THE ANALYTICAL EXPLANATION OF THE METHOD OF MAXIMA AND MINIMA

SUMMARY

Now before the most celebrated isoperimetric problem gained significance [*i.e.* the brachistochrone], certain examples pertaining to this had been produce by Geometers, since from the earliest times it had been found that the circle, amongst all the other figures enclosed by the same perimeter, contained the maximum area ; indeed they had inferred which property from the nature of the circle, truly they had hardly dared to approach the problem directly, in order that, amongst all the figures terminating with an equal perimeter, they might investigate that, which would enclose the maximum area. It is well known that this problem is exceedingly hard, as was the case before the significant advance of the infinitesimal calculus, which allowed one at least to think about that problem. Truly soon the question of the brachistochrone was resolved with the greatest success by the most acute Johan Bernoulli, brought about as if by the first cast of this calculus, to wit from which amongst all the lines drawn from the highest point to a lower point that may be found, upon which a weight may descend in the shortest time, as was found to agree with that outstanding property of the cycloid. But the method, which the most honoured man had used, clearly may be seen to provide the opportunity for his older brother Jacob Bernoulli to contemplate the solution of a great isoperimetric problem, which he treated henceforth. Evidently all questions of this kind have been embraced by this problem to the widest extent, so that amongst all the lines given drawn between the two points, either they must be of the same length (from which indeed the name isoperimetric has arisen) or may be endowed with some other common property that may be investigated, which either shall be a maximum area, a solid of maximum volume rotated about a given axis, or in general may be contained in some maximum or minimum property. But the method, which the greatest Geometer of that time used, leaves us considering in doubt, whether we should admire more his incredible patience expended both in prolixity and in the most tedious calculations, or his great wisdom in setting out satisfactorily the conclusions thence deduced by reason. But on this account, because the conclusions had emerged set out so well, soon it would be able to suppose a less arduous and shorter way be given to produce the same; which also his younger brother Johan approached happily enough, even if it was less successful immediately due to well-hidden difficulties, which yet following the failure was largely compensated, with the whole argument revised more deeply. After a long interceding time, the author of these dissertations applied great enthusiasm to solving the same problem, and since he observed all questions of this kind were referred, in order that a curved line of this kind may be found by expressing an equation between the coordinates x and y, to that in which the formula of the integral $\int Vdx$ for that may reach a maximum or minimum value, in whatever manner the quantity V were given by x and y. But now it is clear that an infinitude of variations can be considered in that quantity V, as in that besides these

variables *x* and *y*, both the differentials of these of any order as well as new integral formulas may be introduced in addition. But if now Bernoulli's solutions may be examined according to this standard, these are found restricted to those cases, in which the quantity *V* involves differentials of the first order only, and besides the cases thence clearly are excluded, in which new integral formulas are present, with very few exceptions, which easily may be freed from this inconvenience according to the nature of the problem. Therefore our author has supplied this deficiency most happily both in these commentaries as well as in the individual works published about this question, as scarcely any shall be found, which may be desired more fully. Yet meanwhile that method itself, even if the whole matter may be put together quickly in a satisfactory manner, still itself has been seen not to be of a satisfactory nature, because the whole strength of the solution was set up thus from the consideration of the elements of the curve requiring to be found, truly the question thus shall be readily equipped , so that it may be recalled absolutely from geometry to pure analysis alone. Indeed the question thus may be proposed, so that for some given quantity V with the two variables *x*, *y* and the differentials of these of any order, why indeed also may that relation between *x* and *y* not be investigated from the integral formulas constructed in whatever manner, from which a maximum or minimum value of the integral formula $\int Vdx$ may be brought about ? In short, the question proposed in this manner, may be freed by geometry ; from which also that natural method of resolution must be freed from geometry ; and as a more difficult analysis would need to be applied to this goal, to that the greater advancement of this science, if the thing were successful, it would deservedly be hoped for.

[A modern view of the resolution of this problem is set out in the paper by Craig G. Fraser, '*The Origins of Euler's Variational Calculus' in the Archives of the Exact Sciences*. Vol. 47, No. 2 (June 1994), pp. 103-141. Essentially, the young Lagrange noted a distinction between two uses of the differential by Euler : one applied to separate the *y* coordinate values of points on neighbouring different curves for the same coordinate *x*, and the other to form a derivative from points on the same curve; subsequently Lagrange introduced the new operator δy to distinguish the former from the latter *dy*. The present paper by Euler resolves this difficulty.]

Moreover, although the author had considered this for a long time and tried to uncover the missing theory with his friends, yet the first glory of finding such a theory was reserved to the young Geometer, Lagrange of Turin, who clearly had arrived at the same solution by using analysis alone, that the author had elicited from geometrical considerations. Truly that solution itself being prepared thus, as plainly constituting a new kind of analysis, and the boundaries of which may be considered to have moved more than just a little ; from which an opportunity is offered to the originator that this science to be enriched by a new kind of calculus known as the *Calculus of Variations* and its elements to be treated here and to constitute a clear explanation. Indeed this calculus likewise depends on differentials and infinitely small increments being compared together, truly it differs greatly from that in the original account of the treatment. For since in the integral calculus, for a given relation of the variable quantities, a relation between the differentials of these of any order may be investigated, but in the calculus of variations a relation itself between infinitely small variables is considered to be changed, thus so that, while following a given relation, for some value of the one variable x certainly the other is given a different value y, the calculus of variations may add a certain infinitely small increment to this value y itself, from which henceforth however both the differentials and integrals of the formula may be varied, it is required to be defined. That increment required to be added to any value of y is called by the writer its variation and, since it may not be confused with differentials, it shall be designated by this character δy ; therefore hence since all formulas, both differential as well as integral, in as much as they involve the quantity y, give rise to certain variations, the writer in the former dissertation established the principles and precepts, with the help of which the variations of all formulas of this kind can be defined: thus if W may denote some formula of this kind, its variation δW is shown to be assigned by special rules. So that with a single calculation put in place then in the following dissertation its application to all problems, which are able to be devised about maxima and minima, is shown most clearly, and in the matter this deserves to be observed in the first place, so that thus the new purely analytic form will supply many fuller and more perfect solutions, than that aimed at before from geometry.

1. What commonly are accustomed to be treated in the elements concerning the method of maxima and minima, that in functions of some single variable quantity are used chiefly, so that for some proposed function V, which had been composed from some variable quantity z and constants, these determinations of the variable z may be required to be investigated, which may lead to a maximum or minimum of the function V. Meanwhile, also functions of two or more variables z, y, x are considered and the values from these individual variables assigned are sought, in which a function may obtain a maximum or minimum value. But the method, by which questions of this latter kind may be resolved, agree with that at once which may be used in general, as with the former kind ; if indeed several variables may be involved, just as with a single variable viewed successively, its value for the maximum or minimum may be found on being produced suitably; which operation if it were carried out by single variables, the values of all would become known, in which the value of the proposed function would return a maximum or minimum.

2. The matter cannot be settled otherwise, if a function of two variables x and y may be proposed and the value attributed to y itself may be sought, so that, when a given magnitude a were put in place for x, that function should achieve a maximum or minimum value ; for a may be written everywhere for x at once, and clearly the question will be reduced to the first kind. Truly if that function of the variables x and y were not disclosed, but may be determined by integration, the questions generally must be referred to be resolved by a separate method, and may require a long digression. Just as if Z were some function of x and y and the integral formula $\int Zdx$ were proposed, the question may be agreed to be enunciated thus:

To define the relation between the two variables x and y, such that the value of this formula, after we may have put $a = x$, shall be generally a maximum or minimum.

3. How much difference there may be between questions of this kind and those, which I have restored to the first kind, will become apparent in the following instances or attended to a little later. Indeed, let *V* be an established function of *x* and *y*, for which the value of *y* must be found, so that on putting $x = a$ the value of the function *V* may emerge a maximum or a minimum ; and towards resolving that question $x = a$ must be put in place at once, with which done the value of *y* thus will be determine by the former method, so that it may not depend on an indefinite value of *x*. But the proposed formula of the integral $\int Zdx$ is not in the formula of the differential *Zdx*, but finally after the integration it is allowed to be attributed that determined value *a* of *x* itself ; nor may a certain determined value be taken for *y* to complete the calculation, so that then the value of the formula $\int Zdx$ may become a maximum or a minimum, but a certain relation between *x* and *y* must be assigned ; on account of which, even if after the integration there may be put $x = a$, yet the value of the integral $\int Zdx$ may depend on an indefinite relation which lies between *x* and *y*, and which shall be determined by all the intermediate values of *y*.

4. Truly such questions about returning the maximum or minimum formula $\int Zdx$ appear much broader, nor are they restricted to the case to the case only, in which *Z* is a function of *x* and *y*, for *Z* can be assumed to be some expression, which may be determined from some assumed relation between *x* and *y*. Hence *Z* will be able to involve besides these variables *x* and *y* also a relation of the differentials of these, nor only of the first order, but also of higher orders of some kind; clearly if these ratios of the differentials may be expressed thus, so that there shall be

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r \quad \text{etc.},$$

the magnitude *Z* will be able to be seen as some general function of these *x, y, p, q, r* etc. Also in addition why may the magnitude *Z* not include within itself some new integral formulas involved in that ; from which many more kinds of questions of this kind arise, to which the method of solving must be adapted.

5. The first problems of this kind were began to be treated on the occasion of that famous isoperimetric problem, at one time pursued by James Bernoulli at the forefront of the advancement of analysis, because the arduous work itself even if it were completed with wonderful shrewdness by that sharpest of men, yet there he used a method extending only

to that, in which the magnitude Z, in addition to x and y involved only first order differentials of these, or the letter p, which in some singular case ought to be found as it were from geometrical considerations. But afterwards I had worked very hard for a long time towards a more fertile explanation of this argument, at last I had come upon a method of the greatest generality, with the help of which all problems of this kind, in which the magnitude Z contained within itself not only differentials of any order but also integral formulas of any kind, which were able to be resolved, as I had pursued the method unique to the book in a liberal manner.

6. But even if this method has been prepared thus, so that its application may require no geometric figures, yet that investigation of the method itself has been demanded from the consideration of curved lines, which for that reason then also seemed to me not to be of a satisfactory nature. For since this question, from which the relation between x and y is sought, so that the formula of the integral $\int Zdx$ may obtain a maximum or minimum value by putting $x = a$ in place after the integration, shall be able to be proposed without any respect to geometry, the solution also to be adequate and deduced from true principles, seen to be exempt from any geometrical consideration. So that since what was desired in my treatment should not be found to be obscure, a certain man De la Grange of Turin, most illustrious and must accomplished in the analytic art, announced by letters given to me from Turin, of his wish for this to be fulfilled, and likewise shared with me kindly his analytical fundamentals. Which since for the most part they have to be considered in a quiet place, I have decided to set out in my customary manner and by developing them more prolifically.

7. Therefore we may consider the integral formula $\int Zdx$, in kind in which Z shall be some function composed from x and y, which also may involve the ratio of the differentials not only of the first but also of higher orders and besides also one or more integral formulas may be included. But we may assume the integral be taken thus for its determination, so that it may vanish on putting $x = 0$; then truly after we may present the integration the value of x given a certain value $x = a$, and the value shall be A, which the formula of the integral then takes. Now the question may be about this, so that a relation may be defined between x and y, from which by these operations themselves a maximum or minimum value may be obtained for A. Hence therefore the relation between x and y, which the question may satisfy, is required to be expressed a certain equation either finite or a differential equation of any order, which was found at the same time, for the problem can be considered to be solved.

8. We may put, as in analysis it is accustomed to happen, this relation which is sought between x and y, now to be agreed on, thus so that, whatever definite value may be assumed for x, thence y also, and hence also the determined value for the function Z may be put arrived at. All the possible values for x may be considered to be substituted from the term $x = 0$ as far as to the term $x = a$, which may be progressing by infinitely small intervals dx, then truly the values of Z, which correspond to the individual values of x, are

to be multiplied by dx, and all these products to be gathered together in a single sum, that quantity may be put in place which we have indicated by the letter A, which must be either a maximum or a minimum. Because it is necessary to be understood thus, that if from some other relation between x and y with the individual values of x agreeing with other values of y and hence of Z, with these for A if it were a maximum, the value certainly would be less, but if it were a minimum, certainly a greater value would be produced, than if the true relation between x and y were used.

9. But if nevertheless these variations, which may be induced by the individual values of y, may be considered to be infinitely small, then by the nature of the maxima and minima thence no change in the magnitude A ought to be in excess ; and from this source itself the determination of the maxima or minima is accustomed to be sought. Clearly since we may attribute a change to the values of y for infinitely small variations, which thence arises in all the values of $Z\,dx$ and thus in the total sum A of these, which by calculation henceforth will give an equation equal to zero in which the nature of the maxima or minima will be contained, and thus the relation sought between x and y ought to be deduced. Therefore by this operation a method of this kind resolves the maxima or minima requiring to be found, which therefore depends on the same principles and common method of maxima and minima ; in whatever manner it can be put in place, which may be expounded by analysis alone without any aids desired from geometry, we may consider more carefully, since now I have been fortunate enough to follow this same matter depending on the principles of geometry.

10. Therefore since infinitely small variations induced in the individual values of y must lead to no change being produced in the value of the magnitude A, and this is required to arise, whatever variations of these may be taken, provided they should be infinitely small, it may suffice to consider a variation of this kind only at a single value of y, and the change thence arising in the magnitude A, to be returned vanishingly small, according to which source my general method of maxima or minima also has demanded. Truly even if from several values of y, indeed why not also plainly from many, some infinitely small variations of this kind may be induced, nevertheless the nature of the maxima or minima will be ascertained, so that the change, which the quantity A thence arrives at, may be returned to zero, and in this use it must come about, whatever these variations may be assumed, clearly all of which are purely arbitrary.

11. But since in my preceding solution I was able to take an infinitely small variation of a single certain value of y, while all the rest might remain unchanged, the meaning may be apparent in this principle of continuity and this precept was the cause, because the whole investigation was unable to be found by analysis alone, but by the consideration of geometrical figures, in which the values of y might be represented by the applied lines of the curve, had to be called into help, so that thence the variations, which entered the ratio of a differential of any order, were able to be elicited more conveniently. On account of which, lest we be too much against the principle of continuity, from which the application of the precepts of pure analysis may be impeded, we may attribute the infinitely small

variations to individual values of *y*, which yet shall thus be indefinite, so that one by one successively to be determined as it pleases and thus all except a single one shall be able to be rendered to nothing, with which agreed upon it is necessary that we may return to my first solutions.

12. But now since we have ascribed not only to one value of *y*, but to innumerable indeed clearly why not to all infinitely small variations, but still arbitrary, there is no doubt, why this method may not appear more general than the preceding, and may lead to the solution of many other problems, for which the first method was either more difficult or even unable to be used. For if these variations may be determined in a certain way, problems of this kind may be able to be resolved with the question translated according to geometry, in which clearly not all the curved lines, but only that number indeed infinite, which may be taken within a certain kind, that may be assigned, which shall give the maxima or minima of some with the property. But such questions may be taken involving more and still more difficulty; truly hence besides at this time it may be permitted to expect many advances in analysis worthy of merit.

13. Therefore since here we may attribute infinitely small variations to the individual values of y, we may consider a two-fold state of the formula $\int Zdx$, in the first of which the individual values of *y* shall be these themselves, which seek the relation required between *x* and *y*, but in the second the same values varied may be contained ; the first state in the cause of distinction I will call the *principal*, the other state the *variation*. Therefore the nature of the maxima and minima postulates that the difference between these two states shall vanish. Therefore just as in the principal state any value of y, while the variable *x* is assumed to increase by the differential *dx*, may be agreed to take the increment *dy*; thus with *x* remaining, while we progress from the principal state to the state of variation, we may put the value of *y* to be increased by the element δy : from which a distinction may properly be noted between these two differential expressions *dy* and δy . Moreover while we have ascribed an increment of this kind δy in the transition made to the varied state with the individual values of *y*, just as that shall be clearly indeterminate nor in any way required to be considered depending on these values of *y*.

14. With these in place it must be investigated, how great an increment any function of *Z* may take for any value of *x*, while it may be transferred from the principal state to the varied state ; because the increment arises from the variation of *y* alone, in as much as in this transfer it may be increased by the element δy . We may indicate this increment by δZ , thus so that the value of *Z* from the principal state transferred to the varied state shall be $= Z + \delta Z$ *;* and in the first place it is apparent at once, if the function Z may depend on the variable *x* alone nor involve the other variable *y*, to become $\delta Z = 0$; nor therefore on the variable *x*, however that may be present in the formation of the function *Z*, whatever may be contributed to δZ , but its value is due to the element δy alone, by which the variable *y* is considered to increase. But here, just as involves Z either the

finite quantities *x* and *y* alone, or also a ratio of the differentials of these, or thus integral formulas, thus the diverse cases will have to be examined.

15. Therefore initially we may put the function *Z* to involve only the finite quantities *x* and *y*, thus so that neither a ratio of the differentials nor any integral formulas may be present in that, and for its variation δZ being define in the function *Z* it will be necessary to write $y+\delta y$ everywhere in place of *y*, with *x* remaining unchanged, and thus the value of the variation $Z+\delta Z$ will be produced, from which if the principal *Z* may be subtracted, the variation δZ will remain. Therefore it is evident to obtain this variation, if the function *Z* may be differentiated in the usual manner by putting *y* alone to be variable, provided for *dy* there may be written δy. Whereby, if by differentiating carried out in the customary manner with each quantity *x* and *y* were to be variable,

$$dZ = Mdx + Ndy,$$

for the transfer from the principle state to the variation there will be

$$\delta Z = N\delta y;$$

therefore the variation will be found, if in the ordinary differential for *dx* there may be written 0, but for *dy* , δy *;* and in this manner we have dealt with the first case easily.

16. Again we have seen, how for this first case, where *Z* is a function of *x* and *y* only, the maximum or minimum value of the integral formula $\int Zdx$ may be able to be found. Therefore since for whatever value of *x* the function *Z* may increase by the element $N\delta y$ and thus *Zdx* by the small amount $Ndx\delta y$, the sum of all these particles from the term $x=0$ as far as to $x=a$ will give the variation of *A*, which if it may be put to be δA, will become

$$\delta A = \int Ndx\delta y;$$

which expression since it may vanish, whichever rule the variations δy may hold, it is necessary, so that for the individual values of *x* there shall be $N=0$. Therefore this equation expresses the relation sought between *x* and *y*, from which the formula $\int Zdx$ arrives at a maximum or minimum value ; and not only will this property have a place in the prescribed case $x=a$, but also, what ever other value of *x* may be given.

