibus curvis in eandem extremitatem desinentibus eundem valorem ipsius W esse proditurum.
Ut itaque per formulam W tota curva *amz*, quatenus toti abscissae *AZ* respondet, definiatur, formulam W ita oportet esse comparatam, ut eius valor ad determinatam curvam *amz* applicatus a positione singulorum elementorum huius curvae intra terminos a et z sitorum pendeat. Hoc autem evenire non potest, nisi quantitas W sit formula integralis indefinita, quae generatim sine assumta aequatione inter x et y integrationem non admittat. Q. E. D.

COROLLARIUM 1

35. Nisi igitur maximi minimive formula W sit quantitas integralis indefinita, nequidem linea curva, in qua valor ipsius W sit maximus vel minimus, determinabitur; atque adeo quaestio de invenienda curva, in qua esset W maximum vel minimum, erit nulla.

COROLLARIUM 2

36. Ut igitur curva assignari possit, in qua valor ipsius W prae aliis sit maximus vel minimus, formula W talem formam $\int Zdx$ habere debet atque quantitatem Z ita comparatam esse oportet, ut differentiale Zdx, nisi aequatio statuatur inter x et y, integrari nequeat.

SCHOLION

37. Quoniam maximi minimive formula W debet esse integrale formulae differentialis indefinitae primi gradus, hoc est cuius integrale fiat quantitas finita, ea formula differentialis semper ad huiusmodi formam Zdx poterit reduci ope litterarum p, q, r etc. Et hanc ob rem in sequentibus maximi minimive formula perpetuo per $\int Zdx$ nobis indicabitur. Erit autem Z functio non solum quantitatum x et y, sed etiam continebit litteras p, q, r etc. Ita si area $AazZ$ debeat esse maxima vel minima, formula W abibit in $\int ydx$; et, si superficies solidi rotundi, quod generatur rotatione curvae amz circa axem AZ debeat esse maxima vel minima, erit $W = \int ydx\sqrt{(1+pp)}$ *;* atque ita porro, quaecunque formula debeat in curva quaesita esse maxima vel minima, ea semper erit huius formae $\int Zdx$, scilicet integrale quantitatis finitae cuiusdam Z in differentiale dx ductae. Debet autem Z eiusmodi esse quantitas, ut, inveniendi lineas curvas si aequatio statuatur inter x et y, integrale $\int Zdx$ determinatum obtineat valorem; ex quo Z erit functio quantitatum x, y et inde pendentium p, q, r etc. vel algebraica sive determinata vel praeterea ipsa in se complectetur formulas integrales indeterminatas; quod discrimen probe est tenendum. Ita si maximi minimive formula W fuerit $\int ydx$ vel $\int ydx\sqrt{(1+pp)}$, quantitas Z erit algebraica, at si sit $W = \int yxdx\int ydx$, tum erit $Z = yx\int ydx$, hoc est ipsa quantitas Z erit indeterminata, cuius valor, nisi relatio inter x et y detur, exhiberi nequit. Quin etiam evenire potest, ut valor ipsius Z huiusmodi formula evoluta exprimi nequeat, sed tantum per aequationem differentialem demum erui debeat, ut si fuerit $dZ = ydx + ZZdx$ *;* ex qua aequatione valor ipsius Z per x et y nequidem exhiberi potest. Hinc igitur tria nascuntur genera formularum $\int Zdx$, quae in curvis quaesitis maxima vel minima fieri debent. Quorum primum eas complectitur formulas, in quibus Z est functio algebraica seu determinata ipsarum x, y et p, q, r etc. Ad secundum genus referimus eas formulas, in quibus quantitas Z ipsa insuper formulas integrales involvit. In tertio autem

genere continentur eae formulae, in quibus valor ipsius *Z* per aequationem differentialem, cuius integratio non constat, determinatur.

