ADDITION IA

ELASTIC CURVES

1. The greatest geometers have recognized the methods treated in this book for some time now, which not only are of the greatest use in analysis itself, but also provide the greatest help in the resolution of physical problems. Indeed since the structure of the universe has been made most perfectly and completely by the wisest creator, generally nothing happens in the universe, into which some account of a maximum or minimum may not be elucidated ; because of which therefore there is no doubt, why all the effects of the universe may not be determined equally well from final causes with the aid of the method of maxima and minima, and from these causes being put into effect. Truly as excellent examples of this affair are apparent everywhere, generally we shall not need confirmation of the truth from many examples, because the matter may be seen to belong to philosophy rather than to mathematics ; but rather an elaboration will be required into this matter, in order that a magnitude may be investigated in any kind of natural question, which may adopt a maximum or minimum value. Therefore since the nature of an effect may appear to be understood in a two-fold way, the one by the effecting causes, which is usually called the direct method [*i.e.* derived from physical principles], the other by the final causes [*i.e.* derived from some general principle], mathematics is used successfully in each part. Without doubt when the affecting causes are very well hidden, but our final understanding escapes our notice less, it is customary to solve the question by the indirect method; but on the contrary the direct method may be used, whenever it is possible to define the effect from the causes bringing about the effect. But a need is raised especially, so that the same solution may be uncovered approached by each way ; for thus not only may the one solution be confirmed especially by the other, but also from the agreement of each, we may derive the greatest pleasure. In this manner the curvature of a suspended rope or chain has been elicited in two ways, the one from first principles from the weight acting, the other truly by the method of maxima and minima, because it ought to be understand for a rope of this kind, that it receives a curvature so that its centre of gravity will reach the lowest position. Similarly the curvatures of rays passing through a transparent medium of varying density can be determined both from first principles, as well as from this principle, so that they may arrive at a given place in the shortest time. Moreover, many other similar examples have been brought forwards by the most celebrated Bernoulli and others, from which both the method of solving from first principles as well as an understanding of the causes effecting the maximum increments received. Therefore although on account of so many outstanding examples, no doubt is left why in all curved lines, which may provide the solution of physico-mathematical problems, the nature of a certain maximum or minimum may not find a place ; yet frequently this maximum or minimum is itself seen with the greatest difficulty, even if a solution were able to be elicited from first principles. Thus even the figure that a curved elastic lamina adopts has been known now for some time, just as still it has not been noticed by anyone at this point in time, that the curvature can be investigated by the method of maxima and minima, that is, by final causes. On account of which, when the most celebrated and observant Daniel Bernoulli, on looking into the general nature of this

problem at the highest level, indicated to me that the general force, which is present in a curved elastic lamina, can be included in a single formula he calls the *force potential*, and this expression is required to be a minimum in the curved elastic plate ; because finding this is shown wonderfully well by my method of maxima and minima treated in this book, and its use is most amply proven, I cannot let this most desired occasion slip away, why by publishing this outstanding property of an elastic curve observed by the celebrated Bernoulli, at the same time may I not make the use of my method clearer. Indeed this same property is contained in a differential equation of the second order, so that thus those methods set out before of solving the isoperimetric problem shall not be sufficient.

2. Let *AB* (Fig. 1) be some curved elastic plate ; calling the arc $AM = s$ and the radius of osculation of the curve $MR = R$; and, following Bernoulli, the potential force present in the portion of the plate *AM* may be expressed by this formula $\int \frac{ds}{RR}$, if indeed the lamina shall be everywhere of equal thickness, width and elasticity and extended sitting in its natural direction. Hence the nature of such a curve *AM* will be, so that in this, this expression may obtain the smallest value of all. Because truly differentials of the second order are present in the radius of osculation *R*, there is a need for four conditions for determining a curve with the given property, which agrees with the nature of the first question. For since infinitely many elastic laminas may be able to be curved through the given ends *A* and *B* and those of the same length, the question will not be determined unless besides the two points *A* and *B* likewise two other points or, what amounts to the same, the position of the tangents at the end points *A* and *B* may be prescribed. In so much as with a longer proposed elastic lamina, than is the separation of the points *A* and *B*, that not only must it be curved thus, so that it may be contained between the ends *A* et *B*, but also so that its tangents at these given points may maintain given directions. With these noted the question about finding the curvatures of the elastic laminas being resolved from this source thus must be proposed : *so that amongst all the curves of the same length, which not only may pass through the points A and B, but also may be tangents at these points with right lines given, that may be defined in which the value of this expression* $\int \frac{ds}{RR}$ *shall be a minimum.*

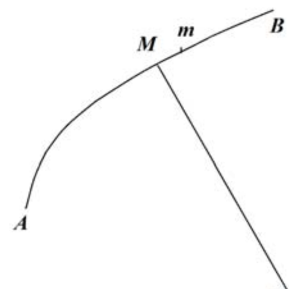

Fig. 1

3. Because it will be convenient to adapt the solution to orthogonal coordinates (Fig. 2), some right line *AD* may be taken for the axis, in which the abscissa shall be $AP = x$, the applied line $PM = y$; there may be put in place, as the method treated orders, $dy = pdx,\ dp = qdx$; an element of the curve will be $Mm = ds = dx\sqrt{(1+pp)}$. Therefore in the first place, because the curves, from which the solution sought must be elicited, are put in place isoperimetric, this expression will have to be considered $\int dx\sqrt{(1+pp)}$, which

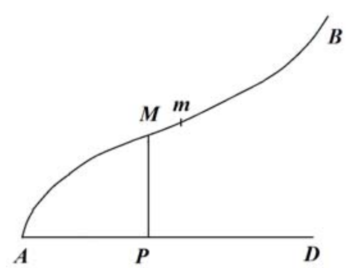

Fig. 2

when compared to the general $\int Zdx$ presents this differential value $\frac{1}{dx}d\cdot\frac{p}{\sqrt{(1+pp)}}$.

Then since the radius of osculation shall be

$$=\frac{dx(1+pp)^{3:2}}{dp}=\frac{(1+pp)^{3:2}}{q}=R,$$

the formula $\int\frac{ds}{RR}$, which must be a minimum, will change into $\int\frac{qqdx}{(1+pp)^{5:2}}$.

This may be compared with the general form $\int Zdx$; there will be $Z=\frac{qq}{(1+pp)^{5:2}}$ and on putting

$$dZ=Mdx+Ndy+Pdp+Qdq\,,$$

there will be

$$M=0,\ N=0,\ P=\frac{-5pqq}{(1+pp)^{7:2}}\ \text{and}\ Q=\frac{2q}{(1+pp)^{5:2}}.$$

Therefore the value of the differential arising from this $\int\frac{qq}{(1+pp)^{5:2}}$ will be $-\frac{dP}{dx}+\frac{ddQ}{dx^2}$

On account of which this equation will be found for the curve sought

$$\frac{\alpha}{dx}d\cdot\frac{p}{\sqrt{(1+pp)}}=\frac{dP}{dx}-\frac{ddQ}{dx^2},$$

which multiplied by *dx* and integrated gives :

$$\frac{\alpha p}{\sqrt{(1+pp)}}+\beta=P-\frac{dQ}{dx}.$$

This equation may be multiplied by $qdx=dp$, so that there may come about :

$$\frac{\alpha pdp}{\sqrt{(1+pp)}}+\beta dp=Pdp-qdQ.$$

But since on account of $M=0$ and $N=0$ there shall be $dZ=Pdp+Qdq$, there becomes $Pdp=dZ-Qdq$, with which value substituted in place of *Pdp* the equation emerges :

$$\frac{\alpha pdp}{\sqrt{(1+pp)}}+\beta dp=dZ-Qdq-qdQ\,;$$

which integrated again gives

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma = Z - Qq.$$

Now since there shall be

$$\mathrm{Z} = \frac{qq}{(1+pp)^{5:2}} \text{ and } Q = \frac{2q}{(1+pp)^{5:2}},$$

there will be

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma = -\frac{qq}{(1+pp)^{5:2}}.$$

The arbitrary constants $\alpha,\ \beta$ and γ may be taken negative and there will be

$$q = (1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)} = \frac{dp}{dx}.$$

Hence the following equation hence is elicited :

$$dx = \frac{dp}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}.$$

Then on account of $dy = pdx$, there will be found also

$$dy = \frac{pdp}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}};$$

which two equations may suffice to construct the curve by quadrature.

4. Thus neither is integrable in the general consideration of these formulas ; but they can be combined in a certain manner, so that the sum may allow an integration. For since there shall be

$$d\cdot\frac{2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}{\sqrt{\sqrt{(1+pp)}}} = \frac{dp\,(\beta-\gamma p)}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}.$$

$$\left[\textit{i.e.} \quad d\cdot\frac{2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}{\sqrt{\sqrt{(1+pp)}}}=d\cdot 2\sqrt{\left(\alpha+\frac{\beta p+\gamma}{\sqrt{(1+pp)}}\right)}=\frac{\frac{\beta}{\sqrt{(1+pp)}}-\frac{p(\beta p+\gamma)}{(1+pp)^{3:2}}}{\sqrt{\left(\alpha+\frac{\beta p+\gamma}{\sqrt{(1+pp)}}\right)}}\cdot dp \right.$$

$$\left. =\frac{\beta(1+pp)-p(\beta p+\gamma)}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}\cdot dp=\frac{(\beta-\gamma p)dp}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}.\right]$$

there will be

$$\frac{2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}{(1+pp)^{1:4}}=\beta x-\gamma y+\delta .$$

Because the position of the axis is arbitrary, the constant δ can be omitted without great deficiency. Then truly the axis thus can be changed, so that the abscissa becomes

$$\frac{\beta x-\gamma y}{\sqrt{(\beta\beta+\gamma\gamma)}}$$

and the applied line will become

$$\frac{\gamma x+\beta y}{\sqrt{(\beta\beta+\gamma\gamma)}};$$

hence also γ can be put equal to zero without risk, because nothing prevents that new abscissa being expressed by x. On this account we will have this equation for the elastic curve :

$$2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p\right)}=\beta x(1+pp)^{1:4},$$

which with the square taken gives :

$$4\alpha\sqrt{(1+pp)}+4\beta p=\beta^2x^2\sqrt{(1+pp)}.$$

Towards introducing homogeneity, let there be $\alpha=\frac{4m}{aa}$ and $\beta=\frac{4n}{aa}$, there will be

$$naap=(nnxx-maa)\sqrt{(1+pp)}\,,$$

from which

$$n^2a^4pp=(nnxx-maa)^2(1+pp)$$

and thus

$$p=\frac{nnxx-maa}{\sqrt{\left(n^2a^4-(nnxx-maa)^2\right)}}=\frac{dy}{dx}.$$

Therefore with the constants changed and with the given abscissa *x* either increased or diminished by a constant, an equation of this kind will be found for the general elastic curve :

$$dy=\frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}}.$$

from which arises

$$ds=\frac{aadx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}};$$

from which equations, the agreement of this curve found with the elastic curve now elicited some time ago [by James Bernoulli.] may be clearly shown.

5. But so that we can see this agreement more clearly, I will investigate the nature of the elastic curve from first principles also ; which even now if it has been done most excellently by the most outstanding James Bernoulli, yet on this suitable occasion I may add a few things about the nature of elastic curves and of their various kinds and figures, which I consider to be either dismissed or only lightly treated by others. *AB* shall be an elastic lamina (Fig. 3) thus firmly fixed at *B* into a wall or pavement, so that this extremity *B* not only may be held firmly, but also the position of the tangent at *B* may be determined. But at *A* the lamina shall have a rigid rod *AC* connected, to which a force $CD=P$ shall be applied normally, by which the lamina may be rendered into a state of curvature *BMA*. This right line *AC* produced may be taken for the axis and by putting $AC=c$ the abscissa shall be $AP=x$, the applied line $PM=y$. But if now the lamina at *M* may suddenly lose all elasticity and become perfectly flexible, it may certainly be bent by the force *P*, with the curvature proceeding from the force *P* with the moment $=P(c+x)$. Therefore so that this bending may not actually follow, the elasticity of the lamina at *M* remains in equilibrium with the moment

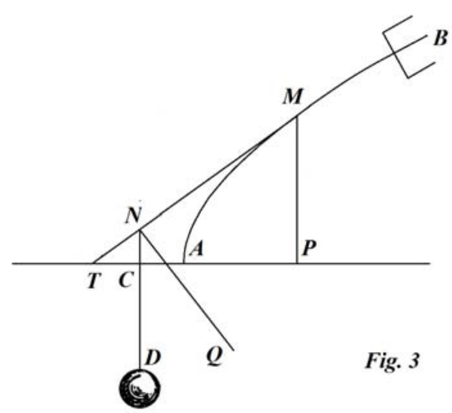

Fig. 3

of the force acting $P(c+x)$. But in the first place the elasticity depends on the nature of the material, on which the lamina depends and as with the same placed everywhere, then truly likewise depends on the bending of the lamina at the point *M*, thus so that it shall be inversely proportional to the radius of osculation at *M*. Therefore the radius of oscillation at *M* shall be $=R=-\frac{ds^3}{dxddy}$, with ds being $=\sqrt{dx^2+dy^2}$ and with *dx* constant, and $\frac{Ekk}{R}$ may express the elastic force of the lamina at *M* , which may be agreed to be in equilibrium with the force acting $P(c+x)$, thus so that there shall be

$$P(c+x)=\frac{Ekk}{R}=-\frac{Ekkdxddy}{ds^3}.$$

This equation multiplied by *dx* shall be integrable and the integral will be

$$P\left(\frac{xx}{2}+cx+f\right)=-\frac{Ekkdy}{\sqrt{dx^2+dy^2}};$$

from which arises

$$dy=\frac{-Pdx\left(\frac{1}{2}xx+cx+f\right)}{\sqrt{\left(E^2k^4-P^2\left(\frac{1}{2}xx+cx+f\right)^2\right)}},$$

which equation generally agrees with that, as by the method of maxima and minima elicited from the principle of Bernoulli
[See : *Opera Omnia Jacobi Bernoulli*, *Curvaturae Laminae Elasticae*, p.576 onwards].

6. From the comparison of this equation with the one found before, the force can be defined which is required to induce a given curvature to the lamina, if indeed the curvature may be contained in the general equation found. Certainly the elastic lamina may maintain the figure *AMB*, the nature of which may be expressed by this equation

$$dy=\frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}};$$

truly *Ekk* expresses the absolute elasticity of this lamina, certainly thus, so that *Ekk* at some place divided by the radius of osculation gives the true elastic force.

