ELEMENTS OF THE CALCULUS OF VARIATIONS

SUMMARY

Now before the most celebrated isoperimetric problem gained significance [*i.e.* the brachistochrone], certain examples pertaining to this had been produce by Geometers, since from the earliest times it had been found that the circle, amongst all the other figures enclosed by the same perimeter, contained the maximum area ; indeed they had inferred which property from the nature of the circle, truly they had hardly dared to approach the problem directly, in order that, amongst all the figures terminating with an equal perimeter, they might investigate that, which would enclose the maximum area. It is well known that this problem is exceedingly hard, as was the case before the significant advance of the infinitesimal calculus, which allowed one at least to think about that problem. Truly soon the question of the brachistochrone was resolved with the greatest success by the most acute Johan Bernoulli, brought about as if by the first cast of this calculus, to wit from which amongst all the lines drawn from the highest point to a lower point that may be found, upon which a weight may descend in the shortest time, as was found to agree with that outstanding property of the cycloid. But the method, which the most honoured man had used, clearly may be seen to provide the opportunity for his older brother Jacob Bernoulli to contemplate the solution of a great isoperimetric problem, which he treated henceforth. Evidently all questions of this kind have been embraced by this problem to the widest extent, so that amongst all the lines given drawn between the two points, either they must be of the same length (from which indeed the name isoperimetric has arisen) or may be endowed with some other common property that may be investigated, which either shall be a maximum area, a solid of maximum volume rotated about a given axis, or in general may be contained in some maximum or minimum property. But the method, which the greatest Geometer of that time used, leaves us considering in doubt, whether we should admire more his incredible patience expended both in prolixity and in the most tedious calculations, or his great wisdom in setting out satisfactorily the conclusions thence deduced by reason. But on this account, because the conclusions had emerged set out so well, soon it would be able to suppose a less arduous and shorter way be given to produce the same; which also his younger brother Johan approached happily enough, even if it was less successful immediately due to well-hidden difficulties, which yet following the failure, with the whole argument revised more deeply, was compensated largely. After a long interceding time, the author of these dissertations applied great enthusiasm to solving the same problem, and since he observed all questions of this kind referred to that, in order that a curved line of this kind may be found by expressing an equation between the coordinates x and y, in which the formula of the integral $\int Vdx$ for that, in whatever manner the quantity V were given by x and y, may reach a maximum or minimum value. But now it is clear that an infinite variation can be considered in that quantity V, as in that besides these variables x and y, both the differentials of these of any order as well as new integral formulas may be

introduced in addition. But if now Bernoulli's solutions may be examined according to this standard, these are found restricted to those cases, in which the quantity V involves differentials of the first order only, and besides the cases, in which new integral formulas are present, thence clearly are excluded, with very few exceptions, which easily may be freed from this inconvenience according to the nature of the problem. Therefore our author has supplied this deficiency most happily both in these commentaries as well as in the individual works published about this question, as scarcely any shall be found, which may be desired more fully. Yet meanwhile that method itself, even if the whole matter may be put together quickly in a satisfactory manner, still itself has been seen not to be of a satisfactory nature, thus because the whole strength of the solution was set up from the consideration of the elements of the curve requiring to be found, truly the question thus shall be readily equipped , so that it may be recalled completely from geometry completely to pure analysis alone. Indeed the question thus may be proposed, so that for some given quantity V with the two variables x, y and the differentials of these of any order, why not also may that relation between x and y be investigated from the integral formulas constructed in whatever manner, from which a maximum or minimum value of the integral formula $\int Vdx$ may be brought about ? In short, the question proposed in this manner, may be freed by geometry ; from which also that natural method of resolution must be freed from geometry ; and so that a more difficult analysis would be applied to this goal, to that the greater advancement of this science, if the thing were successful, it could deservedly be hoped for.

[A modern view of the resolution of this problem is set out in the paper by Craig G. Fraser, '*The Origins of Euler's Variational Calculus' in the Archives of the Exact Sciences*. Vol. 47, No. 2 (June 1994), pp. 103-141. Essentially, the young Lagrange noted a distinction between two uses of the differential by Euler : one applied to separate points on neighbouring different curves, and the other to form a derivative from points on the same curve; subsequently Lagrange introduced the new operator δ to distinguish the former from the latter dx. The present paper by Euler resolves this difficulty.]
Moreover, even if the author had considered this for a long time and tried to uncover the missing theory with his friends, yet the first glory of finding such a theory was reserved to the young Geometer, Lagrange of Turin, clearly who had arrived at the same solution by using analysis only, as the author had elicited from geometrical considerations. Truly that solution itself being prepared thus, so as plainly to constitute a new kind of analysis and the boundaries of which may be considered to have moved more than a little ; from which the chance is offered to the originator that this science being enriched by a new kind of calculus be known as the *Calculus of Variations* and its elements to be treated here and to constitute a clear explanation. Indeed this calculus likewise on differentials and infinitely small increments being compared together, truly in the account of the treatment it differs greatly from that. For since in the integral calculus from a given relation of the variable quantities, a relation between the differentials of these of any order may be investigated, in the calculus of variations itself the relation between infinitely small variables is considered to be changed, thus so that, while following a given relation, for some value of another variable x certainly another different value y is given, the calculus of variations may add a certain infinitely small increment to this

value *y* itself, from which henceforth, however both the differentials and integrals of the formula may be varied, it is required to be defined. That increment required to be added to any value of *y* is called by the writer it variation and, since it shall not be confused with differentials, it may be designated by this character δy ; therefore since hence all formulas both differentials as well as integrals, in as much as they involve the quantity *y*, certain variations are obtained, the writer in the former dissertation establishes the principles and precepts, with the help of which the variations of all formulas of this kind can be defined: thus if *W* may denote some formula of this kind, its variation δW is taught to be assigned by special rules. So that with a single calculation put in place then in the following dissertation its application to all problems, which are able to be devised about maxima and minima, is shown most clearly, and in the matter this deserves to be observed in the first place, so that thus the new purely analytic will supply many fuller and more perfect solutions, than that aimed at before from geometry.

HYPOTHESIS 1

1. Some equation may be give between the two variables *x* and *y*, where a mutual relation of these may be expressed, thus so that thence, whatever determined value of *x* may be attributed, also the determined value for *y* may be defined.

COROLLARY 1

2. Therefore in the equation proposed between the two variables *x* and *y* with the individual values of *x*, whatever they can take, the determined values of *y* will correspond.

COROLLARY 2

3. Therefore by virtue of this proposed equation *y* will be a certain function of *x* and, to whatever *x* may correspond *y*, thus of that value following $x' = x + dx$, $y' = y + dy$ will correspond, of which the differential from the preceding *y*, namely *dy*, will be able to be assigned by the common rules of differentiation.

COROLLARY 3

4. Since *y* shall be a function of *x*, also $\frac{dy}{dx}$ will be n function of *x* by the assignable given relation between *x* and *y*; and if there may be put $\frac{dy}{dx} = p$, in a similar manner

$\frac{dp}{dx}$ will be a certain function of *x;* and if again we may put
$\frac{dp}{dx}=q,\ \frac{dq}{dx}=r,\ \frac{dr}{dx}=s$ etc., these quantities *q, r, s* etc. also will be certain functions of *x* likewise by the given assignable relation between *x* and *y*.

COROLLARY 4

5. Then if *V* shall be some expression constructed from *x* and *y*, that also with the aid of the given relation between *x* and *y* will be able to be prepared, so that for all the values of *x* the determined values may be come upon. And if V' may designate the value following or agreeing with $x+dx$ itself, there will be $V'=V+dV$ or $dV=V'-V$, following the first principles of differential calculus.

HYPOTHESIS 2

6. Whatever relation may be put in place between *x* and *y*, because thence the relation of the differentials *dx* and *dy* becomes known likewise, I may put into the following infinite sequence:

$$\frac{dy}{dx}=p,\ \frac{dp}{dx}=q,\ \frac{dq}{dx}=r,\ \frac{dr}{dx}=s \text{ etc.}$$

and both *p, q, r, s* etc. and *y* in the same way will be functions of *x* by that given assignable relation.

COROLLARY 1

7. Just as the letter *p* contains the relation of the differentials *dx* and *dy*, thus *q* holds the relation of the differentials of the second order, *r* truly of the differentials of the third order, *s* of the fourth order, etc.

COROLLARY 2

8. Therefore in turn also, if differentials which may be present in the expression *V* either of the first, second, or of higher orders, these may be removed from the calculation by introducing these quantities *p, q, r, s* etc.

AXIOM

9. If another relation from the proposed may be put in place between the variables *x* and *y* with an infinitely small difference, its values of *y*, corresponding to the individual values of *x*, will differ by infinitely small amounts from these, which the proposed relation provides.

COROLLARY 1

10. Since a varied relation of this kind shall be able to differ in an infinite number of ways from the proposed relation, thus so that the difference shall be infinitely small, it can come about, that one or more values of *y,* which correspond to certain values of *x*, thence may experience no change.

COROLLARY 2

11. This variation of the relation thus can be taken as general, thus thence all values of *y* may allow some changes. Therefore so that this treatment may extend widely, it will be agreed to understand the variation of this kind of relation most generally.

HYPOTHESIS 3

12. If the proposed relation between *x* and *y* may be changed a little, the value of *y*, which thence corresponds to *x* itself, we may designate by $y+\delta y$, thus so that δy may denote the variation, which *y* undergoes on account of the variation relation.

COROLLARY 1

13. In a similar manner since y' shall be the value of $x+dx$ corresponding to the nature of the proposed relation, whereby its value, which agrees with the same $x+dx$, for the nature of the variation relation, we may express by $y'+\delta y'$, thus so that $\delta y'$ will denote the variation of y', which arises from the variation relation.

COROLLARY 2

14. Therefore since $y' = y+dy$, there will be

$$\delta y' = \delta(y+dy) = \delta y + \delta dy \text{ and } \delta dy = \delta y' - \delta y.$$

But δdy will denote the variation of *dy* arising from the proposed variation between *x* and *y*.

COROLLARY 3

15. But just as y' may denote the following position of *y*, evidently in the position following according to the related $x+dx$, thus $\delta y'$ may denote the position following δy itself, from which $\delta y' - \delta y$ expresses the differential of δy, which is $d\delta y$. Therefore since $\delta dy = \delta y' - \delta y$, there will be $\delta dy = d\delta y$.

COROLLARY 4

16. Hence therefore we obtain this remarkable property : so that the variation of the differential of *y* shall be equal to the differential of the variation of *y*. For δdy is the variation of *dy*, that is of the differential of *y*, and $d\delta y$ is the differential of δy, that is of the variation of *y*.

DEFINITION I

17. If *V* shall be some expression constructed from *x* and *y*, with a certain relation proposed between *x* et *y* the variation of which, which I will indicate by δV, is an increment, which the quantity *V* takes, if the proposed relation between *x* and *y* may be varied an infinitely small amount.

COROLLARY 1

18. Therefore it is required to distinguish properly between the differential *dV* and the variation δV ; for the differential denotes the increment of *V*, while *x* is increased by its own element *dx*, with the proposed relation remaining between *x* and *y* ; but the variation denotes the increment of *V*, while the relation itself is changed with *x* remaining.

COROLLARY 2

19. Since by the variation of the proposed relation between *x* and *y* the quantity *y* may take the increment δy with *x* remaining the same : in whatever manner the quantity *V* were constructed from *x* and *y*, its variation may be found, if in place of *y* everywhere there may be written $y+\delta y$ and hence from the value arising for *V* the value *V* may be taken away.

COROLLARY 3

20. Clearly if in *V* for *y* there may be written everywhere $y+\delta y$, the varied value of *V* will be produced, which is $V+\delta V$; moreover that variation may be found, if the first value *V* may be subtracted from the varied value $V+\delta V$.

DEFINITION II

21. The calculus of variations is a method of finding the variations of some quantities constructed from the two variables *x* and *y*, which they undergo, if the proposed relation between *x* and *y* may be changed in some manner by an infinitely small manner.

COROLLARY 1

22. Therefore from the proposed relation between x and y , if V may denote a quantity depending in some manner on x and y, this calculus instructs how to find the variation of V, or the value of δV .

COROLLARY 2

23. Because we have assumed the given relation between x and y to be changed in some manner, so that y may take some variations for the individual values of x, which also do not depend on each other, this calculus extends the widest and can be adapted to any given conditions.

SCHOLIUM 1

24. So that the use of this calculus may be able to be made clearer, we may add an example. Therefore this relation shall be proposed between x an y

$$aayy - bbxx = aabb,$$

which may change an infinitely small amount by writing $b + db$ in place of b. Now if a certain quantity depending on x and y may be put in place, such as

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}},$$

its variation arising from that change of the relation can be shown ; for since there shall be

$$y = \frac{b}{a}\sqrt{(aa - xx)},$$

there will be

$$\delta y = \frac{db}{a}\sqrt{(aa - xx)},$$

which is the variation of y. Moreover just as from some known variation of the quantity y depending on y and x, thus also the variations of this quantity

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}}$$

thence arising must be determined, is required to be made plain in this calculation;

from which it is apparent everything is contained here, which has been shown everywhere about the variation of the parameters,. Then truly also questions are able to be turned around, just as if from a proposed formula of this kind

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}},$$

this relation may be sought between x and y, from which the variation of its magnitude may provide a given formula or even zero, so that the relation of the proposed formula found in the latter case may produce a maximum or minimum value ; and all problems are to be referred here, which concern curves endowed with the property of a maximum or minimum treated hitherto.

SCHOLIUM 2

25. The precepts of this calculation, by which some proposed formula V may depend on the two variables x and y, are to be adapted for a number of reasons ; which since the diversity shall be infinite, that will be agreed to relate to a number of general precepts. Therefore the first kind shall include these formulas, which have been composed in some manner from the magnitudes x and y themselves and from the derivatives of these :

$$p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx} \text{ etc.},$$

thus yet, so that they involve no integral formulas. These formulas are to be referred to the second kind, which may contain integrals of this kind $\int Zdx$, yet thus, so that the formula Z itself may belong to the first kind. The third kind contains formulas of this kind, in which not only the integrals $\int Zdx$ are present, but where the magnitude Z itself involves integrals in addition. Finally the fourth kind follows, in which the formula V is not required to be varied completely, but at last is defined by a differential equation either of the first or of higher order, so that the kind appears the most general and certainly contains all the preceding kinds within itself. But so that it may include the equation, by which the relation between x and y is expressed, even if I consider that as given, yet not defined, lest the precepts being set out be limited in any way.

THEOREM 1

26. The variation of the differential of any magnitude *V* is equal to the differential of the variation of the same or $\delta dV = d\delta V$.

DEMONSTRATION

Since there shall be $dV = V^{/} - V$ with $V^{/}$ denoting the following value of *V*, which itself agrees with $x + dx$, as *V* itself corresponds to *x*, there will be $\delta dV = \delta V^{/} - \delta V$; truly $d\delta V$ expresses the difference between δV and its following value, which is $\delta V^{/}$, thus so that there shall be $d\delta V = \delta V^{/} - \delta V$, from which there is seen to be $\delta dV = d\delta V$.

COROLLARY 1

27. In the same manner, if in place of *V* we may write *dV*, it is clear that $\delta ddV = d\delta dV$ *;* or $\delta dV = d\delta V$, from which there becomes $d\delta dV = dd\delta V$, and thus these three formulas will be equal to each other :

$$\delta ddV = d\delta dV \ dd\delta V \ .$$

COROLLARY 2

28. Again truly, just as here for *V* if we may write *dV*, we will obtain an equation between the four forms

$$\delta dddV = d\delta ddV = dd\delta dV = ddd\delta V \ ,$$

while truly between these five :

$$\delta d^4V = d\delta d^3V = d^2\delta d^2V = d^3\delta dV = d^4\delta V \ .$$

COROLLARY 3

29. If a differential of any order of *V* may be considered, such as d^nV , its variation being found, there will be

$$\delta d^nV = d^m\delta d^{n-m}V = d^n\delta V$$

clearly equal to the differential of order *n* itself of the variation δV . Hence therefore the variation of the differential is reduced to the differential of the variation.

PROBLEM 1

30. To determine the variations of the quantities *p, q, r, s* etc., in the continuing ratio of the differentials between *x* and *y* into themselves.

SOLUTION

Because the variation may be agreed not to pertain to *x*, there will be $\delta x = 0$ and the variation of *y*, evidently δy , may be considered as known. Hence since there shall be $p = \frac{dy}{dx}$, there will be $\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$
Then on account of $q = \frac{dp}{dx}$ there will be

$$\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx};$$

but with the element *dx* taken constant, there is $d\delta p = \frac{dd\delta y}{dx}$ and hence

$$\delta q = \frac{dd\delta y}{dx^2} \text{ and } d\delta q = \frac{d^3\delta y}{dx^2}$$

But again, since there shall be $r = \frac{dq}{dx}$, its variation becomes

$$\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx} \text{ ideoque } \delta r = \frac{d^3\delta y}{dx^3}$$

from which the variations of the quantities of the derivatives from *x* and *y* : *p, q, r, s* etc. thus themselves will have

$$\delta p = \frac{\delta dy}{dx},\ \delta q = \frac{d^2\delta y}{dx^2},\ \delta r = \frac{d^3\delta y}{dx^3} = \frac{d\delta q}{dx},\ \delta s = \frac{d^4\delta y}{dx^4} \text{ etc.,}$$

if indeed the element *dx* may be assumed constant.