PROPOSITIO II. THEOREMA

38. *Si fuerit amz curva, in qua valor formulae* $\int Zdx$ *sit maximus vel minimus, atque Z sit functio algebraica seu determinata ipsarum x, y, p, q, r* etc., *tum eiusdem curvae quaecunque portio mn eadem gaudebit praerogativa, ut pro ea ad suam abscissam MN relata valor ipsius* $\int Zdx$ *sit pariter maximus vel minimus.*

DEMONSTRATIO

Valor formulae $\int Zdx$ pro abscissa *AZ* est aggregatum omnium valorum eiusdem formulae, qui singulis abscissae *AZ* portionibus respondent. Quodsi ergo abscissa *AZ* in partes quotcunque, quarum una sit *MN*, divisa concipiatur atque ad singulas partes hasce valor formulae $\int Zdx$ exhibeatur, summa omnium horum valorum praebebit valorem formulae $\int Zdx$, qui toti abscissae *AZ* convenit et qui erit maximus vel minimus. Quoniam autem *Z* ponitur functio algebraica ipsarum *x, y, p, q* etc., valor formulae $\int Zdx$ respondens abscissae portioni *MN* a sola portionis curvae respondentis *mn* indole pendebit idemque manebit, utcunque reliquae partes *am* et *nz* varientur; singularum enim litterarum *x, y, p, q* etc. valores per solam curvae portionem *mn* determinantur. Si ergo formulae $\int Zdx$ valores, qui conveniunt abscissae portionibus *AM*, *MN*, *NZ*, ponantur *P*, *Q* et *R*, quantitates hae *P, Q* et R a se mutuo non pendebunt. Quare cum earum aggregatum $P+Q+R$ sit maximum vel minimum, etiam unaquaque maximi minimive proprietate praedita sit necesse est. Hanc ob rem, si in curva *amz* formula $\int Zdx$ maximum minimumve habeat valorem et quantitas *Z* sit functio algebraica ipsarum *x, y, p, q* etc., tum etiam pro qualibet illius curvae portione eadem formula $\int Zdx$ maximi minimive proprietate gaudebit. Q. E. D.

COROLLARIUM 1

39. Quodsi ergo curva fuerit inventa *amz,* quae pro abscissa data *AZ* habeat valorem formulae $\int Zdx$ maximum vel minimum, atque *Z* sit functio algebraica seu determinata, tum etiam eiusdem curvae quaelibet portio, respectu abscissae suae respondentia, eadem maximi minimive proprietate gaudebit.

COROLLARIUM 2

40. In huiusmodi igitur Problematibus, ubi tale maximum minimumve quaeritur, non opus est quantitatem abscissae, cui maximum minimumve respondeat, definire; sed si pro una quacunque abscissa formula $\int Zdx$ sit maximum vel minimum, tum eadem pro quacunque alia abscissa eadem proprietate gaudebit.

COROLLARIUM 3

41. Huiusmodi igitur Problemata resolventur, si singulae curvae quaesitae particulae ita determinentur, ut pro iis valor formulae $\int Zdx$ fiat maximus vel minimus. Tum enim simul tota curva et quaecunque eius portio pariter eadem maximi minimive proprietate erit instructa.

