ADDITION II

CONCERNING THE MOTION OF PROJECTILES IN A NON-RESISTING MEDIUM, BEING DETERMINED BY THE METHOD OF MAXIMA AND MINIMA

1. Because all natural effects follow a certain law of maximum or minimum, there is no reason to doubt in the curved lines which projected bodies may describe, if they may be acted on by some forces, why some property of maximum or minimum may not have a place. But what this property shall be in the first place, may not appear so easy to define from the principles of metaphysics; but since these curves may themselves be allowed to be determined with the help of the direct method; hence attention must be paid to that method itself, so that it can be concluded, what maximum or minimum value is present on these curves themselves. Moreover the effect arising chiefly from the forces acting must be considered ; this effect must be a minimum agreeing with the true motion itself, or rather the sum of all the motions which are present in the projected body. Which conclusion, even if it may not seem to be confirmed very well, if again I have shown that to be in agreement the true motion noted from the start, will follow with so much weight, so that all doubts, which may have arisen may vanish completely. So that, even if the truth of this effect were overturned, it would be easier to inquire into the very heart of the matter and with their final causes, and this assertion to be corroborated by the most firm accounts.

2. The mass of this body projected shall be $=M$, while the small interval $=ds$ is passed through, the speed due to the height *v*, the quantity of motion of the body at this place will be $= M\sqrt{v}$;
[Euler's understanding of the equations of motion under the influence of gravity and the conservation of what we now call energy was a little different then from the modern situation; here Euler, following Galileo, relates the square of the speed of a body *V* to the height *v* it has fallen from rest under gravity, and some quantity to be called the *vis viva* is taken to be conserved : often in the *Mechanica* Euler took the acceleration of gravity to be of such a size that $V^2 = v$, where *v* is the distance fallen, so that at any instant $V = \sqrt{v}$. Part of the unresolved problem at the time was the belief in the *vis viva* introduced by *Leibniz* , which essentially was the kinetic energy of the body, regarded as a point mass, but without the factor of a half, which at the time gave the correct answers for moving point masses under certain conditions : thus, this was a time of incomplete experimental evidence. This quantity, incidentally, does not receive a mention by Newton in the *Principia,* who concerned himself with the time development of physical systems only. In turn, we have essentially Newton's quantity of motion in the formula $M\sqrt{v}$. Goldstine in his book (p.103) here adroitly slips a factor of a half into Euler's equation $V^2 = v$ to make it become $\frac{1}{2}V^2 = v$, which agrees with modern theory if *g* cancels in some ratio; but this is not what Euler said or did, and so is misleading, to say the least. A simple way of getting the correct answers from the *vis visa* difficulty is simply to double the acceleration of gravity *g* at the end of the calculation, if *g* is present, and if *v* above or *a* is given as below, this becomes 2*gv* or 2*ga*, for then the factor of a half is accommodated

and g is accounted for; all of this applying to the case of the earth's uniform gravitational field.]

which multiplied by the small interval ds itself will give the quantity of motion $Mds\sqrt{v}$ of the body acquired through the interval ds. Now I say the line described by the body to be prepared thus, so that among all the other lines contained between the same limits $\int Mds\sqrt{v}$ shall be a minimum or, on account of M being constant, $\int ds\sqrt{v}$ shall be a minimum. So that if the curve may be sought, just as it may be given [by the direct method], it will be considered with the forces acting, the speed $\sqrt{v}$ to be defined by quantities pertaining to the curve and thus the curve itself can be determined by the method of maxima and minima. Moreover this expression will be determined equally by one quantity desired from the quantity of motion, and by another from the *vires vivas* [*i.e.* Leibniz's '*living forces*' ; essentially, from the conservation both of momentum and of the poorly understood concept of energy applied to the body]; for on putting in place a very small time *dt*, in which the element *ds* is run through, because there becomes $ds = dt\sqrt{v}$, the integral becomes $\int ds\sqrt{v} = \int vdt$, so that on the curve described by the projected body the sum of all the *living forces* shall be a minimum, which are present at the individual instants of time. On account of which neither these individuals, who will discover the forces from the speeds themselves, nor those who find the forces by reckoning the squares of the speed required to be put in place, will find anything here that they need be concerned about.
[Thus on this occasion Euler makes a rare sarcastic comment about his contemporaries, for it was a contentious issue at the time.]

3. Therefore at first, if we may consider the body to be acted on absolutely by no forces, also its speed, to which I attend here only (for the method of maxima and minima will include the direction), will experience no change ; and thus v will be a constant quantity, for example, $= b$. Hence the body being disturbed by no forces, if it may be projected in some way, will describe a line of this kind, in which $\int ds\sqrt{b}$ or $\int ds = s$ shall be a minimum. Therefore this way contained among all the ways with the same end points will be the minimum itself, just as the first principles of mechanics postulate. Indeed I bring forwards this case here only this far, so that I can consider my principle to be confirmed ; for I might have assumed some other function of v in place of $\sqrt{v}$, it would have produced the same right line path ; truly by beginning from the most simple cases, it will be able to agree more easily about the method itself.

4. Therefore I progress to the case of uniform gravity or, in which the body projected anywhere (Fig. 26) may be acted on by a force downwards along directions normal to the horizontal directions by a force with a constant acceleration $= g$. *AM* shall be the curve, that the body will describe in this hypothesis, the vertical right line *AP* may be taken for the axis and the abscissa may be put $AP = x$, the applied

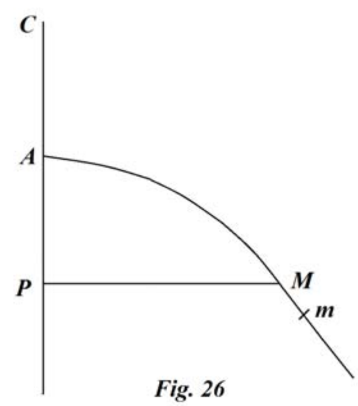

Fig. 26

line $PM = y$ and an element of the curve $Mm = ds$; therefore from the nature of the force acting there will be $dv = gdx$ and $v = a + gx$.
[The first equation contains the term *gdx*, which can be understood as the change in potential energy per unit mass, or the work done, and this must be equated to the change *dv* in the kinetic energy, and so includes the factor 2 in some manner ; thus the equations are physically incorrect by some constant factors that can be corrected as above.]