COROLLARY 1

31. These differentials of the first and of higher orders of the variation δy may be determined by the variations of the value of *y* itself, which agree with the following

values of $x,\ x+dx,\ x+2dx,\ x+3dx$ etc., as it were. For if the following values of y thus may be shown : $y^{\prime},\ y^{\prime\prime},\ y^{\prime\prime\prime},\ y^{\prime\prime\prime\prime}$etc. and the variations of these thus : $\delta y^{\prime},\ \delta y^{\prime\prime},\ \delta y^{\prime\prime\prime}, \delta\, y^{\prime\prime\prime\prime}$etc., from the nature of the differentiations we know to be

$$d\delta y=\delta y^{\prime}-\delta y,$$
$$dd\delta y=\delta y^{\prime\prime}-2\delta y^{\prime}+\delta y,$$
$$d^{3}\delta y=\delta y^{\prime\prime}-3\delta y^{\prime\prime}+3\delta y^{\prime}-\delta y \text{ etc.}$$

COROLLARY 2

32. Therefore if only the value y may be allowed, truly the following $y^{\prime},\ y^{\prime\prime},\ y^{\prime\prime\prime}$ shall not be present, so that there shall be $\delta y^{\prime}=0,\ \delta y^{\prime\prime}=0,\ \delta y^{\prime\prime\prime}=0$ etc., the variation becomes

$$d\delta y=-\delta y,\quad dd\delta y=+\delta y,\quad d^{3}\delta y=-\delta y, d^{4}\delta y=+\delta y \text{ etc.}$$

and thus :

$$\delta p=-\frac{\delta y}{dx},\quad \delta q=+\frac{\delta y}{dx^{2}},\quad \delta r=-\frac{\delta y}{dx^{3}},\quad \delta s=+\frac{\delta y}{dx^{4}},\text{etc.}$$

PROBLEM 2

33. If V were a magnitude constructed in some manner from the variables x and y and whichever differentials of these orders or were made some function of the quantities x, y, p, q, r, s etc., to determine it variation δV .

SOLUTION

This function V may be differentiated in the customary manner and gives rise to

$$dV=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+\text{etc.},$$

which differential is none other, than the increment which the function V takes, if in place of the quantities x, y, p, q, r, s etc. these may be substituted: $x+dx,\ y+dy,\ p+dp,\ q+dq,\ r+dr$ etc. Therefore in a similar manner, if for x, y, p, q, r, s etc. substituting

$$x+0,\ y+\delta y,\ p+\delta p,\ q+\delta q,\ r+\delta r,\ s+\delta s \text{ etc.},$$

the increment, which the function V takes thence, will be its variation

$$\delta V=N\delta y+P\delta p+Q\delta q+R\delta r+S\delta s+\text{etc}.$$

Whereby, if for $\delta p,\ \delta q,\ \delta r$ etc. the above values may be written , the variation sought will be produced :

$$\delta V = N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^2} + \frac{Sd^4\delta y}{dx^4} \text{ etc.}$$

THEOREM 2

34. With some proposed formula integrated $\int Zdx$, its variation is equal to the integral of the variation of the differential *Zdx* or there will be

$$\delta\int Zdx = \int \delta Zdx\,.$$

DEMONSTRATION

Since $\int Zdx$ expresses this sum of all *Zdx*, its variation $\delta\int Zdx$ includes the sum of all the variations of *Z dx* itself, or there will be $\delta\int Zdx = \int \delta Zdx$. So that it can be shown more distinctly in this manner : let $\int Zdx = V$,thus so that it shall be required to define δV *;* therefore since there shall be $dV = Zdx$, there will be $\delta dV = \delta Zdx = d\delta V$, from which with the integral taken it becomes $\delta V = \int \delta Zdx.$.

PROBLEM 3

35. For the proposed formula integrated $\int Zdx$, in which *Z* shall be some magnitude constructed from *x* and *y* themselves and from any differential orders, to investigate its variation $\delta\int Zdx.$

SOLUTION

Therefore since *Z* shall be a function of *x*, *y*, *p*, *q*, *r*, *s* etc. themselves, with its differential obtained in the usual manner, it will have the form

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.,}$$

from which the variation of the same quantity *Z* will be

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} \text{ etc.}$$

Since now there shall be $\delta\int Zdx = \int \delta Zdx$, there becomes

$$\delta\int Zdx = \int N\delta ydx + \int Pd\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{ etc.};$$

now so that in a further reduction the expression δy may change, for the present we may put $\delta y = \omega$, and thus the reductions themselves will be had :

$$\int Pd\omega = P\omega - \int \omega dP$$

$$\int \frac{Qddw}{dx} = \frac{Qd\omega}{dx} - \int \frac{dQ}{dx} d\omega = \frac{Qd\omega}{dx} - \frac{\omega dQ}{dx} + \int \frac{\omega ddQ}{dx}$$

$$\int \frac{Rd^3\omega}{dx^2} = \frac{Rdd\omega}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{\omega ddR}{dx^2} - \int \frac{\omega d^3R}{dx^2}$$

etc.

Now all these values may be gathered together, and δy may be restored for ω, and thus there will be obtained :

$$\delta\int Zdx = \int \delta ydx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{ etc.}\right)$$
$$+\delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)$$

in which differential expression *dx* is assumed constant.

COROLLARY 1

36. Therefore the variation of the integral of the formula $\int Zdx$ may be agreed on from the part of the integral

$$\int \delta ydx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{ etc.}\right)$$

and from the separated parts, which besides the variation δy itself, also include its differentials $d\delta y,\ dd\delta y,\ d^3\delta y$ etc.

COROLLARY 2

37. Thus moreover we have constructed the integral part through the reductions used, so that only the variation itself δy may be included shown from its unchanged differentials, which form is of outstanding use in the application of the calculus of variations.

PROBLEM 4

38. If in the formula for the integral $\int Zdx$ the magnitude *Z* may include not only the letters *x* and *y* with the relations of the differentials *p, q, r, s* etc., but also the integral formula $\Pi = \int \mathfrak{Z}dx$ in some manner, in which moreover $\mathfrak{Z}$ shall be a function of *x, y, p, q, r, s* etc., to define the variation of this formula for the integral $\int Zdx$.

SOLUTION

Since the magnitude *Z* besides the magnitudes *x, y, p, q, r, s* etc. also may involve the integral formula $\Pi = \int \mathfrak{Z}dx$, it will be able to be considered as a function of the quantities $\Pi,\ x,\ y,\ p,\ q,\ r,\ s$ etc., from which, if it may be differentiated in the customary manner, it will give rise to such a form :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.,}$$

from which the variation of *Z* is deduced :

$$\delta Z = L\delta\Pi + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Then since $\mathfrak{Z}$ will be a function of *x, y, p, q, r, s* etc. themselves, there may be put :

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + \text{etc.}$$

and from the preceding problem there will be $\delta\Pi$ or

$$\begin{aligned}\delta\int \mathfrak{Z}dx = \int \delta y dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{ etc.}\right)\\ + \delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ + \frac{d\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ + \frac{dd\delta y}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ + \frac{d^3\delta y}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)\\ + \text{ etc.}\end{aligned}$$

Or rather it may take the previous form :

$$\delta\int \mathfrak{Z}dx = \int \mathfrak{N}\delta y dx + \int \mathfrak{P}d\delta y + \int \frac{\mathfrak{Q}dd\delta y}{dx} + \int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}$$

and on account of $\delta\Pi = \delta\int \mathfrak{Z}dx$

$$\begin{aligned}\delta Z = L\int \mathfrak{N}\delta y dx + L\int \mathfrak{P}d\delta y + L\int \frac{\mathfrak{Q}dd\delta y}{dx} + L\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + L\int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}\\ + N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{ etc.}\end{aligned}$$

Therefore since there shall be $\delta\int Zdx = \int \delta Zdx$, we will have :

$$\begin{aligned}\delta\int Zdx = \int Ldx\int \mathfrak{N}\delta y dx + \int Ldx\int \mathfrak{P}d\delta y + \int Ldx\int \frac{\mathfrak{Q}dd\delta y}{dx}\\ + \int Ldx\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \int Ldx\int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}\\ + \int N\delta y dx + \int Pd\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{ etc.}\end{aligned}$$

Putting $\int Ldx = W$, or $Ldx = dW$, and on account of

$$\int Ldx \int \mathfrak{N}\delta ydx = W\int \mathfrak{N}\delta ydx - \int \mathfrak{N}W\delta ydx$$
$$\int Ldx \int \mathfrak{P}d\delta y = W\int \mathfrak{P}d\delta y - \int \mathfrak{P}Wd\delta y$$
$$\int Ldx \int \frac{\mathfrak{Q}dd\delta y}{dx} = W\int \frac{\mathfrak{Q}dd\delta y}{dx} - \int \frac{\mathfrak{Q}Wdd\delta y}{dx}$$

we will obtain :

$$\delta\int Zdx = W\int \mathfrak{N}\delta ydx + W\int \mathfrak{P}d\delta y + W\int \frac{\mathfrak{Q}dd\delta y}{dx} + W\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \text{ etc.}$$
$$+\int (N-\mathfrak{N}W)\delta ydx + \int (P-\mathfrak{P}W)d\delta y + \int (Q-\mathfrak{Q}W)\frac{dd\delta y}{dx} + \int (R-\mathfrak{R}W)\frac{d^3\delta y}{dx^2} + \text{ etc.}$$

These formulas reduced in the same way as above will give :

$$\delta\int Zdx = W\int \delta ydx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{ etc.}\right)$$
$$+W\delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \text{ etc.}\right)$$
$$+\frac{Wd\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \text{ etc.}\right)$$
$$+\frac{Wd\delta y}{dx^2}(\mathfrak{R} - \text{ etc.})$$
$$+\int \delta ydx\left((N-\mathfrak{N}W) - \frac{d(P-\mathfrak{P}W)}{dx} + \frac{dd(Q-\mathfrak{Q}W)}{dx^2} - \frac{d^3(R-\mathfrak{R}W)}{dx^3} + \text{ etc.}\right)$$
$$+\delta ydx\left((P-\mathfrak{P}W) - \frac{d(Q-\mathfrak{Q}W)}{dx} + \frac{dd(R-\mathfrak{R}W)}{dx^2} - \text{ etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q-\mathfrak{Q}W) - \frac{d(R-\mathfrak{R}W)}{dx} + \text{ etc.}\right)$$
$$+\text{etc.}$$

COROLLARY 1

39. Because the reductions used in any case can be extricated easily, with these set aside the variation sought will be shown more concisely on putting $W = \int Ldx$:

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \frac{\mathfrak{P}d\delta y}{dx} + \frac{\mathfrak{Q}dd\delta y}{dx^2} + \frac{\mathfrak{R}d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+\int dx\left(\left(N-\mathfrak{N}W\right)\delta y + \left(P-\mathfrak{P}W\right)\frac{d\delta y}{dx} + \left(Q-\mathfrak{Q}W\right)\frac{dd\delta y}{dx^2} + \left(R-\mathfrak{R}W\right)\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$

COROLLARY 2

40. But if in addition the quantity Z may involve another integral formula $\Pi' = \int \mathfrak{Z}' dx$, so that there shall be :

$$dZ = Ld\Pi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

then truly:

$$d\mathfrak{Z}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

if there may be put $\int Ldx = W$, $\int L'dx = W'$ and in addition on being abbreviated :

$$N - \mathfrak{N}W - \mathfrak{N}'W' = (N);\ P - \mathfrak{P}W - \mathfrak{P}'W' = (P)$$
$$Q - \mathfrak{Q}W - \mathfrak{Q}'W' = (Q);\ R - \mathfrak{R}W - \mathfrak{R}'W' = (R) \text{ etc.},$$

the variation sought will be :

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+W'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\frac{d\delta y}{dx} + \mathfrak{Q}'\frac{dd\delta y}{dx^2} + \mathfrak{R}'\frac{d^3\delta y}{dx^3} + \text{ etc.}\right).$$
$$+\int dx\left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{ etc.}\right).$$

PROBLEM 5

41. If in formula $\int Zdx$ the magnitude Z besides the letters *x, y, p, q, r* etc. may involve the integral formula $\Pi = \int \mathfrak{Z}dx$, in which the quantity $\mathfrak{Z}$ besides the letters *x, y, p, q, r* etc. in addition includes the integral formula $\pi = \int \mathfrak{z}dx$, but in which $\mathfrak{z}$ shall be a function of the letters *x, y, p, q, r* etc. only, to find the variation of the formula $\int Zdx$.

SOLUTION

Since Z shall be a function of the magnitudes *x, y, p, q, r* etc. and $\Pi = \int \mathfrak{Z} dx$, with its differential taken in the customary manner, it will be of this form :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

and thus its variation

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \text{ etc.}$$

from which the variation sought will be

$$\delta\int Zdx = \int \delta Zdx$$
$$= Ldx\delta\Pi + \int dx\left(N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \text{ etc.}\right).$$

But since $\mathfrak{Z}$ shall be a function of the letters *x, y, p, q, r* etc. and $\pi = \int \mathfrak{z} dx$, there will be on differentiating :

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

and hence the variation of this

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}$$

whereby, since there shall be $\Pi = \int \mathfrak{Z} dx$, there will be $\delta\Pi = \delta\int \mathfrak{Z} dx = \int \delta\mathfrak{Z} dx$ and therefore :

$$\delta\Pi = \int \mathfrak{L}dx\delta\pi + \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right).$$

from which there is found :

$$\int Ldx\delta\Pi = \int Ldx \int \mathfrak{L}dx\delta\pi + \int Ldx \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right).$$

Therefore $\delta\pi$ remains as we have defined ; but there is $\pi = \int \mathfrak{z} dx$, and because $\mathfrak{z}$ is a function of the letters *x*, *y*, *p*, *q*, *r* etc. only, it becomes on differentiation :

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

from which its variation is concluded :

$$\delta\mathfrak{z} = \mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.},$$

then truly on account of $\delta\pi = \delta\int\mathfrak{z}dx = \int\delta\mathfrak{z}dx$ it becomes :

$$\delta\pi = \int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.}\right),$$

On account of which we will have :

$$\int Ldx\int\mathfrak{L}dx\delta\pi = \int Ldx\int\mathfrak{L}dx\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

Now so that we may free this formula from the integral signs, we may put $\int Ldx = W$ and there becomes :

$$\int Ldx\delta\Pi = W\delta\Pi - \int Wd\delta\Pi,$$

truly $d\delta\Pi = \delta\mathfrak{Z}dx,$ from which

$$\int Ldx\delta\Pi = W\delta\Pi - \int W\delta\mathfrak{Z}dx$$

and thus:

$$\int Ldx\delta\Pi = W\int\mathfrak{L}dx\delta\pi + W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$-\int\mathfrak{L}Wdx\delta\pi - \int Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$

Let $\int\mathfrak{L}dx = \mathfrak{W}$, there will be

$$\int\mathfrak{L}dx\delta\pi = \mathfrak{W}\delta\pi - \int\mathfrak{W}\delta\mathfrak{z}dx$$

and hence:

$$\int \mathfrak{L}dx\delta\pi = \mathfrak{W}\int dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right)$$
$$-\int \mathfrak{W}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right).$$

Again putting

$\int \mathfrak{L}Wdx = \int Wd\mathfrak{W} = \mathfrak{V}$, so that there shall be :

$$\int \mathfrak{L}Wdx = \mathfrak{V}\int dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right)$$
$$-\int \mathfrak{V}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right).$$

From all these the variation sought $\delta\int Zdx$ may be deduced

$$=\left(W\mathfrak{W}-\mathfrak{V}\right)\int dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\text{ etc.}\right)$$
$$-W\int \mathfrak{W}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right)$$
$$+\int \mathfrak{V}dx\left(\mathfrak{n}\delta y+\mathfrak{p}\frac{d\delta y}{dx}+\mathfrak{q}\frac{dd\delta y}{dx^2}+\mathfrak{r}\frac{d^3\delta y}{dx^3}+\text{ etc.}\right)$$
$$+W\int dx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{ etc.}\right)$$
$$-\int Wdx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{ etc.}\right)$$
$$+\int dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+\text{ etc.}\right).$$

COROLLARY 1

42. If the variation of the formula $\int Zdx$ is sought from the value $x=0$ as far as to the limit value $x=a$, the integrals may be taken $W=\int Ldx,\ \mathfrak{W}=\int \mathfrak{L}dx$ and $\mathfrak{V}=\int Wd\mathfrak{W}$, thus so that they may vanish on putting $x=0$, then truly on making $x=a$ there comes

about $W = A,\ \mathfrak{W} = \mathfrak{A}$ et $\mathfrak{V} = \mathfrak{B}$, which values will be allowed to put in the formula found in place of the letters W, $\mathfrak{W}$ and $\mathfrak{V}$, in which they occur outside the integral sign.