SCHOLION

42. Proprietas haec, qua gaudent curvae, in quibus istius modi formulae $\int Zdx$, ubi Z est functio algebraica seu determinata ipsarum x, y, p, q etc., sunt maximum vel minimum, est maximi momenti ; ea enim innititur universa methodus huius generis Problemata resolvendi. Ideo autem potissimum hanc Propositionem afferre visum est, ne ea proprietas, quae his tantum formulis $\int Zdx$, ubi Z est functio vel algebraica vel determinata, est propria, omnium omnino formularum, quae proponi possunt, communis esse putetur; in sequente enim Propositione demonstrabimus, si in Z insint formulae integrales, tum eandem proprietatem non amplius locum habere; ex quo simul natura huiusmodi quaestionum clarius intelligetur. Huius autem praesentis Propositionis demonstratio ex eo petita est fundamento, quod valor formulae $\int Zdx$, siquidem Z est functio vel algebraica vel determinata ipsarum x, y, p, q, r etc.; qui convenit cuicunque abscissae portioni *MN*, a sola curvae portione respondente *mn* pendeat neque a reliqua curva vel anteriore *am* vel posteriore *nz* afficiatur; quae ratio cessat, si in Z insint formulae integrales indeterminatae. Valores enim quantitatum x, y, p, q, r etc., qui pro arcu curvae *mn* obtinent, tantum a positione elementorum huius arcus *mn* atque elementis aliquot contiguis, quae arcum finitae quantitatis non constituunt, pendent; ex quo etiam quantitas ex iis litteris utcunque composita per solam arcus *mn* indolem determinabitur, nisi adfuerint quantitates integrales, cuiusmodi sunt $\int ydx$, quae totam aream anteriorem *AamM* introduceret, vel $\int dx\sqrt{(1+pp)}$, quae totum arcum praecedentem *am* involveret. Hinc igitur distinctius intelligitur, quid per functionem determinatam ipsarum x, y, p, q, r etc. denotare velimus: Functio scilicet determinata ita est comparata, ut pro quovis loco a praesentibus valoribus litterarum x, y, p, q etc. tantum pendeat neque valores earum anteriores in se complectatur. Functio autem indeterminata est talis, cuius valor in quovis loco non ex solis valoribus, quos hae litterae x, y, p, q etc. in isto loco obtinent, determinari potest, sed insuper omnes valores ad sui determinationem requirit, quos istae litterae in omnibus locis anterioribus obtinuerunt. Ita patet omnes functiones algebraicas esse simul determinatas; praeterea vero etiam omnes functiones transcendentes, quae a relatione inter x et y non pendent, sunt determinatae, cuiusmodi sunt $l\sqrt{(xx+yy)},\ e^{py}, \text{A}\sin\frac{py}{q}$; quarum valores in quovis loco q ex valoribus litterarum, quos in hoc solo loco obtinent, assignari possunt. Quando autem in functione quapiam insunt formulae integrales indeterminatae, quae a mutua relatione inter x et y, quam ubique tenent, pendent, tum earum valor in dato loco non ex valoribus, quos hae litterae in isto

loco habent, cognosci potest, sed insuper omnes valores in locis quibusque anterioribus nosse oportet, hoc est generalem relationem inter coordinatas x et y; talesque tum functiones vocamus indeterminatas, quippe quae toto coelo diversae sunt ab iis, quas determinatas appellavimus.

PROPOSITIO III. THEOREMA

43. *Si fuerit amz curva abscissae AZ respondens, in qua* $\int Zdx$ *sit maximum vel minimum, in Z autem contineantur formulae integrales indeterminatae, tum eadem maximi minimive proprietas non cadit in quamlibet curvae portionem, sed toti tantum curvae abscissae AZ respondenti propria erit.*

DEMONSTRATIO

Concipiatur tota curva *amz*, pro qua $\int Zdx$ est maximum vel minimum, in duas partes quasque divisa per applicatam *Mm* sitque formulae $\int Zdx$ valor conveniens portioni $am = P$, eiusdem autem formulae valor pro altera portione *mz* sit $= Q$; pro tota igitur curva *amz* valor formulae $\int Zdx$ erit $P+Q$, quem ponimus esse maximum vel minimum. Quo autem omnem ambiguitatem toliamus totamque rem distinctius proponere queamus, ponamus $P+Q$ esse maximum; quod enim de maximo demonstrabitur, idem de minimo facile intelligetur. Quodsi iam valor ipsius Q a valore ipsius P non penderet, tum aggregatum $P+Q$ maximum esse non posset, nisi simul uterque valor P et Q seorsim sit maximus. At nostro casu, quo quantitas Z in se continet formulas integrales indeterminatas, valor ipsius Q non tantum a curvae portione *mz*, ad quam refertur, pendebit, sed simul a tota curva anteriore *am* atque adeo a valore ipsius P. Nunc dicimus ad id, ut $P+Q$ sit maximum, non requiri, ut valor ipsius P sit maximus. Ponamus enim portionem curvae *am* ita esse comparatam, ut pro ea P sit maximum, et aliquantillum mutari concipiatur portio curvae *am*, ita ut valor formulae $\int Zdx$ minor evadat, puta $= P - p$; fieri utique poterit, ut ex hac mutatione valor ipsius Q crescat, quod incrementum ponatur q, eritque mutata aliquantillum portione *am*, ita ut pro ea $\int Zdx$ non amplius sit maximum, valor formulae $\int Zdx$ pro tota curva *amz* $= P - p + Q + q$. Cum igitur evenire queat, ut sit $q > p$, intelligitur formulam $\int Zdx$ pro tota curva *amz* maximam esse posse, etiamsi maxima non sit pro qualibet portione *am*. Q. E. D.