Hence the nature of the curve will be prepared thus, so that on that curve

$$\int ds\sqrt{(a+gx)}$$

shall be a minimum. Putting $dy = pdx$, so that there shall be $ds = dx\sqrt{(1+pp)}$, and

$$\int dx\sqrt{(a+gx)(1+pp)} ;$$

must be a minimum ; which expression compared with the general form $\int Zdx$ gives

$$Z = \sqrt{(a+gx)(1+pp)} ;$$

whereby, since there shall be put $dZ = Mdx + Ndy + Pdp$, there will be

$$N = 0 \text{ and } P = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}}.$$

Therefore because the value of the differential is $N - \frac{dP}{dx}$, on account of $N = 0$ it becomes in the present case $dP = 0$ and $P = \sqrt{C}$. Therefore there will be found :

$$\sqrt{C} = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}} = \frac{dy\sqrt{(a+gx)}}{ds},$$

from which it becomes :

$$Cdx^2 + Cdy^2 = dy^2(a+gx) \text{ and } dy = \frac{dx\sqrt{C}}{\sqrt{(a-C+gx)}}$$

which integrated gives

$$y = \frac{2}{g}\sqrt{C(a-C+gx)}.$$

5. Evidently this equation indeed is for the parabola. But it will help its agreement with the truth to be considered more attentively. Therefore it is apparent in the first place the

tangent of this curve to be horizontal or $dx = 0$, where $a - C + gx = 0$. Therefore since the beginning of the abscissa *A* depends on our choice, it may be taken in this place itself, and there becomes $C = a$; then truly the axis itself may pass through the maximum point of the curve, thus so that on putting $x = 0$ likewise there becomes $y = 0$. From these considerations the equation for the curve will become this :

$$y = 2\sqrt{\frac{ax}{g}} \quad \left[\to 2\sqrt{\frac{2gax}{2g}}\text{ ; or } y^2 = 4ax,\text{ on correcting as suggested above.} \right];$$

that not only appears to be the equation for a parabola, but also, since the speed at the point *A* shall be $\sqrt{a}$, the height *CA*, from which a body by slipping with the same force *g* acting will acquire that same speed, with which it will be moving at the point *A*, will be $= \frac{a}{g}$, that is, it is equal to the fourth part of the parameter; accordingly as it is understood from the theory of projectile motion by the direct method.

[Here Euler equates the initial and final *vis viva* : $0^2 + g\frac{a}{g} = V^2$; also, the equation for the parabola has become $y^2 = 4\frac{a}{g}x$, for which the parameter is $\frac{4a}{g}$. As shown above, this becomes $y^2 = 4ax$]

6. As before the body may be acted on everywhere vertically downwards, but the force itself acting shall not be constant, but may depend in some way on the height *CP*. Clearly by putting the abscissa $CP = x$ the force shall be $= X$, by which the body at *M* is pressed downwards, equal to some function of *x*. Therefore if the applied line may be called $PM = y$, the element of the arc $Mm = ds$ and $dy = pdx$, there will be $dv = Xdx$ et $v = A + \int Xdx$; from which this expression $\int dx\sqrt{\left(A + \int Xdx\right)(1+pp)}$ must be a minimum, from which this expression will be found for the curve described *AM*

$$\sqrt{C} = \frac{p\sqrt{\left(A + \int Xdx\right)}}{\sqrt{(1+pp)}}$$

and

$$p = \frac{\sqrt{C}}{\sqrt{\left(A - C + \int Xdx\right)}} = \frac{dy}{dx}$$

or

$$y = \int \frac{dx\sqrt{C}}{\sqrt{\left(A - C + \int Xdx\right)}}.$$

Therefore the tangent to the curve will be horizontal, where $\int Xdx = C - A$. Truly this same trajectory of the body will be found by the direct method.

7. Now the body at M may be acted on (Fig. 27) by two forces, the one to the horizontal $= Y$ along the direction MP, the other $= X$ to the vertical along the direction MQ. Moreover X shall be some function of the right vertical $MQ = CP = x$ and Y some function of the applied line $PM = y$. Therefore on putting as before $dy = pdx$, there will be

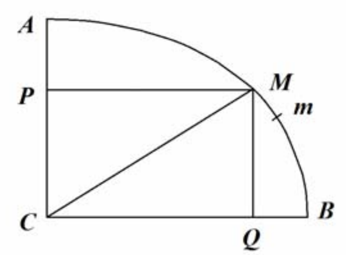

Fig. 27

$$dv = -Xdx - Ydy;$$

[Thus, we are to consider resistive forces acting on the body moving with some initial speed.];
and there becomes

$$v = A - \int Xdx - \int Ydy;$$

from which this formula must be a minimum :

$$\int dx\sqrt{(1+pp)\left(A - \int Xdx - \int Ydy\right)}.$$

$$\sqrt{(1+pp)\left(A - \int Xdx - \int Ydy\right)}$$

may be differentiated, and there will be produced :

$$-\frac{Xdx\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} - \frac{Ydy\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} + \frac{pdp\sqrt{\left(A - \int Xdx - \int Ydy\right)}}{\sqrt{(1+pp)}}$$

Hence on putting

$$N = -\frac{Ydy\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} \quad \text{and} \quad P = \frac{p\sqrt{\left(A - \int Xdx - \int Ydy\right)}}{\sqrt{(1+pp)}}$$

this will be the equation for the curve sought :

$$0 = N - \frac{dP}{dx} \text{ or } Ndx = dP.$$

Hence therefore the equation becomes :

$$-\frac{Ydx\sqrt{(1+pp)}}{2\sqrt{\left(A-\int Xdx-\int Ydy\right)}} = \frac{dp\sqrt{\left(A-\int Xdx-\int Ydy\right)}}{(1+pp)\sqrt{(1+pp)}} - \frac{pXdp+pYdy}{2\sqrt{(1+pp)\left(A-\int Xdx-\int Ydy\right)}}$$

or

$$\frac{pdp\sqrt{\left(A-\int Xdx-\int Ydy\right)}}{(1+pp)\sqrt{(1+pp)}} = \frac{Xdx-Ydy}{2\sqrt{(1+pp)\left(A-\int Xdx-\int Ydy\right)}}$$

and thus

$$\frac{2dp}{1+pp} = \frac{Xdx-Ydy}{A-\int Xdx-\int Ydy}$$

This equation will be shown to be true, if in place of $A-\int Xdx-\int Ydy$ there may be put v; for there will be

$$\frac{2vdp}{(1+pp)^{3:2}} = \frac{Xdx-Ydy}{\sqrt{(1+pp)}}.$$

But the radius of osculation is

$$r = -\frac{(1+pp)^{3:2}\,dx}{dp},$$

with which introduced there is

$$\frac{2v}{r} = \frac{Xdx-Ydy}{ds},$$

where $\frac{2v}{r}\left[\rightarrow \frac{2gv}{r} = \frac{V^2}{r},\right]$ is the centrifugal force, and $\frac{Xdx-Ydy}{ds}$ expresses the normal force arising from the forces acting ; the equality of which forces has a place everywhere in all projectile motion.