COROLLARY 2

43. Therefore there may be put for abbreviating :

$$N + (A - W)\mathfrak{N} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{n} = (N)$$
$$P + (A - W)\,\mathfrak{P} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{p} = (P)$$
$$Q + (A - W)\,\mathfrak{Q} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{q} = (Q)$$
$$R + (A - W)\mathfrak{R} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\,\mathfrak{r} = (R)$$
$$\text{etc.}$$

and the variation of the formula sought $\int Zdx$ as far as to the bounding value $x = a$ will be :

$$\int dx\left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{ etc.}\right).$$

COROLLARY 3

44. But if now here the above reductions may be used, the same variation may be found expressed thus :

$$\begin{aligned}\delta\int Zdx = \int dx\delta y&\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{ etc.}\right)\\ &+ \delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)\\ &+ \frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)\\ &+ \frac{dd\delta y}{dx^2}\left((R) - \text{etc.}\right)\\ &+ \text{ etc.}\end{aligned}$$

COROLLARY 4

45. Since there shall be $\mathfrak{V} = \int Wd\mathfrak{W}$, there will be $A\mathfrak{W} - \mathfrak{V} = \int (A-W)\mathfrak{L}dx$; whereby, if the integral may be put $\int (A-W)\mathfrak{L}dx = X$, thus taken, so that it may vanish on putting $x = 0$, thus truly on making $x = a$ it becomes $X = B$, thus so that there shall be :

$$\int \mathfrak{L}dx = W, \text{ and on putting } x = a \text{ there becomes } W = A,$$
$$\int (A-W)\mathfrak{L}dx = X, \text{ and putting } x = a \text{ makes } X = B,$$

thus the above values shown in Corollary 2 themselves are found :

$$N + (A-W)\mathfrak{N} + (B-X)\mathfrak{n} = (N)$$
$$P + (A-W)\mathfrak{P} + (B-X)\mathfrak{p} = (P)$$
$$Q + (A-W)\mathfrak{Q} + (B-X)\mathfrak{q} = (Q)$$
$$R + (A-W)\mathfrak{R} + (B-X)\mathfrak{r} = (R)$$
etc.

PROBLEM 6

46. If in formula for $\Phi = \int Zdx$ the letter Z besides the letters x, y, p, q, r etc. also may involve the same integral formula Φ, to determine its variation $\delta\Phi = \delta\int Zdx$.

SOLUTION

Since Z shall be a function of the quantities x, y, p, q, r etc. and in addition it may involve the formula $\Phi = \int Zdx$ itself, it may be differentiated in the customary manner and shall give rise to

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

Hence therefore clearly the variation of Z itself will be

$$\delta Z = L\delta\Phi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.}$$

and thus on account of $\delta\Phi = \delta\int Zdx = \int \delta Zdx$

$$\delta\Phi = \int Ldx\delta\Phi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

We may put $\delta\Phi = z$, since it shall be that itself which is sought, and for brevity

$$\int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right) = u,$$

so that there may be had $z = \int Lzdx + u$ and by differentiating, $dz = Lzdx + du$, and here by integrating will become :

$$z = e^{\int Ldx}\int e^{-\int Ldx}du \,;$$

[For $\frac{dz}{dx}e^{-\int Ldt} - Lze^{-\int Ldt} = \frac{du}{dx}e^{-\int Ldt} \therefore \frac{d}{dx}\left(ze^{-\int Ldt}\right) = \frac{du}{dx}e^{-\int Ldt}$ etc.,using the familiar integration factor method originated by Euler.]

putting for the sake of brevity $\int Ldx = W$ and the variation sought will be found :

$$\delta\int Zdx = e^{W}\int e^{-W}dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

If the variation may be desired as far as to the given bound $x = a$ and then it becomes on putting $W = A$, for abbreviation

$$e^{A-W}N = (\mathrm{N}),\quad e^{A-W}P = (\mathrm{P})\ ,\quad e^{A-W}Q = (\mathrm{Q})\ \text{etc.}$$

and from the reductions found above the variation will be :

$$\delta\int Zdx = \int dx\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right)$$
$$+\delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((R) - \text{etc.}\right)$$
$$+ \text{ etc.}$$

COROLLARY

47. Therefore if the quantity requiring to be varies Φ may be defined by this differential equation $d\Phi = Zdx$, in which Z may involve some quantity Φ itself and in addition the letters x, y, p, q, r etc., its variation $\delta\Phi$ will be able to be assigned by this problem.

PROBLEM 7

48. If in formula for the integral $\Phi = \int Zdx$ the magnitude Z besides the letters x, y, p, q, r etc. not only involves the magnitude Φ itself, but in addition involves still another integral formula $\Pi = \int \mathfrak{Z}dx$ in some way, but in which the magnitude $\mathfrak{Z}$ may be given only by the letters x, y, p, q, r etc., to investigate the variation of this formula $\delta\Phi = \delta\int Zdx = \int \delta Zdx.$

SOLUTION

Since Z shall be a function of the magnitudes x, y, p, q, r etc. and in addition of the formulas $\Phi = \int Zdx$ and $\Pi = \int \mathfrak{Z}dx$,by differentiating this shall be produced :

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

from which its variation shall be

$$\delta Z = K\delta\Phi + L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{dd\delta y}{dx^2} + \text{etc.}$$

But again, since $\mathfrak{Z}$ shall be a function of the letters x, y, p, q, r etc. only, there may be put

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

and there will be on account of $\delta\Pi = \int \delta\mathfrak{Z}dx$

$$d\Pi = \int dx\left(\mathfrak{N}dy + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

As before there may be put

$$\delta\Phi = z \;\text{ and }\; L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{dd\delta y}{dx^2} + \text{etc.} = u\;;$$

on account of $\delta\Phi = \int \delta Zdx = z$ there will be $\delta Z = \frac{dz}{dx}$ and thus $\frac{dz}{dx} = Kz + u$*;* from which there becomes:

$$z = e^{\int Kdx}\int e^{-\int Kdx}udx = \delta\Phi\;;$$

let $\int Kdx = V$, and the equation becomes

$$z = e^{-\int Kdx}udx = e^{-V}Ldx\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$- e^{-V}dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$

again there may be put $\int e^{-V}Ldx = W$, and by integrating the variation sought will be

$$\delta\Phi = e^{V}W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$- e^{V}\int Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$+ e^{V}\int e^{-V}dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{ etc.}\right).$$

If the variation as far as the given boundary $x = a$ may be desired and on putting $x = a$ there becomes $V = A$ and $W = B$*,* then there may be put in place for the sake of brevity :

$$e^{A-V}N + e^{A}(B-W)\mathfrak{N} = (N)$$
$$e^{A-V}P + e^{A}(B-W)\mathfrak{P} = (P)$$
$$e^{A-V}Q + e^{A}(B-W)\mathfrak{Q} = (Q)$$
$$e^{A-V}R + e^{A}(B-W)\mathfrak{R} = (R)$$
etc.

[*i.e.* to accommodate the coefficients of the various orders of the derivatives of δy grouped together; recall that at this time the upper and lower bounds of an integral had not been formulated : Euler viewed a definite integral as the solution of a first order d.e. with an added constant.]

with which done the variation of the formula $\Phi = \int Zdx$ extended as far as to the boundary $x = a$:

$$\delta\Phi = \int dx\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right)$$
$$+\delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((R) - \text{etc.}\right)$$

COROLLARY

49. Therefore thus the variation is defined of the magnitude Φ by the given differential equation $d\Phi = Zdx$, in which *Z* not only involves Φ itself besides the letters *x, y, p, q, r* etc., but in addition some integral formula $\int \mathfrak{Z}dx = \Pi$, provided $\mathfrak{Z}$ may be determined by the letters *x, y, p, q, r* etc. alone.

PROBLEM 8

50. If in the formula for the integral $\Phi = \int Zdx$ the magnitude *Z* shall involve the integral formula $\Pi = \int \mathfrak{Z}dx$ besides the letters *x, y, p, q, r* etc., but here the quantity $\mathfrak{Z}$ besides the letters *x, y, p, q, r* etc. may contain the integral formula $\Pi = \int \mathfrak{Z}dx$ itself, to define the variation of the proposed formula $\Phi = \int Zdx$.

SOLUTION

Since Z shall be a function of the magnitudes x, y, p, q, r etc. and of $\Pi = \int \mathfrak{Z} dx$ itself, its differential will be of this kind:

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc. },$$

hence its variation will be

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.},$$

from which on account of $\delta\Phi = \int \delta Z dx$, there will be found :

$$\delta\Phi = \int Ldx\delta\Pi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

But because $\mathfrak{Z}$ is a function of x, y, p, q, r etc. and $\Pi = \int \mathfrak{Z} dx$ themselves, its differential shall be:

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \text{etc.}$$

and thus

$$\delta\mathfrak{Z} = \frac{d\delta\Pi}{dx} = \mathfrak{L}\delta\Pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}$$

There is put $\int \mathfrak{L}dx = \mathfrak{W}$, and there will be:

$$\delta\Pi = e^{\mathfrak{W}}\int e^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

[Here the integration factor is $e^{-\int \mathfrak{L}dx} = e^{-\mathfrak{W}}$.]

Make $\int e^{\mathfrak{W}}Ldx = W$ and there will be obtained :

$$\delta\Phi = W\int e^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$-\int e^{-\mathfrak{W}}Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+\int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

If this variation is required to extend as far as to the boundary $x = a$ and by putting $x = a$ making $W = A$, for brevity there is called :

$$N + e^{-\mathfrak{W}}(A - W)\mathfrak{N} = (N)$$
$$P + e^{-\mathfrak{W}}(A - W)\mathfrak{P} = (P)$$
$$Q + e^{-\mathfrak{W}}(A - W)\mathfrak{Q} = (Q)$$
$$\text{etc.}$$

and the reductions set out above introducing the variation of the integral formula $\Phi = \int Zdx$ will be extended to the bound $x = a$

$$\delta\int Zdx = \int dx\delta y\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right)$$
$$+\delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)$$
$$-\frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((R) - \text{etc.}\right)$$
$$\text{etc.}$$

SCHOLIUM

51. A use of this problem is discerned in the descent of bodies on curved lines in a medium with some resistance, while the bodies may be acted on by some forces, if we wish to define the variation of the time of descent, while the curve may be varied in some manner. In this case Φ may denote the time of descent along an arc, which shall correspond to the abscissa *x*, and the applied line shall be $= y$ and Π must be the height appropriate to the speed acquired; and the time of descent will be :

[*i.e.* according to the *vis viva* theory of the times, whereby $v^2 = h$, which as we have remarked elsewhere has to be replaced by $v^2 = 2gh$ to be correct. Thus, the result used is correct if the acc. of gravity is $\frac{1}{2}$ in some units.]

$$\Phi = \int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$$

on putting $dy = pdx,$ so that $dx\sqrt{(1+pp)}$ may designate an element of arc. Truly from the forces acting there will be

$$d\Pi = Xdx + Ydy - V\sqrt{(dx^2 + dy^2)},$$

where X and Y designate functions of x and y, and V a function of Π,
to which the resistance is proportional. Therefore on account of $dy = pdx$ there will be

$$\Pi = \int \left(X + Yp - V\sqrt{(1+pp)} \right) dx$$

and thus

$$\mathfrak{Z} = X + Yp - V\sqrt{(1+pp)}$$

with Z being given by $\dfrac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$.

COROLLARY

52. If according to the similarity of the values (N), (P), (Q) there may be put

$$M + e^{-\mathfrak{W}}(A-W)\mathfrak{M} = (M),$$

there will be $(M)dx + (N)dy + (P)dp + (Q)dq + (R)dr + \text{etc.}$, truly the differential of this formula:

$$Z + e^{-\mathfrak{W}}(A-W)\mathfrak{Z}.$$

CONCLUSION

53. Therefore whatever formula may be put in place for the integral $\Phi = \int Zdx$, it shall be necessary to find its variation, and its variation extended as far as to the bound $x = a$ may always be expressed in this manner

$$\begin{aligned}
\delta\Phi = \int dx\delta y & \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \frac{d^4(R)}{dx^4} - \text{etc.} \right) \\
& + \delta y \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \frac{d^3(S)}{dx^3} + \text{etc.} \right) \\
& + \frac{d\delta y}{dx} \left((Q) - \frac{d(R)}{dx} + \frac{dd(S)}{dx^2} - \text{etc.} \right) \\
& + \frac{d\delta y}{dx} \left((R) - \frac{d(S)}{dx} + \text{etc.} \right) \\
& + \frac{d^3\delta y}{dx^3} \big((S) - \text{etc.} \big) \\
& \quad \text{etc.}
\end{aligned}$$

with the element *dx* taken constant. But just as the letters (*N*), (*P*), (*Q*), (*R*), (*S*) etc. may be found themselves, that in any case will be apparent.

CASE I

54. If $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds$ etc., there will be

$$(N) = N,\ (P) = P,\ (Q) = Q,\ (R) = R,\ (S) = S \quad \text{etc.}$$

CASE II

55. If $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$ having

$$\Pi = \int \mathfrak{Z}dx \text{ et } d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}\text{dp} + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

let $\int Ldx = W$ and on putting $x = a$ the integral becomes $W = A$, with which done there will be :

$$(N) = N + (A-W)\mathfrak{N} \qquad (P) = P + (A-W)\mathfrak{P}$$
$$(Q) = Q + (A-W)\mathfrak{Q} \qquad (R) = R + (A-W)\mathfrak{R}$$
$$(S) = S + (A-W)\mathfrak{S} \qquad \text{etc.}$$

CASE III

56. If there were

$$dZ = Ld\Pi + L^{/}d\Pi^{/} + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

having $\Pi = \int \mathfrak{Z}dx$ et $\Pi^{/} = \int \mathfrak{Z}^{/}dx$, then truly :

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$
$$d\mathfrak{Z}^{/} = \mathfrak{M}^{/}dx + \mathfrak{N}^{/}dy + \mathfrak{P}^{/}dp + \mathfrak{Q}^{/}dq + \mathfrak{R}^{/}dr + \text{etc.},$$

putting $\int Ldx = W$ and $\int L^{/}dx = W^{/}$, and making $x = a$ there becomes $W = A$ and $W^{/} = A^{/}$, with which done there will be :

$$(N) = N + (A-W)\mathfrak{N} + (A^{/} - W^{/})\mathfrak{N}^{/}$$
$$(P) = P + (A-W)\mathfrak{P} + (A^{/} - W^{/})\mathfrak{P}^{/}$$
$$(Q) = Q + (A-W)\mathfrak{Q} + (A^{/} - W^{/})\mathfrak{Q}^{/}$$
$$(R) = R + (A-W)\mathfrak{R} + (A^{/} - W^{/})\mathfrak{R}^{/}$$
etc.

CASE IV

57. If Z may contain the integral formula $\Pi = \int \mathfrak{Z}dx$, so that there shall be :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

the magnitude $\mathfrak{Z}$ may contain the integral formula $\pi = \int \mathfrak{z}dx$, so that there shall be :

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

but $\mathfrak{z}$ again may involve no integral, thus so that there shall be :

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.}$$

There may be put $\int Ldx = W$ and on putting $x = a$ it becomes $W = A$; then truly there may be put $\int (A-W)\mathfrak{L}dx = \mathfrak{W}$ and in the case $x = a$ the integral becomes $\mathfrak{W} = \mathfrak{A}$, with which done there shall be :

$$(N) = N + (A-W)\mathfrak{N} + (\mathfrak{A}-\mathfrak{W})\mathfrak{n}$$
$$(P) = P + (A-W)\mathfrak{P} + (\mathfrak{A}-\mathfrak{W})\mathfrak{p}$$
$$(Q) = Q + (A-W)\mathfrak{Q} + (\mathfrak{A}-\mathfrak{W})\mathfrak{q}$$
$$(R) = R + (A-W)\mathfrak{R} + (\mathfrak{A}-\mathfrak{W})\mathfrak{r}$$
etc.

CASUS V

58. If Z shall contain the formula itself $\Phi = \int Zdx$, so that there shall be :

$$dZ = Kd\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

putting $\int Kdx = V$ and by making $x = a$ it becomes $V = C$, there will be :

$$(N) = e^{C-V}N,\ (P) = e^{C-V}P,\ (Q) = e^{C-V}Q,\ (R) = e^{C-V}R \text{ etc.}$$

CASE VI

59. If Z besides the formula $\Phi = \int Zdx$ may contain another integral formula $\Pi = \int \mathfrak{Z}dx$, and there shall be :

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

then truly $\mathfrak{Z}$ may involve no integral formula :

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

let $\int Kdx = V$ and on putting $x = a$ it becomes $V = C$. Then there shall be $\int e^{C-V}Ldx = W$ and on putting $x = a$ it becomes $W = A$, and there will be :

$$(N) = e^{C-V} N + (A-W)\mathfrak{N}$$
$$(P) = e^{C-V} P + (A-W)\mathfrak{P}$$
$$(Q) = e^{C-V} Q + (A-W)\mathfrak{Q}$$
$$(R) = e^{C-V} R + (A-W)\mathfrak{R}$$
etc.

