THEOREMATA NONNVLLA GENERALIA DE TRANSLATIONE CORPORVM RIGIDORVM

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1.

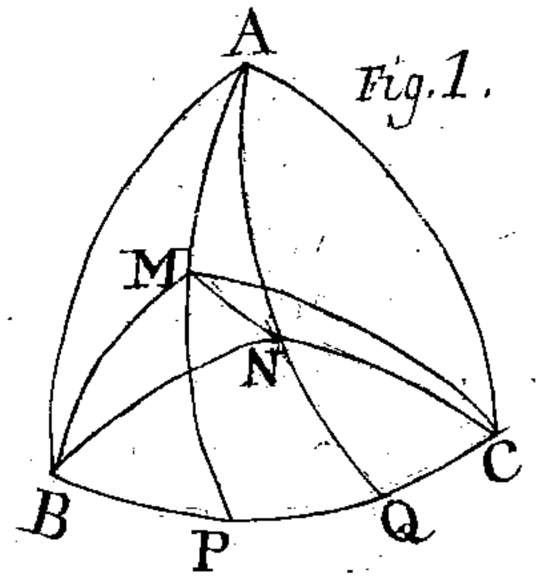
Cum Illustris *Eulerus* de translatione corporum rigidorum nuper agens, elegantissimam hanc detexisset proprietatem, quod in quacunque translatione corporis rigidi, semper detur linea recta, quae situm teneat parallelum ei, quem in statu initiali habuit; simulque ostendisset vt hoc locum inueniat, expressionem quandam Analyticam ad nihilum redigi debere, quod tamen quomodo fiat, ob terminos qui hanc expressionem ingrediuntur complicatiores, aliis penitus enucleandum relinquere est coactus; admonitione itaque Illustri Viri incitatus in animum induxi, vt demonstrationem inuestigarem, qua perspicue ostendi posset, aequationi isti Analyticae in omni translatione corporis rigidi satisfieri, sicque etiam ex principiis Analyticis egregiam istam veritatem prorsus extra dubium poni, quod in situ translato detur talis recta, quae parallelum teneat situm ei, quem in statu initiali habuit. Dum igitur hanc demonstrationem heic explicandam constitui, e re esse duxi nonnullas quoque alias meditationes de translatione corporum praemittere ; partim quia ad istam demonstrationem inueniendam viam mihi praeparararerunt, tum etiam quod inseruire poterunt, ad ea vberius confirmanda, quae Illustris *Eulerus* de hac translatione docuit.

Lemma I.

2. *Sumtis in superficie sphaerica (vide Figuarae 1 & 2) tribus punctis* A, B, C, *quadrantibus a se inuicem distantibus, si bina quaecunque alia puncta* M, N *in eadem supeficie assumta, tam inter se, quam cum illis punctis* A, B, C *arcubus circulorum maximorum iungantur, erit*:

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$

Demonstratio.



Concipiantur arcus AM, AN producti vsque dum circulo maximo BC occurrant in P et Q. Quum igitur sit

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN} \cos \mathrm{MAN},$$

angulus autem MAN sit vel = MAC − NAC, vel MAN + NAC vtrumque hunc casum seorsim considerabimus (Fig. 1). Sit igitur *primo*

$$\mathrm{MAN} = \mathrm{MAC} - \mathrm{NAC}, \text{ eritque}$$

$$\cos \mathrm{MAN} = \cos \mathrm{MAC} \cos \mathrm{NAC} + \sin \mathrm{MAC} \sin \mathrm{NAC},$$

hoc igitur valore pro cos MAN substituto, consequemur

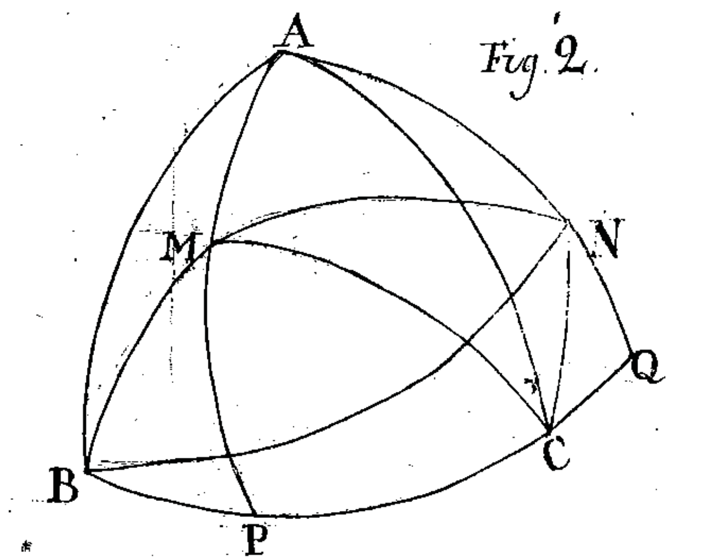
$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN} (\cos \mathrm{MAC} \cos \mathrm{NAC} + \sin \mathrm{MAC} \sin \mathrm{NAC})$$

Est vero,

$$\begin{gathered}
\sin \mathrm{AM} \cos \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{PC} = \cos \mathrm{MC} \\
\sin \mathrm{AN} \cos \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{QC} = \cos \mathrm{NC} \\
\sin \mathrm{AM} \sin \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{BP} = \cos \mathrm{BM} \\
\sin \mathrm{AN} \sin \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{BQ} = \cos \mathrm{BN}
\end{gathered}$$

ex quo euidenter colligitur (Fig. 2),

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$



Secundo si sit MAN = MAC + NAC erit,

$$\cos \mathrm{MAN} = \cos \mathrm{MAC} \cos \mathrm{NAC} - \sin \mathrm{MAC} \sin \mathrm{NAC}, \text{ hinc}$$

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN} \,(\cos \mathrm{MAC} \cos \mathrm{NAC} - \sin \mathrm{MAC} \sin \mathrm{NAC}),$$

tum vero erit

$$\begin{gathered}
\sin \mathrm{AM} \cos \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{CP} = \cos \mathrm{CM} \\
\sin \mathrm{AN} \cos \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{CQ} = \cos \mathrm{CN} \\
\sin \mathrm{AM} \sin \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{BP} = \cos \mathrm{BM} \\
\sin \mathrm{AN} \sin \mathrm{NAC} = -\cos \mathrm{NQ} \cos \mathrm{BQ} = -\cos \mathrm{BN}
\end{gathered}$$

ex quo etiam nunc sit

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$

Theorema I.

3. *Si sphaera circa centrum suum fixum vtcunque conuertatur, vt puncta* A, B, C *quadrantibus inter se distantia perueniant in puncta* a, b, c, *tum punctum quodcunque* Z *ita transferetur in* z, *vt sit*

$$\begin{gathered}
\cos z\mathrm{A} = \cos \mathrm{ZA} \cos \mathrm{A}a + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c \\
\cos z\mathrm{B} = \cos \mathrm{ZA} \cos \mathrm{B}a + \cos \mathrm{ZB} \cos \mathrm{B}b + \cos \mathrm{ZC} \cos \mathrm{B}c \\
\cos z\mathrm{C} = \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC} \cos \mathrm{C}c
\end{gathered}$$

Demonstratio.

Quia situs puncti z respectu punctorum a, b, c idem est, ac situs puncti Z respectu punctorum A, B, C, erit za=ZA; zb=ZB; zc=ZC. Per Lemma autem modo demonstratum constat esse

$$\begin{gathered}\cos z\mathrm{A} = \cos za \cos \mathrm{A}a + \cos zb \cos \mathrm{A}b + \cos zc \cos \mathrm{A}c\\ \cos z\mathrm{B} = \cos za \cos \mathrm{B}a + \cos zb \cos \mathrm{B}b + \cos zc \cos \mathrm{B}c\\ \cos z\mathrm{C} = \cos za \cos \mathrm{C}a + \cos zb \cos \mathrm{C}b + \cos zc \cos \mathrm{C}c\end{gathered}$$

vnde si pro za, zb, zc substituantur ZA, ZB, ZC, prodibunt aequalitates in Theoremate nostro expressae.

4. Consimili quoque ratione ostendi posset esse:

$$\begin{gathered}\cos \mathrm{Z}a = \cos za \cos a\mathrm{A} + \cos zb \cos a\mathrm{B} + \cos zc \cos a\mathrm{C}\\ \cos \mathrm{Z}b = \cos za \cos b\mathrm{A} + \cos zb \cos b\mathrm{B} + \cos zc \cos b\mathrm{C}\\ \cos \mathrm{Z}c = \cos za \cos c\mathrm{A} + \cos zb \cos c\mathrm{B} + \cos zc \cos c\mathrm{C}\end{gathered}$$

per quas formulas puncti Z in statu initiali situs, ad puncta a, b, c determinatur.

5. Si punctum z in ipsum Z incidat, ideoque Z sit punctum quod post conuersionem sphaerae eandem tenet situm, ac in statu initiali, aequationes supra in Theoremate allatae, in has transformabuntur:

$$\begin{gathered}\cos \mathrm{ZA}\ (\cos \mathrm{A}a{-}1) + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c = 0\\ \cos \mathrm{ZA} \cos \mathrm{B}a{+} \cos \mathrm{ZB}\ (\cos \mathrm{B}b{-}1) + \cos \mathrm{ZC} \cos \mathrm{B}c = 0\\ \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC}\ (\cos \mathrm{C}c{-}1) = 0\end{gathered}$$

Nunc si breuitatis gratia, loco harum aequationum sequentes adhibeatur:

$$\text{I.}\quad \alpha x + \beta y + \gamma z = 0 \qquad \text{II.}\quad \alpha' x + \beta' y + \gamma' z = 0 \qquad \text{III.}\quad \alpha'' x + \beta'' y + \gamma'' z = 0 \tag{1}$$

vbi $x = \cos \mathrm{AZ}$; $y = \cos \mathrm{BZ}$; $z = \cos \mathrm{CZ}$; $\alpha = \cos \mathrm{A}a - 1$; $\beta = \cos \mathrm{A}b$; $\gamma = \cos \mathrm{A}c$; $\alpha' = \cos \mathrm{B}a$; $\beta' = \cos \mathrm{B}b - 1$; $\gamma' = \cos \mathrm{B}c$; $\alpha'' = \cos \mathrm{C}a$; $\beta'' = \cos \mathrm{C}b$; $\gamma'' = \cos \mathrm{C}c - 1$, ex prima et secunda inter se comparatis, obtinemus

$$\Big(\frac{\beta}{\alpha} - \frac{\beta'}{\alpha'}\Big)y + \Big(\frac{\gamma}{\alpha} - \frac{\gamma'}{\alpha'}\Big)z = 0, \tag{2}$$

tumque ex prima et tertia

$$\Big(\frac{\beta}{\alpha} - \frac{\beta''}{\alpha''}\Big)y + \Big(\frac{\gamma}{\alpha} - \frac{\gamma''}{\alpha''}\Big)z = 0, \tag{3}$$

vnde per priorem colligitur

$$\frac{y}{z} = \frac{\alpha\gamma - \alpha'\gamma}{\beta\alpha' - \beta'\alpha} \quad \text{et per posteriorem} \quad \frac{y}{z} = \frac{\alpha\gamma'' - \alpha''\gamma}{\beta\alpha'' - \beta''\alpha}, \tag{4}$$

quibus valoribus inter se aequatis, prodit haec aequatio :

$$(\alpha\gamma' - \alpha'\gamma)(\beta\alpha'' - \beta''\alpha) = (\alpha\gamma'' - \alpha''\gamma)(\beta\alpha' - \beta'\alpha) \tag{5}$$

quae euolutione facta in sequentem contrahitur

$$\alpha\beta'\gamma'' - \alpha\gamma'\beta'' - \alpha'\beta\gamma'' - \alpha''\beta'\gamma + \alpha''\beta\gamma' + \alpha'\beta''\gamma = 0. \tag{6}$$

Introductis vero nunc pro α, β, γ etc. valoribus quibus aequales supponebantur, aequatio nostra sic habebitur expressa :

$$\begin{aligned}(\cos \mathrm{A}a - 1)(\cos \mathrm{B}b - 1)(\cos \mathrm{C}c - 1) - \cos \mathrm{B}c \cos \mathrm{C}b(\cos \mathrm{A}a - 1) - \cos \mathrm{B}a \cos \mathrm{A}b(\cos \mathrm{C}c - 1)\\ - \cos \mathrm{C}a \cos \mathrm{A}c(\cos \mathrm{B}b - 1) + \cos \mathrm{C}a \cos \mathrm{A}b \cos \mathrm{B}c + \cos \mathrm{B}a \cos \mathrm{A}c \cos \mathrm{C}b = 0.\end{aligned} \tag{7}$$

Haec autem aequatio perfecte congruit cum illa expressione Analytica, quam Illust. *Eulerus* in Dissertatione de translatione corporum rigidorum §22. adfert, modo obseruatur quantitates per F, G, H etc. ibi expressas, heic ita indigitari, vt sit:

$$\begin{gathered}\mathrm{F} = \cos\ \mathrm{A}a;\ \mathrm{G} = \cos\ \mathrm{B}a;\ \mathrm{H} = \cos\ \mathrm{C}a;\\ \mathrm{F}' = \cos\ \mathrm{A}b;\ \mathrm{G}' = \cos\ \mathrm{B}b;\ \mathrm{H}' = \cos\ \mathrm{C}b;\\ \mathrm{F}'' = \cos\ \mathrm{A}c;\ \mathrm{G}'' = \cos\ \mathrm{B}c;\ \mathrm{H}'' = \cos\ \mathrm{C}c;\end{gathered}$$

Quamuis autem ex ipsa Dissertatione modo allata haec identitas non perfecte innotescat, infra occasio dabitur ostendendi, quod illa omnino locum habeat. Interim obseruare iuuat, relationes illas pro litteris F, G, H etc. ab Illustro *Eulero* allata in §18. Dissertationis suae, omnino locum habere, si his litteris valores mox assignati tribuantur. Est enim

$$\text{I.}\quad \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = 1 \tag{8}$$
$$\text{II.}\quad \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}b + \cos^2 \mathrm{C}b = 1 \tag{9}$$
$$\text{III.}\quad \cos^2 \mathrm{A}c + \cos^2 \mathrm{B}c + \cos^2 \mathrm{C}c = 1 \tag{10}$$

tumque per Lemma nostum I:

$$\text{IV.}\quad \cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{A}a \cos \mathrm{A}b = \cos ab = 0 \tag{11}$$
$$\text{V.}\quad \cos \mathrm{A}a \cos \mathrm{A}c + \cos \mathrm{B}a \cos \mathrm{B}c + \cos \mathrm{C}a \cos \mathrm{C}c = \cos ac = 0 \tag{12}$$
$$\text{VI.}\quad \cos \mathrm{A}b \cos \mathrm{A}c + \cos \mathrm{B}b \cos \mathrm{B}c + \cos \mathrm{C}b \cos \mathrm{C}c = \cos bc = 0. \tag{13}$$
$$\tag{14}$$

6. Quod si iam demonstrari queat, in omni conuersione sphaerae circa centrum suum fixum, expressionem istam

$$\begin{aligned}(\cos \mathrm{A}a - 1)(\cos \mathrm{B}b - 1)(\cos \mathrm{C}c - 1) - \cos \mathrm{B}c \cos \mathrm{C}b(\cos \mathrm{A}a - 1) - \cos \mathrm{B}a \cos \mathrm{A}b(\cos \mathrm{C}c - 1)\\ - \cos \mathrm{C}a \cos \mathrm{A}c(\cos \mathrm{B}b - 1) + \cos \mathrm{C}a \cos \mathrm{A}b \cos \mathrm{B}c + \cos \mathrm{B}a \cos \mathrm{A}c \cos \mathrm{C}b\end{aligned} \tag{15}$$

ad nihilum redigi; patebit omnino existere in eius superficie aliquod punctum Z, quod post conuersionem in eodem sit loco, ac in statu initiali reperiebatur. Verum antequam hanc demonstrationem suscipiamus, operae pretium erit explicare, quomodo formulae Theorematis nostri primi ad exprimendam translationem puncti cuiuslibet in corpore rigido adhiberi queant. Heic autem obseruamus, omnem translationem corporis rigidi duplici esse generis, quorum prius eam complectitur loci mutationem, qua omnes corporis particulae secundum directiones inuicem parallelas feruntur, quae translatio alioquin nomine motus progressiui venire solet. Posterius vero translatione genus in se continet conuersionem corporis circa punctum quoddam fixum, vbi quidem perinde est, siue hoc punctum intra ipsum corpus, seu extra id assumi concipiatur. Mentem igitur si primo abstrahamus a motu progressiuo, solam conuer-

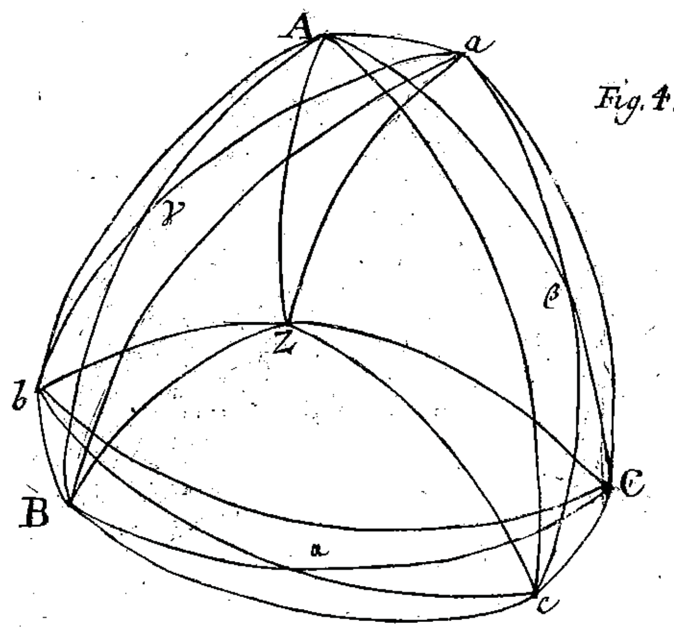
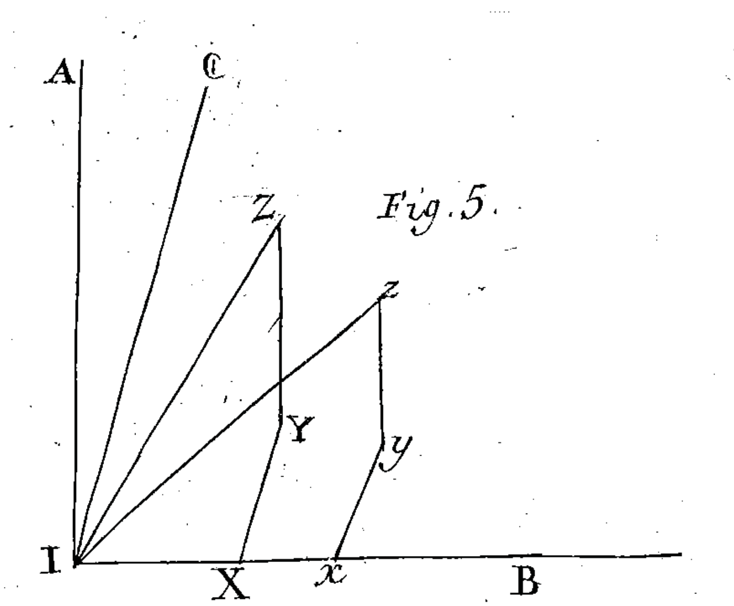


Fig. 4.

sionem circa punctum fixum considerantes (Tab III, Fig. 5), concipiamus punctum illud fixum esse in I et per hoc punctum duci ternos axes IA, IB, IC inter se normales, ad quos situs puncti cuiuslibet corporis Z, in statu initiali referantur per ternas coordinatas IX, XY, YZ, tum vero supponamus conuersione corporis facta circa punctum I, punctum Z peruenisse in z, huiusque situm determinari per coordinatas Ix, xy, yz, existente distantia Iz=IZ, quam distantiam littera s designemus. Si nunc intelligatur, sphaeram cum corpore nostro firmiter esse connexam, cuius centrum sit in I et semidiameter aequalis ipsi s (Fig. 5 & 4), axes vero IA, IB, IC occurrere huic sphaerae in punctis A, B, C, haecque puncta conuersione corporis facta circa I in puncta a, b, c peruenisse; facile patet angulos AIz, BIz, CIz Fig. 5. respectiue aequales esse arcubus Az, Bz, Cz Fig. 4. Atqui est $\frac{\mathrm{I}x}{\mathrm{I}z} = \cos \mathrm{BI}z = \cos \mathrm{B}z$; $\frac{xy}{\mathrm{I}z} = \cos \mathrm{CI}z = \cos \mathrm{C}z$; $\frac{xy}{\mathrm{I}z} = \cos \mathrm{AI}z = \cos \mathrm{A}z$; quare si dicantur,

$$\mathrm{I}x = x; \quad xy = y; \quad yz = z; \quad \text{consequemur} \tag{16}$$

$$\frac{x}{s} = \cos \mathrm{B}z; \quad \frac{y}{s} = \cos \mathrm{C}z; \quad \frac{z}{s} = \cos \mathrm{A}z. \tag{17}$$



Simili modo fiet

$$\frac{\mathrm{IX}}{\mathrm{IZ}} = \cos \mathrm{BZ}; \quad \frac{\mathrm{XY}}{\mathrm{IZ}} = \cos \mathrm{C}Z; \quad \frac{\mathrm{XZ}}{\mathrm{IZ}} = \cos \mathrm{A}z, \tag{18}$$

vnde si IX, XY, YZ respectiue exprimantur per litteras X, Y, Z obtinebimus:

$$\frac{\mathrm{X}}{s} = \cos \mathrm{BZ}; \quad \frac{\mathrm{Y}}{s} = \cos \mathrm{CZ}; \quad \frac{\mathrm{Z}}{s} = \cos \mathrm{A}z, \tag{19}$$

Hinc ope formularum Theorematis nostri primi, sequentes pro x, y, z deducimus expressiones

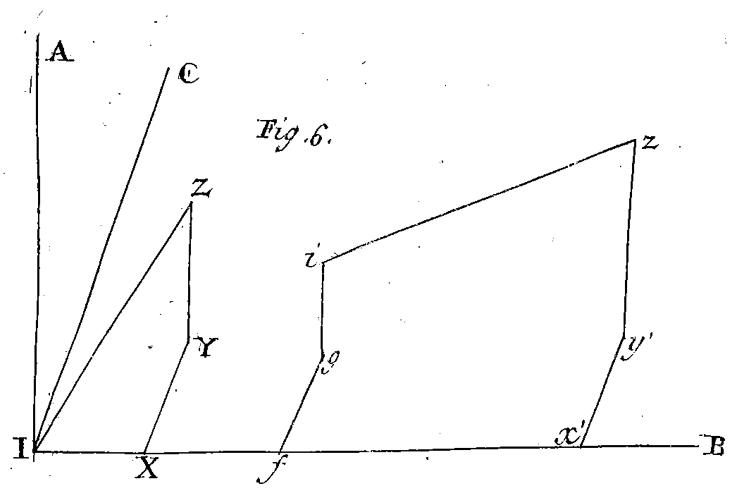
$$z = \mathrm{Z} \cos \mathrm{A}a + \mathrm{X} \cos \mathrm{A}b + \mathrm{Y} \cos \mathrm{A}c \tag{20}$$

$$x = \mathrm{Z} \cos \mathrm{B}a + \mathrm{X} \cos \mathrm{B}b + \mathrm{Y} \cos \mathrm{B}c \tag{21}$$

$$y = \mathrm{Z} \cos \mathrm{C}a + \mathrm{X} \cos \mathrm{C}b + \mathrm{Y} \cos \mathrm{C}c \tag{22}$$

vnde patet modo cognoscantur coordinatae puncti Z pro statu initiali, et translationes punctorum A, B, C in puncta a, b, c, etiam situm puncti Z translatum in z, facile determinari posse.