8. But the equation found

$$\frac{2dp}{1+pp} = \frac{Xdx-Ydy}{A-\int Xdx-\int Ydy}$$

thus generally is integrable, if it may be multiplied by

$$\frac{p\left(A-\int Xdx-\int Ydy\right)}{1+pp},$$

for it becomes

$$\frac{2pdp\left(A-\int Xdx-\int Ydy\right)}{\left(1+pp\right)^2}-\frac{ppXdx-Ydy}{1+pp}=0,$$

which integrated gives

$$\frac{-p^2\int Xdx+\int Ydy-A}{1+pp}=C,$$

or

$$\int Ydy-p^2\int Xdx=A+C+Cpp,$$

from which

$$p=\frac{\sqrt{\left(B+\int Ydy\right)}}{\sqrt{C+\int Xdx}},$$

on putting B for $-A-C$
Therefore since there shall be $p=dx$, there becomes

$$\int\frac{dy}{\sqrt{\left(B+\int Ydy\right)}}=\int\frac{dx}{\sqrt{C+\int Xdx}}$$

the equation for the curve sought, in which the variables *x* and *y* are separated from each other in turn. Or if the constants *B* and *C* may be changed to negative, there will be

$$\int\frac{dy}{\sqrt{\left(B-\int Ydy\right)}}=\int\frac{dx}{\sqrt{C-\int Xdx}}.$$

From which, although the construction of the curve will be had more easily, yet the algebraic equations, however many times they may be contained in these, will not be elicited so easily. *X* and *Y* may be similar functions and indeed powers of *x* and *y*, thus so that there shall be

$$\int\frac{dy}{\sqrt{\left(b^n-y^n\right)}}=\int\frac{dx}{\sqrt{a^n-x^n}},$$

which equation, if $n=1$, gives a parabola, if $n=2$, an ellipse having centre at *C*, and yet in this case each integration requires the quadrature of the circle. Therefore is seen to be plausible also in the other cases, in which neither integration succeeds satisfying algebraic curves ; but for finding which at this stage another method shall be desired.

9. The body *M* may be urged always towards a fixed point along the direction *MC* by a force, which shall be as some function of the distance *MC*. On putting as before $CP = x,\ PM = y$ and $dy = pdx$, there shall be $CM = \sqrt{(x^2+y^2)} = t$, and *T* shall be that function of *t*, which expresses the centripetal force. This force may be resolved into parts along *MQ* and *MP*, the pulling force along MQ shall be $= \frac{Tx}{t}$ and the force along $MP = \frac{Ty}{t}$; from which the acceleration arises

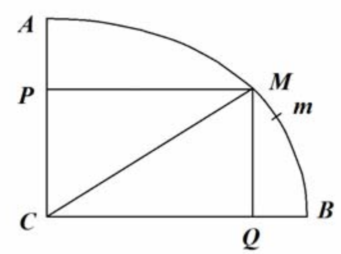

Fig. 27

$$dv = -\frac{Txdx}{t} - \frac{Tydy}{t} = -Tdt,$$

on account of $xdx + ydy = tdt$; from which there becomes

$$v = A - \int Tdt\ .$$

On account of which this expression must be a minimum

$$\int dx\sqrt{(1+pp)\left(A-\int Tdt\right)}.$$

Now following the precept of the rule the quantity

$$\sqrt{(1+pp)\left(A-\int Tdt\right)}$$

may be differentiated, and it will give rise to

$$-\frac{Tdt\sqrt{(1+pp)}}{2\sqrt{\left(A-\int Tdt\right)}} + \frac{pdp\sqrt{\left(A-\int Tdt\right)}}{\sqrt{(1+pp)}}.$$

On account of $dt = \frac{xdx+ydy}{t}$ there will be therefore :

$$N = -\frac{Ty\sqrt{(1+pp)}}{2t\sqrt{\left(A-\int Tdt\right)}} \quad \text{and} \quad P = \frac{p\sqrt{\left(A-\int Tdt\right)}}{\sqrt{(1+pp)}}\ ;$$

from which the equation $Ndx = dP$ is produced for the curve, which gives

$$-\frac{Tydx\sqrt{(1+pp)}}{2t\sqrt{\left(A-\int Tdt\right)}}=\frac{dp\sqrt{\left(A-\int Tdt\right)}}{(1+pp)\sqrt{(1+pp)}}-\frac{pTdt}{2\sqrt{(1+pp)\left(A-\int Tdt\right)}}$$

and this will be reduced into that :

$$\frac{T\left(xdy-ydx\right)}{2t\left(A-\int Tdt\right)}=\frac{dp}{1+pp}.$$

10. Although this equation may contain four different letters, yet with skill it can be integrated. For since there shall be

$$ydy+xdx=tdt= pydx+xdx,$$

there will be

$$dx=\frac{tdt}{x+py} \text{ and } dy=\frac{ptdt}{x+py}.$$

which values substituted into the equation will give

$$\frac{\left(px-y\right)Tdt}{2\left(x+py\right)\left(A-\int Tdt\right)}=\frac{dp}{1+pp}$$

or

$$\frac{Tdt}{2\left(A-\int Tdt\right)}=\frac{dp\left(x+py\right)}{(1+pp)\left(px-y\right)}.$$

Each of these expressions is integrable by logarithms, indeed there is

$$\int\frac{Tdt}{2\left(A-\int Tdt\right)}=-\tfrac{1}{2}l\left(A-\int Tdt\right)$$

and

$$\int\frac{dp\left(x+py\right)}{(1+pp)\left(px-y\right)}$$

is resolved into

$$\int \frac{xdp}{px-y} - \int \frac{pdp}{1+pp} = l\frac{px-y}{\sqrt{(1+pp)}};$$

thus so that there shall be

$$\frac{C}{\sqrt{\left(A-\int Tdt\right)}} = \frac{px-y}{\sqrt{(1+pp)}};$$

from which equation the speed of the body at *M* can be indicated, which is $\sqrt{\left(A-\int Tdt\right)}$, to be inversely as the perpendicular from *C* sent to the tangent ; which is the main property of all [stable central] motions.

[This will become on amending the erroneous *vis viva* formula : $\sqrt{\left(\frac{1}{2}V_o^2-\int Tdt\right)}$.]

11. Truly this same problem can be resolved more conveniently by assuming another variable for the right line *CM*. In truth the method treated above does not demand, that both the variables shall be orthogonal coordinates, provided they shall be two quantities of the same kind, with which determined likewise a point of the curve may be determined. For this reason it will not be allowed to accept the distance *CM* with the perpendicular send from *C* to the tangent for these two variables, because, even if both the distance from the centre and the perpendicular to the tangent may be given, hence the position of a point on the curve is not defined. But nothing prevents the distance *CM* (Fig. 28) and the arc of the circle *BP* described with the centre *C* being substituted in place of the two variables, because with the arc *BP* and the distance *CM* given the point *M* of the curve is determined equally, and by orthogonal coordinates. Therefore with this annotation the use of the method is extended much wider, as may be able to be seen in general.