CASE VII

60. If Z may contain the formula $\Pi = \int \mathfrak{Z} dx$, so that there shall be :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

then truly $\mathfrak{Z}$ anew may involve the same formula $\Pi = \int \mathfrak{Z} dx$, so that there shall be :

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

There may be put $\int \mathfrak{L}dx = \mathfrak{W}$ and on putting $x = a$ it becomes $\mathfrak{W} = \mathfrak{A}$; then there may be put

$$\int e^{-\mathfrak{A}+\mathfrak{W}} Ldx = W$$

and on putting $x = a$ it becomes $W = A$, and there will be

$$(N) = N + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{N}$$
$$(P) = P + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{P}$$
$$(Q) = Q + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{Q}$$
$$(R) = R + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{R}$$
etc.

61. It is possible to extend this investigation in a similar manner to other more complicated formulas, truly since scarcely any such may be accustomed to occur, the labour would be superfluous. Therefore since I should have taught how to define variations both of the simpler formulas of integration as well as of more arranged ones, the calculus of variations may be considered to be almost completely summed up ; for in whatever manner a magnitude should be required to be varied, both form absolute formulas as well as constructed from integrals, with the help of differentiation its variation can be found. Just as if the magnitude U to be varied may contain some integral formulas

$$\Phi = \int Zdx, \quad \Phi' = \int Z'dx, \quad \Phi'' = \int Z''dx \quad \text{etc.,}$$

these may be differentiated in the usual manner and shall give rise to :

$$dU = Kd\Phi + K'd\Phi' + K''d\Phi'' \text{ etc.}$$

then its variation evidently is :

$$\delta U = K\delta\Phi + K'\delta\Phi' + K''\delta\Phi'' \text{ etc.,}$$

but the variations $\delta\Phi$, $\delta\Phi'$, $\delta\Phi''$ etc. by the precepts will be able to be designated in the manner established.
Truly likewise it is apparent the variation δU is always going to be an expression of this kind, so that there shall be:

$$\delta V = \int (A)dx\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} + \text{ etc.,}$$

where (*A*), (*B*), (*C*) etc. are functions come upon from the rules treated above.
Moreover it will be convenient to indicate the use of this calculus of variations in the solution of the most celebrated of isoperimetric problems, taken in the broadest sense.

THE APPLICATION OF THE CALCULUS OF VARIATIONS TO THE SOLUTION OF ISOPERIMETRIC PROBLEMS TO BE TAKEN OF THE BROADEST POSSIBLE SIGNIFICANCE

62. Regarding the most important problem to be set out here thus, so that among all the curves being set forth on the same give base $x = a$ that may be defined, for which a certain formula *U* may reach a maximum or minimum value. For even if the statement of the problem includes only curves of the same length, yet this condition may be ignored conveniently, so that its wider paths may be apparent, nor even the consideration of a single formula *U*, of which the maximum or minimum value must avoid, its nature being agreed to be restricted, as in general I have shown afterwards : if among all the curves being set forth on the same base $x = a$, for which the formula *V* gives rise to the same value, that must be defined, in which the maximum or minimum value of the formula may have to be defined, that must be defined, in which the value of the formula *U* may avoid a maximum or minimum value, here the question be reduced, so that among all the curves plainly produced on the same base $x = a$ that may be defined, for which this composite formula $\alpha V + \beta U$ may follow a maximum or minimum value. Yet meanwhile an account of its reduction can be lucidly explained from the calculus of variations.

63. But this question about the remote consideration of curved lines can be proposed in this way :

Some proposed formula U defines that relation between the two variables x and y, by which if the value of U may be determined and which may be extended from the value $x = 0$ *as far as to the value* $x = a$*, that function is going to give rise to a maximum or minimum.*

Therefore we may consider the relation between *x* and *y* now as found, thus so that a maximum or minimum value of *U* may arise ; and it is clear, if the relation between *x* and *y* may be changed an infinitely small amount, thence no change must arise in the value of *U* ; or, what amounts to the same, a variation of *U* or δU is required equal to zero ; and thus the equation $\delta U = 0$ includes the relation sought between *x* and *y*.

64. But we have taught thence how to define the variation δU, so that for some value of *x* itself its value *y*, which, from the nature of the relation sought may itself happen at the same time, we have assumed to be increased by a certain small amount δy. Therefore since the relation sought between all plainly possible must be endowed with this prerogative, that the variation δU must be equal to zero always, for whatever individual values of *y* may be increased by such small amounts δy, and in whatever way these increases shall have been prepared, because they are hence arbitrary, and in no manner depend on each other. Nor even is there a need, that variations may be attributed to all the values of *y* of this kind, but whether some one or two or perhaps some number for argument's sake may be varied, it is always equally necessary that the variation be redundant, which thence in the whole value of *U*, in as much as it may be extended from the term $x = 0$ as far as to the term $x = a$ shall become zero.

65. Moreover from these, which have been treated above, it is clear the variation of *U* always to be expressed in this way, so that there shall be :

$$\delta U = \int (A) dx \delta y + (B) \delta y + (C) \frac{d\delta y}{dx} + (D) \frac{dd\delta y}{dx^2} \text{ etc.},$$

of which form it is agreed to consider the individual parts separately. But besides the first integral member, the remaining parts $(B)\delta y,\ (C)\frac{d\delta y}{dx}$ etc. depend only on the variation of the final value of y, which corresponds to $x = a$, and do not involve an account of the preceding variations ; so indeed that the whole variation of *U* may be obtained, $x = a$ must be put in place everywhere found in the expression , so that it actually happens in the individual parts besides the first, and thus in these δy will denote the variation, which is attributed to the final value of *y* only, and which is entirely arbitrary and does not depend on the preceding. From which it is clear, unless the integral member is present, plainly nothing can be concluded for the relation between *x* and *y* from the remaining parts.

66. Truly the integral part $\int(A)dx\delta y$ also involves the variations, which are attributed to all the preceding values of y, as well as containing the sum of all the elements $(A)dx\delta y$ arising from the variation of the individual values *y*. Thus, if a single value of its *x* itself, as if it may have a determined value, by considering the corresponding *y* may varied or be increased by the small amount δy, only that integral member becomes $=(A)dx\delta y$, and nothing requiring to be summed may be had ; but if in addition the following value $y^{/}$ of $x+dx$ itself corresponds to the small amount $\delta y^{/}$ may be increased and on putting $x+dx$ in place of *x*, the function (*A*) will be changed into $(A)^{/}$, and the integral part will depend on these two parts :

$$(A)dx\delta y+(A)^{/}dx\delta y^{/}.$$

In a similar manner, if three or more successive values $y, y^{/}, y^{//}, y^{///}, y^{////}$ etc. may be increased by the particles $\delta y, \delta y^{/}, \delta y^{//}, \delta y^{///}$ etc., the integral member will be equivalent to this expression :

$$(A)dx\delta y+(A)^{/}dx\delta y^{/}+(A)^{//}dx\delta y^{//}+(A)^{///}dx\delta y^{///}\text{ etc.}$$

which series can be considered both backwards as far as to the term $x=0$ as well as continued forwards as far as to the term $x=a$.

67. Therefore even if the variation δU may be restricted to the determined term $x=a$, yet on account of the integral member all the intermediate variations are included ; from which, if for the remaining unconditional parts, which are only referring to the final term $x=a$, for the sake of brevity we may write *I*, the variation δU thus will be expressed, so that there shall be :

$$\delta U=(A)dx\delta y+(A)^{/}dx\delta y^{/}+(A)^{//}dx\delta y^{//}+(A)^{///}dx\delta y^{///}\text{ etc.}+I,$$

which, so that it may satisfy the problem, must be equal to zero. But since the variations $\delta y^{/}, \delta y^{//}, \delta y^{///}$ etc. may not depend on each other, but the individuals are purely arbitrary, that annihilation cannot have a place, unless the individual parts may vanish one by one ; from which it is necessary that there shall be :

$$(A)=0,\ (A)^{/}=0,\ (A)^{//}=0,\ (A)^{///}=0\ \text{ etc.},$$

all which small equations may be contained in this one indefinite equation $(A)=0$ or, whatever value may be attributed to *x*, there is required always to be $(A)=0$, and the relation sought between *x* and *y* is contained in this equation.

68. Behold therefore the easy solution of the proposed problem, from which that relation between *x* and *y* is required, out of which for the prescribed formula *U*, after its value had been extended from $x=0$ as far as to $x=a$, a maximum or minimum value may result. Truly the variation of the formula *U* equally is sought from $x=0$ extended as far as $x=a$, which by the precepts treated above must have a form of this kind:

$$\delta U=\int (A)dx\delta y+(B)\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^2}\text{ etc. },$$

and hence from the single member of the integral $\int (A)dx\delta y$ the relation between *x* and *y* sought may be defined thus, so that there shall be $(A)=0$, moreover the remaining parts, because they affect only the final value of *y*, bring nothing to the indefinite relation between *x* and *y*, which may be wished.

69. Yet these latter parts can be of more help in determining the relation found ; for so far only parts of this kind are in agreement, as far as in the integral member $\int (A)dx\delta y$, it involves the function (*A*) , the ratio of the differentials $\frac{dy}{dx}=p$, or also the ratios of the higher differentials, clearly $q=\frac{dp}{dx}$, $r=\frac{dq}{dx}$ etc. But when this arises usually, the equation $(A)=0$ will be a differential either of the first or also of higher orders ; and thus the relation sought between *x* and *y* after one or more integrations is found at last. But since any integration introduces an arbitrary constant quantity, in this way a finite uncertainty will be introduced into the equation and now a new question will arise : how these arbitrary constants ought to be determined, so that the value of *U* itself may arise the maximum or minimum of all. For since now the determination of any of these constants shall be given by the property of the maxima or minima itself of these, here again either the maximum of the maximums or the minimum of the minimums is left to be investigated.

70. Therefore towards resolving this new problem these subsidiary parts unchanged by the integral sign can be used. Clearly it may be agreed to determine the constants added thus from integration, so that on putting $x=a$ the individual coefficients of $\delta y,\ \frac{d\delta y}{dx},\ \frac{dd\delta y}{dx^2}$ etc. themselves separately vanish, or so that in this case it may be satisfied by these conditions :

$$(B)=0,\ (C)=0,\ (D)=0 \quad \text{etc.}$$

Then, because it is allowed to interchange both terms $x=0$, and $x=a$ among themselves, also on putting $x=0$ it will come about, that there becomes $(B)=0,\ (C)=0,\ (D)=0$ etc. For even if the parts, which may be removed from this, may not be contained in our expression, yet these are considered to be contained in the integral part.

71. From the same principles also the problems can be solved, which I have referred to the relative method ; moreover it is permitted generally to enunciate these problems thus :
Amongst all the relations, by which y is defined by x, which are endowed with this common property, so that for the formula $\mathfrak{U}$ *on putting* $x=a$ *they may show the same value, to determine that relation, from which the formula U, if indeed it may be extended from the boundary* $x=0$ *as far as to the boundary* $x=a$, *may follow a maximum or minimum value.*

Therefore here the variations, which are attributed to the individual values of y, not all are arbitrary, but thus are being put in place, so that there becomes $\delta\mathfrak{U}=0$, if indeed its value may be extended from the boundary $x=0$ as far as to $x=a$. Then truly also the nature of the maxima or minima postulates, so that following the same extension as before there shall be $\delta U=0$.

72. Therefore by the method set out before both the formula $\mathfrak{U}$, the variation is sought which may be extended from the boundary $x=0$ as far as to the boundary $x=a$, which must be common, as well as of the formula *U*, which must become a maximum or a minimum ; and the relation sought between *x* and *y* from the equations of these taken together $\delta\mathfrak{U}=0$ and $\delta U=0$ will be investigated. But these variations thus expressed may be found :

$$\delta U=\int (A)\,dx\delta y+(B)\,\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^2} \text{ etc.}$$

where the same are to be understood from the parts freed from the integral sign, which I have now observed above ; and thus the relation sought between *x* and *y* will be required to be derived only from the parts of the integral.

73. And thus hence we may attend to the two following equations :

$$\delta\mathfrak{U}=\int (\mathfrak{A})\,dx\delta y+(\mathfrak{B})\,\delta y+(\mathfrak{C})\frac{d\delta y}{dx}+(\mathfrak{D})\frac{dd\delta y}{dx^2} \text{ etc.}$$

$$\delta U=\int (A)\,dx\delta y+(B)\,\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^2} \text{ etc.}$$

of which the variation assumed from the first δy, $\delta y^{/}$, $\delta y^{//}$ etc. is defined agreeing with the prescribed common condition, which then will be shown in the other related question introduced. Therefore all the variations δy, $\delta y^{/}$, $\delta y^{//}$ etc. beyond one can be considered as arbitrary, clearly which one is required to be defined from the first equation. Now truly it is clear, after one thus has been taken, so that the first equation may be satisfied, then likewise the other is going to be satisfied, if there may be put $(A) = n(\mathfrak{A})$, on taking some constant quantity for *n*.

74. Therefore the proposed problem may be resolved by this equation:

$$\alpha(A) + \beta(\mathfrak{A}) = 0$$

with some constant quantities taken for α and β. But the same solution may be produced, if among all the relations entirely between *x* and *y* that ought to be sought out, from which the formula $\alpha U + \beta\mathfrak{U}$ may attain a maximum or minimum value ; from which likewise it is understood the two formulas $\mathfrak{U}$ and *U* proposed can be interchanged with each other, which I have noted in my treatment, hence they become much more clear. For in a similar manner this matter will be found, if not by a single formula $\mathfrak{U}$, but several must be in common; and thus the *Calculus of Variations* firmly established all problems of this kind can be set out more easily and briefly.

ELEMENTA CALCULI VARIATIONUM

SUMMARIUM

Iam ante celeberrimum problema isoperimetricum insignia quaedam specimina huc pertinentia a Geometris sunt edita, cum antiquissimis iam fuerit exploratum circulum inter omnes alias figuras pari perimetro inclusas maximam aream complecti; quam quidem proprietatem ex circuli natura concluserunt, minime vero ipsam quaestionem directe aggredi sunt ausi, ut inter omnes figuras aequali perimetro terminatas eam investigarent, quae maximam aream includeret. Haec scilicet quaestio nimis est ardua, quam ut ante insignem calculi infinitorum promotionem de ea saltem cogitare licuisset. Mox vero primis quasi iactis huius calculi fundamentis ab acutissimo JOHANNE BERNOULLI quaestio de brachystochronis felicissimo successu est resoluta, quippe qua inter omnes lineas a puncto sublimiori ad humilius ductas ea quaerebatur, super qua grave tempore brevissimo descendat, quam egregiam proprietatem cycloidi competere invenerat. Methodus autem, qua Vir celeberrimus erat usus, fratri ipsius natu maiori JACOBO BERNOULLI manifesto occasionem. praebuisse videtur solutionem magni problematis isoperimetrici, quod deinceps tractavit, meditandi. Latissimo scilicet ambitu omnes huius generis quaestiones in hoc problemate est complexus, ut inter omnes lineas intra data duo puncta ducendas, sive debeant esse eiusdem longitudinis (unde quidem nomen isoperimetrici est natum) sive alia quadam indole communi praeditae, eam investigaret, quae vel maximam aream vel circa datum axem rotata maximum solidum vel in genere quamcumque maximi minimive proprietatem contineret. Methodum autem, qua summus illius temporis Geometra est usus, perpendentes ancipites haeremus, utrum magis eius incredibilem patientiam in prolixissimis et taediosissimis calculis expediendis, an summam sagacitatem in conclusionibus satis concinnis inde deducendis admirari debeamus. Ob hanc ipsam autem causam, quod conclusiones prodierint satis concinnae, mox suspicari licebat viam planiorem ac breviorem dari eodem perducentem; quam etiam eius frater iunior JOHANNES satis feliciter est ingressus, etiamsi statim pro quibusdam casibus nimis absconditis negotium minus successerit, quem tamen defectum deinceps, toto hoc argumento profundius retractato, largiter compensavit. Longo postea interiecto tempore Auctor harum dissertationum in eodem problemate evolvendo summum studium collocavit, et cum perspexisset omnes huius generis quaestiones eo redire, ut eiusmodi linea curva aequatione inter coordinatas x et y exprimenda investigetur, in qua talis formula integralis $\int Vdx$, quomodocunque quantitas V per x et y fuerit data, maximum minimumve valorem obtineat. Nunc autem evidens est in ista quantitate V infinitam varietatem locum habere posse, prout in eam praeter ipsas variabiles x et y tam earum differentialia cuiuscunque ordinis quam novae insuper formulae integrales ingrediuntur. Quodsi iam solutiones BERNOULLIANAE ad hanc normam examinentur, eae tantum ad eos casus, quibus quantitas V sola differentialia primi gradus involvit, restrictae reperiuntur ac praeterea casus, quibus in quantitate V novae formulae integrales insunt, inde penitus excluduntur, paucissimis exceptis, quos facile pro indole quaestionis ab hoc incommodo liberare licet. Hunc igitur defectum noster Auctor felicissime cum in his