7. Iam si in translatione corporis, etiam motus progressiui rationem habere velimus (Fig. 6); concipiamus interea dum corpus conuertitur circa I, ipsum hoc punctum peruenisse in i, eiusque puncti situm determinari per ternas coordinatas If, fg, gi, quas respectiue litteris f, g, h exprimamus. Deinde punctum Z translatione facta, iam in z esse concipiatur, et situs huius puncti z determinetur per coordinatas Ix', $x'y'$, $y'z$, quas per litteras x', y', z' respectiue designemus ; quo facto leui attentione adhibita patebit esse in Fig. 6. I$x'$$-If$=I$x$ Fig. 5, simili modo $x'y' - fg = xy$ tumque $zy' - ig = zy$; quod omnino rigorosa demonstratione firmari poterit, si modo per i ductae concipiantur tres lineae ia, ib, ic parallelae ipsis IA, IB, IC ; quas tamen in nostra figura, ne nimis euaderet complicata, non expressimus. Hinc igitur colligitur $x' = f + x$; $y' = g + y$; $z' = h + z$, quare si valores modo pro x, y, z inuenti adhibeantur, has



expressiones pro x', y', z' consequemur :

$$x' = f + \mathrm{X}\cos \mathrm{B}b + \mathrm{Y}\cos \mathrm{B}c + \mathrm{Z}\cos \mathrm{B}a \tag{23}$$

$$y' = g + \mathrm{X}\cos \mathrm{C}b + \mathrm{Y}\cos \mathrm{C}c + \mathrm{Z}\cos \mathrm{C}a \tag{24}$$

$$z' = h + \mathrm{X}\cos \mathrm{A}b + \mathrm{Y}\cos \mathrm{A}c + \mathrm{Z}\cos \mathrm{A}a \tag{25}$$

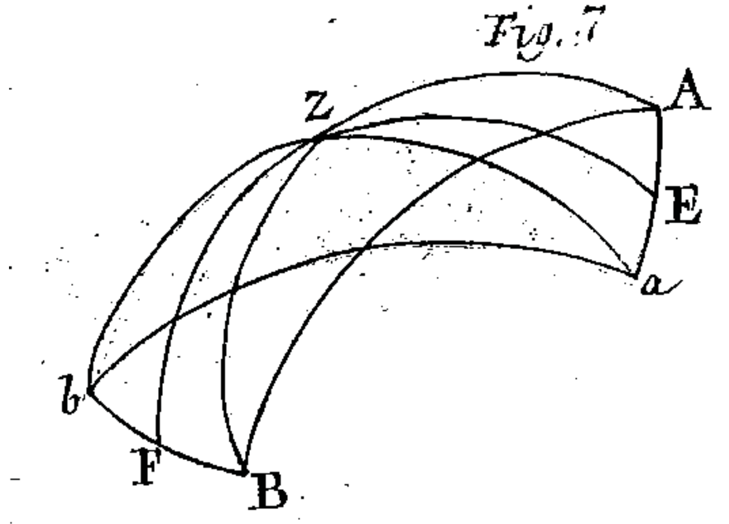
ita vt pro determinandis x', y', z', translationem tam puncti I per ordinatas f, g, h expressam, quam conuersionem corporis circa punctum I per arcus Aa, Bb, Cc determinatam, cognoscere oporteat. Formulae autem hae allatae conferantur cum illis, quas Illustris *Eulerus* inuenerat §10. Dissertationis saepius commemoratae ; nunc omnino euidenter constabit, litteris F, G, H etc. eos tribuendos esse valores, quos ipsis supra assignauimus. In formulis vero his pro x', y', z' licet nouem occurrant quantitates ad translationes punctorum A, B, C spectantes ; tamen quum reliquae ex his tribus cos Aa, cos Bb, cos Cc, vti mox videbimus, determinentur ; situs puncti z per sex eleenta coordinatas scilicet, f, g, h et arcus Aa, Bb, Cc, perfecte determinatus esse censebitur.

Problema.

8. *Pro quacunque conuersione sphaerae circa centrum suum fixum, inuenire punctum in eius superficie, quod post conuersionem in eodem reperiatur loco, ac quo erat ante conuersionem.*

Solutio.

In superficie sphaerae (Fig. 7) accipiatur arcus quicunque AB, qui per conuersionem sphaerae in ab peruenisse supponitur, ita vt punctum A peruenerit in a et B in b, iungatur autem Aa, Bb arcubus circulorum maximorum. Bisecentur hi arcus Aa, Bb in punctis E et F, atque ducantur arcus EZ, FZ normales ad Aa, Bb, quorum mutuus occurius in Z, dabit punctum, quod post conuersionem in eodem reperitur loco, ac in situm initiali erat. Quum nimirum sit AE = aE, ang. AEZ = ang. aEZ = 90° et EZ communis triangulis AEZ, aEZ, fiet AZ = aZ ;



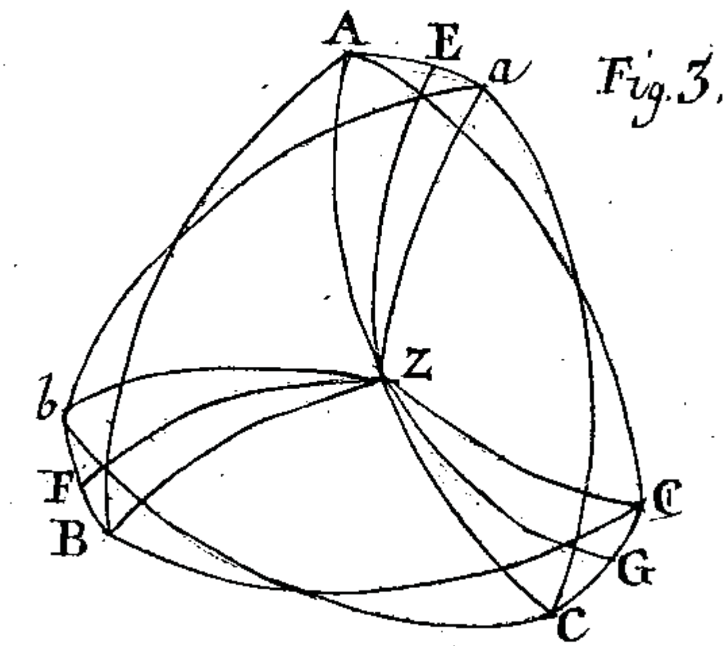
simili modo erit BZ $=$ bZ. Iam in triangulis AZB, aZb, est AZ $=$ aZ, BZ $=$ bZ et AB $=$ ab, erit igitur ang. ABZ $=$ abZ et BAZ $=$ baZ ; ex quo patet punctum Z respectu arcus ab perfecte eundem tenere situm, ac respectu arcus AB, ideoque hoc punctum in eodem esse loco, ac ante conuersionem erat.

9. Mirum forsan alicui videbitur, quod huius Problematis resolutionem adgressus sim, antequam demonstrauerim formulam istam §. 5 allatam ad nihilum redigi ; quippe quum videatur hanc solutionem nequaquam in potestate esse, nisi expressio ista in omni conuersione sphaerae locum habeat. At primum quidem obseruandum est, licet Problema ipsum solutionem admittere non queat, nisi aequatio ista §. 5. vera sit ; tamen impsam solutionem Problematis independenter ab hac aequatione adornari posse, tumque ex ipsa indole solutionis facile perspici, an semper et omni casu locum habeat. In nostra quidem solutione, nullum dubium est, quin arcus EZ et FZ normales ad Aa et Bb se intersecent, vnico excepto casu, quo coincidunt ; tum vero arcus iste EZF transibit per punctum in quo arcus AB, ab se intersecant, immo istud punctum intersectionis iam cum puncto Z coincidet ; dubitare igitur non licet, quin Problema nostrum semper veram et realem admittat solutionem. Deinde ea quidem de caussa praeprimis hanc solutionem heic adornandam censui, vt modum mihi patefaceret, quo ex datis arcubus Aa, Bb, Cc determinationem puncti Z susciperem, quae inuestigatio mihi interuiret ad detegendas relationes, quae inter arcus Aa, Bb, Cc et Ab, Ac, Ba, Bc, Ca, Cb intercedunt (Fig. 3).

10. Vt nunc intelligatur, quomodo punctum Z per arcus Aa, Bb, Cc determinetur, supponamus hos arcus bisecari per arcus EZ, FZ, GZ ipsis respectiue normales, qui arcus EZ, FZ, GZ in eodem puncto Z concurrent; tum vero erit:

$$\sin^2 \mathrm{AZ} \sin^2 \frac{1}{2} \mathrm{AZ}a = \sin^2 \frac{1}{2} \mathrm{A}a \tag{26}$$

$$\sin^2 \mathrm{BZ} \sin^2 \frac{1}{2} \mathrm{BZ}b = \sin^2 \frac{1}{2} \mathrm{B}b \quad \text{et} \tag{27}$$



$$\sin^2 \mathrm{CZ} \sin^2 \frac{1}{2}\mathrm{CZ}c = \sin^2 \frac{1}{2}\mathrm{C}c \tag{28}$$

ideoque ob AZa=BZb=CZc,

$$\sin^2 \frac{1}{2}\mathrm{AZ}a(\sin^2 \mathrm{AZ} + \sin^2 \mathrm{BZ} + \sin^2 \mathrm{CZ}) = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c. \tag{29}$$

Hinc vero deducitur

$$\sin^2 \frac{1}{2}\mathrm{AZ}a(3 - \cos^2 \mathrm{AZ} + \cos^2 \mathrm{BZ} + \cos^2 \mathrm{CZ}) = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c, \tag{30}$$

quae ob

$$\cos^2 \mathrm{AZ} + \cos^2 \mathrm{BZ} + \cos^2 \mathrm{CZ} = 1, \tag{31}$$

in sequentem abit aequationem:

$$2\sin^2 \frac{1}{2}\mathrm{AZ}a = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c. \tag{32}$$

Cuius loco etiam haec adhiberi potest,

$$2\cos^2 \mathrm{AZ}a = \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}b + \cos^2 \mathrm{C}c - 1. \tag{33}$$

Deinde vero colligitur

$$\sin^2 \mathrm{AZ} = \frac{\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{AZ}a} = \frac{2\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c} \tag{34}$$

$$\sin^2 \mathrm{AZ} = \frac{2\sin^2 \frac{1}{2}\mathrm{B}b}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c} \tag{35}$$

$$\sin^2 \mathrm{AZ} = \frac{2\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c}, \tag{36}$$

tumque omnino sponte ista conditio impletur, vt sit

$$\sin^2 \mathrm{AZ} + \sin^2 \mathrm{BZ} + \sin^2 \mathrm{CZ} = 2. \tag{37}$$

Expressiones vero hae sequenti quoque ratione transformari poterunt : quum sit

$$2\sin^2 \mathrm{AZ} = \frac{8\sin^2 \frac{1}{2}\mathrm{A}a}{2\sin^2 \frac{1}{2}\mathrm{A}a + 2\sin^2 \frac{1}{2}\mathrm{B}b + 2\sin^2 \frac{1}{2}\mathrm{C}c}, \tag{38}$$

ob $2\sin^2 \mathrm{AZ} = 1 - \cos 2\mathrm{AZ}$; $2\sin^2 \frac{1}{2}\mathrm{A}a = 1 - \cos \mathrm{A}a$; $2\sin^2 \frac{1}{2}\mathrm{B}b = 1 - \cos \mathrm{B}b$; $2\sin^2 \frac{1}{2}\mathrm{C}c = 1 - \cos \mathrm{C}c$, consequimur

$$1 - \cos 2\mathrm{AZ} = \frac{4(1 - \cos \mathrm{A}a)}{3 - \cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c}, \qquad \text{hincque} \tag{39}$$

$$\cos 2\mathrm{AZ} = \frac{3\cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c - 1}{3 - \cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c}, \tag{40}$$

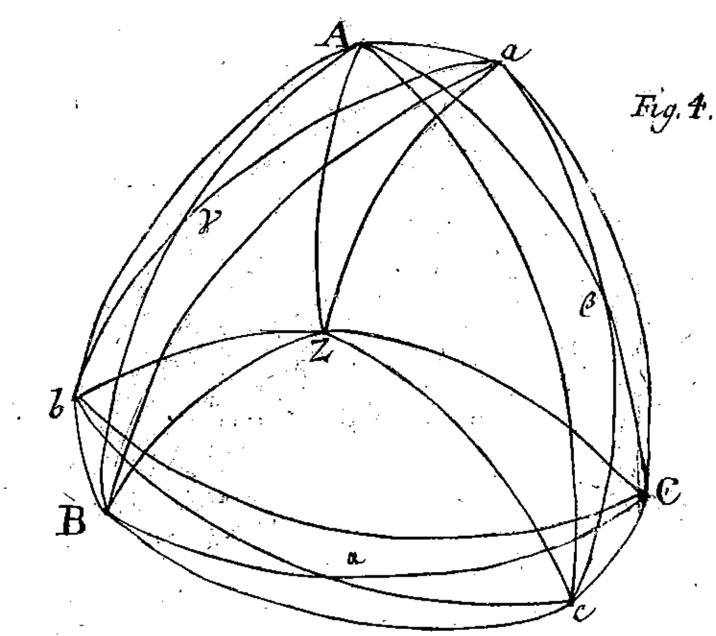
similque modo

$$\cos 2\mathrm{BZ} = \frac{3\cos \mathrm{B}b - \cos \mathrm{A}a - \cos \mathrm{C}c - 1}{3 - \cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c}; \quad \cos 2\mathrm{CZ} = \frac{3\cos \mathrm{C}c - \cos \mathrm{B}b - \cos \mathrm{C}c - 1}{3 - \cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c}; \tag{41}$$

ideoque

$$\cos 2\mathrm{AZ} + \cos 2\mathrm{AZ} + \cos 2\mathrm{AZ} = \frac{\cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c - 3}{3 - \cos \mathrm{A}a - \cos \mathrm{B}b - \cos \mathrm{C}c} = -1, \tag{42}$$

omnino vti esse oportet.



11. Vt vero iam quoque pateat, qua ratione arcus Ab, Ac, Ba, Bc, Ca, Cb per Aa, Bb, Cc determinentur; primum hos arcus, per ipsos AZ, BZ, CZ et angulum AZa determinemus. Constat autem esse

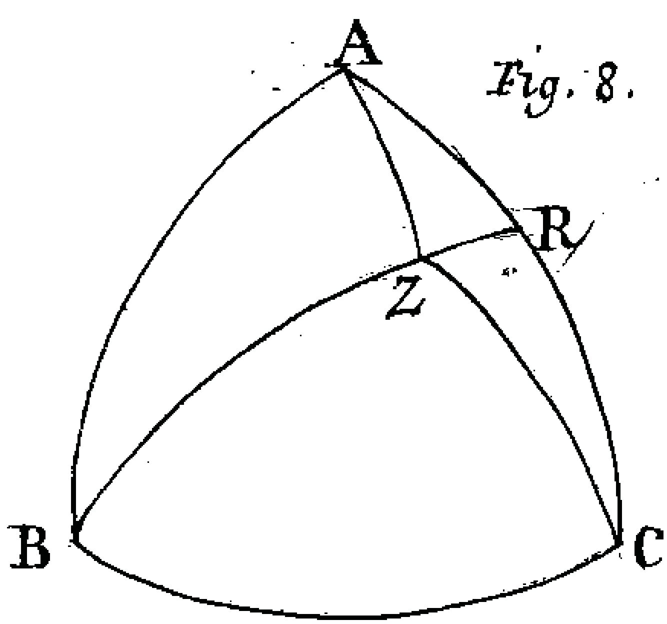
$$\begin{aligned}\cos \mathrm{A}a &= \cos^2 \mathrm{AZ} + \sin^2 \mathrm{AZ} \cos \mathrm{AZ}a && (43)\\ \cos \mathrm{B}b &= \cos^2 \mathrm{BZ} + \sin^2 \mathrm{BZ} \cos \mathrm{AZ}a && (44)\\ \cos \mathrm{C}c &= \cos^2 \mathrm{CZ} + \sin^2 \mathrm{CZ} \cos \mathrm{AZ}a. && (45)\end{aligned}$$

Porro est (vide Tab III, Fig. 4)

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos b\mathrm{Z} + \sin \mathrm{AZ} \sin b\mathrm{Z} \cos \mathrm{AZ}b, \tag{46}$$

vnde ob bZ=BZ et angulum AZb=AZB−BZb, colligimus

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ} + \sin \mathrm{AZ} \cos \mathrm{BZ} (\cos \mathrm{AZB} \cos \mathrm{BZ}b + \sin \mathrm{AZB} \sin \mathrm{BZ}b). \tag{47}$$



Est vero (Fig. 8) cos AZB = −cot AZ cot BZ, ex principiis Sphaericorum, hinc

$$\sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZB} = -\cos \mathrm{AZ} \cos \mathrm{BZ}.$$

Porro habetur quoque

$$\sin \mathrm{AZB} = \frac{\cos c\mathrm{Z}}{\sin \mathrm{AZ} \sin \mathrm{BZ}}, \tag{48}$$

id quod sequentem in modum demonstratur. Producatur arcus BZ, vsque dum circulo maximo AC in R occurrat, eritque

$$\sin \mathrm{AZ} \sin \mathrm{AZB} = \sin \mathrm{AR} = \cos \mathrm{CR} = \frac{\cos \mathrm{CZ}}{\cos \mathrm{ZR}} = \frac{\cos \mathrm{CZ}}{\sin \mathrm{BZ}}. \tag{49}$$

Hinc itaque colligitur

$$\sin \mathrm{AZ} \sin \mathrm{BZ} \sin \mathrm{AZB} = \cos \mathrm{CZ}. \tag{50}$$

Substitutis vero his valoribus, prodit:

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{BZ}b) + \cos \mathrm{CZ} \cos \mathrm{BZ}b. \tag{51}$$

Simili modo, ob

$$\cos \mathrm{B}a = \cos \mathrm{BZ} \cos a\mathrm{Z} + \sin \mathrm{BZ} \cos a\mathrm{Z} \sin \mathrm{BZ}a, \tag{52}$$

quum sit aZ=AZ et BZa=BZA+AZa, colligitur

$$\cos \mathrm{B}a = \cos \mathrm{BZ} \cos \mathrm{AZ} + \sin \mathrm{BZ} \cos \mathrm{AZ}(\cos \mathrm{AZB} \cos \mathrm{BZ}b - \sin \mathrm{AZB} \sin \mathrm{AZ}a) \tag{53}$$

quae ob

$$\sin \mathrm{BZ} \sin \mathrm{AZ} \cos \mathrm{AZB} = -\cos \mathrm{AZ} \cos \mathrm{BZ} \quad \text{et} \quad \sin \mathrm{BZ} \sin \mathrm{AZ} \sin \mathrm{AZB} = \cos \mathrm{CZ}, \tag{54}$$

reducitur ad hanc

$$\cos \mathrm{B}a = \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{AZ}a) - \cos \mathrm{CZ} \sin \mathrm{AZ}a. \tag{55}$$

Simili modo has consequemur expressiones:

$$\cos \mathrm{A}c = \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{CZ}c) - \cos \mathrm{BZ} \sin \mathrm{CZ}c \tag{56}$$
$$\cos \mathrm{C}a = \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{CZ}c) - \cos \mathrm{BZ} \sin \mathrm{CZ}c \tag{57}$$
$$\cos \mathrm{B}c = \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{BZ}c) - \cos \mathrm{AZ} \sin \mathrm{BZ}b \tag{58}$$
$$\cos \mathrm{C}b = \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{BZ}c) - \cos \mathrm{AZ} \sin \mathrm{BZ}b, \tag{59}$$

vbi anguli AZa, BZb, CZc quia inter se aequales, promiscue adhiberi possunt. Porro binas harum formularum inter se combinando obtinebimus:

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 2 \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{AZ}a); \tag{60}$$
$$\cos \mathrm{A}b - \cos \mathrm{B}a = 2 \cos \mathrm{CZ} \sin \mathrm{AZ}a \tag{61}$$
$$\cos \mathrm{A}c + \cos \mathrm{C}a = 2 \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{AZ}a); \tag{62}$$
$$\cos \mathrm{C}a - \cos \mathrm{A}c = 2 \cos \mathrm{BZ} \sin \mathrm{AZ}a \tag{63}$$
$$\cos \mathrm{B}c + \cos \mathrm{C}b = 2 \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{AZ}a); \tag{64}$$
$$\cos \mathrm{B}c - \cos \mathrm{C}b = 2 \cos \mathrm{AZ} \sin \mathrm{AZ}a \tag{65}$$

Denique egregia ista hinc colligitur proprietas, quod sit

$$\cos^2 \mathrm{A}b - \cos^2 \mathrm{B}a = \cos^2 \mathrm{B}c - \cos^2 \mathrm{C}b = \cos^2 \mathrm{C}a - \cos^2 \mathrm{A}c \tag{66}$$
$$= 4 \cos \mathrm{AZ} \cos \mathrm{BZ} \cos \mathrm{CZ} \sin \mathrm{AZ}a(1 - \cos \mathrm{AZ}a), \tag{67}$$

siue

$$\cos 2\mathrm{A}b - \cos 2\mathrm{B}a = \cos 2\mathrm{B}c - \cos 2\mathrm{C}b = \cos 2\mathrm{C}a - \cos 2\mathrm{A}c. \tag{68}$$

12. Si in valoribus supra pro cos Ab, cos Ba, cos Ca, cos Bc, cos Cb inuentis, loco cos AZ, cos BZ, cos CZ, cos AZa et sin AZa substituantur eorum valores per cos Aa, cos Bb, cos Cc expressi, cosinus priorum istorum arcuum, quoque per arcus Aa, Bb, Cc habebuntur definiti. Vt autem hoc facilius fiat, ponamus

$$\sin^2 \frac{1}{2}\mathrm{A}a = \alpha; \sin^2 \frac{1}{2}\mathrm{B}b = \beta; \sin^2 \frac{1}{2}\mathrm{C}c = \gamma; \tag{69}$$

ex quo colligitur

$$\cos \mathrm{AZ} = \sqrt{\frac{\beta+\gamma-\alpha}{\beta+\gamma+\alpha}}; \qquad \cos \mathrm{BZ} = \sqrt{\frac{\alpha+\gamma-\beta}{\alpha+\beta+\gamma}}; \qquad \cos \mathrm{CZ} = \sqrt{\frac{\alpha+\beta-\gamma}{\alpha+\beta+\gamma}}; \tag{70}$$

tumque $1 - \cos$ AZa $= \alpha+\beta+\gamma$; $\sin \frac{1}{2}$ AZa $= \sqrt{\frac{\alpha+\beta+\gamma}{2}}$, hinc $\cos \frac{1}{2}$ BZb $= \sqrt{\frac{2-(\alpha+\beta+\gamma)}{2}}$ et $\sin \mathrm{AZ}a = \sqrt{(\alpha+\beta+\gamma)(2-\alpha-\beta-\gamma)}$.