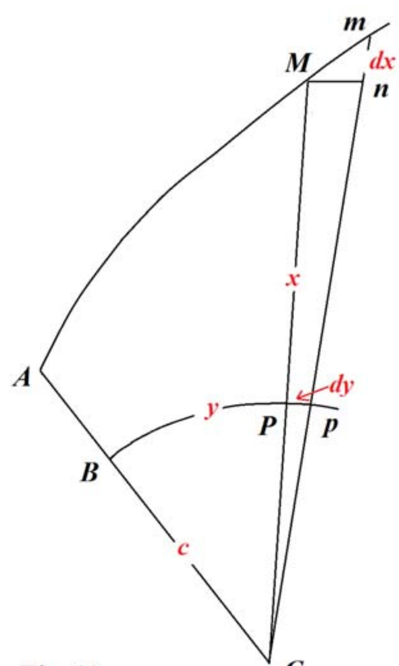

Fig. 28

12. Therefore let the distance of the body from the centre $MC = x$ and the force, by which the body shall be acted on towards *C*, shall be $= X$, some function of *x*. From the centre *C*, with a radius taken as it pleases $BC = c$, a circle is described, of which the arc *BP* may represent the other variable *y*, thus so that there shall be $Pp = dy = pdx$. But from the force acting there is $dv = -Xdx$, from which $v = A - \int Xdx$. With centre *C*, radius $CM = x$, the arc increment may be described *Mn*, there will be

$$mn = dx \text{ and } CP : Pp = CM : Mn,$$

from which there becomes

$$Mn = \frac{pxdx}{c}$$

and the element of the distance

$$Mm = dx\sqrt{\left(1 + \frac{ppxx}{cc}\right)}.$$

On account of which this formula must be the minimum :

$$\int dx\sqrt{\left(A - \int Xdx\right)\left(1 + \frac{ppxx}{cc}\right)},$$

from which the value of the differential arises [Prop. V; see also p.95 Goldstine]:

$$\frac{1}{dx} d \cdot \frac{pxx\sqrt{\left(A - \int Xdx\right)}}{c\sqrt{\left(cc + ppxx\right)}},$$

which, by the rule put equal to zero, will give this equation:

$$\sqrt{C} = \frac{pxx\sqrt{\left(A - \int Xdx\right)}}{c\sqrt{\left(cc + ppxx\right)}}$$

or

$$Cc^4 + Cccppxx = \left(A - \int Xdx\right) ppx^4,$$

from which there becomes

$$p = \frac{cc\sqrt{C}}{\sqrt{\left(\left(A - \int Xdx\right)x^4 - Cccxx\right)}} = \frac{cc\sqrt{C}}{x\sqrt{\left(\left(A - \int Xdx\right)xx - Ccc\right)}}$$

or

$$dy = \frac{ccdx\sqrt{C}}{x\sqrt{\left(\left(A - \int Xdx\right)xx - Ccc\right)}},$$

which same equation also can be found by the direct method.

13. Therefore from these cases the most perfect agreement with the truth of the principles here established has shown forth; but whether this agreement itself in more complicated cases also shall have a place, doubt can remain. On account of which, as this same principle may become widely extended, it will require to be investigated more carefully, so that more cannot be attributed to it than its nature permits. According to this explanation all motions of projectiles must be assigned to two kinds, of which with one the speed of the body, which it may have in some place, may depend only on the position ; thus so that, it if may return to the same position, also it shall have regained the same speed; which arises, if the body may be drawn by forces either towards one or several fixed centres, which shall be as some functions of the distances from these centres. I refer these motions of projectiles to the other kind, in which the speed of the body may not be determined by the place alone, in which it finds itself moving; that which usually comes about either, if that centre about which the body may be acted on were fixed or, if the motion were made in a resisting medium. With this division made it is to be noted, whenever the motion of the body may belong to the first kind, that is, if the body may be acted on towards not only one but to some number of fixed centres of force, so also in the motion this sum of all the motions of the elements becomes a minimum.

14. Moreover this itself postulates the nature of the proposition ; for while that curve may be sought between two given limits, in which $\int ds\sqrt{v}$ shall be a minimum, from that itself the speed of the body at each end is assumed to be the same, whatever curve may constitute the path of the body. But however many fixed centres of force there were, the speed of the body at some location *M* (Fig. 27) is expressed by a determined function of all the variables $CP = x$ and $PM = y$. Therefore *v* shall be some function of *x* and *y*, thus so that there shall be $dv = Tdx + Vdy$, and we may consider whether or not our principle shall be showing the true trajectory of the body. But since there shall be $dv = Tdx + Vdy$, hence the body will be moved, or if it may be acted on at *M* by two forces, the one *T* in the direction parallel to the abscissa *x*, truly the other *V* in the direction parallel to the applied line *y*, from which arises both the tangential force $= \frac{Tdx + Vdy}{ds}$ and the normal force $= \frac{-Vdx + Tdy}{ds}$.
But it must follow, from the free nature the motion to be

$$\frac{2v}{r} = \frac{-Vdx + Tdy}{ds} = \frac{-V + Tp}{\sqrt{(1 + pp)}};$$

if the method of maxima and minima may lead to that equation, it will confirm the truth of our principle everywhere.

15. Therefore since by this principle $\int dx\sqrt{v(1 + pp)}$ must be a minimum,

the quantity $\sqrt{v(1+pp)}$ may be differentiated, and on account of $dv = Tdx + Vdy$ there may arise:

$$\frac{Tdx\sqrt{(1+pp)}}{2\sqrt{v}} + \frac{Vdy\sqrt{(1+pp)}}{2\sqrt{v}} + \frac{pdp\sqrt{v}}{\sqrt{(1+pp)}},$$

from which the following equation may be obtained for the curve sought following the precepts treated

$$\frac{Vdx\sqrt{(1+pp)}}{2\sqrt{v}} = d\cdot\frac{p\sqrt{v}}{\sqrt{(1+pp)}} = \frac{dp\sqrt{v}}{(1+pp)^{3:2}} + \frac{p(Tdx+Vdy)}{2\sqrt{v(1+pp)}}$$

or

$$-\frac{dp\sqrt{v}}{(1+pp)^{3:2}} = \frac{Tpdx - Vdx}{2\sqrt{v(1+pp)}}.$$

But the radius of osculation at *M*

$$= -\frac{(1+pp)dx\sqrt{(1+pp)}}{dp};$$

which if it may be put $= r$, there will be

$$\frac{2v}{r} = \frac{Tp - V}{\sqrt{(1+pp)}},$$

as is found entirely by using the direct method. Therefore provided the forces acting were prepared, so that these may be able to be reduced to two forces *T* and *V* acting along the directions parallel to the coordinates *x* and *y*, which shall be as some functions of these variables *x* and *y*, then always the curve described will be the minimum motion of the body through all the elements gathered together.

16. Therefore although this principle extends widely, so that only the motion from a resisting medium may seem to be excepted ; for the reason of this exception is readily seen, because in that case the body arriving at the same place by different paths therefore will not acquire the same speed. On account of which, with all resistance removed in the motion of projected bodies, this constant property always will have a place, so that the sum of all the motions of the elements shall be a minimum. Nor truly is this property discerned in the motion of a single body only, but also in the motion of several bodies jointly, which in whatever manner they may act amongst themselves, yet the sum of all the motions always is a minimum. So that, since a motion of this kind may be more difficult to refer to calculation, it will be understood more easily from first principles, as from the agreement of the calculation following each method put in place. Because

indeed bodies on account of inertia all may be reluctant to change their states by forces acting as there will be little compliance, as can happen, if indeed they shall be free ; from which it is brought about, that in the motion generated the effect arising must be less, than if the body or bodies were to be moved in any other way. The strength of this reasoning, even if it may not be understood well enough, yet, because it agrees with the truth, I do not doubt, why with the aid of sounder metaphysical principles it may not be able to be raised to become more evident; which matter I leave to others, who profess metaphysics.