[In modern statics, we call *Ekk* the flexural rigidity or bending modulus, defined as the product of the Young modulus (also usually called E: see note 15, page 402 of Truesdell's book below) and the second moment of the area *I*, so that the bending moment $M=\frac{EI}{R}$ is equal to the resisting moment; this moment divided by the radius of curvature at the point gives the elastic stress or force per unit area to which Euler is referring. These matters are considered in texts on the bending of beams, etc. See in particular *C. Truesdell*, who in his book, *The Rational Mechanics of Flexible or Elastic Bodies* 1638-1788, provides a summary of Euler's work on elastic laminas, or bands, as set out in this work.]

Towards putting a comparison in place the numerator and denominator may be multiplied by $\frac{Ekk}{aa}$, so that there may be obtained :

$$dy = \frac{Ekkdx\left(\alpha + \beta x + \gamma xx\right) : aa}{\sqrt{\left(E^2k^4 - \frac{E^2k^4}{a^4}\left(\alpha + \beta x + \gamma xx\right)^2\right)}}.$$

Therefore now there will be:

$$-\tfrac{1}{2}P = \frac{Ekk\gamma}{aa},\ -Pc = \frac{Ekk\beta}{aa},\ -Pf = \frac{Ekk\alpha}{aa}$$

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and thus the force acting $CD = -\frac{2Ekk}{aa}$, the interval $AC = c = \frac{\beta}{2\gamma}$ and the constant

$f = \frac{\alpha}{2\gamma}$.

7. Therefore so that the elastic lamina *AB* attached to the wall by the other end *B* may be curved into the figure *AMB*, whose nature is expressed by this equation

$$dy = \frac{\left(\alpha + \beta x + \gamma xx\right)dx}{\sqrt{\left(\alpha^4 - \left(\alpha + \beta x + \gamma xx\right)^2\right)}},$$

it is necessary, so that this lamina may be acted on in the direction *CD* normal to the axis *AP*, with the distance taken $AC = \frac{\beta}{2\gamma}$, by the force $CD = -\frac{Ekk\gamma}{aa}$; which force certainly acts in the opposite direction, and the figure indicates it will be directed, if γ were a positive quantity. Because $\frac{Ekk}{R}$ is equivalent to the moment of the force acting, the homogeneous expression $\frac{Ekk}{aa}$ will be the force of a true weight, which force $\frac{Ekk}{aa}$ therefore will be recognised from the elastic lamina. This force shall be $= F$, and the bending force *CD* will be to the force *F*, as -2γ to 1; indeed γ will be a pure number.

8. Hence again the force can be defined for the part *BM* of the lamina in place, from its need to be conserved if the portion *AM* evidently may be cut off. With this portion *AM* removed the elastic lamina stops in a rigid rod *MT* with all the elasticity removed, but which shall be connected thus with the lamina, so that it may refer always to the tangent at the point *M*, in whatever manner the lamina may be inclined. With this in place from the preceding it is evident for the conservation of the required curvature *BM* to be maintained, so that the rod *MT* at the point *N* may be drawn in the direction *ND* by a force , which shall be $-\frac{2Ekk\gamma}{aa}$; but the direction *ND* will be normal to the axis *AP* and the interval *AC* will be $=\frac{\beta}{2\gamma}$

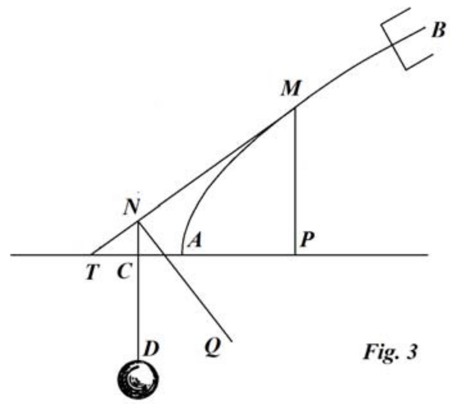

Fig. 3

And thus the distance *MN* becomes

$$=\frac{ds}{dx}CP=\frac{ds}{dx}\cdot\frac{\beta+2\gamma x}{2\gamma}=\frac{(\beta+2\gamma x)ds}{2\gamma dx},$$

indeed there is

$$\frac{ds}{dx}=\frac{aa}{\sqrt{\left(\alpha^4-(\alpha+\beta x+\gamma xx)^2\right)}}.$$

But if this force $ND=-\frac{2Ekk\gamma}{aa}$ may be resolved into the normal *NQ* to the tangent *MT* and to the tangent *NT*, the normal force will be $NQ=-\frac{2Ekk\gamma}{aa}\cdot\frac{dx}{ds}$ and the tangential force $NT=-\frac{2Ekk\gamma}{aa}\cdot\frac{dy}{ds}$.

9. But if the part *BM* may be cut off, with the part *AM* remaining, which with the force $=-\frac{2Ekk\gamma}{aa}$ acting in the direction *CD* as before towards maintaining the curvature *AM* , the end *M*, which may be understood to be connected with a rigid rod with the tangent *MN*, it will have to be acted on at the point *N* by a force equally $=-\frac{2Ekk\gamma}{aa}$ in the opposite direction to that, as we have found in the preceding case. For the forces always applied to each end of the curved lamina mutually cancel each other and thus must have equal and opposite directions. For in general the whole lamina may be moved, to which motion there will be come a need of restraining by a force produced between the forces

acting. Hence therefore the forces applying to any part of the lamina cut can be defined easily, which now lead to the conserved curvature.

10. Let *AM* (Fig.4) be a curved elastic lamina, which may have the rigid rods *AD* and *MN* joined at *A* and *M*, with which the applied forces *DE, NR* shall be applied in directions directly opposite *DE, NR*, which in equilibrium induce the constant curvature of the lamina *AM*, for which it may be necessary to find the equation. Therefore in the first place the right line *AP* may be taken for the axis passing through the point *A* and normal to the direction of the force acting *ER*. The absolute elasticity of the lamina may be put $= Ekk$ and the sine of the angle *CAD*, which the tangent *AD* at *A* makes with the axis and which has been given $= m$, the cosine $= n$, with the whole sin present $= 1$, thus so that there shall be $mm + nn = 1$. Again calling the distance $AC = c$ and the bending force $DE = NR = P$, and on putting the abscissa $AP = x$ with the applied line $PM = y$, the nature of the curve may be expressed by this equation

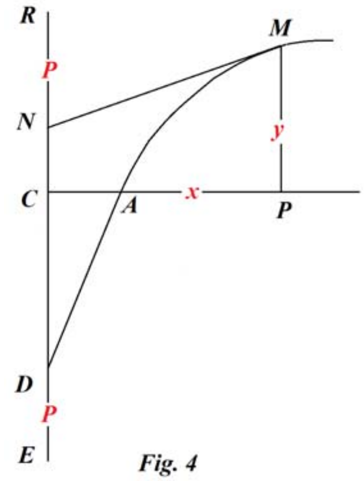

Fig. 4

$$dy = \frac{-Pdx\left(\frac{1}{2}xx + cx + f\right)}{\sqrt{\left(E^2k^4 - P^2\left(\frac{1}{2}xx + cx + f\right)^2\right)}}.$$

Truly because the direction of the tangent at *A* is given, on putting $x = 0$ there must become $\frac{dy}{dx} = \frac{m}{n}$; hence therefore the equation will be found :

$$\frac{m}{n} = \frac{-Pf}{\sqrt{\left(E^2k^4 - P^2f^2\right)}} = \frac{m}{\sqrt{(1-mm)}} \text{ and } m = -\frac{Pf}{Ekk}.$$

Therefore the constant f may be determined, thus so that there shall be

$$f = -\frac{mEkk}{P},$$

and hence thus the whole curve will be determined.

11. Therefore the force $DE = P$ is required to be applied for the curvature of the lamina *AM* expressed by the above equation (Fig. 5) leading to the tangent *AD* at the point *D*, thus so that there shall be $AD = \frac{c}{n}$, whose direction shall be parallel to the applied line *PM*. This force *DE* may be resolved into the two sides *Dd, Df* normal to each other ; there will be the force $Dd = Pn$ and the force

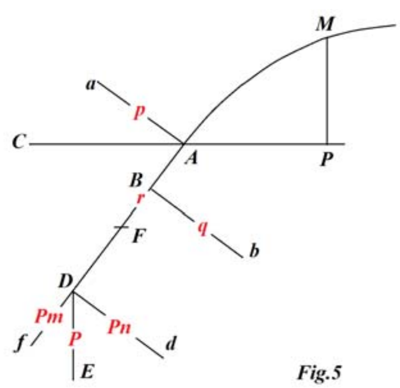

Fig.5

$Df = Pm$. Now so that the consideration of the right line *AD* may be removed from the calculation, in place of the force *Dd* at the given points *A* and *B*, with the interval taken $AB = h$, the two forces $Aa = p,\ Bb = q$ can be substituted, equally normal to the rod *AB*, by taking

$$ph = Pn \cdot BD = nP\left(\frac{c}{n} - h\right) \text{ and } q = p + nP.$$

Because likewise there becomes henceforth, the force of the tangent $Df = mP$ applied at some point of the rod *AD*, that may be applied at that point *A* by putting $AF = nP$. But this force shall be $AF = r$, thus so that the *MA* will be acted on by three forces $Aa = p,\ Bb = q$ and $AF = r$, from which we will investigate such curvature arising.

12. Therefore in the first place, since there shall be $mP = r$ there will be $P = \frac{r}{m}$, which value substituted into the former equations will give

$$ph = \frac{cr}{m} - \frac{nhr}{m} \text{ and } q = p + \frac{nr}{m}.$$

Hence there will be

$$\frac{n}{m} = \frac{q-p}{r};$$

from which equation in the first place the position of the axis *AP* becomes known; clearly then the tangent of the angle $CAD = \frac{r}{q-p}$, hence

$$m = \frac{r}{\sqrt{\left(r^2 + (q-p)^2\right)}} \text{ and } n = \frac{q-p}{\sqrt{\left(r^2 + (q-p)^2\right)}}.$$

Then from the equation

$$hp = \frac{cr}{m} - \frac{nhr}{m} \quad = \frac{cr}{m} - hq + hp$$

there becomes

$$c = \frac{mhq}{r} \text{ or } c = \frac{hq}{\sqrt{\left(r^2 + (q-p)^2\right)}}$$

and

$$P = \sqrt{\left(rr + (q-p)^2\right)}.$$

But since there shall be

$$f = \frac{-mEkk}{P} = \frac{-Ekkr}{rr+(q-p)^2},$$

there will be

$$\tfrac{1}{2}xx + cx + f = \tfrac{1}{2}xx + \frac{hqx}{\sqrt{\left(rr+(q-p)^2\right)}} - \frac{Ekkr}{rr+(q-p)^2};$$

from which this equation will be found for the curve sought

$$dy = \frac{dx\left(\dfrac{Ekkr}{\sqrt{\left(rr+(q-p)^2\right)}} - hqx - \tfrac{1}{2}xx\sqrt{\left(rr+(q-p)^2\right)}\right)}{\sqrt{\left(E^2k^4 - \left(\dfrac{Ekkr}{\sqrt{\left(rr+(q-p)^2\right)}} - hqx - \tfrac{1}{2}xx\sqrt{\left(rr+(q-p)^2\right)}\right)^2\right)}}.$$

But this equation lends itself greatly to the usual manner of curving laminas, while these may be seized by forceps or between two fingers ; of which the one presses the lamina in the direction *Aa*, the other in the direction *Bb* [*i.e.* forming a couple], besides which forces, in addition the lamina can be drawn forwards in the direction *AF*.

13. If the tangential force $AF = r$ may vanish, the axis *AP* will fall on the same tangent *AF* produced and then there will be

$$dy = -\frac{dx\left(hqx + \tfrac{1}{2}(q-p)xx\right)}{\sqrt{\left(E^2k^4 - \left(hqx + \tfrac{1}{2}(q-p)xx\right)^2\right)}}.$$

But if the normal forces *p* and *q* become equal to each other, the axis *AP* will be normal to the tangent *AF* on account of $n = 0$ and this equation will arise for the curve :

$$dy = \frac{dx\left(Ekk - hqx - \tfrac{1}{2}rxx\right)}{\sqrt{\left(2Ekk\left(hqx + \tfrac{1}{2}rxx\right) - \left(hqx + \tfrac{1}{2}rxx\right)^2\right)}}.$$

If here in addition there were $r = 0$, thus so that the lamina may be acted on by equal and opposite forces *Aa, Bb* at the points *A* and *B*, the nature of the curve may be expressed by this equation

$$dy = \frac{dx(Ekk - hqx)}{\sqrt{hq(2Ekkx - hqxx)}},$$

which integrated gives

$$y = \sqrt{\frac{2Ekkx - hqxx}{hq}};$$

which is for a circle, the lamina therefore in this case is curved in the arc of a circle, the radius of which will be $= \frac{Ekk}{hq}$.

ENUMERATION OF ELASTIC CURVES

14. Therefore since we may observe not only a circle to be contained in a class of elastic curves, but also an infinite variety of other curves have a place in these, here it will be worth the effort to put in place a listing of all the verities of forms contained in this kind of curves. For in this manner not only the natures of these curves will be observed thoroughly, but also in some presented case it will be possible to judge from the figure given alone, to what kind of curve the figure ought to be referred. But in the same manner we will put in place here the diversity of forms, by which the kinds of algebraic lines contained in a give order commonly are accustomed to be enumerated.

15. The general equation for the elastic curves

$$dy = \frac{(\alpha + \beta x + \gamma xx)dx}{\sqrt{\left(a^4 - (\alpha + \beta x + \gamma xx)^2\right)}},$$

with the beginning of the abscissas on the axis moved by the interval $\frac{\beta}{2\gamma}$, and on writing aa for $\frac{aa}{\gamma}$, or by putting $\gamma = 1$, will taken this simpler form:

$$dy = \frac{(\alpha + xx)dx}{\sqrt{\left(a^4 - (\alpha + xx)^2\right)}}.$$

Because truly there is

$$a^4 - (\alpha + xx)^2 = (aa - \alpha - xx)(aa + \alpha + xx),$$

putting $aa - \alpha = cc$, so that there shall be $\alpha = aa - cc$, and the equation will be changed into this form:

$$dy = \frac{(aa - cc + xx)dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}.$$

From which the nature of the curve *AMC* may be expressed (Fig. 6) on putting the $AP = x$ and the applied line $PM = y$. Therefore since β shall be $= 0$, the direction of the applied force curving the elastic lamina at some point *A* will be normal to the axis *AP*, and thus *AD* will represent the direction of the force acting, which thus will be $= \frac{2Ekk}{aa}$, with the absolute elasticity expressed by *Ekk*.