His igitur valoribus substitutis, consequemur

$$\cos \mathrm{A}b = \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} + \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{71}$$

$$\cos \mathrm{B}a = \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} - \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{72}$$

$$\cos \mathrm{B}c = \sqrt{(\alpha+\beta-\gamma)(\gamma+\alpha-\beta)} + \sqrt{(\gamma+\beta-\alpha)(2-\alpha-\beta-\gamma)} \tag{73}$$

$$\cos \mathrm{C}b = \sqrt{(\alpha+\beta-\gamma)(\gamma+\alpha-\beta)} - \sqrt{(\gamma+\beta-\alpha)(2-\alpha-\beta-\gamma)} \tag{74}$$

$$\cos \mathrm{C}a = \sqrt{(\beta+\gamma-\alpha)(\beta+\alpha-\gamma)} + \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} \tag{75}$$

$$\cos \mathrm{A}c = \sqrt{(\beta+\gamma-\alpha)(\beta+\alpha-\gamma)} - \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} \tag{76}$$

Vbi quidem obseruare conuenit, hos valores figurae nostrae esse accomodatos, caeterumque binos inter se esse permutabiles, ita vt si pro cos Ab adhibeatur $\sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} - \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)}$, tum statuendum esse

$$\cos \mathrm{B}a = \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} + \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{77}$$

vnde colligitur producta cos Ab cos Ba; cos Bc cos Cb; cos Ca cos Ac, semper rationaliter per α, β, γ exprimi. Fiet autem

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 4\alpha\beta - 2(\alpha+\beta-\gamma), \tag{78}$$

quae valoribus debitis substitutis, simili modo colligetur esse,

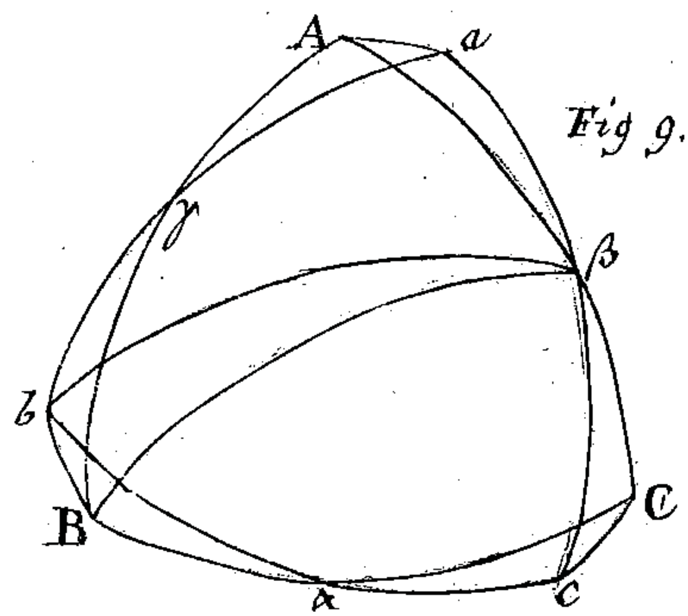
$$\cos \mathrm{B}c + \cos \mathrm{C}b = \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a \quad \text{et} \quad \cos \mathrm{C}a + \cos \mathrm{A}c = \cos \mathrm{C}c \cos \mathrm{A}a - \cos \mathrm{B}b \tag{79}$$

quas aequationes infra independenter a consideratione puncti Z, demonstrabimus.

Lemma II.

13. *Si conuersione sphaerae (Fig. 9) circa centrum suum fixum, puncta* A, B, C *quadrantibus inter se inuicem distantibus, peruenerint in puncta* a, b, c *et arcuum* AB, ab; AC; ac; BC; bc *intersectiones sint in punctis* γ, β, α; *erit arcus* Bb=ang. Aβa = Cβc; Aa = Bαb=Cαc; Cc = Aγa=Bγb;

Demonstratio.



Ducantur arcus Bβ, bβ et ob arcus AB=BC=90°, erit quoque Bβ = 90° et ang. BβA=90°, tum vero ob $ba = bc = 90°$, erit item $b\beta = 90°$ et ang. $b\beta$A=90°, vnde deducitur BβA=$b\beta a$ et Bβb=Aβa; atqui ob B$\beta = b\beta = 90°$, mensura anguli Bβb est arcus Bb; hinc idem arcus aequabitur angulo Aβa. Simili plane ratione ostenditur esse Aa=ang. Bαb et Cc= ang. Aγa.

Lemma III.

14. *In Tetragono* ACcb, *quod componitur ex arcubus circulorum maximorum in superficie sphaerica, si fuerit* AC=bc=90°, erit

$$\cos \mathrm{A}b = \sin \mathrm{AC}c \sin b c\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{80}$$

Demonstratio.

Ducatur arcus bC, eritque cos Ab = sin bC cos ACb, hincque ob ACb = AC$c - b$Cc,

$$\cos \mathrm{A}b = \sin b\mathrm{C}(\cos \mathrm{AC}c \cos b\mathrm{C}c + \sin \mathrm{AC}c \sin b\mathrm{C}c) \tag{81}$$

Iam notetur esse:

$$\sin b\mathrm{C} \sin b\mathrm{C}c = \sin bc\mathrm{C} \text{ et } \cot b\mathrm{C}c = -\cos \mathrm{C}c \cot bc\mathrm{C}, \text{ hincque etiam}$$
$$\sin \mathrm{C}b \cos b\mathrm{C}c = -\cos \mathrm{C}c \cos bc\mathrm{C},$$

his igitur valoribus substitutis, sit

$$\cos \mathrm{A}b = \sin \mathrm{AC}c \sin bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{82}$$

Theorema II.

15. *Si conuersione sphaerae circa centrum suum fixum, puncta* A, B, C *quadrantibus inter se distantia peruenerint in puncta* a, b, c, *et ducti supponantur arcus circulorum maximorum,* Aa, Ab, Ac, Ba, Bb, Bc, Ca, Cb, Cc, *sequens expressio semper ad nihilum redigetur*

$$\begin{aligned}(1-\cos \mathrm{A}a)(1-\cos \mathrm{B}b)(1-\cos \mathrm{C}c) \quad &+ \quad \cos \mathrm{B}c \cos \mathrm{C}b(1-\cos \mathrm{A}a) - \cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a \\ &+ \quad \cos \mathrm{A}c \cos \mathrm{C}a(1-\cos \mathrm{B}c) - \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a \\ &+ \quad \cos \mathrm{C}b \cos \mathrm{B}c(1-\cos \mathrm{C}c)\end{aligned} \tag{83}$$

Demonstratio.

Ex elementis Trigonometriae Sphaericae constat, esse in triangulo $\mathrm{A}\beta a$

$$\cos \mathrm{A}\beta a = \cos \mathrm{A}a \sin \beta\mathrm{A} \sin \beta a\mathrm{A} - \cos \beta\mathrm{A}a \cos \beta a\mathrm{A}, \tag{84}$$

hinc ob $\mathrm{A}\beta a$=Bb; $\beta\mathrm{A}a$=B$\mathrm{A}a - 90°$; $\beta a\mathrm{A}$=$ba\mathrm{A}$+90°, ita vt sit

$$\sin \beta\mathrm{A}a = -\cos \mathrm{BA}a \quad \text{et} \quad \cos \beta\mathrm{A}a = \sin \mathrm{BA}a, \quad \text{tumque} \tag{85}$$

$$\sin \beta a\mathrm{A} = \cos ba\mathrm{A} \quad \text{et} \quad \cos \beta a\mathrm{A} = -\sin ba\mathrm{A}, \quad \text{colligitur} \tag{86}$$

$$\cos \mathrm{B}b = \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A}. \tag{87}$$

Simili ratione ex consideratione trianguli $\mathrm{A}\gamma a$ deducitur,

$$\cos \mathrm{A}\gamma a = \cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{BA}a \cos ba\mathrm{A} \cos ba\mathrm{A}, \tag{88}$$

seu ob ang. $\mathrm{A}\gamma a$=arc. Cc; cos Cc = cos Aa sin BAa sin baA−cos BAa cos baA.
Si prior harum aequationum multiplicetur per cos Aa, obtinebimus

$$\cos \mathrm{A}a \cos \mathrm{B}b = \cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A}, \tag{89}$$

hinc igitur subtracta posteriori nostra aequatione, colligitur

$$\cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c = (1-\cos^2 \mathrm{A}a) \cos \mathrm{BA}a \sin ba\mathrm{A} = \sin^2 \mathrm{A}a \cos \mathrm{BA}a \sin ba\mathrm{A} \tag{90}$$

atqui est sin Aa cos BAa=cos Ba, et sin Aa cos baA=cos Ab, fiet igitur

$$\cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c = \cos \mathrm{A}b \cos \mathrm{B}a. \tag{91}$$

Porro ex consideratione trianguli Bγb colligitur:

$$\cos \mathrm{B}\gamma b = \cos \mathrm{C}c = \sin \mathrm{B}b \sin \mathrm{AB}b \sin ab\mathrm{B} - \cos \mathrm{AB}b \cos ab\mathrm{B}, \tag{92}$$

tumque ex triangulo Bab

$$\cos \mathrm{A}a = \cos \mathrm{B}ab = \cos \mathrm{B}b \sin \mathrm{CB}b \sin cb\mathrm{B} - \cos \mathrm{CB}b \cos cb\mathrm{B}. \tag{93}$$

Prior autem ob ABb=CB$b - 90°$ et abB=cbB+90°, in istam transformatur,

$$\cos \mathrm{C}c = \sin \mathrm{CB} \sin cb\mathrm{B} - \cos \mathrm{B}b \cos \mathrm{CB}b \cos cb\mathrm{B}, \tag{94}$$

multiplicata hac aequatione per cos Bb, et subtracta inde illa pro cos Aa, colligimus demum:

$$\cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a = (1 - \cos^2 \mathrm{B}b) \cos \mathrm{CB}b \cos cb\mathrm{B} = \sin^2 \mathrm{B}b \cos \mathrm{CB}b \cos cb\mathrm{B} \tag{95}$$

Ex quo manifesto deducitur

$$\cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a = \cos \mathrm{B}c \cos \mathrm{C}b. \tag{96}$$

Simili ratione quoque ostenditur esse,

$$\cos \mathrm{A}a \cos \mathrm{C}c - \cos \mathrm{B}b = \cos \mathrm{A}c \cos \mathrm{C}a. \tag{97}$$

Deinde pro quadrilatero ACcb, habemus per Lemma nostrum III:

$$\cos \mathrm{A}b = \cos \mathrm{AC}c \sin bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{98}$$

et pro quadrilatero aCcB, vi eiusdem Lemmatis:

$$\cos \mathrm{B}a = \sin \mathrm{BC}c \sin ac\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{BC}c \cos ac\mathrm{C}. \tag{99}$$

Quum nunc sit BCc=AC$c - 90°$; acC=bcC$-90°$, obtinebimus,

$$\cos \mathrm{B}a = \cos \mathrm{AC}c \cos bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C}, \tag{100}$$

hincque

$$\cos \mathrm{A}b \cos \mathrm{C}c + \cos \mathrm{B}a = (1 - \cos^2 \mathrm{C}c) \cos \mathrm{AC}c \cos bc\mathrm{C} = \sin^2 \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C}, \tag{101}$$

siue ob

$$\cos \mathrm{A}c = \sin \mathrm{C}c \cos \mathrm{AC}c \quad \text{et} \quad \cos \mathrm{C}b = \sin \mathrm{C}c \cos bc\mathrm{C}, \tag{102}$$

$$\cos \mathrm{A}b \cos \mathrm{C}c + \cos \mathrm{B}a = \cos \mathrm{A}c \cos \mathrm{C}b. \tag{103}$$

Simili ratione demonstrabitur esse:

$$\cos \mathrm{A}b \cos \mathrm{B}c = \cos \mathrm{B}b \cos \mathrm{A}c + \cos \mathrm{C}a, \tag{104}$$

multiplicata igitur priori aequatione per cos Ba et posteriori per cos Ca, additisque productis, consequemur:

$$1.\quad \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a = \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos^2 \mathrm{B}a \tag{105}$$
$$2.\quad \cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a = \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{C}a \quad \text{hinc} \tag{106}$$
$$\begin{aligned} 3.\quad & \cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a \\ & = \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a \\ & = \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + 1 - \cos^2 \mathrm{A}a \end{aligned} \tag{107}$$

ob $\cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = 1$.
Introducatur nunc hic valor pro $\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a$, in expressione nostra ad nihilum redigenda, cuius reliquos quoque terminos euolutos statuamus, vt eo melius appareat, quomodo singuli inuicem se destruant. Ad nihilum igitur redigi debebit haec expressio

$$\begin{array}{lll} - & \cos \mathrm{A}a + \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c & - \cos \mathrm{B}c \cos \mathrm{C}b + \cos \mathrm{B}c \cos \mathrm{C}b \cos \mathrm{A}a \\ - & \cos \mathrm{B}b + \cos \mathrm{A}a \cos \mathrm{C}c & - \cos \mathrm{A}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b \\ - & \cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}b & - \cos \mathrm{A}b \cos \mathrm{B}a + \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c \\ & & - \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c \\ & 1 + \cos^2 \mathrm{A}a & - \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b \end{array}$$

Deletis autem terminis, qui manifesto se tollunt, obseruemus etiam per demonstrata esse:

$$\begin{aligned} -\cos \mathrm{A}a + \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{B}c \cos \mathrm{C}b &= 0 \\ -\cos \mathrm{B}b + \cos \mathrm{A}a \cos \mathrm{C}c - \cos \mathrm{A}c \cos \mathrm{C}a &= 0 \\ -\cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{A}b \cos \mathrm{B}a &= 0 \end{aligned}$$

Denique etiam

$$\cos^2 \mathrm{A}a - \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}c \cos \mathrm{C}b = 0 \tag{108}$$

ita vt iam perfecte euictim sit, formulam istam propositam nihilo aequari.

Alia Demonstratio.

16. Quum prior demonstratio Geometrica sit et ex consideratione figurae deducta, nunc aliam insuper addamus Analyticam, quippe quae aequationibus in §. 5 allatis inuitetur. Ex aequatione igitur IV deducimus,

$$\cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{B}a \cos \mathrm{B}b = -\cos \mathrm{C}a \cos \mathrm{C}b \tag{109}$$

et sumendo quadrata,

$$\begin{aligned} \cos^2 \mathrm{A}a \cos^2 \mathrm{A}b \;&+\; \cos^2 \mathrm{B}a \cos^2 \mathrm{B}b + 2 \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{A}b \cos \mathrm{B}a = \cos^2 \mathrm{C}a \cos^2 \mathrm{C}b \\ &= \; (1 - \cos^2 \mathrm{A}a - \cos^2 \mathrm{B}a)(1 - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}b) \end{aligned} \tag{110}$$

per aequationes I et II. Euolutione igitur facta posterioris membri, et deletis terminis, qui se mutuo tollunt, consequemur:

$$\begin{aligned} & 2\cos\mathrm{A}a\cos\mathrm{B}b\cos\mathrm{A}b\cos\mathrm{B}a = 1-\cos^2\mathrm{A}a-\cos^2\mathrm{B}a-\cos^2\mathrm{B}b-\cos^2\mathrm{A}b \qquad (111)\\ + & \cos^2\mathrm{A}a\cos^2\mathrm{B}b+\cos^2\mathrm{A}b\cos^2\mathrm{B}a = \cos^2\mathrm{C}a+\cos^2\mathrm{C}b-1+\cos^2\mathrm{A}a\cos^2\mathrm{B}b+\cos^2\mathrm{A}b\cos^2\mathrm{B}a. \end{aligned}$$

Quum igitur sit (Fig. 4),

$$\cos^2\mathrm{C}a+\cos^2\mathrm{C}b-1 = -\cos^2\mathrm{C}c, \qquad (112)$$

habebimus

$$2\cos\mathrm{A}a\cos\mathrm{B}b\cos\mathrm{A}b\cos\mathrm{B}a = \cos^2\mathrm{A}a\cos^2\mathrm{B}b+\cos^2\mathrm{A}b\cos^2\mathrm{B}a-\cos^2\mathrm{C}c \qquad (113)$$

tumque

$$\begin{aligned} (\cos\mathrm{A}b\cos\mathrm{B}a-\cos\mathrm{A}a\cos\mathrm{B}b)^2 &= \cos^2\mathrm{C}c, \quad \text{et}\\ \cos\mathrm{A}b\cos\mathrm{B}a-\cos\mathrm{A}a\cos\mathrm{B}b &= \pm\cos\mathrm{C}c. \qquad (114) \end{aligned}$$

Quum hic pro cos Ab cos Ba duplex reperiatur valor, ratio istius duplicatis ita explicari potest, vt si concipiatur punctum c' quod ipsi c e diametro est oppositum, esse oporteat:

$$\cos\mathrm{A}b\cos\mathrm{B}a = \cos\mathrm{A}a\cos\mathrm{B}b-\cos\mathrm{C}c \quad \text{et} \quad \cos\mathrm{A}b\cos\mathrm{B}a = \cos\mathrm{A}b\cos\mathrm{B}b+\cos\mathrm{C}c'. \qquad (115)$$

Ex meris autem istis principiis Analyticis minime demonstrari potest, assumi debere

$$\cos\mathrm{A}b\cos\mathrm{B}a = \cos\mathrm{A}a\cos\mathrm{B}b-\cos\mathrm{C}c, \qquad (116)$$

non vero item

$$\cos\mathrm{A}b\cos\mathrm{B}a = \cos\mathrm{A}a\cos\mathrm{B}b+\cos\mathrm{C}c'; \qquad (117)$$

quicquid tamen sit supponamus priorem aequationem locum habere. Tum vero quoque eundem in modum ostendi potest, esse

$$\cos\mathrm{A}c\cos\mathrm{C}a = \cos\mathrm{A}a\cos\mathrm{C}c-\cos\mathrm{B}b \quad \text{et} \quad \cos\mathrm{B}c\cos\mathrm{C}b = \cos\mathrm{B}b\cos\mathrm{C}c+\cos\mathrm{A}a. \qquad (118)$$

Porro obseruetur, praeter illas aequationes §. 5 allatas, etiam sequentes locum habere:

$$\begin{aligned} &\text{VII.} && \cos^2\mathrm{A}a+\cos^2\mathrm{A}b+\cos^2\mathrm{A}c = 1, && (119)\\ &\text{VIII.} && \cos^2\mathrm{B}a+\cos^2\mathrm{B}b+\cos^2\mathrm{B}c = 1, && (120)\\ &\text{IX.} && \cos^2\mathrm{C}a+\cos^2\mathrm{C}b+\cos^2\mathrm{C}c = 1, && (121)\\ &\text{X.} && \cos\mathrm{A}a\cos\mathrm{B}a+\cos\mathrm{A}b\cos\mathrm{B}b+\cos\mathrm{A}c\cos\mathrm{B}c = 0, && (122)\\ &\text{XI.} && \cos\mathrm{B}a\cos\mathrm{C}a+\cos\mathrm{B}b\cos\mathrm{C}b+\cos\mathrm{B}c\cos\mathrm{C}c = 0, && (123)\\ &\text{XII.} && \cos\mathrm{A}a\cos\mathrm{C}a+\cos\mathrm{A}b\cos\mathrm{C}b+\cos\mathrm{A}c\cos\mathrm{C}c = 0. && (124) \end{aligned}$$

Nunc per aequationem X colligimus:

$$\begin{aligned} & \cos^2\mathrm{A}b\cos^2\mathrm{B}b+\cos^2\mathrm{A}c\cos^2\mathrm{B}c+2\cos\mathrm{A}b\cos\mathrm{B}c\cos\mathrm{B}b\cos\mathrm{A}c\\ = & \cos^2\mathrm{A}a\cos^2\mathrm{B}a = (1-\cos^2\mathrm{A}b-\cos^2\mathrm{A}c)(1-\cos^2\mathrm{B}b-\cos^2\mathrm{B}c) \qquad (125) \end{aligned}$$

per aequat. VII et VIII. Hinc facta euolutione prodit:

$$\begin{aligned} & 2\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{B}b \cos \mathrm{A}c = 1 - \cos^2 \mathrm{A}b - \cos^2 \mathrm{A}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{B}c \\ + \; & \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c + \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c = \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a - 1 + \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c \\ + \; & \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c = \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c + \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c - \cos^2 \mathrm{C}a, \end{aligned} \tag{126}$$

per aequat. I. §. 5. Manifesto autem hinc colligitur:

$$(\cos \mathrm{A}b \cos \mathrm{B}c - \cos \mathrm{B}b \cos \mathrm{A}c)^2 = \cos^2 \mathrm{C}a, \tag{127}$$

ideoque

$$\cos \mathrm{A}b \cos \mathrm{B}c - \cos \mathrm{B}b \cos \mathrm{A}c = \pm \cos \mathrm{C}a, \tag{128}$$

vbi quidem ex aequationibus nostris Analyticis minime decidi potest, quodnam signum pro cos Ca valeat. Statuamus tamen esse,

$$\cos \mathrm{A}b \cos \mathrm{B}c = \cos \mathrm{B}b \cos \mathrm{A}c + \cos \mathrm{C}a, \tag{129}$$

ideoque

$$\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a = \cos \mathrm{B}b \cos \mathrm{A}c \cos \mathrm{C}a + \cos^2 \mathrm{C}a. \tag{130}$$

Simili vero modo demonstrari quoque potest esse:

$$\cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a = \cos \mathrm{C}c \cos \mathrm{A}b \cos \mathrm{B}a + \cos^2 \mathrm{B}a. \tag{131}$$

Tum vero reliquum operis in eo consistet, vt hi valores in nostra expressione ad nihilum redigenda, substituantur; negotium in priore demonstratione iam exsecuti sumus, ideoque hic repetere superfluum foret.