COROLLARIUM 1

44. Quando ergo curva fuerit inventa, quae, pro data abscissa *AZ*, habeat valorem formulae $\int Zdx$ maximum vel minimum, et *Z* sit functio indeterminata, tum non sequitur quamlibet curvae inventae portionem eadem maximi minimive proprietate fore praeditam.

COROLLARIUM 2

45. In resolutione igitur huiusmodi Problematum, in quibus curva quaeritur, quae pro data abscissa *AZ* habeat $\int Zdx$ maximum vel minimum, perpetuo ad totius abscissae propositae quantitatem erit respiciendum, atque maximum vel minimum ad eam tantum, non vero ad eius quamlibet portionem, accommodari debebit.

COROLLARIUM 3

46. Maximum igitur hinc patet discrimen, quod inter formulas $\int Zdx$,in quibus *Z* functio est determinata vel indeterminata, intercedit; simulque autem Methodorum diversitas intelligitur, quibus ad resolutiones quaestionum, in quibus huiusmodi formularum maximi minimive valores requiruntur, uti oportebit.

SCHOLION

47. Ex demonstratione huius Propositionis non quidem necessaria sequitur, si pro data abscissa *AZ* curva habeat formulam $\int Zdx$ maximam vel minimam, tum singulas eius portiones eadem hac praerogativa gaudere; verumtamen satis intelligitur, quoties eadem proprietas in singulas portiones competat, id casu evenire. Hincque nihilominus summe necessarium est solutionem perpetuo ad totam propositam abscissam accommodare. Interim tamen in Problematibus ad methodum relativam pertinentibus evenire potest, ut formulas $\int Zdx$, in quibus *Z* sit functio indeterminata, quasi determinata esset tractare liceat. Hoc scilicet accidit, si inter omnes tantum curvas, in quibus formulae illae integrales indeterminatae, quae in *Z* insunt, aequales obtinent valores, ea desideretur, in qua $\int Zdx$ sit maximum vel minimum; hoc enim casu formulae illae integrales indeterminatae fieri censendae sunt determinatae. Ita si inter omnes curvas eiusdem longitudinis determinanda sit ea, in qua sit $\int Zdx$ maximum vel minimum, atque in *Z* praeter quantitates determinatas insit arcus curvae $\int dx\sqrt{(1+pp)}$, hic, quia in omnibus curvis, ex quibus quaesitam definire oportet, eundem obtinet valorem, instar functionis determinatae tractari poterit. Haec autem cuncta in sequentibus clarius explicabuntur.

HYPOTHESIS II

48. *Si curvae* (Fig. 2) *abscissa AZ in elementa innumerabilla infinite parva et inter se aequalia dissecetur, cuiusmodi sunt IK, KL, LM etc., atque portio quaecunque AM*

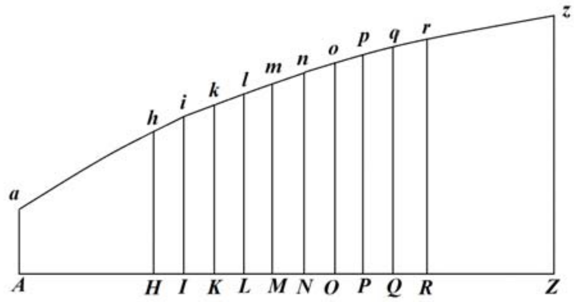

Fig. 2.

vocetur x, cui respondeat functio quaecunque variabilis F, eandem functionem F, quatenus referetur ad puncta abscissae vel sequentia N, O, P, Q etc. vel antecedentia L, K, I etc. ita denotabimus, ut sit valor istius functionis, qui pro puncto M est $=F$ *, ut sequitur*

$$\left.\begin{array}{l} pro\ N = F^{I} \\ pro\ O = F^{II} \\ pro\ P = F^{III} \\ pro\ Q = F^{IV} \\ pro\ R = F^{V} \\ \quad \text{etc.} \end{array}\right\} \textit{pro punctis abscissae sequentibus.}$$

$$\left.\begin{array}{l} pro\ L = F_{I} \\ pro\ K = F_{II} \\ pro\ I = F_{III} \\ pro\ H = F_{IV} \\ \quad \text{etc.} \end{array}\right\} \textit{pro punctis abscissae antecedentibus.}$$

Atque hoc pacto, sine prolixa differentialium scriptione, valor functionis cuiuscunque variabilis, qui in quovis abscissae puncto locum obtinet, commode indicabitur.