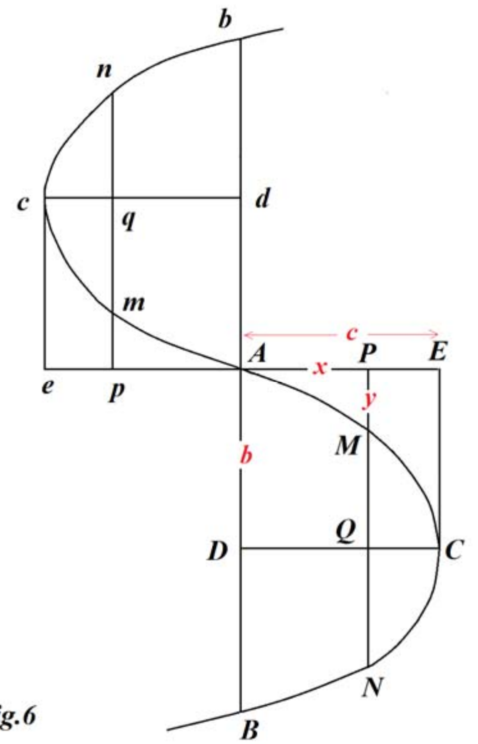

Fig.6

16. If there may be put $x = 0$, there will be

$$\frac{dy}{dx} = \frac{aa - cc}{c\sqrt{(2aa - cc)}};$$

which expression gives the tangent of the angle, which the curve *AM* makes with the axis *AP* at *A*; the sine of this angle will be $= \frac{aa - cc}{aa}$.
[Note that the hypotenuse of the associated right angled triangle is *aa* : as $c^2(2aa - cc) + (aa - cc)^2 = a^4$.]
Whereby, if there were $aa = \infty$, the lamina will be normal to the axis *AP* at the point *A* and it will have no curvature, on account of which the curving force $\frac{2Ekk}{aa}$ vanishes. Therefore in the case in which $a = \infty$ the natural figure of the lamina will be produced, that is a right line ; which therefore constitutes the first kind of elastic lines, which the right line *AB* produced indefinitely in each direction will represent.

17. Before we may enumerate the remaining kinds , it will be convenient in general to put in place some observations about the elastic figure. Moreover it is understood the angle *PAM*, which the curve makes with the axis *AP* at *A*, to decrease, from which a smaller quantity *aa* may arise, that is so that the curving force $\frac{2Ekk}{aa}$ may be stretched out more. And, if $aa = cc$ may arise, then the axis *AP* may be a tangent to the curve at *A*. So that if moreover there were *aa* < *cc*, then the curve *AM*, which at this stage may go off downwards, now may turn up, because there becomes $aa = \frac{1}{2}cc$; in which case the

tangent of the curve lies on the right line *Ab*. But if there were $aa < \frac{1}{2}cc$, then the angle *PAM* clearly becomes imaginary and thus no part of the curve will be present at *A* ; which will constitute a variety of different forms of the case observed.

18. Again it is understood from the equation, because its form does not change, if both the coordinates *x* and *y* may be made negative, the curve about *A* has similar and equal branches *AMC* and *Amc* put in place alternately, thus so that at *A* there shall be a point of contrary inflection ; from which with the known portion of the curve *AMC* likewise its continuation *Amc* beyond *A* will be known, certainly which is similar and equal to that. Thus on taking $Ap = AP$ also there will be $pm = PM$. But on receding from *A*, the curve on both sides may be inclined more from the axis, then with the abscissa taken $= AE = c$ the applied line EC may be a tangent to the curve; for on putting $x = c$ there becomes $\frac{dy}{dx} = \infty$. Moreover it is evident the abscissa *x* cannot increase beyond $AE = c$; for otherwise $\frac{dy}{dx}$ will become imaginary; hence therefore the whole curve will be contained between the outermost applied lines *EC* and *ec*, beyond which boundaries it may not pass. Therefore now generally we have the two known branches of the curve *AC* and *Ac* stretched out on both sides from *A* as far as to the boundaries.

19. Therefore we may know, with the curve running beyond *C* and *c* it may progress to any place. Hence in the end we may take the right line *CD* parallel to *AE* itself for the axis and we may put in place these new coordinates $CQ = t$, $QM = u$; and there will be

$$t + x = AE = CD = c \text{ and } y + u = CE = AD = b;$$

from which there becomes $x = c - t$ and $y = b - u$ or $dy = -du$. With these values substituted, an equation will arise for the curve between the coordinates $CQ = t$ and $QM = u$, which will be :

$$du = \frac{(aa - 2ct + tt)dt}{\sqrt{t(2c - t)(2aa - 2ct + tt)}}.$$

Here it is apparent initially, if *t* may be taken infinitely small, to become :

$$du = \frac{aadt}{2a\sqrt{ct}} \text{ and thus } u = a\sqrt{\frac{t}{c}};$$

which equation indicates the curve beyond *C* begins to progress in a similar manner towards *N*, as it is extended from *C* to *M*. But the ambiguity of the $\sqrt{\ }$ sign in the denominator of the equation clearly declares the applied line *u* equally can take positive and negative values ; from which it is clear the right line *CD* is a diameter of the curve and thus the arc *CNB* is similar and equal to the arc *CMA*.

20. Moreover in a similar manner the right line *cd* drawn through *c* on the other side parallel to the axis *AE* will be a diameter of the curve, because the branch *Acb* is similar and equal to the branch *ACB*. Therefore at the points *B* and *b* there will also be points of contrary inflection entirely as at *A*; from which the curve similarly will proceed further. Therefore the curve will have infinitely many diameters *CD*, *cd* etc. distant from each other by the same interval *Dd* and parallel to each other ; and on this account the curve will be constructed from infinitely many parts equal and similar to each other and thus the whole curve will be known, if only a single part *AMC* were examined.

21. Because at *A* there is a point of contrary inflection, likewise the radius of osculation will be infinitely great ; which is apparent from the nature of the curve. For since the curve at *A* is acted on by a single force $=\frac{2Ekk}{aa}$ in the direction AD, at some place *M* there will be, if the radius of osculation there may be put $=R$, from the nature of the elasticity,

$$\frac{2Ekk}{aa}x=\frac{Ekk}{R};$$

from which there becomes $R=\frac{aa}{2x}$. Therefore at the point *A* the radius of osculation is infinite ; but truly at the points *C*, *c*, on account of $AE=Ae=c$, the radius of osculation will be $=\frac{aa}{2c}$; evidently at these places furthest from the right line *BAb* the curvature is a maximum.

22. But if for the point *C* the abscissa $AE=c$ may be constructed , yet the distance *EC* cannot be found except by integration of the equation

$$dy=\frac{(aa-cc+xx)dx}{\sqrt{(cc-xx)(2aa-cc+xx)}}.$$

For if after the integration there may be put $x=c$, the value of *y* will give the distance *CE*, which taken twice will give the distance *AB* or the interval *Dd* between adjacent diameters. In a like manner with the help of integration the length of the curved lamina *AC* will be able to be determined. For since on putting the arc $AM=s$ there shall be

$$ds=\frac{aadx}{\sqrt{(cc-xx)(2aa-cc+xx)}},$$

the integral of which on putting $x=c$ will give the length of the curve *AC*.

23. But since these formulas may not admit an integration, we may press on to express conveniently by approximating the values of the interval *AD* and the arc of the curve *AC*. To this end we may put $\sqrt{(cc-xx)}=z$, and there will be

$$PM=y=\int\frac{(aa-zz)dx}{z\sqrt{(2aa-zz)}} \text{ and } AM=s=\int\frac{aadx}{z\sqrt{(2aa-zz)}}\ .$$

Truly by the series there becomes

$$\frac{1}{\sqrt{(2aa-zz)}}=\frac{1}{a\sqrt{2}}\left(1+\frac{1}{4}\times\frac{z}{aa}+\frac{1\cdot 3}{4\cdot 8}\times\frac{z^4}{a^4}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^6}{a^6}+\text{ etc.}\right),$$

from which there becomes

$$s=\frac{1}{\sqrt{2}}\int dx\left(\frac{a}{z}+\frac{1}{4}\times\frac{z}{a}+\frac{1\cdot 3}{4\cdot 8}\times\frac{z^3}{a^3}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^5}{a^5}+\text{ etc.}\right)$$

$$s-y=\frac{1}{\sqrt{2}}\int dx\left(\frac{z}{a}+\frac{1}{4}\times\frac{z^3}{a^3}+\frac{1\cdot 3}{4\cdot 8}\times\frac{z^5}{a^5}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^7}{a^7}+\text{ etc.}\right).$$

24. But because we may wish this integration only for the case $x=c$, in which case there shall become $z=0$, that will be able to be expressed conveniently with the aid of the periphery of a circle. For on putting the ratio of the diameter to the periphery $=1:\pi$, there will be

$$\int\frac{dx}{z}=\int\frac{dx}{\sqrt{(cc-xx)}}=\frac{\pi}{2},$$

on putting $x=c$ after the integration. Moreover in a similar manner the following integration thus will be determined, so that there shall be

$$\int zdx=\frac{1}{2}\times\frac{\pi}{2}cc,$$

$$\int z^3dx=\frac{1\cdot 3}{2\cdot 4}\times\frac{\pi}{2}c^4,$$

$$\int z^5dx=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\times\frac{\pi}{2}c^6,$$

$$\int z^7dx=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\times\frac{\pi}{2}c^8,$$

etc.

[e.g., $\int zdx=\int\sqrt{(c^2-x^2)}dx$; putting $x=c\sin\theta\rightarrow\frac{1}{2}c^2\text{A}\sin\frac{x}{c}+\frac{1}{2}x\sqrt{(c^2-x^2)}$, etc.]

Therefore with these integrals called in to help, there will be

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1\cdot 1}{2\cdot 2}\times\frac{cc}{2aa} + \frac{1\cdot 1\cdot 3\cdot 3}{2\cdot 2\cdot 4\cdot 4}\times\frac{c^4}{4a^4} + \text{ etc.}\right)$$

$$AC - AD = \frac{\pi a}{2\sqrt{2}}\left(\frac{cc}{2aa} + \frac{1\cdot 3}{2\cdot 4}\times\frac{c^4}{4a^4} + \frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}\times\frac{c^4}{8a^6} + \text{ etc.}\right).$$

Therefore from these *AD* and *AC* will be found, as follows :

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2}\times\frac{cc}{2aa} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\times\frac{c^4}{4a^4} + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\times\frac{c^6}{8a^6} + \text{ etc.}\right)$$

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2}\times\frac{3}{1}\times\frac{cc}{2aa} - \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\times\frac{5}{3}\times\frac{c^4}{4a^4} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\times\frac{7}{5}\times\frac{c^6}{8a^4} - \text{ etc.}\right).$$

[*C. Truesdell* in *The Rational Mechanics of Flexible or Elastic Bodies* 1638-1788, sets out the above formulas in modern terms, thus on p. 206, eq. (175), we have :

$$AC = f = s(c) = \frac{\pi a}{2\sqrt{2}}\left\{1 + \sum_{n=1}^{\infty}\frac{1}{2^n}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left(\frac{c}{a}\right)^{2n}\right\},$$

$$AD = b = y(c) = \frac{\pi a}{2\sqrt{2}}\left\{1 - \sum_{n=1}^{\infty}\frac{2n+1}{2^n(2n-1)}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left(\frac{c}{a}\right)^{2n}\right\}.$$

And thus if $AE = c$ and $AD = b$ may be given, from these equations both the right line constant *a* and the length of the curve *AC* may be defined. Moreover in turn from the given length of the curve AC and from the right line *a*, by which the bending force may be determined, the right lines *AD* and *CD* will be able to be found.

THE FIRST KIND

25. Therefore because we have put in place the first kind, if in the general equation

$$dy = \frac{(aa - cc + xx)dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$$

there were $c = 0$ or $\frac{a}{c} = \infty,$ in which case a line results representing the natural state of the elastic lamina, we may also refer those cases to that same first kind, in which *c* is a quantity as small as possible, thus so that it may be able to be considered vanishing before a. Therefore because x itself cannot be greater than c, also x will vanish before a and thus this equation will be produced:

$$dy = \frac{adx}{\sqrt{2(cc-xx)}} ,$$

the integral of which is :

$$y = \frac{a}{\sqrt{2}} \mathrm{A}\sin\frac{x}{c},$$

which is the equation for the trochoid curve elongated to infinity. But there will become $AD = \frac{\pi a}{2\sqrt{2}}$, from which the length of that curve may disagree only by an infinitely small amount, on account of which the angle DAM is infinitely small. The length of the lamina $ACB = 2f$ and the absolute elasticity of that $= Ekk$; on account of $f = \frac{\pi a}{2\sqrt{2}}$, the force required for this infinitely small induced curvature of the lamina will be of a finite magnitude and indeed $= \frac{Ekk}{ff}\cdot\frac{\pi\pi}{4}$. Evidently, if the extremities A and B may be connected by the thread AB, this thread must be drawn together by the force $= \frac{Ekk}{ff}\cdot\frac{\pi\pi}{4}$.

THE SECOND KIND.

26. The second kind may constitute the case, in which $c > 0$, but yet $c < a$, evidently, if c may be contained within the limits 0 and a. Indeed in these cases the angle DAM is less than a right angle; for the sine of the angle PAM or the cosine of the angle DAM $= \frac{aa-cc}{aa}$. Therefore in this case the form of the curved line will be almost such a kind, as Figure 6 may show. Therefore because there is $c < a$, there will be $\frac{cc}{2aa} < \frac{1}{2}$; truly since there shall be $\frac{cc}{2aa} > 0$, certainly there will be $AC = f > \frac{\pi a}{2\sqrt{2}}$, from which $aa < \frac{8ff}{\pi\pi}$; whereby the force, by which the ends of the lamina A and B may be drawn towards each other with the aid of a thread AB, will be greater than in the preceding case, Cleary $> \frac{Ekk}{ff}\cdot\frac{\pi\pi}{4}$.

THE THIRD KIND

27. In the third kind I include a single case, where $c = a$, because in this case the axis *AP* touches the curve at the singe point *A*; and this kind possesses the singular name of a rectangular elastic curve. Therefore there will be

$$dy = \frac{xxdx}{\sqrt{\left(a^4 - x^4\right)}} \text{ and } ds = \frac{aadx}{\sqrt{\left(a^4 - x^4\right)}};$$

therefore in this case *AD* and *AC* thus themselves will be found, so that there shall be :

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \times \tfrac{1}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \tfrac{1}{4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \tfrac{1}{8} + \text{ etc.}\right)$$

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \times \frac{3}{1 \cdot 2} - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3 \cdot 4} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5 \cdot 8} - \text{ etc.}\right).$$

But hence although neither *b* nor *f* can be assigned precisely by *a*, yet in another place I have shown a conspicuous relation between these quantities. One may know that it is shown that $4bf = \pi aa$ or the rectangle formed from *AD* and *AC* will be equal to the area of a circle, of which the diameter is $= AE$. But on deducing $f = \frac{5a}{6} \times \frac{\pi}{2}$ approximately from the calculation, thus so that there shall be $a = \frac{12f}{5\pi}$; hence the force, by which the ends *A* and *B* of the lamina must be drawn towards each other, will be

$$= \frac{Ekk}{ff} \times \frac{25}{72}\pi\pi.$$

Truly there will be found closer :

$$f = \frac{\pi a}{2\sqrt{2}} \cdot 1{,}1803206$$

and hence

$$b = \frac{\pi aa}{4f} = \frac{a}{\sqrt{2}} \times 1{,}1803206;$$

from which in pure numbers there will be

$$\frac{f}{a} = 1{,}311006 \text{ and } \frac{b}{a} = 0{,}834612.$$

THE FOURTH KIND.