17. Antequam vlterius progrediar, Geometris haud ingratum fore existimo, si heic dilucide exposuero, quibus ratiociniis ad demonstrationem istam priorem perductus sum. Primum igitur considerata ista expressione ad nihilum redigenda, existimaui in eo potissimum esse elaborandum, vt reliquae quantitates eam ingredientes, per tria Elementa nimirum cos Aa, cos Bb, cos Cc exprimi possent. Eum vero in finem, bina ista quadrilatera BAab, CAac, pro exquirendis valoribus ipsorum cos Ab, cos Ba, consideranda assumti. Quum enim in priori haberem (vide Tab III., Fig. 10) quatuor latera cognita, Aa, Bb, AB=90° et ab = 90°, et in posteriori item quatuor latera cognita Aa, Cc, AC=90° et ac = 90°, iam cogitare coepi, quomodo anguli BAa, baA, CAa, caA, ope arcuum Aa, Bb, Cc determinari possent; ista scilicet relatione in vsum vocata, quod sit CAa=BAa − 90° et caA=baA+90°. Hoc igitur agens ad istas binas perductus sum aequationes:

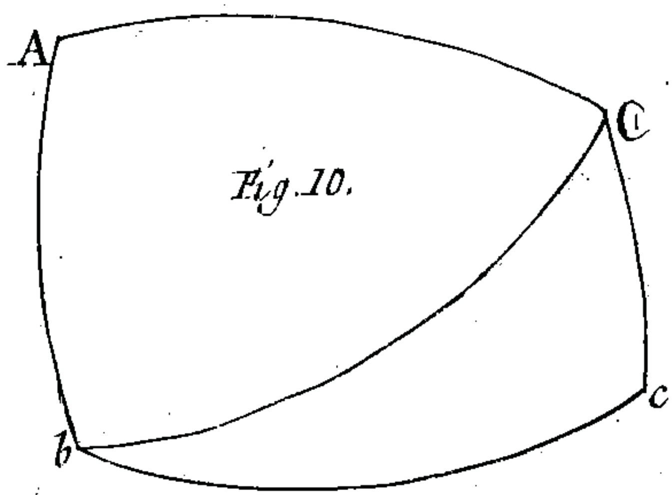
$$\cos \mathrm{B}b = \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} \tag{132}$$

$$\cos \mathrm{C}c = \cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{BA}a \cos ba\mathrm{A}, \tag{133}$$

quarum ope has binas alias collegi

$$\cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c = \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} \tag{134}$$

$$\cos \mathrm{B}b - \cos \mathrm{A}a \cos \mathrm{C}c = \sin^2 \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}. \tag{135}$$



Si ex his aequationibus eliminetur angulus B Aa, obtinebitur aequatio, quae praeter arcus Aa, Bb, Cc, non continet nisi angulum baA, vnde ob cos Ab = sin Aa cos baA, in procliui erat aequationem inuenire, quae praeter Aa, Bb, Cc, non continet nisi Ab. Verum, quum hac ratione ad aequationem perueniretur biquadraticam, leui adhibita attentione, perspexi tanto molimine non esse opus, quia ex ista aequatione:

$$\cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c = \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A}, \tag{136}$$

statim deducitur valor producti cos Ab cos Ba. Simili ratione, quum producta cos Ac cos Ca et cos Bc cos Cb determinentur, nunc id tantum restabat, vt quoque ista producta: cos Ab cos Bc cos Ca; cos Ac cos Cb cos Ba, per elementa tria Aa, Bb, Cc exprimerem; vbi quidem post aliquot tentamina, vidi rem perfici, quia prius aequatur ipsi

$$\cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{C}a,$$

et posterius ipsi

$$\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos^2 \mathrm{B}a,$$

ideoque ambo coniunctim huic expressioni:

$$\begin{aligned} &\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = \\ &\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + 1 - \cos^2 \mathrm{A}a \end{aligned} \tag{137}$$

quae expressio iam ita comparata est, vt facili negotio per meros cosinus ipsorum Aa, Bb, Cc exprimi queat; quod tamen pro demonstratione ipsa exsequenda minime e re est, vti ex antecedentibus patet.

18. Videamus nunc, quomodo valores ipsorum cos Ab, cos Ba etc. per elementa ista cos Aa, cos Bb, cos Cc exprimantur. Hunc in finem, consideremus formulas:

$$\cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c = \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} \tag{138}$$

$$\cos \mathrm{B}b - \cos \mathrm{A}a \cos \mathrm{C}c = \sin^2 \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}, \tag{139}$$

ex quibus sumendo quadrata colligitur:

$$\begin{aligned}\cos^2 \mathrm{B}b \cos^2 \mathrm{A}a + \cos^2 \mathrm{C}c - 2\cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c &= \sin^4 \mathrm{A}a \cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} \qquad (140)\\ \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}a \cos^2 \mathrm{C}c - 2\cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c &= \sin^4 \mathrm{A}a \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A}. \qquad (141)\end{aligned}$$

Sumta igitur differentia harum aequationum sit:

$$\sin^2 \mathrm{A}a(\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b) = \sin^4 \mathrm{A}a(\cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} - \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A}) \qquad (142)$$

et quum sit

$$\cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} - \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A} = \cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A} - 1, \qquad (143)$$

obtinebimus

$$\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b = \sin^2 \mathrm{A}a(\cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A} - 1) \qquad (144)$$

hincque

$$\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b + \sin^2 \mathrm{A}a = \sin^2 \mathrm{A}a(\cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A}) = \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}a. \qquad (145)$$

Erat vero

$$\cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c, \qquad (146)$$

proinde

$$\begin{aligned}(\cos \mathrm{A}b + \cos \mathrm{B}a)^2 &= \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}a + 2\cos \mathrm{A}b \cos \mathrm{B}a\\ &= \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b + \sin^2 \mathrm{A}a + 2\cos \mathrm{A}a \cos \mathrm{B}b - 2\cos \mathrm{C}c\\ &= 1 - 2\cos \mathrm{C}c + \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}a + 2\cos \mathrm{A}a \cos \mathrm{B}b\\ &= (1 - \cos \mathrm{C}c)^2 - (\cos \mathrm{B}b - \cos \mathrm{A}a)^2. \qquad (147)\end{aligned}$$

Si iam vti §. 12 ponamus $1-\cos\ \mathrm{C}c = 2\gamma$, $1-\cos\ \mathrm{B}b = 2\beta$ et $1-\cos\ \mathrm{A}a = 2\alpha$, consequemur

$$(\cos \mathrm{A}b + \cos \mathrm{B}a)^2 = 4\gamma^2 - 4(\alpha - \beta)^2, \qquad (148)$$

ideoque extrahendo radicem

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 2\sqrt{\gamma^2 - (\alpha - \beta)^2} = 2\sqrt{(\gamma - \alpha + \beta)(\gamma + \alpha - \beta)}. \qquad (149)$$

Deinde habebitur

$$\begin{aligned}(\cos \mathrm{A}b - \cos \mathrm{B}a)^2 &= 1 + 2\cos \mathrm{C}c + \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}a - 2\cos \mathrm{A}a \cos \mathrm{B}b\\ &= (1 + \cos \mathrm{C}c)^2 - (\cos \mathrm{A}a + \cos \mathrm{B}b)^2. \qquad (150)\end{aligned}$$

Cum igitur nunc sit

$$1 + \cos \mathrm{C}c = 2(1 - \gamma) \quad \text{et} \quad \cos \mathrm{A}a + \cos \mathrm{B}b = 2(1 - \alpha - \beta), \qquad (151)$$

fiet

$$(\cos \mathrm{A}b - \cos \mathrm{B}a)^2 = 4(1 - \gamma)^2 - 4(1 - \alpha - \beta)^2, \qquad (152)$$

hincque

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 2\sqrt{(1-\gamma)^2 - (1-\alpha-\beta)^2} = 2\sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)}. \tag{153}$$

Nunc igitur colligimus,

$$\cos \mathrm{A}b = \sqrt{(\gamma-\alpha+\beta)(\gamma+\alpha-\beta)} + \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{154}$$

$$\cos \mathrm{B}a = \sqrt{(\gamma-\alpha+\beta)(\gamma+\alpha-\beta)} - \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{155}$$

tumque simili modo:

$$\cos \mathrm{B}c = \sqrt{(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)} + \sqrt{(\beta+\gamma-\alpha)(2-\alpha-\beta-\gamma)} \tag{156}$$

$$\cos \mathrm{C}b = \sqrt{(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)} - \sqrt{(\beta+\gamma-\alpha)(2-\alpha-\beta-\gamma)} \tag{157}$$

$$\cos \mathrm{C}a = \sqrt{(\beta+\gamma-\alpha)(\beta-\gamma+\alpha)} + \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} \tag{158}$$

$$\cos \mathrm{A}c = \sqrt{(\beta+\gamma-\alpha)(\beta-\gamma+\alpha)} - \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} \tag{159}$$

Omnino vti supro §. 12 inuenimus. Praeterea heic quoque haud praeter rem erit, vt obseruemus, ob $(\cos \mathrm{A}b + \cos \mathrm{B}a)^2 = (1-\cos \mathrm{C}c)^2 - (\cos \mathrm{B}b - \cos \mathrm{A}a)^2$, fieri

$$\begin{aligned}\cos \mathrm{A}b + \cos \mathrm{B}a &= 2\sqrt{\sin^4 \frac{\mathrm{C}c}{2} - \sin^2 \frac{\mathrm{A}a+\mathrm{B}b}{2} \sin^2 \frac{\mathrm{A}a-\mathrm{B}b}{2}} \\ &= 2\sqrt{(\sin^2 \frac{\mathrm{C}c}{2} + \sin \frac{\mathrm{A}a+\mathrm{B}b}{2} \sin \frac{\mathrm{A}a-\mathrm{B}b}{2})(\sin^2 \frac{\mathrm{C}c}{2} - \sin \frac{\mathrm{A}a+\mathrm{B}b}{2} \sin \frac{\mathrm{A}a-\mathrm{B}b}{2})}\end{aligned}$$

eodemque modo

$$\begin{aligned}&\cos \mathrm{A}b - \cos \mathrm{B}a = \\ &= 2\sqrt{(\cos^2 \frac{\mathrm{C}c}{2} + \cos \frac{\mathrm{A}a+\mathrm{B}b}{2} \cos \frac{\mathrm{A}a-\mathrm{B}b}{2})(\cos^2 \frac{\mathrm{C}c}{2} - \cos \frac{\mathrm{A}a+\mathrm{B}b}{2} \cos \frac{\mathrm{A}a-\mathrm{B}b}{2})}\end{aligned}$$

19. Hac quoque occasione, operae pretium erit, nonnullas notasse relationes, quae quidem ad institutum nostrum proprie non spectant, elegantia tamen sua se valde commendant. Quum igitur supra inuenissemus

$$\cos \mathrm{BA}a \cos ba\mathrm{A} = \frac{\cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c}{\sin^2 \mathrm{A}a} \quad \text{et} \quad \sin \mathrm{BA}a \sin ba\mathrm{A} = \frac{\cos \mathrm{B}b - \cos \mathrm{A}a \cos \mathrm{C}c}{\sin^2 \mathrm{A}a}, \tag{160}$$

hinc colligimus

$$\begin{aligned}\cos(\mathrm{BA}a + ba\mathrm{A}) &= \cos \mathrm{BA}a \cos ba\mathrm{A} - \sin \mathrm{BA}a \sin ba\mathrm{A} \\ &= -(\cos \mathrm{B}b + \cos \mathrm{C}c)\frac{(1-\cos \mathrm{A}a)}{\sin^2 \mathrm{A}a} = -\frac{(\cos \mathrm{B}b + \cos \mathrm{C}c)}{1+\cos \mathrm{A}a}.\end{aligned} \tag{161}$$

Simili ratione consequemur

$$\begin{aligned}\cos(\mathrm{BA}a - ba\mathrm{A}) &= \cos \mathrm{BA}a \cos ba\mathrm{A} + \sin \mathrm{BA}a \sin ba\mathrm{A} \\ &= (\cos \mathrm{B}b - \cos \mathrm{C}c)\frac{(1+\cos \mathrm{A}a)}{\sin^2 \mathrm{A}a} = \frac{\cos \mathrm{B}b - \cos \mathrm{C}c}{1-\cos \mathrm{A}a}. \qquad (162)\end{aligned}$$

Quum vero sit

$$(\cos \mathrm{B}c + \cos \mathrm{C}b)^2 = (1-\cos \mathrm{A}a)^2 - (\cos \mathrm{B}b - \cos \mathrm{C}c)^2 \quad \text{et} \qquad (163)$$

$$(\cos \mathrm{B}c - \cos \mathrm{C}b)^2 = (1+\cos \mathrm{A}a)^2 - (\cos \mathrm{B}b + \cos \mathrm{C}c)^2, \qquad (164)$$

hinc concludemus esse

$$(\cos \mathrm{B}c+\cos \mathrm{C}b)^2 = (1-\cos \mathrm{A}a)^2(1-\cos^2(\mathrm{BA}a-ba\mathrm{A})) = (1-\cos \mathrm{A}a)^2 \sin^2(\mathrm{BA}a-ba\mathrm{A}), \qquad (165)$$

vnde $\cos \mathrm{B}c + \cos \mathrm{C}b = (1-\cos \mathrm{A}a)\sin(\mathrm{BA}a - ba\mathrm{A})$. Similis expressiones pro $\cos \mathrm{A}c + \cos \mathrm{C}a$, $\cos \mathrm{C}a - \cos \mathrm{A}c$ inueniri possunt, quas hic exponere nihil est necesse.

20. Si Z sit punctum, quod post conuersionem sphaerae eundem occupat locum ac in statu initiali (Fig. 11), supra videmus, esse debere:

$$\begin{aligned}\cos \mathrm{ZA} &= \cos \mathrm{ZA} \cos \mathrm{A}a + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c \\ \cos \mathrm{ZB} &= \cos \mathrm{ZA} \cos \mathrm{B}a + \cos \mathrm{ZB} \cos \mathrm{B}b + \cos \mathrm{ZC} \cos \mathrm{B}c \\ \cos \mathrm{ZC} &= \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC} \cos \mathrm{C}c\end{aligned}$$

In prima harum aequationum, loco cos Aa, cos Ab, cos Ac, introducantur eorum valores, qui sunt:

$$\cos \mathrm{A}a = \cos^2 \mathrm{AZ} + \sin^2 \mathrm{AZ} \cos^2 \mathrm{AZ}a \qquad (166)$$

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ} + \sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZ}b \qquad (167)$$

$$\cos \mathrm{A}c = \cos \mathrm{AZ} \cos \mathrm{CZ} + \sin \mathrm{AZ} \sin \mathrm{CZ} \cos \mathrm{AZ}c, \qquad (168)$$

hocque facto fiet

$$\begin{aligned}\cos \mathrm{AZ} &= \cos^3 \mathrm{AZ} + \cos \mathrm{AZ} \sin^2 \mathrm{AZ} \cos \mathrm{AZ}a + \cos \mathrm{AZ} \cos^2 \mathrm{BZ} \qquad (169) \\ &\quad + \cos \mathrm{BZ} \sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZ}b + \cos \mathrm{AZ} \cos^2 \mathrm{CZ} + \cos \mathrm{CZ} \sin \mathrm{AZ} \sin \mathrm{CZ} \cos \mathrm{AZ}c,\end{aligned}$$

quae ob $\cos^2\mathrm{AZ} + \cos^2\mathrm{BZ} + \cos^2\mathrm{CZ} = 1$, reducitur ad hanc formam:

$$0 = \sin \mathrm{AZ} \cos \mathrm{AZ} \cos \mathrm{AZ}a + \sin \mathrm{BZ} \cos \mathrm{BZ} \cos \mathrm{AZ}b + \sin \mathrm{CZ} \cos \mathrm{CZ} \cos \mathrm{AZ}c, \qquad (170)$$

vel

$$\sin 2\mathrm{AZ} \cos \mathrm{AZ}a + \sin 2\mathrm{BZ} \cos \mathrm{AZ}b + \sin 2\mathrm{CZ} \cos \mathrm{AZ}c = 0. \qquad (171)$$

Simili modo has obtinebimus aequationes:

$$0 = \sin 2\mathrm{AZ} \cos \mathrm{BZ}a + \sin 2\mathrm{BZ} \cos \mathrm{BZ}b + \sin 2\mathrm{CZ} \cos \mathrm{BZ}c \qquad (172)$$

$$0 = \sin 2\mathrm{AZ} \cos \mathrm{CZ}a + \sin 2\mathrm{BZ} \cos \mathrm{CZ}b + \sin 2\mathrm{CZ} \cos \mathrm{CZ}c. \qquad (173)$$

21. Si in superficie sphaerae (Fig. 11), puncta A, B, C, quadrantibus inter se distent, punctumque O sit id ipsum, quod post conuersionem sphaerae in eodem reperiatur situ, ac in statu initiali erat, tumque punctum quodcunque superficiei sphaericae Z post conuersionem peruenisse supponatur in z, quaeritur vt distantiae Az, Bz, Cz per distantias AO, BO, CO et angulum ZOz exprimantur. Heic igitur quum O idem designet punctum, quod §. 11 per Z indicauimus; facilitatis gratia, nunc ponamus AO=α, BO=β, CO=γ et angulum ZO$z = \Phi$, eritque vti §. 11:

$$\begin{aligned}
\cos \mathrm{A}a &= \cos^2\alpha(1-\cos\Phi)+\cos\Phi && (174)\\
\cos \mathrm{B}b &= \cos^2\beta(1-\cos\Phi)+\cos\Phi && (175)\\
\cos \mathrm{C}c &= \cos^2\gamma(1-\cos\Phi)+\cos\Phi && (176)\\
\cos \mathrm{A}b &= \cos\alpha\cos\beta(1-\cos\Phi)+\cos\gamma\sin\Phi && (177)\\
\cos \mathrm{B}a &= \cos\alpha\cos\beta(1-\cos\Phi)-\cos\gamma\sin\Phi && (178)\\
\cos \mathrm{B}c &= \cos\beta\cos\gamma(1-\cos\Phi)+\cos\alpha\sin\Phi && (179)\\
\cos \mathrm{C}b &= \cos\beta\cos\gamma(1-\cos\Phi)-\cos\alpha\sin\Phi && (180)\\
\cos \mathrm{C}a &= \cos\alpha\cos\gamma(1-\cos\Phi)+\cos\beta\sin\Phi && (181)\\
\cos \mathrm{A}c &= \cos\alpha\cos\gamma(1-\cos\Phi)-\cos\beta\sin\Phi && (182)
\end{aligned}$$

Quum igitur sit

$$\begin{aligned}
\cos \mathrm{A}z &= \cos \mathrm{AZ} \cos \mathrm{A}a + \cos \mathrm{BZ} \cos \mathrm{A}b + \cos \mathrm{CZ} \cos \mathrm{A}c\\
\cos \mathrm{B}z &= \cos \mathrm{AZ} \cos \mathrm{B}a + \cos \mathrm{BZ} \cos \mathrm{B}b + \cos \mathrm{CZ} \cos \mathrm{B}c\\
\cos \mathrm{C}z &= \cos \mathrm{AZ} \cos \mathrm{C}a + \cos \mathrm{BZ} \cos \mathrm{C}b + \cos \mathrm{CZ} \cos \mathrm{C}c
\end{aligned}$$

si dicatur AZ=ζ; BZ=η; CZ=θ tumque A$z = \zeta'$; B$z = \eta'$; C$z = \theta'$, consequemur:

$$\begin{aligned}
\cos\zeta' &= \cos\zeta\cos\Phi+(1-\cos\Phi)(\cos\zeta\cos^2\alpha+\cos\eta\cos\alpha\cos\beta+\cos\theta\cos\alpha\cos\gamma)\\
&+ \sin\Phi(\cos\eta\cos\gamma-\cos\theta\cos\beta) && (183)\\
\cos\eta' &= \cos\eta\cos\Phi+(1-\cos\Phi)(\cos\eta\cos^2\beta+\cos\zeta\cos\alpha\cos\beta+\cos\theta\cos\beta\cos\gamma)\\
&+ \sin\Phi(\cos\theta\cos\alpha-\cos\zeta\cos\gamma) && (184)\\
\cos\theta' &= \cos\theta\cos\Phi+(1-\cos\Phi)(\cos\theta\cos^2\gamma+\cos\zeta\cos\alpha\cos\gamma+\cos\eta\cos\beta\cos\gamma)\\
&+ \sin\Phi(\cos\zeta\cos\beta-\cos\eta\cos\alpha) && (185)
\end{aligned}$$

quae etiam sic exprimi possunt:

$$\begin{aligned}
\cos\zeta' &= \cos\zeta(\cos^2\alpha+\sin^2\alpha\cos\Phi)+\cos\eta(\cos\alpha\cos\beta(1-\cos\Phi)+\cos\gamma\sin\Phi)\\
&+ \cos\theta(\cos\alpha\cos\gamma(1-\cos\Phi)-\cos\beta\sin\Phi) && (186)\\
\cos\eta' &= \cos\eta(\cos^2\beta+\sin^2\beta\cos\Phi)+\cos\zeta(\cos\alpha\cos\beta(1-\cos\Phi)-\cos\gamma\sin\Phi)\\
&+ \cos\theta(\cos\beta\cos\gamma(1-\cos\Phi)+\cos\alpha\sin\Phi) && (187)\\
\cos\theta' &= \cos\theta(\cos^2\gamma+\sin^2\gamma\cos\Phi)+\cos\zeta(\cos\alpha\cos\gamma(1-\cos\Phi)+\cos\beta\sin\Phi)\\
&+ \cos\eta(\cos\beta\cos\gamma(1-\cos\Phi)-\cos\alpha\sin\Phi). && (188)
\end{aligned}$$

22. Denique cum supra §. 7 locum translatum puncti z per coordinatas x', y', z' ita definivimus, vt sit

$$x' = f + \mathrm{X}\cos\mathrm{B}b + \mathrm{Y}\cos\mathrm{B}c + \mathrm{Z}\cos\mathrm{B}a \tag{189}$$
$$y' = g + \mathrm{X}\cos\mathrm{C}b + \mathrm{Y}\cos\mathrm{C}c + \mathrm{Z}\cos\mathrm{C}a \tag{190}$$
$$z' = h + \mathrm{X}\cos\mathrm{A}b + \mathrm{Y}\cos\mathrm{A}c + \mathrm{Z}\cos\mathrm{A}a \tag{191}$$

si hic pro cos Aa, cos Ab etc., valores ipsorum substituantur, fiet

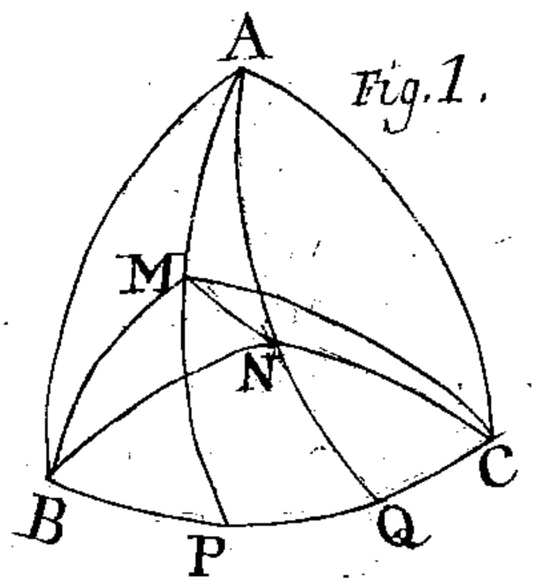
$$\begin{aligned} x' &= f + \mathrm{X}(\cos^2\beta + \sin^2\beta\cos\Phi) + \mathrm{Y}(\cos\beta\cos\gamma(1-\cos\Phi) + \cos\alpha\sin\Phi) \\ &+ \mathrm{Z}(\cos\alpha\cos\beta(1-\cos\Phi) - \cos\gamma\sin\Phi) \qquad (192) \\ y' &= g + \mathrm{X}(\cos\beta\cos\gamma(1-\cos\Phi) - \cos\alpha\sin\Phi) + \mathrm{Y}(\cos^2\gamma + \sin^2\gamma\cos\Phi) \\ &+ \mathrm{Z}(\cos\alpha\cos\gamma(1-\cos\Phi) + \cos\beta\sin\Phi) \qquad (193) \\ z' &= h + \mathrm{X}(\cos\alpha\cos\beta(1-\cos\Phi) + \cos\gamma\sin\Phi) + \mathrm{Y}(\cos\alpha\cos\gamma(1-\cos\Phi) - \cos\beta\sin\Phi) \\ &+ \mathrm{Z}(\cos^2\alpha + \sin^2\alpha\cos\Phi) \qquad (194) \end{aligned}$$

Hincque modo iam cognoscantur, translatio puncti I per ordinatas f, g, h, et distantiae puncti O ab A, B, C, quae per α, β, γ exprimantur, nec non angulus quo conuersio facta est ZO$z = \Phi$, etiam translatio cuiuslibet puncti corporis z expedite assignari potest.