17. The second function *Z* may include besides *x* and *y* also the ratio of the first differential, or on putting $\frac{dy}{dx}=p$, *Z* shall be some function of the magnitudes *x, y* and *p,* with which differentiated in the customary manner there may be produced

$$dZ = Mdx + Ndy + Pdp\,.$$

Hence therefore the variation of *Z* must be sought, while it is transferred from the principal state to the variation state, in which translation the quantity *x* remains the same, truly *y* may be increased by the element δy, but the element by which the quantity *p* increases, shall be δp. But since there shall be $p = \frac{dy}{dx}$, if in the principal state the value of *y*, which may correspond to $x + dx$, we may indicate by y', there will be $p = \frac{y' - y}{dx}$; now in the translation to the varied state y may increase by the element δy and y' by the element $\delta y'$, and there will be

$$\delta p = \frac{\delta y' - \delta y}{dx}$$

But $\delta y' - \delta y$ expresses the increment of δy, while *x* increases by the differential *dx*, thus so that there shall be

$$\delta y' - \delta y = d\delta y$$

then truly also $\delta y' - \delta y$ can be considered as the variation of y' while we are progressing into the varied state, and thus also there will be

$$\delta y' - \delta y = \delta dy\ ;$$

from which there comes about to be

$$d\delta y = \delta dy \text{ and thus } \delta p = \frac{d\delta y}{dx} = \frac{\delta dy}{dx}.$$

18. Moreover in a similar manner, if *Z* besides *x* and *y* also may involve differentials of higher orders, so that on putting

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r \text{ etc.}$$

Z shall be some function of the quantities *x, y, p, q, r* etc. and by differentiating in the customary manner :

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr \text{ etc.},$$

the increments of the quantities *q, r* etc., while they are being transferred from the principal state to the varies state, may be determined. For on account of $q = \frac{dp}{dx}$ there will be :

$$\delta q = \frac{\delta p^{/} - \delta p}{dx} = \frac{d\delta p}{dx} = \frac{\delta dp}{dx} \text{ and equally } \delta r = \frac{\delta q^{/} - \delta q}{dx} = \frac{d\delta q}{dx} = \frac{\delta dq}{dx} \text{ etc.}$$

Truly from the above there is :

$$d\delta p = \frac{dd\delta y}{dx} = \frac{d\delta dy}{dx} \text{ and } \delta dp = \frac{\delta ddy}{dx},$$

thus so that there becomes :

$$\delta q = \frac{dd\delta y}{dx^2} = \frac{d\delta ddy}{dx^2} = \frac{\delta ddy}{dx^2},$$

truly in the same manner there is seen to become :

$$\delta r = \frac{ddd\delta y}{dx^3} = \frac{dd\delta dy}{dx^3} = \frac{d\delta ddy}{dx^3} = \frac{\delta dddy}{dx^3},$$

of which diverse formulas the equality is maintained properly with the form.

19. Therefore while a function *Z* passes from the principal state into the varied state, because the magnitude *x* takes no increment, truly *y* takes the increment δy, while the quantity *p* takes the increment $\frac{d\delta y}{dx}$, the quantity *q* the increment $\frac{dd\delta y}{dx^2}$, the quantity *r* the increment $\frac{ddd\delta y}{dx^3}$ etc., the increment of the function *Z* itself agreeing to this translation may be found by ordinary differentiation, on putting

$$dx = 0,\ dy = \delta y,\ dp = \frac{d\delta y}{dx},\ dq = \frac{dd\delta y}{dx^2},\ dr = \frac{ddd\delta y}{dx^3} \text{ etc.},$$

from which it will be :

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{ etc.}$$

And hence therefore the variation of the function *Z* will be able to be defined for some value of *x*; which form at this point will be shown more, if, just as there is

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr \text{ etc.},$$

there may be observed to be on account of $\delta x = 0$,

$$\delta Z = N\delta y + P\delta p + Q\delta q + R\delta r + \text{ etc.}$$

there truly there shall be on account of $p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx}$ etc.

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}, \quad \delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{ddd\delta y}{dx^3}.$$

20. Since therefore in the translation into the variation state the function *Z* may take the increment δZ, the formula $\int Zdx$ obtains the increment $\int \delta Zdx$, and thus which will be :

$$\int dx(Ndy + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{etc.}),$$

in which if after the integration there may be put $x = a$, the variation of *A* itself or δA will be obtained, which put equal to zero will lead to the maximum or minimum value of the quantity *A*. But in this integration it is no further considered for the transition into the state of variation, but that must be extended through all the increments of *x*, since it may denote the sum of all the variations with the individual variations of *x* agreeing from the end $x = 0$ to $x = a$. Therefore lest the ratio of the indicated differentials may be disturbed by δ, ω may be written for δy, thus so that ω may show some infinitely small arbitrary quantity depending on *x* ; and the above increment being equated to zero will become :

$$\int dx(N\omega + P\frac{d\omega}{dx} + Q\frac{dd\omega}{dx^2} + R\frac{d^3\omega}{dx^3} + \text{etc.}).$$

21. It is evident in these above differentials the element *dx* be assumed constant ; because indeed we have put

$$\frac{d\delta p}{dx} \text{ or } d\frac{\delta p}{dx} = \frac{dd\delta y}{dx^2}$$

on account of $\delta p = \frac{d\delta y}{dx}$, plainly *dx* is assumed constant. Therefore with this observed, if we may integrate the parts of the integral found separately, we will have :

$$\int dx \cdot N\omega = \int N\omega dx$$

$$\int dx \cdot P\frac{d\omega}{dx} = \int Pd\omega = P\omega - \int \omega dP$$

$$\int dx \cdot Q\frac{dd\omega}{dx^2} = \int \frac{Qdd\omega}{dx} = \frac{Qd\omega}{dx} - \frac{\omega dQ}{dx} + \int \frac{\omega ddQ}{dx}$$

$$\int dx \cdot R\frac{d^3\omega}{dx^3} = \int R\frac{d^3\omega}{dx^2} = \frac{Rdd\omega}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{\omega ddR}{dx^2} - \int \frac{\omega d^3R}{dx^2}$$

etc.

[Applying the integration by parts rule once, twice, etc.]

And hence thus the variation sought will be depend partially on the integrals of the parts, partially on absolute [*i.e.* fixed] values, and there will be :

$$\int \omega dx(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.}) + \omega(P + \frac{d\omega}{dx}(Q - \frac{dR}{dx} + \text{etc.})$$
$$+\frac{d\omega}{dx}(Q - \frac{dR}{dx} + \text{etc.}) + \frac{dd\omega}{dx^2}(R - \text{etc.})$$

22. We may restore δy for ω, and of the integral formula $\int Zdx$, with

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.},$$

and

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad s = \frac{dr}{dx} \text{ etc.}$$

the increment, while in some state of variation may be transferred, because $\delta\int Zdx$ can be expressed in this manner, thus so that it will become :

$$\int dx\delta y(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.})$$
$$+\delta y(P-\frac{dQ}{dx}+\frac{ddR}{dx^2}-\frac{d^3S}{dx^3}+\text{etc.})$$
$$+\frac{d\delta y}{dx}(Q-\frac{dR}{dx}+\frac{ddS}{dx^2}-\text{etc.})$$
$$+\frac{dd\delta y}{dx}(R-\frac{dS}{dx}+\text{etc.})$$
$$+\frac{d^3\delta y}{dx^3}(S-\text{etc.})$$
$$+\ \text{etc.},$$

in which formulas, in as much as the differentials of higher orders are involved, for the differential *dx* has been assumed constant. But δy may have an arbitrary value for the individual values of *x*.

23. If therefore for the value $x=a$ the formula $\int Zdx$ must become a maximum or minimum, the increment found in the manner, if $x=a$ may be put into that, is required to equal zero and thus that, so that it may vanish always, just as the variations δy may be assumed. Whereby also, if such a variation may be attributed to a certain single variation of *y*, which agrees with some value of *x* less than *a*, the expression found must become equal to zero. But then they introduce no change thence into the final values of *y*, which themselves correspond to $x=a$; whereby, since on putting $x=a$ the absolute part of the increment

$$\delta y(P-\frac{dQ}{dx}+\frac{ddR}{dx^2}-\frac{d^3S}{dx^3}+\text{etc.})+\frac{d\delta y}{dx}(Q-\frac{dR}{dx}+\frac{ddS}{dx^2}-\text{etc.})$$
$$+\frac{dd\delta y}{dx}(R-\frac{dS}{dx}+\text{etc.})+\ \text{etc.}$$

may depend only on the variation of the final values of *y*, for from these there will be $\delta y=0,\ d\delta y=0,\ dd\delta y=0$ etc. and thus this part will vanish at once. From which it is necessary, that only the separate part of the integral may be returned equal to zero, and thence there must become :

$$\int dx\delta y(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.})=0.$$

24. But this expression of the variation of the whole sum is included, which arises from the variations of the individual variations of y ; but because such a change can be considered to be made from a single value, the whole sum is reduced to this single variation, with all the remaining vanishing; whereby it is necessary, that for this case there shall be :

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.}=0\,;$$

Truly because, in whatever place this variation may be considered to happen, the nature of the maxima or minima postulates this annihilation equally, it is necessary, that for all values of x there shall be

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.}=0\,;$$

which therefore contains the indefinite relation between x and y, with which there comes about, that thence the value arising of the integral formula $\int Zdx$ becomes a maximum or a minimum on putting $x=a$, from which it is apparent this relation does not depend on that magnitude a.

25. Now this is the same equation, which I gave once for the solution of the same problem in my treatment of maxima and minima, but now I have derived from the principles of pure analysis ; so that thus the operation has succeeded so well, as I have assumed the variations to agree with the individual values of y, from which they may be transferred into a varied state. Then truly the reduction of the integral formulas made in paragraph 21 thoroughly completed the operation, by which these were resolved thus into parts, so that others should be free from the summation sign $\int$, but which therefore remained restricted, these were involved with their differentials, only that variation $\omega=\delta y$ itself remained without its differentials; from which itself we have found this conveniently, so that, since any variation singly must be led to nothing, the formula of the integral has provided the equation at once

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.}=0\,,$$

from which the indefinite relation between x and y may be expressed, truly the remaining absolute parts of the increment, as pertaining only to the final values of y, have not entered into the computation.

26. Yet neither these absolute parts have been found in vain, they perform a singular use, for which my first method was less convenient, which provided only the equation

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\text{etc.}=0\ ;$$

on which account this method is to be preferred by far to that. So that I can explain this use more clearly, in the first place the function *Z* shall be of *x* and *y*, not involving the differentials of these, thus so that there shall be $dZ=Mdx+Ndy$, with $P=0,\ Q=0$ etc. being present, and it is evident that in this case the absolute parts vanish at once, and thus the problem is perfectly soluble, and at once we will have made $N=0$. Thus, if $(bb-nxy+\frac{y^3}{c})dx$ should be a maximum or minimum, on account of $N=-nx+\frac{3yy}{c}$ for the question to be satisfied on putting $yy=\frac{1}{3}ncx$, nor here anything further remains to be determined.

27. But if *Z* besides may involve $p=\frac{dy}{dx}$, so that there shall be

$$dZ=Mdx+Ndy+Pdp,$$

then, so that $\int Zdx$ becomes a maximum or minimum, certainly it is necessary that

$$N-\frac{dP}{dx}=0.$$

Truly, because this is a differential equation and thus the differential of a differential, if the function *P* itself may involve $p=\frac{dy}{dx}$, its integration may take one or two arbitrary constants and therefore the relation between *x* and *y* will be determine completely. Now I have therefore observed this relation for the maximum or minimum in my treatment thus can further be desired to be defined, so that on putting $x=a$ the other given variable y may obtain a value, and if that equation $N-\frac{dP}{dx}=0$ were a differential of the second order, in addition one determination to be left to our choice. Therefore from these cases of the condition of the maxima or minima, at this point another condition pertaining to the extreme values of *y* can be added.

28. Furthermore it may be asked, since a relation between *x* and *y* may not be thoroughly determined from these cases, and at this point still can be shown in an infinite number of ways, which may produce a maximum or minimum before all the rest? Truly we will be able to deduce this from the absolute part of the increment ignored before, which in this case is $P\delta y$; therefore its value, which it adopts on putting $x=a$, also must vanish. And

hence we understand in this kind, if $\int Zdx$ must be a maximum or minimum with the equation being

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc. },$$

the equation

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

thus must be determined further, so that on putting $x = a$ it may be satisfied by the following equations:

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0, \quad Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.} = 0,$$
$$R - \frac{dS}{dx} + \text{etc.} = 0, \quad S - \text{etc.} = 0.$$

29. Because this may become clearer by example, the relation between x and y may be sought, so that on putting $x = a$ this formula

$$\int \frac{dx\sqrt{(1+pp)}}{\sqrt{y}} \text{ with the equation present } p = \frac{dy}{dx}$$

may obtain a maximum or minimum value. Therefore since there shall be

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{y}}$$

there will be

$$M = 0, \quad N = -\frac{\sqrt{(1+pp)}}{2y\sqrt{y}} \text{ and } P = \frac{p}{\sqrt{y(1+pp)}}$$

and thus in the first place this equation is fulfilled $N - \frac{dP}{dx} = 0$ or $Ndx - dP = 0,$ which multiplied by p gives $Ndy = pdP$. But on account of $M = 0$ there is $dZ = Ndy + Pdp$, and thus $dZ = pdP + Pdp$, which integrated gives

$$Z = Pp + C \text{ or } \frac{\sqrt{(1+pp)}}{\sqrt{y}} = \frac{pp}{\sqrt{y(1+pp)}} + C$$

that is

$$\frac{1}{\sqrt{y(1+pp)}} = C = \frac{1}{\sqrt{b}}.$$

Hence again we arrive at

$$b = y(1+pp) \text{ and } = \sqrt{\frac{b-y}{y}} = \frac{dy}{dx},$$

so that there shall be

$$dx = \frac{ydy}{\sqrt{(by-yy)}}$$

and on being integrated

$$x = c - \sqrt{(by-yy)} + b\text{Asin } \frac{\sqrt{(by-yy)}}{b}.$$

Truly towards a fuller determination there must be $P = 0$ on putting $x = a$, that is $p = 0$ and $y = b$; from which on putting $x = a$ and $y = b$ thus the constant *c* may be defined, so that there shall be $c = a - \pi b$. And if we may wish, so that on putting $x = 0$ there becomes also $y = 0$, there must be $b = \frac{a}{\pi}$.

30. Before we can apply this analytical investigation to the cases, in which the function *Z* also includes integral formulas within itself, we may examine the analysis itself more carefully which we have used up to now, and we consider the effects more accurately. Moreover this analysis depends on the two variables *x* and *y*, which are referred partially to the state which I have called principal, and partially to the state of variation, thus so that the first of these *x* may pertain equally to each state, truly the other *y*, while it is transferred from the principal state to the variation state, may take the increment δy, but while in the same state it may be moved forwards to the value $x + dx$, it shall be accustomed to take the increase in the differential *dy* ; hence if the variable *y* likewise may be moved forwards from its principal state into the variation state and in the corresponding place itself $x + dx$ may be moved forwards, its increase will be $dy + \delta y$. But since *x* may refer equally to each state , there will be $\delta x = 0$.

31. If now there may be considered some other function *V* related to the place *x* in the principal state and that may be moved to the location $x+dx$ in the same state, its increment, because it will agree with that, we may express in the customary manner by *dV*. But if that, with the value of *x* remaining the same, may be carried forwards from the principal state into the variation, we may set out its increase in the new manner by δV. But if now that function *V* shall be composed from the quantities *x*, *y*, *p*, *q*, *r* etc. in some manner, moreover the letters *p, q, r* etc. may designate quantities of this kind, of which the increments of each set $dp,\ dq,\ dr$ etc. and $\delta p,\ \delta q,\ \delta r$ etc. may be shown, hence by differentiating in the usual manner also both increments of the function *V* will be able to be assigned. If indeed it were for a translation from the place *x* to the place $x+dx$ in the same state by customary

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}\ ,$$

there will be for the translation from the principal state to the variation, truly with the same place *x* present, as we have noted, $\delta x = 0$,

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r \ \text{ etc.}$$

32. Then, if these differentials of the two kinds may be mixed among themselves, from the above there may now be agreed to be

$$\delta dV = d\delta V.$$

Hence if *V* now shall be a differential of the form *dV*, there will be

$$\delta ddU = d\delta dU = dd\delta U,\ \text{on account of}\ \delta dU = d\delta V$$

and in general, in whatever order the two signs of the differentiation *d* and δ were put in place, it will be possible to safely change the order of these without significance ; thus there will be

$$\delta d^3 V = d\delta d^2 V = d^2\delta dV = d^3\delta V.$$

But because here we may consider a single state of variation, for which transition is indicated by the sign δ, this sign at no time can be present more than once in compositions of this kind ; but always regarding that property, δ is the sign in such formulas in the end to advance.

33. The same interchange is extended also to the integral sign ; for if the formula of the integral $\int V$ may be put in place, with $\int$ denoting the sum of all the values in the same state, which correspond to all the values of *x* assumed, there will be also $\delta \int V = \int \delta V$,

because that is self-evident, since the increment of the translation of the whole sum shall be equal to the sum of all the increments of the elements present in the same translation. And it has been deduced from this superior source of analysis ; for since the formula of the integral $\int Zdx$ shall be proposed, its variation was being defined in the variational state, we have assumed to be

$$\delta\int Zdx = \int \delta(Zdx) = \int \delta Z \cdot dx,$$

because

$$\delta(Zdx) = \delta Zdx + Z\delta dx,$$

truly there is $\delta dx = 0$, as $\delta x = 0$. So also, if it should occur in the double integral $\iint V$, in the same manner there would be $\delta\iint V = \int \delta\int V = \iint \delta V$.
[Note Euler's use of the multiplication symbol '$\cdot$' to separate the varied part from the non-varied part of the formula here and above.]

34. Another artifice in the transformation of the integral, when after the integral sign the signs d and δ are joined together in turn, so that at least in the integration the sign of the single δ may be left. Thus for the formula of the proposed integral $\int V\delta dv$ on account of $\delta dv = d\delta v$, by considering δv as a simple quantity, there will be

$$\int V\delta dv = \int Vd\delta v = V\delta v - \int \delta vdV.$$

And again in the same manner there is seen to be :

$$\int Vdd\delta v = Vd\delta v - \delta vdV + \int \delta vddV$$
$$\int Vd^3\delta v = Vdd\delta v - d\delta vdV + \delta vddV - \int \delta vd^3V$$
$$\int Vd^4\delta v = Vd^3\delta v - d^2\delta vdV + d\delta vddV - \delta vd^3V + \int \delta vd^4V$$

etc.,

for there is

$$\int Vdd\delta v = Vd\delta v - \int d\delta vdV,$$

but in turn

$$\int d\delta vdV = \delta vdV - \int \delta vddV,$$

from which the account of these transformations is observed.