COROLLARIUM 1

49. Cum igitur functionis cuiusque valor in loco quocunque sit aequalis suo valori in loco antecedente differentiali suo aucto, erit

$F^{I} = F + dF$	$F = F_{I} + dF_{I},$
$F^{II} = F^{I} + dF^{I}$	$F_{I} = F_{II} + dF_{II}$
$F^{III} = F^{II} + dF^{II}$	$F_{II} = F_{III} + dF_{III}$
$F^{IV} = F^{III} + dF^{III}$	$F_{III} = F_{IV} + dF_{IV}$
etc.	etc.

COROLLARIUM 2

50. Si ex singulis abscissae divisionibus applicatae ducantur atque ea, quae abscissae $AM = x$ respondet, nempe *Mm*, ponatur $= y$, reliquae tam sequentes quam antecedentes ita denotabuntur:

$Mm = y$	$Mm = y$
$Nn = y^{I}$	$Ll = y_{I}$
$Oo = y^{II}$	$Kk = y_{II}$
$Pp = y^{III}$	$Ii = y_{III}$
$Qq = y^{IV}$	$Hh = Y_{IV}$
etc.	etc

COROLLARIUM 3

51. Cum deinde valor ipsius *p* sit $= \frac{dy}{dx} = \frac{Nn - Mm}{dx}$, erit $p = \frac{y^{I} - y}{dx}$; sequentes autem pariter ac antecedentes ipsius *p* valores ita se habunt:

$p = \frac{y^{I} - y}{dx}$	$p = \frac{y^{I} - y}{dx}$
$p^{I} = \frac{y^{II} - y^{I}}{dx}$	$p_{I} = \frac{y_{II} - y_{I}}{dx}$"
$p^{II} = \frac{y^{III} - y^{II}}{dx}$	$p_{II} = \frac{y_{III} - y_{II}}{dx}$
$p^{III} = \frac{y^{IV} - y^{III}}{dx}$	$p_{III} = \frac{y_{IV} - y_{III}}{dx}$
etc.	etc.

COROLLARIUM 4

52. Deinde, quia est $q=\frac{dp}{dx}=\frac{p^{/}-p}{dx}$, erit $q=\frac{y^{//}-2y^{/}+y}{dx^2}$; ex quo quantitatis q valores, cum sequentes tum antecedentes, ita se habebunt:

$q=\frac{y^{//}-2y^{/}+y}{dx^2}$	$q=\frac{y^{//}-2y^{/}+y}{dx^2}$
$q^{/}=\frac{y^{///}-2y^{//}+y^{/}}{dx^2}$	$q_{/}=\frac{y^{/}-2y+y_{/}}{dx^2}$
$q^{//}=\frac{y^{IV}-2y^{///}+y^{//}}{dx^2}$	$q_{//}=\frac{y-2y_{/}+y_{//}}{dx^2}$
etc.	etc.

COROLLARIUM 5

53. Simili igitur modo per ista applicatarum signa poterunt valores quantitatum r, s, t etc., ut has supra assumsimus, determinari atque ex figura definiri. Erit scilicet

$$r=\frac{y^{///}-3y^{//}+3y^{/}-y}{dx^3}$$
$$s=\frac{y^{IV}-4y^{///}+6y^{//}-4y^{/}+y}{dx^4}$$
$$t=\frac{y^{V}-5y^{IV}+10y^{///}-10y^{//}+5y^{/}-y}{dx^5}$$
etc.,

unde harum litterarum valores tam praecedentes quam antecedentes formari possunt.

COROLLARIUM 6

54. Quodsi autem formula $\int Zdx$ ad abscissam $AM=x$ fuerit relata, erit eius valor sequenti abscissae elemento $MN=dx$ respondens $=Zdx$. Hincque simili modo formulae $\int Zdx$ valores singulis abscissae elementis respondentes denotabuntur, ut sequitur:

pro $MN = Zdx$	pro $MN = Zdx$
pro $NO = Z^{I}dx$	pro $LM = Z_{I}dx$
pro $OP = Z^{II}dx$	pro $KL = Z_{II}dx$
pro $PQ = Z^{III}dx$	pro $IK = Z_{III}dx$
etc.	etc.