Some general theorems of translation[1] of a rigid body
by Anders Johan Lexell (translated by Johan Sten)

1.

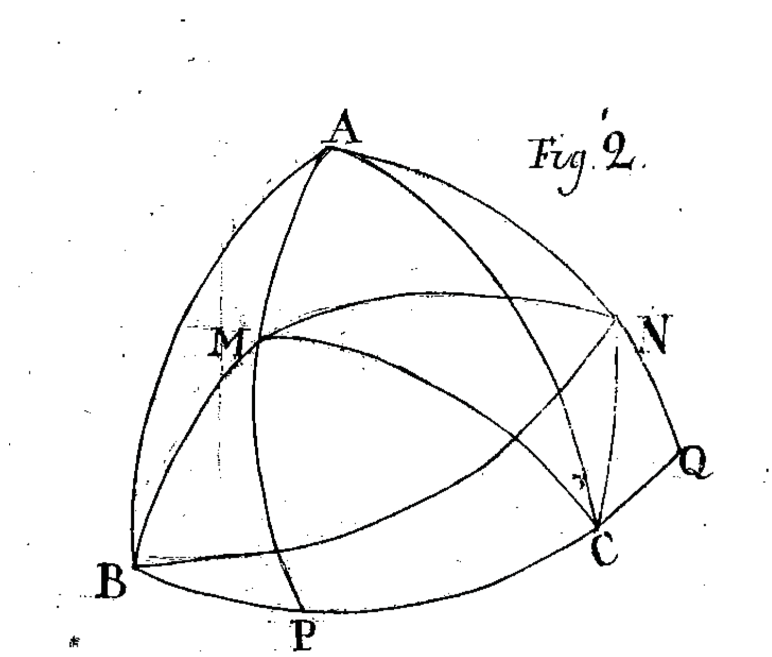
When the illustrious Euler recently worked on the translation of rigid bodies[2], he discovered this very elegant property, that in any translation of a rigid body there can always be found a straight line, which holds the parallel situation with that which it had in the initial state; and showed likewise that in order to find it, a certain analytical expression has to be rendered zero, yet how to do it, due to the complexity of the terms which go into this expression, was left for others to thoroughly examine. And so the illustrious man promptly advised in his mind how to investigate into a proof which could clearly show that this analytical equation is satisfied in every translation of a rigid body, and thus by the principles of analysis to entirely remove every doubt that in a translated state there be given such a straight line, which holds the parallel situation with which it had in the initial state. When I here plan to explain this proof, some other thoughts about translation of bodies are put forward from the considered facts; partly because they were prepared on the way to find this proof, and also since they can be serviceable for further confirming what the illustrious Euler has taught concerning the translation.



Lemma I.

[1]It is reminded that for Lexell, as for Euler, the Latin word 'translatione' can mean both a linear and a rotational movement or change of a body in a broad sense. In the present exposition the same word has been retained in English, since its meaning should be clear from the context.

[2]"Formulae generales pro translatione quacunque corporum rigidorum", *Novi Commentarii Academiae Scientiarum Petropolitanae*, A. MDCCLXXV, Tom. XX, pp. 189-207. E478.



2. *On the surface of a sphere (Fig. 1 & 2), take three points* A, B, C, *their mutual distances being quadrants, if any two other points* M, N *be taken on the same surface, then by joining them to each other as well as with the points* A, B, C *using arcs of a great circle, gives*:

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$

Proof.

Conceive the arcs AM, AN produced until they meet the great circle BC at P and Q. Then, since

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN} \cos \mathrm{MAN},$$

but the angle MAN is either = MAC − NAC, or MAN + NAC these cases are considered separately. Now, if *first*

$$\mathrm{MAN} = \mathrm{MAC} - \mathrm{NAC}, \text{ and then}$$

$$\cos \mathrm{MAN} = \cos \mathrm{MAC} \cos \mathrm{NAC} + \sin \mathrm{MAC} \sin \mathrm{NAC},$$

this value substituted for cos MAN, it follows that

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN} (\cos \mathrm{MAC} \cos \mathrm{NAC} + \sin \mathrm{MAC} \sin \mathrm{NAC})$$

It is true that

$$\begin{gathered}\sin \mathrm{AM} \cos \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{PC} = \cos \mathrm{MC} \\ \sin \mathrm{AN} \cos \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{QC} = \cos \mathrm{NC} \\ \sin \mathrm{AM} \sin \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{BP} = \cos \mathrm{BM} \\ \sin \mathrm{AN} \sin \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{BQ} = \cos \mathrm{BN}\end{gathered}$$

from which one clearly obtains (see Fig. 2),

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$

Second, if MAN = MAC + NAC then,

$$\cos \mathrm{MAN} = \cos \mathrm{MAC} \cos \mathrm{NAC} - \sin \mathrm{MAC} \sin \mathrm{NAC}, \text{ hence}$$

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \sin \mathrm{AM} \sin \mathrm{AN}\,(\cos \mathrm{MAC} \cos \mathrm{NAC} - \sin \mathrm{MAC} \sin \mathrm{NAC}),$$

then

$$\begin{gathered}\sin \mathrm{AM} \cos \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{CP} = \cos \mathrm{CM} \\ \sin \mathrm{AN} \cos \mathrm{NAC} = \cos \mathrm{NQ} \cos \mathrm{CQ} = \cos \mathrm{CN} \\ \sin \mathrm{AM} \sin \mathrm{MAC} = \cos \mathrm{MP} \cos \mathrm{BP} = \cos \mathrm{BM} \\ \sin \mathrm{AN} \sin \mathrm{NAC} = -\cos \mathrm{NQ} \cos \mathrm{BQ} = -\cos \mathrm{BN}\end{gathered}$$

from which it still follows

$$\cos \mathrm{MN} = \cos \mathrm{AM} \cos \mathrm{AN} + \cos \mathrm{BM} \cos \mathrm{BN} + \cos \mathrm{CM} \cos \mathrm{CN}.$$

Theorem I.

3. *If a sphere is turned around its fixed centre in whatever way, so that the points* A, B, C *a quadrant apart from each other arrive at the point* a, b, c, *then any point* Z *will be carried to* z, *so that*

$$\begin{gathered}\cos z\mathrm{A} = \cos \mathrm{ZA} \cos \mathrm{A}a + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c \\ \cos z\mathrm{B} = \cos \mathrm{ZA} \cos \mathrm{B}a + \cos \mathrm{ZB} \cos \mathrm{B}b + \cos \mathrm{ZC} \cos \mathrm{B}c \\ \cos z\mathrm{C} = \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC} \cos \mathrm{C}c\end{gathered}$$

Proof.

Because the situation of the point z with respect to the points a, b, c is the same as the situation of the point Z with respect to the points A, B, C, then za=ZA; zb=ZB; zc=ZC. But with the aid of the Lemma it was shown that

$$\begin{aligned}\cos z\mathrm{A} &= \cos za \cos \mathrm{A}a + \cos zb \cos \mathrm{A}b + \cos zc \cos \mathrm{A}c\\ \cos z\mathrm{B} &= \cos za \cos \mathrm{B}a + \cos zb \cos \mathrm{B}b + \cos zc \cos \mathrm{B}c\\ \cos z\mathrm{C} &= \cos za \cos \mathrm{C}a + \cos zb \cos \mathrm{C}b + \cos zc \cos \mathrm{C}c\end{aligned}$$

whence, if for za, zb, zc be substituted ZA, ZB, ZC, the equalities expressed by our Theorem will turn out.

4. By very similar reasoning it can be shown that :

$$\begin{aligned}\cos \mathrm{Z}a &= \cos za \cos a\mathrm{A} + \cos zb \cos a\mathrm{B} + \cos zc \cos a\mathrm{C}\\ \cos \mathrm{Z}b &= \cos za \cos b\mathrm{A} + \cos zb \cos b\mathrm{B} + \cos zc \cos b\mathrm{C}\\ \cos \mathrm{Z}c &= \cos za \cos c\mathrm{A} + \cos zb \cos c\mathrm{B} + \cos zc \cos c\mathrm{C}\end{aligned}$$

through which formulas the point Z of the situation of the initial state is determined by the points a, b, c.

5. If the point z falls in the very point Z, and thus Z is the point which after the rotation of the sphere has the same situation as in the initial state, the equations given above in the Theorem transform into these:

$$\begin{gathered}\cos \mathrm{ZA}\ (\cos \mathrm{A}a-1) + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c = 0\\ \cos \mathrm{ZA} \cos \mathrm{B}a + \cos \mathrm{ZB}\ (\cos \mathrm{B}b-1) + \cos \mathrm{ZC} \cos \mathrm{B}c = 0\\ \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC}\ (\cos \mathrm{C}c-1) = 0\end{gathered}$$

Now if owing to brevity, in place of these equations the following would be employed:

$$\text{I.}\quad \alpha x + \beta y + \gamma z = 0 \qquad \text{II.}\quad \alpha' x + \beta' y + \gamma' z = 0 \qquad \text{III.}\quad \alpha'' x + \beta'' y + \gamma'' z = 0 \tag{1}$$

where $x = \cos \mathrm{AZ}$; $y = \cos \mathrm{BZ}$; $z = \cos \mathrm{CZ}$; $\alpha = \cos \mathrm{A}a - 1$; $\beta = \cos \mathrm{A}b$; $\gamma = \cos \mathrm{A}c$; $\alpha' = \cos \mathrm{B}a$; $\beta' = \cos \mathrm{B}b - 1$; $\gamma' = \cos \mathrm{B}c$; $\alpha'' = \cos \mathrm{C}a$; $\beta'' = \cos \mathrm{C}b$; $\gamma'' = \cos \mathrm{C}c - 1$, by comparing the first one with the second we obtain

$$\left(\frac{\beta}{\alpha} - \frac{\beta'}{\alpha'}\right) y + \left(\frac{\gamma}{\alpha} - \frac{\gamma'}{\alpha'}\right) z = 0, \tag{2}$$

and then by comparing the first and the third

$$\left(\frac{\beta}{\alpha} - \frac{\beta''}{\alpha''}\right) y + \left(\frac{\gamma}{\alpha} - \frac{\gamma''}{\alpha''}\right) z = 0, \tag{3}$$

from the first one of which we get

$$\frac{y}{z} = \frac{\alpha\gamma - \alpha'\gamma}{\beta\alpha' - \beta'\alpha} \quad \text{and from the second} \quad \frac{y}{z} = \frac{\alpha\gamma'' - \alpha''\gamma}{\beta\alpha'' - \beta''\alpha}, \tag{4}$$

which values equated with each other produce this equation:

$$(\alpha\gamma' - \alpha'\gamma)(\beta\alpha'' - \beta''\alpha) = (\alpha\gamma'' - \alpha''\gamma)(\beta\alpha' - \beta'\alpha) \tag{5}$$

which after expansion contracts in the following

$$\alpha\beta'\gamma'' - \alpha\gamma'\beta'' - \alpha'\beta\gamma'' - \alpha''\beta'\gamma + \alpha''\beta\gamma' + \alpha'\beta''\gamma = 0. \tag{6}$$

Introducing now for α, β, γ etc. the values which they supposed to equal, our equation would have been expressed as such:

$$(\cos \mathrm{A}a - 1)(\cos \mathrm{B}b - 1)(\cos \mathrm{C}c - 1) - \cos \mathrm{B}c \cos \mathrm{C}b(\cos \mathrm{A}a - 1) - \cos \mathrm{B}a \cos \mathrm{A}b(\cos \mathrm{C}c - 1)$$

$$- \cos \mathrm{C}a \cos \mathrm{A}c(\cos \mathrm{B}b - 1) + \cos \mathrm{C}a \cos \mathrm{A}b \cos \mathrm{B}c + \cos \mathrm{B}a \cos \mathrm{A}c \cos \mathrm{C}b = 0. \tag{7}$$

But this equation coincides perfectly with the analytical expression, which the illustrious Euler announced in the article on translation of rigid bodies §22, but observing that the quantities expressed there by F, G, H, etc., are broken down here so that

$$\begin{gathered}\mathrm{F} = \cos\ \mathrm{A}a;\ \mathrm{G} = \cos\ \mathrm{B}a;\ \mathrm{H} = \cos\ \mathrm{C}a;\\ \mathrm{F}' = \cos\ \mathrm{A}b;\ \mathrm{G}' = \cos\ \mathrm{B}b;\ \mathrm{H}' = \cos\ \mathrm{C}b;\\ \mathrm{F}'' = \cos\ \mathrm{A}c;\ \mathrm{G}'' = \cos\ \mathrm{B}c;\ \mathrm{H}'' = \cos\ \mathrm{C}c;\end{gathered}$$

But although these identities are not entirely visible from the article in question, below an occasion will be given to show, that they will entirely take place. Meanwhile it is helpful to note, that the relationships given by the illustrious Euler through the letters F, G, H etc. in §18 of his article, will entirely take place, if these letters be bestowed their assigned values. Namely,

$$\text{I.} \quad \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = 1 \tag{8}$$

$$\text{II.} \quad \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}b + \cos^2 \mathrm{C}b = 1 \tag{9}$$

$$\text{III.} \quad \cos^2 \mathrm{A}c + \cos^2 \mathrm{B}c + \cos^2 \mathrm{C}c = 1 \tag{10}$$

and by our Lemma I:

$$\text{IV.} \quad \cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{A}a \cos \mathrm{A}b = \cos ab = 0 \tag{11}$$

$$\text{V.} \quad \cos \mathrm{A}a \cos \mathrm{A}c + \cos \mathrm{B}a \cos \mathrm{B}c + \cos \mathrm{C}a \cos \mathrm{C}c = \cos ac = 0 \tag{12}$$

$$\text{VI.} \quad \cos \mathrm{A}b \cos \mathrm{A}c + \cos \mathrm{B}b \cos \mathrm{B}c + \cos \mathrm{C}b \cos \mathrm{C}c = \cos bc = 0. \tag{13}$$

$$\tag{14}$$

6. But if it can be shown, that in every rotation of a sphere about its fixed centre, the expression

$$(\cos \mathrm{A}a - 1)(\cos \mathrm{B}b - 1)(\cos \mathrm{C}c - 1) - \cos \mathrm{B}c \cos \mathrm{C}b(\cos \mathrm{A}a - 1) - \cos \mathrm{B}a \cos \mathrm{A}b(\cos \mathrm{C}c - 1)$$

$$- \cos \mathrm{C}a \cos \mathrm{A}c(\cos \mathrm{B}b - 1) + \cos \mathrm{C}a \cos \mathrm{A}b \cos \mathrm{B}c + \cos \mathrm{B}a \cos \mathrm{A}c \cos \mathrm{C}b \tag{15}$$

must vanish; there clearly must exist on its surface some point Z, which after a rotation will be in the same place, as it had in the initial state. But before really undertaking this demonstration, the importance of the work will be explained as how the formulas of our first

Theorem for expressing the translation of any point of a rigid body can be employed. But here we observe that of every translation of a rigid body there are two kinds, the former of which involves change of location, in which all particles of a body follow in directions mutually parallel, which translation besides is usually called by the name of progressive motion. The latter kind of translation contains rotations of bodies about a certain fixed point, where indeed it is the same as if this point be conceived inside the body or outside it. If, therefore, by

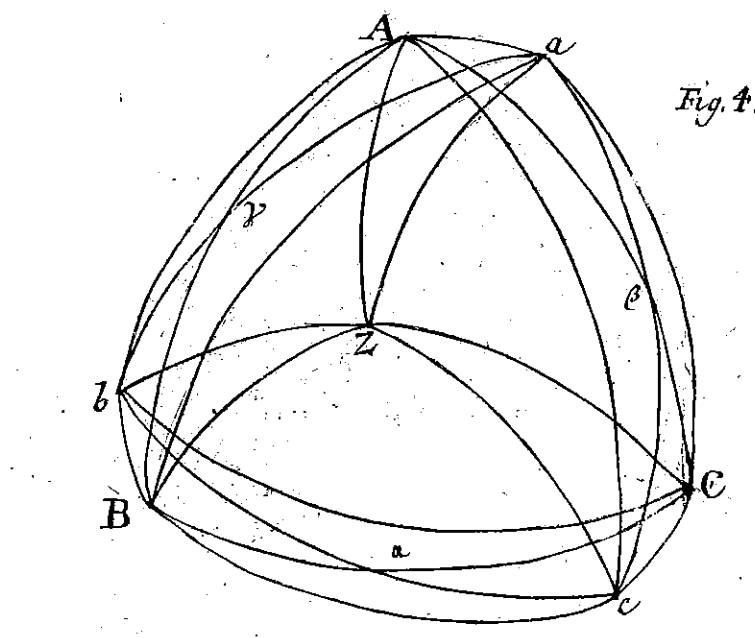


Fig. 4.

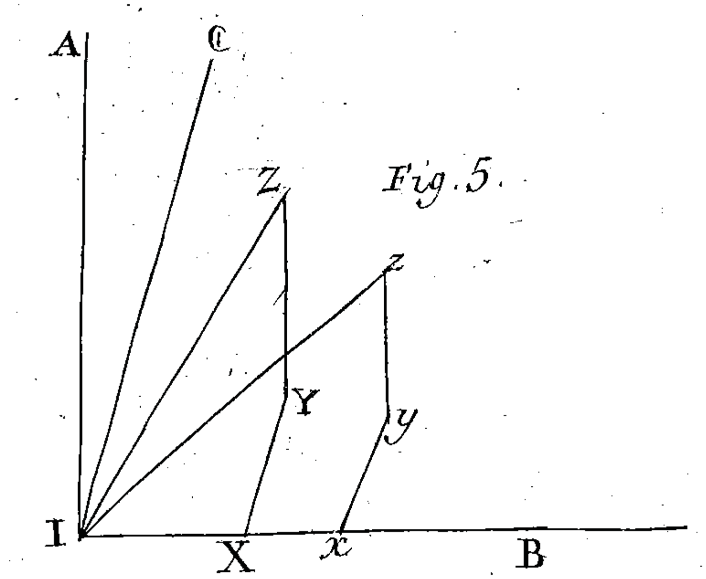
reason we first do away with the progressive motion to consider only rotations about a fixed point, conceiving the fixed point to be in I and drawing through it three axes IA, IB, IC normal to each other, by means of which the situation of any point Z of a body in its initial state is given by the three coordinates IX, XY, YZ, then supposing indeed that the rotation of the body about the point I has been made, the point Z arriving at z, and its coordinates being determined by the coordinates Ix, xy, yz, then there is a distance Iz=IZ, which we denote by the letter s. Now if it be understood that the sphere is firmly connected to our body, whose centre be in I and whose semidiameter equals s itself, the axes IA, IB, IC meet with this sphere at the points A, B, C, and after the turning of the sphere about I these points arrive at a, b, c. It is easily seen that the angles AIz, BIz, CIz in Fig. 5. be equal the respective angles Az, Bz, Cz of Fig. 4. And yet
$\frac{\mathrm{I}x}{\mathrm{I}z} = \cos \mathrm{BI}z = \cos \mathrm{B}z$; $\frac{xy}{\mathrm{I}z} = \cos \mathrm{CI}z = \cos \mathrm{C}z$; $\frac{xy}{\mathrm{I}z} = \cos \mathrm{AI}z = \cos \mathrm{A}z$;
by means of which, if it is asserted,

$$\mathrm{I}x = x; \quad xy = y; \quad yz = z; \quad \text{it follows} \tag{16}$$

$$\frac{x}{s} = \cos \mathrm{B}z; \quad \frac{y}{s} = \cos \mathrm{C}z; \quad \frac{z}{s} = \cos \mathrm{A}z. \tag{17}$$

In like manner it results that

$$\frac{\mathrm{IX}}{\mathrm{IZ}} = \cos \mathrm{BZ}; \quad \frac{\mathrm{XY}}{\mathrm{IZ}} = \cos \mathrm{C}Z; \quad \frac{\mathrm{XZ}}{\mathrm{IZ}} = \cos \mathrm{A}z, \tag{18}$$



whence if IX, XY, YZ be expressed by the respective letters X, Y, Z we obtain:

$$\frac{\mathrm{X}}{s} = \cos \mathrm{BZ}; \quad \frac{\mathrm{Y}}{s} = \cos \mathrm{CZ}; \quad \frac{\mathrm{Z}}{s} = \cos \mathrm{A}z, \tag{19}$$

Hence by the aid of the formulas of our first Theorem, we deduce the following expressions for x, y, z:

$$z = \mathrm{Z} \cos \mathrm{A}a + \mathrm{X} \cos \mathrm{A}b + \mathrm{Y} \cos \mathrm{A}c \tag{20}$$

$$x = \mathrm{Z} \cos \mathrm{B}a + \mathrm{X} \cos \mathrm{B}b + \mathrm{Y} \cos \mathrm{B}c \tag{21}$$

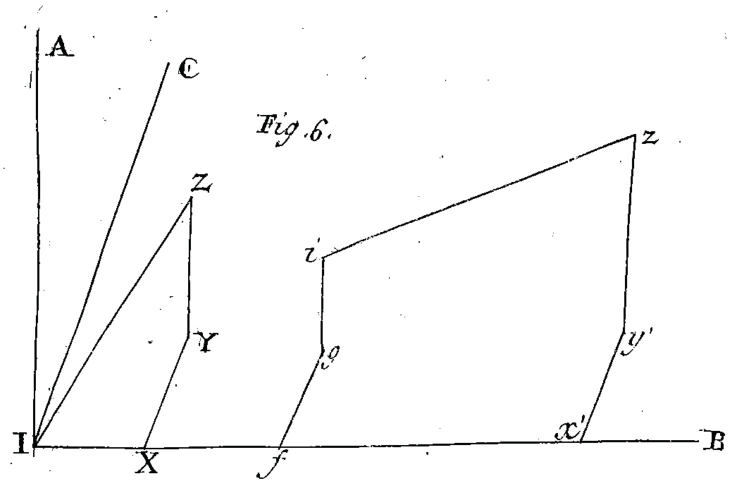
$$y = \mathrm{Z} \cos \mathrm{C}a + \mathrm{X} \cos \mathrm{C}b + \mathrm{Y} \cos \mathrm{C}c \tag{22}$$

from which it is evident that by means of knowing the coordinates of the point Z for the initial state, and the translation of the points A, B, C into the points a, b, c, also the location of the point Z translated into z can be easily determined. 7. Now if we also want the rules for progressive motion in the translation of a body, let us in the meantime, while the body rotates about I and the point itself arriving in i, conceive the location of that point being determined by the three coordinates If, fg, gi, which we express by the respective letters f, g, h. Then, the translation accomplished, the point Z will be contained in z, and the location of this point z is determined by the coordinates Ix', $x'y'$, $y'z$, which we denote by the respective letters x', y', z'; doing this it will be obvious by slight inspection from Fig. 6 I$x'$$-If$=I$x$ in Fig. 5, likewise $x'y' - fg = xy$ and also $zy' - ig = zy$; the rigour of the entire proof could be strengthen by means of the three lines ia, ib, ic parallel with IA, IB, IC ; although we do not show them in our figure, to avoid too much complication. Hence it will then result $x' = f + x$; $y' = g + y$; $z' = h + z$, wherefore by using the found values for x, y, z the following expressions for x', y', z' are obtained:

$$x' = f + \mathrm{X} \cos \mathrm{B}b + \mathrm{Y} \cos \mathrm{B}c + \mathrm{Z} \cos \mathrm{B}a \tag{23}$$

$$y' = g + \mathrm{X} \cos \mathrm{C}b + \mathrm{Y} \cos \mathrm{C}c + \mathrm{Z} \cos \mathrm{C}a \tag{24}$$

$$z' = h + \mathrm{X} \cos \mathrm{A}b + \mathrm{Y} \cos \mathrm{A}c + \mathrm{Z} \cos \mathrm{A}a \tag{25}$$



so that for determining x', y', z', just as for the translation of a point I expressed by the ordinates f, g, h, it would require the rotation of a body about the point I determined by the angles Aa, Bb, Cc, to be known. But if these formulas given here be compared with those found by the illustrious Euler in §10 of his more often commemorated article, then it will be perfectly evident that the letters F, G, H etc. should be attributed the values, which we above assigned for them. In fact, in these formulas for x', y', z' one may see nine quantities occurring for translations of the points A, B, C; Yet, as soon as what remains of these three cos Aa, cos Bb, cos Cc, as will be seen next, would be determined; the point z can be considered perfectly determined through the six elements, namely the coordinates f, g, h and the arcs Aa, Bb, Cc.