35. From these analytical premises there will not be difficulty to resolve all the questions of this kind about maxima and minima, even if in the formula $\int Zdx$ the function *Z* may contain within itself some integral formulas. Clearly the whole business reverts to this, so that the increment $\delta\int Zdx$ may be defined, which the proposed formula $\int Zdx$ takes, while it is transferred from the principle state to the varied state ; evidently so that put equal to nothing it will contain the solution of a maximum or minimum. Moreover I will call this increment the *differential variation* of the formula $\int Zdx$, which is required to be understood to arise, if the individual values of *y* may be increased by these arbitrary small parts δy . Then truly this variation must be seen to be extended through all the values of *x* from the term $x = 0$ as far as to the term $x = a$, for its complete determination it is required to be noted that thus by necessity that must vanish on putting $x = 0$. Hence therefore we may resolve the following problems with the help of this method, concerning which it is required to bear in mind the letters *p, q, r, s* etc., and thus to involve an account of the differentials of the two variables *x* and *y*, thus so that

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad s = \frac{dr}{dx} \text{ etc.}$$

PROBLEM 1

If *Z* shall be some function of the variables *x* and *y* and of their differential quantities involving *p, q, r, s* etc., thus so that its differential shall be of this form :

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

to find the differential variation of the integral formula $\int Zdx$ from the term $x = 0$ extended as far as to $x = a$.

SOLUTION

Therefore $\delta\int Zdx$ must be found, and since there shall be $\delta\int Zdx = \int \delta Zdx$, we will have at once on account of $\delta x = 0$

$$\delta Z = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Truly with the differential *dx* taken constant there is :

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$$
$$\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}$$
$$\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx} = \frac{d^3\delta y}{dx^3}$$
$$\delta s = \frac{\delta dr}{dx} = \frac{d\delta r}{dx} = \frac{d^4\delta y}{dx^4},$$

from which we will obtain

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{d^2\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + S\frac{d^4\delta y}{dx^4} + \text{etc.}$$

Now for the integration of the formula $\int \delta Z dx$, through the parts being put in place we see there is :

$$\int N\delta y dx = \int \delta y dx \cdot N$$
$$\int P d\delta y = P\delta y - \int \delta y dP$$
$$\int Q\frac{dd\delta y}{dx} = Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}dQ + \int \frac{\delta y}{dx}ddQ$$
$$\int R\frac{d^3\delta y}{dx^2} = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2}dR + \frac{\delta y}{dx^2}ddR - \int \frac{\delta y}{dx^2}d^3R$$
$$\int S\frac{d^4\delta y}{dx^3} = S\frac{d^3\delta y}{dx^3} - \frac{dd\delta y}{dx^3}dS + \frac{d\delta y}{dx^3}ddS - \frac{\delta y}{dx^3}d^3S - \int \frac{\delta y}{dx^3}d^4S$$
etc.

Therefore the variation of the differential sought is deduced from these:

$$\delta\int Zdx = \int \delta y dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)$$
$$+\delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)$$
$$+\frac{d^3\delta y}{dx^3}\left(S - \text{etc.}\right)$$
$$+\text{etc.},$$

where the first integral part has to be extended from the term [*i.e.* boundary] $x = 0$ as far as to $x = a$, which therefore includes all the intermediate variations; but in the remaining absolute [*i.e.* constant] parts at once it is allowed to put $x = a$, and δy will denote the increment of the extreme value of *y;* but $d\delta y,\ dd\delta y$ etc. will depend in addition on the values of the neighbouring increments.

COROLLARY 1

Therefore if the formula of the $\int Zdx$ must be a maximum or minimum for the term $x = a$, it is necessary, that the variation of its differential shall vanish, in whatever manner the variations δy may be taken. Therefore in the first place it is required, for all the intermediate values of *x*

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

from which equation the relation sought between *x* and *y* is held.

COROLLARY *2*

Hence moreover, if the terms *P, Q, R* etc. shall be present, on account of the integrations put in place, the relation between *x* and *y* may not be completely determined, because through the individual integrations arbitrary constant quantities are introduced in that. Therefore for these cases for the question of a maximum or minimum, other non zero conditions can be added on, so that as it were for certain given values of *x* the other variable *y* may obtain given values.

COROLLARY 3

But with conditions of this kind laid aside a new question can be formed, just as these constants introduced by integration must be defined, so that either a maximum of the maximas or a minimum of the minimums may be obtained : but for this it is necessary, that on putting $x = a$ it may be satisfied by these equations :

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} = 0$$
$$Q - \frac{dR}{dx} + \frac{ddS}{dx^2} = 0$$
$$R - \frac{dS}{dx} = 0$$
$$S = 0.$$

COROLLARY 4

Then truly on account of these reasons there is a need, so that for the other term $x = 0$ it may be satisfied by these same equations. For since the variation must vanish on putting $x = 0$, the part of the integral of this kind involves a constant, which may fulfill this condition ; but this constant absolute term, if in these there may be put $x = 0$, must be returned to zero. Whereby these formulas

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \text{etc.}, \quad Q - \frac{dR}{dx} + \text{etc.}, \quad R - \frac{dS}{dx} + \text{etc.}$$

must vanish equally in the case $x = 0$ and in the case $x = a$.

PROBLEM 2

If the function Z besides the quantities x, y, p, q, r etc. also may involve some certain integral $\Phi = \int \mathfrak{Z} dx$, so that there shall be

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

moreover in the formula Φ, $\mathfrak{Z}$ shall be some function of x, y, p, q, r etc. with the differential present

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + \text{etc.},$$

and thus by having these themselves, it may be required to define the variation of the differential of this integral formula $\int Zdx$ from the term $x=0$ extended to the term $x=a$.

SOLUTION

Since there shall be $\delta\int Zdx=\int\delta Zdx$, before everything we may seek δZ, and indeed in the first place it will be apparent at once :

$$\delta Z=L\delta\Phi+N\delta y+P\delta p+Q\delta q+R\delta r+\text{etc.},$$

where as it will be had as before

$$\delta p=\frac{d\delta y}{dx},\quad \delta q=\frac{dd\delta y}{dx^2},\quad \delta r=\frac{d^3\delta y}{dx^3},\quad \delta s=\frac{d^4\delta y}{dx^4}\ \text{etc.},$$

truly, on account of $\delta\Phi=\delta\int\mathfrak{Z}dx=\int\delta\mathfrak{Z}dx$ in a similar manner there will be

$$\delta\mathfrak{Z}=\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}$$

and hence

$$\delta\Phi=\int dx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right).$$

Therefore since the first member of the formula δZdx shall be $Ldx\delta\Phi$, there will be [an extra set of brackets has been inserted here by the translator]

$$\int Ldx\delta\Phi=\int Ldx\left(\int\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right)dx\right).$$

Now there may be put $\int Ldx=V$, and the equation will be had

$$\int Ldx\delta\Phi=V\int dx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right)$$
$$-\int Vdx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right).$$

This is the same as by which rule the integral $\int Ldx=V$ may be taken [*i.e.* evaluated between the limits 0 and *a*]; for whatever constant we may add, that may be taken away again in this expression. Therefore we may put that integral to be taken thus, so that it may vanish on putting $x=a$, and because the variation of the differential must be adapted to the term $x=a$, there will be

$$\int Ldx\delta\Phi = -\int Vdx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right),$$

to which if the rest of the parts may be added, we deduce to become

$$\delta\int Zdx = \int dx\left(\left(N - V\mathfrak{N}\right)\delta y + \left(P - V\mathfrak{P}\right)\delta p + \left(Q - V\mathfrak{Q}\right)\delta q + \text{etc.}\right),$$

where, if we may use the reductions indicated above, this variation of the differential now restricted to the term $x = a$ will be produced :

$$\begin{aligned}
&\int \delta y dx\left(\left(N - V\mathfrak{N}\right) - \frac{d\left(P - V\mathfrak{P}\right)}{dx} + \frac{dd\left(Q - V\mathfrak{Q}\right)}{dx^2} - \frac{d^3\left(R - V\mathfrak{R}\right)}{dx^3} + \text{etc.}\right)\\
&+\delta y\left(\left(P - V\mathfrak{P}\right) - \frac{d\left(Q - V\mathfrak{Q}\right)}{dx} + \frac{dd\left(R - V\mathfrak{R}\right)}{dx^2} - \text{etc.}\right)\\
&+\frac{d\delta y}{dx}\left(\left(Q - V\mathfrak{Q}\right) - \frac{d\left(R - V\mathfrak{R}\right)}{dx} + \text{etc.}\right)\\
&+\frac{dd\delta y}{dx^2}\left(\left(R - V\mathfrak{R}\right) - \text{etc.}\right),
\end{aligned}$$

the constitution of which expression by itself is evident.

COROLLARY 1

Therefore this solution arises from the preceding, if in place of the simple quantity *N*, *P, Q, R* etc. these composite expressions may be substituted :

$$N - V\mathfrak{N},\quad P - V\mathfrak{P},\quad Q - V\mathfrak{Q},\quad R - V\mathfrak{R}\ \text{ etc.},$$

where there is $V = \int Ldx$, with which integral taken thus, so that it vanishes on putting $x = a$.

COROLLARY 2

Therefore if the integral formula $\int Zdx$ must be returned a maximum or minimum for the term $x = a$, it is required to be put into place, that from the variation of all the intermediate values of *y* no variation of the differential may result, so that the relation between *x* and *y* may be defined thus, so that there shall be :

$$\left(N-V\mathfrak{N}\right)-\frac{d\left(P-V\mathfrak{P}\right)}{dx}+\frac{dd\left(Q-V\mathfrak{Q}\right)}{dx^2}-\frac{d^3\left(R-V\mathfrak{R}\right)}{dx^3}+\text{etc.}=0,$$

which relation therefore involves the prescribed term $x=a$, thus so that, if another term may be prescribed, also another indefinite relation between *x* and *y* would result, thus because the quantity *V* itself depends on this value $x=a$.

COROLLARY 3

In this manner a relation of this kind is found between *x* and *y*, from which the formula $\int Zdx$ thus may arrive at a maximum or minimum value, so that with the extreme values of *y* remaining the same, in whatever manner the intermediate values may change, the value of the formula $\int Zdx$ going to be produced shall always be in the case of the maximum less and in the case of the minimum greater, if the relation be used correctly.

COROLLARY 4

Truly also if our extreme values may be allowed to be determined, from the variation of the differential found these also may be able to be defined. Clearly the relation found by integrations must be determined thus, so that on putting $x=a$ the absolute part may vanish also. And so hence it is put into effect, so that on putting $x=a$ there shall be

$$\left(P-V\mathfrak{P}\right)-\frac{d\left(Q-V\mathfrak{Q}\right)}{dx}+\frac{dd\left(R-V\mathfrak{R}\right)}{dx^2}-\text{etc.}=0$$

$$\left(Q-V\mathfrak{Q}\right)-\frac{d\left(R-V\mathfrak{R}\right)}{dx}+\frac{dd\left(S-V\mathfrak{S}\right)}{dx^2}-\text{etc}=0$$

$$\left(R-V\mathfrak{R}\right)-\frac{d\left(S-V\mathfrak{S}\right)}{dx}+\text{etc.}=0$$

etc.

COROLLARY 5

Indeed in the case that V becomes $=0$, but yet hence not unless it is permitted to discard these terms, which involve the quantity *V* itself. For where its differentials occur, because here there is $\frac{dV}{dx}=L$, the value must be written for *L*, which it adopts on putting $x=a$, which perhaps in this case does not vanish, which likewise is to be borne in mind with the following differentials :

$$\frac{ddV}{dx^2}=\frac{dL}{dx},\ \frac{d^3V}{dx^3}=\frac{ddL}{dx^2}\ \text{ etc.,}$$

which earlier values are to be taken in general, before putting $x=a$ into these.

COROLLARY 6

But if also our first values of *y* may be left to be determined, then the equation must be satisfied by putting $x=0$ into the same equations, where the same are to required to be observed, which we have noted just now. Clearly it will be required to set out these equations completely before, so that $x=0$ may be substituted into these. But from these conditions only constant quantities may be determined to enter into an indefinite relation between *x* and *y*.

PROBLEM 3

If the function *Z* besides the quantities *x*, *y*, *p*, *q*, *r* etc. may involve also any two integral formulas $\varPhi=\int\mathfrak{Z}dx$ and $\varPhi^{/}=\int\mathfrak{Z}^{/}dx$, so that there shall be

$$dZ=Ld\varPhi+L^{/}d\varPhi^{/}+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.,}$$

moreover in these formulas $\varPhi$ and $\varPhi^{/}$ the functions $\mathfrak{Z}$ and $\mathfrak{Z}^{/}$ may be determined by the quantities *x*, *y*, *p*, *q*, *r* etc. only, so that there shall be

$$d\mathfrak{Z}=\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc.}$$
$$d\mathfrak{Z}^{/}=\mathfrak{M}^{/}dx+\mathfrak{N}^{/}dy+\mathfrak{P}^{/}dp+\mathfrak{Q}^{/}dq+\mathfrak{R}^{/}dr+\text{etc.,}$$

to define the relation between *x* and *y*, so that this integral formula $\int Zdx$ may follow, in as much as it may be extended from the term $x=0$ as far as $x=a$, to have a maximum or minimum value.

SOLUTION

Therefore it is required to define the variation differential of the formula $\int Zdx$, which since there shall be $\delta\int Zdx=\int\delta Zdx$, we have in the first place :

$$\delta Z=L\delta\varPhi+L^{/}\delta\varPhi^{/}+N\delta y+P\delta p+Q\delta q+R\delta r+\text{etc.},$$

then truly there is

$$\delta\varPhi=\delta\int\mathfrak{Z}dx=\int\delta\mathfrak{Z}dx\ \text{ et }\ \delta\varPhi^{/}=\int\delta\mathfrak{Z}^{/}dx$$

and hence on that account :

$$d\mathfrak{Z} = \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$
$$d\mathfrak{Z}' = \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

from which we deduce :

$$\delta\Phi = \int dx\left(\mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}\right)$$
$$\delta\Phi' = \int dx\left(\mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.}\right).$$

Therefore since the variation of the differential shall be sought

$$\delta\int Zdx = \int Ldx\delta\Phi + \int L'dx\delta\Phi' + \int Ndx\delta y + \int Pdx\delta p + \text{etc.},,$$

we may put $\int Ldx = V$ and $\int L'dx = V'$, and there will be as above

$$\int Ldx\delta\Phi = V\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right)$$
$$-\int Vdx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

$$\int L'dx\delta\Phi = V'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right)$$
$$-\int V'dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right).$$

Moreover we may put these integrals $\int Ldx = V$ and $\int L'dx = V'$ to be take thus, so that they may vanish on putting $x = a$, and the preceding parts of the formulas at once will become zero, if indeed the values of these should be taken for the term $x = a$. Therefore with all the parts compounded together we will obtain

$$\delta\int Zdx = \int dx\delta y\left(N - V\mathfrak{N} - V'\mathfrak{N}'\right)$$
$$+\int dx\delta p\left(P - V\mathfrak{P} - V'\mathfrak{P}'\right)$$
$$+\int dx\delta q\left(Q - V\mathfrak{Q} - V'\mathfrak{Q}'\right)$$
$$+\int dx\delta r\left(R - V\mathfrak{R} - V'\mathfrak{R}'\right)$$
etc.

Truly since there shall be

$$\int Pdx\delta p = P\delta y - \int \delta y dP$$
$$\int Qdx\delta q = Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}\, dQ \;+ \int \frac{\delta y}{dx} ddQ$$
$$\int Rdx\delta r = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2} dR + \frac{\delta y}{dx^2} ddR - \int \frac{\delta y}{dx^2} d^3R$$
$$\text{etc.}$$

we may elicit the variation of the differential sought :

$$\delta\int Zdx = \int dx\delta y\left(\left(N - V\mathfrak{N} - V^{/}\mathfrak{N}^{/}\right) - \frac{d\left(P - V\mathfrak{P} - V^{/}\mathfrak{P}^{/}\right)}{dx} + \frac{dd\left(Q - V\mathfrak{Q} - V^{/}\mathfrak{Q}^{/}\right)}{dx^2} - \text{ etc.}\right)$$
$$+\delta y\left(\left(P - V\mathfrak{P} - V^{/}\mathfrak{P}^{/}\right) - \frac{d\left(Q - V\mathfrak{Q} - V^{/}\mathfrak{Q}^{/}\right)}{dx} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(Q - V\mathfrak{Q} - V^{/}\mathfrak{Q}^{/}\right) - \frac{d\left(R - V\mathfrak{R} - V^{/}\mathfrak{R}^{/}\right)}{dx} + \text{etc.}\right)$$

COROLLARY

For the sake of brevity we may put :

$$N - V\mathfrak{N} - V^{/}\mathfrak{N}^{/} = (N),\; P - V\mathfrak{P} - V^{/}\mathfrak{P}^{/} = (P),\; Q - V\mathfrak{Q} - V^{/}\mathfrak{Q}^{/} = (Q) \text{ etc.}$$

and the indefinite relation between *x* and *y* may be expressed by this equation :

$$(N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{ etc.} = 0,$$

which still now involves the prescribed term $x = a$,because the integral formulas $\int Ldx = V$ and $\int L^{/}dx = V^{/}$ thus have been taken, so they may vanish on putting $x = a$.

COROLLARY 2

But since the integration of this equation, if it were differential, would involve arbitrary constants, and if these may be left to our determination, so that the formula $\int Zdx$ arrives at the maximum or minimum value of all, thus it will be agreed to define these, so that on putting $x=0$ as well as $x=a$ also it may be satisfied by these equations :

$$(P)-\frac{d(Q)}{dx}+\frac{dd(R)}{dx^2}-\text{etc.}=0,\quad (Q)-\frac{d(R)}{dx}+\text{etc.}=0,\quad (R)-\text{etc.}=0.$$

COROLLARY 3

If the function *Z* may involve not only the two integral formulas of this kind $\varPhi=\int \mathfrak{Z}dx,\ \varPhi^{/}=\int \mathfrak{Z}^{/}dx$, but also more $\varPhi^{//}=\int \mathfrak{Z}^{//}dx,\ \varPhi^{///}=\int \mathfrak{Z}^{///}dx$ etc., thus yet, so that the letters $\mathfrak{Z},\ \mathfrak{Z}^{/},\ \mathfrak{Z}^{//}$ etc. may denote only functions of the quantities *x, y, p, q, r* etc. nor may involve any further integral formulas, from the solution of the problem also the variations of the differential of formulas of this kind may be assigned readily.