ADDITAMENTUM II

DE MOTU PROIECTORUM IN MEDIO NON RESISTENTE, PER METHODUM MAXIMORUM AC MINIMORUM DETERMINANDO

1. Quoniam omnes naturae effectus sequuntur quandam maximi minimive legem, dubium est nullum, quin in lineis curvis, quas corpora proiecta, si a viribus quibuscunque sollicitentur, describunt, quaepiam maximi minimive proprietas locum habeat. Quaenam autem sit ista proprietas, ex principiis metaphysicis a priori definire non tam facile videtur; cum autem has ipsas curvas ope Methodi directae determinare liceat; hinc debita adhibita attentione id ipsum, quod in istis curvis est maximum vel minimum, concludi poterit. Spectari autem potissimum debet effectus a viribus sollicitantibus oriundus; qui cum in motu corporis genito consistat, veritati consentaneum videtur hunc ipsum motum seu potius aggregatum omnium motuum, qui in corpore proiecto insunt, minimum esse debere. Quae conclusio etsi non satis confirmata videatur, tamen, si eam cum veritate iam a priori nota consentire ostendero, tantum consequetur pondus, ut omnia dubia, quae circa eam suboriri queant, penitus evanescant. Quin-etiam, cum eius veritas fuerit evicta, facilius erit in intimas Naturae leges atque causas finales inquirere hocque assertum firmissimis rationibus corroborare.

2. Sit massa corporis proiecti $=M$ eiusque, dum spatiolum $=ds$ emetitur, celeritas debita altitudini v, erit quantitas motus corporis in hoc loco $=M\sqrt{v}$; quae per ipsum spatiolum ds multiplicata dabit $Mds\sqrt{v}$ motum corporis collectivum per spatiolum ds. Iam dico lineam a corpore descriptam ita fore comparatam, ut inter omnes alias lineas iisdem terminis contentas sit $\int Mds\sqrt{v}$ seu, ob M constans, $\int ds\sqrt{v}$ minimum. Quodsi autem curva quaesita, tanquam esset data, spectetur, ex viribus sollicitantibus celeritas $\sqrt{v}$ per quantitates ad curvam pertinentes definiri ideoque ipsa curva per Methodum maximorum ac minimorum determinari potest. Ceterum haec expressio ex quantitate motus petita aeque ad vires vivas traduci poterit; posito enim tempusculo, quo elementum ds percurritur, dt, quia est $ds=dt\sqrt{v}$, fiet $\int ds\sqrt{V}v=\int vdt$, ita ut in curva a corpore proiecto descripta summa omnium virium vivarum, quae singulis temporis momentis corporis insunt, sit minima. Quamobrem neque ii, qui vires per ipsas celeritates, neque illi, qui per celeritatum quadrata aestimari oportere statuunt, hic quicquam, quo offendantur, reperient.

3. Primum igitur, si corpus a nullis prorsus viribus sollicitari ponamus, eius quoque celeritas, ad quam hic solum attendo (directionem enim ipsa Methodus maximorum et minimorum complectetur), nullam patietur alterationem; eritque ideo v quantitas constans, puta $=b$. Hinc corpus a nullis viribus sollicitatum, si utcunque proiiciatur,

eiusmodi describet lineam, in qua sit $\int ds\sqrt{b}$ vel $\int ds = s$ minimum. Via ergo haec inter omnes iisdem terminis contentas ipsa erit minima atque adeo recta, prorsus uti prima Mechanicae principia postulant. Hunc quidem casum non adeo hic affero, quo principium meum confirmari putem; quamcunque enim loco celeritatis $\sqrt{v}$ aliam assumsissem functionem ipsius *v*, eadem prodiisset via recta; verum a casibus simplicissimis incipiendo facilius ipsa consensus ratio intelligi poterit.

4. Progredior ergo ad casum gravitatis uniformis seu, quo corpus proiectum ubique (Fig. 26) secundum directiones ad horizontem normales deorsum sollicitetur a vi constante acceleratrice $= g$. Sit *AM* curva, quam corpus in hac hypothesi describit, sumatur recta verticalis *AP* pro axe ac ponatur abscissa $AP = x$, applicata $PM = y$ et elementum curvae $Mm = ds$; erit ergo ex natura sollicitationis $dv = gdx$ et $v = a + gx$. Hinc curva ita erit comparata, ut in ea sit

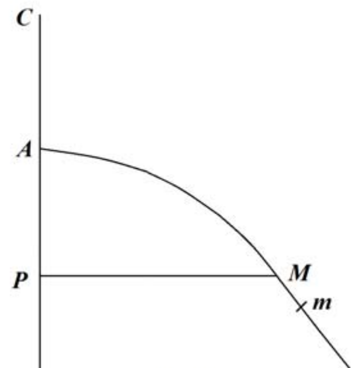

Fig. 26

$$\int ds\sqrt{(a+gx)}$$

minimum. Ponatur $dy = pdx$, ut sit $ds = dx\sqrt{(1+pp)}$, atque minimum esse debet

$$\int dx\sqrt{(a+gx)(1+pp)}\ ;$$

quae expressio cum forma generali $\int Zdx$ comparata dat

$$Z = \sqrt{(a+gx)(1+pp)}\ ;$$

quare, cum positum sit $dZ = Mdx + Ndy + Pdp$, erit

$$N = 0 \ \text{ et } \ P = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}}.$$

Quia ergo valor differentialis est $N - \frac{dP}{dx}$, ob $N = 0$ fiet praesenti casu $dP = 0$ et $P = \sqrt{\mathrm{C}}$. Habebitur ergo

$$\sqrt{C} = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}} = \frac{dy\sqrt{(a+gx)}}{ds},$$

unde fit

$$Cdx^2 + Cdy^2 = dy^2(a + gx) \text{ et } dy = \frac{dx\sqrt{C}}{\sqrt{(a - C + gx)}}$$

quae integrata dat

$$y = \frac{2}{g}\sqrt{C(a - C + gx)}.$$

5. Manifestum quidem est hanc aequationem esse pro Parabola. At eius consensum cum veritate attentius considerasse iuvabit. Primum ergo patet tangentem huius curvae esse horizontalem seu $dx = 0$; ubi est $a - C + gx = 0$. Cum igitur principium abscissarum *A* ab arbitrio nostro pendeat, sumatur id in hoc ipso loco, fietque $C = a$; tum vero ipse axis per hoc punctum curvae summum transeat, ita ut posito $x = 0$ fiat simul $y = 0$. His consideratis aequatio pro curva erit haec $y = 2\sqrt{\frac{ax}{g}}$; quam non solum patet esse pro Parabola, sed etiam, cum celeritas in puncto *A* sit $\sqrt{a}$, altitudo *CA*, ex qua corpus labendo ab eadem vi *g* sollicitatum eam ipsam acquirit celeritatem, qua in puncto *A* movetur, erit $= \frac{a}{g}$, hoc est, quartae parametri parti aequatur; prorsus uti ex doctrina motus proiectorum per Methodum directam intelligitur.