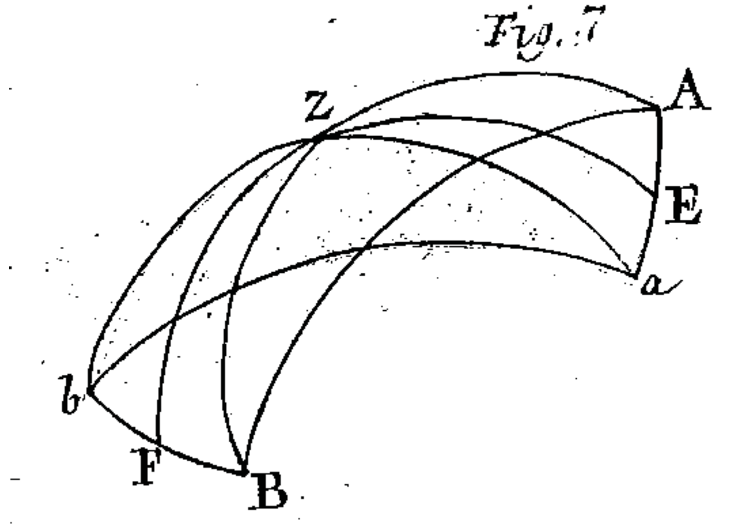
Problem.

8. *For any kind of rotation of a sphere about its fixed centre, find the point on its surface, which after the rotation is found in the same place where it was before the rotation.*

Solution.

Admit on the surface of a sphere (Fig. 7) any arc AB, which through a rotation of the sphere is supposed to arrive at ab, so that the point A comes to a and B to b, then join Aa, Bb by arcs of great circles. The arcs Aa, Bb are bisected in the points E and F, and the arcs EZ, FZ are drawn normal to Aa, Bb, that meet each other in Z, giving the point, which after the rotation will be in the same place as it had in the initial situation. As it is evident that AE $= a$E, the angle AEZ = angle aEZ = 90° and EZ is common for the triangles AEZ, aEZ, it follows that AZ $= a$Z ; in the same manner BZ $= b$Z. Now in the triangles AZB, aZb, there is AZ $= a$Z, BZ $= b$Z and AB $= ab$, therefore the angles ABZ $= ab$Z and BAZ $= ba$Z ; from which it follows that the point Z perfectly holds its place with respect to the arc ab, as also with respect to the arc AB, so that this point is in the same location, as it was before the rotation.

9. It can perhaps seem strange to someone that I should attack this problem before I have proved that the formula given in § 5 vanishes; seeing, of course, that the solution cannot be



valid at all if not this expression in all rotations of the sphere will take place. Indeed, first, it will be noted, that the problem itself cannot admit a solution, if not the equation of §. 5 be true; yet the solution to the problem itself can be prepared independently of this equation, and then from its natural solution it is easily seen, whether it would take place always and in every case. In our solution there certainly is no doubt that the arcs EZ et FZ intersect each other normally at Aa and Bb, one case excepted, that they coincide; then the arc EZF will certainly go across the point in which the arcs AB, ab intersect, more specifically this intersection point coincides with the point Z; therefore one cannot doubt, that our problem would not always admit a true and real solution. Indeed, in this primer I think I now have prepared for the solution, that would open me a way, by which I may undertake to determine the point Z from the given arcs Aa, Bb, Cc, which investigation brings me forth to the hidden relations, which go between the arcs Aa, Bb, Cc et Ab, Ac, Ba, Bc, Ca, Cb.

10. Now to understand, how to determine the point Z through the arcs Aa, Bb, Cc, let us suppose these arcs bisected by the arcs EZ, FZ, GZ themselves normal to the respective arc, the arcs EZ, FZ, GZ which run through the same point Z; then it is true that:

$$\sin^2 \mathrm{AZ} \sin^2 \frac{1}{2}\mathrm{AZ}a = \sin^2 \frac{1}{2}\mathrm{A}a \tag{26}$$

$$\sin^2 \mathrm{BZ} \sin^2 \frac{1}{2}\mathrm{BZ}b = \sin^2 \frac{1}{2}\mathrm{B}b \quad \text{and} \tag{27}$$

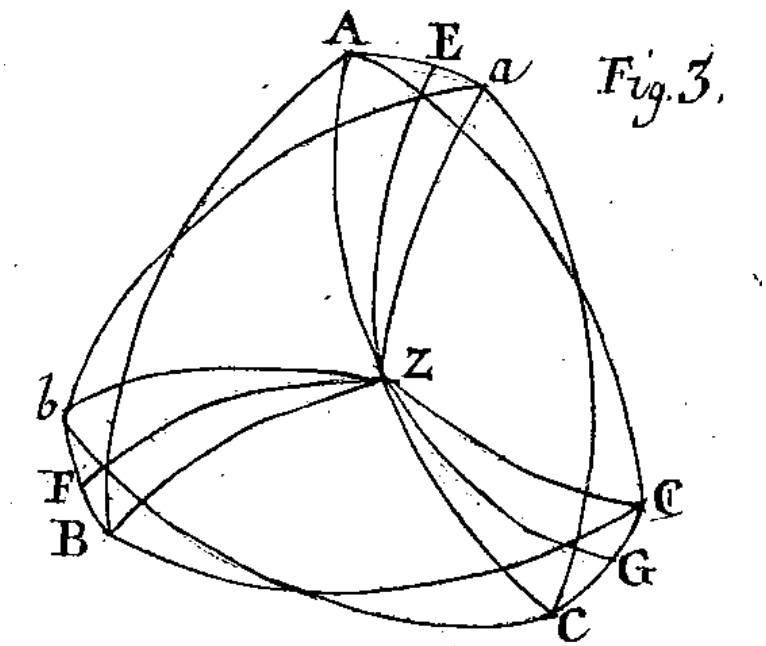
$$\sin^2 \mathrm{CZ} \sin^2 \frac{1}{2}\mathrm{CZ}c = \sin^2 \frac{1}{2}\mathrm{C}c \tag{28}$$

and because AZa=BZb=CZc,

$$\sin^2 \frac{1}{2}\mathrm{AZ}a(\sin^2 \mathrm{AZ} + \sin^2 \mathrm{BZ} + \sin^2 \mathrm{CZ}) = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c. \tag{29}$$

Hence it is concluded

$$\sin^2 \frac{1}{2}\mathrm{AZ}a(3 - \cos^2 \mathrm{AZ} + \cos^2 \mathrm{BZ} + \cos^2 \mathrm{CZ}) = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c, \tag{30}$$



which owing to

$$\cos^2 \mathrm{AZ} + \cos^2 \mathrm{BZ} + \cos^2 \mathrm{CZ} = 1, \tag{31}$$

lead rto the following equations:

$$2\sin^2 \frac{1}{2}\mathrm{AZ}a = \sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c. \tag{32}$$

Instead of this, also this can be employed,

$$2\cos^2 \mathrm{AZ}a = \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}b + \cos^2 \mathrm{C}c - 1. \tag{33}$$

From this it is obtained that

$$\sin^2 \mathrm{AZ} = \frac{\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{AZ}a} = \frac{2\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c} \tag{34}$$

$$\sin^2 \mathrm{AZ} = \frac{2\sin^2 \frac{1}{2}\mathrm{B}b}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c} \tag{35}$$

$$\sin^2 \mathrm{AZ} = \frac{2\sin^2 \frac{1}{2}\mathrm{A}a}{\sin^2 \frac{1}{2}\mathrm{A}a + \sin^2 \frac{1}{2}\mathrm{B}b + \sin^2 \frac{1}{2}\mathrm{C}c}, \tag{36}$$

and so, taken together, the condition is satisfied, that

$$\sin^2 \mathrm{AZ} + \sin^2 \mathrm{BZ} + \sin^2 \mathrm{CZ} = 2. \tag{37}$$

In fact, these expressions can also be transformed in the ratios: when

$$2\sin^2 \mathrm{AZ} = \frac{8\sin^2 \frac{1}{2}\mathrm{A}a}{2\sin^2 \frac{1}{2}\mathrm{A}a + 2\sin^2 \frac{1}{2}\mathrm{B}b + 2\sin^2 \frac{1}{2}\mathrm{C}c}, \tag{38}$$

thanks to $2\sin^2 \mathrm{AZ} = 1-\cos 2\mathrm{AZ}$; $2\sin^2 \frac{1}{2}\mathrm{A}a = 1-\cos \mathrm{A}a$; $2\sin^2 \frac{1}{2}\mathrm{B}b = 1-\cos \mathrm{B}b$; $2\sin^2 \frac{1}{2}\mathrm{C}c = 1-\cos \mathrm{C}c$, it follows

$$1-\cos 2\mathrm{AZ} = \frac{4(1-\cos \mathrm{A}a)}{3-\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c}, \qquad \text{and hence} \tag{39}$$

$$\cos 2\mathrm{AZ} = \frac{3\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c-1}{3-\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c}, \tag{40}$$

in a similar way

$$\cos 2\mathrm{BZ} = \frac{3\cos \mathrm{B}b-\cos \mathrm{A}a-\cos \mathrm{C}c-1}{3-\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c}; \quad \cos 2\mathrm{CZ} = \frac{3\cos \mathrm{C}c-\cos \mathrm{B}b-\cos \mathrm{C}c-1}{3-\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c}; \tag{41}$$

and therefore

$$\cos 2\mathrm{AZ}+\cos 2\mathrm{AZ}+\cos 2\mathrm{AZ} = \frac{\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c-3}{3-\cos \mathrm{A}a-\cos \mathrm{B}b-\cos \mathrm{C}c} = -1, \tag{42}$$

precisely as it should be.

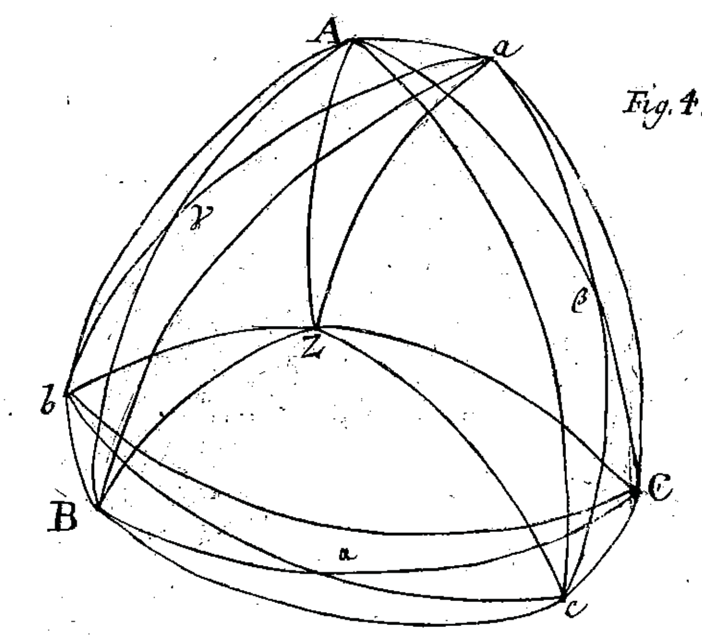


Fig. 4.

11. Now to obtain the relations between the arcs Ab, Ac, Ba, Bc, Ca, Cb in terms of Aa, Bb, Cc; we first determine them through [the arcs] AZ, BZ, CZ and the angle AZa. On the other hand it is agreed that

$$\cos \mathrm{A}a = \cos^2 \mathrm{AZ} + \sin^2 \mathrm{AZ} \cos \mathrm{AZ}a \tag{43}$$

$$\cos \mathrm{B}b = \cos^2 \mathrm{BZ} + \sin^2 \mathrm{BZ} \cos \mathrm{AZ}a \tag{44}$$

$$\cos \mathrm{C}c = \cos^2 \mathrm{CZ} + \sin^2 \mathrm{CZ} \cos \mathrm{AZ}a. \tag{45}$$

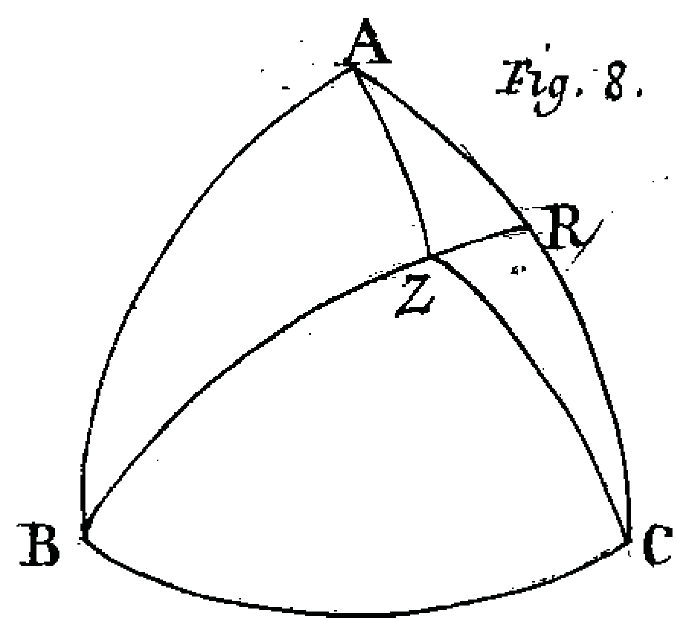
Furthermore (Fig. 4)

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos b\mathrm{Z} + \sin \mathrm{AZ} \sin b\mathrm{Z} \cos \mathrm{AZ}b, \tag{46}$$

from which, owing to bZ=BZ and the angles AZb=AZB−BZb, we get

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ} + \sin \mathrm{AZ} \cos \mathrm{BZ}(\cos \mathrm{AZB} \cos \mathrm{BZ}b + \sin \mathrm{AZB} \sin \mathrm{BZ}b) \tag{47}$$

From the theory of the sphere (see Fig. 8) cos AZB = −cot AZ cot BZ, hence



$$\sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZB} = -\cos \mathrm{AZ} \cos \mathrm{BZ}.$$

Furthermore, we also have

$$\sin \mathrm{AZB} = \frac{\cos c\mathrm{Z}}{\sin \mathrm{AZ} \sin \mathrm{BZ}}, \tag{48}$$

which is showed in the following manner. Let the arc BZ be produced until it occurs with the great circle AC at R, then

$$\sin \mathrm{AZ} \sin \mathrm{AZB} = \sin \mathrm{AR} = \cos \mathrm{CR} = \frac{\cos \mathrm{CZ}}{\cos \mathrm{ZR}} = \frac{\cos \mathrm{CZ}}{\sin \mathrm{BZ}}. \tag{49}$$

Hence

$$\sin \mathrm{AZ} \sin \mathrm{BZ} \sin \mathrm{AZB} = \cos \mathrm{CZ} \tag{50}$$

is obtained. These values substituted produce:

$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{BZ}b) + \cos \mathrm{CZ} \cos \mathrm{BZ}b. \tag{51}$$

In the same way, on account of

$$\cos \mathrm{B}a = \cos \mathrm{BZ} \cos a\mathrm{Z} + \sin \mathrm{BZ} \cos a\mathrm{Z} \sin \mathrm{BZ}a, \tag{52}$$

where aZ=AZ and BZa=BZA+AZa,

$$\cos \mathrm{B}a = \cos \mathrm{BZ} \cos \mathrm{AZ} + \sin \mathrm{BZ} \cos \mathrm{AZ}(\cos \mathrm{AZB} \cos \mathrm{BZ}b - \sin \mathrm{AZB} \sin \mathrm{AZ}a) \tag{53}$$

is obtained, which owing to

$$\sin \mathrm{BZ} \sin \mathrm{AZ} \cos \mathrm{AZB} = -\cos \mathrm{AZ} \cos \mathrm{BZ} \quad \text{and} \quad \sin \mathrm{BZ} \sin \mathrm{AZ} \sin \mathrm{AZB} = \cos \mathrm{CZ}, \tag{54}$$

reduces into this

$$\cos \mathrm{B}a = \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{AZ}a) - \cos \mathrm{CZ} \sin \mathrm{AZ}a. \tag{55}$$

These expressions follow in the same way:

$$\begin{aligned} \cos \mathrm{A}c &= \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{CZ}c) - \cos \mathrm{BZ} \sin \mathrm{CZ}c && (56)\\ \cos \mathrm{C}a &= \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{CZ}c) - \cos \mathrm{BZ} \sin \mathrm{CZ}c && (57)\\ \cos \mathrm{B}c &= \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{BZ}c) - \cos \mathrm{AZ} \sin \mathrm{BZ}b && (58)\\ \cos \mathrm{C}b &= \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{BZ}c) - \cos \mathrm{AZ} \sin \mathrm{BZ}b, && (59) \end{aligned}$$

where the angles AZa, BZb, CZc being equal to each other, may be used as one. Then, two of these formulas combined with each other give us:

$$\begin{aligned} \cos \mathrm{A}b + \cos \mathrm{B}a &= 2 \cos \mathrm{AZ} \cos \mathrm{BZ}(1 - \cos \mathrm{AZ}a); && (60)\\ \cos \mathrm{A}b - \cos \mathrm{B}a &= 2 \cos \mathrm{CZ} \sin \mathrm{AZ}a && (61)\\ \cos \mathrm{A}c + \cos \mathrm{C}a &= 2 \cos \mathrm{AZ} \cos \mathrm{CZ}(1 - \cos \mathrm{AZ}a); && (62)\\ \cos \mathrm{C}a - \cos \mathrm{A}c &= 2 \cos \mathrm{BZ} \sin \mathrm{AZ}a && (63)\\ \cos \mathrm{B}c + \cos \mathrm{C}b &= 2 \cos \mathrm{BZ} \cos \mathrm{CZ}(1 - \cos \mathrm{AZ}a); && (64)\\ \cos \mathrm{B}c - \cos \mathrm{C}b &= 2 \cos \mathrm{AZ} \sin \mathrm{AZ}a && (65) \end{aligned}$$

Finally, we arrive at this outstanding property that

$$\begin{aligned} \cos^2 \mathrm{A}b - \cos^2 \mathrm{B}a &= \cos^2 \mathrm{B}c - \cos^2 \mathrm{C}b = \cos^2 \mathrm{C}a - \cos^2 \mathrm{A}c && (66)\\ &= 4 \cos \mathrm{AZ} \cos \mathrm{BZ} \cos \mathrm{CZ} \sin \mathrm{AZ}a(1 - \cos \mathrm{AZ}a), && (67) \end{aligned}$$

or

$$\cos 2\mathrm{A}b - \cos 2\mathrm{B}a = \cos 2\mathrm{B}c - \cos 2\mathrm{C}b = \cos 2\mathrm{C}a - \cos 2\mathrm{A}c. \tag{68}$$

12. If in the values discovered above for cos Ab, cos Ba, cos Ca, cos Bc, cos Cb, in place of cos AZ, cos BZ, cos CZ, cos AZa and sin AZa be substituted their values expressed by cos Aa, cos Bb, cos Cc, the cosine of the first of these arcs will also be defined by the arcs Aa, Bb, Cc. But to make this easier, let us put

$$\sin^2 \frac{1}{2}\mathrm{A}a = \alpha; \sin^2 \frac{1}{2}\mathrm{B}b = \beta; \sin^2 \frac{1}{2}\mathrm{C}c = \gamma; \tag{69}$$

from which it follows

$$\cos \mathrm{AZ} = \sqrt{\frac{\beta+\gamma-\alpha}{\beta+\gamma+\alpha}}; \qquad \cos \mathrm{BZ} = \sqrt{\frac{\alpha+\gamma-\beta}{\alpha+\beta+\gamma}}; \qquad \cos \mathrm{CZ} = \sqrt{\frac{\alpha+\beta-\gamma}{\alpha+\beta+\gamma}}; \tag{70}$$

then also $1 - \cos$ AZ$a = \alpha+\beta+\gamma$; $\sin \frac{1}{2}$ AZ$a = \sqrt{\frac{\alpha+\beta+\gamma}{2}}$, and hence $\cos \frac{1}{2}$ BZ$b = \sqrt{\frac{2-(\alpha+\beta+\gamma)}{2}}$ and $\sin \mathrm{AZ}a = \sqrt{(\alpha+\beta+\gamma)(2-\alpha-\beta-\gamma)}$.