PROBLEM 4

If the function *Z* besides the quantities *x, y, p, q, r* etc. also involve integral formula $\varPhi=\int \mathfrak{Z}dx$ in some manner, so that there shall be

$$dZ=Ld\varPhi+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.},$$

but the function $\mathfrak{Z}$ also besides *x, y, p, q, r* etc. may involve anew the integral formula $\varPhi=\int \mathfrak{z}dx$, so that

$$d\mathfrak{Z}=\mathfrak{L}d\varPhi+\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc.},$$

truly the function $\mathfrak{z}$ shall be composed only from the quantities *x, y, p, q, r* etc. with the differential present

$$d\mathfrak{z}=\mathfrak{m}dx+\mathfrak{n}dy+\mathfrak{p}dp+\mathfrak{q}dq+\mathfrak{r}dr+\text{etc.},$$

to define the relation between *x* and *y*, so that this formula of the integral $\int Zdx$, in as much as it extends from the term $x=0$ as far as to the term $x=a$, may follow with a maximum or minimum value.

SOLUTION

Therefore it is agreed upon to inquire in this final variation of the differential of the formula $\int Zdx$; which since there shall be $\delta\int Zdx = \int \delta Zdx$, we have in the first place :

$$dZ = Ld\Phi + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

and thus the variation of the differential will be

$$\delta\int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.}$$

But now on account of $\delta\Phi = \delta\int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ and

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

we will have in a similar manner

$$\delta\Phi = \int \mathfrak{L}dx\delta\Phi + \int \mathfrak{N}dx\delta y + \int \mathfrak{P}dx\delta p + \int \mathfrak{Q}dx\delta q + \int \mathfrak{R}dx\delta r + \text{etc.}$$

Then truly there is, $\delta\Phi = \delta\int \mathfrak{z}dx = \int \delta zdx$ and thus on account of

$$d\mathfrak{z} = \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.}$$

there will be

$$\delta\Phi = \int \mathfrak{n}dx\delta y + \int \mathfrak{p}dx\delta p + \int \mathfrak{q}dx\delta q + \int \mathfrak{r}dx\delta r + \text{etc.}$$

Now there shall be $\int \mathfrak{L}dx = v$, and there becomes

$$\int \mathfrak{L}dx\delta\Phi = v\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$- \int vdx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right),$$

from which we acquire

$$\delta\Phi = v\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$+ \int dx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) + \int dx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) + \int dx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}$$

Therefore again we may put $\int Ldx = V$ and $\int Lvdx = T$, and there will be

$$\begin{aligned}\int Ldx\delta\Phi = T\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)\\ -\int Tdx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)\\ +V\int dx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) + V\int dx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) + V\int dx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}\\ -\int Vdx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) - \int Vdx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) - \int Vdx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}\end{aligned}$$

There from all these gathered together the variation of the differential sought will be produce :

$$\begin{aligned}\delta\int Zdx = T\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)\\ +V\int dx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) + V\int dx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) + V\int dx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}\\ +V\int dx\delta y\left(N - V\mathfrak{N} + Vv\mathfrak{n} - T\mathfrak{n}\right)\\ +\int dx\delta p\left(P - V\mathfrak{P} + Vv\mathfrak{p} - T\mathfrak{p}\right)\\ +\int dx\delta q\left(Q - V\mathfrak{Q} + Vv\mathfrak{q} - T\mathfrak{q}\right)\\ + \text{etc.,}\end{aligned}$$

which since it must be extended as far as to the term $x = a$, we may put the integrals $\int Ldx = V$ and $\int Ldx\int \mathfrak{L}dx = T$, whenever the determination of the integration is left to our choice, thus taken, so that they may vanish on putting $x = a$, so that our expression may be returned more easily. Then truly for the sake of brevity we may put :

$$\begin{aligned}N - V\mathfrak{N} + \left(Vv - T\right)\mathfrak{n} = \left(N\right)\\ P - V\mathfrak{P} + \left(Vv - T\right)\mathfrak{p} = \left(P\right)\\ Q - V\mathfrak{Q} + \left(Vv - T\right)\mathfrak{q} = \left(Q\right)\\ R - V\mathfrak{R} + \left(Vv - T\right)\mathfrak{r} = \left(R\right)\\ + \text{etc.}\end{aligned}$$

to be present, so that we may assume $v = \int \mathfrak{L}dx$, and the variation of the differential sought may be reduced to this form :

$$\delta\int Zdx = \int dx\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right)$$
$$+\delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q) - \frac{(dR)}{dx} + \text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left((R) - \text{etc.}\right)$$
$$+\text{etc.},$$

COROLLARY 1

Since there shall be $v = \int \mathfrak{L}dx$, there will be

$$Vv = \int Ldz\int \mathfrak{L}dx \text{ and } Vv - T = \int Ldz\int \mathfrak{L}dx - \int Ldz\int \mathfrak{L}dx = \int \mathfrak{L}dx\int Ldz.$$

Because moreover by the determinations assumed the expression $Vv - T$ vanishes on putting $x = a$, if we may put $\int Ldx = V$ and $\int \mathfrak{L}Vdx = \mathfrak{V}$ thus it is required that both these integrals be taken thus, so that they may vanish on putting $x = a$.

COROLLARY 2

From these formulas $\int Ldx = V$ and $\int \mathfrak{L}Vdx = \mathfrak{V}$ introduced into the computation there will be on putting :

$$N - V\mathfrak{N} + \mathfrak{V}\mathfrak{n} = (N)$$
$$P - V\mathfrak{P} + \mathfrak{V}\mathfrak{p} = (P)$$
$$Q - V\mathfrak{Q} + \mathfrak{V}\mathfrak{q} = (Q)$$
$$R - V\mathfrak{R} + \mathfrak{V}\mathfrak{r} = (R)$$
$$+\text{etc.}$$

and the variation of the differential may be expressed thus by the letters (N), (P), (Q), and in the first place the above case was defined by the letters N, P, Q etc.

COROLLARY 3

From these now it can be deduced easily, if also the function $\mathfrak{z}$ may involve a new integral formula, just as then the variation of the differential may be expressed ; Cleary if there were

$$d\mathfrak{z} = \mathfrak{l}\, d\Phi' + \mathfrak{m}dx + \text{etc.},$$

then for the formulas $\int Ldx = V$ and $\int \mathfrak{L}Vdx = \mathfrak{V}$ above a third $\mathfrak{v} = \int \mathfrak{l}\,\mathfrak{V}dx$ may be added; the rest easily present themselves to be attended to.

PROBLEM 5

If the function *Z* besides the quantities *x, y, p, q, r* etc. also involves the integral formula $\Phi = \int \mathfrak{Z}dx$ in some manner, so that there shall be

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

truly the function $\mathfrak{Z}$ besides the quantities *x, y, p, q, r* etc. may involve anew the integral formula $\Phi = \int \mathfrak{Z}dx$, so that there shall be

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

to define the relation between *x* and *y*, so that this formula of the integral $\int Zdx$, as far as it extends from the term $x = 0$ to the given term $x = a$, arrives at a maximum or minimum value.

SOLUTION

The variation of the differential is as thus far :

$$\delta\int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

then truly we have $\delta\Phi = \delta\int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ and

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\Phi + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

But since there shall be $\Phi = \int \mathfrak{Z}dx$, there becomes

$$\mathfrak{Z} = \frac{d\Phi}{dx} \text{ and } \delta\mathfrak{Z} = \frac{\delta d\Phi}{dx} = \frac{d\delta\Phi}{dx};$$

we may put for the present

$$\delta\Phi = u \text{ and } \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} = \omega, s$$

so that this equation may be obtained

$$\frac{du}{dx} = \mathfrak{L}u + \omega,$$

of which the integral by taking *e* for the number, of which the logarithm $=1$, is

$$e^{-\int \mathfrak{L}dx}u = \int e^{-\int \mathfrak{L}dx}\omega dx,$$

[For $du - \mathfrak{L}udx = \omega dx$; $e^{-\int \mathfrak{L}dx}du - e^{-\int \mathfrak{L}dx}\mathfrak{L}udx = e^{-\int \mathfrak{L}dx}\omega dx$; thus $\frac{d}{dx}\left(ue^{-\int \mathfrak{L}dx}\right) = e^{-\int \mathfrak{L}dx}\omega$ etc.]
and thus

$$\delta\Phi = e^{\int \mathfrak{L}dx}\int e^{-\int \mathfrak{L}dx}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right),$$

from which it is deduced that

$$\int Ldx\delta\Phi = e^{\int \mathfrak{L}dx}Ldx\int e^{-\int \mathfrak{L}dx}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

Now putting $e^{\int \mathfrak{L}dx}Ldx = V$, which with the integral thus taken, so that it may vanish on putting $x = a$, and there shall be $e^{-\int \mathfrak{L}dx}V = U$, there will be

$$\int Ldx\delta\Phi = -\int Udx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right),$$

to which part if the remaining parts may be added and the above reductions made , will produce the variation of the differential sought $\delta\int Zdx$

$$\delta\int Zdx=\int dx\delta y\left(\left(N-U\mathfrak{N}\right)-\frac{d\left(P-U\mathfrak{P}\right)}{dx}+\frac{dd\left(Q-U\mathfrak{Q}\right)}{dx^2}-\frac{d^3\left(R-U\mathfrak{R}\right)}{dx^3}+\text{etc.}\right)$$
$$+\delta y\left(\left(P-U\mathfrak{P}\right)-\frac{d\left(Q-U\mathfrak{Q}\right)}{dx}+\frac{dd\left(R-U\mathfrak{R}\right)}{dx^2}-\text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(Q-U\mathfrak{Q}\right)-\frac{d\left(R-U\mathfrak{R}\right)}{dx}+\text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(\left(R-U\mathfrak{R}\right)-\text{etc.}\right),$$

from which as above that relation is elicited between *x* and *y*, from which a maximum or minimum value may be acquired for the term $x=a$ of the formula of the integral $\int Zdx$; for this relation is expressed by that equation :

$$\left(N-U\mathfrak{N}\right)-\frac{d\left(P-U\mathfrak{P}\right)}{dx}+\frac{dd\left(Q-U\mathfrak{Q}\right)}{dx^2}-\frac{d^3\left(R-U\mathfrak{R}\right)}{dx^3}+\text{etc.}=0$$

Then truly for the determination of the constant introduced by the integration, the individual absolute parts both for the case $x=a$ as well as for the case $x=0$ will be able to be put equal to zero.

COROLLARY

Because we have put $e^{-\int\mathfrak{L}dx}V=U$, there will be $V=e^{\int\mathfrak{L}dx}U$, so that by differentiating the equation becomes

$$dV=e^{\int\mathfrak{L}dx}\left(dU+U\mathfrak{L}dx\right).$$

But since there shall be $dV=e^{\int\mathfrak{L}dx}Ldx$, this differential equation will be had :

$$dU+U\mathfrak{L}dx=Ldx\ ,$$

from which the magnitude *U* thus is necessary to be defined, so that this may vanish on putting $x=a$.

PROBLEM 6

If the function Z besides the quantities x, y, p, q, r etc. may depend on the integral formula $\Phi = \int Zdx$ also, thus so that there shall be

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

do define the relation between x et y, so that this formula $\int Zdx$ may be led to a maximum or minimum valor, indeed as far as it may be extended from the term $x = 0$ as far as to the term $x = a$.

SOLUTION

Since the variation of the differential shall be

$$\delta\int dZ = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

also there will be had $\delta\Phi = \delta\int Zdx$, from which there becomes on differentiation :

$$d\delta\Phi = Ldx\delta\Phi + Ndx\delta y + Pdx\delta p + Qdx\delta q + Rdx\delta r + \text{etc.}$$

and hence there is found as before :

$$\int Ldx\delta\Phi = e^{\int Ldx} Ldx \int e^{-\int Ldx} dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),$$

and from which there is come upon, on account of $\int e^{\int Ldx} Ldx = e^{\int Ldx}$

$$\int Ldx\delta\Phi = e^{\int Ldx} \int e^{-\int Ldx} dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right)$$
$$- \int dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),$$

because the latter member is removed from the remaining parts. Whereby, if we may put $e^{-\int Ldx} = T$, the whole variation of the differential shall be

$$\delta\int Zdx = \frac{1}{T}\int dx\delta y\left(TN - \frac{d\cdot TP}{dx} + \frac{dd\cdot TQ}{dx^2} - \frac{d^3\cdot TR}{dx^3} + \text{etc.}\right)$$
$$+\delta y\left(TP - \frac{d\cdot TQ}{dx} + \frac{dd\cdot TR}{dx^2} - \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(TQ - \frac{d\cdot TR}{dx} + \text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(TR - \text{etc.}\right)$$
$$+\text{etc.},$$

Therefore so that the formula $\int Zdx$ may become a maxima or minima, the indefinite relation between x and y may be expressed by this equation :

$$TN - \frac{d\cdot TP}{dx} + \frac{dd\cdot TQ}{dx^2} - \frac{d^3\cdot TR}{dx^3} + \text{etc} = 0,$$

indeed the absolute individual parts may be taken care of by the constant introduced determined by the integration.

SCHOLIUM

Therefore with no geometrical considerations involved in this analysis, not only have we arrived at the same solutions of all the problems pertaining to this method of maximas and minimas, which I gave in my book on maximas and minimas, but also this special method supplies the determinations of the constants, which were left undetermined in the previous method ; from which innumerable individual problems are able to be resolved readily, to which it may be less suitable to apply the first method. Or to be used if among all the lines it may be required to be drawn from a given point not to another point, but to a certain given line either right or curved, upon which a body descending from that point may arrive at the given line in the shortest time, by the consideration of these absolute parts this problem is resolved easily, while from these with this condition, so that the curve sought may be normal to the given curve. But before finishing, I submit a theorem of outstanding analysis for examination, of which the truth is evident without difficulty from the principles put in place at this point and which may be seen to be of outstanding use in the integral calculus.

THEOREM

The proposed formula of the differential *Zdx*, in which *Z* shall be some function of the quantities $x,\ y,\ \ p=\frac{dy}{dx},\ \ q=\frac{dp}{dx},\ r=\frac{dp}{dx}$ etc., and these may be produced by differentiation:

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

thus so that this differential formula of the differential *Zdx* may include not only the first order, but also higher orders of each, then it will be readily decided, whether this permits an integration or if the differential shall be complete, or not? Indeed this expression may be considered with *dx* constant

$$V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{ etc.},$$

which if it may be taken equal to zero, the formula *Zdx* will be integrable; truly if it were not $V=0$, that will not be integrable.

ANALYTICA EXPLICATIO METHODI MAXIMORUM ET MINIMORUM

SUMMARIUM

Iam ante celeberrimum problema isoperimetricum insignia quaedam specimina huc pertinentia a Geometris sunt edita, cum antiquissimis iam fuerit exploratum circulum inter omnes alias figuras pari perimetro inclusas maximam aream complecti; quam quidem proprietatem ex circuli natura concluserunt, minime vero ipsam quaestionem directe aggredi sunt ausi, ut inter omnes figuras aequali perimetro terminatas eam investigarent, quae maximam aream includeret. Haec scilicet quaestio nimis est ardua, quam ut ante insignem calculi infinitorum promotionem de ea saltem cogitare licuisset. Mox vero primis quasi iactis huius calculi fundamentis ab acutissimo JOHANNE BERNOULLI quaestio de brachystochronis felicissimo successu est resoluta, quippe qua inter omnes lineas a puncto sublimiori ad humilius ductas ea quaerebatur, super qua grave tempore brevissimo descendat, quam egregiam proprietatem cycloidi competere invenerat. Methodus autem, qua Vir celeberrimus erat usus, fratri ipsius natu maiori JACOBO BERNOULLI manifesto occasionem. praebuisse videtur solutionem magni problematis isoperimetrici, quod deinceps tractavit, meditandi. Latissimo scilicet ambitu omnes huius generis quaestiones in hoc problemate est complexus, ut inter omnes lineas intra data duo puncta ducendas, sive debeant esse eiusdem longitudinis (unde quidem nomen isoperimetrici est natum) sive alia quadam indole communi praeditae, eam investigaret, quae vel maximam aream vel circa datum axem rotata maximum solidum vel in genere quamcumque maximi minimive proprietatem contineret. Methodum autem, qua summus illius temporis Geometra est usus, perpendentes ancipites haeremus, utrum magis eius incredibilem patientiam in prolixissimis et taediosissimis calculis expediendis, an summam sagacitatem in conclusionibus satis concinnis inde deducendis admirari debeamus. Ob hanc ipsam autem causam, quod conclusiones prodierint satis concinnae, mox suspicari licebat viam planiorem ac breviorem dari eodem perducentem; quam etiam eius frater iunior JOHANNES satis feliciter est ingressus, etiamsi statim pro quibusdam casibus nimis absconditis negotium minus successerit, quem tamen defectum deinceps, toto hoc argumento profundius retractato, largiter compensavit. Longo postea interiecto tempore Auctor harum dissertationum in eodem problemate evolvendo summum studium collocavit, et cum perspexisset omnes huius generis quaestiones eo redire, ut eiusmodi linea curva aequatione inter coordinatas x et y exprimenda investigetur, in qua talis formula integralis $\int Vdx$, quomodocunque quantitas V per x et y fuerit data, maximum minimumve valorem obtineat. Nunc autem evidens est in ista quantitate V infinitam varietatem locum habere posse, prout in eam praeter ipsas variabiles x et y tam earum differentialia cuiuscunque ordinis quam novae insuper formulae integrales ingrediuntur. Quodsi iam solutiones BERNOULLIANAE ad hanc normam examinentur, eae tantum ad eos casus, quibus quantitas V sola differentialia primi gradus involvit, restrictae reperiuntur ac praeterea casus, quibus in quantitate V novae formulae integrales insunt, inde penitus excluduntur, paucissimis exceptis, quos facile pro indole quaestionis ab hoc

incommodo liberare licet. Hunc igitur defectum noster Auctor felicissime cum in his Commentariis tum in opere singulari de hoc argumento edito supplevit, ut vix quicquam, quod amplius desiderari queat, reperiatur. Interim tamen ipsa methodus, etiamsi totum negotium satis expedite conficiat, tamen ipsi non satis naturalis est visa, propterea quod vis solutionis tota in consideratione elementorum curvae investigandae erat posita, ipsa vero quaestio facile ita adornari possit, ut ex Geometria penitus ad solam Analysin puram revocatur. Quaestio enim ita proposita, ut data quantitate V utcunque ex binis variabilibus x, y earumque differentialibus cuiuscunque ordinis, quin etiam ex formulis integralibus utcunque conflata, ea inter x et y relatio investigari debeat, qua formulae integrali $\int Vdx$ maximus minimusve valor concilietur *?* hoc in quam modo quaestio proposita prorsus a Geometria segregatur; ex quo etiam methodus genuina eam resolvendi a Geometria immunis esse debebat; et quo difficilius Analysis ad hunc scopum accommodari poterat, eo maiora incrementa huius scientiae, si res successerit, merito sperari licebat. Tametsi autem Auctor de hoc diu multumque esset meditatus atque amicis hoc desiderium aperuisset, tamen gloria primae inventionis acutissimo Geometrae Taurinensi LA GRANGE erat reservata, qui sola Analysi usus eandem plane solutionem est adeptus, quam Auctor ex considerationibus geometricis elicuerat. Verum ipsa illa solutio ita erat comparata, ut novam plane Analyseos speciem constituere eiusque fines non mediocriter promovere videretur; ex quo Auctori occasio est oblata hanc scientiam novo Calculi genere locupletandi, quem *Calculum variationum* appellat et cuius elementa hic tradere ac dilucide explicare constituit. Hic quidem calculus perinde ac differentialis in incrementis infinite parvis inter se comparandis versatur, verum in ratione tractationis ab eo maxime discrepat. Cum enim in calculo differentiali ex data quantitatum variabilium relatione relatio inter earum differentialia cuiusque ordinis investigetur, in calculo variationum ipsa relatio inter variabiles infinite parum immutari concipitur, ita ut, dum secundum relationem datam pro quovis alterius variabilis x valore altera y certum valorem sortitur, calculo variationum huic ipsi valori y incrementum quoddam infinite parvum adiiciatur, ex quo deinceps, quemadmodum formulae tam differentiales quam integrales varientur, definiri oportet. Incrementum iillud cuicunque valori y adiectum ab Auctore eius variatio vocatur ac, ne cum differentialibus confundatur, hoc charactere δy designatur; cum igitur hinc omnes formulae tam differentiales quam integrales, quatenus quantitatem y involvunt, certas variationes nanciscantur, auctor in priore dissertatione principia ac praecepta stabilit, quorum ope omnium huiusmodi formularum variationes definiri possunt: ita si W denotet huiusmodi formulam quamcunque, eius variationem δW per regulas peculiares assignare docet. Quo singulari calculo constituto deinceps in sequente dissertatione eius applicationem ad omnia problemata, quae circa maxima et minima excogitari possunt, clarissime ostendit, inque negotio hoc imprimis observari meretur, quod ita nova methodus mere analytica multo pleniores ac perfectiores solutiones suppeditet, quam prior illa ex Geometria petita.