COROLLARIUM 7

55. Si ergo expressio $\int Zdx$ ad abscissam curvae $AM = x$ pertineat, eiusdem expressionis valor, qui conveniet abscissae propositae *AZ*, erit

$$= \int Zdx + Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.}$$

in infinitum, donec perveniatur ad ultimum punctum *Z*.

COROLLARIUM 8

56. Si igitur curva inveniri debeat, quae pro data abscissa *AZ* valorem formulae $\int Zdx$ habeat maximum minimumve, tum posita abscissa quacunque indefinita *A1Jf x* efficiendum est, ut haec expressio $\int Zdx + Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.}$ usque in *Z* fiat maxima vel minima.

SCHOLION

57. Though this hypothesis has been made by choice, yet these signs bring the greatest use to problems, which relate to this method of maxima and minima, requiring to be resolved succinctly. Indeed it is of great worth in calculations of this kind with the convenient choice of these signs and its help not only in shortening the calculation, but also by making it much easier and quicker. Praestabit autem iste signandi modus longe alteri recepto, quo per differentialia valores functionum variabilium proxime sequentes exprimi solent, eo, quod in ipsa resolvendi methodo alius generis differentialia occurrent, quae cum naturalibus quantitatum variabilium differentialibus facile confundi possent, nisi ista assumta signandi methodo naturalia differentialia notatione saltem tollerentur.

PROPOSITIO IV. THEOREMA

58. *Si amnoz fuerit curva* (Fig. 3) *ad abscissam datam AZ relata, in qua formula* $\int Zdx$ *maximum minimumve obtineat valorem, atque alia concipiatur curva amvoz ab ista infinite parum discrepans, tum valor formulae* $\int Zdx$ *pro utraque curva erit idem.*

DEMONSTRATIO

Quando in Analysi formula quaepiam variabilis fit maxima, tum primo crescendo continuo magis ad maximum valorem accedit, deinde vero, cum hunc attigit, iterum decrescendo ab eo recedit. Iste autem accessus ad maximum valorem atque recessus ab eodem ita sit, ut, dum quantitas proxime ad maximum valorem versatur, tum eius incrementa ac decrementa momentanea evanescant; hocque idem de minimo est intelligendum. Dantur quidem etiam eiusmodi maxima et minima, circa quae incrementa et decrementa sint infinite magna; verum huius generis maxima et minima in praesenti instituto raro locum inveniunt, et si inveniunt, facile erit ea determinare. Sufficiat igitur notasse circa maximum et minimum mutationes momentaneas non dari posse finitas. Quodsi ergo in curva *amnoz* expressio $\int Zdx$ maximum minimumve habeat valorem, pro alia curva eiusdem expressionis valor eo magis a maximo minimove recedet, quo magis haec alia curva ab illa discrepet. Sin autem alia curva infinite parum differat ab illa satisfaciente, tum pro utraque formula $\int Zdx$ eundem obtinebit valorem. Huiusmodi autem curvam minime discrepantem concipiemus, si arcum tantum infinite parvum *mno* infinite parum variari eiusque loco arcum *mvo* substitui ponamus. Quamobrem ex curva *az*, pro qua $\int Zdx$ maximum est vel minimum, portionem infinite parvam *m no* exscindi eiusque loco aliam *mv o* infinite parum ab illa discrepantem inseri intelligamus; tum valor formulae $\int Zdx$, qui convenit curvae *amnoz*, aequalis erit valori, qui convenit curvae *amvoz*. Q. E. D.

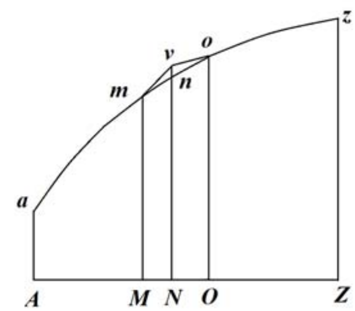

Fig.3

COROLLARIUM 1

59. Quoniam mutatio debet poni quam minima, non sufficiet arcum *mno*, qui immutari ponitur, accipere infinite parvum, sed etiam deviatio *nv* prae arcus longitudine *mno* debet esse infinite parva.