Commentariis tum in opere singulari de hoc argumento edito supplevit, ut vix quicquam, quod amplius desiderari queat, reperiatur. Interim tamen ipsa methodus, etiamsi totum negotium satis expedite conficiat, tamen ipsi non satis naturalis est visa, propterea quod vis solutionis tota in consideratione elementorum curvae investigandae erat posita, ipsa vero quaestio facile ita adornari possit, ut ex Geometria penitus ad solam Analysin puram revocatur. Quaestio enim ita proposita, ut data quantitate V utcunque ex binis variabilibus x, y earumque differentialibus cuiuscunque ordinis, quin etiam ex formulis integralibus utcunque conflata, ea inter x et y relatio investigari debeat, qua formulae integrali $\int Vdx$ maximus minimusve valor concilietur *?* hoc in quam modo quaestio proposita prorsus a Geometria segregatur; ex quo etiam methodus genuina eam resolvendi a Geometria immunis esse debebat; et quo difficilius Analysis ad hunc scopum accommodari poterat, eo maiora incrementa huius scientiae, si res successerit, merito sperari licebat. Tametsi autem Auctor de hoc diu multumque esset meditatus atque amicis hoc desiderium aperuisset, tamen gloria primae inventionis acutissimo Geometrae Taurinensi LA GRANGE erat reservata, qui sola Analysi usus eandem plane solutionem est adeptus, quam Auctor ex considerationibus geometricis elicuerat. Verum ipsa illa solutio ita erat comparata, ut novam plane Analyseos speciem constituere eiusque fines non mediocriter promovere videretur; ex quo Auctori occasio est oblata hanc scientiam novo Calculi genere locupletandi, quem *Calculum variationum* appellat et cuius elementa hic tradere ac dilucide explicare constituit. Hic quidem calculus perinde ac differentialis in incrementis infinite parvis inter se comparandis versatur, verum in ratione tractationis ab eo maxime discrepat. Cum enim in calculo differentiali ex data quantitatum variabilium relatione relatio inter earum differentialia cuiusque ordinis investigetur, in calculo variationum ipsa relatio inter variabiles infinite parum immutari concipitur, ita ut, dum secundum relationem datam pro quovis alterius variabilis x valore altera y certum valorem sortitur, calculo variationum huic ipsi valori y incrementum quoddam infinite parvum adiiciatur, ex quo deinceps, quemadmodum formulae tam differentiales quam integrales varientur, definiri oportet. Incrementum iillud cuicunque valori y adiectum ab Auctore eius variatio vocatur ac, ne cum differentialibus confundatur, hoc charactere δy designatur; cum igitur hinc omnes formulae tam differentiales quam integrales, quatenus quantitatem y involvunt, certas variationes nanciscantur, auctor in priore dissertatione principia ac praecepta stabilit, quorum ope omnium huiusmodi formularum variationes definiri possunt: ita si W denotet huiusmodi formulam quamcunque, eius variationem δW per regulas peculiares assignare docet. Quo singulari calculo constituto deinceps in sequente dissertatione eius applicationem ad omnia problemata, quae circa maxima et minima excogitari possunt, clarissime ostendit, inque negotio hoc imprimis observari meretur, quod ita nova methodus mere analytica multo pleniores ac perfectiores solutiones suppeditet, quam prior illa ex Geometria petita.

HYPOTHESIS 1

1. Detur inter variabiles binas *x* et *y* aequatio quaecunque, qua earum relatio mutua exprimatur, ita ut inde, quicunque valor determinatus ipsi *x* tribuatur, valor quoque determinatus pro *y* definiatur.

COROLLARIUM 1

2. Proposita ergo aequatione inter binas variabiles *x* et *y* singulis valoribus ipsius *x*, quicunque concipi possunt, determinati valores ipsius *y* respondebunt.

COROLLARIUM 2

3. Vi ergo istius aequationis propositae erit *y* certa quaedam functio ipsius *x* et, quemadmodum ipsi *x* respondet *y*, ita illius valori sequenti $x^{/} = x + dx$ respondebit $y^{/} = y + dy$, cuius differentia a praecedente *y*, differentiale nempe *dy*, per vulgares differentiandi regulas assignari poterit.

COROLLARIUM 3

4. Cum *y* sit functio ipsius *x*, etiam $\frac{dy}{dx}$ erit functio ipsius *x* per relationem inter *x* et *y* datam assignabilis; ac si ponatur $\frac{dy}{dx} = p$, simili modo
$\frac{dp}{dx}$ erit certa functio ipsius *x;* ac si porro ponamus $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$, $\frac{dr}{dx} = s$ etc., etiam hae quantitates *q*, *r*, *s* etc. erunt certae functiones ipsius *x* itidem per relationem inter *x* et *y* datam assignabiles.

COROLLARIUM 4

5. Si deinde *V* sit expressio quomodocunque ex *x* et *y* conflata, ea quoque ope relationis inter *x* et *y* datae ita erit comparata, ut pro omnibus valoribus ipsius *x* valores determinatos adipiscatur. Ac si $V^{/}$ designet valorem sequentem seu ipsi $x + dx$ convenientem, erit $V^{/} = V + dV$ sive $dV = V^{/} - V$, secundum prima calculi differentialis principia.

HYPOTHESIS 2

6. Quaecunque proponatur relatio inter *x* et *y*, quia inde simul relatio differentialium *dx* et *dy* innotescit, ponam in sequentibus perpetuo:

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r, \quad \frac{dr}{dx} = s \text{ etc.}$$

eruntque *p, q, r, s* etc. perinde ac *y* functiones ipsius *x* per illam relationemdatam assignabiles.

COROLLARIUM 1

7. Quemadmodum littera *p* relationem differentialium *dx* et *dy* continet, ita *q* complectetur relationem differentialium secundi gradus, *r* vero differentialium tertii gradus, *s* quarti gradus etc.

COROLLARIUM 2

8. Vicissim igitur etiam, si qua in expressione *V* differentialia sive primi sive secundi sive altioris ordinis insint, ea introducendis his quantitatibus *p, q, r, s* etc. ex calculo tolli poterunt.

AXIOMA

9. Si inter variabiles *x* et *y* alia relatio a proposita infinite parum discrepans constituatur, valores ipsius *y*, singulis valoribus ipsius *x* respondentes, ab iis, quos proposita relatio praebet, infinite parum discrepabunt.

COROLLARIUM 1

10. Cum huiusmodi relatio variata infinitis modis a relatione proposita discrepare possit, ita ut discrepantia sit infinite parva, evenire potest, ut unus pluresve valores ipsius *y*, qui certis valoribus ipsius *x* respondent, nullam inde mutationem patiantur.

COROLLARIUM 2

11. Ista variatio relationis ita generalis concipi potest, ut inde omnes valores ipsius *y* mutationes quascunque patiantur, quo nullo modo a se invicem pendeant. Quo igitur haec tractatio latissime pateat, huiusmodi variationem relationis generalissime conceptam intelligi conveniet.

HYPOTHESIS 3

12. Si relatio inter x et y proposita parum mutetur, valorem ipsius y, qui inde ipsi x respondet, per $y+\delta y$ designemus, ita ut δy variationem denotet, quam y ob variatam relationem patitur.

COROLLARIUM 1

13. Simili modo cum y' sit valor ipsi $x+dx$ vi relationis propositae respondens, eius valorem, qui eidem $x+dx$ vi relationis variatae convenit, per $y'+\delta y'$ exprimamus, ita ut $\delta y'$ variationem ipsius y' denotet, quae ex variatione relationis oritur.

COROLLARIUM 2

14. Cum igitur sit $y'=y+dy$, erit

$$\delta y'=\delta(y+dy)=\delta y+\delta dy \quad \text{et} \quad \delta dy=\delta y'-\delta y.$$

Denotabit autem δdy variationem ipsius dy ex variatione inter x et y propositae ortam.

COROLLARIUM 3

15. Quemadmodum autem y' statum sequentem ipsius y denotat, statu scilicet sequente ad $x+dx$ relato, ita $\delta y'$ statum sequentem ipsius δy denotat, ex quo $\delta y'-\delta y$ exprimet differentiale ipsius δy, quod est $d\delta y$. Cum ergo sit $\delta dy=\delta y'-\delta y$ erit $\delta dy=d\delta y$.

COROLLARIUM 4

16. Hinc ergo consequimur istam insignem proprietatem: quod variatio differentialis ipsius y aequalis sit differentiali variationis ipsius y. Est enim δdy variatio ipsius dy, hoc est differentialis ipsius y, et $d\delta y$ est differentiale ipsius δy, hoc est variationis ipsius y.

DEFINITIO I

17. Si V sit expressio utcunque ex x et y conflata, proposita quadam relatione inter x et y eius variatio, quam per δV indicabo, est incrementum, quod quantitas V capit, si relatio inter x et y proposita infinite parum varietur.

COROLLARIUM 1

18. Probe ergo distingui oportet differentiale dV et variatio δV *;* differentiale enim denotat incrementum ipsius *V*, dum *x* suo elemento *dx* augetur, manente relatione inter *x* et *y* proposita; variatio autem denotat incrementum ipsius *V*, dum ipsa relatio variatur manente *x*.

COROLLARIUM 2

19. Cum per variationem relationis inter *x* et *y* propositae quantitas *y* incrementum capiat δy manente *x* eadem: quomodocunque quantitas *V* ex *x* et *y* fuerit conflata, eius variatio reperietur, si loco *y* ubique scribatur $y+\delta y$ et a valore hinc pro *V* oriundo ipse valor *V* subtrahatur.

COROLLARIUM 3

20. Scilicet si in *V* ubique pro *y* scribatur $y+\delta y$, prodibit valor variatus ipsius *V*, qui est $V+\delta V$ *;* ipsa autem variatio reperitur, si a valore variato $V+\delta V$ valor primitivus *V* subtrahatur.

DEFINITIO II

21. Calculus variationum est methodus inveniendi variationes quantitatum utcunque ex binis variabilibus *x* et *y* conflatarum, quas patiuntur, si relatio inter *x* et *y* proposita infinite parum quomodocunque immutetur.

COROLLARIUM 1

22. Proposita ergo relatione inter *x* et *y* , si *V* denotet quantitatem quomodocunque ab *x* et *y* pendentem, hic calculus docet invenire variationem ipsius *V* seu valorem ipsius δV .

COROLLARIUM 2

23. Quia relationem inter *x* et *y* datam utcunque immutari assumimus, ut *y* pro singulis valoribus ipsius *x* variationes quascunque, quae etiam a se invicem non pendeant, accipiat, hic calculus latissime patet atque ad quasvis conditiones variationum datas accommodari poterit.

SCHOLION 1

24. Quo usus huius calculi clarius perspici queat, exemplum afferamus. Proposita ergo sit haec relatio inter *x* et *y*

$$aayy-bbxx=aabb,$$

quae scribendo $b+db$ loco b infinite parum immutetur. Iam si proponatur quantitas quaepiam ab x et y pendens, veluti

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}},$$

huius variatio ex illa immutatione relationis oriunda ope istius calculi exhiberi poterit; cum enim sit

$$y = \frac{b}{a}\sqrt{(aa - xx)},$$

erit

$$\delta y = \frac{db}{a}\sqrt{(aa - xx)},$$

quae est variatio ipsius y. Quemadmodum autem ex cognita variatione ipsius y quantitatum utcunque ab y et x pendentium ideoque etiam huius

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}}$$

variationes inde natae determinari debeant, in hoc calculo est ostendendum; unde patet omnia, quae de variabilitate parametrorum passim sunt tradita, hic contineri. Deinde vero etiam quaestiones inverti possunt, veluti si proposita huiusmodi formula

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{y}}$$

ea relatio inter x et y quaeratur, unde variatio istius formulae datae prodeat magnitudinis vel etiam nulla, quo posteriori casu relatio inventa formulae propositae maximum minimumve valorem comparabit; atque huc referenda erunt omnia problemata, quae circa curvas maximi minimive proprietate gaudentes adhuc sunt tractata.

SCHOLION 2

25. Praecepta huius calculi ad diversitatem rationis, qua formula quaepiam proposita V a binis variabilibus x et y pendet, sunt accommodanda, quae diversitas cum sit infinita, eam ad aliquot genera praecipua revocari conveniet. Primum ergo genus complectatur eas formulas, quae ex ipsis quantitatibus x et y earumque derivatis

$$p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx} \text{ etc.}$$

utcunque sunt compositae, ita tamen, ut nullas formulas integrales involvant. Ad secundum genus refero eas formulas, quae integralia huiusmodi $\int Zdx$ contineant, ita tamen, ut ipsa formula Z ad primum genus pertineat. Tertium genus comprehendet eiusmodi formulas, in quibus non solum integralia $\int Zdx$ insunt, sed ubi quantitas Z ipsa insuper integralia involvit. Tandem sequetur quartum genus, in quo formula varianda V non absolute, sed demum per aequationem differentialem vel primi vel adeo altioris gradus definitur, quod genus utique latissime patet ac praecedentia omnia in se complectitur. Quod autem ad aequationem, qua relatio inter x et y exprimitur, attinet, etsi eam ut datam specto, tamen non definio, ne praecepta tradenda ullo modo limitentur.

THEOREMA 1

26. Variatio differentialis cuiusvis quantitatis V aequalis est differentiali variationis eiusdem seu est $\delta dV = d\delta V$.

DEMONSTRATIO

Cum sit $dV = V^{/} - V$ denotante $V^{/}$ valorem sequentem ipsius V, qui ipsi $x+dx$ convenit, uti V ipsi x respondet, erit $\delta dV = \delta V^{/} - \delta V$; verum $d\delta V$ exprimit differentiam inter δV eiusque valorem sequentem, qui est $\delta V^{/}$, ita ut sit $d\delta V = \delta V^{/} - \delta V$, unde perspicuum est esse $\delta dV = d\delta V$.

COROLLARIUM 1

27. Eodem modo, si loco V scribamus dV, patet esse $\delta ddV = d\delta dV$; sed $\delta dV = d\delta V$, unde fit $d\delta dV = dd\delta V$, sicque aequales inter se erunt hae tres formae

$$\delta ddV = d\delta dV \ \ dd\delta V \ .$$

COROLLARIUM 2

28. Porro vero, si et hic pro V scribamus dV, obtinebimus aequalitatem inter has quatuor formas

$$\delta dddV = d\delta ddV = dd\delta dV = ddd\delta V \,,$$

tum vero inter has quinque

$$\delta d^4V = d\delta d^3V = d^2\delta d^2V = d^3\delta dV = \ d^4\delta V \ .$$

COROLLARIUM 3

29. Si habeatur differentiale cuiuscunque ordinis ipsius V, nempe d^nV , cuius variatio sit investiganda, erit

$$\delta d^nV = d^m\delta d^{n-m}V = \ d^n\delta V$$

aequatur scilicet differentiali ordinis *n* ipsius variationis δV. Hinc ergo reducitur variatio differentialium ad differentiationem variationis.

PROBLEMA 1

30. Determinare variationes quantitatum *p, q, r, s* etc. rationem differentialium ipsarum *x* et *y* in se continentium.

SOLUTIO

Quia variatio non ad *x* pertinere censetur, erit $\delta x = 0$ et variatio ipsius *y*, nempe δy, tanquam cognita spectatur. Hinc cum sit $p = \frac{dy}{dx}$, erit $\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$

Deinde ob $q = \frac{dp}{dx}$ erit

$$\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx};$$

sumto autem elemento *dx* constante est $d\delta p = \frac{dd\delta y}{dx}$ hincque

$$\delta q = \frac{dd\delta y}{dx^2} \quad \text{et} \quad d\delta q = \frac{d^3\delta y}{dx^2}$$

Porro autem, cum sit $r = \frac{dq}{dx}$, erit

$$\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx} \quad \text{ideoque} \quad \delta r = \frac{d^3\delta y}{dx^3}$$

unde variationes quantitatum ex *x* et *y* derivatarum *p, q, r, s* etc. ita se habebunt

$$\delta p = \frac{\delta dy}{dx}, \ \delta q = \frac{d^2\delta y}{dx^2}, \ \delta r = \frac{d^3\delta y}{dx^3} = \frac{d\delta q}{dx}, \ \delta s = \frac{d^4\delta y}{dx^4} \ \text{etc.,}$$

siquidem elementum *dx* pro constante assumatur.