1. Quae vulgo in elementis de methodo maximorum et minimorum tradi solent, ea in functionibus unius cuiuspiam quantitatis variabilis potissimum consumuntur, ita ut

proposita functione quacunque *V*, quae utcunque ex quantitate variabili *z* et constantibus fuerit composita, eas variabilis *z* determinationes investigari oporteat, quae functioni *V* maximum minimumve valorem inducant. Interdum etiam functiones duarum pluriumve variabilium *z, y, x* considerantur valoresque iis singulis assignandi quaeruntur, quibus functio maximum vel minimum valorem consequatur. Methodus autem, qua huius posterioris generis quaestiones resolvuntur, prorsus convenit cum ea, quae in genere priori adhibetur; si enim plures implicentur variabiles, successive unica tanquam variabilis spectatur eiusque valor pro maximo minimove producendo idoneus indagatur; quae operatio si per singulas variabiles fuerit instituta, omnium valores innotescunt, quibus valor functionis propositae vel maximus vel minimus reddatur.

2. Haud aliter res se habet, si proponatur functio duarum variabilium *x* et *y* quaeraturque valor ipsi *y* tribuendus, ut, cum pro *x* data quantitas *a* fuerit posita, ipsa functio maximum minimumve valorem impetret; statim enim ubique pro *x* scribatur *a*, et quaestio manifesto ad primum genus erit reducta. Verum si illa functio variabilium *x* et *y* non fuerit evoluta, sed per integrationem determinetur, quaestiones ad genus omnino diversum erunt referendae methodumque solvendi longe diversam requirunt. Veluti si *Z* fuerit functio quaecunque ipsarum *x* et *y* ac proponatur formula integralis $\int Zdx$, quaestionem ita enunciari conveniet: *Definire relationem inter binas variabiles x et y, ut valor istius formulae, postquam posuerimus* $a = x$, *fiat omnium maximus vel minimus.*

3. Quantum inter huiusmodi quaestiones et eas, quas ad prius genus retuli, intersit, ad sequentia momenta vel leviter attendenti mox patebit. Sit enim *V* functio evoluta ipsarum *x* et *y*, pro qua investigari debeat valor ipsius *y*, ut posito $x = a$ valor functionis *V* evadat maximus minimusve; atque ad hanc quaestionem solvendam statim poni potest $x = a$, quo facto valor ipsius *y* per methodum priorem ita determinabitur, ut non pendeat a valore indefinito ipsius *x*. At proposita formula integrali $\int Zdx$ non in formula differentiali *Zdx*, sed demum post integrationem ipsi *x* valorem illum determinatum *a* tribuere licet; neque, ut tum formulae $\int Zdx$ valor evadat maximus minimusve, valor quidam determinatus pro *y* sumendus negotium conficit, sed relatio quaedam inter *x* et *y* assignari debet; propterea quod, etiamsi post integrationem ponatur *x a*, tamen valor integralis $\int Zdx$ a relatione indefinita, quae inter *z* et *y* intercedit, pendeat et per omnes valores medios ipsius *y* determinetur.

4. Verum tales quaestiones circa formulam $\int Zdx$ maximam minimamve reddendam multo latius patent neque tantum ad casus, quibus *Z* est functio ipsarum *x* et *y*, restringuntur, sed pro *Z* assumi potest expressio quaecunque, quae relatione quapiam inter *x* et *y* assumta determinetur. Hinc *Z* involvere poterit praeter ipsas variabiles *x* et *y* etiam relationem differentialium earum, neque solum primi ordinis, sed etiam altiorum ordinum quorumcunque; scilicet si hae differentialium rationes ita exprimantur, ut sit

$$\frac{dy}{dx}=p,\ \frac{dp}{dx}=q,\ \frac{dq}{dx}=r\ \text{ etc.},$$

quantitas Z spectari poterit ut functio quaecunque omnium harum x, y, p, q, r etc. Quin etiam quantitas Z praeterea in se complecti potest novas formulas integrales utcunque in ea involutas; unde plura genera huiusmodi quaestionum nascuntur, ad quae methodus solvendi accommodari debet.

5. Huiusmodi problemata primum occasione famosi illius Problematis Isoperimetrici a IACOBO BERNOULLIO olim in summum Analyseos incrementum agitati tractari sunt coepta, quod arduum negotium etsi mira sagacitate a Viro illo acutissimo est expeditum, tamen methodus ab eo adhibita tantum ad casus, quibus quantitas Z praeter x et y solum earum differentialia prima seu litteram p involvebat, extendebatur et quovis casu singulari quasi ex considerationibus geometricis repeti debebat. Postquam autem in uberiori huius argumenti enodatione diu desudassem, methodum tandem maxime generalem sum adeptus, cuius ope omnia huiusmodi problemata, quibus quantitas Z non solum differentialia cuiusque ordinis sed etiam formulas integrales quascunque in se contineret, resolvi possunt, quam methodum libro singulari ample sum persecutus.

6. Etsi autem haec methodus ita est comparata, ut eius applicatio nullas figuras geometricas requirat, tamen ipsa istius methodi investigatio ex contemplatione linearum curvarum est petita, quam ob causam mihi etiam tum non satis naturalis est visa. Cum enim haec quaestio, qua relatio inter x et y quaeritur, ut formula integralis $\int Zdx$ posito post integrationem $x=a$ maximum minimumve valorem obtineat, sine ullo respectu ad geometriam proponi possit, solutio etiam adaequata et ex veris principiis deducta ab omni geometrica consideratione immunis esse debere videtur. Quod desiderium cum in meo tractatu non obscure essem testatus, Vir quidam Clarissimus et in arte Analytica versatissimus DE LA GRANGE Tournier litteris Taurini ad me datis nunciavit se huius voti compotem esse factum simulque fundamenta suae Analyseos mecum benevole communicavit. Quae cum plurimum in recessu habere videantur, meo more explicanda et uberius excolenda statui.

7. Consideremus igitur in genere formulam integralem $\int Zdx$, in qua sit Z functio utcunque per x et y composita, quae etiam rationem differentialium non solum primi sed etiam superiorum ordinum involvat ac praeterea quoque unam pluresve formulas integrales complectatur. Pro eius autem determinatione assumamus integrale ita capi, ut evanescat posito $x=0$; tum vero post integrationem tribuamus ipsi x valorem quendam datum $x=a$, sitque A valor, quem formula integralis tum recipit. Iam quaestio in hoc versatur, ut definiri debeat ea relatio inter x et y, ex qua per istas operationes maximus vel minimus valor pro A obtineatur. Hanc igitur relationem inter x et y, quae quaesito satisfaciat, aequatione quadam sive finita sive differentiali cuiuscunque ordinis exprimi oportet, quae simulac fuerit inventa, problema pro soluto erit habendum.

8. Ponamus, uti in Analysi fieri solet, hanc relationem inter x et y, quae quaeritur, iam constare, ita ut, quicunque valor definitus pro x assumatur, inde y quoque ac proinde etiam functio Z valorem determinatum adipiscatur. Concipiantur hoc modo successive pro x omnes possibiles valores a termino $x = 0$ usque ad terminum $x = a$ substitui, qui intervallis infinite parvis dx progrediantur, tum vero valores ipsius Z, qui his singulis valoribus ipsius x respondent, per dx multiplicari, haecque omnia producta in unam summam collecta eam quantitatem, quam littera A indicavimus, constituent, quae maxima vel minima esse debet. Quod ita est intelligendum, ut, si ex alia relatione inter x et y quacunque singulis valoribus ipsius x alii valores ipsi y hincque ipsi Z conveniant, ex iis pro A, si fuerit maximum, valor certe minor, sin autem fuerit minimum, certe maior sit prodituros, quam si iusta relatio inter x et y fuisset adhibita.

9. Quodsi autem hae variationes, quae singulis valoribus ipsius y inducuntur, infinite parvae concipiantur, tum per indolem maximorum et minimorum inde nulla mutatio in quantitatem A redundare debet; atque ex hoc ipso fonte determinatio maximorum et minimorum peti solet. Cum scilicet valoribus ipsius y pro arbitrio variationes infinite parvas tribuerimus, mutatio, quae inde in valoribus omnibus ipsius $Z\,dx$ ac proinde in eorum summa tota A oritur, calculo colligi debet, quae deinceps nihilo aequalis posita praebebit aequationem, in qua natura maximi minimive ideoque quaesita relatio inter x et y continebitur. Hac igitur operatione methodus huiusmodi maxima vel minima investigandi absolvitur, quae idcirco iisdem principiis atque vulgaris methodus maximorum ac minimorum innititur; quae quomodo per sola Analyseos praecepta, sine ullis ex Geometria petitis subsidiis, institui possit, accuratius perpendamus, quandoquidem hoc idem negotium principiis geometricis innixus iam satis prospero successu sum executus.

10. Cum igitur variationes infinite parvae singulis valoribus ipsius y inductae nullam mutationem in valore quantitatis A producere debeant hocque fieri oporteat, utcunque illae variationes accipiantur, dummodo fuerint infinite parvae, sufficiet in unico tantum quodam valore ipsius y huiusmodi variationem concipere et mutationem, quae inde in quantitate A oritur, evanescentem reddere, ex quo fonte etiam universa mea methodus maximorum et minimorum est petita. Verum etiamsi pluribus valoribus ipsius y, quin etiam plane omnibus, huiusmodi variationes infinite parvae quaecunque inducantur, nihilominus natura maximorum et minimorum exigit, ut mutatio, quam quantitas A inde adipiscitur, ad nihilum redigatur, atque hoc usu venire debet, utcunque illae variationes, quippe quae omnes mere sunt arbitrariae, assumantur.

11. Sed quoniam in mea praecedente solutione unicum quendam valorem ipsius y variationem infinite parvam accipere posui, dum reliqui omnes immutati manerent, in hoc principium continuitatis vim patiebatur haecque praecipua erat causa, quod tota investigatio per sola Analyseos praecepta expediri nequiverit, sed contemplatio figurae geometricae, in qua valores ipsius y per applicatas lineae curvae repraesentarentur, in subsidium vocari debuerit, quo inde variationes, quas ratio differentialium cuiusque ordinis subiret, commodius elici possent. Quamobrem, ne nimis principio continuitatis

adversemur, quo applicatio praeceptorum mere Analyticorum impediebatur, singulis valoribus ipsius y variationes infinite parvas tribuamus, quae tamen ita sint indefinitae, ut singulae deinceps ad libitum determinari atque adeo omnes praeter unam ad nihilum redigi possint, quo pacto ad solutiones meas priores devolvamur necesse est.

12. Cum autem nunc non solum uni valori ipsius y, sed innumerabilibus quin plane omnibus variationes infinite parvas quidem, sed tamen arbitrarias, tribuimus, dubium est nullum, quin haec methodus multo latius pateat quam praecedens, atque ad solutionem plurium aliorum problematum manuducat, ad quae prior methodus vel difficilius vel etiam frustra adhiberetur. Si enim illae variationes certo quodam modo determinentur, quaestione ad Geometriam translata huiusmodi problemata resolvi poterunt, in quibus non inter omnes plane lineas curvas, sed tantum eas numero quidem infinitas, quae sub certa quadam specie comprehendantur, ea debeat assignari, quae maximi minimive cuiuspiam proprietate sit praedita. Tales autem quaestiones plerumque plurimum difficultatis implicare deprehenduntur; verum praeterea hinc adhuc plura incrementa in Analysi merito expectare licet.

13. Cum igitur hic singulis valoribus ipsius y variationes infinite parvas tribuamus, duplicem statum formulae $\int Zdx$ consideramus, in quorum altero singuli valores ipsius y sint ii ipsi, quos quaesita relatio inter x et y requirit, in altero autem iidem valores variati contineantur; priorem statum distinctionis causa *principalem*, alterum vero statum *variatum* appellabo. Natura ergo maximorum et minimorum postulat, ut differentia inter hos duos status evanescat. Quemadmodum igitur in statu principali valor ipsius y quicunque, dum variabilis x differentiali dx crescere sumitur, incrementum dy capere censetur, ita manente x, dum a statu principali ad statum variatum progredimur, valorem ipsius y elemento δy augeri statuamus; unde discrimen inter has duas expressiones differentiales dy et δy probe notetur. Dum autem singulis valoribus ipsius y, transitu ad statum variatum facto, huiusmodi incrementa δy tribuimus, ea tanquam plane indeterminata neque ullo modo ab ipsis valoribus ipsius y pendentia sunt spectanda.

14. His positis indagari debet, quantum incrementum quaecunque functio Z pro quolibet valore ipsius x, dum a statu principali ad variatum transfertur, capiat; quod incrementum a sola variatione ipsius y, quatenus hac translatione elemento δy augetur, proficiscitur. Indicemus hoc incrementum per δZ, ita ut valor ipsius Z a statu principali ad variatum translatus sit $= Z + \delta Z$; ac primo statim patet, si functio Z a sola variabili x penderet neque alteram y implicaret, fore $\delta Z = 0$; neque igitur variabilis x, utcunque ea in formationem functionis Z ingrediatur, quicquam ad δZ conferet, sed eius valor a solo elemento δy, quo variabilis y crescere concipitur, resultat. Hic autem, prout Z vel solas quantitates finitas x et y vel etiam earum differentialium rationem vel adeo formulas integrales involvit, ita diversi casus erunt examinandi.

15. Ponamus ergo primo functionem Z tantum ipsas quantitates finitas x et y involvere, ita ut neque ratio differentialium neque ullae formulae integrales in eam ingrediantur atque ad eius variationem δZ definiendam in functione Z ubique loco y scribi oportet $y+\delta y$, relicto x invariato, sicque prodibit valor variatus $Z+\delta Z$, a quo si principalis Z subtrahatur, remanebit variatio δZ. Manifestum ergo est hanc variationem obtineri, si functio Z more solito differentietur posita sola y variabili, dummodo pro dy scribatur δy. Quare, si differentiatione more solito instituta sumta utraque quantitate x et y variabili fuerit

$$dZ = Mdx + Ndy\,,$$

erit pro translatione a statu principali ad variatum $\delta Z = N\delta y$; haec ergo variatio reperitur, si in differentiali ordinario pro dx scribatur 0, pro dy autem δy; hocque modo casum primum facillime expedivimus.

16. Videamus autem porro, quomodo pro hoc casu primo, quo Z est functio ipsarum x et y tantum, formulae integralis $\int Zdx$ valor maximus vel minimus inveniri queat. Cum igitur pro quovis valore ipsius x functio Z crescat elemento $N\delta y$ ideoque Zdx particula $Ndx\delta y$, summa omnium harum particularum a termino $x=0$ usque ad $x=a$ dabit variationem ipsius A, quae si ponatur δA, erit

$$\delta A = \int Ndx\delta y\,;$$

quae expressio cum debeat evanescere, quamcunque legem variationes δy teneant, necesse est, ut pro singulis valoribus ipsius x sit $N=0$. Haec ergo aequatio exprimit relationem inter x et y quaesitam, ex qua formula $\int Zdx$ adipiscitur valorem vel maximum vel minimum; neque haec proprietas tantum locum habebit casu praescripto $x=a$, sed etiam, quicunque alius valor ipsi x tribuatur.