28. If $c > a$, the fourth kind will be apparent as far as that point, so that there becomes $AD = b = 0$; whereby the other limit of c may be defined by this equation :

$$1 = \frac{1^2}{2^2} \times \frac{3}{1} \times \frac{cc}{2aa} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3} \times \frac{c^4}{4a^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5} \times \frac{c^6}{8a^6} + \text{ etc.}$$

Therefore in this kind (Fig. 7), since there shall be $c > a$, the curve at A rises above the axis AE and puts in place the angle PAM, of which the sine will be $= \frac{cc - aa}{aa}$, [See § 16.] ; but soon we will see this angle PAM to be less than 40° 41′; because, if it may acquire this value, the interval AD will vanish, which case I refer to the fifth kind. Hence in the fourth kind the cases will be contained, in which is understood $\frac{cc}{aa}$ lies between the limits 1 and 1,651868. Moreover the form of this curvature is understood from the figure, provided that it is noted, where the closer $\frac{cc}{aa}$ will have approached to the lower limit 1,651868, that a smaller interval AD is going to be present and with that closer to the ends of the lamina A and B for themselves to be removed. There it can happen, so that the gibbousness of the lamina m and R likewise M and r not only are mutual tangents, but also will intersect, and intersections of this kind will be multiplied indefinitely, then all the diameters DC, dc coincide and are combined with the axis AE.

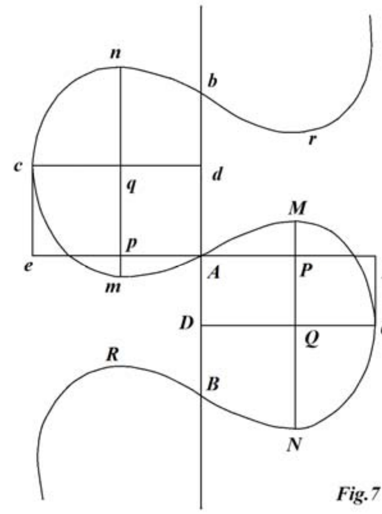

Fig.7

THE FIFTH KIND

29. If this has happened, the fifth kind will arise (Fig. 8), of which the nature will be expressed by this equation between the coordinates $AP = x$ and $PM = y$:

$$dy = \frac{(cc - aa - xx)\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}} ,$$

with this relation present between a and c, so that the interval $AD = b$ shall be $= 0$.
Putting $\frac{cc}{2aa} = v$, and v must be defined from this infinite equation

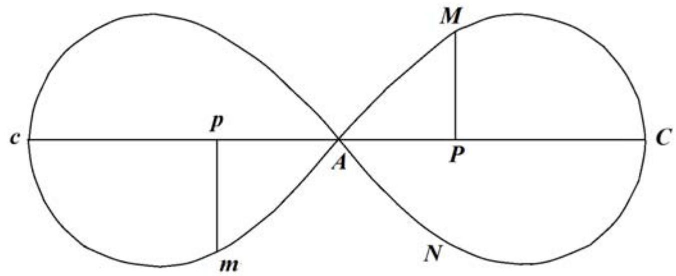

Fig.8

$$1=\frac{1\cdot 3}{2\cdot 2}v+\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}v^2+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}v^3+\text{ etc.}$$

$$i.e.\left[\begin{array}{l} AD=b=\frac{\pi a}{2\sqrt{2}}\left(1-\frac{1^2}{2^2}\times\frac{3}{1}\times v-\frac{1^2\cdot 3^2}{2^2\cdot 4^2}\times\frac{5}{3}\times v^2-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\times\frac{7}{5}\times v^3-\text{ etc.}\right)=0, \\ \qquad\qquad \text{where } v=\frac{cc}{2aa}. \end{array}\right]$$

In the first place the limits are sought by the usual methods for that, or perhaps by testing, between which the true value of *v* may be contained, and limits of this kind $v=0,824$ and $v=0,828$ will be found. But if now each may be substituted into the equation, from the two errors arising there will be concluded finally to become

$$v=0,825934=\frac{cc}{2aa};$$

from which there comes about

$$\frac{cc}{aa}=1,651868 \text{ and } \frac{cc-aa}{aa}=0,651868;$$

which expression since it shall be the sine of the angle *PAM*, from tables this angle will be found to be $=40°\ 41'$; and thus the double of this, or the angle *MAN* will be $=81^{\circ}\ 22'$. Whereby, if the ends of the elastic lamina may be contracted so far, that they may be touching each other, then the curve *AMCNA* will be formed and both ends at *A* will make an angle of 81° 22'.

THE SIXTH KIND

30. If both ends *A* and *B* of the lamina (Fig. 9), after they had been induced in turn, by a force increased in the opposite direction may

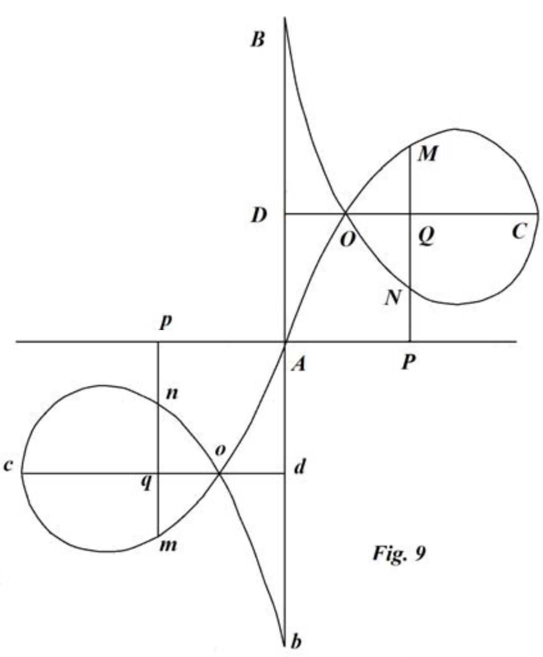

Fig. 9

be drawn away from each other, a curve of this form *AMCNB*, which may constitute the sixth kind. Therefore with curves belonging to this form, there will be

$$\frac{cc}{2aa} > 0,825934,$$

yet thus, so that $\frac{cc}{2aa} < 1$. But if now there shall be $cc = 2aa$, the seventh kind will need to be explained next. Therefore with these curves the angle *PAM*, which the curve makes with the axis at , will be greater than $40° \ 41'$, yet less than a right angle ; for since its sine shall be $= \frac{cc}{2aa}$, on account of $cc < 2aa$, this sine by necessity is less than the whole sine and therefore neither can the angle *PAM* become a right angle, unless there may be put $cc = 2aa$.

THE SEVENTH KIND

31. Now there shall be $cc = 2aa$, in which case the seventh kind is put in place, and the nature of the curve will be expressed by this equation

$$dy = \frac{(aa - xx)dx}{x\sqrt{(2aa - xx)}};$$

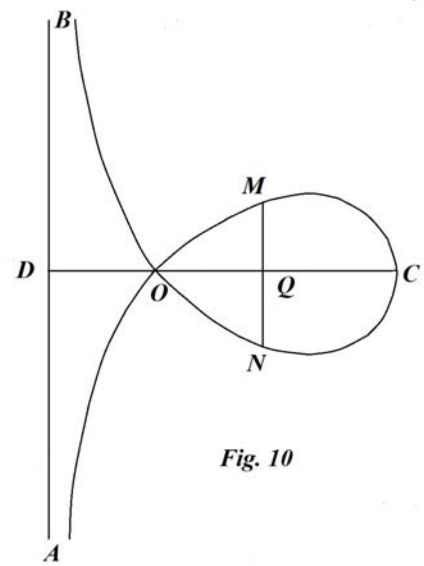

Fig. 10

from which it is deduced the branches of the curve (Fig. 10) *A* and *B* thus extend to infinity, so that the right line *AB* becomes an asymptote of the curve. Therefore each infinite branch of the curve *AMC* and *BNC* comes about, that which is understood from the series found above for the arc *AC*; indeed there will be

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{ etc.}\right),$$

the sum of which series is infinite. Because therefore if the length of the lamina *AC* were finite $= f$, it is necessary that there shall be $a = 0$ and hence also $CD = c = 0$; therefore the lamina, after it had curved at the node, in this case will be extended in a direction, for which extension there will be a need for an infinite force. But if the lamina were infinitely long, the curve will form a node and converge to the asymptote *AB*, with $CD = c$. Moreover the equation for this curve can be integrated with the help of logarithms, and indeed the curve will be found :

$$y = \sqrt{(cc-xx)} - \frac{c}{2} l \frac{\sqrt{(cc-xx)}}{x},$$

with the abscissa x taken on that diameter DC, thus so that there shall be $DQ = x$ and $QM = y$; for the applied line y will vanish on putting $x = CD = c$. But at the node O the applied line y equally will vanish ; towards finding that place the equation is put :

$$\frac{2\sqrt{(cc-xx)}}{c} = l \frac{c+\sqrt{(cc-xx)}}{x}.$$

Let φ be the angle, of which the cosine $= \frac{x}{c}$ and the sine $= \frac{\sqrt{(cc-xx)}}{c}$, there will be

$$2\sin\varphi = l \text{ tang } \left(45° + \tfrac{1}{2}\varphi\right),$$

whereby the logarithm must be taken of the hyperbolic kind; if tables of this kind may be lacking, the logarithm of the tangent of the angle $45° + \frac{1}{2}\varphi$ may be taken from tables of common logarithms, from which the characteristic of ten may be taken, and the remainder will be $= \omega$; with which done there will be $2 \sin\varphi = \omega \cdot 2,30258509$ [$i.e. \ln 10 = 2,30258509$]; with common logarithms taken again there will be

$$l2 + l \sin \varphi = l\omega + 0,3622156886$$

or

$$l \sin \varphi = l\, \omega + 0,0611856930.$$

With this method adopted soon truly a nearer value of the angle φ may be elicited ; from which again by the trial and error rule the true value of the angle φ will be defined and from that the $x = DO$. Moreover the angle φ is found in this way $= 73°14'12''$, from which there is produced

$$\frac{x}{c} = 0,2884191 \text{ et } \frac{\sqrt{(cc-xx)}}{c} = 0,9575042 \;;$$

and thus the angle QOM truly shall become $= 2\varphi - 90 = 56°\ 28'\ 24''$ and thus the angle $MON = 112°\ 56'\ 48''$. Therefore since with the fifth kind the angle of the node shall be 81^{o} 22', in the sixth kind the angle of the node MON will be contained between the limits 81° 22' and 112° 56' 48''. But in the fourth kind, if indeed a node may be given, the angle of this will be less than 81° 22'.

THE EIGHTH KIND

32. Now there shall be $cc > 2aa$, on putting $cc = 2aa + gg$; the equation for the curve will be, on account of $aa = \frac{cc - gg}{2}$:

$$dy = \frac{\left(xx - \frac{1}{2}cc - \frac{1}{2}gg\right)dx}{\sqrt{(cc - xx)(xx - gg)}} ,$$

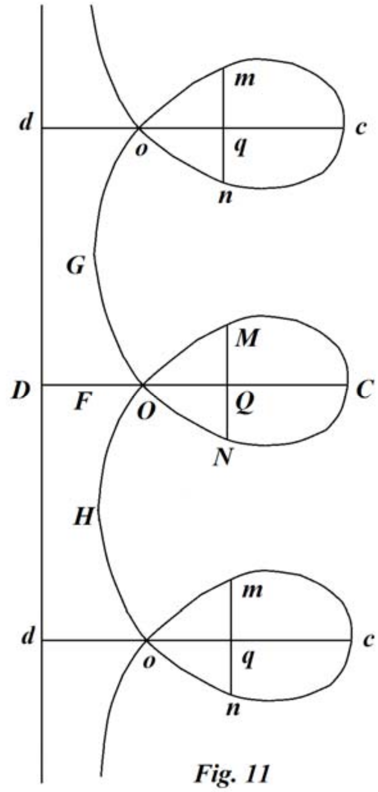

Fig. 11

by which equation the eighth kind will be contained, and there will be (Fig.11), if the right line *dDd* may represent the direction of the force acting, $x = DQ$ and $y = QM$. Therefore in the first place it is apparent the applied line *y* cannot be real, unless there shall be $x > g$, then truly *x* cannot exceed the right line $DC = c$, from which with the line taken $DF = g$ the whole curve will be contained between the right lines parallel to *dd* drawn through the points *C* and *F*, which likewise touch the curve. But it is the same, whichever of the right lines *c* and *g* shall be greater, provided they shall be unequal, that the equation may not be varied, if the right lines *c* and *g* may be interchanged between each other. Then truly this curve also will have infinite diameters between the parallel line themselves *DC, dc, dc* and which equally may be drawn normal to *dDd* through the individual points *G* and *H*; but at no time through the whole curve will a point of contrary inflection be given and thus it will progress to infinity with a continued curvature on both sides to infinity, as the figure shows ; but the angles *MON*, which may be put in place at the nodes, will be greater than 112° 56' 42".

THE NINTH KIND

33. Since in this above kind not only the case may be contained, in which $gg < cc$, but also, in which $gg > cc$, at this stage a singe case remains, in which $c = g$, in which indeed the whole curve is returned vanishing in space, on account of $CF = 0$. But if moreover we may put both *c* and *g* infinite, thus still, so that the difference of these becomes finite, the curve will occupy a finite space. Towards finding this curve therefore there may be put $g = c - 2h$ and $x = c - h - t$, and on account of $c = \infty$, the quantities *h* and *t* truly are finite, there will be

$$\tfrac{1}{2}cc + \tfrac{1}{2}gg = cc - 2ch \text{ and } xx - \tfrac{1}{2}cc - \tfrac{1}{2}gg = -2ct ;$$

then truly

$$cc - xx = 2c(h + t) \text{ and } xx - gg = 2c(h - t);$$

from which the following equation will be produced

$$dy = \frac{tdt}{\sqrt{(hh - tt)}}$$

for a circle. The elastic lamina therefore in this case is curved in a circle, as now we have indicated above; therefore a circle will constitute the ninth and final kind of curve.