COROLLARIUM 1

31. Haec differentialia primi altiorumque graduum variationis δy determinantur per variationes valorum ipsius *y*, qui conveniunt sequentibus valoribus ipsius *x*, scilicet $x+dx,\ x+2dx,\ x+3dx$ etc. Si enim sequentes valores ipsius *y* ita exhibeantur: $y^{/},\ y^{//},\ y^{///},\ y^{////}$etc. eorumque variationes ita: $\delta y^{/},\ \delta y^{//},\ \delta y^{///},\ \delta y^{////},$

ex natura differentialium novimus esse

$$\begin{aligned} &d\delta y = \delta y^{/} - \delta y, \\ &dd\delta y = \delta y^{//} - 2\delta y^{/} + \delta y, \\ &d^3\delta y = \delta y^{//} - 3\delta \mathrm{y}^{//} + 3\delta y^{/} - \delta y \ \text{ etc.} \end{aligned}$$

COROLLARIUM 2

32. Si ergo solus valor *y* variationem pateretur, sequentes vero $y^{/},\ y^{//},\ y^{///}$ nulli essent obnoxiae, ut esset $\delta y^{/} = 0,\ \delta y^{//} = 0,\ \delta y^{///} = 0$ etc., foret

$$d\delta y = -\delta y,\ \ dd\delta y = +\delta y,\ \ d^3\delta y = -\delta y, d^4\delta y = +\delta y \ \text{ etc.}$$

ideoque:

$$\delta p = -\frac{\delta y}{dx},\ \ \delta q = +\frac{\delta y}{dx^2},\ \ \delta r = -\frac{\delta y}{dx^3},\ \ \delta s = +\frac{\delta y}{dx^4}, \text{etc.}$$

PROBLEMA 2

33. Si *V* fuerit quantitas quomodocunque ex variabilibus *x* et *y* earumque differentialibus cuiuscunque ordinis conflata seu fi fuerit functio quaecunque quantitatum *x, y, p, q, r, s* etc., determinare eius variationem δV .

SOLUTIO

Differentietur more consueto haec functio *V* prodeatque

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

quod differentiale nil aliud est, nisi incrementum, quod functio *V* capit, si loco quantitatum *x, y, p, q, r, s* etc. substituantur istae $x+dx,\ y+dy,\ p+dp,\ q+dq,\ r+dr$ etc. Simili ergo modo, si pro *x, y, p, q, r, s* etc. substituantur

$$x+0,\ y+\delta y,\ p+\delta p,\ q+\delta q,\ r+\delta r,\ s+\delta s \ \text{ etc.},$$

incrementum, quod inde functio *V* capit, erit eius variatio

$$\delta V = N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc}\,.$$

Quare, si pro δp, δq, δr etc. valores supra inventi scribantur, prodibit variatio quaesita

$$\delta V = N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^2} + \frac{Sd^4\delta y}{dx^4} \text{ etc.}$$

THEOREMA 2

34. Proposita formula integrali quacunque $\int Zdx$ eius variatio aequalis est integrali variationis differentialis *Zdx* seu erit

$$\delta\int Zdx = \int \delta Zdx\,.$$

DEMONSTRATIO

Cum $\int Zdx$ exprimat summam omnium *Zdx*, eius variatio $\delta\int Zdx$ comprehendet summam omnium variationum ipsius *Z dx* seu erit $\delta\int Zdx = \int \delta Zdx$. Quod etiam hoc modo distinctius ostendi potest: sit $\int Zdx = V$, ita ut definiri oporteat δV *;* cum igitur sit $dV = Zdx$, erit $\delta dV = \delta Zdx = d\delta V$, unde sumtis integralibus fiet $\delta V = \int \delta Zdx.$.

PROBLEMA 3

35. Proposita formula integrali $\int Zdx$, in qua *Z* quantitas quomodocunque ex ipsis *x* et *y* earumque differentialibus cuiuscunque ordinis conflata, investigare eius variationem $\delta\int Zdx$.

SOLUTIO

Cum ergo *Z* sit functio ipsarum *x, y, p, q, r, s* etc., eius differentiale more consueto sumtum huiusmodi formam habebit

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.,}$$

unde eiusdem quantitatis *Z* variatio erit

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} \text{ etc.}$$

Cum nunc sit $\delta\int Zdx = \int \delta Zdx$, erit

$$\delta\int Zdx = \int N\delta ydx + \int Pd\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{ etc.};$$

ne iam in ulteriori reductione expressio δy turbet, ponamus tantisper $\delta y = \omega$, et reductiones ita se habebunt

$$\int Pd\omega = P\omega - \int \omega dP$$

$$\int \frac{Qddw}{dx} = \frac{Qd\omega}{dx} - \int \frac{dQ}{dx} d\omega = \frac{Qd\omega}{dx} - \frac{\omega dQ}{dx} + \int \frac{\omega ddQ}{dx}$$

$$\int \frac{Rd^3\omega}{dx^2} = \frac{Rdd\omega}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{\omega ddR}{dx^2} - \int \frac{\omega d^3R}{dx^2}$$

etc.

Colligantur omnes isti valores et pro ω restituatur δy, sicque obtinebitur

$$\begin{aligned}\delta\int Zdx = \int \delta ydx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{ etc.}\right) \\ + \delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right) \\ + \frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right) \\ + \frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)\end{aligned}$$

in qua expressione differentiale *dx* sumtum est constans.

COROLLARIUM 1

36. Constat ergo variatio formulae integralis $\int Zdx$ ex parte integrali

$$\int \delta y dx \left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{ etc.} \right)$$

et partibus absolutis, quae praeter ipsam variationem δy etiam eius differentialia $d\delta y,\ dd\delta y,\ d^3\delta y$ etc. complectuntur.

COROLLARIUM 2

37. Partem autem integralem per reductiones adhibitas ita instruximus, ut tantum ipsam variationem δy complecteretur ab eiusque differentialibus immunis exhiberetur, quae forma in applicatione calculi variationum maximam praestat utilitatem.

PROBLEMA 4

38. Si in formula integrali $\int Zdx$ quantitas Z non solum litteras x et y cum relationibus differentialium p, q, r, s etc., sed etiam formulam integralem $\Pi = \int \mathfrak{Z} dx$ utcunque complectatur, in qua autem sit $\mathfrak{Z}$ functio ipsarum x, y, p, q, r, s etc., definire variationem formulae illius integralis $\int Zdx$.

SOLUTIO

Cum quantitas Z praeter quantitates x, y, p, q, r, s etc. etiam formulam integralem $\Pi = \int \mathfrak{Z} dx$ involvat, spectari poterit tanquam functio quantitatum $\Pi,\ x,\ y,\ p,\ q,\ r,\ s$ etc., unde, si more solito differentietur, prodibit talis forma

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.},$$

unde colligitur variatio ipsius Z

$$\delta Z = L\delta\Pi + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Cum deinde sit $\mathfrak{Z}$ functio ipsarum x, y, p, q, r, s etc., ponatur

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + \text{etc.}$$

atque ex praecedente problemate erit $\delta\Pi$ seu

$$\delta\int \mathfrak{Z}dx = \int \delta ydx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{ etc.}\right)$$
$$+\delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right)$$
$$+\frac{d^3\delta y}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)$$
$$+\text{ etc.}$$

Vel sumatur potius prior forma

$$\delta\int \mathfrak{Z}dx = \int \mathfrak{N}\delta ydx + \int \mathfrak{P}d\delta y + \int \frac{\mathfrak{Q}dd\delta y}{dx} + \int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}$$

eritque ob $\delta\Pi = \delta\int \mathfrak{Z}dx$

$$\delta Z = L\int \mathfrak{N}\delta ydx + L\int \mathfrak{P}d\delta y + L\int \frac{\mathfrak{Q}dd\delta y}{dx} + L\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + L\int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}$$
$$+N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{ etc.}$$

Cum igitur sit $\delta\int Zdx = \int \delta Zdx$, habebimus:

$$\delta\int Zdx = \int Ldx\int \mathfrak{N}\delta ydx + \int Ldx\int \mathfrak{P}d\delta y + \int Ldx\int \frac{\mathfrak{Q}dd\delta y}{dx}$$
$$+\int Ldx\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \int Ldx\int \frac{\mathfrak{S}d^4\delta y}{dx^3} + \text{ etc.}$$
$$+\int N\delta ydx + \int Pd\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{ etc.}$$

Ponatur $\int Ldx = W$, seu $Ldx = dW$, et ob

$$\int Ldx\int \mathfrak{N}\delta ydx = W\int \mathfrak{N}\delta ydx - \int \mathfrak{N}W\delta ydx$$
$$\int Ldx\int \mathfrak{P}d\delta y = W\int \mathfrak{P}d\delta y - \int \mathfrak{P}Wd\delta y$$
$$\int Ldx\int \frac{\mathfrak{Q}dd\delta y}{dx} = W\int \frac{\mathfrak{Q}dd\delta y}{dx} - \int \frac{\mathfrak{Q}Wdd\delta y}{dx}$$

obtinebimus:

$$\delta\int Zdx = W\int \mathfrak{N}\delta ydx + W\int \mathfrak{P}d\delta y + W\int \frac{\mathfrak{Q}dd\delta y}{dx} + W\int \frac{\mathfrak{R}d^3\delta y}{dx^2} + \text{ etc.}$$
$$+\int(N-\mathfrak{N}W)\delta ydx + \int(P-\mathfrak{P}W)d\delta y + \int(Q-\mathfrak{Q}W)\frac{dd\delta y}{dx} + \int(R-\mathfrak{R}W)\frac{d^3\delta y}{dx^2} + \text{ etc.}$$

Hae formulae eodem modo ut supra reductae dabunt:

$$\delta\int Zdx = W\int \delta ydx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{ etc.}\right)$$
$$+W\delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \text{ etc.}\right)$$
$$+\frac{Wd\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \text{ etc.}\right)$$
$$+\frac{Wd\delta y}{dx^2}(\mathfrak{R} - \text{ etc.})$$
$$+\int \delta ydx\left((N-\mathfrak{N}W) - \frac{d(P-\mathfrak{P}W)}{dx} + \frac{dd(Q-\mathfrak{Q}W)}{dx^2} - \frac{d^3(R-\mathfrak{R}W)}{dx^3} + \text{ etc.}\right)$$
$$+\delta ydx\left((P-\mathfrak{P}W) - \frac{d(Q-\mathfrak{Q}W)}{dx} + \frac{dd(R-\mathfrak{R}W)}{dx^2} - \text{ etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q-\mathfrak{Q}W) - \frac{d(R-\mathfrak{R}W)}{dx} + \text{ etc.}\right)$$
$$+\text{etc.}$$

COROLLARIUM 1

39. Quia reductiones adhibitae quovis casu facile expediri possunt, iis praetermissis variatio quaesita hoc modo succinctius exhibitur posito $W = \int Ldx$:

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \frac{\mathfrak{P}d\delta y}{dx} + \frac{\mathfrak{Q}dd\delta y}{dx^2} + \frac{\mathfrak{R}d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+\int dx\left((N-\mathfrak{N}W)\delta y + (P-\mathfrak{P}W)\frac{d\delta y}{dx} + (Q-\mathfrak{Q}W)\frac{dd\delta y}{dx^2} + (R-\mathfrak{R}W)\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$

COROLLARIUM 2

40. Ac si quantitas Z involvat insuper aliam formulam integralem $\varPi' = \int \mathfrak{Z}'dx$, ut sit:

$$dZ = Ld\varPi + Ld\varPi + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

tum vero:

$$d\mathfrak{Z}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

si ponatur $\int Ldx = W$, $\int L'dx = W'$ insuperque ad abbreviandum:

$$N - \mathfrak{N}W - \mathfrak{N}'W' = (N);\; P - \mathfrak{P}W - \mathfrak{P}'W' = (P)$$
$$Q - \mathfrak{Q}W - \mathfrak{Q}'W' = (Q);\; R - \mathfrak{R}W - \mathfrak{R}'W' = (R) \text{ etc.},$$

erit variatio quaesita:

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+W'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\frac{d\delta y}{dx} + \mathfrak{Q}'\frac{dd\delta y}{dx^2} + \mathfrak{R}'\frac{d^3\delta y}{dx^3} + \text{ etc.}\right) .$$
$$+\int dx\left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{ etc.}\right).$$

PROBLEMA 5

41. Si in formula $\int Zdx$ quantitas Z praeter litteras *x, y, p, q, r* etc. involvat formulam integralem $\varPi = \int \mathfrak{Z}dx$, in qua quantitas $\mathfrak{Z}$ praeter litteras *x, y, p, q, r* etc. insuper complectatur formulam integralem $\pi = \int \mathfrak{z}dx$, ubi $\mathfrak{z}$ autem sit functio solarum litterarum *x, y, p, q, r* etc., invenire variationem formulae $\int Zdx$.

SOLUTIO

Cum Z sit functio quantitatum *x, y, p, q, r* etc. et $\varPi = \int \mathfrak{Z}dx$, eius differentiale more consueto sumtum erit huius formae:

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

ideoque eius variatio

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \text{ etc.}$$

unde variatio quaesita erit

$$\delta\int Zdx = \int \delta Zdx$$
$$= Ldx\delta\Pi + \int dx\left(N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \text{ etc.}\right).$$

At cum sit $\mathfrak{Z}$ functio quantitatum *x, y, p, q, r* etc. et $\pi = \int \mathfrak{z}dx$, erit differentiando:

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

hincque variatio eius

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}.$$

quare, cum sit $\Pi = \int \mathfrak{Z}dx$, erit $\delta\Pi = \delta\int \mathfrak{Z}dx = \int \delta\delta\mathfrak{Z}dx$ ac propterea:

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}$$

$$\delta\Pi = \int \mathfrak{L}dx\delta\pi + \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right).$$

unde reperitur:
$\int Ldx\delta\Pi = \int Ldx\int \mathfrak{L}dx\delta\pi + \int Ldx\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$. Superest ergo ut definiamus $\delta\pi$ est autem $\pi = \int \mathfrak{z}dx$, et quia $\mathfrak{z}$ est functio litterarum *x, y, p, q, r* etc. tantum, fiat differentiando:

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

ex quo concluditur eius variatio:

$$\delta\mathfrak{z} = \mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.},$$

tum vero ob $\delta\pi = \delta\int \mathfrak{z}dx = \int \delta\mathfrak{z}dx$ erit:

$$\delta\pi = \int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right),$$

Quamobrem habebimus

$$\int Ldx\int \mathfrak{L}dx\delta\pi = \int Ldx\int \mathfrak{L}dx\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right).$$

Ut iam hanc formulam a signis integralibus multiplicatis liberemus, ponamus $\int Ldx = W$ eritque:

$$\int Ldx\delta\Pi = W\delta\Pi - \int Wd\delta\Pi,$$

verum $d\delta\Pi = \delta\mathfrak{Z}dx,$ unde

$$\int Ldx\delta\Pi = W\delta\Pi - \int W\delta\mathfrak{Z}dx$$

ideoque:

$$\begin{aligned}\int Ldx\delta\Pi &= W\int \mathfrak{L}dx\delta\pi + W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.} \right) \\ &\quad - \int \mathfrak{L}Wdx\delta\pi - \int Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.} \right)\end{aligned}$$

Sit $\int \mathfrak{L}dx = \mathfrak{W}$, erit

$$\int \mathfrak{L}dx\delta\pi = \mathfrak{W}\delta\pi - \int \mathfrak{W}\delta\mathfrak{z}dx$$

hincque:

$$\begin{aligned}\int \mathfrak{L}dx\delta\pi &= \mathfrak{W}\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.} \right) \\ &\quad - \int \mathfrak{W}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.} \right).\end{aligned}$$

Porro ponatur

$$\int \mathfrak{L}Wdx = \int Wd\mathfrak{W} = \mathfrak{V},\ \text{ut sit :}$$

$$\int \mathfrak{L}Wdx = \mathfrak{V}\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$-\int \mathfrak{V}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right).$$

Ex his omnibus colligetur variatio quaesita $\delta\int Zdx$

$$= \left(W\mathfrak{W} - \mathfrak{V}\right)\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$-W\int \mathfrak{W}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+\int \mathfrak{V}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{ etc.}\right)$$
$$+W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$-\int Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$+\int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{ etc.}\right).$$

COROLLARIUM 1

42. Si quaeratur variatio formulae $\int Zdx$ a valore $x = 0$ usque ad valorem determinatum $x = a$, sumantur integralia $W = \int Ldx,\ \mathfrak{W} = \int \mathfrak{L}dx$ et $\mathfrak{V} = \int Wd\mathfrak{W}$, ita ut evanescant posito $x = 0$, tum vero facto $x = a$ fiat $W = A,\ \mathfrak{W} = \mathfrak{A}$ et $\mathfrak{V} = \mathfrak{B}$, quos valores in formula inventa loco litterarum W, $\mathfrak{W}$ et $\mathfrak{V}$, ubi extra signum integrale occurrunt, ponere licebit.