17. Complectatur secundo functio Z praeter x et y etiam rationem differentialium primorum, seu posito $\frac{dy}{dx}=p$ sit Z functio quaecunque quantitatum x, y et p, qua more solito differentiata prodeat

$$dZ = Mdx + Ndy + Pdp\,.$$

Hinc igitur quaeri debet variatio ipsius Z, dum a statu principali in statum variatum transfertur, qua translatione quantitas x manet eadem, y vero augetur elemento δy, elementum autem, quo quantitas p crescit, sit δp. Cum autem sit $p=\frac{dy}{dx}$, si in statu principali valorem ipsius *y*, qui ipsi $x+dx$ respondet, per $y^{/}$ indicemus, erit $p=\frac{y^{/}-y}{dx}$; crescat iam in translatione in situm variatum *y* elemento δy et $y^{/}$ elemento $\delta y^{/}$, eritque

$$\delta p=\frac{\delta y^{/}-\delta y}{dx}$$

At $\delta y^{/}-\delta y$ exprimit incrementum ipsius δy, dum *x* crescit differentiali *dx*, ita ut sit

$$\delta y^{/}-\delta y=d\delta y$$

tum vero etiam $\delta y'-\delta y$ spectari potest ut variatio ipsius $y^{/}$ dum in statum variatum progredimur, sicque erit quoque

$$\delta y^{/}-\delta y=\delta dy\ ;$$

unde perficitur esse

$$d\delta y=\delta dy \text{ ideoque } \delta p=\frac{d\delta y}{dx}=\frac{\delta dy}{dx}.$$

18. Simili autem modo, si *Z* praeter *x* et *y* etiam differentialia superiorum ordinum involvat, ut positis

$$\frac{dy}{dx}=p,\quad \frac{dp}{dx}=q,\quad \frac{dq}{dx}=r \text{ etc.}$$

sit *Z* functio quaecunque quantitatum *x, y, p, q, r* etc. et more solito differentiando

$$dZ=Mdx+Ndy+Pdp+Qdq+Rdr \text{ etc.},$$

incrementa quantitatum *q, r* etc., dum a statu principali in variatum transferuntur, determinantur. Nam ob $q=\frac{dp}{dx}$ erit

$$\delta q=\frac{\delta p^{/}-\delta p}{dx}=\frac{d\delta p}{dx}=\frac{\delta dp}{dx} \text{ pariterque } \delta r=\frac{\delta q^{/}-\delta q}{dx}=\frac{d\delta q}{dx}=\frac{\delta dq}{dx} \text{ etc.}$$

Verum ex superioribus est

$$d\delta p = \frac{dd\delta y}{dx} = \frac{d\delta dy}{dx} \text{ et } \delta dp = \frac{\delta ddy}{dx},$$

ita ut sit:

$$\delta q = \frac{dd\delta y}{dx^2} = \frac{d\delta ddy}{dx^2} = \frac{\delta ddy}{dx^2},$$

eodem vero modo perspicitur fore:

$$\delta r = \frac{ddd\delta y}{dx^3} = \frac{dd\delta dy}{dx^3} = \frac{d\delta ddy}{dx^3} = \frac{\delta dddy}{dx^3},$$

quarum formularum specie diversarum aequalitas probe est tenenda.

19. Dum igitur functio Z ex statu principali in variatum transit, quia quantitas x incrementum nullum capit, y vero incrementum δy, tum quantitas p incrementum $\frac{d\delta y}{dx}$, quantitas q incrementum $\frac{dd\delta y}{dx^2}$, quantitas r incrementun $\frac{ddd\delta y}{dx^3}$ etc., ipsius functionis Z incrementum huic translationi conveniens reperietur per ordinariam differentiationem ponendo

$$dx = 0,\; dy = \delta y,\; dp = \frac{d\delta y}{dx},\; dq = \frac{dd\delta y}{dx^2},\; dr = \frac{ddd\delta y}{dx^3} \text{ etc.},$$

unde id erit:

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{ etc.}$$

Hincque ergo variatio functionis Z pro quovis ipsius x valore definiri poterit; quae forma adhuc magis illustrabitur, si, quemadmodum est

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr \text{ etc.},$$

observetur esse oportere ob $\delta x = 0$

$$\delta Z = N\delta y + P\delta p + Q\delta q + R\delta r + \text{ etc.}$$

tum vero esse ob $p = \frac{dy}{dx},\quad q = \frac{dp}{dx},\quad r = \frac{dq}{dx}$ etc.

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}, \quad \delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{ddd\delta y}{dx^3}.$$

20. Cum ergo translatione in statum variatum functio Z incrementum capiat δZ, ipsa formula $\int Zdx$ incrementum nanciscetur $\int \delta Zdx$, quod itaque erit:

$$\int dx(Ndy + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{etc.}),$$

in quo si post integrationem ponatur $x = a$, obtinebitur variatio ipsius A seu δA, quae nihilo aequalis posita inducet quantitati A valorem maximum seu minimum. In hac autem integratione non amplius ad transitum in statum variatum respicitur, sed ea per omnia incrementa ipsius x extendi debet,cum denotet summam omnium variationum singulis valoribus ipsius x a termino $x = 0$ usque ad $x = a$ convenientium. Ne igitur ratio differentialium per δ indicatorum turbet, pro δy scribatur ω, ita ut ω exhibeat quantitatem infinite parvam arbitrariam utcunque ab x pendentem; ac superius incrementum nihilo aequandum erit:

$$\int dx(N\omega + P\frac{d\omega}{dx} + Q\frac{dd\omega}{dx^2} + R\frac{d^3\omega}{dx^3} + \text{etc.}).$$

21. Perspicuum est in his differentialibus superioribus elementum dx
assumi constans; quia enim posuimus

$$\frac{d\delta p}{dx} \text{ seu } d\frac{\delta p}{dx} = \frac{dd\delta y}{dx^2}$$

ob $\delta p = \dfrac{d\delta y}{dx}$, aperte dx constans est assumtum. Hoc ergo observato, si singulas partes integralis inventi seorsim integremus, habebimus:

$$\int dx \cdot N\omega = \int N\omega dx$$

$$\int dx \cdot P\frac{d\omega}{dx} = \int Pd\omega = P\omega - \int \omega dP$$

$$\int dx \cdot Q\frac{dd\omega}{dx^2} = \int \frac{Qdd\omega}{dx} = \frac{Qd\omega}{dx} - \frac{\omega dQ}{dx} + \int \frac{\omega ddQ}{dx}$$

$$\int dx \cdot R\frac{d^3\omega}{dx^3} = \int R\frac{d^3\omega}{dx^2} = \frac{Rdd\omega}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{\omega ddR}{dx^2} - \int \frac{\omega d^3R}{dx^2}$$

etc.

Hinc itaque variatio quaesita partim ex membris integralibus, partim ex absolutis constabit, eritque:

$$\int \omega dx(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\text{etc.})+\omega(P+\frac{d\omega}{dx}(Q-\frac{dR}{dx}+\text{etc.})$$
$$+\frac{d\omega}{dx}(Q-\frac{dR}{dx}+\text{etc.})+\frac{dd\omega}{dx^2}(R-\text{etc.})$$

22. Restituamus δy pro ω, ac formulae integralis $\int Zdx$, existente

$$dZ = Mdx+Ndy+Pdp+Qdq+Rdr+Sds \text{ etc.},$$

et

$$p=\frac{dy}{dx},\quad q=\frac{dp}{dx},\quad r=\frac{dq}{dx}\quad s=\frac{dr}{dx}\ \text{ etc.}$$

incrementum, dum in statum quemcunque variatum transfertur, quod hoc modo $\delta\int Zdx$ exprimere licet, ita se habebit:

$$\int dx\delta y(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.})$$
$$+\delta y(P-\frac{dQ}{dx}+\frac{ddR}{dx^2}-\frac{d^3S}{dx^3}+\text{etc.})$$
$$+\frac{d\delta y}{dx}(Q-\frac{dR}{dx}+\frac{ddS}{dx^2}-\text{etc.})$$
$$+\frac{dd\delta y}{dx}(R-\frac{dS}{dx}+\text{etc.})$$
$$+\frac{d^3\delta y}{dx^3}(S-\text{etc.})$$
$$+\ \text{etc.},$$

in quibus formulis, quatenus differentialia superiorum graduum involvunt, differentiale dx constans est assumtum. At δy pro singulis valoribus ipsius x valorem habet arbitrarium.

23. Si igitur pro valore $x=a$ formula $\int Zdx$ maxima vel minima fieri debeat, incrementum modo inventum, si in eo statuatur $x=a$, nihilo aequale poni oportet hocque ita, ut semper evanescat, quomodocunque variations δy assumantur. Quare etiam, si talis variatio unico cuipiam valori y,qui convenit valori cuicunque ipsius x minori quam a,

tribuatur, expression inventa in nihilum abire debet. Tum autem nulla mutatio inde valoribus ultimis ipsius *y*, qui ipsi $x = a$ respondent, inducuntur; quare, cum posito $x = a$ pars incrementi absoluta

$$\delta y(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}) + \frac{d\delta y}{dx}(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.})$$
$$+ \frac{dd\delta y}{dx}(R - \frac{dS}{dx} + \text{etc.}) + \text{ etc.}$$

tantum ab ultimorum ipsius *y* valorum variatione pendeat, pro iis erit $\delta y = 0,\ d\delta y = 0,\ dd\delta y = 0$ etc. sicque haec pars sponte evanescit. Ex quo necesse est, ut sola pars integralis seorsim nihilo aequalis reddatur indeque fieri debeat:

$$\int dx\delta y(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}) = 0.$$

24. Haec autem expressio summam omnium variationum, quae ex singulorum ipsius *y* variationibus nascuntur, complectitur; sed quia talis mutatioin unico valore fieri concipitur, tota summa ad hanc unam variationem reducitur, reliquis omnibus evanescentibus; quare necesse est, ut pro hoc casu sit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0\ ;$$

Quoniam vero, in quocunque loco haec ; variatio fieri concipiatur, natura maximi minimive aeque hanc annihilationem postulat, necesse est, ut pro omnibus valoribus ipsius *x* sit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0\ ;$$

quae ergo aequatio continet relationem indefinitam inter *x* et *y*, qua efficitur, ut inde oriundus valor formulae integralis $\int Zdx$ fiat maximus vel minim usposito $x = a$, unde patet hanc relationem non ab ista quantitate *a* pendere.

25. Haec iam est eadem aequatio, quam pro solutione eiusdem problematis olim in Tractatu meo de Maximis ac Minimis dedi, nunc autem ex meris principiis analyticis derivavi; quod negotium ideo commode successit, quod singulis valoribus ipsius *y* variationes accedere assumsi, quibus in statum variatum transferantur. Deinde vero reductio formularum integralium paragrapho 21 facta negotium penitus confecit, qua illae

ita fuerunt in partes resolutae, ut aliae a signo summatorio $\int$ essent liberae, quae autem eo manserunt adstrictae, eae tantum ipsam variationem $\omega = \delta y$ sine eius differentialibus involverent; quo ipso hoc commodi sumus nacti, ut, cum quaelibet variatio seorsim ad nihilum perduci debeat, formula integralis statim praebuerit aequationem

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

qua indefinite relatio inter x et y exprimeretur, reliquae vero incrementi partes absolutae, utpote ad ultimos tantum ipsius y valores pertinentes, non in computum venirent.

26. Neque tamen hae partes absolutae frustra sunt inventae, sed singularem praestant usum, ad quem methodus mea prior, quae tantum aequationem $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.} = 0$ suppeditavit, minus est accommodata; quam ob causam haec methodus illi longe est anteferenda. Quem usum quo clarius exponam, sit primo Z functio tantum ipsarum x et y, earum differentialia non involvens, ita ut sit $dZ = Mdx + Ndy$, existentibus $P = 0,\ Q = 0$ etc., ac manifestum est hoc casu partes absolutas sponte evanescere, atque adeo problema perfecte esse solutum, statim ac fecerimus $N = 0$. Ita, si $(bb - nxy + \frac{y^3}{c})dx$ debeat esse maximum vel minimum, ob $N = -nx + \frac{3yy}{c}$ quaestioni satisfit statuendo $yy = \frac{1}{3}ncx$, neque hic quicquam ultra determinandum superest.

27. At si Z praeterea involvat $p = \frac{dy}{dx}$, ut sit

$$dZ = Mdx + Ndy + Pdp,$$

tum, ut $\int Zdx$ fiat maximum vel minimum, utique necesse est, sit $N - \frac{dP}{dx} = 0.$
Verum, quia haec aequatio est differentialis atque adeo differentio-differentialis, si functio P ipsam quantitatem $p = \frac{dy}{dx}$ involvat, integratio eius unam vel duas constantes arbitrarias accipiet neque propterea relatio inter x et y penitus determinabitur. Observavi igitur iam in meo tractatu hanc relationem maximo minimove convenientem ita praeterea ad lubitum definiri posse, ut posito $x = a$ altera variabilis y datum valorem obtineat, ac si illa aequatio $N - \frac{dP}{dx} = 0$ fuerit differentialis secundi gradus, insuper unam determinationem arbitrio nostro relinqui. His igitur casibus conditioni maximi vel minimi adhuc alia conditio ad valores extremos ipsius y pertinens adiungi potest.

28. Porro igitur quaeri potest, cum his casibus relatio inter *x* et *y* non penitus determinetur eaque adhuc infinitis modis exhiberi queat, quinam prae omnibus reliquis maximum minimumve producat. Hoc vero colligere poterimus ex parte incrementi absoluta ante neglecta, quae hoc casu est $P\delta y$; cuius igitur valor, quem induit posito $x = a$, etiam evanescere debet. Atque hinc in genere intelligimus, si $\int Zdx$ debeat esse maximum vel minimum existente

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc. },$$

aequationem

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

ita ulterius determinari debere, ut posito $x = a$ sequentibus satisfiat aequationibus:

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0, \quad Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.} = 0,$$
$$R - \frac{dS}{dx} + \text{etc.} = 0, \quad S - \text{etc.} = 0.$$

29. Quia exemplo haec clariora evadent, quaeratur relatio inter *x* et *y*, ut posito $x = a$ haec formula

$$\int \frac{dx\sqrt{(1+pp)}}{\sqrt{y}} \text{ existente } p = \frac{dy}{dx}$$

maximum minimumve obtineat valorem. Cum ergo sit

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{y}}$$

erit

$$M = 0, \quad N = -\frac{\sqrt{(1+pp)}}{2y\sqrt{y}} \text{ et } P = \frac{p}{\sqrt{y(1+pp)}}$$

sicque primo adimplenda est haec aequatio $N-\frac{dP}{dx}=0$ seu $Ndx-dP=0,$ quae per p multiplicata dat $Ndy=pdP$. At ob $M=0$ est $dZ=Ndy+Pdp$, ideoque $dZ=pdP+Pdp$, quae integrata praebet

$$Z=Pp+C \quad \text{seu} \quad \frac{\sqrt{(1+pp)}}{\sqrt{y}}=\frac{pp}{\sqrt{y(1+pp)}}+C$$

hoc est

$$\frac{1}{\sqrt{y(1+pp)}}=C=\frac{1}{\sqrt{b}}.$$

Hinc porro nanciscimur

$$b=y(1+pp) \quad \text{et} \quad p=\sqrt{\frac{b-y}{y}}=\frac{dy}{dx},$$

ita ut sit

$$dx=\frac{ydy}{\sqrt{(by-yy)}}$$

et integrando

$$x=c-\sqrt{(by-yy)}+b\text{Asin}\,\frac{\sqrt{(by-yy)}}{b}.$$

Verum ad pleniorem determinationem debet esse $P=0$ posito $x=a$, hoc est $p=0$ et $y=b$; unde positis $x=a$ et $y=b$ constans c ita definitur, ut sit $c=a-\pi b$. Ac si velimus, ut posito $x=0$ fiat et $y=0$, debet esse $b=\frac{a}{\pi}$.

30. Antequam hanc investigationem analyticam ad casus, quibus functio Z etiam formulas integrales in se complectitur, accommodemus, ipsam Analysin, qua hactenus sumus usi, diligentius examinemus ac momenta, quibus innititur, accuratius perpendamus. Versatur autem haec Analysis circa duas variabiles x et y, quae partim ad statum, quem vocavi principalem, partim ad statum variatum referuntur, ita ut earum altera x ad utrumque statum aeque pertineat, altera vero y, dum a statu principali ad variatum transfertur, incrementum capiat δy, dum autem in eodem statu ad valorem $x+dx$ promovetur, augmentum differentiale consuetum dy accipiat; hinc si variabilis y simul a statu principali in variatum et locum ipsi $x+dx$ respondentem promoveatur, augmentum eius erit $dy+\delta y$. Cum autem x ad utrumque statum aeque referatur, erit $\delta x=0$.

31. Si iam habeatur alia quaecunque functio *V* ad locum *x* in statu principali relata eaque in eodem statu ad locum $x+dx$ promoveatur, eius incrementum, quod ei accedet, more solito per *dV* exprimamus. Sin autem ea, manente valore ipsius *x* eodem, e statu principali in variatum proferatur, eius augmentum novo more per δV exponamus. Quodsi iam functio illa *V* sit ex quantitatibus *x, y, p, q, r* etc. utcunque composita, litterae autem *p, q, r* etc. eiusmodi quantitates designent, quarum utraque incrementa $dp,\ dq,\ dr$ etc. et $\delta p,\ \delta q,\ \delta r$ etc. exhiberi queant, hinc consueto differentiandi modo etiam ambo incrementa functionis *V* assignari poterunt. Si enim fuerit pro translatione a loco *x* ad locum $x+dx$ in eodem statu ex differentiatione consueta

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}\ ,$$

erit pro translatione a statu principali in variatum, eodem vero loco *x* existente, uti notavimus, $\delta x = 0$,

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r \quad \text{etc.}$$

32. Deinde, si haec duplicis generis differentialia inter se permisceantur, ex superioribus iam constat esse

$$\delta dV = d\delta V.$$

Hinc si *V* iam sit differentiale formae *dV,* erit

$$\delta ddU = d\delta dU = dd\delta U,\ \text{ob}\ \ \delta dU = d\delta V$$

atque in genere, quocunque ordine bina differentiationis signa *d* et δ fuerint constituta, eorum ordo pro lubitu permutari potest salva significatione; sic erit

$$\delta d^3V = d\delta d^2V = d^2\delta dV = d^3\delta V.$$

Quia autem hic unicum statum variatum consideramus, ad quem transitus signo δ indicatur, hoc signum nunquam plus quam semel in huiusmodi compositionibus inesse potest; semper autem e re est signum δ in talibus formulis in ultimum promovere.

33. Eadem permutatio quoque ad signa integralia extenditur; si enim proponatur formula integralis $\int V$, denotante $\int$ summam omnium valorum in eodem statu, qui omnibus valoribus ipsius *x* respondent, sumtorum, erit etiam $\delta\int V = \int \delta V$, id quod per se est perspicuum, cum incrementum translatitium totius summae aequale sit summae omnium

incrementorum elementarium in eadem translatione existentium. Atque ex hoc ipso fonte superior Analysis est deducta; nam cum proposita esset formula integralis $\int Zdx$, cuius variatio in statum variatum erat definienda, assumsimus esse

$$\delta\int Zdx=\int\delta(Zdx)=\int\delta Z\cdot dx,$$

quia

$$\delta(Zdx)=\delta Zdx+Z\delta dx,$$

est vero $\delta dx=0$, uti $\delta x=0$. Quin etiam, si occurreret integratio geminata $\iint V$, foret eodem modo $\delta\iint V=\int\delta\int V=\iint\delta V$.