ADDITAMENTUM IA

DE CURVIS ELASTICUS

1. Iam pridem summi quique Geometrae agnoverunt Methodi in hoc Libro traditae non solum maximum esse usum in ipsa Analysi, sed etiam eam ad resolutionem Problematum physicorum amplissimum subsidium afferre. Cum enim Mundi universi fabrica sit perfectissima atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat; quamobrem dubium prorsus est nullum, quin omnes Mundi effectus ex causis finalibus ope Methodi maximorum et minimarum aeque feliciter determinari queant, atque ex ipsis causis efficientibus. Huius rei vero passim tam eximia extant specimina, ut ad veritatis confirmationem pluribus Exemplis omnino non indigeamus; quin potius in hoc erit elaborandum, ut in quovis Quaestionum naturalium genere ea investigetur quantitas, quae maximum minimumve induat valorem; quod negotium ad Philosophiam potius quam ad Mathesin pertinere videtur. Cum igitur duplex pateat via effectus Naturae cognoscendi, alter per causas efficientes, quae Methodus directa vocari solet, alter per causas finales, Mathematicus utraque pari successu utitur. Quando scilicet causae efficientes nimis sunt absconditae, finales autem nostram cognitionem minus effugiunt, per Methodum indirectam Quaestio solet resolvi; e contrario autem Methodus directa adhibetur, quoties ex causis efficientibus effectum definire licet. Inprimis autem opera est adhibenda, ut per utramque viam aditus ad Solutionem aperiatur; sic enim non solum altera Solutio per alteram maxime confirmatur, sed etiam ex utriusque consensu summam percipimus voluptatem. Hoc modo curvatura funis seu catenae suspensae duplici via est eruta, altera a priori ex sollicitationibus gravitatis, altera vero per Methodum maximorum ac minimorum, quoniam funis eiusmodi curvaturam recipere debere intelligebatur, cuius centrum gravitatis infimum obtineret locum. Similiter curvatura radiorum per medium diaphanum variae densitatis transeuntium tam a priori est determinata, quam etiam ex hoc principio, quod tempore brevissimo ad datum locum pervenire debeant. Plurima autem alia similia exempla a Viris Celeberrimis BERNOULLIIS aliisque sunt prolata, quibus tam Methodus solvendi a priori quam cognitio causarum efficientium maxima accepit incrementa. Quanquam igitur ob haec tam multa ac praeclara specimina dubium nullum relinquitur, quin in omnibus lineis curvis, quas Solutio Problematum physico-mathematicorum suppeditat, maximi minimive cuiuspiam indoles locum obtineat, tamen saepenumero hoc ipsum maximum vel minimum difficillime perspicitur, etiamsi a priori Solutionem eruere licuisset. Sic etsi figura, quam lamina Elastica incurvata induit, iam pridem est cognita, tamen, quemadmodum ea curva per Methodum maximorum et minimorum, hoc est, per causas finales investigari possit, a nemine adhuc est animadversum. Quamobrem, cum Vir Celeberrimus atque in hoc sublimi naturam scrutandi genere perspicacissimus DANIEL BERNOULLI mihi indicasset se universam vim, quae in lamina Elastica incurvata insit, una quadam formula, quam vim potentialem appellat, complecti posse, hancque expressionem in curva Elastica minimam esse oportere, quoniam hoc invento Methodus mea maximorum ac minimorum hoc Libro tradita mirifice illustratur eiusque usus amplissimus maxime evincitur, hanc occasionem exoptatissimam praetermittere non possum, quin hanc insignem curvae Elasticae

proprietatem a Celeberrimo BERNOULLIO observatam publicando simul Methodi meae usum clarius patefaciam. Continet enim ista proprietas in se differentialia secundi gradus, ita ut ei evolvendae Methodi Problema isoperimetricum solvendi ante traditae non sufficiant.

2. Sit AB (Fig. 1) lamina Elastica utcunque incurvata; vocetur arcus $AM = s$ et radius osculi curvae $MR = R$; atque, secundum BERNOULLIUM, exprimetur *vis potentialis* in laminae portione *AM* contenta hac formula $\int \frac{ds}{RR}$, siquidem lamina sit ubique aequaliter crassa, lata et Elastica atque in statu naturali in directum extensa. Hinc ista erit curvae *AM* indoles, ut in ea haec expressio omnium minimum obtineat valorem. Quoniam vero in radio osculi *R* differentialia secundi gradus insunt, ad curvam hac proprietate praeditam determinandam quatuor opus erit conditionibus, id quod cum quaestionis natura apprime convenit. Cum enim Fig.I per datos terminos *A* et *B* infinitae laminae Elasticae eaeque eiusdem longitudinis inflecti queant, quaestio non erit determinata, nisi praeter duo puncta *A* et *B* simul alia duo puncta seu, quod eodem redit, positio tangentium in punctis extremis *A* et *B* praescribatur. Proposita namque lamina Elastica longiori, quam est distantia punctorum *A* et *B*, ea non solum ita incurvari potest, ut intra terminos *A* et *B* contineatur, sed etiam ut eius tangentes in punctis hisce datas teneant directiones. His notatis quaestio de invenienda curvatura laminae Elasticae ex hoc fonte resolvenda ita debet proponi: *ut inter omnes curvas eiusdem longitudinis, quae non solum per puncta A et B transeant, sed etiam in his punctis a rectis positione datis tangantur, definiatur ea, in qua sit valor huius expressionis* $\int \frac{ds}{RR}$ *minimus.*

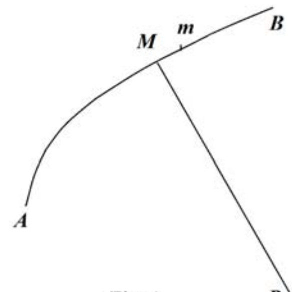

Fig. 1

3. Quia solutionem ad coordinatas orthogonales (Fig. 2) accommodari convenit, sumatur recta quaecunque AD pro axe, in qua sit abscissa $AP = x$, applicata $PM = y$; ponatur, uti Methodus tradita iubet, $dy = pdx$, $dp = qdx$; erit elementum curvae $Mm = ds = dx\sqrt{(1+pp)}$. Primum ergo, quia curvae, ex quibus quaesita erui debet, isoperimetrae statuuntur, habebitur ista expressio consideranda $\int dx\sqrt{(1+pp)}$, quae cum generali $\int Zdx$ comparata hunc praebet valorem differentialem $\frac{1}{dx} d \cdot \frac{p}{\sqrt{(1+pp)}}$. Deinde cum sit radius osculi

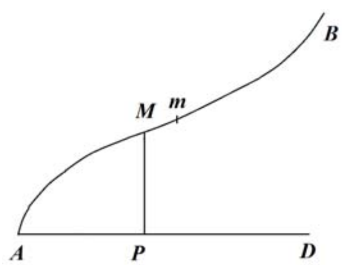

Fig. 2

$$= \frac{dx(1+pp)^{3:2}}{dp} = \frac{(1+pp)^{3:2}}{q} = R,$$

formula $\int \frac{ds}{RR}$, quae minimum esse debet, abit in $\int \frac{qqdx}{(1+pp)^{5:2}}$. Comparetur haec cum forma generali $\int Zdx$; erit $Z = \frac{qq}{(1+pp)^{5:2}}$ et posito

$$dZ = Mdx + Ndy + Pdp + Qdq\,,$$

erit

$$M = 0,\ N = 0,\ P = \frac{-5pqq}{(1+pp)^{7:2}} \text{ et } Q = \frac{2q}{(1+pp)^{5:2}}.$$

Valor ergo differentialis ex hac formula $\int \frac{qq}{(1+pp)^{5:2}}$ oriundus erit $-\frac{dP}{dx} + \frac{ddQ}{dx^2}$ Quamobrem pro curva quaesita haec habebitur aequatio

$$\frac{\alpha}{dx} d \cdot \frac{p}{\sqrt{(1+pp)}} = \frac{dP}{dx} - \frac{ddQ}{dx^2},$$

quae per *dx* multiplicata et integrata dat

$$\frac{\alpha p}{\sqrt{(1+pp)}} + \beta = P - \frac{dQ}{dx}.$$

Multiplicetur haec aequatio per $qdx = dp$, ut prodeat

$$\frac{\alpha pdp}{\sqrt{(1+pp)}} + \beta dp = Pdp - qdQ.$$

Cum autem ob $M = 0$ et $N = 0$ sit $dZ = Pdp + Qdq$, erit $Pdp = dZ - Qdq$, quo valore loco *Pdp* substituto emerget

$$\frac{\alpha pdp}{\sqrt{(1+pp)}} + \beta dp = dZ - Qdq - qdQ\,;$$

quae denuo integrata dat

$$\alpha \sqrt{(1+pp)} + \beta p + \gamma = Z - Qq.$$

Iam cum sit

$$Z = \frac{qq}{(1+pp)^{5:2}} \text{ et } Q = \frac{2q}{(1+pp)^{5:2}}\,,$$

erit

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=-\frac{qq}{(1+pp)^{5:2}}.$$

Sumantur constantes arbitrariae $\alpha,\ \beta$ et γ negative eritque

$$q=(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}=\frac{dp}{dx}.$$

Hinc ergo elicitur sequens aequatio

$$dx=\frac{dp}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}.$$

Deinde ob $dy=pdx$, habebitur quoque

$$dy=\frac{pdp}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}};$$

quae duae aequationes sufficerent ad curvam per quadraturas construendam.

4. Harum formularum sic in genere spectatarum neutra est integrabilis; combinari autem certo quodam modo possunt, ut aggregatum integrationem admittat. Cum enim sit

$$d\cdot\frac{2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}{\sqrt{\sqrt{(1+pp)}}}=\frac{dp(\beta-\gamma p)}{(1+pp)^{5:4}\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}.$$

erit

$$\frac{2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p+\gamma\right)}}{(1+pp)^{1:4}}=\beta x-\gamma y+\delta.$$

Quoniam axis positio est arbitraria, constans δ sine defectu amplitudinis omitti potest. Deinde vero etiam axis ita mutari potest, ut fiat

$$\frac{\beta x-\gamma y}{\sqrt{(\beta\beta+\gamma\gamma)}}$$

abscissa, eritque applicata

$$\frac{\gamma x+\beta y}{\sqrt{(\beta\beta+\gamma\gamma)}};$$

hinc etiam tuto γ nihilo aequalis poni potest, quia nihil impedit, quominus illa nova abscissa per x exprimatur. Hanc ob rem habebimus pro curva Elastica istam aequationem

$$2\sqrt{\left(\alpha\sqrt{(1+pp)}+\beta p\right)}=\beta x(1+pp)^{1:4},$$

quae sumptis quadratis dat

$$4\alpha\sqrt{(1+pp)}+4\beta p=\beta^2x^2\sqrt{(1+pp)}.$$

Sit ad homogeneitatem introducendam $\alpha=\frac{4m}{aa}$ et $\beta=\frac{4n}{aa}$ erit

$$na^2p=(nnxx-maa)\sqrt{(1+pp)},$$

unde

$$n^2a^4pp=(nnxx-maa)^2(1+pp)$$

ideoque

$$p=\frac{nnxx-maa}{\sqrt{\left(n^2a^4-(nnxx-maa)^2\right)}}=\frac{dy}{dx}.$$

Mutatis ergo constantibus atque abscissam x data constante sive augendo sive minuendo habebitur huiusmodi aequatio pro curva Elastica generalis

$$dy=\frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}}.$$

ex qua oritur

$$ds=\frac{aadx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}};$$

ex quibus aequationibus consensus huius curvae inventae cum curva Elastica iam pridem eruta manifesto elucet.

5. Quo autem iste consensus clarius ob oculos ponatur, naturam curvae Elasticae a priori quoque investigabo; quod etsi iam a Viro summo JACOBO BERNOULLIO

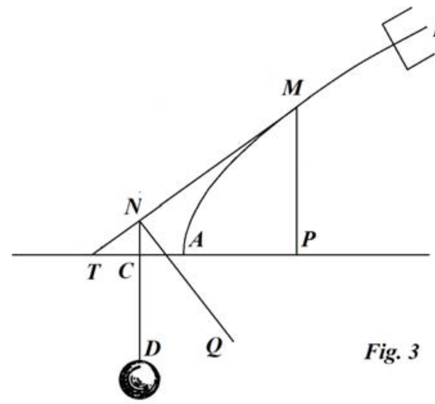

Fig. 3

excellentissime est factum, tamen hac idonea occasione oblata nonnulla circa indolem curvarum Elasticarum earumque varias species et figuras adiiciam, quae ab aliis vel praetermissa vel leviter tantum pertractata esse video. Sit lamina Elastica (Fig. 3) *AB* in *B* ita muro seu pavimento firmo infixa, ut haec extremitas *B* non solum firmiter retineatur, sed etiam tangentis in *B* positio determinetur. In *A* autem lamina connexam habeat virgam rigidam *AC*, cui normaliter applicata sit vis $CD = P$, qua lamina in statum incurvatum *BMA* redigatur. Sumatur haec recta *AC* producta pro axe ac posita $AC = c$ sit abscissa $AP = x$, applicata $PM = y$. Quodsi iam lamina in *M* omnem elasticitatem subito amitteret ac perfecte flexilis evaderet, a vi *P* utique inflecteretur, inflexione proficiscente a vis *P* momento $= P(c+x)$. Quominus ergo haec inflexio actu sequatur, elasticitas laminae in *M* in aequilibrio consistit cum vis sollicitantis momento $P(c+x)$. Elasticitas autem primo ab indole materiae, ex qua lamina constat et quam ubique eandem statuo, pendet, tum vero simul ab incurvatione laminae in puncto *M*, ita ut sit reciproce proportionalis radio osculi in *M*. Sit ergo radius osculi in

$M = R = -\dfrac{ds^3}{dxddy}$, existente $ds = \sqrt{dx^2 + dy^2}$ et *dx* constante, atque exprimat $\dfrac{Ekk}{R}$ vim Elasticam laminae in M, quae cum momento vis sollicantis

$P(c+x)$ in aequilibrio consistat, ita ut sit $P(c+x) = \dfrac{Ekk}{R} = -\dfrac{Ekkdxddy}{ds^3}$.

Aequatio haec per *dx* multiplicata fit integrabilis eritque integrale

$$P\left(\frac{xx}{2} + cx + f\right) = -\frac{Ekkdy}{\sqrt{dx^2 + dy^2}};$$

unde oritur

$$dy = \frac{-Pdx\left(\tfrac{1}{2}xx + cx + f\right)}{\sqrt{\left(E^2k^4 - P^2\left(\tfrac{1}{2}xx + cx + f\right)^2\right)}},$$

quae aequatio omnino convenit cum ea, quam modo per Methodum maximorum ac minimarum ex principio BERNOULLIANO elicui.