COROLLARIUM 2

43. Ponatur ergo ad abbreviandum:

$$N+(A-W)\mathfrak{N}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{n}=(N)$$
$$P+(A-W)\mathfrak{P}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{p}=(P)$$
$$Q+(A-W)\mathfrak{Q}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{q}=(Q)$$
$$R+(A-W)\mathfrak{R}+(A\mathfrak{A}-\mathfrak{B}-A\mathfrak{W}+\mathfrak{V})\mathfrak{r}=(R)$$
etc.

et variatio quaesita formulae $\int Zdx$ usque ad valorem determinatum $x=a$ erit;

$$\int dx\left((N)\delta y+(P)\frac{d\delta y}{dx}+(Q)\frac{dd\delta y}{dx^2}+(R)\frac{d^3\delta y}{dx^3}+\text{ etc.}\right).$$

COROLLARIUM 3

44. Quodsi iam hic reductiones superiores adhibeantur, reperietur eadem variatio ita expressa :

$$\delta\int Zdx=\int dx\delta y\left((N)-\frac{d(P)}{dx}+\frac{dd(Q)}{dx^2}-\frac{d^3(R)}{dx^3}+\text{ etc.}\right)$$
$$+\delta y\left((P)-\frac{d(Q)}{dx}+\frac{dd(R)}{dx^2}-\text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left((Q)-\frac{d(R)}{dx}+\text{etc.}\right)$$
$$+\frac{dd\delta y}{dx^2}\left((R)-\text{etc.}\right)$$
$$+\text{ etc.}$$

COROLLARIUM 4

45. Cum sit $\mathfrak{V}=\int Wd\mathfrak{W}$, erit $A\mathfrak{W}-\mathfrak{V}=\int(A-W)\mathfrak{L}dx$; quare, si ponatur integrale $\int(A-W)\mathfrak{L}dx=X$, ita sumtum, ut evanescat posito $x=0$, tum vero facto $x=a$ fiat $X=B$, ita ut sit:

$$\int\mathfrak{L}dx=W,\text{ et posito } x=a \text{ fiat } W=A,$$
$$\int(A-W)\mathfrak{L}dx=X,\text{ et posito } x=a \text{ fiat } X=B,$$

superiores valores Corollario 2 exhibiti ita se habebunt:

$$N+(A-W)\mathfrak{N}+(B-X)\mathfrak{n}=(N)$$
$$P+(A-W)\mathfrak{P}+(B-X)\mathfrak{p}=(P)$$
$$Q+(A-W)\mathfrak{Q}+(B-X)\mathfrak{q}=(Q)$$
$$R+(A-W)\mathfrak{R}+(B-X)\mathfrak{r}=(R)$$
etc.

PROBLEMA 6

46. Si in formula integrali $\Phi=\int Zdx$ quantitas *Z* praeter litteras *x, y, p, q, r* etc. etiam ipsam formulam integralem Φ involvat, determinare eius variationem $\delta\Phi=\delta\int Zdx$.

SOLUTIO

Cum *Z* sit functio quantitatum *x, y, p, q, r* etc. insuperque ipsam formulam integralem $\Phi=\int Zdx$ involvat, differentietur more soli to ac prodeat

$$dZ=Ld\Phi+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.}$$

Hinc igitur erit variatio ipsius *Z* scilicet

$$\delta Z=L\delta\Phi+N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+R\frac{d^3\delta y}{dx^3}+\text{etc.}$$

ideoque ob $\delta\Phi=\delta\int Zdx=\int\delta Zdx$

$$\delta\Phi=\int Ldx\delta\Phi+\int dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+R\frac{d^3\delta y}{dx^3}+\text{etc.}\right).$$

Ponamus *of* $\delta\Phi=z$, cum sit id ipsum, quod quaeritur, et brevitatis gratia

$$\int dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+\text{etc.}\right)=u,$$

ut habeatur $z=\int Lzdx+u$ et differentiando $dz=Lzdx+du$, eritque integrando

$$z=e^{\int Ldx}\int e^{-\int Ldx}$$

statuatur brevitatis gratia $\int Ldx=W$ et habebitur variatio quaesita

$$\delta\int Zdx = e^{W}\int e^{-W}dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

Si desideretur variatio usque ad datum terminum $x = a$ fiatque tum $W = A$, ponatur ad abbreviandum

$$e^{A-W}N = (\mathrm{N}),\quad e^{A-W}P = (\mathrm{P})\ ,\quad e^{A-W}Q = (\mathrm{Q})\ \text{etc.}$$

eritque reductionibus ut supra factis variatio

$$\begin{aligned}\delta\int Zdx = \int dx\delta y&\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right)\\ &+\delta y\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right)\\ &+\frac{d\delta y}{dx}\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right)\\ &+\frac{d\delta y}{dx}\left((R) - \text{etc.}\right)\\ &+\ \text{etc.}\end{aligned}$$

COROLLARIUM

47. Si ergo quantitas varianda Φ definiatur per hanc aequationem differentialem $d\Phi = Zdx$, in qua Z involvat utcunque ipsam quantitatem Φ et insuper litteras *x, y, p, q, r* etc., eius variatio $\delta\Phi$ per hoc problema assignari poterit.

PROBLEMA 7

48. Si in formula integrali $\Phi = \int Zdx$ quantitas *Z* praeter litteras *x, y, p, q, r* etc. non solum ipsam quantitatem Φ, sed insuper adhuc aliam formulam integralem $\Pi = \int \mathfrak{Z}dx$ quomodocunque implicet, in qua autem quantitas $\mathfrak{Z}$ tantum per litteras *x, y, p, q, r* etc. detur, investigare variationem huius formulae $\delta\Phi = \delta\int Zdx = \int \delta Zdx$.

SOLUTIO

Cum Z sit functio quantitatum x, y, p, q, r etc. insuperque formularum $\Phi = \int Zdx$ et $\Pi = \int \mathfrak{Z}dx$, praebeat ea differentiando

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

unde eius variatio erit

$$\delta Z = K\delta\Phi + L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{dd\delta y}{dx^2} + \text{etc.}$$

Porro autem, cum $\mathfrak{Z}$ sit functio litterarum x, y, p, q, r etc. tantum, ponatur

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

eritque ob $\delta\Pi = \int \delta\mathfrak{Z}dx$

$$d\Pi = \int dx\left(\mathfrak{N}dy + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

Ponatur ut ante

$$\delta\Phi = z \text{ et } L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + \frac{dd\delta y}{dx^2} + \text{etc.} = u\ ;$$

ob $\delta\Phi = \int \delta Zdx = z$ erit $\delta Z = \frac{dz}{dx}$ ideoque $\frac{dz}{dx} = Kz + u$; unde oritur

$$z = e^{\int Kdx}\int e^{-\int Kdx}udx = \delta\Phi\ ;$$

sit $\int Kdx = V$, eritque

$$z = e^{-\int Kdx}udx = e^{-V}Ldx\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$
$$- e^{-V}dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{ etc.}\right)$$

statuatur porro $\int e^{-V}Ldx = W$, eritque integrando variatio quaesita

$$\begin{aligned}\delta\Phi = e^{V}W\int dx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{ etc.}\right)\\ -e^{V}\int Wdx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{ etc.}\right)\\ +e^{V}\int e^{-V}dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+\text{ etc.}\right).\end{aligned}$$

Si variatio usque ad datum terminum $x=a$ desideretur ac posito $x=a$ fiat $V=A$ et $W=B$, tum statuatur brevitatis gratia:

$$\begin{aligned}e^{A-V}N+e^{A}\left(B-W\right)\mathfrak{N}&=\left(N\right)\\ e^{A-V}P+e^{A}\left(B-W\right)\mathfrak{P}&=\left(P\right)\\ e^{A-V}Q+e^{A}\left(B-W\right)\mathfrak{Q}&=\left(Q\right)\\ e^{A-V}R+e^{A}\left(B-W\right)\mathfrak{R}&=\left(R\right)\\ &\text{etc.}\end{aligned}$$

quo facto erit variatio formulae $\Phi=\int Zdx$ usque ad terminum $x=a$ extensa:

$$\begin{aligned}\delta\Phi=\int dx\delta y\left(\left(N\right)-\frac{d\left(P\right)}{dx}+\frac{dd\left(Q\right)}{dx^2}-\frac{d^3\left(R\right)}{dx^3}+\text{etc.}\right)\\ +\delta y\left(\left(P\right)-\frac{d\left(Q\right)}{dx}+\frac{dd\left(R\right)}{dx^2}-\text{etc.}\right)\\ +\frac{d\delta y}{dx}\left(\left(Q\right)-\frac{d\left(R\right)}{dx}+\text{etc.}\right)\\ +\frac{d\delta y}{dx}\left(\left(R\right)-\text{etc.}\right)\end{aligned}$$

COROLLARIUM

49. Sic ergo variatio definitur quantitatis Φ per aequationem differentialem $d\Phi=Zdx$ datae, in qua Z non solum praeter litteras x, y, p, q, r etc. ipsam Φ, sed insuper formulam integralem $\int \mathfrak{Z}dx=\Pi$ utcunque involvit, dummodo $\mathfrak{Z}$ per solas litteras x, y, p, q, r etc. determinetur.

PROBLEMA 8

50. Si in formula integrali $\Phi = \int Zdx$ quantitas Z praeter litteras x, y, p, q, r etc. formulam integralem $\Pi = \int \mathfrak{Z}dx$ involvat, hic autem quantitas $\mathfrak{Z}$ praeter litteras x, y, p, q, r etc. ipsam formulam integralem $\Pi = \int \mathfrak{Z}dx$ contineat, definire variationem formulae propositae $\Phi = \int Zdx$.

SOLUTIO

Cum Z sit functio quantitatum x, y, p, q, r etc. et ipsius $\Pi = \int \mathfrak{Z}dx$, eius differentiale erit huiusmodi:

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

hinc eius variatio erit

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.},$$

ex quo ob $\delta\Phi = \int \delta Zdx$ habebitur:

$$\delta\Phi = \int Ldx\delta\Pi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

At quia $\mathfrak{Z}$ est functio ipsarum x, y, p, q, r etc. et $\Pi = \int \mathfrak{Z}dx$, sit eius differentiale:

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \text{etc.}$$

eritque

$$\delta\mathfrak{Z} = \frac{d\delta\Pi}{dx} = \mathfrak{L}\delta\Pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}$$

Ponatur $\int \mathfrak{L}dx = \mathfrak{W}$, eritque:

$$\delta\Pi = e^{\mathfrak{W}}\int e^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$

Fiat $\int e^{\mathfrak{W}}Ldx = W$ et obtinebitur:

$$\delta\Phi = W\int e^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$-\int e^{-\mathfrak{W}}Wdx\left(\mathfrak{N}\delta y+\mathfrak{P}\frac{d\delta y}{dx}+\mathfrak{Q}\frac{dd\delta y}{dx^2}+\text{etc.}\right)$$
$$+\int dx\left(N\delta y+P\frac{d\delta y}{dx}+Q\frac{dd\delta y}{dx^2}+\text{etc.}\right).$$

Si hanc variationem ad terminum $x=a$ usque extendi oporteat ac posito $x=a$ fiat $W=A$, vocetur brevitatis gratia

$$N+e^{-\mathfrak{W}}\left(A-W\right)\mathfrak{N}=\left(N\right)$$
$$P+e^{-\mathfrak{W}}\left(A-W\right)\mathfrak{P}=\left(P\right)$$
$$Q+e^{-\mathfrak{W}}\left(A-W\right)\mathfrak{Q}=\left(Q\right)$$
$$\text{etc.}$$

eritque reductiones supra expositas introducendo variatio formulae integralis $\Phi=\int Zdx$ ad terminum $x=a$ extensa

$$\delta\int Zdx=\int dx\delta y\left(\left(N\right)-\frac{d\left(P\right)}{dx}+\frac{dd\left(Q\right)}{dx^2}-\frac{d^3\left(R\right)}{dx^3}+\text{etc.}\right)$$
$$+\delta y\left(\left(P\right)-\frac{d\left(Q\right)}{dx}+\frac{dd\left(R\right)}{dx^2}-\text{etc.}\right)$$
$$-\frac{d\delta y}{dx}\left(\left(Q\right)-\frac{d\left(R\right)}{dx}+\text{etc.}\right)$$
$$+\frac{d\delta y}{dx}\left(\left(R\right)-\text{etc.}\right)$$
$$\text{etc.}$$

SCHOLION

51. Usus huius problematis cernitur in descensu corporum super lineis curvis in medio quocunque resistente, dum corpora a viribus quibuscunque sollicitantur, si variationem temporis descensus definire velimus, dum curva quomodocunque variatur. Denotet hoc casu Φ tempus descensus per arcum, qui abscissae x respondeat, sitque applicata $=y$ et Π altitudo celeritati acquisitae debita; ac tempus descensus erit

$$\varPhi = \int \frac{dx\sqrt{(1+pp)}}{\sqrt{\varPi}}$$

posito $dy = pdx$, ut $dx\sqrt{(1+pp)}$ elementum arcus designet. Verum ex sollicitationibus erit

$$d\varPi = Xdx + Ydy - V\sqrt{(dx^2 + dy^2)},$$

ubi X et Y significant functiones ipsarum x et y, et V functionem ipsius $\varPi$, cui resistentia est proportionalis. Erit ergo ob $dy = pdx$

$$\varPi = \int\left(X + Yp - V\sqrt{(1+pp)}\right)dx$$

ideoque

$$\mathfrak{Z} = X + Yp - V\sqrt{(1+pp)}$$

existente $Z = \dfrac{\sqrt{(1+pp)}}{\sqrt{\varPi}}$.

COROLLARIUM

52. Si ad similitudinem valorum (N), (P), (Q) ponatur

$$M + e^{-\mathfrak{W}}(A - W)\mathfrak{M} = (M),$$

erit $(M)dx + (N)dy + (P)dp + (Q)dq + (R)dr + \text{etc.}$, differentiale verum huius formulae :

$$Z + e^{-\mathfrak{W}}(A - W)\mathfrak{Z}.$$

CONCLUSIO

53. Quaecunque ergo formula integralis $\varPhi = \int Zdx$ proponatur, cuius variationem investigari oporteat, eius variatio usque ad terminum $x = a$ extensa semper exprimetur hoc modo

$$\begin{aligned}\delta\Phi = \int dx\delta y & \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \frac{d^4(R)}{dx^4} - \text{etc.} \right) \\ & + \delta y \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \frac{d^3(S)}{dx^3} + \text{etc.} \right) \\ & + \frac{d\delta y}{dx} \left((Q) - \frac{d(R)}{dx} + \frac{dd(S)}{dx^2} - \text{etc.} \right) \\ & + \frac{d\delta y}{dx} \left((R) - \frac{d(S)}{dx} + \text{etc.} \right) \\ & + \frac{d^3\delta y}{dx^3} \left((S) - \text{etc.} \right) \\ & \quad \text{etc.}\end{aligned}$$

sumto elemento *dx* constante. Quemodmodum autem litterae (N), (P), (Q), (R), (S) etc. se habeant, id quovis casu patebit.

CASUS I

54. Si $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds$ etc., erit

$$(N) = N,\ (P) = P,\ (Q) = Q,\ (R) = R,\ (S) = S \ \text{ etc.}$$

CASUS II

55. Si $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$ existente

$$\Pi = \int \mathfrak{Z}dx \text{ et } d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}\mathrm{dp} + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

sit $\int Ldx = W$ ac posito $x = a$ fiat $W = A$, quo facto erit :

$$\begin{aligned} (N) &= N + (A - W)\mathfrak{N} & (P) &= P + (A - \mathrm{W})\mathfrak{P} \\ (Q) &= Q \ + (A - W)\mathfrak{Q} & (R) &= R + (A - W)\mathfrak{R} \\ (S) &= S \ \ + (A - W)\mathfrak{S} & & \text{etc.} \end{aligned}$$

CASUS III

56. Si fuerit

$$dZ = Ld\Pi + L'd\Pi' + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

existente $\Pi = \int \mathfrak{Z}dx$ et $\Pi' = \int \mathfrak{Z}'dx$, tum vero :

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$
$$d\mathfrak{Z}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

ponatur $\int Ldx = W$ et $\int L'dx = W'$, ac facto $x = a$ $W = A$ fiat et $W' = A'$, quo facto erit:

$$(N) = N + (A - W)\mathfrak{N} + (A' - W')\mathfrak{N}'$$
$$(P) = P + (A - W)\mathfrak{P} + (A' - W')\mathfrak{P}'$$
$$(Q) = Q + (A - W)\mathfrak{Q} + (A' - W')\mathfrak{Q}'$$
$$(R) = R + (A - W)\mathfrak{R} + (A' - W')\mathfrak{R}'$$
etc.

CASUS IV

57. Si Z contineat formulam integralem $\Pi = \int \mathfrak{Z}dx$, ut sit :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

quantitas $\mathfrak{Z}$ vero formulam integralem $\pi = \int \mathfrak{z}dx$, ut sit:

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

at $\mathfrak{z}$ nullum porro integrale involvat, ita ut sit:

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.}$$

Ponatur $\int Ldx = W$ et posito $x = a$ fiat $W = A$; tum vero ponatur $\int (A - W)\mathfrak{L}dx = \mathfrak{W}$ casuque $x = a$ fiat $\mathfrak{W} = \mathfrak{A}$, quo facto erit:

$$(N) = N + (A - W)\mathfrak{N} + (\mathfrak{A} - \mathfrak{W})\mathfrak{n}$$
$$(P) = P + (A - W)\mathfrak{P} + (\mathfrak{A} - \mathfrak{W})\mathfrak{p}$$
$$(Q) = Q + (A - W)\mathfrak{Q} + (\mathfrak{A} - \mathfrak{W})\mathfrak{q}$$
$$(R) = R + (A - W)\mathfrak{R} + (\mathfrak{A} - \mathfrak{W})\mathfrak{r}$$
etc.