6. Sollicitetur ut ante corpus ubique verticaliter deorsum, at ipsa vis sollicitans non sit constans, sed pendeat utcunque ab altitudine *CP*. Scilicet posita abscissa $CP = x$ sit vis, qua corpus in *M* deorsum nititur, $= X$ functioni cuicunque ipsius *x*. Si ergo vocetur applicata $PM = y$, elementum arcus $Mm = ds$ et $dy = pdx$, erit $dv = Xdx$ et $v = A + \int Xdx$; unde minimum esse debet haec expressio $\int dx\sqrt{\left(A + \int Xdx\right)(1+pp)}$, ex qua pro curva descripta *AM* obtinebitur haec aequatio

$$\sqrt{C} = \frac{p\sqrt{\left(A + \int Xdx\right)}}{\sqrt{(1 + pp)}}$$

et

$$p = \frac{\sqrt{C}}{\sqrt{\left(A - C + \int Xdx\right)}} = \frac{dy}{dx}$$

seu

$$y = \int \frac{dx\sqrt{C}}{\sqrt{\left(A - C + \int Xdx\right)}}.$$

Tangens ergo curvae erit horizontalis, ubi $\int Xdx = C - A$. Haec vero eadem aequatio traiectoriae corporis per Methodum directam reperitur.

7. Sollicitetur nunc corpus (Fig. 27) in M a duabus viribus, altera horizontali $= Y$ secundum directionem MP, altera verticali $= X$ secundum directionem MQ. Sit autem X functio quaecunque rectae verticalis $MQ = CP = x$ et Y functio quaecunque applicatae $PM = y$. Positis ergo ut ante $dy = pdx$, erit

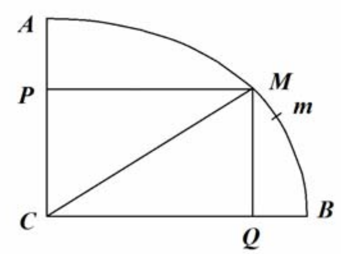

Fig. 27

$$dv = -Xdx - Ydy$$

fietque

$$v = A - \int Xdx - \int Ydy\,;$$

unde minimum esse debet haec formula

$$\int dx\sqrt{(1+pp)\left(A - \int Xdx - \int Ydy\right)}..$$

Differentietur

$$\sqrt{(1+pp)\left(A - \int Xdx - \int Ydy\right)}$$

atque prodibit

$$-\frac{Xdx\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} - \frac{Ydy\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} + \frac{pdp\sqrt{\left(A - \int Xdx - \int Ydy\right)}}{\sqrt{(1+pp)}}$$

Hinc posito

$$N = -\frac{Ydy\sqrt{(1+pp)}}{2\sqrt{\left(A - \int Xdx - \int Ydy\right)}} \quad \text{et} \quad P = \frac{p\sqrt{\left(A - \int Xdx - \int Ydy\right)}}{\sqrt{(1+pp)}}$$

erit pro curva quaesita haec aequatio

$$0 = N - \frac{dP}{dx} \text{ seu } Ndx = dP.$$

Hinc ergo fit

$$-\frac{Ydx\sqrt{(1+pp)}}{2\sqrt{\left(A-\int Xdx-\int Ydy\right)}}=\frac{dp\sqrt{\left(A-\int Xdx-\int Ydy\right)}}{(1+pp)\sqrt{(1+pp)}}-\frac{pXdp+pYdy}{2\sqrt{(1+pp)\left(A-\int Xdx-\int Ydy\right)}}$$

seu

$$\frac{pdp\sqrt{\left(A-\int Xdx-\int Ydy\right)}}{(1+pp)\sqrt{(1+pp)}}=\frac{Xdx-Ydy}{2\sqrt{(1+pp)\left(A-\int Xdx-\int Ydy\right)}}$$

ideoque

$$\frac{2dp}{1+pp}=\frac{Xdx-Ydy}{A-\int Xdx-\int Ydy}$$

Hanc aequationem veritati esse consentaneam patebit, si loco $A-\int Xdx-\int Ydy$ ponatur v; erit enim

$$\frac{2vdp}{(1+pp)^{3:2}}=\frac{Xdx-Ydy}{\sqrt{(1+pp)}}.$$

At est radius osculi

$$r=-\frac{(1+pp)^{3:2}dx}{dp}$$

quo introducto est

$$\frac{2v}{r}=\frac{Xdx-Ydy}{ds},$$

ubi est $\frac{2v}{r}$ vis corporis centrifuga et $\frac{Xdx-Ydy}{ds}$, exprimit vim normalem ex viribus sollicitantibus ortam; quarum virium aequalitas utique in omni motu proiectorum locum habet.

8. Aequatio autem inventa

$$\frac{2dp}{1+pp}=\frac{Xdx-Ydy}{A-\int Xdx-\int Ydy}$$

ita generaliter est integrabilis, si multiplicetur per

$$\frac{p\left(A-\int Xdx-\int Ydy\right)}{1+pp},$$

fiet enim

$$\frac{2pdp\left(A-\int Xdx-\int Ydy\right)}{(1+pp)^2}-\frac{ppXdx-Ydy}{1+pp}=0,$$

quae integrata dat

$$\frac{-p^2\int Xdx+\int Ydy-A}{1+pp}=C,$$

seu

$$\int Ydy-p^2\int Xdx=A+C+Cpp,$$

unde

$$p=\frac{\sqrt{\left(B+\int Ydy\right)}}{\sqrt{C+\int Xdx}}$$

posito B pro $-A-C$
Cum ergo sit $p=dx$, erit

$$\int\frac{dy}{\sqrt{\left(B+\int Ydy\right)}}=\int\frac{dx}{\sqrt{C+\int Xdx}}$$

aequatio pro curva quaesita, in qua variabiles x et y sunt a se invicem separatae.
Vel si constantes B et C in negativas convertantur, erit

$$\int\frac{dy}{\sqrt{\left(B-\int Ydy\right)}}=\int\frac{dx}{\sqrt{C-\int Xdx}}.$$

Ex quibus, etsi curvae constructio facilis habetur, tamen aequationes algebraicae, quoties quidem in ipsis continentur, non tam facile eruuntur. Sint X et Y functiones similes et quidem potestates ipsarum x et y, ita ut sit

$$\int\frac{dy}{\sqrt{\left(b^n-y^n\right)}}=\int\frac{dx}{\sqrt{a^n-x^n}},$$

quae aequatio, si $n=1$, praebet Parabolam, sin $n=2$, Ellipsin centrum in C habentem, etsi hoc casu utraque integratio quadraturam Circuli requirit. Verisimile ergo videtur etiam aliis casibus, quibus neutra integratio succedit, curvas algebraicas satisfacere; quarum autem inveniendarum Methodus adhuc desideratur.