COROLLARIUM 2

60. Posita igitur tali mutatione in curva, mutatio inde etiam in valore formulae $\int Zdx$ orietur, quae autem per demonstrationem erit evanescens. Atque hoc modo ex tali assumta mutatione orietur aequatio, quae simul curvae quaesitae naturam praebebit.

SCHOLION

61. In hac Propositione continetur universa methodus resolvendi Problemata, quibus curva desideratur, in qua valor formulae cuiusdam indeterminatae, ut $\int Zdx$, sit maximus vel minimus. Semper enim concipitur portio curvae infinite parva, uti *mno*, aliquantillum variari in *mvo*, atque tum quaeritur differentia valorum, quos formula $\int Zdx$ cum pro curva vera *amnoz* tum pro ficta *amvoz* sortitur, eaque differentia nihilo aequalis posita dat naturam curvae quaesitae. Mutatio autem ista in loco indefinito fieri debet, ut ad totam curvam pertineat atque ad singula loca pateat. Potest autem ista mutatio utcunque institui, dummodo sit infinite parva, atque vel ad duo vel plura curvae elementa extendi; semper enim eadem resultare debet aequatio finalis. Interim tamen calculi commoditas postulat, ut mutatio in tam paucis elementis instituatur, quae sufficiat ad solutionem absolvendam. Ita si inter omnes omnino curvas eidem abscissae respondentes ea determinari debeat, in qua sit $\int Zdx$ maximum vel minimum, tum sufficiet bina tantum curvae elementa mutata concipere. At si non inter omnes curvas, sed eas tantum, quae unam pluresve expressiones communes habeant, ea definiri debeat, in qua quaepiam quantitas sit maxima vel minima, tum mutationem non quamcunque *mvo* accipere licet, sed talem statui oportet, ut illae proprietates omnibus curvis communes conserventur. His igitur casibus duo elementa non sufficient, sed plura accipi debebunt, ut. omnibus conditionibus satisfieri queat.

DEFINITIO V

62. *Valor Differentialis* datae maximi minimive formulae respondens est differentia inter valores, quos haec formula cum in ipsa curva quaesita tum in eadem infinite parum immutata obtinet.

COROLLARIUM 1

63. In curva igitur, pro qua data formula, puta $\int Zdx$, maximum minimumve esse debet, huius formulae valor differentialis respondens evanescet. Atque hanc ob rem, si valor differentialis nihilo aequalis ponatur, habebitur aequatio, qua curvae quaesitae natura exprimetur.

COROLLARIUM 2

64. Ex invento igitur valore differentiali, qui propositae maximi minimive formulae respondeat, statim habebitur aequatio exprimens naturam eius curvae, in qua formula illa proposita maximum minimumve habeat valorem.

COROLLARIUM 3

65. Totum igitur negotium ad curvas inveniendas, quae maximi minimive proprietate gaudeant, eo est reductum, ut pro quaque maximi minimive formula eius conveniens valor differentialis investigetur.

SCHOLION

66. Cum igitur in genere tradita sit idea non solum naturae quaestionum, quibus curvae maximi minimive proprietate praeditae quaeruntur, sed etiam methodi, qua ad eas resolvendas uti oporteat, ad ipsam tractationem progrediemur. Ac primo quidem Methodum absolutam, qua curvae quaeruntur, quae inter omnes omnino curvas ad eandem abscissam relatas maximi minimive proprietate quapiam sint praeditae, trademus. Deinde pergemus ad Methodum maximorum ac :minimorum relativam, ad quam tales pertinent quaestiones, quae non inter omnes curvas datae abscissae respondentes, sed eas tantum, quae data quadam communi proprietate una pluribusve gaudent, eam determinari iubent, cui maximi minimive praerogativa quaepiam conveniat. In has autem tractationes natura formulae $\int Zdx$, quae maximum minimumve esse debet, ingens discrimen infert, prout Z fuerit functio vel determinata vel indeterminata, quemadmodum iam observavimus.