34. Alterum artificium in transformatione integralium, quando post signum integrale signa *d* et δ invicem coniunguntur, ut saltem in integratione signum δ solitarium relinquatur. Ita proposita formula integrali $\int V\delta dv$ ob $\delta dv=d\delta v$, considerando δv uti quantitatem simplicem, erit

$$\int V\delta dv=\int Vd\delta v=V\delta v-\int\delta vdV.$$

Eodemque porro modo perspicitur fore:

$$\int Vdd\delta v=Vd\delta v-\delta vdV+\int\delta vddV$$
$$\int Vd^3\delta v=Vdd\delta v-d\delta vdV+\delta vddV-\int\delta vd^3V$$
$$\int Vd^4\delta v=Vd^3\delta v-d^2\delta vdV+d\delta vddV-\delta vd^3V+\int\delta vd^4V$$

etc.,

est enim

$$\int Vdd\delta v=Vd\delta v-\int d\delta vdV,$$

at est

$$\int d\delta vdV=\delta vdV-\int\delta vddV,$$

unde ratio harum transformationum perspicitur.

35. His regulis analyticis praemissis non erit difficile omnes quaestiones huiusmodi circa maxima et minima resolvere, etiamsi in formula $\int Zdx$ functio *Z* formulas integrales

quascunque in se contineat. Totum negotium scilicet huc redit, ut incrementum d $\delta\int Zdx$, quod formula proposita $\int Zdx$, dum a statu principali in variatum transfertur, accipit, definiatur; quippe quod nihilo aequale positum solutionem maximi minimive continebit. Vocabo autem hoc incrementum *variationem differentialem* formulae $\int Zdx$, quae oriri intelligenda est, si singuli valores ipsius *y* particulis infinite parvis *dy* iisque arbitrariis augeantur. Tum vero hanc variationem per omnes valores ipsius *x* a termino $x=0$ usque ad terminum $x=a$ extendi debere perspicuum est, pro cuius completa determinatione observandum est eam ita sumi oportere, ut posito $x=0$ ea evanescat. Hinc igitur sequentia problemata ope huius methodi resolvamus, circa quae tenendum est litteras *p, q, r, s* etc. rationem differentialium binarum variabilium *x* et *y* ita involvere, ut sit

$$p=\frac{dy}{dx},\quad q=\frac{dp}{dx},\quad r=\frac{dq}{dx}\quad s=\frac{dr}{dx}\ \text{etc.}$$

PROBLEMA 1

Si *Z* sit functio quaecunque variabilium *x* et *y* quantitatumque earum differentialia involventium *p, q, r, s* etc., ita ut eius differentiale sit huiusmodi:

$$dZ=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+\text{etc.},$$

invenire variationem differentialem formulae integralis $\int Zdx$ a termino $x=0$ usque ad $x=a$ extensam.

SOLUTIO

Quaeri ergo debet $\delta\int Zdx$, et cum sit $\delta\int Zdx=\int\delta Zdx$, habebimus statim ob $\delta x=0$

$$\delta Z=M\delta x+N\delta y+P\delta p+Q\delta q+R\delta r+S\delta s+\text{etc.}$$

Est vero sumto differentiali *dx* constante :

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$$
$$\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}$$
$$\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx} = \frac{d^3\delta y}{dx^3}$$
$$\delta s = \frac{\delta dr}{dx} = \frac{d\delta r}{dx} = \frac{d^4\delta y}{dx^4},$$

unde obtinebimus

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{d^2\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + S\frac{d^4\delta y}{dx^4} + \text{etc.}$$

Iam pro integratione formulae $\int \delta Z dx$, per partes instituenda vidimus esse

$$\int N\delta y dx = \int \delta y dx \cdot N$$
$$\int P d\delta y = P\delta y - \int \delta y dP$$
$$\int Q\frac{dd\delta y}{dx} = Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}dQ + \int \frac{\delta y}{dx}ddQ$$
$$\int R\frac{d^3\delta y}{dx^2} = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2}dR + \frac{\delta y}{dx^2}ddR - \int \frac{\delta y}{dx^2}d^3R$$
$$\int S\frac{d^4\delta y}{dx^3} = S\frac{d^3\delta y}{dx^3} - \frac{dd\delta y}{dx^3}dS + \frac{d\delta y}{dx^3}ddS - \frac{\delta y}{dx^3}d^3S - \int \frac{\delta y}{dx^3}d^4S$$
etc.

Ex his ergo colligitur variatio differentialis quaesita:

$$\begin{aligned}\delta\int Zdx = \int \delta y dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)\\ +\delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)\\ +\frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)\\ +\frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)\\ +\frac{d^3\delta y}{dx^3}\left(S - \text{etc.}\right)\\ +\text{etc.},\end{aligned}$$

ubi pars prima integralis a termino $x = 0$ usque ad $x = a$ extendi debet, quae ergo omnes variationes intermedias complectitur; in reliquis autem partibus absolutis statim ponere licet $x = a$, et δy denotabit incrementum extremi valoris ipsius *y;* at $d\delta y$, $dd\delta y$ etc. pendebunt insuper ab incrementis valorum contiguorum.

COROLLARIUM 1

Si ergo formula integralis $\int Zdx$ debeat esse maximum vel minimum pro termino $x = a$, necesse est, ut eius variatio differentialis evanescat, quomodocunque variationes δy accipiantur. Primum ergo oportet, sit pro omnibus valoribus intermediis ipsius *x*

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

qua aequatione relatio requisita inter *x* et *y* continetur.

COROLLARIUM *2*

Hinc autem, si termini *P, Q, R* etc. adsint, ob integrationes instituendas relatio inter *x* et *y* non penitus determinatur, quia in eam per singulas integrationes constantes quantitates arbitrariae ingrediuntur. His igitur casibus ad quaestionem maximi vel minimi aliae nonnullae conditiones adiungi possunt, veluti ut pro datis quibusdam valoribus ipsius *x* altera variabilis *y* datos valores obtineat.

COROLLARIUM 3

Omissis autem huiusmodi conditionibus nova quaestio formari potest, quemadmodum constantes illae per integrationem introductae definiri debeant, ut vel maximum maximorum vel minimum minimorum obtineatur: pro hoc autem necesse est, ut posito $x = a$ his aequationibus satisfiat:

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} = 0$$
$$Q - \frac{dR}{dx} + \frac{ddS}{dx^2} = 0$$
$$R - \frac{dS}{dx} = 0$$
$$S = 0.$$

COROLLARIUM 4

Deinde vero ob easdem rationes opus est, ut pro altero termino $x = 0$ iisdem hisce aequationibus satisfiat. Nam cum variatio differentialis evanescere debeat posito $x = 0$, pars integralis eiusmodi constantem involvit, quae hanc conditionem adimpleat; haec autem constans terminos absolutos, si in iis ponatur $x = 0$, ad nihilum redigere debet. Quare formulae illae

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \text{etc.}, \quad Q - \frac{dR}{dx} + \text{etc.}, \quad R - \frac{dS}{dx} + \text{etc.}$$

aeque evanescere debent casu $x = 0$ atque casu $x = a$.

PROBLEMA 2

Si functio Z praeter quantitates x, y, p, q, r etc. etiam formulam quandam integralem $\Phi = \int \mathfrak{Z} dx$ utcunque implicet, ut sit

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

in formula autem Φ sit $\mathfrak{Z}$ functio quaecunque ipsarum x, y, p, q, r etc. existente

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + \text{etc.},$$

atque his ita se habentibus oporteat definiri variationem differentialem huius formulae integralis $\int Zdx$ a termino $x=0$ ad terminum $x=a$ extensam.

SOLUTIO

Cum sit $\delta\int Zdx = \int \delta Zdx$, quaeramus ante omnia δZ, ac primo quidem statim patet esse

$$\delta Z = L\delta\Phi + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

ubi ut ante habebitur

$$\delta p = \frac{d\delta y}{dx},\quad \delta q = \frac{dd\delta y}{dx^2},\quad \delta r = \frac{d^3\delta y}{dx^3},\quad \delta s = \frac{d^4\delta y}{dx^4}\ \text{etc.},$$

verum, ob $\delta\Phi = \delta\int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ erit simili modo

$$\delta\mathfrak{Z} = \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

hincque

$$\delta\Phi = \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

Cum igitur primum membrum formulae δZdx sit $Ldx\delta\Phi$, erit

$$\int Ldx\delta\Phi = \int Ldx\int\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right)dx.$$

Ponatur nunc $\int Ldx = V$, ac habebitur

$$\int Ldx\delta\Phi = V\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right)$$
$$-\int Vdx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

Perinde hic est, qua lege integrale $\int Ldx = V$ capiatur; quamcunque enim constantem adiiceremus, ea in hac expressione iterum tolleretur. Ponamus ergo istud integrale ita capi, ut evanescat posito $x=a$, et quia variatio differentialis ad terminum $x=a$ accommodari debet, erit

$$\int Ldx\delta\Phi = -\int Vdx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right),$$

ad quod si addantur reliquae partes, colligimus fore

$$\delta\int Zdx=\int dx\big((N-V\mathfrak{N})\delta y+(P-V\mathfrak{P})\delta p+(Q-V\mathfrak{Q})\delta q+\text{etc.}\big),$$

ubi, si reductiones supra indicatas adhibeamus, prodibit ista variatio differentialis iam ad terminum $x=a$ adstricta:

$$\begin{aligned}&\int\delta ydx\left((N-V\mathfrak{N})-\frac{d(P-V\mathfrak{P})}{dx}+\frac{dd(Q-V\mathfrak{Q})}{dx^2}-\frac{d^3(R-V\mathfrak{R})}{dx^3}+\text{etc.}\right)\\&+\delta y\left((P-V\mathfrak{P})-\frac{d(Q-V\mathfrak{Q})}{dx}+\frac{dd(R-V\mathfrak{R})}{dx^2}-\text{etc.}\right)\\&+\frac{d\delta y}{dx}\left((Q-V\mathfrak{Q})-\frac{d(R-V\mathfrak{R})}{dx}+\text{etc.}\right)\\&+\frac{dd\delta y}{dx^2}\big((R-V\mathfrak{R})-\text{etc.}\big),\end{aligned}$$

cuius expressionis constitutio per se est manifesta.

COROLLARIUM 1

Haec ergo solutio ex praecedente oritur, si loco quantitatum simplicium *N, P, Q, R* etc. substituantur hae compositae :

$$N-V\mathfrak{N},\quad P-V\mathfrak{P},\quad Q-V\mathfrak{Q},\quad R-V\mathfrak{R}\ \text{ etc.},$$

ubi est $V=\int Ldx$, integrali hoc ita sumto, ut evanescat posito $x=a$.

COROLLARIUM 2

Si igitur formula integralis $\int Zdx$ debeat reddi maximum vel minimum pro termino $x=a$, efficiendum est, ut ex variatione omnium valorum intermediorum ipsius *y* nulla variatio differentialis resultet, unde relatio inter *x* et *y* ita definitur, ut sit :

$$(N-V\mathfrak{N})-\frac{d(P-V\mathfrak{P})}{dx}+\frac{dd(Q-V\mathfrak{Q})}{dx^2}-\frac{d^3(R-V\mathfrak{R})}{dx^3}+\text{etc.}=0,$$

quae ergo relatio iam terminum praescriptum $x = a$ involvit, ita ut, si alius terminus praescribatur, alia quoque relatio indefinita inter *x* et *y* esset resultatura, propterea quod quantitas *V* hunc valorem $x = a$ in se complectitur.

COROLLARIUM 3

Hoc modo eiusmodi relatio inter *x* et *y* invenitur, ex qua formula $\int Zdx$ ita maximum minimumve valorem adipiscatur, ut manentibus valoribus ipsius *y* extremis iisdem, quomodocunque valores intermedii immutentur, formulae $\int Zdx$ valor proditurus sit semper vel minor casu maximi vel maior casu minimi, quam si iusta relatio adhiberetur.

COROLLARIUM 4

Si vero etiam valores extremi determinationi nostrae permittantur, ex variatione differentiali inventa etiam hos definire licet. Relatio scilicet inventa per integrationes ita determinari debet, ut posito $x = a$ etiam pars absoluta evanescat. Hinc itaque efficiendum est, ut posito $x = a$ sit

$$\left(P - V\mathfrak{P}\right) - \frac{d\left(Q - V\mathfrak{Q}\right)}{dx} + \frac{dd\left(R - V\mathfrak{R}\right)}{dx^2} - \text{etc.} = 0$$
$$\left(Q - V\mathfrak{Q}\right) - \frac{d\left(R - V\mathfrak{R}\right)}{dx} + \frac{dd\left(S - V\mathfrak{S}\right)}{dx^2} - \text{etc} = 0$$
$$\left(R - V\mathfrak{R}\right) - \frac{d\left(S - V\mathfrak{S}\right)}{dx} + \text{etc.} = 0$$
etc.

COROLLARIUM 5

Hoc quidem casu fit $V = 0$, verumtamen hinc nonnisi eos terminos, qui ipsam quantitatem *V* involvunt, eiicere licet. Ubi enim eius differentialia occurrunt, quia est $\frac{dV}{dx} = L$, pro *L* scribi debet valor, quem induit posito $x = a$, qui forte hoc casu non evanescit, quod idem tenendum est de differentialibus sequentibus :

$$\frac{ddV}{dx^2} = \frac{dL}{dx},\ \frac{d^3V}{dx^3} = \frac{ddL}{dx^2}\ \text{ etc.},$$

qui valores prius in genere sunt capiendi, antequam in iis ponatur $x = a$.

COROLLARIUM 6

Sin autem quoque valores primi ipsius *y* nostrae determinationi relinquantur, tum iisdem aequationibus satisfieri debet ponendo $x = 0$, ubi eadem sunt observanda, quae modo notavimus. Aequationes scilicet has ante penitus evolvi oportet, quam in iis statuatur $x = 0$. His autem conditionibus quantitates tantum constantes in relationem indefinitam inter *x* et *y* ingressae determinantur.

PROBLEMA 3

Si functio *Z* praeter quantitates *x, y, p, q, r* etc. etiam duas formulas integrales $\Phi = \int \mathfrak{Z} dx$ et $\Phi^{/} = \int \mathfrak{Z}^{/} dx$ utcunque involvat, ut sit

$$dZ = Ld\Phi + L^{/} d\Phi^{/} + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

in iis autem formulis Φ et $\Phi^{/}$ functiones $\mathfrak{Z}$ et $\mathfrak{Z}^{/}$ tantum per quantitates *x, y, p, q, r* etc. determinentur, ut sit

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$
$$d\mathfrak{Z}^{/} = \mathfrak{M}^{/}dx + \mathfrak{N}^{/}dy + \mathfrak{P}^{/}dp + \mathfrak{Q}^{/}dq + \mathfrak{R}^{/}dr + \text{etc.},$$

definire relationem inter *x* et *y*, ut haec formula integralis $\int Zdx$, quatenus a termino $x = 0$ usque ad $x = a$ extenditur, maximum minimumve valorem consequatur.

SOLUTIO

Oportet igitur variationem differentialem formulae $\int Zdx$ definiri, quae cum sit $\delta \int Zdx = \int \delta Zdx$, habemus primo:

$$\delta Z = L\delta\Phi + L^{/}\delta\Phi^{/} + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

deinde vero est

$$\delta\Phi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx \text{ et } \delta\Phi^{/} = \int \delta\mathfrak{Z}^{/}dx$$

hincque propterea :

$$d\mathfrak{Z} = \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$
$$d\mathfrak{Z}' = \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

ex quibus colligimus :

$$\delta\Phi = \int dx\left(\mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}\right)$$
$$\delta\Phi' = \int dx\left(\mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.}\right).$$

Cum igitur sit variatio differentialis quaesita

$$\delta\int Zdx = \int Ldx\delta\Phi + \int L'dx\delta\Phi' + \int Ndx\delta y + \int Pdx\delta p + \text{etc.},,$$

ponamus $\int Ldx = V$ et $\int L'dx = V'$, eritque ut supra

$$\int Ldx\delta\Phi = V\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right)$$
$$-\int Vdx\left(\mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}\right).$$

$$\int L'dx\delta\Phi = V'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right)$$
$$-\int V'dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}\right).$$

Ponamus autem haec integralia $\int Ldx = V$ et $\int L'dx = V'$ ita capi, ut evanescant posito $x = a$, ac praecedentium formularum partes priores sponte in nihilum abibunt, siquidem earum valores pro termino $x = a$ capi debent. Omnibus igitur partibus coniungendis obtinebimus

$$\delta\int Zdx = \int dx\delta y\left(N - V\mathfrak{N} - V'\mathfrak{N}'\right)$$
$$+\int dx\delta p\left(P - V\mathfrak{P} - V'\mathfrak{P}'\right)$$
$$+\int dx\delta q\left(Q - V\mathfrak{Q} - V'\mathfrak{Q}'\right)$$
$$+\int dx\delta r\left(R - V\mathfrak{R} - V'\mathfrak{R}'\right)$$
etc.

Cum vero sit

$$\int Pdx\delta p = P\delta y - \int \delta y dP$$
$$\int Qdx\delta q = Q\frac{d\delta y}{dx} - \frac{\delta y}{dx}\, dQ \ + \int \frac{\delta y}{dx} ddQ$$
$$\int Rdx\delta r = R\frac{dd\delta y}{dx^2} - \frac{d\delta y}{dx^2}dR + \frac{\delta y}{dx^2}ddR - \int \frac{\delta y}{dx^2}d^3R$$
etc.

eliciemus variationem differentialem quaesitam

$$\delta\int Zdx = \int dx\delta y\left(\left(N - V\mathfrak{N} - V'\mathfrak{N}'\right) - \frac{d\left(P - V\mathfrak{P} - V'\mathfrak{P}'\right)}{dx} + \frac{dd\left(Q - V\mathfrak{Q} - V'\mathfrak{Q}'\right)}{dx^2} - \text{ etc.}\right)$$
$$+\delta y\left(\left(P - V\mathfrak{P} - V'\mathfrak{P}'\right) - \frac{d\left(Q - V\mathfrak{Q} - V'\mathfrak{Q}'\right)}{dx} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(Q - V\mathfrak{Q} - V'\mathfrak{Q}'\right) - \frac{d\left(R - V\mathfrak{R} - V'\mathfrak{R}'\right)}{dx} + \text{etc.}\right)$$

COROLLARIUM I

Ponamus brevitatis gratia:

$$N - V\mathfrak{N} - V'\mathfrak{N}' = (N),\ P - V\mathfrak{P} - V'\mathfrak{P}' = (P),\ Q - V\mathfrak{Q} - V'\mathfrak{Q}' = (Q) \text{ etc.}$$
ac relatio indefinita inter x et y exprimetur hac aequatione :

$$(N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{ etc.} = 0,$$

quae tamen iam involvit terminum praescriptum $x = a$, quia formulae integrales $\int Ldx = V$ et $\int L'dx = V'$ ita sunt captae, ut evanescant posito $x = a$.