CASUS V

58. Si Z contineat ipsam formulam $\Phi = \int Zdx$, ut sit:

$$dZ = Kd\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

ponatur $\int Kdx = V$ et facto $x = a$ sit $V = C$, erit:

$$(N) = e^{C-V}N,\ (P) = e^{C-V}P,\ (Q) = e^{C-V}Q,\ (R) = e^{C-V}R \text{ etc.}$$

CASUS VI

59. Si Z praeter formulam $\Phi = \int Zdx$ contineat aliam formulam integralem $\Pi = \int \mathfrak{Z}dx$, sitque :

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

tum vero $\mathfrak{Z}$ nullam formulam integralem involvat :

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

sit $\int Kdx = V$ et posito $x = a$ fiat $V = C$. Deinde sit $\int e^{C-V}Ldx = W$ et posito $x = a$ fiat $W = A$, eritque :

$$(N) = e^{C-V}N + (A-W)\mathfrak{N}$$
$$(P) = e^{C-V}P + (A-W)\mathfrak{P}$$
$$(Q) = e^{C-V}Q + (A-W)\mathfrak{Q}$$
$$(R) = e^{C-V}R + (A-W)\mathfrak{R}$$
etc.

CASUS VII

60. Si Z contineat formulam $\Pi = \int \mathfrak{Z}dx$, ut sit :

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

tum vero $\mathfrak{Z}$ denuo eandem formulam $\Pi = \int \mathfrak{Z}dx$ involvat, ut sit :

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

Ponatur $\int \mathfrak{L}dx = \mathfrak{W}$ et posito $x = a$ fiat $\mathfrak{W} = \mathfrak{A}$; deinde ponatur

$$\int e^{-\mathfrak{A}+\mathfrak{W}} Ldx = W$$

et posito $x = a$ fiat $W = A$, eritque

$$\begin{aligned}
(N) &= N + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{N} \\
(P) &= P + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{P} \\
(Q) &= Q + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{Q} \\
(R) &= R + e^{\mathfrak{A}-\mathfrak{W}}(A-W)\mathfrak{R}
\end{aligned}$$

etc.

61. Simili modo hanc investigationem ad alias formulas complicatas extendere licet, verum cum tales vix unquam occurrere soleant, labor superfluus foret. Cum igitur formularum integralium tam simpliciorum quam magis compositarum variationes definire docuerim, calculus variationum fere penitus absolutus videtur ; quomodocunque enim quantitas varianda fuerit, tam ex formulis absolutis quam integralibus confiata, ope differentiationis ordinariae eius variatio reperiri poterit. Veluti si quantitas varianda U contineat formulas integrales quascunque

$$\Phi = \int Zdx, \quad \Phi' = \int Z'dx, \quad \Phi'' = \int Z''dx \quad \text{etc.},$$

differentietur ea more solito prodeatque :

$$dU = Kd\Phi + K'd\Phi' + K''d\Phi'' \quad \text{etc.}$$

tum evidens est fore eius variationem:

$$\delta U = K\delta\Phi + K'\delta\Phi' + K''\delta\Phi'' \quad \text{etc.},$$

at variationes $\delta\Phi$, $\delta\Phi'$, $\delta\Phi''$ etc. per praecepta modo exposita assignabuntur. Simul vero patet variationem δU semper huiusmodi forma expressum iri, ut sit:

$$\delta V = \int (A)dx\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} + \text{ etc.},$$

ubi (A), (B), (C) etc. sunt functiones ex regulis supra traditis inveniendae. Istius autem calculi variationum usum in solutione celeberrimi problematis isoperimetrici, latissima significatione ACCEPTIi, breviter indicari conveniet.

APPLICATIO CALCULI VARIATIONUM AD SOLUTIONEM PROBLEMATIS ISOPERIMETRICI LATISSIMA SIGNIFICATIONE ACCEPTI

62. Problema primarium huc spectans ita enunciari potest, ut inter omnes curvas super eadem data basi $x = a$ extruendas ea definiatur, pro qua formula quaepiam U maximum minimumve valorem obtineat. Etsi enim enunciatio problematis curvas tantum eiusdem longitudinis complectitur, tamen haec conditio commode, ut eius ambitus latius pateat, omittitur neque etiam commemoratione unicae formulae U, cuius valor maximus minimusve evadere debet, eius vis restringi est censenda, postquam in genere demonstravi: si inter omnes curvas super eadem basi $x = a$ extruendas, pro quibus formula V eundem nanciscatur valorem, definiri debeat ea, in qua valor formulae U maximus evadat minimusve, quaestionem huc reduci, ut inter omnes plane curvas super basi $x = a$ extruendas ea definiatur, pro qua haec formula composita $\alpha V + \beta U$ consequatur maximum minimumve valorem. Interim tamen et huius reductionis ratio ex ipsis calculi variationum principiis dilucide explicari potest.

63. Quaestio autem haec a consideratione linearum curvarum remota hoc modo proponi potest :
Proposita formula quacunque U definire eam relationem inter binas variabiles x et y, per quam si valor ipsius U determinetur atque a valore $x = 0$ *usque ad valorem* $x = a$ *extendatur, is proditurus sit sive maximus sive minimus.*

Spectemus ergo relationem inter x et y tanquam iam inventam, ita ut inde oriatur valor ipsius U maximus vel minimus; atque manifestum est, si relatio inter x et y infinite parum immutetur, nullam inde mutationem in valore ipsius U nasci debere; seu, quod eodem redit, variationem ipsius U seu δU nihilo aequalem esse oportere; sicque aequatio $\delta U = 0$ relationem quaesitam inter x et y complectetur.

64. Variationem autem δU inde definire docuimus, quod pro quovis valore ipsius x valorem ipsius y, qui ipsi vi relationis quaesitae competeret, particula quapiam δy augeri assumsimus. Cum igitur relatio quaesita inter omnes plane possibiles hac praerogativa gaudere debeat, variatio δU semper esse debet nihilo aequalis, quomodocunque singuli valores ipsius y talibus particulis δy augeantur et quomodocunque haec augmenta fuerint comparata, quoniam prorsus sunt arbitraria neque ullo modo a se invicem pendentia. Neque etiam opus est, ut omnibus valoribus ipsius y huiusmodi variationes tribuantur, sed sive unicus quispiam sive duo sive quotcunque pro lubitu varientur, semper aeque necesse est, ut variatio, quae inde in totum valorem ipsius U, quatenus is a termino $x = 0$ usque ad terminum $x = a$ extenditur, redundat, in nihilum abeat.

65. Ex iis autem, quae supra sunt tradita, manifestum est variationem ipsius U semper hoc modo exprimi, ut sit :

$$\delta U=\int(A)dx\delta y+(B)\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^2}\text{ etc.},$$

cuius formae singulas partes seorsim considerari convenit. At praeter primum membrum integrale reliquae partes $(B)\delta y,(C)\frac{d\delta y}{dx}$ etc. tantum a variatione ultimi valoris *y*, qui valori *x a* convenit, pendent neque rationem variationis praecedentium implicant; ut enim tota variatio ipsius *U* obtineatur, in expressione inventa ubique statui debet $x=a$, quod in singulis partibus praeter primam actu fieri potest, sicque in iis δy denotabit variationem, quae soli ultimo valori ipsius *y* tribuitur et quae omnino est arbitraria neque a praecedentibus pendet. Unde perspicuum est, nisi membrum integrale adesset, ex reliquis partibus nihil plane pro relatione inter *x* et *y* concludi posse.

66. Verum membrum integrale $\int(A)dx\delta y$ etiam variationes, quae omnibus praecedentibus valoribus ipsius *y* tribuuntur, involvit, dum continet summam omnium elementorum $(A)dx\delta y$ ex variatione singulorum valorum *y* oriundorum. Ita, si unicus eius valor ipsi *x*, quasi determinatum valorem haberet, spectato respondens varietur seu particula δy augeatur, membrum iillud integrale tantum esset $=(A)dx\delta y$ nihilque summandum haberetur; sin autem insuper sequens valor $y^{/}$ ipsi $x+dx$ respondens particula $\delta y^{/}$ augeatur positoque $x+dx$ loco *x* functio (*A*) abeat in $(A)^{/}$, membrum integrale constabit his duabus partibus :

$$(A)dx\delta y+(A)^{/}dx\delta y^{/}.$$

Simili modo, si tres pluresve valores successivi $y,\ y^{/},\ y^{//},\ y^{///},\ y^{////}$ etc. particulis $\delta y,\ \delta y^{/},\ \delta y^{//},\ \delta y^{///}$ etc. augeantur, membrum integrale aequivalebit huic expressioni:

$$(A)dx\delta y+(A)^{/}dx\delta y^{/}+(A)^{//}dx\delta y^{//}+(A)^{///}dx\delta y^{///}\text{ etc.}$$

quae series tam retrorsum usque ad terminum $x=0$ quam antrorsum usque ad terminum *x a* continuata concipi potest.

67. Etsi igitur variatio δU ad terminum determinatum $x=a$ adstringitur, tamen ob membrum integrale omnes variationes intermedias complectitur ;· unde, si pro reliquis partibus absolutis, quae tantum ad terminum ultimum $x=a$ referuntur, brevitatis gratia scribamus *I*, variatio δU ita erit expressa, ut sit :

$$\delta U=(A)dx\delta y+(A)^{/}dx\delta y^{/}+(A)^{//}dx\delta y^{//}+(A)^{///}dx\delta y^{///}\text{ etc.}+I\ ,$$

quae, ut problemati satisfiat, nihilo aequari debet. Cum autem variationes δy^{I}, δy^{II}, δy^{III} etc. non a se invicem pendeant, sed singulae mere sint arbitrariae, illa annihilatio locum habere nequit, nisi singulae partes sigillatim evanescant; ex quo necesse est, ut sit :

$$(A)=0\ ,\ (A)^{I}=0\ ,\ (A)^{II}=0\ ,\ (A)^{III}=0\ \text{ etc. },$$

quae aequatiunculae omnes in hac una indefinita $(A)=0$ continentur seu, quicunque valor ipsi *x* tribuatur, perpetuo esse oportet $(A)=0$, hacque aequatione relatio quaesita inter *x* et *y* continetur.

68. En igitur solutionem facilem problematis propositi, quo ea relatio inter *x* et *y* requiritur, ex qua pro formula praescripta *U*, postquam eius valor a termino $x=0$ usque ad $x=a$ fuerit extensus, maximus minimusve valor resultet. Quaeratur scilicet variatio formulae *U* pariter a termino $x=0$ usque ad $x=a$ extensa, quae per praecepta supra tradita huiusmodi formam habere debet :

$$\delta U=\int(A)dx\delta y+(B)\delta y+(C)\frac{d\delta y}{dx}+(D)\frac{dd\delta y}{dx^{2}}\ \text{ etc. },$$

hincque ex solo membro integrali $\int(A)dx\delta y$ relatio inter *x* et *y* quaesita ita definietur, ut sit $(A)=0$, reliquae autem partes, quia ultimum tantum valorem ipsius *y* afficiunt, nihil conferunt ad relationem indefinitam inter *x* et *y*, quae desideratur.

69. Istae tamen partes posteriores relationi inventae magis determinandae inservire possunt ; eatenus enim tantum huiusmodi partes accedunt, quatenus in membro integrali $\int(A)dx\delta y$ functio (*A*) differentialium rationem $\frac{dy}{dx}=p$ vel etiam rationes differentialium superiorum, nempe $q=\frac{dp}{dx},\ r=\frac{dq}{dx}$ etc. involvit. Quando autem hoc usu venit, aequatio $(A)=0$ erit differentialis vel primi vel etiam altioris gradus ; sicque relatio quaesita inter *x* et *y* post unam pluresve demum integrationes reperitur. Cum autem quaelibet integratio quantitatem constantem arbitrariam invehat, hoc modo ad aequationem finitam vagam pervenietur atque nunc nova quaestio existet, quomodo has constantes arbitrarias determinari oporteat, ut valor ipsius *U* omnium maximus minimusve prodeat. Cum enim quaelibet illarum constantium determinatio iam per se maximi minimive proprietate sit praedita, hic porro vel maximum maximorum vel minimum minimorum investigandum relinquitur.

70. Ad hoc igitur novum problema accessorium resolvendum partes illae a signo integrali immunes adhiberi poterunt. Constantes scilicet per integrationes invectas ita determinari

conveniet, ut posito $x = a$ coefficientes ipsarum $\delta y,\ \frac{d\delta y}{dx},\ \frac{dd\delta y}{dx^2}$ etc. singuli seorsim evanescant, sive ut hoc casu satisfiat his conditionibus :

$$(B) = 0,\ (C) = 0,\ (D) = 0 \quad \text{etc.}$$

Deinde, quia ambos terminos $x = 0$, et $x = a$ inter se permutare licet, etiam posito $x = 0$ efficiendum erit, ut fiat $(B) = 0,\ (C) = 0,\ (D) = 0$ etc. Etsi enim partes, quae hoc exigant, in nostra expressione non continentur, tamen eae in membro integrali contineri sunt censendae.

71. Ex iisdem principiis etiam problemata, quae ad methodum relativam retuli, solvi possunt; haec autem problemata ita generaliter enunciare licet:
Inter omnes. relationes, quibus y per x definitur, quae hac communi gaudent proprietate, ut pro formula $\mathfrak{U}$ posito $x = a$ eundem valorem exhibeant, determinare eam relationem, ex qua formula U, siquidem a termino $x = 0$ usque ad terminum $x = a$ extendatur, maximum vel minimum consequatur valorem.

Hic igitur variationes, quae singulis valoribus ipsius *y* tribuuntur, non omnes sunt arbitrariae, sed ita statuendae sunt, ut fiat $\delta\mathfrak{U} = 0$, siquidem eius valor a termino $x = 0$ usque ad $x = a$ extendatur. Tum vero etiam natura maximi minimive postulat, ut secundum eandem extensionem sit ut ante $\delta U = 0$.

72. Per methodum ergo ante expositam tam formulae $\mathfrak{U}$, quae debet esse communis, quam formulae *U*, quae maxima fieri debet vel minima, quaeratur variatio a termino $x = 0$ usque ad terminum $x = a$ extendenda ; atque relatio quaesita inter *x* et *y* ex coniunctione harum duarum aequationum $\delta\mathfrak{U} = 0$ et $\delta U = 0$ erit investiganda. At hae variationes ita expressae reperientur :

$$\delta U = \int (A)dx\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} \text{ etc.}$$

ubi de membris a signo integrali liberis eadem sunt tenenda, quae supra iam observavi; ideoque relationem inter *x* et *y* quaesitam tantum ex membris integralibus derivari oportebit.

73. Hinc itaque binas sequentes consequemur aequationes :

$$\delta\mathfrak{U} = \int (\mathfrak{A})dx\delta y + (\mathfrak{B})\delta y + (\mathfrak{C})\frac{d\delta y}{dx} + (\mathfrak{D})\frac{dd\delta y}{dx^2} \text{ etc.}$$

$$\delta U = \int (A)dx\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} \text{ etc.}$$

quarum priore assumtio variationum δy, $\delta y^{/}$, $\delta y^{//}$ etc. conditioni communi praescriptae conveniens definitur, quae deinde in alteram introducta relationem quaesitam manifestabit. Omnes ergo variationes δy, $\delta y^{/}$, $\delta y^{//}$ etc. praeter unam ut arbitrariae spectari possunt, quippe quae una ex priori aequatione est definienda. Iam vero evidens est, postquam una ita fuerit sumta, ut priori aequationi satisfiat, tum simul alteri satisfactum iri, si statuatur $(A) = n(\mathfrak{A})$, sumendo pro *n* quantitatem quamcunque constantem.

74. Problema igitur propositum hac resolvetur aequatione:

$$\alpha(A) + \beta(\mathfrak{A}) = 0$$

sumtis pro α et β quantitatibus quibusvis constantibus. Eadem autem solutio prodiisset, si inter omnes omnino relationes inter *x* et *y* ea exquiri debuisset, unde formula $\alpha U + \beta\mathfrak{U}$ maximum minimumve valorem impetraret ; ex quo simul intelligitur binas formulas $\mathfrak{U}$ et *U* propositas inter se permutari posse eaque omnia, quae in Tractatu meo annotavi, hinc multo magis fiunt perspicua. Simili enim modo res se habebit, si non una formula $\mathfrak{U}$, sed plures debeant esse communes ; sicque stabilito *Calculo Variationum* omnia huius generis problemata facillime et brevissime expediuntur.