6. Ex comparatione huius aequationis cum ante inventa definiri poterit vis, quae requiritur ad datam laminae curvaturam inducendam, siquidem curvatura contineatur in aequatione generali inventa. Teneat scilicet lamina Elastica figuram *AMB*, cuius natura exprimatur hac aequatione

$$dy = \frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}};$$

exprimat vero *Ekk* huius laminae elastricitatem absolutam, ita scilicet, ut *Ekk* in quovis loco per radium osculi divisa praebeat vim Elasticam veram. Ad comparationem instituendam multiplicetur numerator et denominator per $\frac{Ekk}{aa}$, ut habeatur

$$dy = \frac{Ekkdx(\alpha+\beta x+\gamma xx):aa}{\sqrt{\left(E^2k^4-\frac{E^2k^4}{a^4}(\alpha+\beta x+\gamma xx)^2\right)}}.$$

Nunc ergo erit

$$-\tfrac{1}{2}P = \frac{Ekk\gamma}{aa},\ -Pc = \frac{Ekk\beta}{aa},\ -Pf = \frac{Ekk\alpha}{aa}$$

–

ideoque vis *CD* sollicitans $= -\frac{2Ekk}{aa}$, intervallum $AC = c = \frac{\beta}{2\gamma}$ et constans $f = \frac{\alpha}{2\gamma}$.

7. Ut igitur lamina Elastica *AB* altero termino *B* muro infixa incurvetur in figuram *AMB*, cuius natura exprimitur hac aequatione

$$dy = \frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(\alpha^4-(\alpha+\beta x+\gamma xx)^2\right)}},$$

necesse est, ut haec lamina sollicitetur in directione *CD* normali ad axem *AP*, sumpta distantia $AC = \frac{\beta}{2\gamma}$, a vi $CD = -\frac{Ekk\gamma}{aa}$; quae vis scilicet in plagam contrariam, ac figura indicat, dirigetur, si γ fuerit quantitas positiva. Quia $\frac{Ekk}{R}$ aequivalet momento vis sollicitantis, expression $\frac{Ekk}{aa}$ homogenea erit ponderi seu vi purae, quae vis propterea $\frac{Ekk}{aa}$ cognoscetur ex elasticitate laminae. Sit haec vis $= F$, atque erit vis flectens *CD* ad hanc vim *F*, ut -2γ ad 1; erit enim γ numerus purus.

8. Hinc porro definiri potest vis ad laminae portionem *BM* in statu suo conservandam requisita, si portio *AM* prorsus rescindatur. Rescissa hac portione *AM* desinat lamina Elastica in virgam rigidam *MT* omnis flexionis expertem, quae autem cum lamina ita sit connexa, ut perpetuo tangentem in puncto *M* referat, utcunque lamina inclinetur. Hoc

posito ex antecedentibus manifestum est ad conservationem curvaturae *BM* requiri, ut virga *MT* in puncto *N* trahatur in directione *ND* vi, quae sit

$$-\frac{2Ekk\gamma}{aa};$$

directio autem *ND* erit normalis ad axem *AP* atque intervallum *AC* erit $=\frac{\beta}{2\gamma}$

Distantia itaque *MN* fiet

$$\frac{ds}{dx}CP=\frac{ds}{dx}\cdot\frac{\beta+2\gamma x}{2\gamma}=\frac{(\beta+2\gamma x)ds}{2\gamma dx},$$

est vero

$$\frac{ds}{dx}=\frac{aa}{\sqrt{\left(\alpha^4-(\alpha+\beta x+\gamma xx)^2\right)}}.$$

Quodsi haec vis $ND=-\frac{2Ekk\gamma}{aa}$ resolvatur in normalem *NQ* ad tangentem *MT* et tangentialem *NT*, erit vis normalis $NQ=-\frac{2Ekk\gamma}{aa}\cdot\frac{dx}{ds}$ et vis tangentialis $NT=-\frac{2Ekk\gamma}{aa}\cdot\frac{dy}{ds}$.

9. Sin autem pars *BM* rescindatur, relicta parte *AM*, quae in directione *CD* sollicitatur ut ante vi $=-\frac{2Ekk\gamma}{aa}$,
ad curvaturam *AM* conservandam extremitas *M*, quae connexa intelligatur cum virga rigida tangente *MN*, sollicitari debebit in puncto *N* a vi pariter $=-\frac{2Ekk\gamma}{aa}$ sed in directione contraria ei, quam casu praecedente invenimus. Perpetuo enim vires utrique extremitati laminae incurvatae applicandae se mutuo destruere atque adeo aequales et directiones oppositas habere debent. Alioquin enim tota lamina moveretur, ad quem motum compescendum opus foret vi aequilibrium inter vires sollicitantes producente. Hinc ergo vires cuicunque portioni laminae resectae applicandae facillime definiri possunt, quae iam inductam curvaturam conservent.

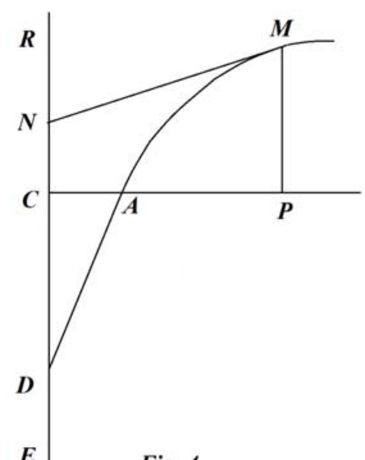

Fig. 4

10. Sit *AM* (Fig.4) lamina Elastica incurvata, quae in *A* et *M* annexas habeat virgas rigidas *AD, MN*, quibus in directionibus directe oppositis *DE, NR* applicatae sint vires aequales *DE, NR*, quae in aequilibrio consistentes laminae curvaturam *AM* inducant, pro qua aequationem quaeri oporteat. Primum ergo pro axe sumatur recta *AP* per punctum *A* transiens atque ad directionem vis sollicitantis *ER* normalis. Ponatur Elasticitas laminae absoluta $=Ekk$ sitque anguli *CAD*, quem tangens *AD* in *A* cum axe constituit et qui est datus, sinus $=m$, cosinus $=n$, existente sinu toto $=1$, ita ut sit $mm+nn=1$. Vocetur porro distantia $AC=c$ et vis flectens $DE=NR=P$, ac positis abscissa $AP=x$ applicata $PM=y$ natura curvae hac exprimetur aequatione

$$dy=\frac{-Pdx\left(\frac{1}{2}xx+cx+f\right)}{\sqrt{\left(E^2k^4-P^2\left(\frac{1}{2}xx+cx+f\right)^2\right)}}.$$

Quoniam vero directio tangentis in *A* datur, posito $x=0$ fieri debet $\frac{dy}{dx}=\frac{m}{n}$;
hinc ergo obtinebitur

$$\frac{m}{n}=\frac{-Pf}{\sqrt{\left(E^2k^4-P^2f^2\right)}}=\frac{m}{\sqrt{(1-mm)}} \text{ and } m=-\frac{Pf}{Ekk}.$$

Determinatur ergo hinc constans *f*, ita ut sit

$$f=-\frac{mEkk}{P},$$

ideoque hinc tota curva determinatur.

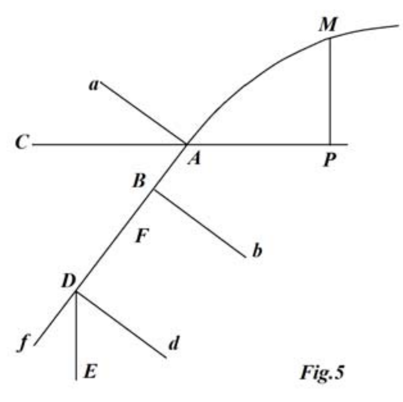

Fig.5

11. Ad curvaturam ergo superiori aequatione expressam (Fig. 5) laminae *AM* inducendam tangenti *AD* in puncto *D*, ita ut sit $AD=\frac{c}{n}$, applicatam esse oportet vim $DE=P$, cuius directio sit parallela et applicatis *PM*. Resolvatur haec vis *DE* in duas laterales *Dd, Df* inter se normales ; erit vis $Dd=Pn$ et vis $Df=Pm$.
Quo iam consideratio rectae *AD* ex computo expellatur, loco vis *Dd* in datis punctis *A* et *B*, sumpto intervallo $AB=h$, duae vires substitui possunt $Aa=p$, $Bb=q$, normales pariter ad virgam *AB*, sumendo

$$ph=Pn\cdot BD=nP\left(\frac{c}{n}-h\right) \text{ et } q=p+nP.$$

Quia deinceps perinde est, in quonam virgae *AD* puncto applicetur vis tangentialis

$Df = mP$, applicetur ea in ipso puncto A ponendo $AF = nP$. Sit autem haec vis $AF = r$, ita ut lamina MA a tribus viribus $Aa = p,\ Bb = q$ et $AF = r$ sollicitetur, a quibus qualis incurvatio oriatur, investigemus.

12. Primo ergo, cum sit $mP = r$ erit $P = \frac{r}{m}$, qui valor substitutus in prioribus aequationibus dabit

$$ph = \frac{cr}{m} - \frac{nhr}{m} \text{ et } q = p + \frac{nr}{m}.$$

Hinc erit

$$\frac{n}{m} = \frac{q-p}{r};$$

ex qua aequatione primum positio axis AP innotescit; erit nempe tangens anguli $CAD = \frac{r}{q-p}$, hinc

$$m = \frac{r}{\sqrt{\left(r^2 + (q-p)^2\right)}} \text{ et } n = \frac{q-p}{\sqrt{\left(r^2 + (q-p)^2\right)}}.$$

Deinde ex aequatione

$$hp = \frac{cr}{m} - \frac{nhr}{m} = \frac{cr}{m} - hq + hp$$

fit

$$c = \frac{mhq}{r} \text{ seu } c = \frac{hq}{\sqrt{\left(r^2 + (q-p)^2\right)}}$$

atque

$$P = \sqrt{\left(rr + (q-p)^2\right)}.$$

Cum autem sit

$$f = \frac{-mEkk}{P} = \frac{-Ekkr}{rr + (q-p)^2},$$

erit

$$\tfrac{1}{2}xx + cx + f = \tfrac{1}{2}xx + \frac{hqx}{\sqrt{\left(rr + (q-p)^2\right)}} - \frac{Ekkr}{rr + (q-p)^2};$$

unde pro curva quaesita ista obtinebitur aequatio

$$dy = \frac{dx\left(\frac{Ekkr}{\sqrt{\left(rr+(q-p)^2\right)}} - hqx - \frac{1}{2}xx\sqrt{\left(rr+(q-p)^2\right)}\right)}{\sqrt{\left(E^2k^4 - \left(\frac{Ekkr}{\sqrt{\left(rr+(q-p)^2\right)}} - hqx - \frac{1}{2}xx\sqrt{\left(rr+(q-p)^2\right)}\right)^2\right)}}.$$

Haec autem aequatio maxime est accommodata ad modum maxime consuetum laminas incurvandi, dum eae vel forcipe vel duobus digitis apprehenduntur; quorum alter laminam in directione *Aa*, alter in directione *Bb* urget, praeter quas vires lamina insuper in directione *AF* protrahi potest.

13. Si vis tangentialis $AF = r$ evanescat, incidet axis *AP* in ipsam tangentem *AF* productam eritque tum

$$dy = -\frac{dx\left(hqx + \frac{1}{2}(q-p)xx\right)}{\sqrt{\left(E^2k^4 - \left(hqx + \frac{1}{2}(q-p)xx\right)^2\right)}}.$$

Sin autem vires normales *p* et *q* :fiant inter se aequales, erit axis *AP* normalis ad tangentem *AF* ob $n = 0$ et pro curva orietur haec aequatio

$$dy = \frac{dx\left(Ekk - hqx - \frac{1}{2}rxx\right)}{\sqrt{\left(2Ekk\left(hqx + \frac{1}{2}rxx\right) - \left(hqx + \frac{1}{2}rxx\right)^2\right)}}.$$

Hic si praeterea fuerit $r = 0$, ita ut lamina in punctis *A* et *B* urgeatur a viribus aequalibus *Aa, Bb*, contrariis tantum, natura curvae exprimetur hac aequatione

$$dy = \frac{dx(Ekk - hqx)}{\sqrt{hq(2Ekkx - hqxx)}},$$

quae integrata dat

$$y = \sqrt{\frac{2Ekkx - hqxx}{hq}};$$

quae est pro Circulo, lamina ergo hoc casu in arcum Circuli incurvatur, cuius radius erit

$$= \frac{Ekk}{hq}.$$

ENUMERATIO CURVARUM ELASTICARUM

14. Cum igitur videamus non solum Circulum in curvarum Elasticarum classe contineri, sed etiam in ipsis infinitam varietatem locum habere, operae pretium erit hic enumerationem omnium variarum specierum in hoc curvarum genere contentarum instituere. Hoc enim modo non solum indoles harum curvarum penitius perspicietur, sed etiam casu quocunque oblato ex sola figura diiudicare licebit, ad quamnam speciem curva formata referri debeat. Eodem autem modo hic specierum diversitatem constituemus, quo vulgo linearum algebraicarum species in dato ordine contentae enumerari solent.

15. Aequatio generalis pro curvis Elasticis

$$dy=\frac{(\alpha+\beta x+\gamma xx)dx}{\sqrt{\left(a^4-(\alpha+\beta x+\gamma xx)^2\right)}},$$

initio abscissarum in axe per intervallum $\frac{\beta}{2\gamma}$ promoto et pro $\frac{aa}{\gamma}$ scribendo aa seu ponendo $\gamma=1$ accipiet hanc formam simpliciorem:

$$dy=\frac{(\alpha+xx)dx}{\sqrt{\left(a^4-(\alpha+xx)^2\right)}}.$$

Quia vero est

$$a^4-(\alpha+xx)^2=(aa-\alpha-xx)(aa+\alpha+xx),$$

ponatur $aa-\alpha=cc$, ut sit $\alpha=aa-cc$, atque aequatio transibit in hanc formam

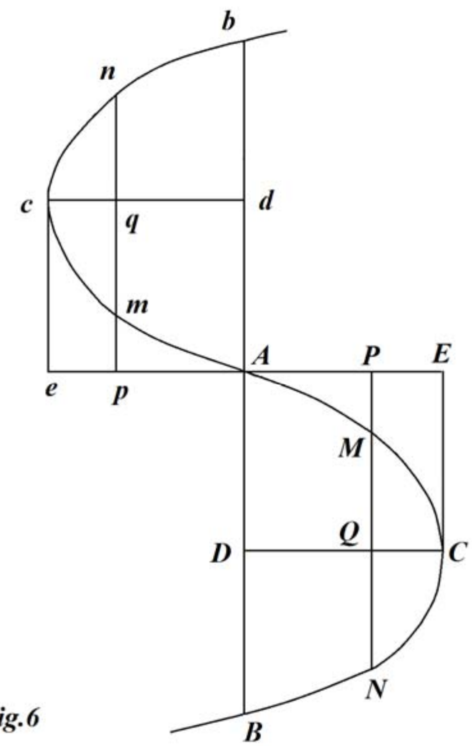

Fig.6

$$dy=\frac{(aa-cc+xx)dx}{\sqrt{(cc-xx)(2aa-cc+xx)}}.$$

Qua aequatione exprimatur (Fig. 6) natura curvae AMC posita abscissa $AP=x$ et applicata $PM=y$. Cum ergo sit $\beta=0$, directio vis laminam Elasticam incurvans erit ad axem AP in ipso puncto A normalis, ideoque AD repraesentabit directionem vis sollicitantis, quae ipsa erit $=\frac{2Ekk}{aa}$ exprimente Ekk elasticitatem absolutam.