Then, by substituting these values it follows

$$\begin{aligned}
\cos \mathrm{A}b &= \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} + \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} && (71)\\
\cos \mathrm{B}a &= \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} - \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} && (72)\\
\cos \mathrm{B}c &= \sqrt{(\alpha+\beta-\gamma)(\gamma+\alpha-\beta)} + \sqrt{(\gamma+\beta-\alpha)(2-\alpha-\beta-\gamma)} && (73)\\
\cos \mathrm{C}b &= \sqrt{(\alpha+\beta-\gamma)(\gamma+\alpha-\beta)} - \sqrt{(\gamma+\beta-\alpha)(2-\alpha-\beta-\gamma)} && (74)\\
\cos \mathrm{C}a &= \sqrt{(\beta+\gamma-\alpha)(\beta+\alpha-\gamma)} + \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} && (75)\\
\cos \mathrm{A}c &= \sqrt{(\beta+\gamma-\alpha)(\beta+\alpha-\gamma)} - \sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)}. && (76)
\end{aligned}$$

Here it is noteworthy that these values are adjusted to our figure, but the other two are mutually exchangible, so that if for cos Ab $\sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)}-\sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)}$ is employed, then it must be stated that

$$\cos \mathrm{B}a = \sqrt{(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)} + \sqrt{(\alpha+\beta-\gamma)(2-\alpha-\beta-\gamma)} \tag{77}$$

from which it is obtained that the products cos Ab cos Ba; cos Bc cos Cb; cos Ca cos Ac, are always given as relations of α, β, γ. But

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 4\alpha\beta - 2(\alpha+\beta-\gamma), \tag{78}$$

the required values substituted in the same way give

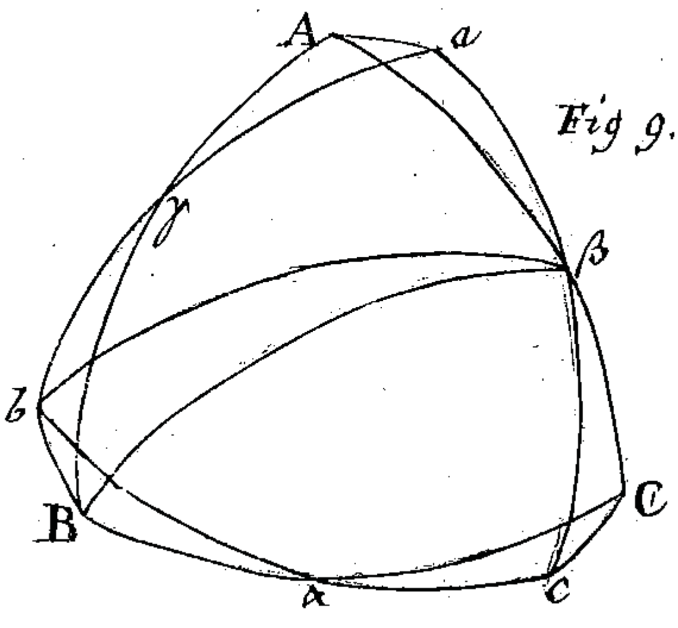
$$\cos \mathrm{B}c + \cos \mathrm{C}b = \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a \quad \text{and} \quad \cos \mathrm{C}a + \cos \mathrm{A}c = \cos \mathrm{C}c \cos \mathrm{A}a - \cos \mathrm{B}b \tag{79}$$

which equations will be shown below to be independent of the consideration of the point Z.

Lemma II.

13. *If in the rotation of a sphere (see Fig. 9) about its fixed centre, the points* A, B, C *their mutual distances being quadrants, arrive at the points* a, b, c *and the intersections of the arcs* AB, ab; AC; ac; BC; bc *are in the points* γ, β, α; *the arcs* Bb=angle Aβa = Cβc; Aa = angle Bαb=Cαc; Cc =angle Aγa=Bγb;

Proof.



Considering the arcs Bβ, bβ, owing to the arcs AB=BC=90°, also B$\beta = 90°$ and the angle BβA=90°, then thanks to $ba = bc = 90°$, likewise $b\beta = 90°$ and the angle $b\beta$A=90°, from which it is deduced that BβA=$b\beta a$ and Bβb=Aβa; but due to B$\beta = b\beta = 90°$, the measure of the angle Bβb is the arc Bb; hence also the arc would be equal to the angle Aβa. By similar plain reasoning it may be shown that Aa=angle Bαb and Cc= angle Aγa.

Lemma III.

14. *In a tetragon (quadrangle)* ACcb, *which is built from four great circles on a spherical surface, if* AC=bc=90°, *then*

$$\cos \mathrm{A}b = \sin \mathrm{AC}c \sin bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{80}$$

Proof.

Considering the arc bC, because cos Ab = sin bC cos ACb, hence due to ACb = ACc − bCc,

$$\cos \mathrm{A}b = \sin b\mathrm{C}(\cos \mathrm{AC}c \cos b\mathrm{C}c + \sin \mathrm{AC}c \sin b\mathrm{C}c) \tag{81}$$

Now it is noted that:

$$\sin b\mathrm{C} \sin b\mathrm{C}c = \sin bc\mathrm{C} \text{ and } \cot b\mathrm{C}c = -\cos \mathrm{C}c \cot bc\mathrm{C}, \text{ hence also}$$
$$\sin \mathrm{C}b \cos b\mathrm{C}c = -\cos \mathrm{C}c \cos bc\mathrm{C},$$

these values substituted would then give

$$\cos \mathrm{A}b = \sin \mathrm{AC}c \sin bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{82}$$

Theorem II.

15. *If in a rotation of a sphere about its fixed centre, the points* A, B, C *having quadrants as the mutual distances, arrive at* a, b, c, *with the arcs of great circle*, Aa, Ab, Ac, Ba, Bb, Bc, Ca, Cb, Cc, *being drawn, the following expression always has to vanish*

$$\begin{aligned}(1-\cos \mathrm{A}a)(1-\cos \mathrm{B}b)(1-\cos \mathrm{C}c) &+ \cos \mathrm{B}c \cos \mathrm{C}b(1-\cos \mathrm{A}a)-\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a \\ &+ \cos \mathrm{A}c \cos \mathrm{C}a(1-\cos \mathrm{B}c)-\cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a \\ &+ \cos \mathrm{C}b \cos \mathrm{B}c(1-\cos \mathrm{C}c) \qquad (83)\end{aligned}$$

Proof.

From the elements of spherical trigonometry it is known that in a triangle Aβa

$$\cos \mathrm{A}\beta a=\cos \mathrm{A}a \sin \beta \mathrm{A} \sin \beta a \mathrm{A}-\cos \beta \mathrm{A}a \cos \beta a \mathrm{A}, \qquad (84)$$

hence owing to Aβa=Bb; βAa=BA$a-90°$; βaA=baA+90°, so that

$$\sin \beta \mathrm{A}a=-\cos \mathrm{BA}a \quad \text{and} \quad \cos \beta \mathrm{A}a=\sin \mathrm{BA}a, \quad \text{as well as} \qquad (85)$$

$$\sin \beta a \mathrm{A}=\cos ba\mathrm{A} \quad \text{and} \quad \cos \beta a \mathrm{A}=-\sin ba\mathrm{A}, \quad \text{giving} \qquad (86)$$

$$\cos \mathrm{B}b=\sin \mathrm{BA}a \sin ba\mathrm{A}-\cos \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A}. \qquad (87)$$

By similar reasoning from consideration of the triangle Aγa one deduces

$$\cos \mathrm{A}\gamma a=\cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}-\cos \mathrm{BA}a \cos ba\mathrm{A} \cos ba\mathrm{A}, \qquad (88)$$

or because of the angle Aγa=arc Cc; cos Cc = cos Aa sin BAa sin baA−cos BAa cos baA. If the first of these equations be multiplied by cos Aa, we obtain

$$\cos \mathrm{A}a \cos \mathrm{B}b=\cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}-\cos^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A}, \qquad (89)$$

hence by subtracting our latter equation gives

$$\cos \mathrm{A}a \cos \mathrm{B}b-\cos \mathrm{C}c=(1-\cos^2 \mathrm{A}a) \cos \mathrm{BA}a \sin ba\mathrm{A}=\sin^2 \mathrm{A}a \cos \mathrm{BA}a \sin ba\mathrm{A} \qquad (90)$$

but as sin Aa cos BAa=cos Ba, and sin Aa cos baA=cos Ab, gives therefore

$$\cos \mathrm{A}a \cos \mathrm{B}b-\cos \mathrm{C}c=\cos \mathrm{A}b \cos \mathrm{B}a. \qquad (91)$$

Furthermore, inspection of the triangle Bγb gives:

$$\cos \mathrm{B}\gamma b=\cos \mathrm{C}c=\sin \mathrm{B}b \sin \mathrm{AB}b \sin ab\mathrm{B}-\cos \mathrm{AB}b \cos ab\mathrm{B}, \qquad (92)$$

as well as of the triangle Bab

$$\cos \mathrm{A}a=\cos \mathrm{B}ab=\cos \mathrm{B}b \sin \mathrm{CB}b \sin cb\mathrm{B}-\cos \mathrm{CB}b \cos cb\mathrm{B}. \qquad (93)$$

However, since ABb=CB$b - 90°$ and abB=cbB+90°, the former is transformed into

$$\cos \mathrm{C}c = \sin \mathrm{CB} \sin cb\mathrm{B} - \cos \mathrm{B}b \cos \mathrm{CB}b \cos cb\mathrm{B}, \tag{94}$$

multiplying this equation by cos Bb, and subtracting from that cos Aa, we get finally:

$$\cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a = (1 - \cos^2 \mathrm{B}b) \cos \mathrm{CB}b \cos cb\mathrm{B} = \sin^2 \mathrm{B}b \cos \mathrm{CB}b \cos cb\mathrm{B} \tag{95}$$

From this one may deduce

$$\cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a = \cos \mathrm{B}c \cos \mathrm{C}b. \tag{96}$$

By similar reasoning one may also show

$$\cos \mathrm{A}a \cos \mathrm{C}c - \cos \mathrm{B}b = \cos \mathrm{A}c \cos \mathrm{C}a. \tag{97}$$

Next, for the quadrilateral ACcb, we have by our third Lemma:

$$\cos \mathrm{A}b = \cos \mathrm{AC}c \sin bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C} \tag{98}$$

and for the quadrilateral aCcB, due to the same Lemma:

$$\cos \mathrm{B}a = \sin \mathrm{BC}c \sin ac\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{BC}c \cos ac\mathrm{C}. \tag{99}$$

Now, because BCc=AC$c - 90°$; acC=bcC−90°, we obtain,

$$\cos \mathrm{B}a = \cos \mathrm{AC}c \cos bc\mathrm{C} - \cos \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C}, \tag{100}$$

and hence

$$\cos \mathrm{A}b \cos \mathrm{C}c + \cos \mathrm{B}a = (1 - \cos^2 \mathrm{C}c) \cos \mathrm{AC}c \cos bc\mathrm{C} = \sin^2 \mathrm{C}c \cos \mathrm{AC}c \cos bc\mathrm{C}, \tag{101}$$

or in virtue of

$$\cos \mathrm{A}c = \sin \mathrm{C}c \cos \mathrm{AC}c \quad \text{and} \quad \cos \mathrm{C}b = \sin \mathrm{C}c \cos bc\mathrm{C}, \tag{102}$$

$$\cos \mathrm{A}b \cos \mathrm{C}c + \cos \mathrm{B}a = \cos \mathrm{A}c \cos \mathrm{C}b. \tag{103}$$

By like reasoning it can be shown that

$$\cos \mathrm{A}b \cos \mathrm{B}c = \cos \mathrm{B}b \cos \mathrm{A}c + \cos \mathrm{C}a, \tag{104}$$

having multiplied the former equation by cos Ba and the latter by cos Ca, and having added the products, leads to:

1. $\cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a = \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos^2 \mathrm{B}a$ (105)
2. $\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a = \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{C}a$ hinc (106)
3. $\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a$
$= \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a$
$= \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + 1 - \cos^2 \mathrm{A}a$ (107)

owing to $\cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = 1$.
Now, introducing this value for $\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a$ in our expression to be rendered zero, the remaining terms of which are expanded in order to become more clear, how each of them destroy each other. It is, then, this expression

$$\begin{array}{clr}
- & \cos \mathrm{A}a + \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c & - \cos \mathrm{B}c \cos \mathrm{C}b + \cos \mathrm{B}c \cos \mathrm{C}b \cos \mathrm{A}a \\
- & \cos \mathrm{B}b + \cos \mathrm{A}a \cos \mathrm{C}c & - \cos \mathrm{A}c \cos \mathrm{C}a + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b \\
- & \cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}b & - \cos \mathrm{A}b \cos \mathrm{B}a + \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c \\
 & & - \cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c \\
 & 1 + \cos^2 \mathrm{A}a & - \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b
\end{array}$$

that is required to be zero. But deleting terms, which manifestly destroy each other, also noting that it has been shown that:

$$\begin{aligned}
- \cos \mathrm{A}a + \cos \mathrm{B}b \cos \mathrm{C}c - \cos \mathrm{B}c \cos \mathrm{C}b &= 0 \\
- \cos \mathrm{B}b + \cos \mathrm{A}a \cos \mathrm{C}c - \cos \mathrm{A}c \cos \mathrm{C}a &= 0 \\
- \cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{A}b \cos \mathrm{B}a &= 0
\end{aligned}$$

Finally also

$$\cos^2 \mathrm{A}a - \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c + \cos \mathrm{A}a \cos \mathrm{B}c \cos \mathrm{C}b = 0 \tag{108}$$

so that now it is clearly established that the proposed formula must be equal to zero.

Another Proof.

16. While the previous proof was geometrical and deduced from a figure, we will now add yet another analytic one, by naturally inviting the equations given in §. 5. By the equation IV we thus deduce,

$$\cos \mathrm{A}a \cos \mathrm{A}b + \cos \mathrm{B}a \cos \mathrm{B}b = - \cos \mathrm{C}a \cos \mathrm{C}b \tag{109}$$

and taking the square,

$$\begin{aligned}
\cos^2 \mathrm{A}a \cos^2 \mathrm{A}b \;+\;& \cos^2 \mathrm{B}a \cos^2 \mathrm{B}b + 2 \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{A}b \cos \mathrm{B}a = \cos^2 \mathrm{C}a \cos^2 \mathrm{C}b \\
=\;& (1 - \cos^2 \mathrm{A}a - \cos^2 \mathrm{B}a)(1 - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}b) \qquad (110)
\end{aligned}$$

by the equations I and II. Having performed the expansion of the latter member, and having omitted terms that delete each other mutually, it follows:

$$\begin{aligned}
& 2 \cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{A}b \cos \mathrm{B}a = 1 - \cos^2 \mathrm{A}a - \cos^2 \mathrm{B}a - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}b \qquad (111) \\
+ \; & \cos^2 \mathrm{A}a \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}b \cos^2 \mathrm{B}a = \cos^2 \mathrm{C}a + \cos^2 \mathrm{C}b - 1 + \cos^2 \mathrm{A}a \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}b \cos^2 \mathrm{B}a.
\end{aligned}$$

Because of (see Fig. 4),

$$\cos^2 \mathrm{C}a + \cos^2 \mathrm{C}b - 1 = - \cos^2 \mathrm{C}c, \tag{112}$$

we have

$$2\cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{A}b \cos \mathrm{B}a = \cos^2 \mathrm{A}a \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}b \cos^2 \mathrm{B}a - \cos^2 \mathrm{C}c \tag{113}$$

as well as

$$\begin{aligned} (\cos \mathrm{A}b \cos \mathrm{B}a - \cos \mathrm{A}a \cos \mathrm{B}b)^2 &= \cos^2 \mathrm{C}c, \quad \text{and} \\ \cos \mathrm{A}b \cos \mathrm{B}a - \cos \mathrm{A}a \cos \mathrm{B}b &= \pm \cos \mathrm{C}c. \end{aligned} \tag{114}$$

While here two values is found for cos Ab cos Ba, the reason for these two values can be explained as such, that if the point c' be conceived as the one being diametrically opposite of c, then it has to be:

$$\cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c \quad \text{and} \quad \cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}b \cos \mathrm{B}b + \cos \mathrm{C}c'. \tag{115}$$

By these analytical principles only it is not possible to decide whether

$$\cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c, \tag{116}$$

must be assumed, but likewise not

$$\cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}a \cos \mathrm{B}b + \cos \mathrm{C}c'; \tag{117}$$

yet whichever it be, the equations previously assumed will take place. In the same manner it can then be shown that

$$\cos \mathrm{A}c \cos \mathrm{C}a = \cos \mathrm{A}a \cos \mathrm{C}c - \cos \mathrm{B}b \quad \text{and} \quad \cos \mathrm{B}c \cos \mathrm{C}b = \cos \mathrm{B}b \cos \mathrm{C}c + \cos \mathrm{A}a. \tag{118}$$

Furthermore it is noted that beyond the equations given in §. 5 also the following will take place:

$$\begin{aligned}
&\text{VII.} && \cos^2 \mathrm{A}a + \cos^2 \mathrm{A}b + \cos^2 \mathrm{A}c = 1, && (119)\\
&\text{VIII.} && \cos^2 \mathrm{B}a + \cos^2 \mathrm{B}b + \cos^2 \mathrm{B}c = 1, && (120)\\
&\text{IX.} && \cos^2 \mathrm{C}a + \cos^2 \mathrm{C}b + \cos^2 \mathrm{C}c = 1, && (121)\\
&\text{X.} && \cos \mathrm{A}a \cos \mathrm{B}a + \cos \mathrm{A}b \cos \mathrm{B}b + \cos \mathrm{A}c \cos \mathrm{B}c = 0, && (122)\\
&\text{XI.} && \cos \mathrm{B}a \cos \mathrm{C}a + \cos \mathrm{B}b \cos \mathrm{C}b + \cos \mathrm{B}c \cos \mathrm{C}c = 0, && (123)\\
&\text{XII.} && \cos \mathrm{A}a \cos \mathrm{C}a + \cos \mathrm{A}b \cos \mathrm{C}b + \cos \mathrm{A}c \cos \mathrm{C}c = 0. && (124)
\end{aligned}$$

Now, by the equation X we have:

$$\begin{aligned} & \cos^2 \mathrm{A}b \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}c \cos^2 \mathrm{B}c + 2\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{B}b \cos \mathrm{A}c \\ = \ & \cos^2 \mathrm{A}a \cos^2 \mathrm{B}a = (1 - \cos^2 \mathrm{A}b - \cos^2 \mathrm{A}c)(1 - \cos^2 \mathrm{B}b - \cos^2 \mathrm{B}c) \end{aligned} \tag{125}$$

on equations VII and VIII. Hence, expansion produces:

$$\begin{aligned} & 2\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{B}b \cos \mathrm{A}c = 1 - \cos^2 \mathrm{A}b - \cos^2 \mathrm{A}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{B}c \\ + \ & \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c + \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c = \cos^2 \mathrm{A}a + \cos^2 \mathrm{B}a - 1 + \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c \\ + \ & \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c = \cos^2 \mathrm{A}b \cos^2 \mathrm{B}c + \cos^2 \mathrm{B}b \cos^2 \mathrm{A}c - \cos^2 \mathrm{C}a, \end{aligned} \tag{126}$$

on equation I. of §. 5. But hence it clearly results:

$$(\cos \mathrm{A}b \cos \mathrm{B}c - \cos \mathrm{B}b \cos \mathrm{A}c)^2 = \cos^2 \mathrm{C}a, \tag{127}$$

and thus

$$\cos \mathrm{A}b \cos \mathrm{B}c - \cos \mathrm{B}b \cos \mathrm{A}c = \pm \cos \mathrm{C}a, \tag{128}$$

where indeed from our analytical equations it is by no means clear, what sign is valid for cos Ca. Nevertheless, stating it to be

$$\cos \mathrm{A}b \cos \mathrm{B}c = \cos \mathrm{B}b \cos \mathrm{A}c + \cos \mathrm{C}a, \tag{129}$$

thus

$$\cos \mathrm{A}b \cos \mathrm{B}c \cos \mathrm{C}a = \cos \mathrm{B}b \cos \mathrm{A}c \cos \mathrm{C}a + \cos^2 \mathrm{C}a. \tag{130}$$

In the same way one can also show that:

$$\cos \mathrm{A}c \cos \mathrm{C}b \cos \mathrm{B}a = \cos \mathrm{C}c \cos \mathrm{A}b \cos \mathrm{B}a + \cos^2 \mathrm{B}a. \tag{131}$$

Then the remaining work will consist in substituting these values in our expression to be rendered zero; this job has already be done in the previous proof, thus it would have been superfluous to repete it here.

17. Before proceeding further, I think that Geometers will not value it as ungrateful if I would give a distinctive account of the reasoning I have conducted in this previous proof. Having then first considered the expression to be rendered zero, I had judged it to be best to elaborate it so that the remaining of its quantities could be clearly expressed by three elements cos Aa, cos Bb, cos Cc. To this end, then, the two quadrilaterals BAab, CAac will be taken to examination, for searching the values of cos Ab, cos Ba themselves (see Fig. 10). In fact, because in the first one there were four known sides Aa, Bb, AB=90° and $ab = 90°$, and in the latter likewise four known sides Aa, Cc, AC=90° and $ac = 90°$, one can now start considering, how the angles BAa, baA, CAa, caA, could be determined by means of Aa, Bb, Cc; These relations are of course used with CAa=BAa − 90° and caA=baA+90°. Thus, by doing this, I have been brought to the following two equations:

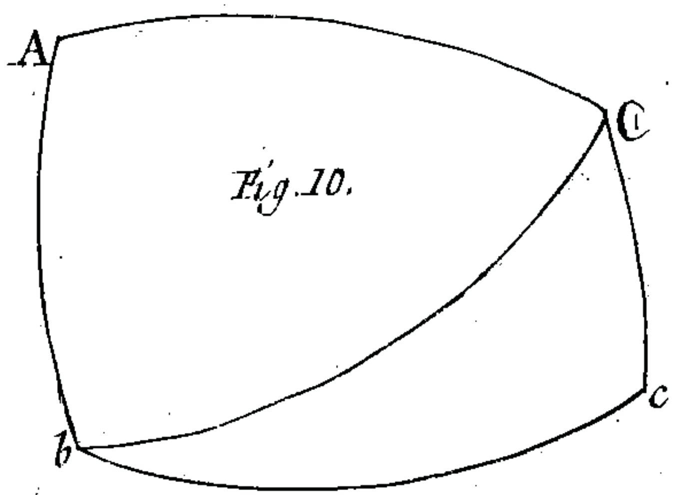
$$\begin{aligned} \cos \mathrm{B}b &= \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} && (132)\\ \cos \mathrm{C}c &= \cos \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A} - \cos \mathrm{BA}a \cos ba\mathrm{A}, && (133)\end{aligned}$$

of which, by the aid of these two others, I obtain

$$\begin{aligned} \cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c &= \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} && (134)\\ \cos \mathrm{B}b - \cos \mathrm{A}a \cos \mathrm{C}c &= \sin^2 \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}. && (135)\end{aligned}$$

If the angle BAa be eliminated from these equations, one obtains an equation, which besides the arcs Aa, Bb, Cc, does not contain but the angle baA, whence by cos Ab = sin Aa cos baA, it will be easy to find an equation, which beyond Aa, Bb, Cc, do not contain but Ab. In fact, because this would lead to biquadratic equation, by paying some attention, I noticed that the work will not be so bulky, because from this equation:

$$\cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c = \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} \tag{136}$$



the value of the product cos Ab cos Ba is determined immediately. By similar reasoning, when determining the products cos Ac cos Ca and cos Bc cos Cb, it now only remains, that also these products: cos Ab cos Bc cos Ca; cos Ac cos Cb cos Ba, would be expressed by the three elements Aa, Bb, Cc; whereby indeed after some trials, I have seen this thoroughly, because the former equals this one

$$\cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{C}a,$$

and the latter this one

$$\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos^2 \mathrm{B}a,$$

and thus both jointly this expression:

$$\begin{aligned}&\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + \cos^2 \mathrm{B}a + \cos^2 \mathrm{C}a = \\ &\cos \mathrm{A}b \cos \mathrm{B}a \cos \mathrm{C}c + \cos \mathrm{A}c \cos \mathrm{C}a \cos \mathrm{B}b + 1 - \cos^2 \mathrm{A}a \qquad (137)\end{aligned}$$

which is now compared such that it can be easily expressed in terms of cosines of Aa, Bb, Cc; nevertheless, this would be of the smallest things to demonstrate, as is evident from the preceding.