COROLLARIUM 2

Cum autem integratio huius aequationis, si fuerit differentialis, constantes arbitrarias involvat, si et hae determinationi nostrae relinquantur, ut formula $\int Zdx$ valorem omnium maximum vel minimum adipiscatur, eas ita definiri convenit, ut tam posito $x = 0$ quam $x = a$ etiam his aequationibus satisfiat

$$(P)-\frac{d(Q)}{dx}+\frac{dd(R)}{dx^2}-\text{etc.}=0,\quad (Q)-\frac{d(R)}{dx}+\text{etc.}=0,\quad (R)-\text{etc.}=0.$$

COROLLARIUM 3

Si functio *Z* non solum duas huiusmodi formulas integrales $\Phi=\int \mathfrak{Z}dx,\ \Phi^{/}=\int \mathfrak{Z}^{/}dx$, sed etiam plures $\Phi^{//}=\int \mathfrak{Z}^{//}dx,\ \Phi^{///}=\int \mathfrak{Z}^{///}dx$ etc. involvat, ita tamen, ut litterae $\mathfrak{Z},\ \mathfrak{Z}^{/},\ \mathfrak{Z}^{//}$ etc. denotent tantum functiones quantitatum *x, y, p, q, r* etc. neque ultra ullas formulas integrales involvant, ex solutione problematis etiam huiusmodi formularum variationes differentiales facile assignantur.

PROBLEMA 4

Si functio *Z* praeter quantitates *x, y, p, q, r* etc. etiam formulam integralem $\Phi=\int \mathfrak{Z}dx$ utcunque implicet, ut sit $dZ=Ld\Phi+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.},$ functio autem $\mathfrak{Z}$ etiam praeter *x, y, p, q, r* etc. aliam denuo formulam integralem $\Phi=\int \mathfrak{z}dx$ involvat, ut

$$d\mathfrak{Z}=\mathfrak{L}d\Phi+\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc.},$$

functio vero $\mathfrak{z}$ tantum ex quantitatibus *x, y, p, q, r* etc. sit composita existente

$$d\mathfrak{z}=\mathfrak{m}dx+\mathfrak{n}dy+\mathfrak{p}dp+\mathfrak{q}dq+\mathfrak{r}dr+\text{etc.},$$

definire relationem inter *x* et *y*, ut haec formula integralis $\int Zdx$, quatenus a termino $x=0$ usque ad terminum $x=a$ extenditur, maximum minimumve valorem consequatur.

SOLUTIO

In hunc igitur finem variationem differentialem formulae $\int Zdx$ exquiri convenit; quae cum sit $\delta\int Zdx=\int \delta Zdx$, habemus primo:

$$dZ=Ld\Phi+Ndy+Pdp+Qdq+Rdr+\text{etc.}$$

ideoque variatio differentialis erit

$$\delta\int Zdx=\int Ldx\delta\Phi+\int Ndx\delta y+\int Pdx\delta p+\int Qdx\delta q+\int Rdx\delta r+\text{etc.}$$

Nunc autem ob $\delta\Phi=\delta\int \mathfrak{Z}dx=\int \delta\mathfrak{Z}dx$ et

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

habebimus simili modo

$$\delta\Phi = \int \mathfrak{L}dx\delta\Phi + \int \mathfrak{N}dx\delta y + \int \mathfrak{P}dx\delta p + \int \mathfrak{Q}dx\delta q + \int \mathfrak{R}dx\delta r + \text{etc.}$$

Denique vero est, $\delta\Phi = \delta\int \mathfrak{z}dx = \int \delta zdx$ ideoque ob

$$d\mathfrak{z} = \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.}$$

erit

$$\delta\Phi = \int \mathfrak{n}dx\delta y + \int \mathfrak{p}dx\delta p + \int \mathfrak{q}dx\delta q + \int \mathfrak{r}dx\delta r + \text{etc.}$$

Sit iam $\int \mathfrak{L}dx = v$, ac fiet

$$\int \mathfrak{L}dx\delta\Phi = v\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$-\int vdx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right),$$

unde acquirimus

$$\delta\Phi = v\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$+\int dx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) + \int dx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) + \int dx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}$$

Ponamus ergo porro $\int Ldx = V$ et $\int Lvdx = T$, eritque

$$\int Ldx\delta\Phi = T\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$-\int Tdx\left(\mathfrak{n}\delta y + \mathfrak{p}\delta p + \mathfrak{q}\delta q + \mathfrak{r}\delta r + \text{etc.}\right)$$
$$+V\int dx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) + V\int dx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) + V\int dx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}$$
$$-\int Vdx\delta y\left(\mathfrak{N} - v\mathfrak{n}\right) - \int Vdx\delta p\left(\mathfrak{P} - v\mathfrak{p}\right) - \int Vdx\delta q\left(\mathfrak{Q} - v\mathfrak{q}\right) + \text{etc.}$$

His igitur omnibus colligendis prodibit variatio differentialis quaesita

$$\delta\int Zdx = T\int dx\left(\mathfrak{n}\delta y+\mathfrak{p}\delta p+\mathfrak{q}\delta q+\mathfrak{r}\delta r+\text{etc.}\right)$$
$$+V\int dx\delta y\left(\mathfrak{N}-v\mathfrak{n}\right)+V\int dx\delta p\left(\mathfrak{P}-v\mathfrak{p}\right)+V\int dx\delta q\left(\mathfrak{Q}-v\mathfrak{q}\right)+\text{etc.}$$
$$+V\int dx\delta y\left(N-V\mathfrak{N}+Vv\mathfrak{n}-T\mathfrak{n}\right)$$
$$+\int dx\delta p\left(P-V\mathfrak{P}+Vv\mathfrak{p}-T\mathfrak{p}\right)$$
$$+\int dx\delta q\left(Q-V\mathfrak{Q}+Vv\mathfrak{q}-T\mathfrak{q}\right)$$
$$+\text{etc.},$$

quae cum ad terminum usque $x=a$ extendi debeat, ponamus integralia $\int Ldx = V$ et $\int Ldx\int\mathfrak{L}dx = T$, quandoquidem determinatio integrationis nostro arbitrio relinquitur, ita capi, ut evanescant posito $x=a$, quo nostra expressio facilior reddatur. Deinde vero ponamus brevitatis gratia :

$$N-V\mathfrak{N}+\left(Vv-T\right)\mathfrak{n}=\left(N\right)$$
$$P-V\mathfrak{P}+\left(Vv-T\right)\mathfrak{p}=\left(P\right)$$
$$Q-V\mathfrak{Q}+\left(Vv-T\right)\mathfrak{q}=\left(Q\right)$$
$$R-V\mathfrak{R}+\left(Vv-T\right)\mathfrak{r}=\left(R\right)$$
$$+\text{etc.}$$

existente, ut assumsimus, $v=\int\mathfrak{L}dx$, atque variatio differentialis quaesita reducetur ad hanc formam :

$$\delta\int Zdx = \int dx\delta y\left(\left(N\right)-\frac{d\left(P\right)}{dx}+\frac{dd\left(Q\right)}{dx^2}-\frac{d^3\left(R\right)}{dx^3}+\text{etc.}\right)$$
$$+\delta y\left(\left(P\right)-\frac{d\left(Q\right)}{dx}+\frac{dd\left(R\right)}{dx^2}-\text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(Q\right)-\frac{\left(dR\right)}{dx}+\text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(\left(R\right)-\text{etc.}\right)$$
$$+\text{etc.},$$

COROLLARIUM 1

Cum sit $v=\int\mathfrak{L}dx$, erit

$$Vv = \int Ldz \int \mathfrak{L}dx \text{ et } Vv - T = \int Ldz \int \mathfrak{L}dx - \int Ldz \int \mathfrak{L}dx = \int \mathfrak{L}dx \int Ldz.$$

Quia autem per determinationes assumtas expressio $Vv - T$ evanescit posito $x = a$, si ponamus $\int Ldx = V$ et $\int \mathfrak{L}Vdx = \mathfrak{V}$ ambo haec integralia ita capi oportet, ut evanescant posito $x = a$.

COROLLARIUM 2

His igitur formulis $\int Ldx = V$ et $\int \mathfrak{L}Vdx = \mathfrak{V}$ in computum introductis ponendum erit :

$$\begin{aligned} N - V\mathfrak{N} + \mathfrak{V}\mathfrak{n} &= (N) \\ P - V\mathfrak{P} + \mathfrak{V}\mathfrak{p} &= (P) \\ Q - V\mathfrak{Q} + \mathfrak{V}\mathfrak{q} &= (Q) \\ R - V\mathfrak{R} + \mathfrak{V}\mathfrak{r} &= (R) \\ &+ \text{etc.} \end{aligned}$$

et variatio differentialis per litteras (N), (P), (Q) etc. perinde exprimetur, ac supra casu primo per litteras *N, P, Q* etc. erat definita.

COROLLARIUM 3

Ex his iam facile colligere licet, si etiam functio $\mathfrak{z}$ novam formulam integralem involvat, quemadmodum tum variatio differentialis exprimatur; si scilicet fuerit

$$d\mathfrak{z} = \mathfrak{l}\, d\Phi' + \mathfrak{m}dx + \text{etc.},$$

tum ad formulas $\int Ldx = V$ et $\int \mathfrak{L}Vdx = \mathfrak{V}$ insuper tertia $\mathfrak{v} = \int \mathfrak{l}\,\mathfrak{V}dx$ accederet ; reliqua attendenti facile se offerent.

PROBLEMA 5

Si functio *Z* praeter quantitates *x, y, p, q, r* etc. etiam formulam integralam $\Phi = \int \mathfrak{Z}dx$ utcunque implicet, ut sit

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

functio vero $\mathfrak{Z}$ praeter quantitates *x, y, p, q, r* etc. eandem denuo formulam integralem $\Phi = \int \mathfrak{Z}dx$ involvat, ut sit

$$d\mathfrak{Z} = \mathfrak{L}d\Phi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

definire relationem inter *x* et *y*, ut haec formula integralis $\int Zdx$, quatenus a termino $x = 0$ ad terminum datum $x = a$ extenditur, maximum minimumve valorem adipiscatur.

SOLUTIO

Variatio differentialis est ut hactenus

$$\delta\int Zdx = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

deinde vero habemus $\delta\Phi = \delta\int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ et

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\Phi + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

Cum autem sit $\Phi = \int \mathfrak{Z}dx$, erit

$$\mathfrak{Z} = \frac{d\Phi}{dx} \text{ et } \delta\mathfrak{Z} = \frac{\delta d\Phi}{dx} = \frac{d\delta\Phi}{dx};$$

ponamus tantisper

$$\delta\Phi = u \text{ et } \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.} = \omega,$$

ut obtineatur haec aequatio

$$\frac{du}{dx} = \mathfrak{L}u + \omega,$$

cuius integrale sumto *e* pro numero, cuius logarithmus 1, est

$$e^{-\int \mathfrak{L}dx}u = \int e^{-\int \mathfrak{L}dx}\omega dx,$$

ideoque

$$\delta\Phi = e^{\int \mathfrak{L}dx}\int e^{-\int \mathfrak{L}dx}dx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right),$$

unde deducitur

$$\int Ldx\delta\Phi = e^{\int \mathfrak{L}dx}Ldx\int e^{-\int \mathfrak{L}dx}dx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right).$$

Ponatur iam $e^{\int \mathfrak{L}dx}Ldx = V$, quod integrale ita capiatur, ut evanescat posito $x = a$, sitque $e^{-\int \mathfrak{L}dx}V = U$, erit

$$\int Ldx\delta\Phi = -\int Udx\left(\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.}\right),$$

cui parti si reliquae partes addantur reductionesque superiores fiant, prodibit variatio differentialis quaesita $\delta\int Zdx$

$$\delta\int Zdx = \int dx\delta y\left(\left(N-U\mathfrak{N}\right)-\frac{d\left(P-U\mathfrak{P}\right)}{dx}+\frac{dd\left(Q-U\mathfrak{Q}\right)}{dx^2}-\frac{d^3\left(R-U\mathfrak{R}\right)}{dx^3}+\text{etc.}\right)$$
$$+\delta y\left(\left(P-U\mathfrak{P}\right)-\frac{d\left(Q-U\mathfrak{Q}\right)}{dx}+\frac{dd\left(R-U\mathfrak{R}\right)}{dx^2}-\text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(Q-U\mathfrak{Q}\right)-\frac{d\left(R-U\mathfrak{R}\right)}{dx}+\text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(\left(R-U\mathfrak{R}\right)-\text{etc.}\right),$$

ex qua ut supra ea relatio inter *x* et *y* elicitur, qua formulae integrali $\int Zdx$ pro termino $x = a$ valor maximus vel minimus conciliatur; haec enim relatio exprimetur ista aequatione:

$$\left(N-U\mathfrak{N}\right)-\frac{d\left(P-U\mathfrak{P}\right)}{dx}+\frac{dd\left(Q-U\mathfrak{Q}\right)}{dx^2}-\frac{d^3\left(R-U\mathfrak{R}\right)}{dx^3}+\text{etc.}=0$$

Tum vero pro constantium per integrationem invectarum determinatione singulae partes absolutae tam pro casu $x = a$ quam pro casu $x = 0$ nihilo aequales effici poterunt.

COROLLARIUM

Quia posuimus $e^{-\int \mathfrak{L}dx}V = U$, erit $V = e^{\int \mathfrak{L}dx}U$, unde differentiando fiet

$$dV = e^{\int \mathfrak{L}dx}\left(dU + U\mathfrak{L}dx\right).$$

Cum autem sit $dV = e^{\int \mathfrak{L}dx} Ldx$, habebitur ista aequatio differentialis:

$$dU + U\mathfrak{L}dx = Ldx \ ,$$

ex qua quantitatem *U* ita definiri oportet, ut ea evanescat posito $x = a$.

PROBLEMA 6

Si functio *Z* praeter quantitates *x, y, p, q, r* etc. etiam formulam integralem $\Phi = \int Zdx$ involvat, ita ut sit

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

definire relationem inter *x* et *y*, ut haec formula $\int Zdx$ maximum minimumve valorem induat, quatenus quidem a termino $x = 0$ usque ad terminum $x = a$ extenditur.

SOLUTIO

Cum variatio differentialis sit

$$\delta\int dZ = \int Ldx\delta\Phi + \int Ndx\delta y + \int Pdx\delta p + \int Qdx\delta q + \int Rdx\delta r + \text{etc.},$$

habebitur etiam $\delta\Phi = \delta\int Zdx$, unde fit differentiando :

$$d\delta\Phi = Ldx\delta\Phi + Ndx\delta y + Pdx\delta p + Qdx\delta q + Rdx\delta r + \text{etc.}$$

hincque invenitur ut ante

$$\int Ldx\delta\Phi = e^{\int Ldx} Ldx \int e^{-\int Ldx} dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),$$

ex quo adipiscimur ob $\int e^{\int Ldx} Ldx = e^{\int Ldx}$

$$\begin{aligned}\int Ldx\delta\Phi = e^{\int Ldx} &\int e^{-\int Ldx} dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right)\\ &-\int dx\left(N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right),\end{aligned}$$

quod postremum membrum a reliquis partibus tollitur. Quare, si ponamus

$e^{-\int Ldx} = T$, erit tota variatio differentialis

$$\begin{aligned}\delta\int Zdx = \frac{1}{T}\int dx\delta y\left(TN - \frac{d\cdot TP}{dx} + \frac{dd\cdot TQ}{dx^2} - \frac{d^3\cdot TR}{dx^3} + \text{etc.}\right)\\ +\delta y\left(TP - \frac{d\cdot TQ}{dx} + \frac{dd\cdot TR}{dx^2} - \text{etc.}\right)\\ +\frac{d\delta y}{dx}\left(TQ - \frac{d\cdot TR}{dx} + \text{etc.}\right)\\ +\frac{dd\delta y}{dx^2}\left(TR - \text{etc.}\right)\\ +\text{etc.},\end{aligned}$$

Quo igitur formula $\int Zdx$ evadat maxima vel minima, relatio indefinita inter x et y hac aequatione exprimetur :

$$TN - \frac{d\cdot TP}{dx} + \frac{dd\cdot TQ}{dx^2} - \frac{d^3\cdot TR}{dx^3} + \text{etc} = 0.$$

partes vero absolutae singulae inservient constantibus per integrationem ingressis determinandis.

SCHOLION

Hac igitur Analysi nullas considerationes geometricas involvente non solum omnium problematum ad hanc maximorum et minimorum methodum pertinentium easdem adepti sumus solutiones, quas iam in libro meo de maximis et minimis dedi, sed etiam haec methodus peculiarem suppeditavit determinationem constantium, quae priori methodo indeterminatae relinquuntur; unde innumera problemata singularia expedite resolvi possunt, ad quae prior methodus minus congrue accommodatur. Vel uti si inter omnes lineas a dato puncto non ad aliud punctum, sed ad lineam quandam datam sive rectam sive curvam ducendas ea requiratur, super qua corpus ab illo puncto descendens tempore brevissimo ad hanc lineam perveniat, per considerationem illarum partium absolutarum hoc problema facile resolvitur, dum iis ista conditio praescribitur, ut curva quaesita ad datam sit normalis. Antequam autem finiam, examini Analystarum egregium Theorema subiiciam, cuius veritas ex principiis hactenus positis haud difficulter perspicitur et quod in calculo integrali eximium usum praestare videtur.

THEOREMA

Proposita formula differentiali *Zdx*, in qua *Z* sit functio quaecunque quantitatum $x,\ y,\ \ p=\frac{dy}{dx},\ q=\frac{dp}{dx},\ r=\frac{dp}{dx}$ etc., eaque differentiata prodeat:

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

ita ut haec formula differentialis *Zdx* differentialia non solum prima, sed etiam altiora cuiusque ordinis complectatur, tum facile diiudicari poterit, utrum ista formula integrationem admittat sive sit differentiale completum, nec ne? Consideretur enim ista expressio sumto *dx* constante

$$V = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{ etc.},$$

quae si reperiatur nihilo aequalis, formula *Zdx* erit integrabilis; verum si non fuerit $V = 0$, ea non erit integrabilis.