16. Si ponatur $x = 0$, erit

$$\frac{dy}{dx} = \frac{aa - cc}{c\sqrt{(2aa - cc)}};$$

quae expressio praebet tangentem anguli, quem curva *AM* in *A* cum axe *AP* constituit ; cuius anguli sinus erit $= \frac{aa-cc}{aa}$. Quare, si fuerit $aa = \infty$, lamina in puncto *A* erit normalis ad axem *AP* nullamque habebit curvaturam, propterea quod vis incurvans $\frac{2Ekk}{aa}$ evanescit. Casu ergo quo $a = \infty$ prodit laminae figura naturalis, hoc est linea recta; quae ergo primam speciem linearum Elasticarum constituit, quam repraesentabit recta *AB* utrimque in infinitum producta.

17. Antequam reliquas species enumeremus, conveniet in genere circa figuram Elasticae quasdam observationes instituere. Intelligitur autem angulum *PAM*, quem curva in *A* cum axe AP constituit, decrescere, quo minor evadat quantitas *aa*, hoc est quo magis vis incurvans $\frac{2Ekk}{aa}$ intendatur. Atque, si evadat $aa = cc$, tum axis *AP* ipse curvam in *A* tanget. Quodsi autem fuerit $aa < cc$, tum curva *AM*, quae adhuc deorsum excurrebat, nunc sursum verget, quoad fiat $aa = \frac{1}{2}cc$; quo casu tangens curvae in rectam *Ab* incidet. At si fiat $aa < \frac{1}{2}cc$, tum angulus *PAM* prorsus fiet imaginarius ideoque in *A* nulla existet curvae portio; qui diversi casus specierum varietatem constituent.

18. Ex aequatione porro intelligitur, quia formam suam non mutat, si coordinatae *x* et *y* ambae negativae statuantur, curvam circa *A* ramos habere similes et aequales *AMC* et *Amc* alternatim dispositos, ita ut in *A* sit punctum flexus contrarii; unde cognita curvae portione *AMC* simul eius continuatio *Amc* ultra *A* cognoscetur, quippe quae illi est similis et aequalis. Sic sumpta $Ap = AP$ erit quoque $pm = PM$. Recedendo autem ab *A*, curva utrimque magis ab axe reclinatur, donec sumpta abscissa $= AE = c$ applicata *EC* curvam tangat; namque posito $x = c$ fit $\frac{dy}{dx} = \infty$. Perspicuum autem est abscissam *x* ultra $AE = c$ excrescere non posse; alioquin enim fieret $\frac{dy}{dx}$ imaginarium; hinc ergo tota curva continebitur inter applicatas extremas *EC* et *ec*, ultra quos cancellos egredi non queat. Iam ergo generatim cognitos habemus binos curvae ramos *AC* et *Ac* utrimque ab *A* usque ad cancellos protensos.

19. Videamus ergo, quonam cursu curva ultra *C* et *c* progrediatur. Hunc in finem sumamus rectam *CD* ipsi *AE* parallelam pro axe ac ponamus has novas coordinatas $CQ = t,\ QM = u$; eritque

$$t + x = AE = CD = c \text{ et } y + u = CE = AD = b;$$

unde fit $x = c - t$ et $y = b - u$ seu $dy = -du$. His valoribus substitutis, orietur aequatio pro curva inter coordinatas $CQ = t$ et $QM = u$, quae erit

$$du = \frac{(aa - 2ct + tt)\,dt}{\sqrt{t(2c - t)(2aa - 2ct + tt)}}.$$

Hic primum patet, si sumatur *t* infinite parvum, fore

$du = \frac{aadt}{2a\sqrt{ct}}$ ideoque $u = a\sqrt{\frac{t}{c}}$;

quae aequatio indicat curvam ultra *C* simili modo versus *N* progredi incipere, quo ex *C* ad *M* extenditur. Ambiguitas autem signi $\sqrt{\ }$ in denominatore aequationis luculenter declarat applicatam *u* aeque negative accipi posse atque affirmative; unde manifestum est rectam *CD* esse curvae diametrum atque adeo arcum *CNB* similem et aequalem fore arcui *CMA*.

20. Simili autem modo recta *cd* ex alteraparte axi *AE* per *c* parallela ducta erit curvae diameter, propterea quod ramus *Acb* similis et aequalis est ramo *ACB*. In punctis ergo *B* et *b* erunt quoque puncta flexus contrarii omnino uti in *A*; unde curva similiter ulterius progredietur. Habebit ergo curva infinitas diametras *CD*, *cd* etc. intervallo eodem *Dd* a se invicem distantes ac parallelas inter se; hancque ob rem curva constabit ex infinitis partibus inter se similibus et aequalibus atque ideo tota curva cognoscetur, si unica tantum portio *AMC* fuerit perspecta.

21. Quia in *A* est punctum flexus contrarii, ibidem erit radius osculi infinite magnus; id quod ex ipsa curvae natura patet. Cum enim curva in *A* sollicitur a vi $= \frac{2Ekk}{aa}$; in directione AD , erit in quovis loco M , si radius osculi ibi ponatur $= R$, ex natura elasticitatis

$$\frac{2Ekk}{aa}x = \frac{Ekk}{R};$$

unde fit $R = \frac{aa}{2x}$. In puncto ergo *A* radius osculi est infinitus; at vero in punctis *C, c*, ob $AE = Ae = c$, erit radius osculi $= \frac{aa}{2c}$; in his scilicet locis maxime a recta *BAb* remotis curvatura est maxima.

22. Etsi autem pro puncto *C* constat abscissa $AE = c$, tamen distantia *EC* nisi per integrationem aequationis

$$dy = \frac{(aa - cc + xx)\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$$

definiri non potest. Si enim post integrationem ponatur $x = c$, valor ipsius *y* dabit distantiam *CE*, quae bis sumpta praebebit distantiam *AB* seu intervallum *Dd* inter diametros interiacens. Simili modo integratione opus erit ad laminae incurvatae *AC* longitudinem determinandam. Cum enim posito arcu $AM = s$
sit

$$ds = \frac{aadx}{\sqrt{(cc-xx)(2aa-cc+xx)}} ,$$

huius integrale posito $x = c$ dabit longitudinem curvae *AC*.

23. Cum autem istae formulae integrationem non admittant, per approximationem valores intervalli *AD* et arcus curvae *AC* commode exprimere nitamur. Ponamus in hunc finem $\sqrt{(cc-xx)} = z$, eritque

$$PM = y = \int \frac{(aa-zz)dx}{z\sqrt{(2aa-zz)}} \text{ et } AM = s = \int \frac{aadx}{z\sqrt{(2aa-zz)}} .$$

Est vero per seriem

$$\frac{1}{\sqrt{(2aa-zz)}} = \frac{1}{a\sqrt{2}}\left(1 + \frac{1}{4}\times\frac{zz}{aa} + \frac{1\cdot 3}{4\cdot 8}\times\frac{z^4}{a^4} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^6}{a^6} + \text{ etc.}\right),$$

unde fiet

$$s = \frac{1}{\sqrt{2}}\int dx\left(\frac{a}{z} + \frac{1}{4}\times\frac{z}{a} + \frac{1\cdot 3}{4\cdot 8}\times\frac{z^3}{a^3} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^5}{a^5} + \text{ etc.}\right)$$

$$s - y = \frac{1}{\sqrt{2}}\int dx\left(\frac{z}{a} + \frac{1}{4}\times\frac{z^3}{a^3} + \frac{1\cdot 3}{4\cdot 8}\times\frac{z^5}{a^5} + \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\times\frac{z^7}{a^7} + \text{ etc.}\right).$$

24. Quia autem haec integralia tantum pro casu $x = c$ desideramus, quo casu fit $z = 0$, ea commode ope peripheriae Circuli exprimi poterunt. Posita enim ratione diametri ad peripheriam $= 1:\pi$, erit

$$\int \frac{dx}{z} = \int \frac{dx}{\sqrt{(cc-xx)}} = \frac{\pi}{2}$$

posito post integrationem $x = c$. Pari modo autem sequentia integralia ita determinabuntur, ut sit

$$\int zdx = \frac{1}{2} \times \frac{\pi}{2} cc,$$

$$\int z^3dx = \frac{1 \cdot 3}{2 \cdot 4} \times \frac{\pi}{2} c^4,$$

$$\int z^5dx = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \times \frac{\pi}{2} c^6,$$

$$\int z^7dx = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \times \frac{\pi}{2} c^8,$$

etc.

His ergo integralibus in subsidium vocatis erit

$$AC = \frac{\pi a}{2\sqrt{2}} \left(1 + \frac{1 \cdot 1}{2 \cdot 2} \times \frac{cc}{2aa} + \frac{1 \cdot 1 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \times \frac{c^4}{4a^4} + \text{ etc.} \right)$$

$$AC - AD = \frac{\pi a}{2\sqrt{2}} \left(\frac{cc}{2aa} + \frac{1 \cdot 3}{2 \cdot 4} \times \frac{c^4}{4a^4} + \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} \times \frac{c^4}{8a^6} + \text{ etc.} \right).$$

Ex his ergo reperiuntur AD et AC, ut sequitur:

$$AC = \frac{\pi a}{2\sqrt{2}} \left(1 + \frac{1^2}{2^2} \times \frac{cc}{2aa} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{c^4}{4a^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{c^6}{8a^6} + \text{ etc.} \right)$$

$$AD = \frac{\pi a}{2\sqrt{2}} \left(1 - \frac{1^2}{2^2} \times \frac{3}{1} \times \frac{cc}{2aa} - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3} \times \frac{c^4}{4a^4} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5} \times \frac{c^6}{8a^4} - \text{ etc.} \right).$$

Si itaque detur $AE = c$ et $AD = b$, ex his aequationibus et recta constans *a* et longitudo curvae *AC* definietur. Vicissim autem ex data longitudine curvae AC et recta *a*, per quam vis inflectens determinatur, reperiri poterunt rectae *AD* et *CD*.

SPECIES PRIMA

25. Quoniam igitur speciem primam constituimus, si in aequatione generali

$$dy = \frac{(aa - cc + xx)dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$$

fuerit $c = 0$ seu $\frac{a}{c} = \infty,$ quo casu linea resultat repraesentans statum laminae Elasticae naturalem, ad eandem speciem primam referamus quoque eos casus, quibus *c* est quantitas quamminima, ita ut prae a pro evanescente haberi queat. Quia ergo *x* ipsam *c* superare nequit, etiam *x* prae *a* evanescet ideoque ista prodibit aequatio

$$dy = \frac{adx}{\sqrt{2(cc - xx)}} ,$$

cuius integrale est

$$y = \frac{a}{\sqrt{2}} \operatorname{A sin} \frac{x}{c},$$

quae est aequatio pro curva Trochoide in infinitum elongata. Fiet autem $AD = \frac{\pi a}{2\sqrt{2}},$ a qua ipsa curvae longitudo infinite parum tantum discrepat, propterea quod angulus *DAM* est infinite parvus. Sit longitudo laminae $ACB = 2f$ eiusque elasticitas absoluta $= Ekk$; ob $f = \frac{\pi a}{2\sqrt{2}},$ erit vis ad hanc curvaturam infinite parvam laminae inducendam requisita finitae magnitudinis et quidem $= \frac{Ekk}{ff} \cdot \frac{\pi\pi}{4}.$ Scilicet, si extremitates *A* et *B* colligentur filo *AB*, hoc filum contrahi debebit vi $= \frac{Ekk}{ff} \cdot \frac{\pi\pi}{4}.$

SPECIES SECUNDA.

26. Secundam speciem constituat casus, quo $c > 0$, attamen $c < a$, scilicet, si *c* contineatur intra limites 0 et *a*. His enim casibus angulus *DAM* recto erit minor; est namque anguli *PAM* sinus seu anguli *DAM* cosinus $= \frac{aa - cc}{aa}.$ Hoc ergo casu forma lineae curvae talis fere erit, qualem Figura 6 repraesentat. Quia igitur est $c < a$, erit $\frac{cc}{2aa} < \frac{1}{2}$; cum vero sit $\frac{cc}{2aa} > 0$, erit utique $AC = f > \frac{\pi a}{2\sqrt{2}},$ unde $aa < \frac{8ff}{\pi\pi}$; quare vis, qua extremitates laminae *A* et *B* ope fili *AB* ad se invicem attrahuntur, maior erit quam casu praecedente, nempe $> \frac{Ekk}{ff} \cdot \frac{\pi\pi}{4}.$

SPECIES TERTIA

27. In tertia specie unicum complector casum, quo $c = a$, quia hoc casu axis *AP* curvam in puncto *A* tangit; haecque species singulare nomen curvae Elasticae rectangulae obtinuit. Erit ergo

$$dy = \frac{xxdx}{\sqrt{\left(a^4 - x^4\right)}} \text{ et } ds = \frac{aadx}{\sqrt{\left(a^4 - x^4\right)}};$$

hoc igitur casu *AD* et *AC* ita se habebunt, ut sit

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \times \tfrac{1}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \tfrac{1}{4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \tfrac{1}{8} + \text{ etc.}\right)$$

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \times \frac{3}{1 \cdot 2} - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3 \cdot 4} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5 \cdot 8} - \text{ etc.}\right).$$

Quanquam autem hinc neque *b* neque *f* per *a* accurate assignari potest, tamen alibi insignem relationem inter has quantitates locum habere demonstravi. Scilicet ostendi esse $4bf = \pi aa$ seu rectangulum ex *AD* et *AC* formatum erit aequale areae Circuli, cuius diameter est $= AE$. Reperietur autem calculum subducendo proxime $f = \frac{5a}{6} \times \frac{\pi}{2}$, ita ut sit $a = \frac{12f}{5\pi}$; hinc vis, qua laminae extremitates *A*, *B* ad se invicem contrahi debent, erit

$$= \frac{Ekk}{ff} \times \frac{25}{72}\pi\pi.$$

Propius vero reperitur

$$f = \frac{\pi a}{2\sqrt{2}} \cdot 1{,}1803206$$

hincque

$$b = \frac{\pi aa}{4f} = \frac{a}{\sqrt{2}} \times 1{,}1803206;$$

unde in numeris puris erit

$$\frac{f}{a} = 1{,}311006 \text{ et } \frac{b}{a} = 0{,}834612.$$

SPECIES QUARTA

28. Si $c > a$, orietur species quarta eousque patens, quoad fiat $AD = b = 0$; qui alter limes ipsius *c* definietur per hanc aequationem:

$$1 = \frac{1^2}{2^2} \times \frac{3}{1} \times \frac{cc}{2aa} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3} \times \frac{c^4}{4a^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5} \times \frac{c^6}{8a^6} + \text{ etc.}$$

In hae ergo specie (Fig. 7), cum sit $c > a$, curva in *A* supra axem *AE* ascendet angulumque constituet *PAM*, cuius sinus erit $=\frac{cc-aa}{aa}$; mox autem videbimus hunc angulum *PAM* minorem esse quam 40° 41′; quoniam, si hunc valorem aequirit, intervallum *AD* evanescit, quem casum ad speciem quintam refero. Hinc in specie quarta continentur casus, quibus $\frac{cc}{aa}$ inter hos limites 1 et 1,651868 comprehenditur. Harum autem curvarum forma ex figura intelligitur, dummodo notetur, quo propius $\frac{cc}{aa}$ ad posteriorem limitem 1,651868 accesserit, eo minus esse futurum intervallum *AD* eoque propius laminae terminos *A* et *B* ad se invicem adduci. Fieri ergo potest, ut laminae gibbositates *m* et *R* item *M* et *r* se mutuo non solum tangant, sed etiam intersecent, atque huiusmodi intersectiones in infinitum multiplicabuntur, donec omnes diametri *DC, dc* coincidant atque cum axe *AE* confundantur.