18. Let us no see, how to express the values of cos Ab, cos Ba etc. themselves by these elements cos Aa, cos Bb, cos Cc. To this end, we consider the formulas:

$$\begin{aligned}\cos \mathrm{B}b \cos \mathrm{A}a - \cos \mathrm{C}c &= \sin^2 \mathrm{A}a \cos \mathrm{BA}a \cos ba\mathrm{A} \qquad (138)\\ \cos \mathrm{B}b - \cos \mathrm{A}a \cos \mathrm{C}c &= \sin^2 \mathrm{A}a \sin \mathrm{BA}a \sin ba\mathrm{A}, \qquad (139)\end{aligned}$$

from which by thaking the squares we infer:

$$\begin{aligned}\cos^2 \mathrm{B}b \cos^2 \mathrm{A}a + \cos^2 \mathrm{C}c - 2\cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c &= \sin^4 \mathrm{A}a \cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} \quad (140)\\ \cos^2 \mathrm{B}b + \cos^2 \mathrm{A}a \cos^2 \mathrm{C}c - 2\cos \mathrm{A}a \cos \mathrm{B}b \cos \mathrm{C}c &= \sin^4 \mathrm{A}a \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A}. \quad (141)\end{aligned}$$

Thus, taking the difference between these equations gives:

$$\sin^2 \mathrm{A}a(\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b) = \sin^4 \mathrm{A}a(\cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} - \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A}), \qquad (142)$$

and when

$$\cos^2 \mathrm{BA}a \cos^2 ba\mathrm{A} - \sin^2 \mathrm{BA}a \sin^2 ba\mathrm{A} = \cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A} - 1, \tag{143}$$

we get

$$\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b = \sin^2 \mathrm{A}a(\cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A} - 1) \tag{144}$$

and hence

$$\cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b + \sin^2 \mathrm{A}a = \sin^2 \mathrm{A}a(\cos^2 \mathrm{BA}a + \cos^2 ba\mathrm{A}) = \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}a. \tag{145}$$

It is true that

$$\cos \mathrm{A}b \cos \mathrm{B}a = \cos \mathrm{A}a \cos \mathrm{B}b - \cos \mathrm{C}c, \tag{146}$$

hence

$$\begin{aligned}
(\cos \mathrm{A}b + \cos \mathrm{B}a)^2 &= \cos^2 \mathrm{A}b + \cos^2 \mathrm{B}a + 2\cos \mathrm{A}b \cos \mathrm{B}a \\
&= \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b + \sin^2 \mathrm{A}a + 2\cos \mathrm{A}a \cos \mathrm{B}b - 2\cos \mathrm{C}c \\
&= 1 - 2\cos \mathrm{C}c + \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}a + 2\cos \mathrm{A}a \cos \mathrm{B}b \\
&= (1 - \cos \mathrm{C}c)^2 - (\cos \mathrm{B}b - \cos \mathrm{A}a)^2.
\end{aligned} \tag{147}$$

If now as in §. 12 we put $1-\cos\ \mathrm{C}c = 2\gamma$, $1-\cos\ \mathrm{B}b = 2\beta$ and $1-\cos\ \mathrm{A}a = 2\alpha$, it follows

$$(\cos \mathrm{A}b + \cos \mathrm{B}a)^2 = 4\gamma^2 - 4(\alpha - \beta)^2, \tag{148}$$

thus, extracting the root

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 2\sqrt{\gamma^2 - (\alpha - \beta)^2} = 2\sqrt{(\gamma - \alpha + \beta)(\gamma + \alpha - \beta)}. \tag{149}$$

Then, it gives

$$\begin{aligned}
(\cos \mathrm{A}b - \cos \mathrm{B}a)^2 &= 1 + 2\cos \mathrm{C}c + \cos^2 \mathrm{C}c - \cos^2 \mathrm{B}b - \cos^2 \mathrm{A}a - 2\cos \mathrm{A}a \cos \mathrm{B}b \\
&= (1 + \cos \mathrm{C}c)^2 - (\cos \mathrm{A}a + \cos \mathrm{B}b)^2.
\end{aligned} \tag{150}$$

Now since

$$1 + \cos \mathrm{C}c = 2(1 - \gamma) \quad \text{et} \quad \cos \mathrm{A}a + \cos \mathrm{B}b = 2(1 - \alpha - \beta), \tag{151}$$

it gives

$$(\cos \mathrm{A}b - \cos \mathrm{B}a)^2 = 4(1 - \gamma)^2 - 4(1 - \alpha - \beta)^2, \tag{152}$$

and hence

$$\cos \mathrm{A}b + \cos \mathrm{B}a = 2\sqrt{(1 - \gamma)^2 - (1 - \alpha - \beta)^2} = 2\sqrt{(\alpha + \beta - \gamma)(2 - \alpha - \beta - \gamma)}. \tag{153}$$

Thus, we now obtain,

$$\cos \mathrm{A}b = \sqrt{(\gamma - \alpha + \beta)(\gamma + \alpha - \beta)} + \sqrt{(\alpha + \beta - \gamma)(2 - \alpha - \beta - \gamma)} \tag{154}$$

$$\cos \mathrm{B}a = \sqrt{(\gamma - \alpha + \beta)(\gamma + \alpha - \beta)} - \sqrt{(\alpha + \beta - \gamma)(2 - \alpha - \beta - \gamma)}, \tag{155}$$

as well as in the same way:

$$\begin{aligned}
\cos \mathrm{B}c &= \sqrt{(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)}+\sqrt{(\beta+\gamma-\alpha)(2-\alpha-\beta-\gamma)} && (156)\\
\cos \mathrm{C}b &= \sqrt{(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)}-\sqrt{(\beta+\gamma-\alpha)(2-\alpha-\beta-\gamma)} && (157)\\
\cos \mathrm{C}a &= \sqrt{(\beta+\gamma-\alpha)(\beta-\gamma+\alpha)}+\sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)} && (158)\\
\cos \mathrm{A}c &= \sqrt{(\beta+\gamma-\alpha)(\beta-\gamma+\alpha)}-\sqrt{(\alpha+\gamma-\beta)(2-\alpha-\beta-\gamma)}. && (159)
\end{aligned}$$

Everything is as we have found above in §. 12. Beyond these there is nothing else but to observe, owing to $(\cos \mathrm{A}b+\cos \mathrm{B}a)^2=(1-\cos \mathrm{C}c)^2-(\cos \mathrm{B}b-\cos \mathrm{A}a)^2$, giving

$$\cos \mathrm{A}b+\cos \mathrm{B}a = 2\sqrt{\sin^4\frac{\mathrm{C}c}{2}-\sin^2\frac{\mathrm{A}a+\mathrm{B}b}{2}\sin^2\frac{\mathrm{A}a-\mathrm{B}b}{2}}$$

$$= 2\sqrt{(\sin^2\frac{\mathrm{C}c}{2}+\sin\frac{\mathrm{A}a+\mathrm{B}b}{2}\sin\frac{\mathrm{A}a-\mathrm{B}b}{2})(\sin^2\frac{\mathrm{C}c}{2}-\sin\frac{\mathrm{A}a+\mathrm{B}b}{2}\sin\frac{\mathrm{A}a-\mathrm{B}b}{2})}$$

and in the same way

$$\cos \mathrm{A}b-\cos \mathrm{B}a =$$

$$= 2\sqrt{(\cos^2\frac{\mathrm{C}c}{2}+\cos\frac{\mathrm{A}a+\mathrm{B}b}{2}\cos\frac{\mathrm{A}a-\mathrm{B}b}{2})(\cos^2\frac{\mathrm{C}c}{2}-\cos\frac{\mathrm{A}a+\mathrm{B}b}{2}\cos\frac{\mathrm{A}a-\mathrm{B}b}{2})}$$

19. On this occasion it will also be worthwhile noting some relationships, which certainly cannot be seen to belong to our original scope, yet they lend themselves to very much elegance. Now since above we have found that

$$\cos \mathrm{BA}a\cos ba\mathrm{A} = \frac{\cos \mathrm{A}a\cos \mathrm{B}b-\cos \mathrm{C}c}{\sin^2 \mathrm{A}a} \quad \text{and} \quad \sin \mathrm{BA}a\sin ba\mathrm{A} = \frac{\cos \mathrm{B}b-\cos \mathrm{A}a\cos \mathrm{C}c}{\sin^2 \mathrm{A}a}, \tag{160}$$

hence we get

$$\begin{aligned}
\cos(\mathrm{BA}a+ba\mathrm{A}) &= \cos \mathrm{BA}a\cos ba\mathrm{A}-\sin \mathrm{BA}a\sin ba\mathrm{A}\\
&= -(\cos \mathrm{B}b+\cos \mathrm{C}c)\frac{(1-\cos \mathrm{A}a)}{\sin^2 \mathrm{A}a} = -\frac{(\cos \mathrm{B}b+\cos \mathrm{C}c)}{1+\cos \mathrm{A}a}. && (161)
\end{aligned}$$

By similar reasoning it follows

$$\begin{aligned}
\cos(\mathrm{BA}a-ba\mathrm{A}) &= \cos \mathrm{BA}a\cos ba\mathrm{A}+\sin \mathrm{BA}a\sin ba\mathrm{A}\\
&= (\cos \mathrm{B}b-\cos \mathrm{C}c)\frac{(1+\cos \mathrm{A}a)}{\sin^2 \mathrm{A}a} = \frac{\cos \mathrm{B}b-\cos \mathrm{C}c}{1-\cos \mathrm{A}a}. && (162)
\end{aligned}$$

As it is true that

$$\begin{aligned}
(\cos \mathrm{B}c+\cos \mathrm{C}b)^2 &= (1-\cos \mathrm{A}a)^2-(\cos \mathrm{B}b-\cos \mathrm{C}c)^2 \quad \text{and} && (163)\\
(\cos \mathrm{B}c-\cos \mathrm{C}b)^2 &= (1+\cos \mathrm{A}a)^2-(\cos \mathrm{B}b+\cos \mathrm{C}c)^2, && (164)
\end{aligned}$$

we hence conclude

$$(\cos \mathrm{B}c+\cos \mathrm{C}b)^2 = (1-\cos \mathrm{A}a)^2(1-\cos^2(\mathrm{BA}a-ba\mathrm{A})) = (1-\cos \mathrm{A}a)^2 \sin^2(\mathrm{BA}a-ba\mathrm{A}), \quad (165)$$

whence $\cos \mathrm{B}c + \cos \mathrm{C}b = (1 - \cos \mathrm{A}a) \sin(\mathrm{BA}a - ba\mathrm{A})$. Similar expressions can be found for $\cos \mathrm{A}c + \cos \mathrm{C}a$, $\cos \mathrm{C}a - \cos \mathrm{A}c$, but it is not necessary to expose them here.

20. Let Z be the point, which after the rotation of the sphere occupies the same place as it had in the initial state, we can see above, that it has to be:

$$\begin{aligned}
\cos \mathrm{ZA} &= \cos \mathrm{ZA} \cos \mathrm{A}a + \cos \mathrm{ZB} \cos \mathrm{A}b + \cos \mathrm{ZC} \cos \mathrm{A}c \\
\cos \mathrm{ZB} &= \cos \mathrm{ZA} \cos \mathrm{B}a + \cos \mathrm{ZB} \cos \mathrm{B}b + \cos \mathrm{ZC} \cos \mathrm{B}c \\
\cos \mathrm{ZC} &= \cos \mathrm{ZA} \cos \mathrm{C}a + \cos \mathrm{ZB} \cos \mathrm{C}b + \cos \mathrm{ZC} \cos \mathrm{C}c
\end{aligned}$$

In the first of these equations, in place of $\cos \mathrm{A}a$, $\cos \mathrm{A}b$, $\cos \mathrm{A}c$, is introduced their values, which are:

$$\cos \mathrm{A}a = \cos^2 \mathrm{AZ} + \sin^2 \mathrm{AZ} \cos^2 \mathrm{AZ}a \quad (166)$$
$$\cos \mathrm{A}b = \cos \mathrm{AZ} \cos \mathrm{BZ} + \sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZ}b \quad (167)$$
$$\cos \mathrm{A}c = \cos \mathrm{AZ} \cos \mathrm{CZ} + \sin \mathrm{AZ} \sin \mathrm{CZ} \cos \mathrm{AZ}c, \quad (168)$$

and having done this gives

$$\begin{aligned}
\cos \mathrm{AZ} &= \cos^3 \mathrm{AZ} + \cos \mathrm{AZ} \sin^2 \mathrm{AZ} \cos \mathrm{AZ}a + \cos \mathrm{AZ} \cos^2 \mathrm{BZ} \quad (169) \\
&\quad + \cos \mathrm{BZ} \sin \mathrm{AZ} \sin \mathrm{BZ} \cos \mathrm{AZ}b + \cos \mathrm{AZ} \cos^2 \mathrm{CZ} + \cos \mathrm{CZ} \sin \mathrm{AZ} \sin \mathrm{CZ} \cos \mathrm{AZ}c,
\end{aligned}$$

which due to $\cos^2\mathrm{AZ} + \cos^2\mathrm{BZ} + \cos^2\mathrm{CZ} = 1$, reduces into this form:

$$0 = \sin \mathrm{AZ} \cos \mathrm{AZ} \cos \mathrm{AZ}a + \sin \mathrm{BZ} \cos \mathrm{BZ} \cos \mathrm{AZ}b + \sin \mathrm{CZ} \cos \mathrm{CZ} \cos \mathrm{AZ}c, \quad (170)$$

or

$$\sin 2\mathrm{AZ} \cos \mathrm{AZ}a + \sin 2\mathrm{BZ} \cos \mathrm{AZ}b + \sin 2\mathrm{CZ} \cos \mathrm{AZ}c = 0. \quad (171)$$

In the same way we obtain these equations:

$$0 = \sin 2\mathrm{AZ} \cos \mathrm{BZ}a + \sin 2\mathrm{BZ} \cos \mathrm{BZ}b + \sin 2\mathrm{CZ} \cos \mathrm{BZ}c \quad (172)$$
$$0 = \sin 2\mathrm{AZ} \cos \mathrm{CZ}a + \sin 2\mathrm{BZ} \cos \mathrm{CZ}b + \sin 2\mathrm{CZ} \cos \mathrm{CZ}c. \quad (173)$$

21. If A, B, C be points on the surface of the sphere (see Fig. 11), the distances between them being quadrants, and the point O be the one that after the rotation of the sphere is found on the same location as it had in the initial state, and moreover Z be some point on the surface which after the rotation is supposed to arrive at z, it is now asked how to express the distances Az, Bz, Cz by the distances AO, BO, CO as well as the angle ZOz. Thus, the

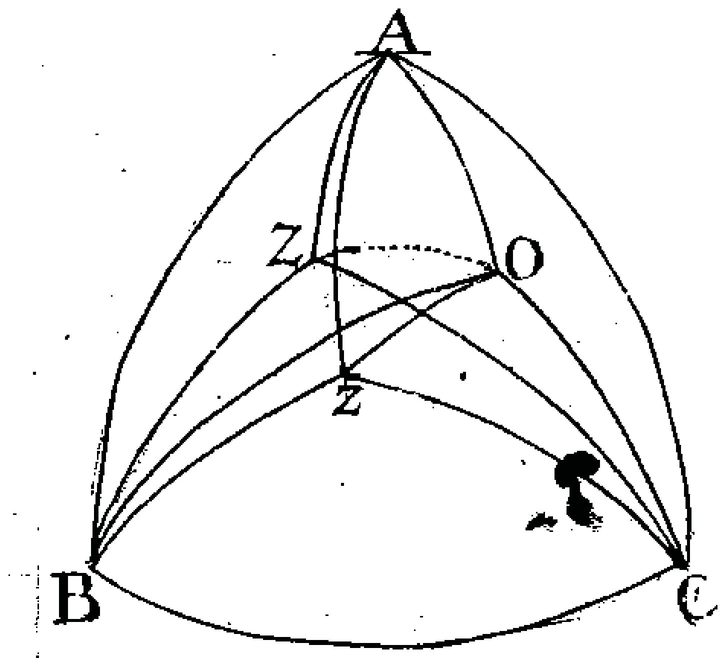


Fig. 11.

point O is then the same as that which we denoted by Z in §. 11; for simplicity, we now put AO=α, BO=β, CO=γ and the angle ZO$z = \Phi$, and as in §. 11:

$$\cos \mathrm{A}a = \cos^2\alpha(1-\cos\Phi)+\cos\Phi \tag{174}$$
$$\cos \mathrm{B}b = \cos^2\beta(1-\cos\Phi)+\cos\Phi \tag{175}$$
$$\cos \mathrm{C}c = \cos^2\gamma(1-\cos\Phi)+\cos\Phi \tag{176}$$
$$\cos \mathrm{A}b = \cos\alpha\cos\beta(1-\cos\Phi)+\cos\gamma\sin\Phi \tag{177}$$
$$\cos \mathrm{B}a = \cos\alpha\cos\beta(1-\cos\Phi)-\cos\gamma\sin\Phi \tag{178}$$
$$\cos \mathrm{B}c = \cos\beta\cos\gamma(1-\cos\Phi)+\cos\alpha\sin\Phi \tag{179}$$
$$\cos \mathrm{C}b = \cos\beta\cos\gamma(1-\cos\Phi)-\cos\alpha\sin\Phi \tag{180}$$
$$\cos \mathrm{C}a = \cos\alpha\cos\gamma(1-\cos\Phi)+\cos\beta\sin\Phi \tag{181}$$
$$\cos \mathrm{A}c = \cos\alpha\cos\gamma(1-\cos\Phi)-\cos\beta\sin\Phi \tag{182}$$

Thus, when

$$\cos\ \mathrm{A}z = \cos\ \mathrm{AZ}\cos\ \mathrm{A}a + \cos\ \mathrm{BZ}\cos\ \mathrm{A}b + \cos\ \mathrm{CZ}\cos\ \mathrm{A}c$$
$$\cos\ \mathrm{B}z = \cos\ \mathrm{AZ}\cos\ \mathrm{B}a + \cos\ \mathrm{BZ}\cos\ \mathrm{B}b + \cos\ \mathrm{CZ}\cos\ \mathrm{B}c$$
$$\cos\ \mathrm{C}z = \cos\ \mathrm{AZ}\cos\ \mathrm{C}a + \cos\ \mathrm{BZ}\cos\ \mathrm{C}b + \cos\ \mathrm{CZ}\cos\ \mathrm{C}c$$

and denoting AZ=ζ; BZ=η; CZ=θ as well as A$z=\zeta'$; B$z=\eta'$; C$z=\theta'$, it follows:

$$\cos\zeta' = \cos\zeta\cos\Phi+(1-\cos\Phi)(\cos\zeta\cos^2\alpha+\cos\eta\cos\alpha\cos\beta+\cos\theta\cos\alpha\cos\gamma)$$

$$
\begin{aligned}
& + \quad \sin\Phi(\cos\eta\cos\gamma - \cos\theta\cos\beta) && (183)\\
\cos\eta' &= \cos\eta\cos\Phi + (1-\cos\Phi)(\cos\eta\cos^2\beta + \cos\zeta\cos\alpha\cos\beta + \cos\theta\cos\beta\cos\gamma)\\
& + \quad \sin\Phi(\cos\theta\cos\alpha - \cos\zeta\cos\gamma) && (184)\\
\cos\theta' &= \cos\theta\cos\Phi + (1-\cos\Phi)(\cos\theta\cos^2\gamma + \cos\zeta\cos\alpha\cos\gamma + \cos\eta\cos\beta\cos\gamma)\\
& + \quad \sin\Phi(\cos\zeta\cos\beta - \cos\eta\cos\alpha) && (185)
\end{aligned}
$$

which also can be expressed as:

$$
\begin{aligned}
\cos\zeta' &= \cos\zeta(\cos^2\alpha + \sin^2\alpha\cos\Phi) + \cos\eta(\cos\alpha\cos\beta(1-\cos\Phi) + \cos\gamma\sin\Phi)\\
& + \cos\theta(\cos\alpha\cos\gamma(1-\cos\Phi) - \cos\beta\sin\Phi) && (186)\\
\cos\eta' &= \cos\eta(\cos^2\beta + \sin^2\beta\cos\Phi) + \cos\zeta(\cos\alpha\cos\beta(1-\cos\Phi) - \cos\gamma\sin\Phi)\\
& + \cos\theta(\cos\beta\cos\gamma(1-\cos\Phi) + \cos\alpha\sin\Phi) && (187)\\
\cos\theta' &= \cos\theta(\cos^2\gamma + \sin^2\gamma\cos\Phi) + \cos\zeta(\cos\alpha\cos\gamma(1-\cos\Phi) + \cos\beta\sin\Phi)\\
& + \cos\eta(\cos\beta\cos\gamma(1-\cos\Phi) - \cos\alpha\sin\Phi). && (188)
\end{aligned}
$$

22. Finally, when we have expressed above in §. 7 the location of the translated point z by the coordinates x', y', z', such that

$$
\begin{aligned}
x' &= f + \mathrm{X}\cos\mathrm{B}b + \mathrm{Y}\cos\mathrm{B}c + \mathrm{Z}\cos\mathrm{B}a && (189)\\
y' &= g + \mathrm{X}\cos\mathrm{C}b + \mathrm{Y}\cos\mathrm{C}c + \mathrm{Z}\cos\mathrm{C}a && (190)\\
z' &= h + \mathrm{X}\cos\mathrm{A}b + \mathrm{Y}\cos\mathrm{A}c + \mathrm{Z}\cos\mathrm{A}a && (191)
\end{aligned}
$$

if here for cos Aa, cos Ab etc., their values be substituted, gives

$$
\begin{aligned}
x' &= f + \mathrm{X}(\cos^2\beta + \sin^2\beta\cos\Phi) + \mathrm{Y}(\cos\beta\cos\gamma(1-\cos\Phi) + \cos\alpha\sin\Phi)\\
& + \mathrm{Z}(\cos\alpha\cos\beta(1-\cos\Phi) - \cos\gamma\sin\Phi) && (192)\\
y' &= g + \mathrm{X}(\cos\beta\cos\gamma(1-\cos\Phi) - \cos\alpha\sin\Phi) + \mathrm{Y}(\cos^2\gamma + \sin^2\gamma\cos\Phi)\\
& + \mathrm{Z}(\cos\alpha\cos\gamma(1-\cos\Phi) + \cos\beta\sin\Phi) && (193)\\
z' &= h + \mathrm{X}(\cos\alpha\cos\beta(1-\cos\Phi) + \cos\gamma\sin\Phi) + \mathrm{Y}(\cos\alpha\cos\gamma(1-\cos\Phi) - \cos\beta\sin\Phi)\\
& + \mathrm{Z}(\cos^2\alpha + \sin^2\alpha\cos\Phi) && (194)
\end{aligned}
$$

In this way the translation of a point I by the ordinates f, g, h, and the distances of the point O into A, B, C, expressed by α, β, γ, as well as the angle of rotation ZOz = Φ, becomes known, and also the translation of any point z can be most easily given.