9. Urgeatur corpus M perpetuo versus punctum fixum secundum directionem MC vi, quae sit ut functio quaecunque distantiae MC. Positis ut ante $CP=x$, $PM=y$ et $dy=pdx$ sit

$CM = \sqrt{\left(x^2 + y^2\right)} = t$, atque sit T ea functio ipsius t, quae exprimit vim centripetam. Resolvatur haec vis in laterales secundum MQ et MP, erit vis trahens secundum $MQ = \frac{Tx}{t}$ et vis secundum $MP = \frac{Ty}{t}$; ex quibus oritur acceleratio

$$dv = -\frac{Txdx}{t} - \frac{Tydy}{t} = -Tdt,$$

ob $xdx + ydy = tdt$; unde fit

$$v = A - \int Tdt .$$

Quamobrem minimum esse debet haec expression

$$\int dx\sqrt{(1+pp)\left(A-\int Tdt\right)}.$$

Iam secundum Regulae praeceptum differentietur quantitas

$$\sqrt{(1+pp)\left(A-\int Tdt\right)},$$

prodibitque

$$-\frac{Tdt\sqrt{(1+pp)}}{2\sqrt{\left(A-\int Tdt\right)}} + \frac{pdp\sqrt{\left(A-\int Tdt\right)}}{\sqrt{(1+pp)}}.$$

Ob $dt = \frac{xdx+ydy}{t}$ erit ergo

$$N = -\frac{Ty\sqrt{(1+pp)}}{2t\sqrt{\left(A-\int Tdt\right)}} \quad \text{et} \quad P = \frac{p\sqrt{\left(A-\int Tdt\right)}}{\sqrt{(1+pp)}};$$

ex quibus efficitur aequatio pro curva $Ndx = dP$, quae praebet

$$-\frac{Tydx\sqrt{(1+pp)}}{2t\sqrt{\left(A-\int Tdt\right)}} = \frac{dp\sqrt{\left(A-\int Tdt\right)}}{(1+pp)\sqrt{(1+pp)}} - \frac{pTdt}{2\sqrt{(1+pp)\left(A-\int Tdt\right)}}$$

haecque reducta abibit in istam

$$\frac{T(xdy-ydx)}{2t\left(A-\int Tdt\right)}=\frac{dp}{1+pp}.$$

10. Quamvis haec aequatio quatuor contineat litteras diversas, tamen debita dexteritate integrari potest. Cum enim sit

$$ydy+xdx=tdt=pydx+xdx,$$

erit

$$dx=\frac{tdt}{x+py} \text{ et } dy=\frac{ptdt}{x+py}.$$

qui valores in aequatione substituti dabunt

$$\frac{(px-y)Tdt}{2(x+py)\left(A-\int Tdt\right)}=\frac{dp}{1+pp}$$

seu

$$\frac{Tdt}{2\left(A-\int Tdt\right)}=\frac{dp(x+py)}{(1+pp)(px-y)}.$$

Harum expressionum utraque per logarithmos est integrabilis, est enim

$$\int\frac{Tdt}{2\left(A-\int Tdt\right)}=-\tfrac{1}{2}l\left(A-\int Tdt\right)$$

et

$$\int\frac{dp(x+py)}{(1+pp)(px-y)}$$

resolvitur in

$$\int\frac{xdp}{px-y}-\int\frac{pdp}{1+pp}=l\frac{px-y}{\sqrt{(1+pp)}};$$

ita ut sit

$$\frac{C}{\sqrt{\left(A-\int Tdt\right)}}=\frac{px-y}{\sqrt{(1+pp)}};$$

qua aequatione declaratur celeritatem corporis in M, quae est $\sqrt{\left(A-\int Tdt\right)}$, esse reciproce ut perpendiculum ex C in tangentem demissum; quae est proprietas palmaria horum motuum.

11. Hoc vero idem Problema commodius resolvi potest ipsam rectam *CM* pro altera variabili assumendo. Verum Methodus supra tradita non postulat, ut ambae variablies sint coordinatae orthogonales, dummodo sint eiusmodi binae quantitates, quibus determinatis simul curvae punctum determinetur. Hanc ob causam non liceret distantiam *CM* cum perpendiculo ex *C* in tangentem demisso pro binis illis variabilibus accipere, quoniam, etiamsi detur et distantia a centro et perpendiculum in tangentem, hinc tamen locus puncti curvae non definitur. Nihil autem impedit, quominus distantia *CM* (Fig. 28) et arcus circuli *BP* centro *C* descripti in locum duarum variabilium substituantur, quia dato arcu *BP* et distantia *CM* curvae punctum *M* aeque determinatur, ac per coordinatas orthogonales. Hac ergo annotatione usus Methodi multo latius extenditur, quam alioquin videri queat.

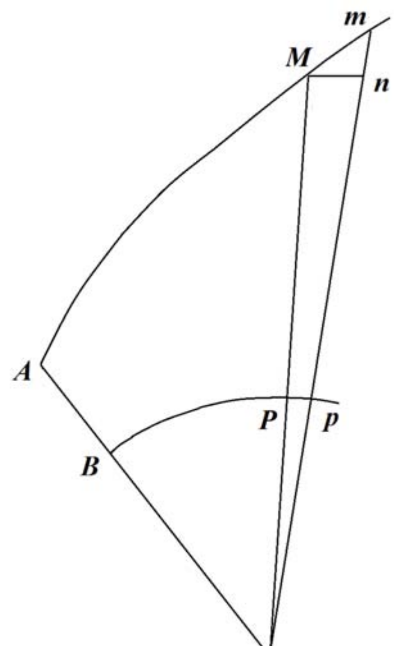

Fig. 28

12. Sit igitur distantia corporis a centro $MC = x$ et vis, qua corpus ad centrum *C* sollicitatur, sit $= X$ functioni cuicunque ipsius *x*. Centro *C*, radio pro lubitu assumpto $BC = c$, describatur circulus, cuius arcus *BP* teneat locum alterius variabilis *y*, ita ut sit $Pp = dy = pdx$. Ex vi autem sollicitante est $dv = -Xdx$, unde $v = A - \int Xdx$. Centro *C*, radio $CM = x$, describatur arculus *Mn*, erit

$$mn = dx \text{ et } CP : Pp = CM : Mn,$$

unde fit

$$Mn = \frac{pxdx}{c}$$

et elementum spatii

$$Mm = dx\sqrt{\left(1 + \frac{ppxx}{cc}\right)}.$$

Quamobrem minimum esse debet haec formula

$$\int dx\sqrt{\left(A - \int Xdx\right)\left(1 + \frac{ppxx}{cc}\right)},$$

ex qua oritur valor differentialis

$$\frac{1}{dx} d \cdot \frac{pxx\sqrt{\left(A - \int Xdx\right)}}{c\sqrt{(cc + ppxx)}},$$