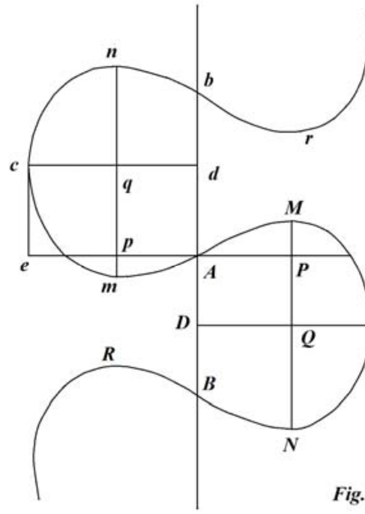

Fig.7

SPECIES QUINTA

29. Hoc si evenerit, orietur (Fig. 8) species quinta, cuius natura hac exprimetur aequatione inter coordinatas $AP = x$ et $PM = y$:

$$dy=\frac{(cc-aa-xx)dx}{\sqrt{(cc-xx)(2aa-cc+xx)}},$$

existente hac inter *a* et *c* relatione, ut sit intervallum $AD=b=0$. Ponatur $\frac{cc}{2aa}=v$, atque *v* ex hac aequatione infinita definiri debet

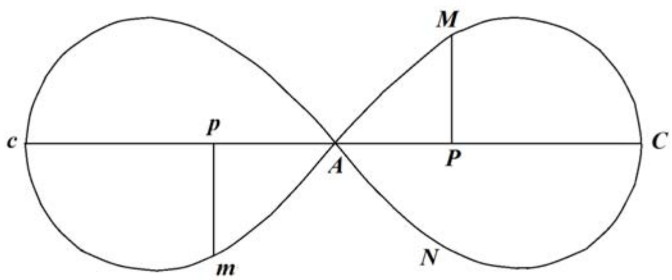

Fig.8

$$1=\frac{1\cdot 3}{2\cdot 2}v+\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}v^{2}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}v^{3}+\text{ etc.}$$

Quaerantur primum per methodos cuique solitas, vel saltem tentando, limites, inter quos verus valor ipsius v contineatur, atque huiusmodi limites reperientur
$v = 0,824$ et $v = 0,828$.

Quodsi iam uterque substituatur in aequatione ex erroribus binis oriundis, concludetur tandem fore

$$v = 0,825934 = \frac{cc}{2aa};$$

unde fit

$$\frac{cc}{aa} = 1,651868 \text{ et } \frac{cc-aa}{aa} = 0,651868;$$

quae expressio cum sit sinus anguli *PAM*, ex Tabulis reperietur hic angulus = 40° 41′; ideoque huius duplum seu angulus *MAN* erit = 81° 22′. Quare, si laminae Elasticae extremitates eousque ad se invicem adducantur, ut se contingant, tum curvam *AMCNA* formabunt et ambae extremitates in *A* angulum constituent 81° 22'.

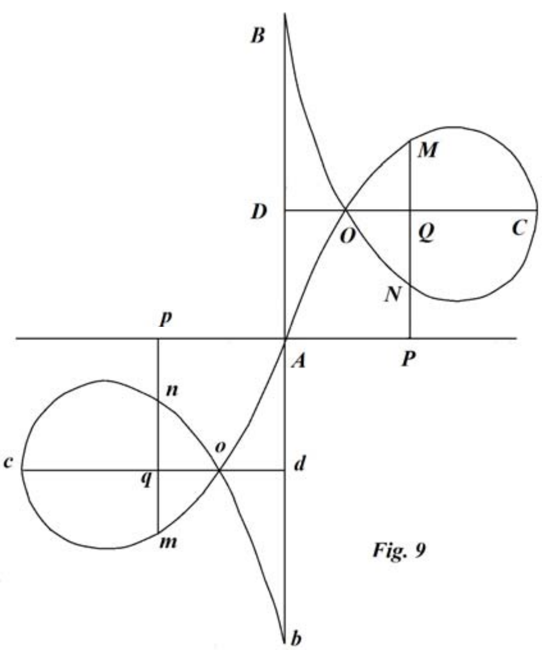

Fig. 9

SPECIES SEXTA

30. Si ambae extremitates laminae *A* et *B* (Fig. 9), postquam ad se invicem fuerint adductae, aucta vi in plagas contrarias a se invicem diducantur, orietur curva huius formae *AMCNB*, quae speciem sextam constituat. In curvis ergo ad hanc speciem pertinentibus, erit

$$\frac{cc}{2aa} > 0,825934,$$

ita tamen, ut sit $\frac{cc}{2aa} < 1$. Quodsi enim sit $cc = 2aa$, orietur species septima mox explicanda. Erit ergo in his curvis angulus *PAM*, quem curva in *A* cum axe constituit, maior quam 40° 41′, minor tamen recto; cum enim eius sinus sit $= \frac{cc}{2aa}$, ob $cc < 2aa$ sinus iste necessario est minor sinu toto neque ergo angulus *PAM* rectus fieri potest, nisi ponatur $cc = 2aa$.

SPECIES SEPTIMA

31. Sit iam $cc = 2aa$, quo casu species septima constituitur, atque natura curvae exprimetur hac aequatione

$$dy = \frac{(aa - xx)dx}{x\sqrt{(2aa - xx)}};$$

ex qua colligitur curvae (Fig. 10) ramos *A* et *B* infinitum extendi ita, ut recta *AB* fiat curvae asymtota. Fiet ergo uterque ramus *AMC* et *BNC* infinitus, id quod ex serie supra pro arcu *AC* inventa intelligitur; erit enim

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{ etc.}\right),$$

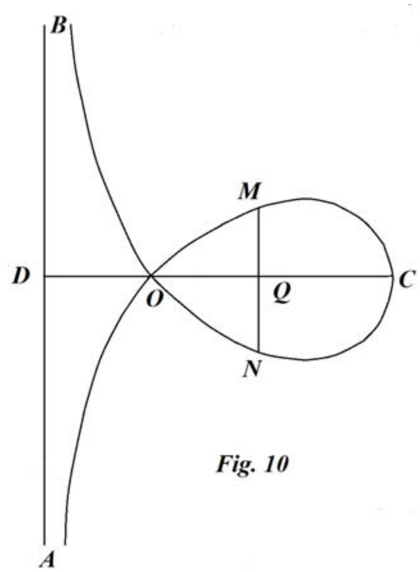

Fig. 10

cuius seriei summa est infinita. Quodsi igitur laminae longitudo *AC* fuerit finita $= f$, necesse est, ut sit $a = 0$ hincque etiam $CD = c = 0$; lamina ergo, postquam in nodum fuerit incurvata, hoc casu iterum in directum extendetur, ad quam extensionem opus erit vi infinita. Sin autem lamina fuerit infinite longa, curvam formabit nodatam ad asymtotam *AB* convergentem, existente $CD = c$. Aequatio autem pro hac curva ope logarithmorum integrari potest, obtinebitur enim

$$y = \sqrt{(cc - xx)} - \frac{c}{2} l \frac{\sqrt{(cc - xx)}}{x},$$

sumptis abscissis *x* in ipsa diametro *DC*, ita ut sit $DQ = x$ et $QM = y$; evanescit enim applicata y posito $x = CD = c$. In nodo autem *O* applicata *y* pariter evanescit; ad quem locum inveniendum ponatur

$$\frac{2\sqrt{(cc - xx)}}{c} = l \frac{c + \sqrt{(cc - xx)}}{x}.$$

Sit φ angulus, cuius cosinus $= \frac{x}{c}$ et sinus $= \frac{\sqrt{(cc - xx)}}{c}$, erit

$$2\sin\varphi = l \text{ tang } \left(45^\circ + \tfrac{1}{2}\varphi\right),$$

qui logarithmus ex hyperbolicorum genere sumi debet; cuiusmodi Canon si deficiat, sumatur ex Canone vulgari logarithmus tangentis anguli $45^\circ + \frac{1}{2}\varphi$, a cuius characteristica denarius auferatur, sitque residuum $= \omega$; quo facto erit $2 \sin\varphi = \omega \cdot 2{,}30258509$; sumendis ergo iterum logarithmis vulgaribus, erit

$$l2 + l \sin \varphi = l\omega + 0{,}3622156886$$

seu

$$l\sin\varphi = l\,\omega + 0,0611856930.$$

Hoc artificio tentando mox vero proximus valor anguli φ elicietur; unde porro per regulam falsi verus valor anguli φ ex eoque abscissa $x = DO$ definietur. Reperitur autem hoc modo angulus $\varphi = 73°14'12''$, unde prodit

$$\frac{x}{c} = 0,2884191 \text{ et } \frac{\sqrt{(cc-xx)}}{c} = 0,9575042\ ;$$

angulus vero *QOM* fit $= 2\varphi - 90 = 56°\ 28'\ 24''$
ideoque

$$\text{angulus } MON = 112°\ 56'\ 48''.$$

Cum igitur specie quinta angulus nodi esset 81^o 22', in specie sexta angulus nodi *MON* continebitur inter limites 81° 22' et 112° 56' 48''. In specie quarta autem, siquidem detur nodus, erit eius angulus minor quam 81° 22'.

SPECIES OCTAVA

32. Sit iam $cc > 2aa$, puta $cc = 2aa + gg$; erit aequatio pro curva
Ob $aa = \frac{cc-gg}{2}$

$$dy = \frac{\left(xx - \tfrac{1}{2}cc - \tfrac{1}{2}gg\right)dx}{\sqrt{(cc-xx)(xx-gg)}}\ ,$$

qua aequatione species octava continetur, eritque (Fig.11), si recta *dDd* repraesentet directionem vis sollicitantis, $x = DQ$ et $y = QM$. Primum ergo patet applicatam *y* realem esse non posse, nisi sit $x > g$, tum vero *x* non potest excedere rectam $DC = c$, unde sumpta $DF = g$ tota curva continebitur inter

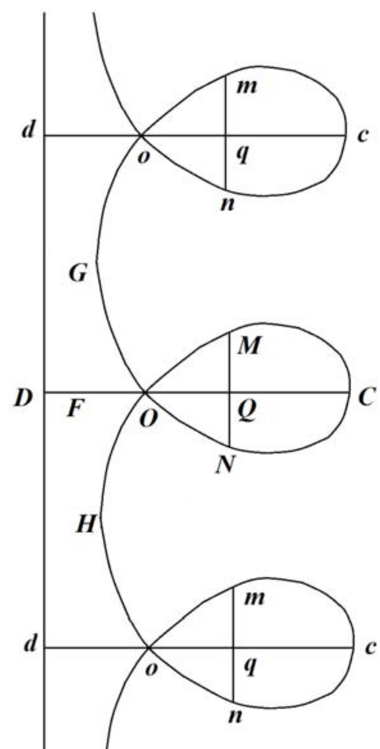

Fig. 11

rectas ipsi *dd* parallelas per puncta *C* et *F* ductas, quae curvam simul tangent. Perinde autem est, utra rectarum *c* et *g* sit maior, dummodo fuerint inaequales, aequatio enim non variatur, si rectae *c* et *g* inter se permutentur. Deinde vero haec curva quoque habebit infinitas diametras inter se parallelas *DC*, *dc*, *dc* et quae per singula puncta *G* et *H* ducuntur rectae pariter ad *dDd* normales; nusquam autem per totam curvam dabitur punctum flexus contrarii ideoque continua curvatura utrimque in infinitum progreditur, uti figura indicat ; anguli autem, qui in nodis constituuntur, *MON* maiores erunt quam 112° 56' 42".

SPECIES NONA

33. Cum in hac specie non solum contineantur casus, quibus $gg < cc$, sed etiam, quibus $gg > cc$, unicus adhuc superest casus, quo $c = g$, quo quidem tota curva in spatium evanescens, ob $CF = 0$, redigitur. Quodsi autem utramque *c* et *g* statuamus infinitam, ita tamen, ut earum differentia fiat finita, curva finitum spatium occupabit. Ad eam ergo inveniendam ponatur $g = c - 2h$ et $x = c - h - t$, atque ob $c = \infty$, quantitates *h* et *t* vero finitae, erit

$$\tfrac{1}{2}cc + \tfrac{1}{2}gg = cc - 2ch \text{ et } xx - \tfrac{1}{2}cc - \tfrac{1}{2}gg = -2ct;$$

tum vero

$$cc - xx = 2c(h+t) \text{ et } xx - gg = 2c(h-t);$$

ex quibus sequens prodibit aequatio

$$dy = \frac{tdt}{\sqrt{(hh-tt)}}$$

pro Circulo. Lamina Elastica ergo hoc casu in Circulum incurvatur, uti supra iam annotavimus; Circulus ergo speciem nonam atque ultimam constituet.