qui, per Regulam nihilo aequalis positus, praebebit hanc aequationem:

$$\sqrt{C} = \frac{pxx\sqrt{\left(A - \int Xdx\right)}}{c\sqrt{(cc + ppxx)}}$$

seu

$$Cc^4 + Cccppxx = \left(A - \int Xdx\right) ppx^4,$$

ex qua fit

$$p = \frac{cc\sqrt{C}}{\sqrt{\left(\left(A - \int Xdx\right)x^4 - Cccxx\right)}} = \frac{cc\sqrt{C}}{x\sqrt{\left(\left(A - \int Xdx\right)xx - Ccc\right)}}$$

seu

$$dy = \frac{ccdx\sqrt{C}}{x\sqrt{\left(\left(A - \int Xdx\right)xx - Ccc\right)}},$$

quae eadem aequatio etiam per Methodum directam invenitur.

13. Ex his igitur casibus perfectissimus consensus principii hic stabiliti cum veritate elucet; utrum autem iste consensus in casibus magis complicatis locum quoque sit habiturus, dubium superesse potest. Quamobrem, quam late pateat istud principium, diligentius erit investigandum, quo plus ipsi non tribuatur, quam eius natura permittit. Ad hoc explicandum omnis motus proiectorum in duo genera distribui debet, quarum altero celeritas corporis, quam in quavis loco habet, a solo situ pendet; ita ut, si ad eundem situm revertatur, eandem quoque sit recuperaturum celeritatem; quod evenit, si corpus vel ad unum vel ad plura centra fixa trahatur viribus, quae sint ut functiones quaecunque distantiarum ab his centris. Ad alterum genus refero eos proiectorum motus, quibus celeritas corporis per solum locum, in quo haeret, non determinatur; id quod usu venit vel, si centra illa, ad quae corpus sollicitatur, fuerint mobilia, vel, si motus fiat in medio resistente. Hac facta divisione notandum est, quoties motus corporis ad prius genus pertineat, hoc est, si corpus non solum ad unum, sed ad quotcunque centra fixa sollicitetur viribus quibuscunque, toties in motu hoc summam omnium motuum elementarium fore minimam.

14. Hoc ipsum autem postulat indoles Propositionis; dum enim inter datos terminos ea quaeritur curva, in qua sit $\int ds\sqrt{v}$ minimum, eo ipso assumitur celeritatem corporis in utroque termino eandem esse, quaecunque curva corporis viam constituat. Quotcunque autem fuerint centra virium fixa, celeritas corporis in quovis loco *M* (Fig. 27) exprimitur

functione determinata ambarum variabilium $CP = x$ et $PM = y$. Sit igitur *v* functio quaecunque ipsarum *x* et *y*, ita ut sit $dv = Tdx + Vdy$, atque videamus, an principium nostrum veram exhibiturum sit proiectoriam corporis. Cum autem sit $dv = Tdx + Vdy$, corpus perinde movebitur, ac si sollicitetur in *M* a duabus viribus, altera *T* in directione abscissis *x* parallela, altera vero *V* in directione parallela applicatis *y*, ex quibus oritur et vis tangentialis $= \frac{Tdx + Vdy}{ds}$ et vis normalis $= \frac{-Vdx + Tdy}{ds}$.

Debet autem, ex natura motus liberi esse

$$\frac{2v}{r} = \frac{-Vdx + Tdy}{ds} = \frac{-V + Tp}{\sqrt{(1+pp)}};$$

ad quam aequationem si Methodus maximorum ac minimarum deducat, erit utique principium nostrum veritati conforme.

15. Cum igitur per hoc principium debeat esse $\int dx\sqrt{v(1+pp)}$ minimum, differentietur quantitas $\sqrt{v(1+pp)}$, atque ob $dv = Tdx + Vdy$ orietur:

$$\frac{Tdx\sqrt{(1+pp)}}{2\sqrt{v}} + \frac{Vdy\sqrt{(1+pp)}}{2\sqrt{v}} + \frac{pdp\sqrt{v}}{\sqrt{(1+pp)}},$$

ex quo obtinetur pro curva quaesita sequens aequatio secundum praecepta tradita

$$\frac{Vdx\sqrt{(1+pp)}}{2\sqrt{v}} = d\cdot\frac{p\sqrt{v}}{\sqrt{(1+pp)}} = \frac{dp\sqrt{v}}{(1+pp)^{3:2}} + \frac{p(Tdx+Vdy)}{2\sqrt{v(1+pp)}}$$

seu

$$-\frac{dp\sqrt{v}}{(1+pp)^{3:2}} = \frac{Tpdx - Vdx}{2\sqrt{v(1+pp)}}.$$

At est radius osculi in *M*

$$= -\frac{(1+pp)dx\sqrt{(1+pp)}}{dp};$$

qui si ponatur $= r$, erit

$$\frac{2v}{r} = \frac{Tpd - V}{\sqrt{(1+pp)}},$$

omnino uti per Methodum directam invenitur. Dummodo ergo vires sollicitantes ita fuerint comparatae, ut eae reduci queant ad duas vires *T* et *V* secundum directiones coordinatis x et y parallelas sollicitantes, quae sint ut functiones quaecunque harum variabilium x et y, tum semper in curva descripta erit motus corporis per omnia elementa collectus minimus.

16. Tam late ergo hoc principium patet, ut solus motus a resistentia medii perturbatus excipiendus videatur; cuius quidem exceptionis ratio facile perspicitur, propterea quod hoc casu corpus per varias vias ad eundem locum perveniens non eandem acquirit celeritatem. Quamobrem, sublata omni resistentia in motu corporum proiectorum, perpetuo haec constans proprietas locum habebit, ut summa omnium motuum elementarium sit minima. Neque vero haec proprietas in motu unius corporis tantum cernetur, sed etiam in motu plurium corporum coniunctim, quae quomodocunque in se invicem agant, tamen semper summa omnium motuum est minima. Quod, cum huiusmodi motus difficulter ad calculum revocentur, facilius ex primis principiis intelligitur, quam ex consensu calculi secundum utramque Methodum instituti. Quoniam enim corpora ob inertiam omni status mutationi reluctantur, viribus sollicitantibus tam parum obtemperabunt, quam fieri potest, siquidem sint libera; ex quo efficitur, ut in motu genito effectus a viribus ortus minor esse debeat, quam si ullo alio modo corpus vel corpora fuissent promota. Cuius ratiocinii vis, etiamsi nondum satis perspiciatur, tamen, quia cum veritate congruit, non dubito, quin ope principiorum sanioris Metaphysicae ad maiorem evidentiam evehi queat; quod negotium aliis, qui Metaphysicam profitentur, relinquo.